European University Institute
DEPARTMENT OF ECONOMICS

Asset Prices, Disagreement and Trade Volume

Fabian Schuetze

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

Florence, 09 May 2018

European University Institute
Department of Economics

Asset Prices, Disagreement and Trade Volume

Fabian Schuetze

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

Examining Board

Prof. Ramon Marimon, EUI, Supervisor
Prof. Piero Gottardi, EUI
Prof. Rodolfo Manuelli, Washington University in St. Louis
Prof. Christian Hellwig, University of Toulouse

Researcher declaration to accompany the submission of written work

I Fabian Schuetze certify that I am the author of the work "Asset Prices, Disagreement and Trade Volume" I have presented for examination for the PhD thesis at the European University Institute. I also certify that this is solely my own original work, other than where I have clearly indicated, in this declaration and in the thesis, that it is the work of others.

I warrant that I have obtained all the permissions required for using any material from other copyrighted publications.

I certify that this work complies with the *Code of Ethics in Academic Research* issued by the European University Institute (IUE 332/2/10 (CA 297).

The copyright of this work rests with its author. [quotation from it is permitted, provided that full acknowledgement is made.] This work may not be reproduced without my prior written consent. This authorisation does not, to the best of my knowledge, infringe the rights of any third party.n

Statement of inclusion of previous work (if applicable):

I confirm that chapter 3 was jointly co-authored with João Brogueira and I contributed 50% of the work.

Signature and Date: Fabian Schuetze, 28/04/2018

Contents

Chapter 1

Acknowledgements

I very much enjoyed writing this thesis and I want to thank several people without which I could not have written this thesis. Most importantly, I cannot thank my supervisor Ramon Marimon for all his guidance and support enough. Without all the hours I spend talking to him, working on my thesis would have been much less interesting and my thesis would have been much worse. I also want to thank my supervisor Piero Gottardi. He supported me throughout my entire thesis, was always available for discussions, and helped to identify many weaknesses in my arguments. The discussions I had with Rody Manuelli and Axelle Ferriere during my job-search were very helpful. I also want to thank Christian Hellwig for becoming part of my PhD committee and for sending me so many interesting remarks. I learned most from working so closely with Joao Brogueira — this was certainly the greatest joy while working on my PhD, thank you!

I also want to thank Mathijs Jansen and Matthias Schmidtblaicher. Mathijs always made inspiring comments about every theory I had the fortune to discussed with him and I learned much about econometrics and empirical asset pricing by working with Matthias. I also want to thank Benedikt Dengler, Luis Rojas and Pawel Doligalski, and Lukas Mayr for all the discussions we had. The comments I received from the participants of Ramon's working group were important to refine my work. I also want to thank each member of my cohort at EUI. I learned a lot from working at NYU and LSE and want to thank Thomas Sargent and Wouter den Haan for supporting my visit. Finally, I want to thank Maria Arenas Schmidt-Kessen.

Chapter 2

Summary

In this thesis, I discuss how asset prices are influenced by the decisions of heterogeneous investors. Asset prices are conventionally explained through a representative-investor whose risk-aversion fluctuates or who faces fluctuating fundamental uncertainty. Much progress has been made in describing how such an investor influences prices. Yet, such work poses considerable difficulties. In particular, empirical studies document that trade volume predicts asset prices and investors infer information from prices. Furthermore, the burgeoning household finance literature documents patterns in portfolio allocations across investors. While models with heterogeneous investors can address such shortcomings, more work is needed to understand them. In particular, little is known about how differentially informed investors learn in financial markets and how their opinions affect prices. I describe how disagreement affects volatility in my chapter one of my thesis. I also examine how asymmetrically informed investors learn from prices in chapter two. Finally, Joao Brogueira and I made a theoretical contribution in our published paper which is contained in chapter 3 of my thesis. I describe each chapter briefly.

In chapter one, I provide conditions under which disagreement about dividend growth forecasts amplifies stock market volatility, in line with empirical evidence. In a frictionless economy with two Epstein-Zin investors, I model disagreement as exogenous heterogeneity in beliefs: one investor is pessimistic, the other is not. I show that disagreement amplifies volatility only if investors switch beliefs, that is if an investor is only temporarily optimistic. If instead one investor is permanently pessimistic,

prices are less volatile than dividends, and higher disagreement lowers volatility âĂŤ in contradiction with evidence. Finally, I provide empirical support for switching beliefs among investors, using cross-sectional data from the Survey of Professional Forecasters.

In chapter two I discuss the relationship between trade volume and stock market returns. There is substantial evidence that high trading volume predicts low returns, both in the cross-section but also across several years. To permit information-based trade among asymmetrically informed investors, economic models conventionally include noise traders. However, these models cannot explain the observed relationship between trade and returns. As noise traders demand random quantities, they generate a too volatile trade volume compared to the empirical low-frequency variations. I argue in "Trade Volume, Noise Traders and Information Acquisition with Neural Networks" that neural networks can be used to describe the empirical evidence. I first characterize elementary properties of neural networks. I then show in a model of trade among differentially informed investors, that neural networks permit information inference from prices at arbitrary precision but that information asymmetry can persist even without noise traders. Finally, I outline how such models might be able to explain why trade volume predicts excess years ahead.

Finally, chapter 3 contains a paper I wrote together with Joao Brogueira. Our note presents a proof of the existence of a unique equilibrium in a Lucas (1978) economy when the utility function displays constant relative risk aversion, and the logarithm of dividends follow a normally distributed autoregressive process of order one with positive autocorrelation. We provide restrictions on the coefficient of relative risk aversion, the discount factor and the conditional variance of the consumption process that ensure the existence of a unique equilibrium.

Chapter 3

Disagreement, Changing Beliefs, and Stock Market Volatility

Fabian Schuetze[1]

3.1 Introduction

The influence of disagreement among investors on asset prices has been discussed for a long time. Already Keynes (1936) discussed if stock markets should be closed because disagreement among investors sparks so much volatility to make long-term investments untenable. Savage (1954) examined how disagreement among equally informed investors arises. Similar debates continue until today. The president of the NY-Fed, Dudley (2017), expects higher market volatility due to widespread disagreement. The European Central Bank (2017) warns about its adverse effects on financial stability. This paper investigates how disagreement affects portfolio allocations and asset prices.

Despite the long tradition of discussing the influence of disagreement, asset prices

[1] I am deeply indebted to Piero Gottardi, Rody Manuelli and, in particular, Ramon Marimon for encouragement and discussion. I also want to thank Caio Almeida, Hajorat Bhamra, Joao Brogueira, Wei Cui, Darell Duffie, Erik Eyster, Axelle Ferriere, William Fuchs, Wouter den Haan, Julien Hugonnier, Felipe Iachan, Marcin Kacperczyk, Loukas Karabarbounis, Albert Marcet, and Dimitri Vayanos.

are currently commonly explained through a representative-investor whose effective risk-aversion fluctuates or who faces changing fundamental volatility and growth rates. Although such models replicate asset pricing statistics superbly, they face considerable difficulties in domains related to asset prices. For example, a puzzle in the macro-finance literature is that trade volume predicts excess returns years ahead but representative-investor models do not generate any trade. Cochrane (2016) calls models of heterogeneous investors trading assets to be the "great unresolved problem of financial economics" and my paper attempts some steps in illuminating them. As I document in Section 3.2, increases in disagreement among investors are significantly related to future volatility and expected excess returns even when controlling for measures of time-varying risk-aversion or fundamental volatility. Disagreement thus seems to be related to asset prices in a way that is not captured by the conventional explanations.

I then examine if the empirical evidence can be replicated theoretically. I concentrate on the effects of disagreement on volatility. To understand the findings presented below, I briefly outline how volatility is calculated in general. In any closed exchange economy, stock prices equal dividends relative to the average consumption-wealth ratio. The average is calculated using relative wealth as weights. Suppose there is a positive dividend shock. As investors disagree about expected growth, they hold different portfolios and one investor gains relative wealth. For prices to rise beyond the level indicated by higher dividends, the average consumption-wealth ratio needs to fall — the investor with a lower consumption-wealth ratio needs to gain relative wealth. The empirical evidence that higher disagreement is associated with higher volatility thus constrains the role of disagreement in asset pricing models.

As I show in Section 3.3, constant disagreement, in which one investor forecasts average dividend growth correctly whilst the other is pessimistic, is not compatible with the empirical evidence.[2] The more optimistic investor shoulders more aggregate risk, and, as excess returns are positive, reaps higher portfolio returns. To avoid that

[2] I assume the investor with subjective beliefs is pessimistic, in accordance with the literature analysing the role of expectations for asset prices (the disaster-risk and ambiguity-aversion literature). To illustrate, suppose the average dividend growth rate is 2%. The investor with accurate (pessimistic) beliefs expects a growth rate of 2% (1.5%).

this investor eventually owns (almost) all wealth and determines asset prices alone, both investors need to have identical wealth growth rates. As wealth growth equals the portfolio returns minus the consumption-wealth ratio, the pessimistic investor compensates for her relatively poor portfolio returns through a lower consumption-wealth ratio. In sum, as the optimistic investor is more exposed to aggregate risk she gains relative wealth in response to positive dividend shocks and has a higher consumption-wealth ratio, which lowers volatility.

I provide conditions in Section 3.4 so that disagreement is compatible with the empirical evidence. Suppose one investor is temporarily more optimistic. Then, she temporarily reaps higher portfolio returns and gains relative wealth in response to dividend shocks. Whether she has a lower consumption-wealth ratio as well depends on how disagreement affects the consumption-wealth ratios of both investors. I characterize in an approximate log-linear solution three channels through which disagreement influences the consumption-wealth ratios:

1. *Portfolio Channel:* As market prices reflect average beliefs, each investor thinks she can form a more profitable portfolio than the other. If an investor is more pessimistic (optimistic) than the average belief reflected in market prices, she shorts (holds a long position in) the asset.

2. *Expected Returns Channel:* In contrast to the *Portfolio Channel*, the *Expected Returns Channel* depends on the type of disagreement: For a fixed portfolio, a more optimistic (pessimistic) investor expects higher (lower) returns from holding the asset.

3. *Utility Distortion channel:* The previous two channels affect the consumption-wealth ratio indirectly through expected portfolio returns. Subjective beliefs affect the consumption-wealth ratio also directly: Expected utilities under subjective beliefs can be expressed as expectations of a distorted utility under objective beliefs. If the investor expects high (low) dividend growth, her consumption-wealth ratio decreases (increases).

The *portfolio channel* (if the elasticity of intertemporal substitution exceeds one, which I assume) biases each investor's consumption-wealth ratio downwards, irrespective of whether she is optimistic or pessimistic. To, however, guarantee that the optimistic investor has a particularly low consumption-wealth ratio, the *portfolio channel* must not be too influential. Proposition 2 states that if risk aversion exceeds disagreement by some margin, disagreement amplifies volatility, as investors do not re-structure their portfolios too aggressively. Taken together, Section 3.3 and Section 3.4 document that disagreement amplifies volatility only if investors switch beliefs, that is if an investor is only temporarily optimistic.

I examine the empirical predictions of the model discussed above in Section 3.5 numerically. The main finding is that higher disagreement is indeed associated with higher volatility and expected excess returns: An increase in disagreement by one standard deviation leads to an increase of volatility (expected excess return) by one-third (one-tenth) of a standard deviation. The model broadly matches historical asset prices. The annual risk premium is 7.4% and its volatility is 15%. The average risk-free rate is 2.3% with a volatility of 1.7%.

As argued, disagreement amplifies volatility only if investors are temporarily optimistic. Using data from the Survey of Professional Forecasters, I show in Section 3.6 that such expectations are compatible with the empirical evidence. If a forecaster's expectation about next quarters profit growth exceeds the median forecast, she switches to a negative forecast (a forecast below the median) in the subsequent period with a probability of 40%. Furthermore, she switches to a negative forecast over the next two (three) periods at least once with a probability of 60% (70%).

Finally, I compare my paper to the literature in Section 3.7 but summarize my contribution below. The literature explaining the effects of disagreement on volatility relies on different auxiliary assumptions. One common assumption are short-sale constraints which imply that prices reflect only the views of optimistic investors. Another assumption are finite-lives of investors which ensures that all investors have enough wealth to trade assets irrespective of their past portfolio returns. Without these assumptions, the literature does not generate a non-degenerate wealth distribution in which disagreement amplifies volatility. I exploit the analysis of Borovicka (2015) and use Epstein-Zin preferences to avoid a non-degenerate wealth distribution when infinitely-lived investors with heterogeneous beliefs trade assets and examine market volatility and excess returns in a frictionless economy. The first main finding of the paper in Section 3.3 is that disagreement lowers volatility when investors do not switch beliefs, which I show to be in contradiction with empirical evidence I provide in Section 3.2, The second main finding of the paper is that I provide conditions under which disagreement amplifies volatility when investors switch beliefs in Section 3.4 and Section 3.5 Finally, I provide empirical support for switching beliefs in Section 3.6.

3.2 Empirical Motivation

In this section, I present the empirical motivation for studying the effects of disagreement on stock market volatility and expected excess returns. I show that higher disagreement is associated with higher volatility and expected returns, even when controlling for measures of time-varying risk aversion or business cycle conditions. I measure disagreement as the standard deviation among professional forecasters' expected aggregate profit growth. I use data from the Survey of Professional Forecasters because of the long duration of the series. I show that an increase of disagreement by one standard deviation about next quarter's profit growth is associated with an increase by one-fifth (one-third) of a standard deviation of next quarter's volatility (expected excess returns). All estimates are significant at the 5 % level, even when controlling for measures of time-varying risk aversion, fundamental volatility and business cycle conditions. I describe the individual series in Section 3.2.1 and the regression results in Section 3.2.2.

3.2.1 Data Description

I describe the data for individual forecasts, followed by excess returns and close with volatility. All data is quarterly from 1969Q4–2015Q4 which spans the availability of forecasting data[3].

[3]The data and associated computer programs can be downloaded from my website.

Disagreement In the following, I analyse forecasts for dividend growth and its relationship with the actual series. Dividend growth, labelled D_t, is the one-period growth rate of corporate dividends. Every quarter, private sector forecasters submit beliefs about corporate after-tax profits. I denote the expectation of dividend growth as the growth rate of these forecasts. I label as F_t the median forecast and by dis_t, the standard deviation among forecasters. Panel A of Table 3.1 contains selected summary statistics. The first column of Table 3.1 shows summary statistic for historical

Table 3.1: **Summary Statistics**

	D_t	F_t	dis_t	$RMSFE$	vol_t	μ_R
Panel A Univariate Summary Statistics						
Mean	1.93%	1.41%	2.89%	4.69%	6.87%	1.16%
Stdv	4.84%	2.07%	1.282%	3.77%	3.5%	0.66%
Autocorr.	-0.24	0.67	0.42	0.68	0.55	0.98
Panel B Correlation						
D_t	1.00	0.11	-0.15	-0.10	-0.25	-0.01
F_t		1.00	-0.11	-0.14	-0.24	0.01
dis_t			1.00	0.32	0.23	0.38
$RMSFE_t$				1.00	0.13	-0.00
vol_t					1.00	-0.16
$\mu_{R,t}$						1.00

The table states summary statistic for the dividend growth, D_t, the median of the dividend growth forecasts, F_t, the disagreement among forecasters dis_t, and the root-mean-squared forecast error, $RMSFE$. The top panel shows the estimates of the mean, standard deviation and autocorrelation. The bottom panel shows cross-correlations of the variables.

dividend growth. Dividends grew, on average, at close to 2%. The standard deviation is more than twice its mean, and the autocorrelation is negative. On average, the median forecast is slightly lower than the actual mean, as seen in the second column. The median forecasts do not exhibit much volatility and evolve smoothly. That forecasts are smoother than the forecasted data has also been observed by Piazzesi et al. (2009). Moments about disagreement are stated in the third column. On average, the standard deviation of forecasters is almost 2%. Disagreement is not very volatile, but spikes occasionally and is moderately persistent. Finally, the fourth column reports the median root-mean-squared forecast error. Panel B states the correlations across the series. Most notably, the median forecast is positively correlated with the actual dividend growth and disagreement among forecasters is low whenever actual dividend growth is high. Such counter-cyclicality among forecasts has been mentioned extensively, see, for example, Van Nieuwerburgh et al. (2006) or Veronesi (1999). Furthermore, the median forecast error is low whenever growth is high and high disagreement usually entails larger errors.

Excess Returns In this paragraph, I describe how I estimated expected excess returns. I used quarterly data from 1969Q1–2016Q4 for real stock returns, the risk-free interest rate, and the dividend-price ratio. The data comes from Robert Shiller's website and has been used first in his book "Market Volatility", Robert J Shiller (1992). Expected excess returns are the fitted values of regressing excess returns at time $t+1$ on a constant and the (log) dividend-price ratio at time t. Using the dividend-price ratio as forecast for excess returns was pioneered by Campbell and Robert J. Shiller (1988) and Fama et al. (1988). The univariate statistics for quarterly data and their correlations with the other variables can be seen in the last column of Table 3.1.

Volatility I describe how I estimated stock market volatility in this paragraph. The data comes from Yahoo Finance and contains daily stock market prices for 1969Q4–2016Q4. Volatility is calculated, in general, by estimating from daily volatility a measure of raw quarterly volatility. As this estimate is a noisy measure of the underlying fundamental volatility, I filtered the data. In more detail, I calculated raw quarterly volatility as the squared sum of the demeaned log price changes within the quarter. This measure of raw volatility is called realized volatility and was calculated first by Schwert (1989). I transformed the estimate into logs to reduce the kurtosis and skewness of the data. Bollerslev et al. (2016) argue that an $ARMA(1,1)$ model for realized volatility delivers good estimates of underlying volatility in the presence of such noisy raw volatility. For robustness purposes, I also estimated the underlying volatility with an $AR(1)$ process and by using the seasonal de-trending procedure of Gallant et al. (1992). The estimates for the underlying volatility were similar and are omitted. The univariate statistics and the correlation of volatility are stated in the fifth column of Table 3.1. A striking result is that the estimate for volatility and the expected excess return are negatively correlated, with an estimated value of -0.16. In theory, one expects the correlation to be positive. However, econometric evidence for a positive correlation is difficult to obtain, see French et al. (1987).

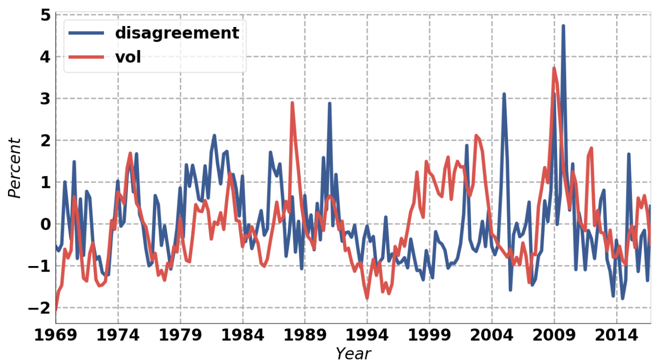
Control Variables: Time-Varying Risk Aversion, GDP Growth and Fundamental Volatility I used several variables to check if the correlation between disagreement, volatility and expected excess returns are due to a correlation with an omitted variable. Disagreement, volatility, and expected excess returns are all countercyclical. I thus control for GDP growth in the regressions in Section 3.2.2. Furthermore, Campbell and Cochrane (1999) explain asset prices through time-varying risk aversion. I calculated a measure for time-varying risk aversion (the surplus consumption-ration) following Cochrane (2011). The variable has a correlation of 0.53 (0.00) with expected excess returns (stock market volatility). Finally, Bansal et al. (2004) explain asset prices through fluctuating conditional volatility. I calculated the volatility of consumption growth as the fitted values of an $AR(1)$ process of the absolute value of the residual of an $AR(1)$ process of GDP growth. This estimate is motivated by Schwert (1989) who estimated the conditional volatility of monthly production growth similarly. The correlation between fundamental volatility and expected excess returns (stock market volatility) is 0.3 (0.09).

3.2.2 Facts

In this subsection, I discuss the empirical evidence about the relationship between disagreement among professional forecasters, expected excess returns and volatility. I show that an increase of disagreement by one standard deviation about next quarter's dividend growth is associated with an increase by one-fifth (one-third) of a standard deviation of next quarter's volatility (expected excess returns). The estimates are significant at the 5 % level, even when controlling for measures of time-varying risk aversion, fundamental volatility and business cycle conditions.

Fact 1: Higher Disagreement is associated with higher Volatility In this paragraph, I consider the relationship between volatility and disagreement among forecasters. The time series for volatility and disagreement are plotted in Figure 3.1. I regress volatility at time t on disagreement about time t dividend growth (which is

Figure 3.1: **Disagreement and Volatility**



The figure shows the standard deviation of the forecasts for next quarters aggregate profit growth and the estimated underlying volatility. At any point in time, the data refers to the forecast *for* that time and the estimate of the underlying volatility *at* that time. The plot is standardized to facilitate inspecting the co-movement between the series.

measured at time $t-1$). Some regressions allow for covariates. In formal terms:

$$\hat{vol}_t = \beta_0 + \beta_1 disagreement_{t|t-1} + \beta_2 control_t + \epsilon_t \tag{3.1}$$

The estimates of this regression are presented in Panel A of Table 3.2. An increase in disagreement by one standard deviation is associated with an increase of volatility by 0.23 standard deviations. The estimate is significant at the 5% level and remains significant when I control for GDP growth (regression number 2), time-varying risk aversion (regression number 3), or fluctuating conditional volatility (row number 4). Interestingly, the estimate for time-varying risk aversion is not significantly different from 0.

The relationship between disagreement and stock market volatility has been examined in great detail. Most prominently, Bloom (2009) showed that uncertainty is associated with higher stock market volatility; one indicator for uncertainty was disagreement among forecasters. Following the work of Bloom, many researchers studied the implications for uncertainty in financial markets. Brogaard et al. (2015) document that uncertainty about governmental policy leads to higher market volatility. Jurado et al. (2015) studied different measures of uncertainty, among them disagreement of forecasters, and showed that they are closely related. Carlin et al. (2014) use

Table 3.2: **Regression Results**

#	$Disagreement_{t\|t-1}$	GDP_t	$Risk_t$	$GDP-Vol_t$	R^2
		Panel A: Volatility			
1	**0.23** (2.81)				0.05
2	**0.19** (2.5)	**-0.22** (-2.89)			0.09
3	**0.25** (2.90)		-0.08 (-0.77)		0.05
4	**0.22** (2.84)			0.06 (0.77)	0.05
		Panel B: Expected Excess Returns			
1	**0.38** (3.69)				0.14
2	**0.4** (3.67)	0.6 (0.74)			0.14
3	**0.25** (2.84)		**0.47** (6.2)		0.34
4	**0.34** (3.59)			**0.25** (4.03)	0.2

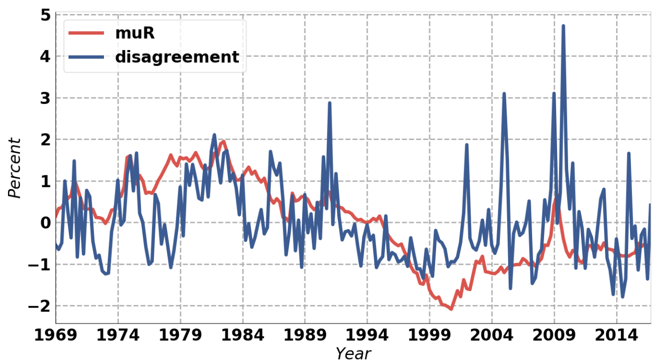
Standard errors are calculated with 4 HAC White (1980) correction. The data is standardized to facilitate comparison between regressors. The last column contains the adjusted R^2. Estimates which are significant at the five-percent level are printed in bold-face. T-values are in parenthesis.

micro-data about expected repayment in an Mortage-Backed-Security market and document that disagreement among forecasters is associated with higher volatility.

Fact 2: Expected Returns increase with larger Disagreement The time series for expected excess returns and disagreement is shown in Figure 3.2. The estimation results are shown in Panel B of Table 3.2. An increase in disagreement by one standard deviation is associated with an increase of expected returns by roughly 0.3 standard deviations and remains significant in the presence of the four control variables.

Expected returns have long been studied in relation to individual forecasts. I follow Greenwood et al. (2014) and related econometric expectations of excess returns, cal-

Figure 3.2: **Disagreement and Stock Market Volatility**



The figure shows the standard deviation of the forecasts for next quarters aggregate profit growth and the estimated underlying volatility. At any point in time, the data refers to the forecast *for* that time and the estimate of the underlying volatility *at* that time. The plot is standardized to facilitate inspecting the co-movement between the series.

culated by using the price-dividend ratio as forecaster, to expectations of individual forecasters. Carlin et al. (2014) document that excess returns in the Mortage-Backed-Security market increase with disagreement about repayment rates. Anderson et al. (2009) use the same data as I use, but a more elaborate procedure to uncover disagreement among forecasters, to show that disagreement is a statistically significant forecaster for excess returns. While these papers document a positive relationship between returns and disagreement, several other papers document a negative relationship. For instance, Diether et al. (2002) sort stock according to the extent of disagreement about their returns and show that average returns are decreasing in disagreement. The finding validates Miller (1977) who suggests stock prices reflect the opinion of optimistic investors when short-sales are prohibited: Optimistic investors' bid-up prices of stocks leading to high contemporaneous but low future returns. Yu (2011) indeed documents that disagreement is positively associated with high contemporaneous but low future aggregate returns. This pattern is especially strong for growth stocks (the price of these stocks is sensitive to changes in discount rates, Campbell and Vuolteenaho (2004)). Sadka et al. (2007) confirm the negative empirical relationship between disagreement and future returns and document furthermore that such future returns are particularly low for stock which are difficult to trade. In contrast to the papers above, Buraschi et al. (2013) studies how a stock return's exposure to disagreement is compensated. While Diether et al. (2002) and Yu (2011) sort stocks according to disagreement, Buraschi et al. (2013) sorts stocks according to how sensitive their returns are to changes of disagreement. The authors document that returns increase for stocks which are more sensitive to changes in disagreement. In contrast to the literature cited above, I estimate the relationship between expected excess returns (instead of excess returns) and disagreement. As a robustness check[4], I also sort excess returns according to the disagreement measure I use and show that disagreement is not systematically related to future excess returns. I think, the data I use does not contain a significantly negative relationship between excess returns and disagreement because I measure disagreement based on forecasts about aggregate corporate profit growth and do not construct a disagreement measure as a weighted sum of disagreement about each stock, as done by Yu (2011).

3.3 A Frictionless Economy with two Types

I document in this section that the empirical evidence that higher disagreement is associated with higher volatility and expected returns cannot be replicated in a frictionless economy with two types of investors. Instead, the return volatility is below (i) dividend volatility, (ii) the return volatility when beliefs are homogeneous and that (iii) higher disagreement causes lower volatility. The finding is obtained as follows: The model is set in continuous-time with Epstein-Zin investors. One investor forecasts mean dividend growth accurately, the other expects lower growth rates. The investors trade a risky-asset in positive net supply and borrow and lend from each other. As in any closed economy, stock prices equal dividends relative to the average consumption-wealth ratio. The average is calculated using relative wealth as weights. Suppose there is a positive dividend shock. As the investors disagree about expected growth, they hold different portfolios and one investor gains relative wealth. For prices to rise beyond the level indicated by higher dividends, the average consumption-wealth ratio needs to fall — the investors with a lower consumption-wealth ratio needs to gain relative wealth. However, instead, the investor with a higher consumption-wealth ratio gains relative wealth in response to a

[4] whose results can be seen on the publically available computer program on my website

dividend shock, lowering volatility: The optimistic investor shoulders more aggregate risk, and, as excess returns are positive, reaps higher portfolio returns. The individual wealth growth rates need to be identical for both investors to retain a wealth share bounded away from zero. As wealth growth equals the portfolio returns minus the consumption-wealth ratio, the pessimistic investor compensates for her relatively poor portfolio return through a lower consumption-wealth ratio. I document in numerical simulations that whenever the investor with pessimistic beliefs forecasts a 12% lower annual dividend growth rate than the investor with accurate beliefs, the return volatility is 10% below the dividend volatility. When disagreement doubles, return volatility is 13% lower than dividend volatility. I begin by describing the economy in detail.

Dividend Growth The model is set in continuous time and dividends evolve as:

$$\frac{dD}{D} = \mu_D dt + \sigma_D dW_t \tag{3.2}$$

with W_t being a standard Brownian motion, σ_D the instantaneous standard deviation of dividend growth, and μ_D its mean.

Investors' Beliefs There are two types of investors in the standard economy. Type "A" expects dividends to evolve according to the actual process, (3.2). The other type thinks dividends follow:

$$\frac{dD}{D} = \tilde{\mu}_D dt + \sigma_D d\tilde{W}. \tag{3.3}$$

The investor presumes dividends grow on average with $\tilde{\mu}_D$ instead of μ_D. While she believes $\tilde{W}$ has a mean of zero, it actually contains a drift term:

$$d\tilde{W} = \frac{\mu_D - \tilde{\mu}_D}{\sigma_D} dt + dW \equiv edt + dW$$

The drift term guarantees that the dividend realizations according to the subjective process (3.3) equal the realization of the actual process (3.2). Although there are infinitely many investors of each type, each investor of one type makes the same choice. Thus, I only differentiate between types.

Preferences Each investor has Epstein-Zin preferences, characterized in continuous time by Duffie et al. (1992):

$$V_t = E \int_t^\infty f(c_s, V_s) ds, \tag{3.4}$$

with V_t being the current value and f:

$$f(c, V) = \frac{\varphi}{\varphi - 1}\beta \left[c^{1-1/\varphi}((1-\gamma)V)^{1+\dfrac{1-\varphi}{(1-\gamma)\varphi}} - (1-\gamma)V \right].$$

The coefficient of relative risk aversion is γ, the Elasticity of Intertemporal Substitution (EIS) φ, and the time discount factor is β. When $\varphi = \gamma^{-1}$, preferences resemble CRRA utility.

Traded Assets and Budget Constraint The dynamic budget constraint for wealth w faced by an type "A" investor is:

$$dw_t = w_t \left\{ \left[(1-\theta_t) r_t + \theta_t (\mu_{R,t} + r_t) \right] dt + \theta_t \sigma_{R,t} dW \right\} - c_t dt \tag{3.5}$$

The budget constraint can be interpreted as follows: The part in curly brackets describe wealth gains from the investor's portfolio and $-c_t$ wealth losses due to consumption. The investor invests a fraction $1 - \theta_t$ of her wealth into a riskless asset and receives, with certainty, a payoff of rdt. The other fraction θ is invested in a risky asset which delivers, in expectations, a return of $\mu_{R,t} + r_t$, the excess return plus the return on riskless assets. However, as dividend payments are risky, returns fluctuate by $\sigma_{R,t} dW$, the conditional volatility of returns. The budget constraint for an investor of type "B" is identical, except that wealth, consumption and the fraction of wealth invested in risky assets are denoted by $\tilde{w}_t$, $\tilde{c}_t$ and $\tilde{\theta}_t$ respectively.

Optimizing A type "A" investors maximizes her life time utility (3.4) by choosing the optimal fraction of wealth invested in risky assets θ and her consumption c:

$$\sup_{\theta_i, c} \mathbb{E} \int_0^\infty f(c_s, v_s) ds \tag{3.6}$$

subject to (3.5). In contrast, a type "B" investor maximizes her expected utility, subject to her budget constraint, under her subjective expectations. Her expectation operator is denoted as $\tilde{\mathbb{E}}$. Girsanov's theorem implies that the maximization under the subjective and true measure are related as:

$$\sup_{\theta,c} \tilde{\mathbb{E}} \int_0^\infty f(c_s, v_s) ds = \sup_{\theta,c} \mathbb{E} \int_0^\infty M(s) f(c_s, v_s) ds \tag{3.7}$$

with $M(s)$ being the Radon-Nikodym derivative:

$$M(t) = \exp\left(-\int_0^t e_s dW_s - \frac{1}{2}\int_0^t {e_s}^2 ds\right) \tag{3.8}$$

The Value Function The value function for solving (3.4) can be written as:

$$V_t = \beta^{-\dfrac{\varphi(1-\gamma)}{1-\varphi}} cw_t^{\dfrac{1-\varphi}{1-\gamma}} \frac{w_t^{1-\gamma}}{1-\gamma} \tag{3.9}$$

with cw being the consumption-wealth ratio, $cw = c_t/w_t$. Whenever $cw^{(1-\varphi)/(1-\gamma)}$ is high, utility is high for a given level of net worth w_t. The consumption-wealth ratio evolves as:

$$dcw_t = \mu_{cw_t} dt + \sigma_{cw_t} dW \tag{3.10}$$

Equilibrium Denote by P_t the price of a claim on the dividends of the risky asset. The equilibrium in this economy is defined as:

Definition 1. *A competitive equilibrium is a set of stochastic processes for the risk free interest rate* $\{r_t\}_0^\infty$, *the excess return* $\{\mu_{R,t}\}_0^\infty$, *its volatility* $\{\sigma_{R,t}\}_0^\infty$ *the consumption and asset holdings for each agent* $\{c_t, \tilde{c}_t\}_0^\infty$, $\left\{\theta_t, \tilde{\theta}_t\right\}_0^\infty$ *and a process for relative wealth* $\{\lambda_t\}_0^\infty$ *such that:*

i The investors choose $c_t(\tilde{c}_t)$, $\theta_t(\tilde{\theta}_t)$ *to maximize (3.4) subject to the budget constraint (3.5), taking processes for* r, σ_R *and* μ_R *as given.*

ii Markets clear, i.e.:

$$c_t + \tilde{c}_t = D_t$$
$$w_t\theta_t + \tilde{w}_t\tilde{\theta}_t = P_t$$

By Walras' law, the market for risk-free debt clear automatically. I endow both investors at time zero with the same number of shares, which influences how relative wealth evolves. However, I concentrate my analysis around the stochastic steady state of the relative wealth distribution which is independent of the initial allocation of stocks.

Solving the Equilibrium In equilibrium, all variables are Markovian functions of relative wealth, which is:

$$\lambda = \frac{w}{w + \tilde{w}} \tag{3.11}$$

In equilibrium, the relative wealth share evolves as the Ito process

$$d\lambda = \mu_\lambda(\lambda)dt + \sigma(\lambda)dW. \tag{3.12}$$

The sign of $\sigma_\lambda(\lambda)$ represents which investor gains relative wealth after a positive shock dW: If $\sigma_\lambda(\lambda) > 0$, investor "A" gains. I now describe asset prices.

3.3.1 Asset Prices

The equilibrium predictions are stated in the following Proposition

Proposition 1. *Suppose there exists an equilibrium. Then, the conditional excess return and its variance are:*

$$\mu_R(\lambda) = \gamma\sigma_R(\lambda)^2 + (1-\lambda)e\sigma_R(\lambda) - \frac{1-\gamma}{1-\varphi}\overline{\sigma_{cw}}(\lambda)\sigma_R(\lambda) \tag{3.13}$$

$$\sigma_R(\lambda) = \sigma_D - \frac{\Delta cw(\lambda) + \overline{cw_\lambda}(\lambda)}{\overline{cw}(\lambda)}\sigma_\lambda(\lambda) \tag{3.14}$$

with $\overline{\sigma_{cw}}(\lambda) = \lambda\sigma_{cw} + (1-\lambda)\tilde{\sigma}_{\tilde{cw}}$*,* $\overline{cw}(\lambda)$*, and* $\overline{cw_\lambda}(\lambda)$ *being similarly defined weighted averages. The risk-free interest rate is defined in the Appendix Section A*

I describe at first the volatility and then excess returns.

The conditional return volatility The price of a claim on the risky asset equals the dividend relative to the average consumption-wealth ratio:

$$P = \frac{D}{\lambda cw(\lambda) + (1-\lambda)\tilde{cw}(\lambda)}. \tag{3.15}$$

A shock dW changes the price due to a *fundamental effect*, a *change in allocation*, and a *change in valuation*: The shock changes dividends by σ_D and the numerator of the pricing equation fluctuates; the *fundamental effect*. The shock also changes relative wealth by $\sigma_\lambda(\lambda)$ which affects the average consumption-wealth ratio $\overline{cw} = \lambda cw(\lambda) + (1-\lambda)\tilde{cw}(\lambda)$ in two ways. First, fix $cw(\lambda)$ and $\tilde{cw}(\lambda)$. Then, $\overline{cw}(\lambda)$

changes by $cw(\lambda) - \tilde{cw}(\lambda) = \Delta cw(\lambda)$; a *change in allocation*. Second, because the consumption-wealth ratios are functions of λ, $\overline{cw}(\lambda)$ changes by $\overline{cw_\lambda}(\lambda)$; a *change in valuation*. The *change in allocation* and *change in valuation* affect the denominator of the pricing equation (3.15). Suppose there is a positive dividend shock. Then D increases which exerts upward pressure on the price. If the average consumption-wealth ratio falls, prices increase even more. The derivative of the price with respect to the shock dW equals the volatility of returns:

$$\sigma_R(\lambda) = \underbrace{\sigma_D}_{\text{Fundamental}} - \underbrace{\frac{\Delta cw(\lambda)}{\overline{cw}(\lambda)}\sigma_\lambda(\lambda)}_{\text{Change in Allocation}} - \underbrace{\frac{\overline{cw_\lambda}(\lambda)}{\overline{cw}(\lambda)}\sigma_\lambda(\lambda)}_{\text{Change in Valuation}} \tag{3.16}$$

Numerical simulations across a wide range of parameter values documented that the *change in valuation* term is of neglegible size. I thus approximate return volatility as:

$$\sigma_R(\lambda) \approx \underbrace{\sigma_D}_{\text{Fundamental}} - \underbrace{\frac{\Delta cw(\lambda)}{\overline{cw}(\lambda)}\sigma_\lambda(\lambda)}_{\text{Change in Allocation}}$$

The sign of $\sigma_\lambda(\lambda)$ represents which investor gains relative wealth after a positive dividend shock: If $\sigma_\lambda(\lambda) > 0$, investor "A" gains. The volatility of returns $\sigma_R(\lambda)$ exceeds the volatility of dividends σ_D if the investor gaining from the shock has a lower consumption-wealth ratio, i.e. $\Delta cw(\lambda)\sigma_\lambda(\lambda) < 0$. I document below that investor "A" gains from a positive dividend shock but that she has a lower consumption-wealth ratio. Thus, the return volatility is below the volatility of dividends, contrary to the empirical evidence conveyed by Robert J. Shiller (1981). Before examining the return volatility in detail, I describe the excess returns.

Excess Returns The excess return has three components, which can be classified as:

$$\mu_R(\lambda) = \underbrace{\gamma\sigma_R(\lambda)^2}_{\text{Aggregate Risk}} + \underbrace{(1-\lambda)e\sigma_R(\lambda)}_{\text{Skewed Beliefs}} - \underbrace{\frac{1-\gamma}{1-\varphi}\overline{\sigma_{cw}}(\lambda)\sigma_R(\lambda)}_{\text{Average Hedging Demand}}$$

The first part, $\gamma\sigma_R(\lambda)^2$, is a compensation for *aggregate risk*. Holding the risky assets exposes each investor's wealth to dividend fluctuations. The investors are compensated (in proportion to their risk aversion) by positive excess returns for facing such risks. The *Skewed Beliefs* element, $(1-\lambda)e\sigma_R(\lambda)$, originates from the subjective beliefs of type "B". If she is pessimistic $e > 0$. For any given price of the asset, lower expected dividend growth reduces the allure of holding the asset. To induce the pessimistic investor to hold the asset, the price must be reduced which increases returns. The increase is proportional to her influence on the asset's price as measured by her wealth $1-\lambda$. The last element, $\overline{\sigma_{cw}} = \lambda\sigma_{cw} + (1-\lambda)\tilde{\sigma}_{cw}$, is a compensation for *average hedging demands*. Suppose each investor's hedging demand is positive. As the asset is valued so much, the price for the asset increases which lowers its return.

3.3.2 Why Return Volatility is below Dividend Volatility

As discussed, $\sigma_R(\lambda)$ exceeds σ_D if the investor gaining from a dividend shock has a lower consumption-wealth ratio, i.e. $\Delta cw(\lambda)\sigma_\lambda(\lambda) < 0$. The investor with accurate beliefs (which is also more optimistic) invests more of her wealth in the risky asset than the pessimistic investor. As the excess return is positive, this investors gains, on average, higher returns from her portfolio than the other investor. Furthermore due to the larger exposure to the risky asset, she gains relative wealth in response to a dividend shock. However, for both investors to retain a relative wealth share bounded away from zero, the individual wealth growth rates need to be equal. The pessimistic investor can compensate for her lower portfolio return by a lower consumption-wealth ratio, lowering volatility. The conditions under which both investors hold a positive wealth share have been analysed by Borovicka (2015); documenting its implications for volatility and excess returns is one contribution of my paper. I discuss this mechanism in detail below.

Economic Mechanism that guarantees a positive Wealth Share for both Investors I document how each investor retains a wealth share bounded away from zero by describing the different components of the diffusion equation for λ:

corollary 1. *The process for relative wealth is a diffusion process:*

$$d\lambda = \lambda(1-\lambda)\left\{\mu_\lambda(\lambda)dt + \sigma_\lambda(\lambda)dW\right\}$$

with the coefficients satisfying the following equations:

$$\sigma_\lambda(\lambda) = \Delta\theta(\lambda)\sigma_R(\lambda) \tag{3.17}$$

$$\mu_\lambda(\lambda) = \Delta\theta(\lambda)\mu_R(\lambda) - \Delta cw(\lambda) - \sigma_\lambda(\lambda)\sigma_R(\lambda) \tag{3.18}$$

To guarantee existence of a stable steady state (a steady state with $0 < \lambda < 1$) the drift term $\mu_\lambda(\lambda)$ needs to cross zero from above: Suppose there exist a point $\bar{\lambda}$: $\mu_\lambda(\bar{\lambda}) = 0$. At that point, relative wealth does not vary deterministically. If $\mu_\lambda(\lambda) > 0$ ($\mu_\lambda(\lambda) < 0$) for $\lambda < \bar{\lambda}$ ($\lambda > \bar{\lambda}$), type "A" gains (loses) relative wealth below (above) $\bar{\lambda}$. The elements of the drift term can be grouped as:

$$\mu_\lambda(\lambda) = \underbrace{\Delta\theta(\lambda)\mu_R(\lambda)}_{\text{Portfolio Gains}} - \underbrace{\Delta cw(\lambda)}_{\text{consumption-wealth}} - \underbrace{\sigma_\lambda(\lambda)\sigma_R(\lambda)}_{\text{Ito Term}} \tag{3.19}$$

The difference in *Portfolio Gains* influence relative welath. Relative wealth also changes due to different *consumption-wealth* ratios. If a "type A" investor consumes more out of her wealth than the other, $\Delta cw > 0$ and she loses relative wealth. The *Ito Term* is small in magnitude. Such a situation is depicted in the left pane of Figure 3.3.[5] The solid line shows $\mu_\lambda(\lambda)$ as function of λ. As can be seen, the line crosses zero from above. The dashed line shows $\Delta cw(\lambda)$.

Effect of the Mechanism that guarantees a positive Wealth share on Volatility

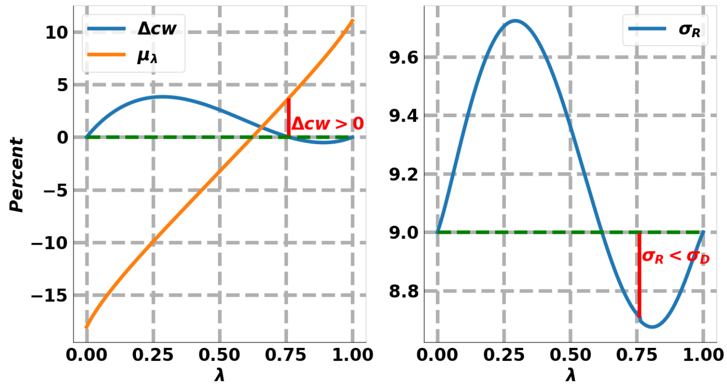
The volatility of excess returns is, again:

$$\sigma_R(\lambda) = \underbrace{\sigma_D}_{\text{Fundamental}} - \underbrace{\frac{\Delta cw(\lambda)}{\overline{cw(\lambda)}}\sigma_\lambda(\lambda)}_{\text{Change in Allocation}} - \underbrace{\frac{\overline{cw_\lambda(\lambda)}}{\overline{cw(\lambda)}}\sigma_\lambda(\lambda)}_{\text{Change in Valuation}}$$

As type "B" investor is pessimistic $\Delta\theta(\lambda) > 0$. As $\sigma_\lambda(\lambda) = \lambda(1-\lambda)\Delta\theta(\lambda)\sigma_R(\lambda)$, $\sigma_\lambda(\lambda) > 0$ as well. As $\Delta cw(\lambda) > 0$ $\Delta, -cw(\lambda)\sigma_\lambda(\lambda) < 0$. Return volatility is thus below dividend volatility. Such situation is depicted in Figure 3.3. Around the point $\bar{\lambda}$, $\sigma_R(\lambda) < \sigma_D$. The return volatility also equals dividend volatility when both investors have the same expectation.

[5] The parameters for the figure can be found in Table 3.3.

Figure 3.3: **The Relationship between** $\mu_\lambda(\lambda)$ **and** $\sigma_R(\lambda)$



This figure plots the functions for $\mu_\lambda(\lambda)$, $\Delta cw(\lambda)$ and $\sigma_R(\lambda)$ as function of λ. The right line indicates the attracting point of the differential equations. As can be seen, the difference in consumption-wealth ratio is positive and the return volatility is below the dividend volatility.

The argument so far relied on the difference in consumption-wealth ratios. The difference can be explained, approximately, as function of the underlying preference parameters and the equilibrium prices. The explanation highlights why the optimistic investor decides to consume more than the other. As derived in the appendix, the consumption wealth ratio satisfies the following differential equation:

$$
\begin{aligned}
\tilde{cw}(\lambda) &= \beta\varphi \\
&+ (1-\varphi)\left\{ \underbrace{r(\lambda) + \tilde{\theta}(\lambda)\left[\mu_R(\lambda) - \sigma_R(\lambda)e\right]}_{\text{Expected Portolio Return}} - \underbrace{\frac{1}{2}\gamma\left[\sigma_R(\lambda)\tilde{\theta}(\lambda)\right]^2}_{\text{Risk adjustment}} + \underbrace{\frac{1-\gamma}{1-\varphi}\sigma_{\tilde{cw}}(\lambda)\sigma_R(\lambda)\tilde{\theta}(\lambda)}_{\text{Hedging Motive}} \right\} \\
&+ \mu_{\tilde{cw}}(\lambda) + \frac{1}{2}\frac{\varphi-\gamma}{1-\varphi}\sigma_{\tilde{cw}}(\lambda)^2 - \sigma_{\tilde{cw}}(\lambda)e
\end{aligned}
\tag{3.20}
$$

At first, the consumption-wealth ratio is equal to the time discount factor times the elasticity of intertemporal substitution. The second line equals one minus the EIS times the certainty equivalent of the expected portfolio returns. The portfolio returns are calculated under the subjective probability measure, as indicated by the presence of e. The last line contains the derivatives $\tilde{cw}'(\lambda)$ and $\tilde{cw}''(\lambda)$. Through log-linear approximations, Campbell (1993), Campbell and Viceira (1999) and Hansen et al. (2007) show that the consumption-wealth ratio equals the present-value of risk-adjusted expected portfolio returns. Although in general no analytical solution exists for the consumption-wealth ratio exist, at $\lambda \to \{0, 1\}$ one can, however, derive the consumption-wealth ratio analytically. The ratio then equals the certainty equivalent of expected portfolio returns. One could thus interpret the finding that investor A has a higher consumption ratio as saying that investor A expects lower risk-adjusted portfolio returns from holding the asset. The interpretation is confirmed by the numerical solution: At the stable point $\lambda \approx 0.75$, the investor with pessimistic beliefs expects a negative excess return $\mu_R(\lambda) - \sigma_R(\lambda)e < 0$. She shorts the asset and invests a fraction of -2.6 times her wealth in the risky asset. The high expected returns, with an EIS larger than one, lead her to consume very little.

Numerical Simulation when Disagreement is constant The finding that return volatility is below dividend volatility is at odds with historical patterns of asset prices, Robert J. Shiller (1981). Empirically, the model predicts a volatility of 8.9% in the baseline calibration of $e = 0.1$ as seen in Figure 3.3 and Table 3.3. As can be seen in

Table 3.3: **Empirical Evaluation when Disagreement is constant**

	Model		Data	
Variable	$e = 0.1$	$e = 0.2$	Mean	Stderr
Panel A: Asset Prices				
$\mathbb{E}(r_e)$	0.5%	0.77%	6.3%	2.15%
$\sigma(r_e)$	8.9%	8.88%	19.42%	3.07%

The calibration is chosen to obtain a positive wealth share of both investors and realistic aggregate consumption growth rate: The EIS and RA are $\varphi = 3.3$ and $\gamma = 2.$ respectively. The annual mean dividend growth rate is 2% with standard deviation of 3%. The investor with subjective beliefs expects a growth rate of 1.25%. In the appendix, I show what other preference parameters satisfy a non-degenerate wealth distribution. All asset pricing values are reported with a leverage ratio of 3, as chosen by Bansal et al. (2004).

Table 3.3, increasing disagreement from $e = 0.1$ to $e = 0.2$ lowers volatility slightly, contrary to the empirical motivation in Section 3.2.

Excess Return As term $\sigma_R(\lambda)$ enters all three components of the excess return, a low volatility immediately leads to a low excess return. The excess return cannot be large because the "portfolio gains" for the more optimistic investor would be too large to be conformable with guaranteeing a positive wealth share for the other investor. The excess return is virtually zero in numerical simulations. As e increases from 0.1 to 0.2, the investor with subective beliefs becomes more pessimistic, which lowers the price of the risky asset and thus increases returns.

3.4 Investors with switching Beliefs

I extend the model in this section so that disagreement amplifies volatility. Suppose one investor is temporarily more optimistic. Then, she temporarily reaps higher portfolio returns and gains relative wealth in response to dividend shocks. Whether she has a lower consumption-wealth ratio as well depends on how disagreement affects the consumption-wealth ratios of both investors. I characterize, in an approximate log-linear solution, three channels through which disagreement influences the consumption-wealth ratios:

1. *Portfolio Channel:* As market prices reflect average beliefs, each investor thinks she can form a more profitable portfolio than the other. If an investor is more pessimistic (optimistic) than the average belief reflected in market prices, she shorts (holds a long position in) the asset.

2. *Expected Returns Channel:* In contrast to the *Portfolio Channel*, the *Expected Returns Channel* depends on the type of disagreement: For a fixed portfolio, a more optimistic (pessimistic) investor expects higher (lower) returns from holding the risky asset.

3. *Utility Distortion channel:* The previous two channels affect the consumption-wealth ratio indirectly through expected portfolio returns. Subjective beliefs affect the consumption-wealth ratio also indirectly: Expected utilities under subjective beliefs can be expressed as expectations of a distorted utility under objective beliefs. If the investor expects high (low) dividend growth, her consumption-wealth ratio decreases (increases).

The *portfolio channel* (if the elasticity of intertemporal substitution exceeds one, which I chose as baseline calibration) biases each investor's consumption-wealth ratio downwards. To, however, guarantee that the investor gaining wealth in response to a dividend shock has a particularly low consumption-wealth ratio, the *portfolio channel* must not be too important. Proposition 2 states that if risk aversion exceeds disagreement by some margin, disagreement amplifies volatility, as investors do not restructure their portfolios too aggressively. I begin by describing the extended economy, which allows each investor to be temporarily more optimistic.

Modified Dividend Growth The mean dividend growth rate is not fixed anymore, as:

$$\frac{dD}{D} = \mu_D(x)dt + \sigma_D dW \tag{3.21}$$

and that x varies as the continuous time analogue to a $AR(1)$ process:

$$dx = \kappa_x(\bar{x} - x)dt + \sigma_x dW. \tag{3.22}$$

The variable x reverts to its mean $\bar{x}$ at speed κ_x. Its conditional volatility is σ_x. In accordance with Bansal et al. (2004), I define $\mu_D(x) = x$.

Modified Beliefs The investor with subjective beliefs thinks dividends grow as:

$$\frac{dD}{D} = \tilde{\mu}_D(x)dt + \sigma_D d\tilde{W}. \tag{3.23}$$

The investors presumes the dividend grow in the mean at rate $\tilde{\mu}_D(x)$ instead of $\mu_D(x)$. While investor "B" believes $\tilde{W}$ has a mean of zero, it actually contains a drift term:

$$d\tilde{W} = \frac{\mu_D(x) - \tilde{\mu}_D(x)}{\sigma_D}dt + dW \equiv e(x)dt + dW$$

All other elements of Section 3.3 remain as before.

3.4.1 The Condition under which Return Volatility Exceeds Dividend Volatility

Disagreement amplifies volatility if the investor with a lower consumption-wealth ratio gains relative wealth in response to a dividend shock. Both the consumption-wealth ratio and the function for relative wealth gains do not admit an analytical solution. To examine how disagreement affects volatility, I instead obtain approximate solutions of both functions. The approximation proceeds in three steps: First, I argue that the optimistic investor gains relative wealth after a dividend shock. Second, I discuss how to obtain the consumption-wealth ratios as a function of disagreement. Finally, I discuss the three channels through which disagreement affects the consumption-wealth ratio.

Why the optimistic investor gains relative wealth from a dividend shock The investors who gains relative wealth in response to a dividend shock is effectively determined by who is optimistic. The function for relative wealth gains is:

$$\sigma_\lambda(x,\lambda) = \lambda(1-\lambda)\sigma_R(x,\lambda)\Delta\theta(x,\lambda)$$

The investor with accurate beliefs gains (loses) relative wealth if $\sigma_\lambda(x,\lambda)$ is positive (negative). The sign of $\sigma_\lambda(x,\lambda)$ equals the sign of $\Delta\theta(x,\lambda)$, which is:

$$\Delta\theta(x,\lambda) = \frac{1}{\gamma\sigma_R(x,\lambda)^2}\left[\underbrace{\frac{1-\gamma}{1-\varphi}\Delta\sigma_{cw}(x,\lambda)}_{\text{Differece in hedging Demands}} + \underbrace{\sigma_R(x,\lambda)e}_{\text{Disagreement}}\right]$$

The *Difference in hedging demands* captures long-term investment motives, see Campbell and Viceira (1999). The *Difference in hedging demands* does not admit an analytical solution. I did many numerical simulations across a broad range of preference parameters and the *Difference in the hedging demands* (induced by disagreement e) was always smaller than the *Disagreement* term $e\sigma_R(x,\lambda)$. Hence, I presume the sign $\sigma_\lambda(x,\lambda)$ equals the sign of disagreement e.

Finding an approximate solution when wealth is concentrated in the hands of one investor The consumption-wealth ratios are, in general, a solution to a system of coupled partial differential equations. However, when (almost) all wealth is concentrated in the hands of one investor, that investor holds all risk and prices the asset, as in Lucas (1978). The investor with marginal wealth takes price as given and forms an optimal portfolio without repercussions on prices, as in Merton (1971). The consumption-wealth ratios are then functions of the state x and linear and quadratic terms of disagreement. Using log-linear approximation to study asset prices when one investor prices the asset is an established technique in the literature, see Bansal et al. (2004). In contrast to Bansal and Yaron, I describe the effect of disagreement on consumption-wealth ratios.

The three channels through which disagreement affect the consumption-wealth ratios The effects of disagreement depend on which investor holds almost all wealth in the economy and whether the investors with subjective beliefs is optimistic or pessimistic. To illustrate the effects of disagreement, suppose the investor with accurate beliefs holds (almost) all wealth. Her consumption-wealth ratio is:

$$\begin{aligned} cw(x) &= \beta\varphi \\ &+ (1-\varphi)\left\{ \underbrace{r(x) + \mu_R(x)\theta(x)}_{\text{Expected Portfolio Return}} - \underbrace{\frac{\gamma}{2}[\sigma_R(x)\theta(x)]^2}_{\text{Risk Adjustment}} + \underbrace{\frac{1-\gamma}{1-\varphi}\sigma_{cw}(x)\sigma_R(x)\theta(x)}_{\text{Hedging Motive}} \right\} \\ &+ \mu_{cw}(x) + \frac{1}{2}\frac{\varphi-\gamma}{1-\varphi}\sigma_{cw}(x)^2 \end{aligned} \tag{3.24}$$

The elements of the consumption-wealth ratio can be described similarly as above, (3.20). To describe the consumption-wealth ratio, I log-linearly approximate the left hand side of the equation above by $cw \approx h_0 + h_1 \log \hat{cw}$, similarly to Chacko et al. (2005). I guess the solution to the approximated equation is:

$$\hat{cw} = \exp\left(A_0 + A_1 x\right).$$

Under the approximation, the excess return and the volatility are constant, as in Bansal et al. (2004). Given these returns, there exist values of A_0, A_1 solving the approximated equation. To describe whether the optimistic investor has a lower consumption-wealth ratio, I solve the consumption-wealth ratio of the other investor below.

The consumption-wealth ratio for the investor with subjective beliefs is:

$$
\begin{aligned}
\tilde{cw}(x,e) &= \beta\varphi \\
&+ (1-\varphi)\left\{ r(x) + \underbrace{\tilde{\theta}(e)\left[\mu_R - \sigma_R e\right]}_{\text{Expected Portolio Return}} - \underbrace{\frac{1}{2}\gamma\left[\sigma_R\tilde{\theta}(e)\right]^2}_{\text{Risk adjustment}} + \underbrace{\frac{1-\gamma}{1-\varphi}\sigma_{\tilde{cw}}\sigma_R\tilde{\theta}(e)}_{\text{Hedging Motive}} \right\} \\
&+ \mu_{\tilde{cw}}(x) + \frac{1}{2}\frac{\varphi-\gamma}{1-\varphi}\sigma_{\tilde{cw}}^2 - \sigma_{\tilde{cw}} e
\end{aligned} \tag{3.25}
$$

The marginal investor chooses a portfolio without influencing prices. Suppose (after the log-linear approximation) her consumption-wealth ratio is:

$$\hat{\tilde{cw}} = \exp(\tilde{A}_0 + \tilde{A}_1 x + \tilde{A}_2 e + \tilde{A}_3 e^2)$$

The quadratic terms of disagreement are captured by the *portfolio channel* and equal:

$$\Delta\tilde{\theta}(e)\Delta\mu_R(e) - \frac{1}{2}\left[\sigma_R\Delta\tilde{\theta}(e)\right]^2,$$

with $\Delta\tilde{\theta}(e)$ ($\Delta\mu_R(e)$) being the difference in demands (expected returns) among two investors. The difference in risk-adjusted returns due to the different portfolios is $0.5e^2/\gamma$. If the EIS exceeds one, higher portfolio returns reduce the consumption-wealth ratio by $(1-\varphi)$. The term $\tilde{A}_3$ is:

$$\tilde{A}_3 = (1-\varphi)\frac{1}{h_1}\frac{1}{2}\frac{1}{\gamma}$$

As the term is negative, the *portfolio channel* lowers the consumption-wealth ratio. The linear effects of disagreement are captured by the *expected returns channel* and the *utility distortion channel*. Whilst the actual expected excess return is μ_R, the investor with subjective beliefs expects $\mu_R - \sigma_R e$. For the same portfolio as the investor with accurate beliefs, disagreement thus creates a difference in expected portfolio returns; *expected returns channel*. The last line of the equation (3.25) shows that the consumption-wealth ratio is correlated with disagreement through the term $-\sigma_F e$; *utility distortion channel*. The sum of the two channels determines the parameter $\tilde{A}_2$. After inserting the equilibrium solutions for returns, one gets:

$$\tilde{A}_2 = -(1-\varphi)\left[\sigma_D + \frac{1}{h_1+\kappa_x}\sigma_x\right]\frac{1}{h_1}$$

If the EIS is larger then one, the coefficient $\tilde{A}_2$ is positive. The coefficients $\tilde{A}_0, \tilde{A}_1$ equal the coefficient for the investor with accurate beliefs.

After the individual consumption-wealth ratios have been approximated, one can examine if the optimistic investor has a lower consumption-wealth ratio. Consider first the situation when the investor with subjective beliefs is temporarily optimistic ($e < 0$). As $\tilde{A}_3$ is negative and $\tilde{A}_2$ is positive, her consumption-wealth decreases. Thus the optimistic investor has a lower consumption-wealth ratio, increasing volatility. On the other hand, the difference in consumption-wealth ratios is ambiguous when the investor is pessimistic. While the *return channel* and *belief distortion channel* increase her consumption-wealth ratio, the *portfolio channel* decreases her consumption-wealth ratio. To guarantee that the optimistic investor (which is the investor with accurate beliefs) has a lower consumption-wealth ratio, the *portfolio channel* must not be too important. Proposition 2 states under which conditions the *portfolio channel* is not too important.

Proposition 2. *Suppose $\lambda \to \{0, 1\}$, that the economy is solved trough a log-linear approximation, that the elasticity of intertemporal substitution is larger than one, and that the optimistic investor gains relative wealth after a dividend shock. Assume furthermore, that the following inequality is satisfied:*

$$2\gamma \left(\sigma_D + \frac{\sigma_x}{h_1 + \kappa_x} \right) > |e| \tag{3.26}$$

with h_1 being the average consumption-wealth ratio of the investor pricing the asset, $h_1 = \exp(\overline{\log(c) - \log(w)})$. Then, disagreement amplifies return volatility σ_R.

One can interpret the parameters as follows: If risk-aversion exceeds disagreement by some margin, investors do not re-structure their portfolios too aggressively. The proof of Proposition 2 is in the appendix. Figure 3.5 shows the parameter values for the coeficient of risk-aversion and the extent of disagreement which are compatible with amplified volatility. The blue area denotes the set of comaptabile parameters and the green are the set of incompatible parameters. The red area shows the set of parameters in which disagreement amplifies volatility in the non-linear numerical solution which I describe in Section 3.5.

3.4.2 Asset Prices

After having described under which conditions return volatility exceeds dividend volatility in a log-linear approximation, I briefly present the equilibrium prices in Proposition 3:

Proposition 3. *Suppose there exists an equilibrium. Then, the conditional excess return and its variance are:*

$$\mu_R(\lambda, x) = \gamma\sigma_R(\lambda, x)^2 + (1-\lambda)e(x)\sigma_R(\lambda, x) - \frac{1-\gamma}{1-\varphi}\overline{\sigma_{cw}(\lambda, x)}\sigma_R(\lambda, x) \tag{3.27}$$

$$\sigma_R(\lambda, x) = \sigma_D - \frac{\Delta cw(\lambda, x) + \overline{cw_\lambda}(\lambda, x)}{\overline{cw}(\lambda, x)}\sigma_\lambda(\lambda, x) + \frac{\overline{cw_x}(\lambda, x)}{\overline{cw}(\lambda, x)}\sigma_x \tag{3.28}$$

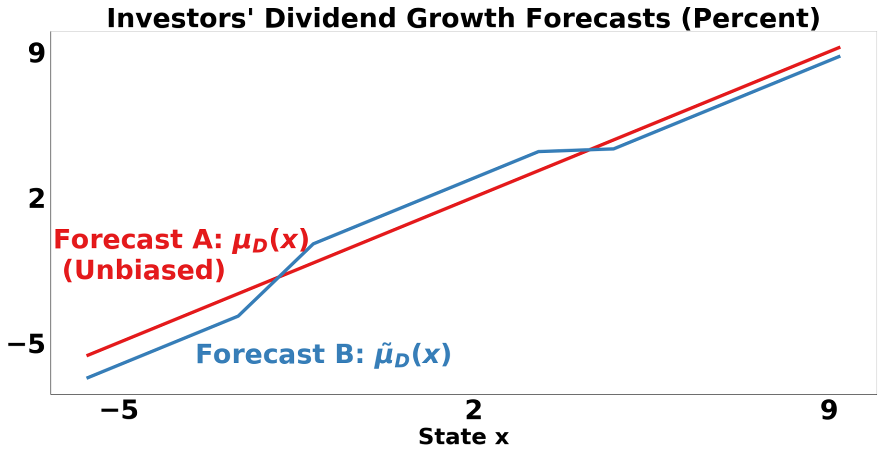
The interpretation of the function for the asset prices and the behaviour of individual investors remains as before. The only difference is that the results are now Markovian function of (x, λ) instead of only λ.

3.5 Simulation

In this section, I describe the quantitative evaluation of the extended model. I describe at first which parameters for disagreement I use and how the numerical solution compares to the approximate solution of Section 3.4. Finally, I describe the quantitative predictions of the model.

Calibration I calibrate the model at a monthly frequency and report annualized values. The average annual consumption growth is 1.8% with a standard deviation of 3.9%. I assume the investors forecast growth as plotted in Figure 3.4. I set the elasticity of intertemporal substitution to 1.75. Figure 3.5 shows under which conditions the consumption-wealth ratio of the investor gaining relative wealth in response to a dividend shock is indeed smaller. The x-axis denotes the extent of disagreement and the y-axis the coefficient of risk-aversion. The blue area highlight the set of parameters in which the log-linear solution, Proposition 2, predicts that the investor gaining wealth in response to a dividend shock has a lower consumption-wealth ratio. The red area, in contrast, denotes the range for which the numerical solution predicts

Figure 3.4: **Dividend Growth Forecasts**

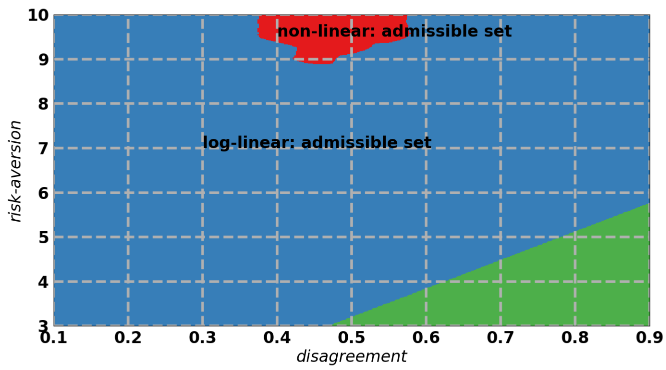


The figure plots the individual forecasts of both investors. The state x is plotted on the horizontal axis, and the two forecasts are plotted as the two lines. Investor "A" forecasts growth accurately, investor "B" is for low values of x too pessimistic, then optimistic and finally pessimistic again.

such outcome. The red area is smaller because, numerically, the consumption-wealth ratios, when the investor with accurate beliefs prices the asset, do not cross as often as predicted by the log-linear solution. In fact, for the numerical solution, I required the investor gaining from a dividend shock to have a lower consumption-wealth ratio only when $x < 0.04$ and not over the entire interval. When the investor with accurate beliefs prices the asset, only the consumption-wealth ratio of the investor with subjective beliefs is affected by disagreement. I suspect such narrow impact of disagreement leads to the limited effect of disagreement. In contrast, when the investor with subjective beliefs prices the asset, the currently optimistic investor has a lower consumption-wealth ratio, as predicted by the log-linear solution.

Estimation Results The simulation results of the model are presented in Table 3.4. The model broadly matches asset prices. The volatility of the risky asset is 15.4%, slightly lower than in the data. In contrast, the simulated excess return is 8% and exceeds the historical excess return of 6.3%. Both estimates are, however, within two standard deviations of the historical averages. The moments about the risk-free rate are matched broadly too. Both the mean and its standard deviation are slightly

Figure 3.5: **Admissible Sets**



The x-axis denotes the extent of disagreement and the y-axis the coefficient of risk-aversion. The blue area highlights the set of parameters in which the log-linear solution, Proposition 2, predicts that the investor gaining wealth in response to a dividend shock has a lower consumption-wealth ratio. The red area, in contrast, denotes the range for which the numerical solution predicts such outcome. The green area denotes the range of parameters which are not compatible with Proposition 2.

higher than in the data. The impact of disagreement on the volatility and expected excess return are presented in Table 3.5. As one can see, increases in disagreement lead to a larger volatility and a large expected excess return. I computed the excess return, as in the empirical motivation, Section 3.2, by using the price-dividend ratio as forecasting variable. An increase of disagreement by one standard deviation increases volatility (expected excess returns) by one-third (one-tenth) of a standard deviation. To facilitate comparison, I repeated empirical estimates from Table 3.2 in the second row. The model predicts relatively too strong results.

3.6 Professional Forecaster are only temporarily more optimistic than their peers

The previous section assumed one investor is only temporarily more optimistic than the other. I document that such behaviour resembles the empirical evidence as observed in the Survey of Professional Forecasters. The estimates presented in this section differ from previous studies. While Mankiw et al. (2003) or Bhandari et al. (2016) examine the accuracy of individual forecasts, little is known about whether

Table 3.4: **Empirical Evaluation of the Extended Model**

Variable	Model	Data	
	$e = 0.45$	Mean	Stderr
Panel A: Asset Prices			
$\mathbb{E}(r_e)$	8.0%	6.3%	2.15%
$\mathbb{E}(r_f)$	1.9%	0.8%	0.42
$\sigma(r_e)$	15.4%	19.42%	3.07%
$\sigma(r_f)$	1.7%	0.97%	0.28%
Panel B: Individual Statistics			
$\mathbb{E}(\lambda)$	0.18		
$\sigma(\lambda)$	0.1		

This table shows the simulation results of the extended model. The coefficient of EIS is 1.75 and the risk-aversion coefficient is 10. The annual consumption growth has a mean of 1.8 and a standard deviation of 3.9. I simulated the economy twice over $10,000$ years, discarded the initial $1,000$ years and computed the average of both sample means. Historical data are taken from Bansal et al. (2004).

individual forecasters remain consistently more optimistic than their peers. The only other papers studying persistence of optimism relative to peers is Patton et al. (2010). However, these authors conclude that forecasters remain indeed constantly more optimistic. The authors use data from Consensus Economics. Their Figure 3 highlights that Bear Sterns (the Economic Intelligence Unit) was constantly more optimistic (pessimistic) than the median forecaster. I interpret the difference between my and their paper with strategic behaviour: Whilst the forecasters in the sample I use are anonymous, the forecasters surveyed by Patton et al. (2010) communicate forecasts under a brand name.

To investigate if forecasters switch from being optimistic, I calculate the probability that a currently optimistic forecaster remains optimistic in the subsequent periods. In particular, for each date, $t, t+1, \ldots$, I assign each forecaster into a set of optimistic (pessimistic) forecasters if her forecast exceeds (is below) the median forecast. I then calculate the switching probabilities as the probability of switching from the

Table 3.5: **Regression Results of the Extended Model**

#	$Disagreement_{t\|t-1}$	GDP_t	R^2
	Panel A: Volatility		
Model:	**0.37** (13.6)	**-0.22** (-9.1)	0.16
Data:	**0.19** (2.5)	**-0.22** (-2.89)	0.09
	Panel B: Expected Returns		
Model:	**0.11** (9.91)	**-0.57** (-50.7)	0.44
Data:	**0.4** (3.67)	0.6 (0.747)	0.14

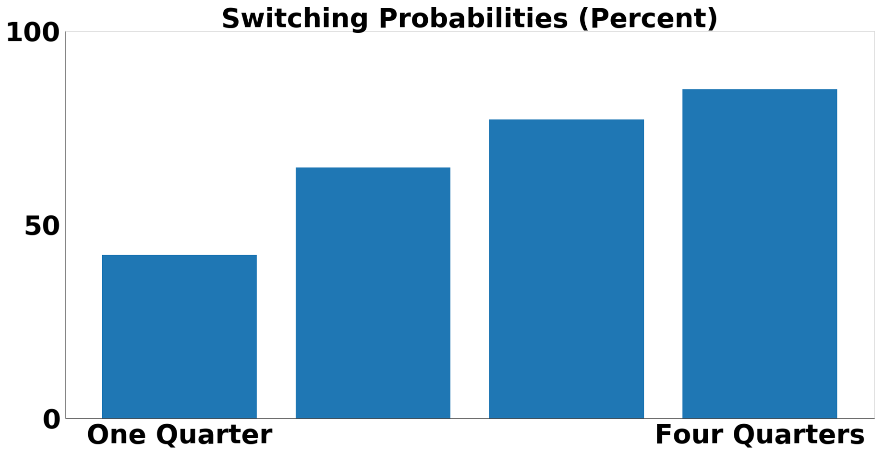
The data is standardized to facilitate comparison between regressors. The last column contains the R^2.

set of optimistic (pessimistic) to pessimistic (optimistic) forecasters in two consecutive periods. I calculate switching probabilities for a horizon of three (four) quarters as the probability of switching at least once from an optimistic (pessimistic) set to an pessimistic (optimistic) set. The estimates of the calculation are plotted in Figure 3.6. The probability of switching from being relatively optimistic (pessimistic) to pessimistic (optimistic) forecast increases over time. While only 40% of currently optimistic forecasters become pessimistic in the next period, more than 70% switch at least once over three periods. In fact, the switching probability within one year is roughly 80%.

3.7 Literature Review

In this section, I review the literature. I organize the discussion among the three groups my paper is related to the literature of *asset pricing under belief disagreement*, *asset pricing with heterogeneous preferences* and *representative-agent asset pricing*. Additionally, I relate my paper to previous work studying the effects of *disagreement on the wealth distribution*. I discuss several papers in the first group, as they are the core reference.

Figure 3.6: **Probabilities of Remaining Optimistic or Pessimistic**



The figure plots the switching from an optimistic (pessimistic) forecast to an pessimistic (optimistic) forecast. The x-axis denotes the horizon and the y-axis shows the probability of switching.

Asset pricing under disagreement In general, my work relates to other papers describing asset pricing under disagreement as follows: I illustrate that disagreement amplifies volatility when the investors gaining from a dividend shock has a lower consumption-wealth ratio in frictionless economy. I provide conditions for such amplifications and discuss how investors retain a positive wealth share. No other paper has examined the role of disagreement, volatility, and the wealth distribution in such a way. Several papers discuss how disagreement affects asset prices in various other economies. Building on the seminal contributions by Miller (1977) and Harrison et al. (1978), Scheinkman et al. (2003) illuminate the effect of disagreement under short sale-constraints on volatility. Two investors disagree about a time-varying dividend growth rate. Investors are risk neutral and have infinite wealth, so each is willing to pay at least her subjectively expected dividend stream for the stock. Due to the short sale constraint, investors are willing to pay more as they expect to re-sell the stock to a more optimistic investor. As a consequence, prices commonly exceed each investors' valuation. Scheinkman et al. (2003) show that the resulting stock prices are more volatile than the underlying dividends. I assume constraints contrary to the ones used in these papers: I allow short selling and introduce budget constraints.

Several papers characterize various effects of disagreement on asset prices. For example, Barberis et al. (2015) study its pricing implications when a rational investor trades with an investor extrapolating returns from recent observations. In response to a positive dividend shock, extrapolators expect higher returns, which amplifies the pressure on prices to appreciate. In contrast to the mechanism discussed in my paper, price volatility does not stem from the interaction among investors but the behaviour of the extrapolator. Cujean et al. (2017) explain why stock returns can be forecasted better in recessions than in expansions. The authors use a continuous-time model with two investors which use different models to forecast future returns. Cujean et al. (2017) discuss under which conditions the resulting disagreement is counter-cyclical. The authors show that returns are functions of disagreement and that elasticity or returns with respect to disagreement increases with higher disagreement. Because disagreement is largest in recessions and persistent, the autocorrelation in returns increases in recessions. While the authors explain how disagreement affects returns, it is not clear how both investors can retain a positive wealth share over time. In contrast, I describe how disagreement affects returns in a stationary steady state. Kubler et al. (2012) discuss the effects of disagreement on volatility in an OLG model. Due to the OLG structure, cohorts have different demands for the financial asset. Due to the differential demands, return volatility exceeds dividend volatility in an OLG economy with homogeneous beliefs, as shown by Huffman (1987). Kubler et al. (2012) document that stock market volatility is amplified further if investors within cohorts disagree about future returns and markets are complete.

Asset pricing with heterogeneous preferences Instead of using heterogeneous beliefs to induce different portfolio holdings, one can also assume heterogeneous preferences, as done by Garleanu et al. (2015). The authors the same economy as I do in Section 3.3. In the model, the less risk-averse investor holds the majority of aggregate risk and reaps high portfolio returns. The equation for return volatility equals (3.14). As I described in this paper, return volatility exceeds dividend volatility if the investor gaining relative wealth in response to a dividend shock has a lower consumption-wealth ratio. Because the less risk-averse investor has a higher EIS, her consumption-wealth ratio is indeed below the one of the other investor. To prevent the less risk-averse investor to own all wealth in the economy, Garleanu et al. (2015) impose random deaths among investors. Gomez (2017) highlights that such mechanism leads to a counter-factual wealth distribution among living investors.

Representative-Agent asset pricing Conventionally asset prices have been explained in representative-agent model through time-varying risk aversion or fluctuating conditional volatilities of dividend growth rates. Campbell and Cochrane (1999) assume that investors have so-called "habit-formation" preferences which value consumption relative to a moving average of past consumption (the habit). Whenever current consumption is close to the historical average, marginal utility is high. High marginal utility can equivalently be thought of a temporarily high coefficient of risk aversion. Campbell and Cochrane (1999) show that a representative-agent model with such preferences can replicate observed asset prices. Motivated by such a successful finding, other papers evaluate whether habit-formation preferences are compatible with micro-data. These preferences predict that an investor holds more risky asset when her wealth increases. Brunnermeier et al. (2008) do not find evidence for such behaviour in the Survey of Consumer Finance. However, Guiso et al. (2013) indeed measure an increase in risk-aversion among Italian retail investors after the recent financial crisis but explained such behaviour due to the emotional stress investors' experienced. Another explanation for historical asset prices has been proposed by Bansal et al. (2004). The authors explain historical asset prices in a model in which the average dividend growth rate fluctuates (and is very persistent) and the volatility of dividend growth is stochastic. As argued by Beeler et al. (2012) the model by Basal and Yaron counter-factually suggests that the price-dividend ratio is a good predictor of dividend growth.

Disagreement and the Wealth Distribution Investors assembly different portfolios when they disagree. The portfolios earn different returns which influence the wealth distribution. Sandroni (2000) shows that investors with wrong beliefs lose constantly wealth when investors have expected utility preferences and aggregate resources are bounded. When investors have Epstein-Zin preferences (and the coefficient of risk-aversion does not equal the inverse of the elasticity of intertemporal substitution) investors with wrong beliefs may however indefinitely hold a positive wealth share, as shown by Borovicka (2015). I follow Borovicka and use Epstein-Zin preferences to ensure that the wealth distribution is indeed non-degenerate. Borovicka obtains his results by studying a social planner solution and derives asset prices in the limiting case when (almost) all wealth is held by one investor. I instead derive asset prices in a competitive economy and characterize asset prices as a function of relative wealth.

3.8 Conclusion

I provided conditions under which disagreement about dividend growth forecasts amplifies stock market volatility, in line with empirical evidence. In a frictionless economy with two Epstein-Zin investors, I model disagreement as exogenous heterogeneity in beliefs: one investor is pessimistic, the other is not. I show that disagreement amplifies volatility only if/ investors switch beliefs, that is if an investor is only temporarily optimistic. If instead one investor is permanently pessimistic, prices are less volatile than dividends, and higher disagreement lowers volatility — in contradiction with evidence. Finally, I provide empirical support for switching beliefs among investors, using cross-sectional data from the Survey of Professional Forecasters. While disagreement seems a fruitful channel to describe much work remains to be done: At first, evaluating the empirical validity of different channels which supposedly determine asset prices would be interesting. Second, my work assumes exogenous forecasts and does not allow for any learning. Why investor disagree and how to best describe such disagreement remains an open challenge.

Chapter 4

Trade Volume, Noise Traders and Information Acquisition with Neural Networks

Fabian Schuetze[1]

4.1 Introduction

There is substantial evidence that high stock market trading volume (and related measures) predict low stock returns. Measures of trade volume predict cross-sectional stock returns, Amihud and Mendelson (1986) or Pastor et al. (2003). Trade volume predicts returns years ahead too, Amihud (2002) or Jones (2002). Existing models of trade in financial markets face considerable difficulties explaining why trade volume predicts returns. This paper discusses existing work of trade in financial markets and describes how neural networks can be used to motivate trade among asymmetrically informed investors. I argue why neural network could describe the empirical relationship between trade volume and excess returns.

Differences in preferences, endowments or information are common mechanisms to

[1] I am deeply indebted to Ramon Marimon for constant encouragement and discussion. I also want to thank Joao Brogeueira, Piero Gottardi, Wouter den Haan, Mathijs Janssen, and Marcin Kacperczyk.

generate trade among investors. Salient patterns of trade are, however, difficult to reconcile through preferences or endowments. Such patterns are, for instance, that trade of stocks is pro-cyclical and increases around public announcements, Kandel et al. (1995). Alternatively, investors trade because their assessment about the returns of stocks differs. Yet, Milgrom et al. (1982) argue such trades should not occur: Suppose one investor receives information which leads her to believe a stock is undervalued. As soon as she attempts to purchase the stock, the potential seller reasons private information motivated the offer and the stock is undervalued. Prices should increase until no investor wants to trade. Several mechanisms prevent such information inferences. For example, Kandel et al. (1995) and Harris et al. (1993) assume investors interpret information differently and refuse to learn from prices. While such mechanisms serve as shortcuts to generate trade, Weller (2016) and Santosh (2016) argue empirical evidence suggests investors indeed learn from prices. If prices reflect other influences besides private information, investors can learn from prices while differences in information persist. For example, Grossman et al. (1980) introduce traders demanding random quantities which shade the informativeness of prices: Prices could be high because investors with private information value the asset highly, or due to high random demands. Alternatively, Wang (1993) suggests investors have private investment opportunities whose returns are correlated with commonly available assets and thus influence the demand for these assets.

Although noise traders or private investment opportunities generate trade among asymmetrically informed investors, such mechanisms face considerable difficulties explaining why trade volume predicts returns. Trade volume does not predict excess returns if asymmetric information is sustained exclusively by noise traders: Suppose there is high trade volume because noise traders demand much (or little, i.e. supply the asset), then current prices rise (falls) but will fall (rise) in the next periods. Thus high trade volume is accompanied by high and low returns, contrary to the empirical evidence. Wang (1993) sustains asymmetric information by private investment opportunities. He shows theoretically high trade volume accompanied by high (low) returns predicts low (high) returns if trade occurs because investment opportunities change. Campbell, Grossman, et al. (1993) expand Wang's model to explain daily autocorrelations among returns. The authors document that daily returns are in general positively correlated but the correlations fall in periods of high trade volume.

Although the authors explain daily correlations of returns and its relationship with trade volume, it is not clear how one can explain the empirical cyclicality of trade volume.

To explain why trade volume predicts excess returns, I propose to model information acquisition by neural networks. As I document in Section 4.2, a neural network can be thought of as a continuous piecewise linear function which can approximate any continuous function on a compact domain to an arbitrary degree. Neural networks permit trade among asymmetrically informed investors without any further elements (such as noise traders) as follows: Suppose the pricing function (with private information as argument) is non-linear. As neural networks are piecewise linear, uninformed investors infer private information imperfectly but to an arbitrary degree. I describe a related example in Section 4.3. I argue neural networks can be used to generate persistent information inference. If such persistent inference induce persistent demands, one might be able to describe why trade volume predicts excess returns.

Neural Networks have been discussed earlier: Sargent (1993) describes its elementary structure and relate it to bounded rationality. Cho (1995) introduce neural networks in a Prisoner's Dilemma, and Marimon et al. (1990) study equilibrium selection in exchange economies. Recent advances in machine learning greatly increase the practical usefulness of neural networks. I describe the elementary capabilities of a state-of-the-art neural network in the next section.

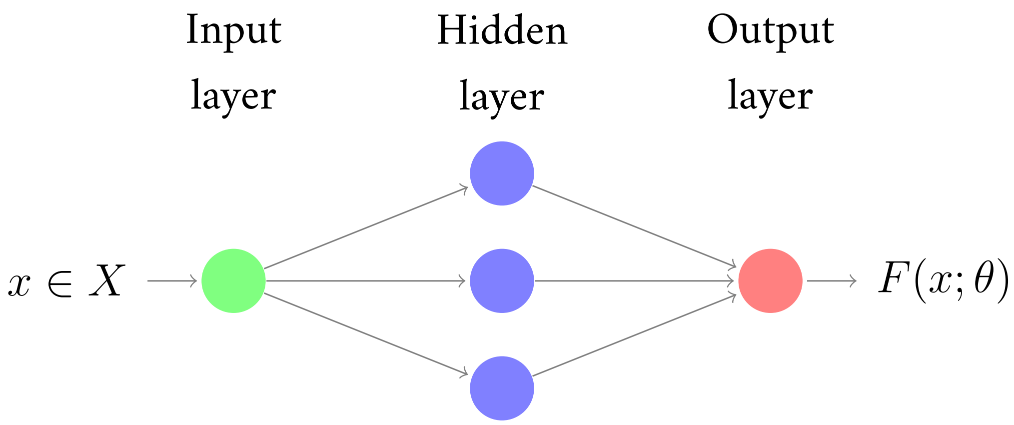
4.2 Description of Neural Networks

This section contains an illustration of elementary capabilities of a neural network used in the current standard software such as Theano and TensorFlow and discussed in corresponding textbooks, Goodfellow et al. (2016). A neural network is a function

$$f : P \to S \tag{4.1}$$

used for estimating $s \in S$ given a vector of $p \in P$ as precisely as possible. In the context of information acquisition among asymmetrically informed investors, p refers to prices and s to private information. The function f is a composition of affine and

Figure 4.1: **Neural Network with One Hidden Layer**



non-affine functions. A prototypical neural network is depicted in Figure 4.1. The idea behind the neural network is as follows: There is a signal which is processed in the "input layer" and send to elements of the "hidden layer". Elements of the hidden layer are neurons. A subset of these neurons will be activated (depending on the strength of the input signal). Active neurons will establish a connection among the input layer and the output layer. The mathematical representation of the network is:

$$f(p) = f_{out} \circ g \circ f_{in}(p). \tag{4.2}$$

The function $f_{in} : P \rightarrow \mathbf{R}^H$ is affine and called a "pre-activation function". The function sends input to the "hidden layer" and is:

$$f_{in,j}(p) = w_j p + b_j$$

The elements of the second function $g : \mathbf{R}^H \rightarrow \mathbf{R}^H$ constitute the "hidden layer" and are called "activation functions" as they are meant to represent the neural activities. A commonly used form are "rectifier activation functions", Nair et al. (2010):

$$g_j(p) = \max\left\{0, f_{in,j}(p)\right\}. \tag{4.3}$$

The function can be understood as follows: A neuron is activated upon a strong enough stimulus $f_{in,j}(p)$. Finally, the output function $f_{out} : \mathbf{R}^H \rightarrow \mathbf{R}$ transforms activities into output:

$$f_{out}(p) = \beta_0 + \sum_{j=1}^{H} \beta_j g_j(p). \tag{4.4}$$

The parameters of the network are $\theta = \{w, b, \beta\}$. Proposition 4 and Proposition 5 describe the properties of a neural network. The first Proposition documents that the neural network can be thought of as a piecewise linear function:

Proposition 4. *Suppose the neural networks has H neurons and that the input dimension is $P \subset \mathbf{R}$. Then, one can find parameters θ such that the $f : P \to S$ is continuous and piecewise linear with at most $H + 1$ changes of the slope.*

Proof. Set all $w_j = 1$ and order the different "pre-activation functions" f_{in} so that if $f_{in,j}(p) > 0$, $f_{in,j-1}(p) > 0$ as well; that is when neuron j is activated, neuron $j - 1$ is activated as well. I will now go through the domain of P and document how the function f_{out} behaves. First, for all $p < -b_1$, $f_{out} = \beta_0$. Consider now instead $p \in P : f_{in,1} > 0, f_{in,2}(p) < 0$. Define $\bar{p}_1 = -b_1$ and $\Delta p_1 = p - \bar{p}_1$. Then, the output of the neural network is:

$$\begin{aligned} f_{out}(p) &= \beta_0 + \beta_1 (p + b_1) \\ &= \beta_0 + \beta_1 \Delta p_1 \end{aligned}$$

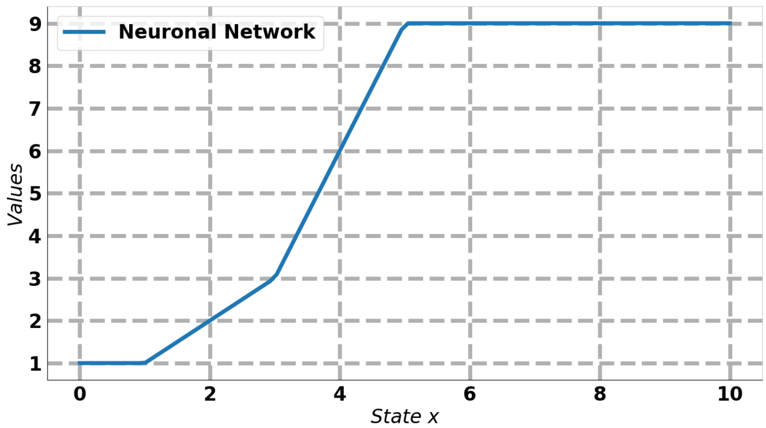
That is, the neural network is β_0 for all values $p < -b_1$ and then increases with slope β_1 afterwards. Now, consider the value $p \in P : f_{in,2} > 0, f_{in,3} < 0$. Define, similar as before $\bar{p}_2 = -b_1$ and $\Delta p_2 = p - \bar{p}_1$. With a slight abuse of notation, re-define the values considered above to $\Delta p_1 = \bar{p}_2 - \bar{p}_1$. Then output of the neural network is then:

$$\begin{aligned} f_{out}(p) &= \beta_0 + \beta_1 (p + b_1) + \beta_2 (p + b_2) && (4.5) \\ &= \beta_0 + \beta_1 \Delta p_2 + \beta_2 (\Delta p_2 + \Delta p_1) && (4.6) \\ &= \beta_0 + \beta_1 \Delta p_1 + (\beta_1 + \beta_2) \Delta p_2 && (4.7) \end{aligned}$$

The neural network is $\beta_0 + \beta_1 \Delta_1$ up to the point where the second neuron is activated and the slope than changes to $\beta_1 + \beta_2$. The proof can be continued for every $j > 2$. □

A neural network with three neurons is illustrated in Figure 4.2. Intuitively, the network accumulates slope coefficients as p increases. Its domain is $P = [0, 10]$. All $w_i = 1$, and $b_1 = -1, b_2 = -3, b_3 = -5$, and $\beta_0 = 1, \beta_1 = 1, \beta_2 = 2, \beta_3 = -3$. The idea behind the piecewise linearity is that the output function is linear and the

Figure 4.2: **A Neuronal Network with Rectifier Activation units**



transformation function g is piecewise linear (zero for some inputs and with a different constant slope for other values). The functional form of the neural network has been analysed in great generality by Montufar et al. (2014). The next property shows that neural networks can approximate any continuous function to arbitrary precision. The proof is a special case of the more general Theorem 1 presented in Hornik (1991).

Proposition 5. *Take any $\epsilon > 0$ and any continuous and bounded function g. Then there exits a number of neurons $H^\star \in \mathbf{R}_+ : \forall H > H^\star$*

$$\sup_{p \in P} |g(p) - f(p)| < \epsilon \tag{4.8}$$

Proof. The proof is constructive. I show how to construct a sequence of functions $f_i(p)$ differing in the number of neurons. Increasing the amount of neurons monotonically increases the approximation capacity. Take a function g. Such a function has a finite number of local maxima and minima. Define the parameters θ such that $f_0(p)$ linearly interpolates the minima and maxima. The maximum error of the interpolation occurs at the point

$$p_1^\star = \arg\max_{p \in P} |g(p) - f_0(p)|,$$

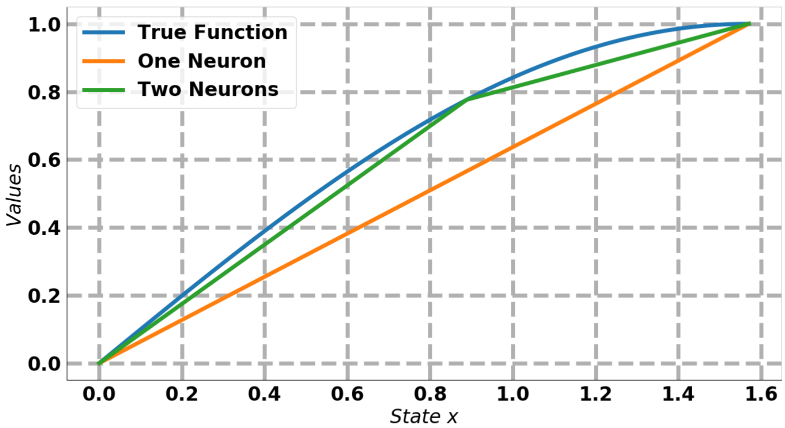
with an associated error of δ_0. Now, if one adds an interpolation point at $p^\star$ the error at the point is zero and the maximum error falls to

$$\delta_1 = \sup_{p \in P} |g(p) - f_1(p)| \tag{4.9}$$

Again, one can define the to δ_1 corresponding point $p_2^\star$ and generate a new interpolation. The process continues until $\delta_i < \epsilon$. □

An example of the Proposition is illustrated in Figure 4.3 The function which is to be

Figure 4.3: **A Neuronal Network with Rectifier Activation units**



approximated is plotted in blue. The yellow function has one neuron, with $\beta_0 = 0$. A neural network with one neuron corresponds to a linear function. I then calculated the point

$$p_1^\star = \arg\max_{p \in P} |g(p) - f_0(p)|,$$

which denotes the $\arg\max$ of the approximation error of the neural network. The activation threshold for the second neuron are set at this point $b_2 = -p_1^\star$. The approximation error at this point thus collapses to zero.

4.3 Inference from Prices with Neural Networks

In this section, I show through an example that neural networks can be used to describe information acquisition in a model of asymmetric information with arbitrary precision without noise traders. The uninformed investor infers private information from prices to an arbitrary degree and stock prices not fully reveal private information (except in the limit of full information acquisition). The economy can be summarized as follows: There are two types of investors which trade a claim to an outstanding risky-asset and borrow and lend from each other. Each investor chooses how much risky assets to hold, how much to invest in the risk-free asset and how much to consume to maximize expected discounted utility subject to a budget constraint. The investors have CRRA preferences and the economy evolves in discrete time. I show that the equilibrium pricing function is a non-linear function of the fundamental and uninformed investors approximate the fundamental.

Dividend Growth At each time t the risky-asset pays a dividend d_t which evolves as:

$$d_t = f_t + \sigma_d u_t \tag{4.10}$$

$$\log f_{t+1} = \alpha \log f_t + \sigma_f \epsilon_{t+1}, \tag{4.11}$$

with u_t and ϵ_t being i.i.d. normally distributed. The dividend d_t equals a fundamental f_t plus a noise term u_t. The fundamental evolves according to an $AR(1)$ log-normal process.

Information Set The two investors are differently informed about the fundamental f_t. The informed investor knows the fundamental f_t. The uninformed investor does not observe the fundamental but infers it from the price.

Traded Asset and Budget Constraint The budget constraint of each investor is:

$$c_t^i + \pi_{t+1}^i p_t + b_{t+1}^i \leq \pi_t^i d_t + \pi_t^i p_t + R_t b_t^i \tag{4.12}$$

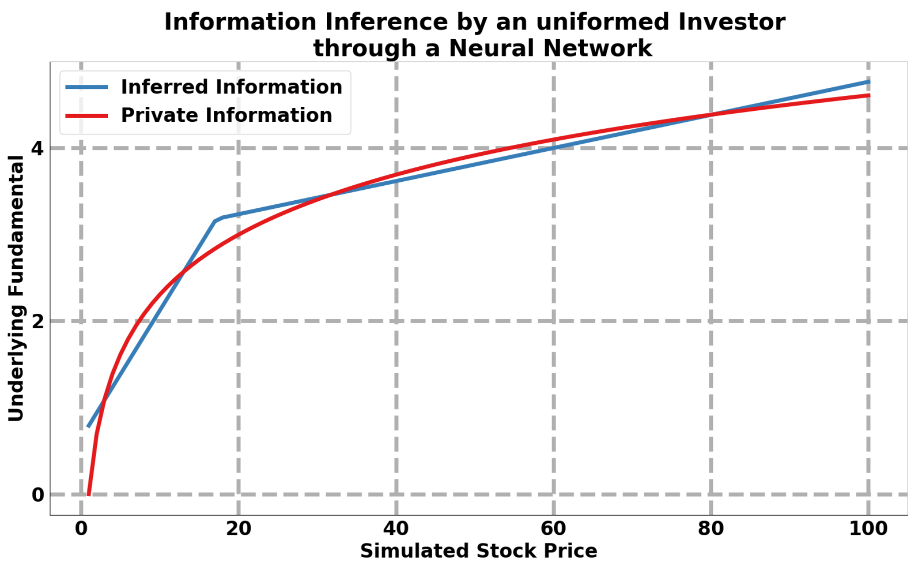
The investor allocates her resources to consumption c_t^i, risky-asset π_{t+1}^i, and bond b_{t+1}^i purchases. The superscript $i = \{I, U\}$ represents the informed or the uninformed investor. The price of the risky-asset is p_t and the return for borrowing and lending is R_t.

Optimizing The investors have CRRA preferences with a risk-aversion coefficient γ and discount factor β. The investor maximizes her expected life-time utility by choosing consumption and asset allocations subject to their budget constraint.

Relative Wealth The equilibrium variables are functions of the relative wealth of each investor. For now, I avoid this by assuming that almost all wealth is concentrated in the hands of the informed investor. This means the informed investor assumes (almost) all aggregate risk and thus prices the asset, as in Lucas (1978). In contrast, the choice of assets of the uninformed investor does not influence asset prices, as in Merton (1971). While such an assumption is clearly a simplification, it allows me to concentrate on the description of information acquisition.

Information Inference from Prices Because of the assumption that almost all wealth is concentrated in the hands of the informed investors, the economy can be solved as in Lucas (1978). The stock price is then exclusively a function of the fundamental f. Such a price is plotted in Figure 4.4. Figure 4.4 shows the equilibrium

Figure 4.4: **Example for Filtering**



The figure plots the fundamental on the y-axis as function of the equilibrium price on the x-axis in red. The figure also plots the approximated fundamental by the uninformed investors in blue. The inference in done by a neural-network with one hidden layer and two neurons.

price on the horizontal axis and the underlying fundamental on the vertical axis in red. The blue line shows what information the uninformed investor infers from the price. The inference is done by a neural network with one layer and two neurons, as described in Section 4.2. One can see that the inferred information is more precise in when prices are high than when prices are low.

4.4 Future Work

In showed through an example that neural networks can be used to describe information acquisition in a model of asymmetric information at arbitrary precision without noise traders. As stated in the introduction, I would like to use neural networks to describe why trade volume predicts returns. In comparison to a model with noise traders, I hope information acquisition with neural networks allows to make individual demands more persistent and therefore smooth trade volume. If prices are autocorrelated, uniformed investors infer similar information within a specific amount of time which might lead to more persistent demand functions. My next step will be to characterize the equilibrium. In general, the uniformed investor infers private information for prices which influences her demand function. This demand function, in turn, influences the pricing function again. Such repercussions need to be addressed in a definition of the equilibrium.

Chapter 5

Existence and uniqueness of equilibrium in Lucas' Asset Pricing model when utility is unbounded

João Brogueira and Fabian Schütze

5.1 Introduction

We prove the existence of a unique equilibrium in a Lucas (1978) economy when the utility function displays constant relative risk aversion (CRRA) and log dividends follow a normally distributed AR(1) process with positive auto-correlation. The equilibrium in the economy is characterized by a pricing function for the Lucas tree and a value function for the representative consumer. Our result is obtained after we restict the set of candidate equilibria to a space of functions which are bounded with respect to a particular weigthed supremum norm, as specified in Section 5.3. Under the assumption of a bounded utility function, Lucas proves the existence of a unique equilibrium by showing that the pricing and value functions are fixed points of functional equations. Lucas resorts to the sufficient conditions of Blackwell (1965) to document that Banach's fixed point theorem (e.g. p. 176 of Ok (2007)) guarantees the existence of a unique solution to each of the functional equations. Alas, Blackwell's conditions do not hold when the utility function displays the CRRA property. The conditions require utility being bounded in the sup-norm, which does not hold when the consumption space is equal to the positive real numbers and the investor displays CRRA preferences. Fortunately, Blackwell's conditions are only sufficient. We exploit the extension of Blackwell's conditions by Boyd (1990) and document under which circumstances Banach's theorem can be applied. In particular, we provide a joint restriction on the coefficient of relative risk aversion, the discount factor and the conditional variance of the consumption process under which an equilibrium in the economy exists and is unique.

Our solution method serves the same purpose as the local contraction methods (see Martins-da-Rocha et al. (2010), Matkowski et al. (2011) and the references therein), which provide conditions under which a functional equation has a unique solution in an unbounded setting. In short, local contraction arguments rewrite the domain of the elements of the functional space under consideration as a countable union of always increasing compact subsets. A function is then said to be bounded if it is bounded in every such subset of its domain. Local boundedness is implied by restrictions on the co-domain of the transition function for the state variable. In non-stochastic problems, the strictest version of such restrictions is simply to require tomorrow's state to be an element of the same subset of the state space as today's state variable. For stochastic problems, Matkowski et al. (2011) write the state space as a sequence of increasing (in the sense of inclusion) subspaces. The probability distribution that characterizes the transition of the state is such that with probability one tomorrow's state lies in the strictly larger but smallest subspace relative to the smallest subspace that includes today's state. These assumptions about how the state traverses imply that if today's value function is locally bounded, tomorrow's value function is locally bounded too. In our application, the state traverses according to a log-normal distribution. This probability distribution has an unbounded support, precluding the application of the argument outlined above.

Calin et al. (2005) suggest yet another way to solve a variant of Lucas' asset pricing model with unbounded utility. Nevertheless, our paper and their paper differ slightly in their methodology and in their focus. Our paper supposes that the log of dividend growth follows and AR(1) process whilst their paper considers the (commonly used) specification of the dividend growth rate adherring to an AR(1) process. Following Lucas' we phrase the euqilibirum asset pricing function as a solution to a functional equation. To guarantee a solution to such equation, we draw on extensions to Blackswell's sufficient conditions. Our approach thus conforms with the practice of solving such asset pricing model models by iterating until convergence of an initial guess. Calin et al. (2005) pursue a very inspiring path: They show that a unique equilibrium exists when the set of candidate equilibrium pricing functions is restricted to functions which are integrable with respect to a particular measure. Under a condition on model parameters, the equilibrium price-dividend ratio is shown to be uniquely defined as an analytic function The authors also such that such a solution needs be an analytical function. Analytic functions can be approximated to an abitrary precision by a covergent power series. Calin et al. show that the coefficients of such a power series can be found by solving a system of linear equations.

This note is a complement to Kamihigashi (1998), who provides sufficient conditions for uniqueness of equilibrium prices in a Lucas (1978) economy, presuming existence of the equilibrium. He shows that utility functions which are continuous and unbounded below can lead to non-uniqueness of asset prices, and that a transversality condition is not sufficient to guarantee uniqueness. The sufficient conditions for uniqueness of equilibrium in Kamihigashi (1998) are in the form of a bound on the growth rate of marginal utility when consumption goes to zero and to infinity, independently of the process governing dividends. CRRA utility functions satisfy such a bound on the growth rate of marginal utility and hence the equilibrium is unique. In contrast, we formulate the optimization problem recursively as in Lucas (1978). We show that, for a particular endowment process, an equilibrium exists and is unique: the value of the problem and the asset pricing function are uniquely defined, and a transversality condition on the value function holds. Alvarez et al. (1998) analyses dynamic programming problems with homogeneous return functions and transition functions that are homogeneous of degree one. Using similar arguments as theirs, we consider an optimization problem with a transition function that is not homogeneous of degree one.

The structure of this note is as follows: In the next Section, we provide a brief description of the economy and a definition of equilibrium in the asset pricing problem. Section 3 presents the main result of the note and discusses an extension to Blackwell's sufficient conditions for a given operator on a metric space to be a contraction that is useful in our argument. We prove the existence and uniqueness of the equilibrium and study properties of the pricing function in Section 4. Section 5 concludes.

5.2 The economy and definition of equilibrium

We describe the equilibrium as in Lucas (1978). Let us denote next period's share holdings by $x' \in \mathbf{X}$, with $\mathbf{X} = [\underline{x}, \bar{x}]$, $0 < \underline{x} < 1 < \bar{x}$, and current consumption by $c \in \mathbb{R}_+$. Let $y \in \mathbf{Y} = \mathbb{R}_{++}$ be the current dividend of a Lucas-tree. The transition equation for next period's dividend is $G(y, z') = y^\alpha z'$ with $\alpha \in (0, 1)$ and $\log z' \sim N(0, \sigma^2)$, $\sigma > 0$. Let Q be the probability density over z'. Instantaneous utility is given by $u(c) = c^{1-\gamma}/(1-\gamma)$, with $\gamma > 0$; $\beta \in (0, 1)$ is a discount factor. We use the following equilibrium definition:

Definition 1. *An equilibrium is a continuous function $p(y) : \mathbf{Y} \to \mathbb{R}_+$ and a continuous function $v(y, x) : \mathbf{Y} \times \mathbf{X} \to \mathbb{R}_+$ such that:*

$$v(y, x) = \max_{c, x' \in \Gamma(y,x)} \left\{ u(c) + \beta \int_Z v(G(y, z'), x') Q(dz') \right\} \tag{5.1}$$

with

$$\Gamma(y, x) = \{(c, x') \in \mathbf{Y} \times \mathbf{X} : c + p(y)x' \leq yx + p(y)x\},$$

and

$$\textit{for each } \ y, \ v(y, 1) \textit{ is attained by } c = y \textit{ and } x' = 1, \textit{and satisfies}$$

$$\lim_{t \to \infty} \mathbb{E}_0 \left[\beta^t v(x_t, y_t) \right] = 0$$

where $\mathbb{E}_0$ is the expectation operator conditional on the initial period information, which is the initial endowment $y_0 \in \mathbf{Y}$ and asset holdings $x_0 \in \mathbf{X}$.

5.3 Sufficient conditions for the existence of a unique equilibrium

The following proposition is the crux of our note:

Proposition 1. *Take $\beta \in (0,1)$, $\sigma \in (0,\infty)$. Suppose that for all $\gamma \in (0,1)$:*

$$\beta \left[0.5 + \int_1^\infty (z')^{1-\gamma} Q(dz') \right] < 1. \tag{5.2}$$

Alternatively, for all $\gamma > 1$, suppose that:

$$\beta \left[\int_0^1 (z')^{1-\gamma} Q(dz') + 0.5 \right] < 1. \tag{5.3}$$

Finally, suppose that $\log(\beta) + (1-\gamma)^2\sigma^2/2 < 0$. Then there exists a unique equilibrium. That is:

(1) There exists a unique non-negative continuous pricing function $p(y)$,

(2) There exists a unique function $v(y,x)$,

in accordance with Definition 1.

The inequalities (5.2) and (5.3) will be described at the end of this section. Given a value function v, we first study the existence of a unique pricing function. We begin by deriving a variant of an Euler equation following Lucas (1978), given by (5.4) below. Let us assume that for each y, $v(y,x)$ is an increasing, concave and differentiable function with respect to x. Defining $f(y) = p(y)\partial u(y)/\partial y$ and using the equilibrium conditions $x = x' = 1$ and $c = y$ allows formulating the stochastic Euler equation as:

$$f(y) = h(y) + \beta \int_Z f(G(y,z'))Q(dz'), \tag{5.4}$$

with $h(y) = \beta \int_Z \left[\frac{\partial u(G(y,z'))}{\partial y'} G(y,z') \right] Q(dz') = \beta y^{\alpha(1-\gamma)} \exp\left((1-\gamma)^2\sigma^2/2\right).$

Lucas uses Blackwell's sufficient conditions to show that the operator T, defined such that (5.4) is equivalent to $Tf = f$, is a contraction and then applies Banach's fixed point theorem. To employ Blackwell's conditions, Lucas assumes that the utility function u and thereby the function h is bounded with the sup-norm. With CRRA utility and a dividend process in $\mathbb{R}_{++}$, the function h is unbounded with the sup-norm. Importantly, boundedness is a characteristic that is closely linked to the employed metric. In the following subsection we study a norm with respect to which the function h in (5.4) is bounded.

5.3.1 A weighted norm approach

Boyd (1990) extends Blackwell's sufficient conditions by generalizing the metric from a sup-norm to a weighted sup-norm. We denote the set of continuous functions $f : \mathbf{Y} \to \mathbb{R}_+$ by $\mathbf{S}$, and take $\varphi \in \mathbf{S}$ with $\varphi > 0$. Then f is φ-bounded with respect to the weighted sup-norm $||f||_\varphi = \sup_{y \in \mathbf{Y}} \{|f(y)|/\varphi(y)\}$ if $\exists\, B \in \mathbb{R}_+$ such that $||f||_\varphi < B$. Let $\mathbf{S}_\varphi \subset \mathbf{S}$ be the set of continuous and φ-bounded functions. Let us define the metric $d_\varphi(f, g) := ||f - g||_\varphi$ on $\mathbf{S}_\varphi$. Note that $(\mathbf{S}_\varphi, d_\varphi)$ is a complete metric space (see e.g. Theorem 12.2.8 in Stachurski (2009)). We have the following lemma, which is a corollary to the Weighted Contraction Mapping Theorem in Boyd (1990):

Lemma 1 (Boyd's sufficient conditions)**.** *Let $T : \mathbf{S}_\varphi \to \mathbf{S}$ and suppose:*

1. *(monotonicity) T is monotone, that is $\forall f, g \in \mathbf{S}_\varphi$, $f \geq g$ implies $Tf \geq Tg$;*
2. *(discounting) For any $A \in \mathbb{R}_{++}$, there exists $\theta \in (0, 1)$ such that: $T(f + A\varphi) \leq Tf + \theta A\varphi$;*
3. *(self-map) $T(0) \in \mathbf{S}_\varphi$.*

Then T is a contraction in $(\mathbf{S}_\varphi, d_\varphi)$ with modulus θ.

The discounting and self-map property of operator T involve the weighting function explicitly. The self-map property requires the function h to be bounded with a weighted sup-norm: $||h||_\varphi < B$ for some $B \in \mathbb{R}_+$. To develop some intuition about our proposal for such a φ, we consider the functional form of $h(y) = \kappa y^{\alpha(1-\gamma)}$, where $\kappa = \beta \exp\left((1 - \gamma)^2\sigma^2/2\right)$. For a given $\alpha \in (0, 1)$, if $\gamma < 1$, h is a strictly increasing and concave function; alternatively, if $\gamma > 1$, h is strictly decreasing and convex. Any positive continuous function φ which is weakly above h in $\mathbf{Y}$ is a weighting function that makes h φ-bounded. One such function is given by:

$$\varphi(y) = \kappa \cdot \max\left\{1, y^{1-\gamma}\right\}. \tag{5.5}$$

When $\gamma \in (0, 1)$, the function above is equal to κ while $0 < y \leq 1$ and then grows strictly above h for all $y > 1$. When $\gamma > 1$, the weighting function in (5.5) is above h while $0 < y < 1$ and stays at κ when $y \geq 1$.

In our framework, the restriction implied by Boyd's discounting property is equivalent to:

$$\beta \frac{\int_Z \varphi(G(y,z'))Q(dz')}{\varphi(y)} < 1, \forall y \in \mathbf{Y}. \tag{5.6}$$

Similarly to the discussion of dynamic programming techniques with homogeneous return functions in Alvarez et al. (1998), inequality (5.6) places a bound on the expected growth rate of the weighting function. If φ was a constant function, (5.6) would be trivially satisfied (since $\beta < 1$). However, as discussed above, in order to bound h the weight φ has to be above h, that decreases or increases with y depending on γ. This observation motivates using another weighting function arbitrarily close to and greater than h: relative to the weight in (5.5), such a weighting function increases the range of parameters β, γ, σ that satisfy condition (5.6). Instead, for the sake of simplicity, we proceed with our analysis using (5.5) and illustrate the restriction on the parameter space imposed by (5.6). To that end, take $\gamma \in (0,1)$. To evaluate (5.6), for a given $y \in \mathbf{Y}$, consider realizations of z' in the interval $(0, y^{-\alpha})$. Since in this interval $G(y,z') = y^\alpha z' < 1$, $\varphi(G(y,z')) = 1$. Conversely, for $z' \in [y^{-\alpha}, \infty)$, $\varphi(G(y,z')) = G(y,z')^{1-\gamma}$. The following lines place an upper bound on the left hand side of (5.6):

$$\begin{aligned}
\beta \frac{\int_Z \varphi(G(y,z'))Q(dz')}{\varphi(y)} &= \beta \frac{\int_0^{y^{-\alpha}} Q(dz') + y^{\alpha(1-\gamma)} \int_{y^{-\alpha}}^{\infty} (z')^{1-\gamma} Q(dz')}{\varphi(y)} \\
&\leq \begin{cases} \dfrac{\beta \left[\int_0^{y^{-\alpha}} Q(dz') + \int_{y^{-\alpha}}^{\infty} z'^{1-\gamma} Q(dz')\right]}{1} & \text{if } 0 < y < 1, \\ \dfrac{y^{\alpha(1-\gamma)} \beta \left[\int_0^{y^{-\alpha}} Q(dz') + \int_{y^{-\alpha}}^{\infty} z'^{1-\gamma} Q(dz')\right]}{y^{(1-\gamma)}} & \text{if } y \geq 1, \end{cases} \\
&\leq \beta \left[\int_0^{y^{-\alpha}} Q(dz') + \int_{y^{-\alpha}}^{\infty} z'^{1-\gamma} Q(dz')\right] \quad \forall y \in \mathbf{Y} \\
&\leq \beta \left[\int_0^{1} Q(dz') + \int_{1}^{\infty} z'^{1-\gamma} Q(dz')\right] \quad \forall y \in \mathbf{Y} \\
&= \beta \left[0.5 + \int_{1}^{\infty} z'^{1-\gamma} Q(dz')\right].
\end{aligned}$$

In the numerator, after the first equality sign, we split the support of z' in two intervals to consider the two branches in (5.5) separately. The inequalities in the second line hold as $1 > y^{\alpha(1-\gamma)}$ for $0 < y < 1$ and $y^{\alpha(1-\gamma)} \geq 1$ for $y \geq 1$. The inequality

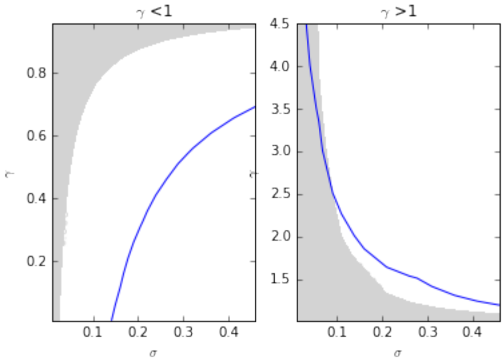


Figure 5.1: Illustration of the set of (γ, σ) values satisfying the conditions in Proposition 1. The grey regions represent the combinations of (γ, σ) satisfying inequalities 5.2 and 5.3 in Proposition 1. The region above (respectively below) the blue line corresponds to parameter combinations that satisfy the additional restriction in Proposition 1, when $\gamma < 1$ (resp. $\gamma > 1$). β is 0.99.

in the third line follows since $y^{\alpha(1-\gamma)} < y^{1-\gamma}$ for $y > 1$. Finally, the last inequality holds because the term in square brackets is the greatest at $y = 1$. Accordingly, inequality (5.2) of Proposition 1 guarantees that, if $\gamma < 1$, Boyd's discounting property is satisfied. A similar arguments holds for values $\gamma > 1$, when inequality (5.3) holds. As an illustration, the shaded region in Figure 5.1 documents which parameter pairs (σ, γ) satisfy conditions (5.2) and (5.3), when β is 0.99. When $\gamma = 1$, $u(c) = \log(c)$ and a solution to (5.4) can be calculated analytically. A lower value of β enlarges the admissible region. In the following, the proof of Part (1) of Proposition 1 will be completed.

5.4 Proof of the Proposition

In this section we formally prove the existence and uniqueness of equilibrium in the economy described in Section 2. Following Proposition 1,we proceed in two steps. We first solve for a unique pricing function, p, taking as given a value function v (part (1) in Proposition 1). Then, taking p as given, we complete the proof by showing that there is a unique value function v (part (2)), in accordance with Definition 1.

Proof. (Part (1) of Proposition 1.) The proof begins by showing that T is a map from $\mathbf{S}_\varphi$ to $\mathbf{S}$. We show first that for any $f \in \mathbf{S}_\varphi$, Tf is continuous. Take a sequence $y_n \to y$ and some $f \in \mathbf{S}_\varphi$. We define the difference $|Tf(y_n) - Tf(y)| = |h(y_n) - h(y) +$

$\beta \int (f(G(y_n, z')) - f(G(y, z')))\, Q(dz')|$. By the triangle inequality, the right hand side expression is majored by:

$$|h(y_n) - h(y)| + \beta| \int (f(G(y_n, z')) - f(G(y, z')))\, Q(dz')|.$$

The first term above converges to zero as $y_n \to y$ since the h is a continuous function. Now we define $g_n(z) \equiv f(G(y_n, z))$ and $g(z) \equiv f(G(y, z))$. Because f is a continuous function, we know that $g_n(z)$ converges pointwise to $g(z)$. Let $u(z)$ denote the supremum of the sequence of functions $\varphi_n(z) \equiv \varphi(G(y_n, z))$. Since $f \in S_\varphi$, it is bounded by φ, and we have that $|g_n(z)| \leq u(z)$. Since $u(z)$ is integrable, by Lebesgue's dominated convergence theorem:

$$\lim_{n\to\infty} \int f(G(y_n, z')) - f(G(y, z'))\; Q(dz') = 0$$

and hence the second term multiplying β above converges to zero. Hence, $Tf(y_n)$ converges to $Tf(y)$: the operator T is continuous and we write $T : \mathbf{S}_\varphi \to \mathbf{S}$. We now study the monotonicity of T. The operator T is monotone since for any $f, g \in \mathbf{S}_\varphi$ with $f \geq g$, $\int f(G(y, z'))Q(dz') \geq \int g(G(y, z'))Q(dz')$, so $Tf \geq Tg$. Therefore condition 1 of Lemma 1 holds. Under assumption (5.2), condition 2 of Lemma 1 is satisfied for $0 < \gamma < 1$, as the argument in section 5.3.1 shows. It remains to be shown that this condition holds for $\gamma > 1$. For these values of γ, we observe the following:

$$\begin{aligned}
\beta \frac{\int_Z \varphi(G(y, z'))Q(dz')}{\varphi(y)} &= \beta \frac{y^{\alpha(1-\gamma)} \int_0^{y^{-\alpha}} (z')^{1-\gamma} Q(dz') + \int_{y^{-\alpha}}^{\infty} Q(dz')}{\varphi(y)} \\
&\leq \begin{cases} \beta \dfrac{y^{\alpha(1-\gamma)} \left[\int_0^{y^{-\alpha}} z'^{1-\gamma} Q(dz') + \int_{y^{-\alpha}}^{\infty} Q(dz')\right]}{y^{1-\gamma}} & \text{if } 0 < y < 1, \\ \beta \dfrac{\left[\int_0^{y^{-\alpha}} z'^{1-\gamma} Q(dz') + \int_{y^{-\alpha}}^{\infty} Q(dz')\right]}{1} & \text{if } y \geq 1, \end{cases} \\
&\leq \beta \left[\int_0^{y^{-\alpha}} z'^{1-\gamma} Q(dz') + \int_{y^{-\alpha}}^{\infty} Q(dz')\right] \forall y \in \mathbf{Y} \\
&\leq \beta \left[\int_0^{1} z'^{1-\gamma} Q(dz') + \int_{1}^{\infty} Q(dz')\right] \quad \forall y \in \mathbf{Y} \\
&= \beta \left[\int_0^{1} z'^{1-\gamma} Q(dz') + 0.5\right].
\end{aligned}$$

The reasoning for each condition is analogous to the one made for the case $0 < \gamma < 1$. Under (5.3), condition 2 in Lemma 1 holds for $\gamma > 1$. The third condition of Lemma 1 requires h to be bounded with the weighted sup-norm. Hence, since:

$$||h||_\varphi = \beta \exp\left((1-\gamma)^2\sigma^2/2\right) \sup_{y\in\mathbf{Y}}\left\{\frac{y^{\alpha(1-\gamma)}}{\varphi(y)}\right\} = \beta\exp\left((1-\gamma)^2\sigma^2/2\right),$$

$T(0) \in \mathbf{S}_\varphi$ and point 3 of Lemma 1 holds. Note that, under Lemma 1, the operator T is a self-map (maps the space $\mathbf{S}_\varphi$ into itself).[1] Concluding, since T is a contraction over $(\mathbf{S}_\varphi, d_\varphi)$, Banach's fixed point theorem guarantees that a unique function $f \in \mathbf{S}_\varphi$ satisfying (5.4) exists. The solution is non-negative and continuous. Therefore the pricing function $p(y) = f(y)/u'(y)$ is non-negative and continuous as well. □

After having shown that there exists a unique pricing function p given v, the converse remains to be shown. The following argument completes the proof of Proposition 1.

Proof. (Part (2) of Proposition 1). We define the operator H such that:

$$Hv(y,x) = \max_{c,x'\in\Gamma(y,x)}\left\{u(c) + \beta\int_Z v(G(y,z'),x')Q(dz')\right\}. \tag{5.7}$$

The weighting function, now denoted by ϕ, can be defined similarly as:

$$\phi(y,x) = \kappa\cdot\max\{1, y^{1-\gamma}\}. \tag{5.8}$$

Let us assume that the function p is as in part (1) of Proposition 1, and $p \in \mathbf{S}_\phi$.[2] To prove that a unique fixed point H exists, one can resort to Lemma 1 to show that H is a contraction and then use Banach's theorem to establish existence and uniqueness of the fix point. We begin by showing $H : \mathbf{S}_\phi \to \mathbf{S}$. For any $v \in \mathbf{S}_\phi$ and u continuous, $u(c) + \beta\int_Z v(G(y,z'),x)Q(dz')$ is continuous. Since for each (y,x) the budget correspondence is compact valued and continuous, Berge's theorem (Theorem 3.6 in Stokey et al. (1989)) guarantees that Hv is continuous. Hence $H : \mathbf{S}_\phi \to \mathbf{S}$. Monotonicity of H holds. Discounting of H can then be established as in the proof of part (1) of Proposition 1, above. Finally one needs to show that H has the self-map property. In mathematical terms, $H(0) \in \mathbf{S}_\phi$ if there is some $B \in \mathbb{R}_+$ such that:

$$\sup_{(y,x)\in\mathbf{Y}\times\mathbf{X}}\left\{\frac{|\max_{c,x'\in\Gamma(y,x)} u(c)|}{\phi(y,x)}\right\} < B. \tag{5.9}$$

[1] See proof in p.6, Section 3, in Boyd (1990).
[2] $\mathbf{S}_\phi$ is defined as $\mathbf{S}_\varphi$, with φ replaced by ϕ.

To show that (5.9) holds we consider two cases: $\gamma \in (0, 1)$ and $\gamma > 1$. For any $\gamma \in (0, 1)$ condition (5.9) is equivalent to:

$$\sup_{(y,x)\in \mathbf{Y}\times\mathbf{X}} \left\{ \frac{(p(y)x + xy)^{1-\gamma}}{\phi(y, x)} \right\} < B \Leftrightarrow \sup_{y\in[1,\infty)} \left\{ \left(\frac{p(y)}{y} + 1 \right)^{1-\gamma} \right\} < B. \tag{5.10}$$

Now we note that, by definition, $p(y) = f(y)y^{\gamma}$ and as shown in Part (1) of the proof, for all $y \in [1, \infty)$ (and all $x \in \mathbf{X}$): $f(y)y^{\gamma} \leq \phi(y, x)y^{\gamma} = \kappa y$. Therefore it follows that

$$\sup_{y\in[1,\infty)} \left\{ \left(\frac{p(y)}{y} + 1 \right)^{1-\gamma} \right\} \leq (\kappa + 1)^{1-\gamma} < B \tag{5.11}$$

holds for some $B \in \mathbb{R}_+$, which proves that (5.9) holds for all $\gamma \in (0, 1)$. The proof for the complementary case $\gamma > 1$ follows directly from (5.9) and for brevity we omit the argument.[3] Concluding, we showed that H is a contraction and Banach's fixed point theorem establishes that it has a unique solution. □

At this stage, a characterization of function f that solves (5.4) is in order. The following lemma documents some of its properties:

Lemma 2. *Take $f \in \mathbf{S}_{\varphi}$ such that $Tf = f$. Then:*

1. *For any $f_0 \in \mathbf{S}_{\varphi}$, $||T^n f_0 - f||_{\varphi} \to 0$ as $n \to \infty$;*

2. *Suppose $0 < \gamma < 1$. Then, both h and f are strictly increasing and concave. Suppose otherwise $1 < \gamma$. Then both h and f are strictly decreasing and convex.*

Proof. Point 1 of Lemma 2 follows directly from the fact that T is a contraction on a complete metric space and hence, for brevity, will not be proved here. Point 2 is proved in the Appendix of this note. □

Properties of v assumed in Section 5.3 can be shown by arguments similar to Lucas (1978) (see propositions 1 and 2). The proof that the fixed point v of the operator H satisfies the limit condition in Proposition (1) is in Appendix B. This concludes the proof of Proposition 1.

[3]In the case $\gamma > 1$ the assumption $x \geq \underline{x}$ with $\underline{x} > 0$ is needed to show that H has the self-map property.

5.5 Conclusion

We present a proof of the existence and uniqueness of equilibrium in a pure exchange economy of Lucas (1978), when the utility function takes the CRRA form and the dividend stream follows an autoregressive process of order one with positive autocorrelation. An interesting extension of the argument presented in this note, that we leave for future research, is to consider the case in which innovations affect the growth rate of dividends instead of the level.

Appendices

Appendix for "Disagreement, Changing Beliefs, and Stock Market Volatility"

For the sake of a brief notation, I will use the following definitions $\xi_1 = \rho/(1-\rho)$, $\xi_2 = \frac{1-\varphi}{1-\gamma}$ and $\xi_3 = 1/(1-\rho)$ in the appendix. At first, I will document how the value functions of both investors can be rewritten. Take any two states, y and w, the value function of the informed investor is, by definition:

$$V(y_t, w_t) = \sup_{C,\theta} \mathbb{E}\left[\int_t^{t+h} f(c,v)ds + v(y_{t+h}, w_{t+h})\right]$$

Ito's integral rule allows to rewrite the future value function as:

$$\begin{aligned} v(y_{t+h}, w_{t+h}) = v(y_t, w_t) \\ &+ \int_t^{t+h} \frac{\partial v(y_s, w_s)}{\partial w_s} dw_s + \frac{1}{2}\frac{\partial^2 v(y_s, w_s)}{\partial w_s^2}(dw_s)^2 \\ &+ \frac{\partial v(y_s, w_s)}{\partial y_s} dy_s + \frac{1}{2}\frac{\partial^2 v(y_s, w_s)}{\partial y_s^2}(dy_s)^2 \\ &+ \frac{\partial v(y_s, w_s)}{\partial y_s w_s} dw_s dy_s \end{aligned}$$

The value function for the second agent need to be adapated to account for the subjective beliefs. This is done by using Girsanov's theorem. By Girsanov's theorem, one can write the value function as:

$$M_t v(x_t, w_t) = \sup_{C,\theta} \mathbb{E}\left[\int_t^{t+h} M_s f(c_s, v_s)ds + M_{t+h} v(x_{t+h}, w_{t+h})\right] \tag{12}$$

with M being the Radon-Nikdoym derivative (**??**). One can again write the future value function by Ito's integral rule in a similar way.

A Proof of Proposition 1

Guess that the value function can be written as:

$$V(\lambda, w) = \beta^{-\varphi/\xi_2} F(\lambda)^{1/\xi_2} w^{1-\gamma} \frac{1}{1-\gamma}$$

With F a function who's functional form is unknown. After rewriting the value function, the HJB equation for the agent with accurate beliefs can be written as:

$$\begin{aligned}
0 = \sup_{C,\theta} & f(c, V) + \beta^{-\varphi/\xi_2} F(\lambda)^{1/\xi_2} w^{1-\gamma} (r - cw + \theta\mu_R) \qquad (13) \\
& + \beta^{-\varphi/\xi_2} F(\lambda)^{1/\xi_2} w^{1-\gamma} \frac{1}{1-\varphi} \left\{ \frac{F_\lambda(\lambda)}{F(\lambda)} \mu_\lambda + \frac{1}{2} \frac{F_{\lambda,\lambda}(\lambda)}{F(\lambda)} \sigma_\lambda^2 \right\} \\
& + \beta^{-\varphi/\xi_2} F(\lambda)^{1/\xi_2} w^{1-\gamma} \frac{1}{2} \frac{1}{1-\varphi} \left(\frac{1}{\xi_2} - 1 \right) \left(\frac{F_\lambda(\lambda)}{F(\lambda)} \right)^2 \sigma_\lambda^2 \\
& - \beta^{-\varphi/\xi_2} F(\lambda)^{1/\xi_2} w^{1-\gamma} \frac{1}{2} \gamma \theta^2 \sigma_R^2 \\
& + \beta^{-\varphi/\xi_2} F(\lambda)^{1/\xi_2} w^{1-\gamma} \frac{1}{\xi_2} \frac{F_\lambda(\lambda)}{F(\lambda)} \sigma_\lambda \sigma_R \theta \qquad (14)
\end{aligned}$$

The optimal consumption choice is:

$$c = wF(x, \lambda) \qquad (15)$$

Inserting optimal consumption and the guessed value function into the felicity function $f(C, V)$ gives:

$$f(C, V) = F(\lambda)^{1/\xi_2} w^{1-\gamma} \beta^{-\varphi/\xi_2} \left(F(\lambda) \frac{\varphi}{1-\varphi} - \beta \frac{\varphi}{1-\varphi} \right) \qquad (16)$$

Inserting this felicity function, the optimal consumption choice, and making use of the following two definitions:

$$\begin{aligned}
\mu_F(\lambda) &= \frac{F_\lambda(\lambda)}{F(\lambda)} \mu_\lambda + \frac{1}{2} \frac{F_{\lambda,\lambda}(\lambda)}{F(\lambda)} \sigma_\lambda^2 \\
\sigma_F(\lambda) &= \frac{F_\lambda(\lambda)}{F(\lambda)} \sigma_\lambda
\end{aligned}$$

into the HJB equation gives:

$$\begin{aligned}
0 = \sup_{\theta} F(\lambda)^{1/\xi_2} \beta^{\gamma/\rho} w^{1-\gamma} & \left\{ F(\lambda) \frac{1}{\varphi - 1} - \beta \frac{\varphi}{\varphi - 1} + r + \mu_R \theta - \frac{1}{2} \gamma \sigma_R^2 \theta^2 \right. \\
& \left. + \frac{1}{1-\varphi} \mu_F + \frac{1}{2} \frac{1}{1-\varphi} \left(\frac{1}{\xi_2} - 1 \right) \sigma_F^2 + \frac{1}{\xi_2} \sigma_F \sigma_R \theta \right\} \qquad (17)
\end{aligned}$$

The HJB equation for the second agent is very similar. Apart from potential differences in preferences, subjective beliefs introduce covariances. The consumption decision is again $\tilde{c} = \tilde{F}\tilde{w}$ The HJB equation for the agent can be written as:

$$
\begin{aligned}
0 = \sup_{\theta} \tilde{F}(\lambda)^{1/\xi_2} \beta^{\gamma/\rho} w^{1-\gamma} \bigg\{ & \tilde{F}(\lambda)\frac{1}{\varphi-1} - \beta\frac{\varphi}{\varphi-1} + r + \mu_R\tilde{\theta} - \frac{1}{2}\gamma\sigma_R^2\tilde{\theta}^2 \\
& + \frac{1}{1-\varphi}\mu_{\tilde{F}} + \frac{1}{2}\frac{1}{1-\varphi}\left(\frac{1}{\xi_2}-1\right)\sigma_{\tilde{F}}^2 + \frac{1}{\xi_2}\sigma_{\tilde{F}}\sigma_R\tilde{\theta} - \frac{1}{1-\varphi}\sigma_{\tilde{F}}e(x) - \tilde{\theta}\sigma_R e(x) \bigg\}
\end{aligned} \tag{18}
$$

The optimal portfolio choice for the agents is then:

$$
\begin{aligned}
\theta &= \frac{1}{\gamma\sigma_R^2}\left[\mu_R + \frac{1}{\xi_2}\sigma_F\sigma_R\right] \\
\tilde{\theta} &= \frac{1}{\gamma\sigma_R^2}\left[\mu_R + \frac{1}{\xi_2}\tilde{\sigma}_F\sigma_R - \sigma_R e(x)\right]
\end{aligned}
$$

The differential equations for the consumption-wealth ratio of the investor with accurate beliefs can then be written as:

$$
\begin{aligned}
F(\lambda) = \beta\varphi + (1-\varphi)&\left\{ r + \mu_R\theta - \frac{1}{2}\gamma\sigma_R^2\theta^2 + \frac{1}{\xi_2}\sigma_F\sigma_R\theta \right\} \\
&+ \mu_F + \frac{1}{2}\left(\frac{1}{\xi_2}-1\right)\sigma_F^2
\end{aligned}
$$

And the consumption-wealth ratio for the investor with subjective beliefs is:

$$
\begin{aligned}
\tilde{F}(\lambda) = \beta\varphi + (1-\varphi)&\left\{ r + \mu_R\tilde{\theta} - \tilde{\theta}\sigma_R e - \frac{1}{2}\gamma\sigma_R^2\tilde{\theta}^2 + \frac{1}{\xi_2}\sigma_F\sigma_R\theta \right\} \\
&+ \mu_{\tilde{F}} + \frac{1}{2}\left(\frac{1}{\xi_2}-1\right)\sigma_{\tilde{F}}^2 - \sigma_{\tilde{F}}e
\end{aligned}
$$

Finally, by inserting the demand functions, the consumption-wealth ratios can be written even shorter:

$$
\begin{aligned}
F(\lambda) &= \beta\varphi + (1-\varphi)\left(r + \frac{1}{2}\gamma\sigma_R^2\theta^2\right) + \mu_F + \frac{1}{2}\left(\frac{1}{\xi_2}-1\right)\sigma_F^2 \\
\tilde{F}(\lambda) &= \beta\varphi + (1-\varphi)\left(r + \frac{1}{2}\gamma\sigma_R^2\tilde{\theta}^2\right) + \mu_{\tilde{F}} + \frac{1}{2}\left(\frac{1}{\xi_2}-1\right)\sigma_{\tilde{F}}^2 - \sigma_{\tilde{F}}e
\end{aligned}
$$

These two equations need to be solved. Unfortunately, there exist no analytical solution to these two equations but they can be solved numerically using finite difference methods with upwinding, as in Achdou et al. (2015). I will now describe how one can obtain the excess return, volatility, and the risk-free rate.

$$
\lambda\theta + (1-\lambda)\tilde{\theta} = 1 \tag{19}
$$

Inserting the demand functions and solving for $\mu_R(\lambda)$ gives the excess return, as provided in equation (3.3.1). The market clearing condition for the risky asset prescribes asset prices of

$$P = \frac{D}{\lambda cw(\lambda) + (1-\lambda)c\tilde{w}(\lambda)}$$

The price is used to calculate the conditional return volatility σ_R and the risk free interest rate. In equilibrium, returns satisfy:

$$\mathbb{E}\left[\frac{dP}{P} + \frac{D}{P}\right] = \mu_R + r \tag{20}$$

Ito's lemma provides the following diffusion equation for the price: $dP/P = \mu_P dt + \sigma_P dW$ with $\sigma_P = \sigma_R$. The terms σ_R is as in equation XX in the main text. The equation for μ_P is:

$$\begin{aligned}
\mu_P(\lambda) &= \mu_D - \frac{\Delta cw(\lambda) + \overline{cw}_\lambda(\lambda)}{\overline{cw}(\lambda)}\mu_\lambda(\lambda) \\
&+ \left[\left(\frac{\Delta cw(\lambda) + \overline{cw}_\lambda(\lambda)}{\overline{cw}(\lambda)}\right)^2 - \frac{2\Delta cw_\lambda(\lambda) + \overline{cw}_{\lambda\lambda}(\lambda)}{\overline{cw}(\lambda)}\right]\sigma_\lambda(\lambda)^2 \\
&+ -\frac{\Delta cw(\lambda) + \overline{cw}_\lambda(\lambda)}{\overline{cw}(\lambda)}\sigma_\lambda(\lambda)\sigma_D
\end{aligned} \tag{21}$$

Reorganizing the equation for the return and using the fact that $\mathbb{E}(dP/P + D/P) = \mu_P + D/P$ gives the risk-free rate.

B Proof of Corollary 1

The relative wealth share is: $\lambda = w/(w+\tilde{w})$. Ito's lemma allows calculating the diffusion equation for wealth as:

$$d\lambda = \lambda(1-\lambda)\left\{\frac{dw}{w} - \frac{d\tilde{w}}{\tilde{w}} + (1-\lambda)\left(\frac{d\tilde{w}}{\tilde{w}}\right)^2 - \lambda\left(\frac{dw}{w}\right)^2 + (2\lambda-1)\frac{dw}{w}\frac{d\tilde{w}}{\tilde{w}}\right\}$$

The dt terms in this equation are:

$$\mu_\lambda = (1-\lambda)\lambda\left[\frac{dw}{w} - \frac{d\tilde{w}}{\tilde{w}} + \sigma_R^2\left((1-\lambda)\tilde{\theta}^2 - \lambda\theta + (2\lambda-1)\theta\tilde{\theta}\right)\right]$$

Inserting the solution (3.3.1) into the demand equations for the risky asset gives:

$$\theta = \frac{1}{(1-\gamma)\sigma_R^2}\left[(1-\gamma)\sigma_R^2 + (1-\lambda)\left(\sigma_R e(x) + \sigma_R\sigma_\lambda\left(\frac{F_\lambda(x,\lambda)}{F(x,\lambda)} - \frac{\tilde{F}_\lambda(x,\lambda)}{\tilde{F}(x,\lambda)}\right)\right)\right]$$

$$\tilde{\theta} = \frac{1}{(1-\gamma)\sigma_R^2}\left[(1-\gamma)\sigma_R^2 - \lambda\left(\sigma_R e(x) + \sigma_R\sigma_\lambda\left(\frac{F_\lambda(x,\lambda)}{F(x,\lambda)} - \frac{\tilde{F}_\lambda(x,\lambda)}{\tilde{F}(x,\lambda)}\right)\right)\right]$$

The difference between these two terms is:

$$\theta - \tilde{\theta} = \frac{1}{(1-\gamma)\sigma_R}\left[e(x) + \sigma_\lambda \left(\frac{F_\lambda(x,\lambda)}{F(x,\lambda)} - \frac{\tilde{F}_\lambda(x,\lambda)}{\tilde{F}(x,\lambda)}\right)\right]$$

Inserting the solutions for θ and $\tilde{\theta}$ in $(1-\lambda)\tilde{\theta}^2 - \lambda\theta + (2\lambda - 1)\theta\tilde{\theta}$ gives:

$$\begin{aligned}
&- \frac{1}{(1-\gamma)\sigma_R}\left[e(x) + \sigma_\lambda \left(\frac{F_\lambda(x,\lambda)}{F(x,\lambda)} - \frac{\tilde{F}_\lambda(x,\lambda)}{\tilde{F}(x,\lambda)}\right)\right] \\
=& - (\theta - \tilde{\theta})
\end{aligned}$$

The drift term of the relative wealth equation can thus be written as (**??**). Inserting the difference between the demand functions and solving the equation for σ_λ gives (**??**).

C Proof of Proposition 3

The proof of this proposition is very similar to the one above. The main difference is that the function F in the two value function are now functions of x and λ and not only λ anymore. This difference implies that the function μ_F and σ_F are now:

$$\begin{aligned}
\mu_F(x,\lambda) =& \frac{F_\lambda(x,\lambda)}{F(x,\lambda)}\mu_\lambda(x,\lambda) + \frac{F_x(x,\lambda)}{F(x,\lambda)}\mu_x(x) && (22) \\
&+ \frac{1}{2}\frac{F_{\lambda,\lambda}(x,\lambda)}{F(x,\lambda)}\sigma_\lambda(x,\lambda)^2 + \frac{1}{2}\frac{F_{x,x}(x,\lambda)}{F(x,\lambda)}{\sigma_x}^2 + \frac{F_{\lambda,x,}(x,\lambda)}{F(x,\lambda)}\sigma_\lambda(x,\lambda)\sigma_x && (23)
\end{aligned}$$

and

$$\sigma_F(x,\lambda) = \frac{F_\lambda(x,\lambda)}{F(x,\lambda)}\sigma_\lambda(x,\lambda) + \frac{F_x(x,\lambda)}{F(x,\lambda)}\sigma_x \tag{24}$$

inserting these variables in the equations (17) and (18) allows one to obtain the equilibrium variables as

D Proof of Proposition 2

This section proves Proposition 2. To do so, I derive the asset prices and consumption-wealth ratio for the two investors when $\lambda \to \{0,1\}$. As shown by Borovicka (2015) when relative wealth approaches the boundaries, the consumption-wealth ratios approach the consumption-wealth ratios at the boundaries. At the boundaries, the consumption-wealth ratios are functions of x. The consumption-wealth ratio for the investors with accurate beliefs is:

$$
\begin{aligned}
cw(x) &= \beta\varphi \\
&+ (1-\varphi)\left\{\underbrace{r(x) + \mu_R(x)\theta(x)}_{\text{Expected Portfolio Return}} - \underbrace{\frac{\gamma}{2}[\sigma_R(x)\theta(x)]^2}_{\text{Risk Adjustment}} + \underbrace{\frac{1-\gamma}{1-\varphi}\sigma_F(x)\sigma_R(x)\theta(x)}_{\text{Hedging Motive}}\right\} \\
&+ \mu_{cw}(x) + \frac{1}{2}\frac{\varphi-\gamma}{1-\varphi}\sigma_{cw}(x)^2
\end{aligned}
\tag{25}
$$

and the consumption-wealth ratio for the investor with subjective beliefs is:

$$
\begin{aligned}
\tilde{cw}(x,e) &= \beta\varphi \\
&+ (1-\varphi)\left\{r(x) + \underbrace{\tilde{\theta}(e)\left[\mu_R - \sigma_R e\right]}_{\text{Expected Portolio Return}} - \underbrace{\frac{1}{2}\gamma\left[\sigma_R\tilde{\theta}(e)\right]^2}_{\text{Risk adjustment}} + \underbrace{\frac{1-\gamma}{1-\varphi}\sigma_{\tilde{cw}}\sigma_R\tilde{\theta}(e)}_{\text{Hedging Motive}}\right\} \\
&+ \mu_{\tilde{cw}}(x) + \frac{1}{2}\frac{\varphi-\gamma}{1-\varphi}\sigma_{\tilde{cw}}^2 - \sigma_{\tilde{cw}}e
\end{aligned}
\tag{26}
$$

I proceed in the similar way as Chacko et al. (2005): I log-linearly approximate the left hand side of the two equations above by $cw \approx h_0 + h_1 \log \hat{cw}$ and the use a equation for cw which solves the resulting differential equation. I look at first at $\lambda \to 1$ and then at the other end.

Rational Investor prices the Asset

Suppose the linearly approximated equation (25) can be solved by the following equation:

$$\hat{cw} = \exp\left(A_0 + A_1 x\right)$$

As $\lambda \to 1$, the investor with objective beliefs assumes all aggregate risk. The volatility and excess return can then be written (the terms are derived using the same methods as in the proof of Proposition 1):

$$
\begin{aligned}
\sigma_R &= \sigma_D - A_1\sigma_x \\
\mu_R &= \gamma\sigma_R^2 - \frac{1-\gamma}{1-\varphi}A_1\sigma_x\sigma_R
\end{aligned}
$$

Such constant excess return and volatility have also been derived by Bansal et al. (2004) in their log-linear approximation. The risk-free asset can be derived from the pricing equation

$$\mu_{R,t} = \mathbb{E}_t\left[\frac{dP_t}{P_t} + \frac{D_t}{P_t}dt\right] - r_t \tag{27}$$

By rewriting the terms in expectations, one can write the risk-free rate as $r(x) = \mu_p(x) + F(x) - \mu_R$ with μ_P being:

$$\mu_P(x) = \mu_D(x) - \mu_F(x) + \sigma_F^2 - \sigma_F \sigma_D \tag{28}$$

Inserting the consumption-wealth ratio into the pricing equation for the risk-free asset gives:

$$r = \frac{1}{\varphi} x + \Phi \tag{29}$$

with Φ being a constant.

cw ratio for the agent with accurate beliefs The guess for the consumption-wealth ratios of the investor with subjective beliefs allowed calculating the excess return, volatility and risk-free rate. The guess can now be verified. I describe at first how to solve for A_1. Using equation (25), there are two instances involving the state x, the risk-free rate and the term for μ_F. The value A_1 that sets all terms to zero is:

$$A_1 = \frac{1-\varphi}{\varphi} \frac{1}{h_1 + \kappa} \tag{30}$$

All other terms are constants and A_0 can be chosen appropriately.

cw for the Investor with Subjective Beliefs The consumption-wealth ratio for the agent with subjective beliefs is slightly different. First, the agent can choose the optimal demand θ taking prices as given. Second, the consumption-wealth ratio also contains elements for the disagreement e. Suppose the function for the cw ratio is:

$$\hat{c\tilde{w}} = \exp(\tilde{A}_0 + \tilde{A}_1 x + \tilde{A}_2 e + \tilde{A}_3 e^2) \tag{31}$$

The terms $\tilde{A}_0, \tilde{A}_1$ are as before. As seen from the equation (26) the quadratic effects of belief disagreement are contained in the terms

$$\underbrace{\tilde{\theta}(e)\left[\mu_R - \sigma_R e\right]}_{\text{Expected Portolio Return}} - \underbrace{\frac{1}{2}\gamma\left[\sigma_R \tilde{\theta}(e)\right]^2}_{\text{Risk adjustment}} \tag{32}$$

Disagreement increases the risk-adjusted returns by $0.5e^2/\gamma$. With an EIS larger than one, the higher risk-adjusted portfolio return reduces the consumption-wealth ratio by $(1-\varphi)$. The term $\tilde{A}_3$ is:

$$\tilde{A}_3 = (1-\varphi)\frac{1}{h_1}\frac{1}{2}\frac{1}{\gamma} \tag{33}$$

The linear effects can be summarized by the *expected returns channel* and the *utility distortion channel*. Whilst the actual expected excess return is μ_R, the investor with subjective beliefs expects an excess return of $\mu_R - \sigma_R e$; the effect on disagreement on the difference between the expected returns represent the *expected returns channel*. Belief distortion also correlates with the consumption-wealth ratio through $-\sigma_F e$. The sum of the two channels determines the parameter $\tilde{A}_2$:

$$\tilde{A}_2 = -[(1-\varphi)\sigma_R + \sigma_{\tilde{F}}]\frac{1}{h_1}$$

The difference in log consumption-wealth ratios The inequality $\log \Delta cw(x)e < 0$ is:

$$\left(-\tilde{A}_2 e - \tilde{A}_3 e^2\right) e < 0 \tag{34}$$

which can be written as: $\tilde{A}_2 + \tilde{A}_3 e > 0$. The term $\tilde{A}_2$ contains endogenous variables, the return volatility σ_R and the volatility of the consumption-wealth ratio σ_F. Inserting the approximate solutions transforms the inequality above to:

$$\frac{1}{2}\frac{1}{\gamma}e - \left[\sigma_D + \sigma_x \frac{1}{h_1 + \kappa_x}\right] < 0 \tag{35}$$

The term in square brackets is always positive. Thus whenever the investor with subjective beliefs is optimistic ($e < 0$) the inequality is satisfied for all values of e. However, when the investor is pessimistic, the inequality is only satisfied if the following inequality holds:

$$2\gamma \left[\sigma_D + \sigma_x \frac{1}{h_1 + \kappa_x}\right] > e \tag{36}$$

Agent with subjective beliefs prices the asset

One can proceed in the same way as above when $\lambda \to 0$. However, the asset prices are now a function of the disagreement term:

$$\sigma_R = \sigma_D - A_1 \sigma_x \tag{37}$$

$$\mu_R = \gamma\sigma_R^2 - \frac{1-\gamma}{1-\varphi}A_1\sigma_x\sigma_R + e\sigma_R \tag{38}$$

$$r = \frac{1}{\varphi}(x - \sigma_D e) + \Phi \tag{39}$$

Cw for rational agent Because prices contain a part e, the cw ratio of the agent with rational beliefs contains a part for e as well:

$$\hat{cw} = \exp(A_0 + A_1 x + A_2 e + A_3 e^2) \tag{40}$$

The terms for A_0, A_1, A_3 are the same as $\tilde{A}_0, \tilde{A}_1, \tilde{A}_3$. However, the terms for the linear effects of disagreement A_2 are different from before. That term is:

$$A_2 = (1-\varphi)\frac{1}{h_1}[\sigma_R - \frac{1}{\varphi}\sigma_D] \tag{41}$$

Cw for agent with subjective beliefs The cw-ratio of the investor with subjective beliefs does not contain a quadratic element for e any more, as the investor cannot choose the optimal risky-share any more. The cw ratio is now:

$$\hat{cw} = \exp(A_0 + A_1 x + A_2 e) \tag{42}$$

The linear effects for disagreement are:

$$A_2 = \frac{1}{h_1}\left[\frac{\varphi - 1}{\varphi}\sigma_D - \sigma_F\right]$$

The difference in consumption-wealth ratios The condition $\Delta \log cw(x)e < 0$ can be written as:

$$\left(A_2 e + A_3 e^3 - \tilde{A}_2 e\right) e < 0 \tag{43}$$

Inserting the solution for the terms, one can rewrite the condition to:

$$\sigma_D + \sigma_x \frac{1}{h_1 + \kappa_x} + \frac{1}{2}\frac{1}{\gamma} e > 0 \tag{44}$$

The equation is always satisfied whenever the investor with subjective beliefs is pessimistic, $e > 0$. However, when the investor is optimistic, the inequality is only satisfied when the condition stated in Proposition 2 holds.

Appendix for "Existence and uniqueness of equilibrium in Lucas' Asset Pricing model when utility is unbounded"

E Proof of Part 2 of Lemma 2

Proof. (Part 2 of Lemma 2) As in the main text, denote by S_φ the set of continuous and φ-bounded functions. The set S'_φ is the set of continuous, φ-bounded, non-decreasing and concave functions, and $S''_\varphi \subset S'_\varphi$ imposes additionally strict monotonicity and concavity. We want to show that the contraction operator T maps any function $\tilde{f} \in S'_\varphi$ into the subset S''_φ. As the solution to the functional equation is characterized by $Tf = f$ and S'_φ is a closed set, if the operator T transforms any non-decreasing and concave function into a strictly increasing and concave function, then f is strictly increasing and concave (Corollary 1 of the Contraction Mapping Theorem in Stokey et al. 1989, p.52). To show the desired result, we suppose first that h is strictly increasing and concave and pick any $\tilde{f} \in S'_{\varphi}$. To begin, let us study whether $T\tilde{f}$ is strictly increasing. For any pair $\hat{y}, y \in Y$ with $\hat{y} > y$, the function $T\tilde{f}$ satisfies:

$$
\begin{aligned}
T\tilde{f}(\hat{y}) &= h(\hat{y}) + \beta \int_Z \tilde{f}(G(\hat{y}, z'))Q(dz') \\
&> h(y) + \beta \int_Z \tilde{f}(G(y, z'))Q(dz') \\
&= T\tilde{f}(y).
\end{aligned}
$$

The inequality holds because G and h are strictly increasing and $\tilde{f}$ is non-decreasing. Hence, $T\tilde{f}$ is strictly increasing. To analyse concavity, define $y_\omega = \omega y + (1-\omega)y'$, for any $y, y' \in Y$, $y \neq y'$, and $0 < \omega < 1$. The strict concavity form of h and G, together with $\tilde{f}$ being concave, ensure that:

$$
\begin{aligned}
T\tilde{f}(y_\omega) &= h(y_\omega) + \beta \int_Z \tilde{f}(G(y_\omega, z'))Q(dz') \\
&> \omega \left[h(y) + \beta \int_Z \tilde{f}(G(y, z'))Q(dz')\right] + (1-\omega) \left[h(y') + \beta \int_Z \tilde{f}(G(y', z'))Q(dz')\right] \\
&= \omega T\tilde{f}(y) + (1-\omega)T\tilde{f}(y').
\end{aligned}
$$

The function $T\tilde{f}$ is strictly concave. Taken together, we know that for any $\tilde{f} \in S'_{\varphi}, T\tilde{f} \in S''_{\varphi}$. Hence, f (such that $Tf = f$) must be an element of the set S''_{φ}, guaranteeing that f has the same functional form as h. Now, suppose h is convex and falling. We could again define the operator T as $Tf(y) = h(y) + \beta \int_Z f(G(y, z'))Q(dz')$ and study into which subset a candidate solution is mapped into. To facilitate analysis though, take a different route. Look at the modified operator $Tf_- = h_- + \beta \int_Z f_-(G(y, z'))Q(z')$, with $h_- = -h$ and $f_- = -f$. Under the same assumptions guaranteeing a unique solution to the original contraction mapping, there exists a unique solution to the modified contraction mapping. As h_- is strictly increasing and concave, the proof above applies to the modified contraction mapping. As f_- is strictly increasing and concave, f is strictly decreasing and convex and inherits the properties of h.

□

F Limit condition on v

Let us take $v \in S_\phi$ sucht that $Hv = v$, with the operator H as defined in Section 5.4 of the main text. Our initial aim is to characterise lower and upper bounds on v in the functional space S_ϕ. We will now argue that when $v \geq 0$ (respectively ≤ 0), v can be bounded below (resp. above) using the zero function and above (resp. below) using function ϕ (respec. $-\phi$). To this end, we consider any $\gamma \in (0, 1)$. Define the set $S'_\phi = \{f \in S_\phi : f \geq 0\}$. This is a closed subset of S_ϕ (its complement in S_ϕ is open). We pick any $f \in S'_\phi$. Since the utility function u takes on positive values, $Hf \geq 0$ Thus, since f was arbitrary, $H : S'_\phi \to S'_\phi$. Then by Corollary 1 of the CMT (p.52) in Stokey et al. 1989 $v \in S'_\phi$, i.e. $v \geq 0$. A similar argument shows that for any $\gamma > 1$, $v \leq 0$. The remainder of this section shows that the discounted expected vale of the upper bound converges to zero, implying that $\lim_{t\to\infty} \mathbb{E}_0 \left[\beta^t v(x_t, y_t)\right] = 0$. We consider any $\gamma \in (0, 1)$ (For $\gamma > 1$, the condition is equivalent, as the constant in the bound changes the sign of the bound.) Then for any t, $x_t \in X$ and $y_t \in Y$:

$$\begin{aligned}
\mathbb{E}_0 \left[\beta^t v(x_t, y_t)\right] &\leq \mathbb{E}_0 \left[\beta^t \phi\right] \\
&= \kappa\beta^t\, \mathbb{E}_0 \left[\mathbb{E}_{t-1} \left[max \left\{1, y_t^{1-\gamma}\right\}\right]\right] \\
&\leq \kappa\beta^t\, \mathbb{E}_0 \left[0.5 + y_{t-1}^{\alpha(1-\gamma)} \int_0^\infty Q(dz')(z')^{1-\gamma}\right] \\
&\leq \kappa\beta^t \left[0.5 + y_0^{1-\gamma} \exp\left(t(1-\gamma)^2\sigma^2/2\right)\right] \\
&= \kappa \left[\beta^t 0.5 + y_0^{1-\gamma} \exp\left(t\left[\log(\beta) + (1-\gamma)^2\sigma^2/2\right]\right)\right]
\end{aligned}$$

The first line follows from the fact that $v \in S_\phi$. The second line uses the definition of ϕ and the law of iterated expectations. The third line bounds the the term in brackets. The strategy is identical to the one used in the proof of Proposition 1. The fourth lines iterates until time zero and uses the fact that $\alpha \in (0, 1)$. The fifth line factors β^t in. The entire sums converges to zero with $t \to \infty$ if $\log(\beta) + (1-\gamma)^2\sigma^2/2 < 0$. The proof for $\gamma > 1$ is analogous.

Bibliography

[1] Yves Achdou et al. “Heterogeneous agent models in continuous time”. 2015 (cit. on p. 66).

[2] Fernando Alvarez and Nancy L Stokey. “Dynamic Programming with Homogeneous Functions”. In: *Journal of Economic Theory* 82.1 (Sept. 1998), pp. 167–189 (cit. on pp. 54, 58).

[3] Yakov Amihud. “Illiquidity and stock returns: cross-section and time-series effects”. In: *Journal of Financial Markets* 5.1 (Jan. 2002), pp. 31–56 (cit. on p. 42).

[4] Yakov Amihud and Haim Mendelson. “Asset pricing and the bid-ask spread”. In: *Journal of Financial Economics* 17.2 (Dec. 1986), pp. 223–249 (cit. on p. 42).

[5] Evan W. Anderson, Eric Ghysels, and Jennifer L. Juergens. “The impact of risk and uncertainty on expected returns”. In: *Journal of Financial Economics* 94.2 (Nov. 2009), pp. 233–263 (cit. on p. 16).

[6] European Central Bank. *Financial Stability Review*. Feb. 2017 (cit. on p. 6).

[7] Ravi Bansal and Amir Yaron. “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles”. en. In: *The Journal of Finance* 59.4 (Aug. 2004), pp. 1481–1509 (cit. on pp. 13, 26, 28, 30, 36, 40, 69).

[8] Nicholas Barberis et al. “X-CAPM: An extrapolative capital asset pricing model”. In: *Journal of Financial Economics* 115.1 (Jan. 2015), pp. 1–24 (cit. on p. 39).

[9] Jason Beeler and John Y. Campbell. “The Long-Run Risks Model and Aggregate Asset Prices: An Empirical Assessment”. English. In: *Critical Finance Review* 1.1 (Jan. 2012), pp. 141–182 (cit. on p. 40).

[10] Anmol Bhandari, Jaroslav Borovicka, and Paul Ho. *Identifying Ambiguity Shocks in Business Cycle Models Using Survey Data*. Working Paper 22225. DOI: 10.3386/w22225. National Bureau of Economic Research, May 2016 (cit. on p. 35).

[11] David Blackwell. "Discounted Dynamic Programming". EN. In: *The Annals of Mathematical Statistics* 36.1 (Feb. 1965), pp. 226–235 (cit. on p. 52).

[12] Nicholas Bloom. "The Impact of Uncertainty Shocks". In: *Econometrica* 77.3 (2009), pp. 623–685 (cit. on p. 14).

[13] Tim Bollerslev, Andrew J. Patton, and Rogier Quaedvlieg. "Exploiting the errors: A simple approach for improved volatility forecasting". In: *Journal of Econometrics* 192.1 (May 2016), pp. 1–18 (cit. on p. 12).

[14] Jaroslav Borovicka. *Survival and long-run dynamics with heterogeneous beliefs under recursive preferences*. Tech. rep. 2015 (cit. on pp. 9, 23, 40, 68).

[15] John III Boyd. "Recursive utility and the Ramsey problem". In: *Journal of Economic Theory* 50.2 (1990), pp. 326–345 (cit. on pp. 53, 57, 61).

[16] Jonathan Brogaard and Andrew Detzel. "The Asset-Pricing Implications of Government Economic Policy Uncertainty". In: *Management Science* 61.1 (Jan. 2015), pp. 3–18 (cit. on p. 14).

[17] Markus K. Brunnermeier and Stefan Nagel. "Do Wealth Fluctuations Generate Time-Varying Risk Aversion? Micro-evidence on Individuals". In: *American Economic Review* 98.3 (June 2008), pp. 713–36 (cit. on p. 40).

[18] Andrea Buraschi, Fabio Trojani, and Andrea Vedolin. "Economic Uncertainty, Disagreement, and Credit Markets". In: *Management Science* 60.5 (Dec. 2013), pp. 1281–1296 (cit. on pp. 16, 17).

[19] Ovidiu L. Calin et al. "Solving Asset Pricing Models when the Pricefffdfffdfffd-Dividend Function Is Analytic". en. In: *Econometrica* 73.3 (May 2005), pp. 961–982 (cit. on pp. 53, 54).

[20] John Y. Campbell. "Intertemporal Asset Pricing without Consumption Data". In: *The American Economic Review* 83.3 (1993), pp. 487–512 (cit. on p. 25).

[21] John Y. Campbell and John H. Cochrane. "By Force of Habit: A ConsumptionfffdfffdfffdBased Explanation of Aggregate Stock Market Behavior". In: *Journal of Political Economy* 107.2 (1999), pp. 205–251 (cit. on pp. 13, 40).

[22] John Y. Campbell, Sanford J. Grossman, and Jiang Wang. "Trading Volume and Serial Correlation in Stock Returns". en. In: *The Quarterly Journal of Economics* 108.4 (Nov. 1993), pp. 905–939 (cit. on p. 43).

[23] John Y. Campbell and Robert J. Shiller. "Stock Prices, Earnings, and Expected Dividends". In: *The Journal of Finance* 43.3 (1988), pp. 661–676 (cit. on p. 12).

[24] John Y. Campbell and Luis M. Viceira. "Consumption and Portfolio Decisions when Expected Returns are Time Varying". In: *The Quarterly Journal of Economics* 114.2 (May 1999), pp. 433–495 (cit. on pp. 25, 29).

[25] John Y. Campbell and Tuomo Vuolteenaho. "Bad Beta, Good Beta". In: *American Economic Review* 94.5 (Dec. 2004), pp. 1249–1275 (cit. on p. 16).

[26] Bruce I. Carlin, Francis A. Longstaff, and Kyle Matoba. "Disagreement and asset prices". en. In: *Journal of Financial Economics* 114.2 (Nov. 2014), pp. 226–238 (cit. on pp. 14, 16).

[27] George Chacko and Luis M. Viceira. "Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets". In: *The Review of Financial Studies* 18.4 (Dec. 2005), pp. 1369–1402 (cit. on pp. 30, 69).

[28] In-Koo Cho. "Perceptrons Play the Repeated Prisonerfffdfffdfffds Dilemma". In: *Journal of Economic Theory* 67.1 (Oct. 1995), pp. 266–284 (cit. on p. 44).

[29] John H. Cochrane. "Presidential Address: Discount Rates". en. In: *The Journal of Finance* 66.4 (Aug. 2011), pp. 1047–1108 (cit. on p. 13).

[30] John H. Cochrane. *Volume and Information*. 2016 (cit. on p. 7).

[31] Julien Cujean and Michael Hasler. "Why Does Return Predictability Concentrate in Bad Times?" en. In: *The Journal of Finance* (), n/a–n/a (cit. on p. 39).

[32] Karl B. Diether, Christopher J. Malloy, and Anna Scherbina. "Differences of Opinion and the Cross Section of Stock Returns". In: *The Journal of Finance* 57.5 (2002), pp. 2113–2141 (cit. on pp. 16, 17).

[33] William Dudley. *Transcript of the Cornell College of Business Annual New York City Predictions Event: February 15, 2017 - FEDERAL RESERVE BANK of NEW YORK*. Feb. 2017 (cit. on p. 6).

[34] Darrell Duffie and Larry G. Epstein. "Stochastic Differential Utility". In: *Econometrica* 60.2 (Mar. 1992), pp. 353–394 (cit. on p. 18).

[35] Eugene F Fama and Kenneth R French. "Dividend yields and expected stock returns". In: *Journal of Financial Economics* 22.1 (Oct. 1988), pp. 3–25 (cit. on p. 12).

[36] Kenneth R. French, G. William Schwert, and Robert F. Stambaugh. "Expected stock returns and volatility". In: *Journal of Financial Economics* (1987), pp. 3–30 (cit. on p. 13).

[37] A. Ronald Gallant, Peter E. Rossi, and George Tauchen. "Stock Prices and Volume". en. In: *Review of Financial Studies* 5.2 (Apr. 1992), pp. 199–242 (cit. on p. 12).

[38] Nicolae Garleanu and Stavros Panageas. "Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing". In: *Journal of Political Economy* 123.3 (June 2015), pp. 670–685 (cit. on p. 39).

[39] Matthieu Gomez. "Asset Prices and Wealth Inequality". PhD thesis. Princeton, 2017 (cit. on p. 39).

[40] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. English. Cambridge, Massachusetts: The MIT Press, Nov. 2016 (cit. on p. 44).

[41] Robin Greenwood and Andrei Shleifer. "Expectations of Returns and Expected Returns". In: *The Review of Financial Studies* 27.3 (Mar. 2014), pp. 714–746 (cit. on p. 15).

[42] Sanford J. Grossman and Joseph E. Stiglitz. "On the Impossibility of Informationally Efficient Markets". In: *American Economic Review* 70.3 (1980), pp. 393–408 (cit. on p. 43).

[43] Luigi Guiso, Paola Sapienza, and Luigi Zingales. *Time Varying Risk Aversion*. Working Paper 19284. DOI: 10.3386/w19284. National Bureau of Economic Research, Aug. 2013 (cit. on p. 40).

[44] Lars Peter Hansen et al. "Chapter 61 Intertemporal Substitution and Risk Aversion". In: *Handbook of Econometrics*. Ed. by James J. Heckman and Edward E. Leamer. Vol. Volume 6, Part A. Elsevier, 2007, pp. 3967–4056 (cit. on p. 25).

[45] Milton Harris and Artur Raviv. "Differences of Opinion Make a Horse Race". In: *The Review of Financial Studies* 6.3 (1993), pp. 473–506 (cit. on p. 43).

[46] J. Michael Harrison and David M. Kreps. “Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations”. In: *The Quarterly Journal of Economics* 92.2 (May 1978), pp. 323–336 (cit. on p. 38).

[47] Kurt Hornik. “Approximation capabilities of multilayer feedforward networks”. In: *Neural Networks* 4.2 (Jan. 1991), pp. 251–257 (cit. on p. 47).

[48] Gregory W. Huffman. “A Dynamic Equilibrium Model of Asset Prices and Transaction Volume”. In: *Journal of Political Economy* 95.1 (1987), pp. 138–159 (cit. on p. 39).

[49] Charles M. Jones. *A Century of Stock Market Liquidity and Trading Costs*. SSRN Scholarly Paper ID 313681. Rochester, NY: Social Science Research Network, May 2002 (cit. on p. 42).

[50] Kyle Jurado, Sydney C. Ludvigson, and Serena Ng. “Measuring Uncertainty”. In: *American Economic Review* 105.3 (Mar. 2015), pp. 1177–1216 (cit. on p. 14).

[51] Takashi Kamihigashi. “Uniqueness of asset prices in an exchange economy with unbounded utility”. en. In: *Economic Theory* 12.1 (June 1998), pp. 103–122 (cit. on p. 54).

[52] Eugene Kandel and Neil D. Pearson. “Differential Interpretation of Public Signals and Trade in Speculative Markets”. In: *Journal of Political Economy* 103.4 (1995), pp. 831–872 (cit. on p. 43).

[53] John Maynard Keynes. *The general theory of employment, interest, and money*. Palgrave Macmillan for the Royal Economic Society, 1936 (cit. on p. 6).

[54] Felix Kubler and Karl Schmedders. “Financial Innovation and Asset Price Volatility”. In: *American Economic Review* 102.3 (May 2012), pp. 147–151 (cit. on p. 39).

[55] Robert E. Lucas Jr. “Asset Prices in an Exchange Economy”. In: *Econometrica* 46.6 (Nov. 1978). ArticleType: research-article / Full publication date: Nov., 1978 / Copyright fffdfffd 1978 The Econometric Society, pp. 1429–1445 (cit. on pp. 29, 50, 52, 54–56, 62, 63).

[56] N. Gregory Mankiw, Ricardo Reis, and Justin Wolfers. *Disagreement about Inflation Expectations*. Working Paper 9796. DOI: 10.3386/w9796. National Bureau of Economic Research, June 2003 (cit. on p. 35).

[57] Ramon Marimon, Ellen McGrattan, and Thomas J. Sargent. “Money as a medium of exchange in an economy with artificially intelligent agents”. In: *Journal of Economic Dynamics and Control* 14.2 (May 1990), pp. 329–373 (cit. on p. 44).

[58] V. Filipe Martins-da-Rocha and Yiannis Vailakis. “Existence and Uniqueness of a Fixed Point for Local Contractions”. en. In: *Econometrica* 78.3 (May 2010), pp. 1127–1141 (cit. on p. 53).

[59] Janusz Matkowski and Andrzej S. Nowak. “On discounted dynamic programming with unbounded returns”. en. In: *Economic Theory* 46.3 (2011), pp. 455–474 (cit. on p. 53).

[60] Robert C Merton. “Optimum consumption and portfolio rules in a continuous-time model”. In: *Journal of Economic Theory* 3.4 (Dec. 1971), pp. 373–413 (cit. on pp. 29, 50).

[61] Paul Milgrom and Nancy Stokey. “Information, trade and common knowledge”. In: *Journal of Economic Theory* 26.1 (Feb. 1982), pp. 17–27 (cit. on p. 43).

[62] Edward M. Miller. “Risk, Uncertainty, and Divergence of Opinion”. In: *The Journal of Finance* 32.4 (1977), pp. 1151–1168 (cit. on pp. 16, 38).

[63] Guido Montufar et al. “On the Number of Linear Regions of Deep Neural Networks”. In: *arXiv:1402.1869 [cs, stat]* (Feb. 2014). arXiv: 1402.1869 (cit. on p. 47).

[64] Vinod Nair and Geoffrey E. Hinton. “Rectified linear units improve restricted boltzmann machines”. In: 2010, pp. 807–814 (cit. on p. 45).

[65] Efe A. Ok. *Real Analysis with Economic Applications*. English. Princeton: Princeton University Press, Jan. 2007 (cit. on p. 52).

[66] Lubos Pastor and RobertfffdfffdF. Stambaugh. “Liquidity Risk and Expected Stock Returns”. In: *Journal of Political Economy* 111.3 (June 2003), pp. 642–685 (cit. on p. 42).

[67] Andrew J. Patton and Allan Timmermann. “Why do forecasters disagree? Lessons from the term structure of cross-sectional dispersion”. In: *Journal of Monetary Economics* 57.7 (Oct. 2010), pp. 803–820 (cit. on p. 36).

[68] Monika Piazzesi and Martin Schneider. *Trend and cycle in bond premia*. Staff Report 424. Federal Reserve Bank of Minneapolis, 2009 (cit. on p. 11).

[69] Ronnie Sadka and Anna Scherbina. “Analyst Disagreement, Mispricing, and Liquidity*”. en. In: *The Journal of Finance* 62.5 (Oct. 2007), pp. 2367–2403 (cit. on p. 16).

[70] Alvaro Sandroni. “Do Markets Favor Agents Able to Make Accurate Predictions?” In: *Econometrica* 68.6 (2000), pp. 1303–1341 (cit. on p. 40).

[71] Shrihari Santosh. *The Speed of Price Discovery: Trade Time vs Clock Time*. SSRN Scholarly Paper ID 2567486. Rochester, NY: Social Science Research Network, Nov. 2016 (cit. on p. 43).

[72] Thomas J. Sargent. *Bounded rationality in macroeconomics: The Arne Ryde memorial lectures*. Oxford [England] : Oxford ; New York: Clarendon Press; Oxford University Press, 1993 (cit. on p. 44).

[73] Leonard J. Savage. *The Foundations of Statistics*. English. 1st edition. John Wiley & Sons, 1954 (cit. on p. 6).

[74] Jose A. Scheinkman and Wei Xiong. “Overconfidence and Speculative Bubbles”. In: *Journal of Political Economy* 111.6 (2003), pp. 1183–1219 (cit. on p. 38).

[75] G. William Schwert. “Why Does Stock Market Volatility Change Over Time?” en. In: *The Journal of Finance* 44.5 (Dec. 1989), pp. 1115–1153 (cit. on pp. 12, 13).

[76] Robert J. Shiller. “Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?” In: *The American Economic Review* 71.3 (June 1981), pp. 421–436 (cit. on pp. 22, 26).

[77] Robert J Shiller. *Market volatility*. English. Cambridge, Mass.: MIT Press, 1992 (cit. on p. 12).

[78] John Stachurski. *Economic Dynamics: Theory and Computation*. English. Cambridge, Mass: The MIT Press, Jan. 2009 (cit. on p. 57).

[79] Nancy L. Stokey and Robert E. Lucas. *Recursive methods in economic dynamics*. Cambridge, Mass: Harvard University Press, 1989 (cit. on pp. 61, 73, 74).

[80] Stijn Van Nieuwerburgh and Laura Veldkamp. “Learning asymmetries in real business cycles”. In: *Journal of Monetary Economics* 53.4 (May 2006), pp. 753–772 (cit. on p. 12).

[81] Pietro Veronesi. “Stock Market Overreactions to Bad News in Good Times: A Rational Expectations Equilibrium Model”. In: *The Review of Financial Studies* 12.5 (Oct. 1999), pp. 975–1007 (cit. on p. 12).

[82] Jiang Wang. “A Model of Intertemporal Asset Prices Under Asymmetric Information”. In: *The Review of Economic Studies* 60.2 (Apr. 1993), pp. 249–282 (cit. on p. 43).

[83] Brian M. Weller. *Efficient Prices at Any Cost: Does Algorithmic Trading Deter Information Acquisition?* SSRN Scholarly Paper ID 2662254. Rochester, NY: Social Science Research Network, May 2016 (cit. on p. 43).

[84] Halbert White. “A Heteroskedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroskedasticity”. In: *Econometrica* 48.4 (1980), pp. 817–38 (cit. on p. 15).

[85] Jialin Yu. “Disagreement and return predictability of stock portfolios”. In: *Journal of Financial Economics* 99.1 (Jan. 2011), pp. 162–183 (cit. on pp. 16, 17).