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Abstract

This paper makes use of optimal control relaxed problems to prove the absence of optimal trajectory in continuous time models with social increasing returns to scale where indeterminacy occurs. Although an efficient optimal policy does not exist, some chattering stabilization policies can mimic trajectories whose criterion functional approximates the supremum of the relaxed problem. This configuration is closely related to indeterminacy: by contrast, when the steady state is determined, an optimal policy is likely to exist.

Key words: Increasing returns, Indeterminacy, Stabilization policy, Relaxed problems.
JEL classification: C61, C62, E32, E6, H61, O4.

1 Introduction

In the literature on sunspot equilibria in a representative agent framework, various authors proposed economic policies that can pin down self-fulfilling fluctuations when the non-convexities of the social production set are sufficient to make the stationary equilibrium locally indeterminate.[1] Guo and Lansing [1998], for instance, specify a tax policy in a one-sector model *à la* Benhabib and Farmer [1994] that is able to stabilize the economy. They show that a sufficiently strong progressive tax on output is necessary to get local determinacy of the stationary equilibrium. This policy plays the role of an automatic stabilizer: it taxes away extra returns resulting from belief-driven fluctuations in expansion phases (taxes increase as the production externality raises) whereas in recession phases taxes decrease in order to dampen the

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[1] These social increasing returns to scale come from productive externalities in the aggregate level of production whereas returns to scale are assumed to be constant in the private level of the firm. As already shown by Benhabib and Farmer [1994], an equivalent pattern can be obtained in a different setup emboying monopoly powers.

diminution of the after-tax income. Consequently, the tax schedule smoothes business cycle fluctuations and stabilizes the economy.

In framing their stabilization policy, Guo and Lansing [1998] pay attention to the local properties of the stationary equilibrium but do not estimate the impact of the policy on the representative agent's welfare. As pointed out by Christiano and Harrison [1999], due to the concavity of the utility function, for the same level of productive externalities fluctuations in expectations are welfare-reducing (*concavity effect*) in contrast with increasing returns to scale that improve welfare: by bunching hard work agents may jointly increase the average level of consumption and decrease the average level of labor effort (*bunching effect*). This leads to an uncertain conclusion as to the overall welfare effect of the stabilization policy. They give examples in which consumers are better-off in a situation of stochastic sunspot equilibria than in a situation of purely deterministic equilibrium. Thus, Guo and Lansing's objective to break the concavity effect and prevent the economy from fluctuations may deteriorate the welfare-improving bunching effect.

The first-best allocation of their economy, simulated by Dupor and Lehnert [2002] in a discrete time framework, provides an important benchmark for judging the desirability of any stabilization policy. The purpose of this paper is to prove the absence of such an allocation in a slightly different setup. [2] We use a continuous time model that exhibits indeterminacy under plausible values of the increasing returns to scale, i.e. returns to scale compatible with the earlier estimates: those of Basu and Fernald [1995, 1997], Burnside, Eichenbaum and Rebelo [1995] or Burnside [1997], among others. The selected framework is the Wen model [1998] modifying the original Benhabib and Farmer one-sector model by integrating a capacity utilization. In the continuous time version developed in this paper, indeterminacy occurs for almost constant returns to scale provided the elasticity of labor supply is perfectly elastic and the cost structure on capacity utilization is barely convex.

Although an optimal allocation does not exist in a continuous time framework, it will be shown however that there is a sequence of "chattering" consumption/investment plans converging to a supremum. This supremum is not a feasible trajectory of the original problem but is the optimum of an artificially convexified problem, usually denoted by *generalized* or *relaxed problem.* [3] This

[2] The conclusions we find also apply to the canonical continuous time Benhabib and Farmer model [1994] studied in discrete time by Guo and Lansing [1998] or Dupor and Lehnert [2002].

[3] Relaxed systems have been extensively studied, for instance by Gamkrelidz [1965], McShane [1967], Young [1969] and Warga [1972], and more recently by Cesari [1983]. They regularly appear in environmental economics, especially in fishing or harvesting puzzles, as in Clark [1976] or Davidson and Harris [1981]. Lewis and Schmatensee [1982] provides a good survey of models of renewable resources with nonconvexities

problem may be viewed as a limiting case of the original problem since the difference between the optimal relaxed solution and some specific feasible allocation can be made uniformly as small as we wish: both criterion functionals have approximately the same value. Thus, even though any trajectory of the initial problem is welfare-dominated by another trajectory closer to the supremum, we are able to determine the behavior-type of any *near-optimal* solution of this economy and compare it to the standard stabilization policies proposed in the literature.

We can conclude from this comparison that standard stabilization policies, like those of Guo and Lansing [1998, 2002], appear to be inefficient in twofold. On the one hand, in absence of optimal trajectory any stabilizing policy is dominated by another stabilizing policy that makes the agents better-off. On the other hand, since the near-optimal trajectories consist of arbitrary fast jumps of the labor effort (switching between zero and full labor effort) that mimic the optimal relaxed solution, a smooth stabilization scheme fails to replicate this erratic behavior. Whilst the standard policy rules were designed to break up the topological stability of the steady state and make a (monotonic) saddle-path equilibrium to appear, it will be seen in the paper that the trajectories mimicking the optimal relaxed solution are actually cycling. Thus, focusing a particular topological structure of the dynamic system is likely to coordinate expectations in case of indeterminacy but does not help to resorb the market failure inherited from the existence of external effects. It is noteworthy however that this inefficiency of the economic policy only occurs under indeterminacy. Under perfect determinacy of the steady state an optimal trajectory may be proved to exist in the initial problem and a Guo and Lansing policy [2002] consisting in subsidizing the economy with a constant subsidy rate that eliminates the wedge between the social and private marginal products of capital and labor manages to pin down expectations on the optimal trajectory. Agents are compelled to internalize the non-convexities of the production set: in this context, the aim of economic policy is not stabilization but Pareto-efficiency. Thus, when the economic policy is efficient there is no need to stabilize the expectations. But when indeterminacy occurs, stabilization policies fail to reach efficiency.

These results are based on the absence of optimal trajectory, which occurs only in the continuous time model. However, the conclusions concerning the standard (smooth) stabilization policies are robust whether the model is discrete or continuous, and whether the condition for indeterminacy requires a high or low level of increasing returns.

The remainder of this paper is organized as follows. Section 2 describes the model setup. The definitions of the maximization problem and the associated

in production.

relaxed problem are exposed in section 3 and an existence theorem of a relaxed optimal solution is provided. We show in section 4 that although there is no optimal trajectory in the original framework, the optimal relaxed trajectory can be approximated by a sequence of admissible trajectories for the non-relaxed problem. The economic consequences for stabilization policies are formulated in section 5 where it will be seen that inefficiency of the (stabilization) economic policy relies intrinsically on the presence of indeterminacy. Section 6 concludes.

2 The model

The framework is a continuous-time version of Wen [1998], which integrates capacity utilization in the standard Benhabib and Farmer model [1994].

The economy is characterized by a continuum of identical competitive firms, with the total number normalized to one, whose production function has constant returns to scale and depends on two factors, capital (K) and labor (L):

$$\begin{aligned} Y(t) &= A(t)[u(t)K(t)]^a L(t)^b && \text{with } 0 < a < 1 \text{ and } a+b=1, \\ A(t) &= [\bar{u}(t)\bar{K}(t)]^{a\gamma}\bar{L}(t)^{b\gamma} && \text{with } \gamma > 0, \end{aligned}$$

where Y denotes the total output, $u \in (0,1)$ capacity utilization and A is a productive externality expressed as a function of the average economy-wide levels of capital ($\bar{K}$), labor ($\bar{L}$) and capacity utilization ($\bar{u}$). In a symmetric equilibrium $K = \bar{K}$, $L = \bar{L}$ and $u = \bar{u}$ then the aggregate production function is:

$$Y(t) = [u(t)K(t)]^{a(1+\gamma)} L(t)^{b(1+\gamma)} \equiv [u(t)K(t)]^{\alpha} L(t)^{\beta}, \qquad (1)$$

which obviously exhibits increasing returns to scale since $\alpha + \beta > 1$.

The economy is populated by a unit measure of identical infinitely lived consumers. The representative consumer, owner of capital, is endowed with one unit of time ($L \leq 1$) and maximizes:

$$\max_{u,C,L} \int_0^\infty U(C(t), L(t)) e^{-\rho t} dt, \qquad (2)$$

with:

$$U(C(t), L(t)) \equiv \log C(t) - \frac{L(t)^{1+\chi}}{1+\chi},$$

where C and L are the consumer's consumption and hours worked. Under the assumption that market factors are perfectly competitive, the budget constraint faced by the representative consumer is:

$$\dot{K}(t) = (r(t) - \delta(t))\, K(t) + w(t)L(t) - C(t), \tag{3}$$
$$\delta(t) = \tau u(t)^{\theta} \quad \text{with } 0 < \tau < 1 \text{ and } \theta > 1, \tag{4}$$

where $\chi \geq 0$ is the inverse of the Frisch elasticity of labor supply, $\rho > 0$ is the discount rate and $\delta(t)$ the depreciation rate at time t. Consumers derive income by supplying capital and labor services to firms, taking factor prices r, the rate of return on capital, and w, the real wage, as given.

The restriction $\theta > 1$ differs from Baxter and King [1990] or Benhabib and Farmer [1994]: when $\theta < 1$ the optimal capacity utilization is always $u = 1$ and the depreciation rate is constant in contrast with our setup. Introducing capacity utilization amplifies and propagates business cycle: capital is more intensively used during economic booms when its marginal product is high. Empirical analysis of the key role of capacity utilization can be found in Greenwood, Hercowitz and Krusell [1988], Shapiro [1993], or Burnside, Eichenbaum, and Rebelo [1995]. However, the introduction of this component in the model is mainly a matter for plausibility of the condition for indeterminacy, requiring an amount of increasing returns coherent with the recent estimates, by contrast with others models dealing with multiple equilibria.

¿From now, it will be assumed that the model exhibits indeterminacy. This means that a continuum of equilibrium paths converges to the steady state (which is topologically stable) and that public intervention is required to coordinate agents and pin down their expectations on a unique equilibrium path. This assumption holds under specific values of the parameters:

$$\beta \frac{\theta}{\theta - \alpha} > 1 + \chi. \tag{5}$$

This condition for indeterminacy although only necessary in the discrete time model of Wen [1998] is necessary and sufficient in the continuous time version and makes indeterminacy more likely to appear with smaller levels of the increasing returns. In the limit case when $\chi = 0$ (Hansen's [1985] indivisible labor) and θ tends to 1 the condition for indeterminacy collapses to $\gamma > 0$: indeterminacy occurs for almost constant returns to scale.

3 Relaxed and non-relaxed optimization problem

Assume a central planner maximizing the representative consumer's utility (2) subject to the aggregate law of motion of capital, that is:

$$\dot{K}(t) = [u(t)K(t)]^{\alpha} L(t)^{\beta} - \tau u(t)^{\theta} K(t) - C(t),$$

with the control variable restriction:

$$x(t) \equiv (u(t), C(t), L(t)) \in \Upsilon \subset [0,1] \times \mathbb{R}_+ \times [0,1].$$

Since Mangasarian [1966], it is well know that the necessary conditions are also sufficient for a global maximum if the maximand (here the utility function) and the constraint (the law of motion of capital) are both differentiable and jointly concave in the variables (K, u, C, L) and if the costate $\Lambda(t) \geqslant 0$ at any period. The traditional sufficiency condition does not hold here since the law of motion of capital is no longer jointly concave in K, u and L (the production function is quasi-concave). Arrow and Kurz [1970] provide a generalized sufficiency condition for optimality that can be used in some problems where the traditional concavity assumptions do not hold. [4] It can be shown that this theorem does not apply in our setup. [5]

To solve the model, we then introduce a more general problem in which convex combinations of the initial production set vectors are authorized. More precisely: in the original problem, for any triple $x \in \Upsilon$ and for a predetermined stock of capital K, the representative firm can produce up to $[uK]^\alpha L^\beta$ units of output. We now extend the production set and consider that the firm is able to produce $\pi[u_1K]^\alpha L_1^\beta + (1-\pi)[u_2K]^\alpha L_2^\beta$ units of output with the vector of inputs $\pi x_1 + (1-\pi)x_2$, for any $\pi \in [0,1]$ and any $(x_1, x_2) \in \Upsilon^2$. We then force the production set to satisfy the definition of a convex set (while the production function is only quasi-concave) and make the optimization problem easier. This method of *relaxed* or *generalized* problem gave raise to an extended literature in the field of mathematics, like Young [1969], Warga [1972] or Cesari [1983] for the most basic results, and appears in the economics literature under the guise of formal derivations in Davidson and Harris [1981]. [6] Once the initial production set has been "convexified" by adding its convex hull to the set of feasible allocations, the relaxed problem of the social planner

[4] For a formal proof of the theorem, see Seierstad and Sydsaester [1977] or Seierstad and Sydsaester [1987], theorems (2.5) and (3.14).

[5] The maximized Hamiltonian we define further is only piecewise convex but not globally convex when the condition for indeterminacy does not hold. Derivations are available from the author upon request.

[6] Clark [1976] alluded the possibility of optimal chattering solutions but does not provide a sufficient rigorous treatment of such solutions.

becomes: [7]

$$\max_{v} \int_0^{\infty} \sum_{i=0}^{2} p_i U(x_i(t)) e^{-\rho t} dt, \tag{6}$$

subject to:

$$\dot{K}(t) = \sum_{i=0}^{2} p_i [u_i(t)K(t)]^{\alpha} L_i(t)^{\beta} - \tau p_i u_i(t)^{\theta} K(t) - p_i C_i(t) \tag{7}$$

with the control variable restrictions:

$$v(t) \equiv (x_1(t), x_2(t), p_1(t), p_2(t)) \in V \equiv \Upsilon^2 \times [0,1]^2 \tag{8}$$

$$p_1(t) + p_2(t) = 1. \tag{9}$$

Since the convexity of the control set is so easily obtained for relaxed problems, the following proposition can be shown:

Proposition 1 *There exists an optimal pair $(K^*(t), v^*(t))$ to the optimization problem (6)-(9).*

Proof. This is an application of the Filippov-Cesari theorem. [8] See Appendix 8.2. ■

While the assumptions of the Filippov-Cesari theorem hold in the case of the relaxed problem, no existence theorem can apply in the case of the non-relaxed problem due to the large non-convexities of the production set. However, it will be seen in the next section that there exists in the original optimization problem a sequence of trajectories whose criterion functional converges to the criterion of the optimal relaxed solution. This relationship will be used to prove the absence of optimal solution in the non-relaxed problem.

[7] When the objective function is not concave, it is necessary to convexify the control set to prove the existence of a solution in the relaxed maximization problem. In our model, the utility function is already concave: this procedure is not necessary. However, we express the relaxed problem in the more general way to fit the formulation usually adopted by the relevant literature. An alternative formulation consists in maximizing $\int_0^{\infty} U(v(t))e^{-\rho t}dt$. It can be proved that the optimal solution is a chattering corner solution: the (limit) value of the maximand will then be the same whatever the objective function we chose.

[8] For a complete proof of the theorem can be found in Cesari [1983], chapter 9. The simplified version presented in this paper is due to Seierstad and Sydsaeter [1987], theorem 2.8.

4 Pseudo-optimal trajectories

The following proposition establishes the possible approximation of relaxed trajectories, whether optimal or not, by ordinary trajectories of the initial problem.

Proposition 2 *Let $\{K^*(t), v^*(t)\}$ be an admissible pair for the relaxed optimal control problem. Then, there exists a sequence $\{K_i(t), u_i(t)\}_{i=1}^{\infty}$ of admissible pairs for the initial non-relaxed problem such that the sequence of admissible trajectories $\{K_i(t)\}$ converges uniformly to $K^*(t)$ on compact subsets of $[0, +\infty)$.*

Proof. This is an extension by Carlson [1993], theorem 4.2, of Berkovitz [1974] and Cesari [1983] to the case of infinite-horizon problems. See appendix 8.3 for the application. ∎

The proposition above is based on the proof that when maximizing on $[0, T] \subset [0, +\infty)$, the difference between the functional criterion of the optimal relaxed solution and the criterion of an approximate non-relaxed trajectory tends to zero as the upper bound T tends to infinity. In other words, the relaxed optimal solution is the limit of a sequence of suboptimal trajectories for any finite interval problem defined on $[0, T]$. It is then possible to enlarge the compact set as much as possible to get a sequence of trajectories whose criterion functional has the same value as the supremum of the relaxed problem.

It is worth noting that the set of admissible trajectories for the initial problem can be expressed as a set of degenerated trajectories in the relaxed problem, with $p_i = 1$ and $p_j = 0$, $i, j = \{1, 2\}$ and $j \neq i$. Thus, an optimal non-relaxed trajectory, if any, performs at the very most as good as the optimal relaxed trajectory. In case of indeterminacy, as shown in appendix 8.1, the non-relaxed trajectory must exhibit an alternation of periods of full labor effort and periods of zero labor effort. The argument can be proved by reducing it to absurdity: assume the optimal trajectory is an “interior” solution and embodies quantities of labor $L \in (0, 1)$. For optimal values of the state and costate variables, by choosing either $L = 0$ or $L = 1$ we can increase the criterion functional. Then the trajectory with $0 < L < 1$ cannot be optimal.

Can this chattering non-relaxed trajectory constitute the optimal solution of the relaxed problem?

Proposition 3 *The optimal relaxed solution cannot be a degenerated solution when indeterminacy occurs.*

Proof. See appendix 8.4. ∎

The immediate consequence of propositions 2 and 3 is that there is no optimal trajectory in the original problem since for any admissible trajectory approximating the optimal relaxed solution one may select another trajectory whose criterion functional gets closer to the supremum: actually, the sequence of non-relaxed trajectories never reaches the supremum. The intuition for these findings relies on the welfare improvement properties of the chattering solutions. By switching from periods of zero labor effort to periods of full labor effort, the social planner may manage to mimic more or less faithfully the optimal relaxed trajectory. Since this trajectory is not degenerated, it is clear that a faster labor switching at some periods of time can make the economy closer to the relaxed optimal solution.

However, due to the convergence of the sequence of trajectories, it is also clear that adding more switchings to an already highly chattering trajectory has very few impact on welfare improvement. For an artificially low error ε and a supremum value $\hat{J}$ we can define a set of original trajectories whose criterion functional is included in $[\hat{J} - \varepsilon, \hat{J})$. These trajectories will be said *pseudo-optimal*: their criterion functional have almost the same value as the supremum and they exhibit a similar behavior for capital, consumption and labor.

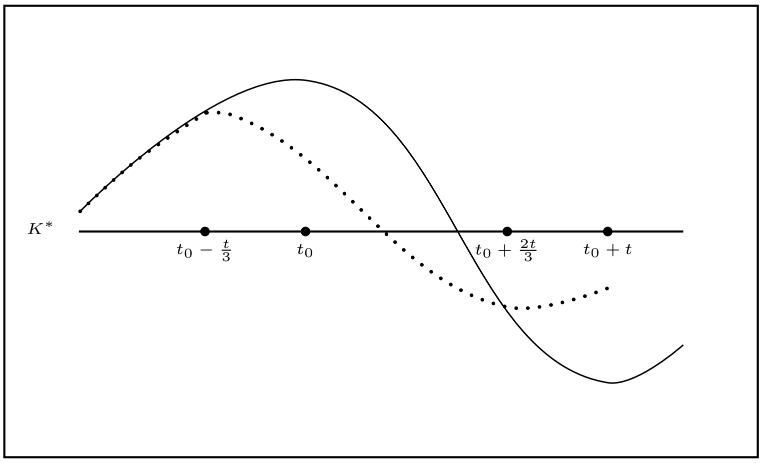
5 Continuous vs. discrete time model

In a discrete time version of the Benhabib and Farmer model [1994], the existence of a non-relaxed solution has been established by Dupor and Lehnert [2002] for a perfectly elastic labor supply.[9] They show for $\alpha < 1$ that the optimal investment, employment and consumption policies feature discontinuous jumps and endogenous cycles. So do the pseudo-optimal trajectories in our continuous time model. The main difference comes from the absence of optimal trajectory in the continuous time model while there is a solution in discrete time.

The intuition for the role played by time continuity in this apparent contradiction is tantamount to understanding how the supremum of the relaxed problem can be approximated. Since lotteries or convex combinations, although optimal in the relaxed problem, are not permitted in the original non-relaxed problem, the social planner can try to mimic the supremum using chattering solutions. Assume for simplicity that the supremum consists in targeting a constant optimal stock of capital K^* (the argument remains of course valid

[9] The proof has been generalized in Saïdi [2007] for alternative values of the Frisch elasticity of labor supply. Results can easily been extended to the discrete time Wen framework [1998].

when the optimal capital stock is cycling). In the non-relaxed problem maintaining $K(t) = K^*$ for any $t \in [0, +\infty)$ is not possible since optimal labor effort may only switch between zero and 1. When the capital stock at period t_0 is higher than K^* it is optimal for the representative agent to stop working so as to decrease the capital stock down to K^* or even below: the optimal labor effort is $L^*(t_0) = 0$. When it is lower than K^* at period $t_0 + t$ it is optimal for the representative agent to work hard so as to increase the capital stock up to K^* or even higher: $L^*(t_0 + t) = 1$. It should be noticed however that the representative agent could mimic the supremum even better if she stopped working at period $t_0 - t/3$ where K is already greater than K^* and if she started working again at period $t_0 + 2t/3$ where K is also lower than K^*.



By switching faster and faster labor effort, she can manage to get closer to the supremum and minimize the departure from $K = K^*$. In the continuous time framework there is no limit to time division: any solution is then welfare-dominated by another for which the switch between zero and full labor effort is made faster. As a consequence, there is no optimal solution. By contrast, in a discrete time model the number of switches in labor effort on a finite interval of time is by definition limited and so is the ability of a pseudo-optimal trajectory to mimic and approximate the relaxed supremum. The optimal solution is then straightforward: at time t, if the capital stock is greater than K^* the optimal labor effort is $L_t = 0$; at time $t + 1$ if the capital stock is still greater than K^* the optimal labor effort is $L_{t+1} = 0$, otherwise it is $L_{t+1} = 1$, etc.

The ability to increase the chattering behavior of a pseudo-optimal trajectory and then improve welfare is related to time continuity. Thus, while there is a non-relaxed optimal solution in the discrete time framework there is none in the continuous time model.

6 Optimal economic policy, indeterminacy and continuous time

Real indeterminacy associated with time continuity makes economic policies inefficient. This inefficiency is twofold.

First, in absence of optimal solution, each pseudo-optimal trajectory is welfare-dominated by another one closer to the optimal relaxed solution. Then, for any stabilization policy there exists another policy performing better by pinning down the expectations on an equilibrium path that improves the representative agent's welfare.

Second, the difference in utility between two pseudo-optimal trajectories is small. One can think that stabilizing the economy on a path close enough to the optimal relaxed solution is likely to make the agents better-off compared to a *laissez-faire* policy. However, the chattering pattern of the pseudo-optimal trajectories requires to adopt a central planning in which prices are fixed at any time. While Bambi and Saïdi [2008] have shown that a subsidy/tax policy constraining prices at only one period of time can pin down expectations on a predetermined equilibrium path, such a policy must be replicated at any period of time in the case of a chattering trajectory. Furthermore, the tax scheme to be designed is non-standard: it consists in fully taxing capital and labor in periods of zero labor effort and subsidizing labor in periods of full labor effort so as to equalize the real wage to the marginal productivity of labor. Thus, the main weakness of the stabilization policies, whether time is continuous or discrete, lies in their inability to be realistically replicated in a decentralized economy unless they ignore the chattering pattern of the pseudo-optimal trajectories. More realistic stabilization policies have extensively been used in the literature, among others Guo and Lansing [1998, 2001] or Guo and Harrison [2001], but they fail to faithfully approximate the cycling behavior of capital, consumption and labor. They smooth the consumption/investment plans of the representative agent, performing worse than some sunspot stochastic trajectories. [10]

However, it is worth noting that such conclusions occurs only under indeterminacy. When the degree of increasing returns to scale is not sufficient to imply indeterminacy, a realistic and decentralized optimal economic policy can be found. [11] In that case, existence and uniqueness of the optimal trajectory in the non-relaxed problem is insured by the Arrow sufficient condition since the *Maximized Hamiltonian* is strictly concave.

The Maximized Hamiltonian $\hat{H}$ is the value of the Hamiltonian once the con-

[10] See Christiano and Harrison [1999] for specific estimates.

[11] This policy is of course not a stabilization policy since there is no coordination failures in absence of indeterminacy.

trols have been replaced by their maximized values, obtained by the first-order conditions (11) to (14): [12]

$$\hat{H}(K,\Lambda) = -\ln\Lambda + \Phi\left(1+\chi-\beta\frac{\theta}{\theta-\alpha}\right)K^{\alpha\frac{\theta-1}{\theta-\alpha}\frac{1+\chi}{1+\chi-\beta\frac{\theta}{\theta-\alpha}}} - 1,$$

with $\Phi \equiv \frac{1}{\beta\frac{\theta}{\theta-\alpha}(1+\chi)}\left[\Lambda\beta\left(\frac{\alpha}{\theta\tau}\right)\right]^{\frac{1+\chi}{1+\chi-\beta\frac{\theta}{\theta-\alpha}}} > 0$.

When the condition for indeterminacy does not hold, the maximized Hamiltonian is strictly concave if and only if the second derivative respect to K is strictly negative:

$$\frac{\partial^2\hat{H}}{\partial K^2}(K,\Lambda) = \Phi\alpha(1+\chi)\frac{\theta-1}{\theta-\alpha}\left[\alpha\frac{\theta-1}{\theta-\alpha}\frac{1+\chi}{1+\chi-\beta\frac{\theta}{\theta-\alpha}} - 1\right]K^{\alpha\frac{\theta-1}{\theta-\alpha}\frac{1+\chi}{1+\chi-\beta\frac{\theta}{\theta-\alpha}}-2}.$$

The following proposition holds after rearranging the terms of the inequality:

Proposition 4 *$\beta < (1-\alpha)(1+\chi)$ is a sufficient condition for existence and uniqueness of an interior optimal solution to the optimization problem. Under this condition, equations (11) to (14) describe the behavior of the optimal solution.*

It can be easily shown by applying the Blanchard-Kahn criterion that the optimal trajectory is a saddle-path equilibrium. Furthermore, it can be noticed that the condition for determinacy collapses to the condition of proposition 4 as θ tends to 1. Thus, when the cost structure on capacity utilization is barely convex and indeterminacy does not occur, the original maximization programme has a unique interior solution that can be replicated by an optimal economic policy. [13]

This policy consists in eliminating the wedge between the social and private marginal production products of capital and labor: it imposes a constant subsidy rate on production (or equivalently on both capital and labor incomes), namely $\sigma = \gamma$. The subsidy is financed by a lump sum tax $T(t)$ at period t. The law of motion of capital then becomes:

[12] As shown in appendix 8.1, provided the condition for indeterminacy is not met, the values of the controls obtained by way of the first order conditions clearly maximize the Hamiltonian for optimal values of K and Λ.

[13] This policy has been extensively studied by Guo and Lansing [2002] in the context of non-linear dynamics. For some parameters, such a policy may lead to Hopf or Flip bifurcations.

$$
\begin{aligned}
\dot{K}(t) &= [(1+\sigma)r(t) - \delta(t)]\,K(t) + (1+\sigma)w(t)L(t) - C(t) - T(t),\\
\delta(t) &= \tau u(t)^{\theta}\\
T(t) &= \sigma \bar{Y}(t),
\end{aligned}
$$

where $\bar{Y}(t)$ represents the average economy-wide level of output at time t. From the representative agent's viewpoint, the programme respects the different convexity assumptions: there is a unique solution whose dynamics coincides exactly with equations (11) to (14).

Consequently, existence (and uniqueness) of an optimal non-relaxed solution in the model − and more generally, in continuous time models where the production set is not convex − relies on the degree of concavity of the utility function. When disutility of labor is high enough, there always exists a limit to the labor effort above which it is no longer welfare-improving to keep bunching hard work. Then, an interior solution is likely to exist and an efficient economic policy is able to replicate its behavior. To the contrary, indeterminacy occurs when the utility function is not sufficiently concave and implies welfare-improving chattering trajectories.

7 Conclusion

It has been proved in this paper that, by contrast with a discrete time framework, there is no Pareto optimal trajectory in continuous time models with social increasing returns to scale whose degree is sufficient to induce indeterminacy. Results are derived within the Wen model [1998] but could be easily extended to the Benhabib and Farmer models [1994, 1996].[14] Although there is no optimal trajectory, a continuum of suboptimal trajectories may converge to a supremum which is the solution of a generalized optimization program where the original production set has been "convexified". This means that no economic policy is able to optimally pin down expectations: any stabilization policy is welfare-dominated by another that leads the economy closer to the supremum trajectory. Furthermore, those stabilization policies must be centralized by a social planner targeting prices at any period. Interventionism is then a *sine qua non condition* to minimize the welfare loss between the stabilizing trajectory and the supremum. The second best (stabilizing) trajectory so obtained is an equilibrium path along which agents alternate periods of full labor effort with periods of zero labor effort. The chattering pattern of this solution differs from the traditional stabilization policies proposed in the standard literature, leading to smooth consumption and investment over

[14] Actually, the limit case $\theta = +\infty$ and $\delta(t) = \delta$ collapses to the Benhabib and Farmer one-sector model.

time. It appears impossible for economic policy to insure both stabilization and efficiency.

However, efficient economic policies may appear provided the degree of social increasing returns to scale is compatible with a determined stationary equilibrium. Then, an optimal economic policy consists in forcing the agents to internalize the productive externality.

8 Appendices

8.1 Hessian matrix of $H(K^*, u, C, L, \Lambda^*)$

Let H be the Hamiltonian of the central planner's program:

$$H(K, u, C, L, \Lambda) = \ln C - \frac{L^{1+\chi}}{1+\chi} + \Lambda \left([uK]^\alpha L^\beta - \tau u^\theta K - C \right). \tag{10}$$

Using the first-order conditions:

$$\frac{\partial H}{\partial u}(K, u, C, L, \Lambda) = 0 \Longleftrightarrow u = \left(\frac{\alpha}{\theta \tau} K^{\alpha - 1} L^\beta \right)^{\frac{1}{\theta - \alpha}} \tag{11}$$

$$\frac{\partial H}{\partial C}(K, u, C, L, \Lambda) = 0 \Longleftrightarrow C = \frac{1}{\Lambda} \tag{12}$$

$$\frac{\partial H}{\partial L}(K, u, C, L, \Lambda) = 0 \Longleftrightarrow L = (\Lambda \beta [uK]^\alpha)^{\frac{1}{1+\chi-\beta}} \tag{13}$$

$$\frac{\partial H}{\partial K}(K, u, C, L, \Lambda) = -\dot{\Lambda} - \rho \Lambda \Longleftrightarrow \frac{\dot{\Lambda}}{\Lambda} = \rho + \tau u^\theta - \frac{\alpha Y}{K} \tag{14}$$

$$\lim_{t \to \infty} \Lambda K e^{-\rho t} = 0,$$

and after some algebra, the Hessian matrix of $H(K^*, u, C, L, \Lambda^*)$ can be written as follows: [15]

$$\begin{pmatrix} -1/C^2 & 0 & 0 \\ 0 & (\beta - 1 - \chi) L^{\chi - 1} & \alpha L^\chi / u \\ 0 & \alpha L^\chi / u & \alpha L^{1+\chi} (\alpha - \theta)/(\beta u^2) \end{pmatrix}.$$

Since $C > 0$ it is straightforward that the first eigenvalue of the Hessian matrix is negative while the others are the eigenvalues of the lower right submatrix, say H_r, whose determinant is:

$$Det(H_r) = -\frac{\alpha L^{2\chi}}{\beta(\theta - \alpha) u^2} \left(\beta \frac{\theta}{\theta - \alpha} - 1 - \chi \right).$$

By definition $\theta > 1 > \alpha$. Thus, if the condition for indeterminacy, equation (5), holds the determinant of H_r is negative. The determinant is equal to

[15] The Hessian matrix is computed for optimal values of K and Λ, respectively K^* and Λ^*.

the product of the eigenvalues: one is strictly positive, the other is strictly negative.

For optimal values of K and Λ, the Hamiltonian $H(K^*, u, C, L, \Lambda^*)$ is then concave in C but not jointly concave in (u, C, L). Once the optimal values of K and Λ have been found, the vectors (u, C) maximizing the Hamiltonian are corner solutions. Provided $K > 0$ and $L > 0$, $\partial H/\partial u$ has a solution between zero and 1 and $\partial^2 H/\partial u^2 < 0$ when $\theta > 1$: the optimal value of u must be interior. Thus, the value of L maximizing the Hamiltonian is either zero or 1. Notice that for $L = 0$ it is clear that the optimal value of u maximizing (10) is also zero.

8.2 Proof of proposition 1

According to the Filippov-Cesari theorem, there exists an optimal pair $(K^*(t), v^*(t))$ to the optimization problem (6)-(9) provided for all $t \in \mathbb{R}_+$ and all admissible pairs $(K(t), v(t))$:

i. there exists an admissible pair $(K(t), v(t))$,
ii. for each (K,t) the set $N(K, V, t) \in \mathbb{R}^2$ defined by

$$N(K, V, t) = \{(\sum_{i=0}^{2} U(x_i(t))e^{\rho t} + \eta, g(K, v, t)) : \eta \geq 0, v \in V\}$$

and

$$g(t, K, u, C, L) \equiv \sum_{i=0}^{2} p_i[u_i(t)K(t)]^\alpha L_i(t)^\beta - \tau p_i u_i(t)^\theta K(t) - p_i C_i(t)$$

is convex,
iii. Υ is closed and bounded,
iv. there exist piecewise continuous functions h and j such that $|\sum_{i=0}^{2} U_i(t)e^{\rho t}| \leq h(t)|K| + j(t)$ for all (K, t), $v \in V$.

Conditions i. and ii. are straightforwardly satisfied: the relaxed problem has been built so as to specifically satisfy condition ii.

Since $f : \mathbb{R} \times V \times \mathbb{R}_+ \to \mathbb{R}$, defined by $f(K, v, t) = \sum_{i=0}^{2} U_i(t)e^{\rho t}$ is a continuously differentiable mapping, if κ is a compact subset then the restriction $f : \kappa \times V \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the Lipschitz continuity condition required by iv. From equation (7), it can be noticed that for all $t \in \mathbb{R}_+$ and all admissible pairs $(K(t), v(t))$, $\partial g/\partial K = 0$ for $K = \breve{K}$ while $\partial^2 g/\partial K^2 < 0$. Thus, since for any $i \in \{1, 2\}$, u_i and L_i are bounded above and C_i is bounded below, g has a maximum in $K \in (\breve{K}, \infty)$ where g is strictly decreasing and tends to $-\infty$

as K tends to infinity. We have: $K \in [0, \bar{K}] \equiv \kappa$.

We have shown that $K \in [0, \bar{K}]$ and by definition, for $i \in \{1, 2\}$, $(u_i, L_i, p_i) \in [0, 1]^3$. Then, $C_i \leq \bar{K}^\alpha + \bar{K}$. Furthermore, we will assume that $C_i = 0$ is feasible: in that case, we define the utility function by $U_i(0, L_i) = -\infty$. Thus $C_i \in [0, \bar{C}]$ and Υ is closed and bounded.

8.3 Proof of proposition 2

According to Carlson [1993], there exists a sequence $\{K_i(t), u_i(t)\}_{i=1}^{\infty}$ of admissible pairs for the initial non-relaxed problem such that the sequence of admissible trajectories $\{K_i(t)\}$ converges uniformly to $K^*(t)$ on compact subsets of $[0, +\infty)$ provided:

i. Υ is closed and bounded,

ii. the set given by

$$M = \{(t, K, x) : (t, K) \in [0, \infty) \times [0, \bar{K}], x \in \Upsilon\}$$

is closed,

iii. $F = (g, f) : M \to \mathbb{R}^2$ be a given continuous vector-valued function, and let h be a piecewise continuous function from $[0, +\infty)$ into $\mathbb{R}$ such that

$$|F(t, K, x) - F(t, K', x)| < h(t)|K - K'|$$

holds for almost all $t \geq 0$, $(t, K, x) \in M$ and $(t, K', x) \in M$.

Condition i. has already been proved.

The closure of $[0, \infty)$, $[0, \bar{K}]$ and Υ implies condition ii.

The argument to prove condition iii. is the same as for condition iv. above.

8.4 Proof of proposition 3

Notice first that a degenerated trajectory with $L = 1$ at every period is suboptimal since for high capital levels the net production is negative or null. For these levels, the optimum labor effort maximizing the Hamiltonian (10) is unambiguously $L = 0$.

Assume then that the strategy $L_2 = 0$ is chosen by the social planner as a generated trajectory at time t. In other words: $p_1 = 0$ while $p_2 = 1$. It is recalled that for such a trajectory $\dot{K} < 0$.

To the contrary, assume that a mixed strategy with $p_1, p_2 > 0$ and $L_1 = 1$ is chosen at time t such that $\dot{K} = 0$, which implies according to equation (7):

$$C_1 - C_2 = K^\alpha - \tau u_1^\theta K > 0.$$

The instantaneous welfare gain for the social planner is:

$$\begin{aligned}
\Delta U &= p_1 \ln C_1 + (1 - p_1) \ln C_2 - \frac{1}{1+\chi} - \ln C_2 \\
&= p_1 \left[\ln C_1 - \ln C_2 - \frac{1}{1+\chi} \right] \\
&\sim p_1 \left[\frac{\ln C_1 - \ln C_2}{C_2} - \frac{1}{1+\chi} \right] \\
&= p_1 \left[\ln C_1 - \ln C_2 - \frac{1}{1+\chi} \right] \\
&\sim p_1 \left[\frac{K^\alpha - \tau u_1^\theta K}{C_2} - \frac{1}{1+\chi} \right].
\end{aligned}$$

To get $\dot{K} = 0$ we must have $K^\alpha L_1^\beta - \tau u_1^\theta K - C_1 > 0$ since $K^\alpha L_2^\beta - \tau u_2^\theta K - C_2 < 0$. Then, $K^\alpha - \tau u_1^\theta K > C_1 > C_2$ and:

$$\frac{K^\alpha - \tau u_1^\theta K}{C_2} > 1 \geq \frac{1}{1+\chi}.$$

We deduce that $\Delta U > 0$: the social planner has no incentive to choose a degenerate strategy.

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