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STATIONARY EQUILIBRIUM DISTRIBUTIONS IN ECONOMIES WITH LIMITED COMMITMENT

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Stationary equilibrium distributions in economies with limited commitment

Tobias Broer*

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Abstract

Limited commitment to contracts can explain imperfect risk sharing even when individuals have access to complete insurance markets. Past contributions have focused on the resulting cross-sectional distribution of consumption (Cordoba 2008, Krueger and Perri 2006). In contrast, this paper looks at the joint dynamics of income, consumption and wealth implied by the asymmetric nature of partial insurance under limited commitment, where negative income shocks are largely insured but positive shocks can lead to large rises in consumption. A theoretical section proves the existence and uniqueness of equilibrium in a limited commitment continuum economy where incomes follow a standard markov process, and solves analytically for the joint equilibrium distribution of consumption, income and wealth. I show that individual consumption follows, at least locally, a left-skewed geometric distribution. Also, the conditional distributions of consumption and wealth are highly non-linear and have a characteristic form of heteroscedasticity, with declining conditional variances as income increases. In a quantitative part, the paper compares the exact distributions in the Krueger and Perri (2006) model to non-parametric estimates of their counterparts in US micro-data, and in a simple Ayagari economy.

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1 Introduction

The economist's toolbox has two classical ways of modelling the relation between individual incomes and consumption: on the one hand, the assumption of complete insurance markets is especially convenient for macro-economists, as it provides a rationale for their customary focus on a "representative" consumer. On the other, the permanent income hypothesis, that individuals smooth consumption of their expected lifetime resources by simple saving and borrowing, is appealing as it puts minimal requirements on the assets and information available to individuals. However, empirically, there is evidence against both perfect risk-sharing (see e.g. Attanasio and Davis 1996) and simple self-insurance (see e.g. Hall and Mishkin 1982). Moreover, conceptually, the permanent income hypothesis lacks a micro-foundation for the absence of assets other than non-contingent bonds, while the complete markets model requires enforcement of very large and persistent net transfers between individuals, as well as detailed public information on individual contingencies. More recent alternatives to the classical benchmarks, on the other hand, do not restrict asset markets a priori, but take seriously the information and enforcement problems of the complete markets model. Particularly, a growing literature has looked at economies with "limited commitment", where individuals have the option to "default" on contracts. As long as default is unattractive, for example because it leads to exclusion from financial trade in the future, this setup allows for some, but not perfect, risk-sharing even against very persistent shocks to income.

Two recent papers analyse the implications of limited commitment for the cross-sectional distribution of agents in an economy with many agents. Krueger and Perri (2006) show that the model can help reconcile the substantial rise in US income inequality over the last 20 years with the more stable inequality of consumption. Cordoba (2008) concludes, however, that the model captures the concentration of wealth at the top of the distribution less well than a simple Ayagari economy. This paper takes a different strategy. Rather than concentrating on particular moments of marginal distributions, it analyses, both theoretically and in a calibrated version of the model, the non-parametric characteristics of the joint distribution of consumption, wealth and income under limited commitment.

Particularly, I show how the asymmetry of insurance under limited commitment, where negative income risks are pooled but positive shocks lead to idiosyncratic rises in consumption if participation constraints bind, implies a characteristic form of non-linearity and heteroscedasticity of the joint distributions. The main theoretical contribution of the paper is to prove existence and uniquencess of a stationary equilibrium in a continuum economy with limited commitment to contracts, and to provide an analytical characterisation of the distribution of consumption and income, including a closed form solution for an example with two income states and CRRA preferences. The theory shows how the asymmetric nature of insurance implies declining conditional variances of wealth and consumption along the income distribution, and a negative relationship between wealth and income on average. The quantitative part of the paper looks at an economy with capital and a more general income process, to confront the joint equilibrium distribution, and its characteristic form of non-linearity and heteroscedasiticity, with the data. For this, I calculate the exact joint distributions in the Krueger and Perri (2006) calibration of the model, and compare them to non-parametric estimates of their counterparts from US micro-data, and to those from a simple Ayagari economy. The results show that, even with a more realistic income process featuring both near-permanent and transitory shocks, the limited commitment economy still produces very asymmetric joint distributions: consumption growth has a floor slightly below zero, but an upward tail that becomes more important for stronger positive income shocks. And both the mean and variance of wealth fall with income. Both the data and the Ayagari model produce less heteroscedastic distributions, and mean wealth that rises with income.

This work contributes to a large literature that analyses insurance contracts with limited commitment. In early work, Thomas and Worrall (1988) looked at self-enforcing long-term contracts between a firm and a risk-neutral worker, when both can costlessly renege on past commitments to take advantage of random fluctuations in the price of labour. In equilibrium, wages can fluctuate, but only to remain within a time-varying interval of values that satisfies participation constraints of both parties. Kehoe et al (1993) prove the first welfare theorem in an endowment economy with complete markets where participation-constraints on consumption sets prevent default. Competitive equilibria are thus constrained efficient, but may feature less than perfect risk sharing unless discount factors are high enough. Kocherlakota (1996) shows that, with a finite number of agents, relative marginal utilities are a sufficient description of the state of the economy, and equilibrium contracts have "amnesia": constrained agents' consumption is independent

of past income realisations. Ligan, Thomas and Worrall (1998) show how this implies asymmetry in the consumption paths of participation-constrained and unconstrained individuals: all unconstrained agents share (in a marginal utility sense) the same drop in consumption, while constrained agents experience relative consumption increases depending on their individual income realisations. Alvarez and Jermann (2000) prove the second welfare theorem and consider asset pricing.
In a similar manner to the present paper, Krueger and Perri (2005) are interested in participation constrained risk sharing in large western economies, and thus look at a setting with a continuum of agents who receive finite income realisations according to an identical Markov process. They use a dual method *à* la Atkeson and Lucas (1992, 1995) to show that, for any given interest rate, there exists a unique stationary consumption distribution, and that aggregate excess demand for consumption increases in interest rates. And, based on a conjecture about the existence of a market clearing interest rate, they characterise the consumption distribution for the special case with 2 iid income values.
Relative to this literature, the theoretical contribution of the present paper is three-fold: First, I am able to show the existence of a unique stationary equilibrium in a limited commitment continuum economy with standard markov uncertainty, under standard assumptions. Second, I provide a closed form for the stationary distribution of consumption, income and wealth with 2 income values and CRRA preferences. The marginal distribution of consumption is a left-skewed geometric, and the conditional variances of both consumption and wealth decline with income. Third, I characterise analytically the joint distribution for an N-state markov income process. The geometric nature of consumption continues to hold, but only locally. And with i.i.d. uncertainty, both consumption and wealth still see their conditional variances decline with income.

The empirical literature has tested the implications of limited commitment models using, for example, data on consumption and income in rural villages (Townsend 1994, Ligan et al 1998, Eozenou 2008), or from experimental settings (Barr 2008, Albarran 2003). More directly relevant to this paper is the work of Krueger and Perri (2006), who analyse the performance of the limited commitment model, relative to more standard incomplete markets models, in explaining why consumption volatility has increased much less than income risk in the United States over the last 30 years. They find that incomplete market models have too limited risk sharing, while the limited commitment model slightly underpredicts the change in consumption volatility implied by the observed rise in income risk. However, they focus mainly on the relative change in inequality mea-

sures. Cordoba (2008) uses numerical simulations to argue that models with - in his case - exogenous, debt-constraints can potentially reproduce key features of the cross-sectional distribution of consumption, but capture the wealth distribution much less well than simple incomplete markets models.
In its quantitative section, this paper looks at the shape of joint, rather than marginal, distributions. This is because the non-linear, heteroscedastic shape of the distributions results directly from the asymmetric nature of insurance under limited commitment. It is thus more robust to changes in the calibration or specification of the model than, for example, the shape of right hand tails of marginal distributions. Particularly, I compare the joint densities of consumption, wealth and income in the Krueger and Perri (2006) limited commitment economy to, on the one hand, non-parametric estimates of its counterparts in US micro-data, and, on the other, the distributions in a simple self-insurance economy. The results show that the data does not reproduce the floor in consumption growth or the declining conditional variances of consumption and wealth at higher income values that the limited commitment model predicts. Rather, the shape of the distributions in the data, where mean consumption inceases more or less linearly with income, and wealth increases, rather than falls, as income rises, seem more in line with the distributions from the simple Ayagari self-insurance economy.

Section II describes the environment of a continuum economy with debt-constrained domestic financial markets. Section III derives some characteristics of dynamic equilibria on the basis of the associated planner's problem. Section IV gives the analytical characterisation of the stationary joint distribution of consumption, income and wealth, and proves the existence and uniqueness of equilibrium. Section IV reports the results from a calibration of the model to the US economy and compares them to those from a simple self-insurance economy, and US micro-data. An appendix contains most proofs.

2 A continuum economy with debt-constrained complete financial markets

This section presents a simple economy with complete asset markets where insurance against idiosyncratic income shocks is constrained by individual default, and defines the competitive equilibrium.

2.1 The economic environment

The economy consists of a large number of individuals of unit mass. Individuals are indexed by i, located on a unit-interval $i \in I = [0,1]$ with Sigma-Algebra $\mathbb{I}$. Denote as $\Phi_I : \mathbb{I} \to [0,1]$ the (constant) non-atomic measure of individuals. Time is discrete $t \in \{0,1,2,...,\infty\}$ and a unique perishable endowment good is used for consumption.
The consumption endowment of agent i in period t, $z_{i,t}$, takes values in a finite set Z: $z_{i,t} \in Z = \{z^1 > z^2 > ... > z^N\}, N \geq 2$. Let $\mathbb{Z}$ be the power set of Z, and denote as $\Phi_{Z,t} : \mathbb{Z} \to [0,1]$ the measure of agents at all (subsets of) income realisations in period t. Endowments follow a Markov process that is independent of i, and $\mathbb{I}$-measurable (i.e. $\{i : z_{i,t+1} = z^k | z_{i,t} = z^j\} \in \mathbb{I}, \ \forall z^j, z^k$). Specifically, it is described by a Markov transition matrix F that has strictly positive entries $f_{i,j} > 0, \forall i, j$, is monotone (in the sense that the conditional expectation of an increasing function of tomorrow's income is itself an increasing function of today's income), and has a unique ergodic distribution $\Phi_Z : \mathbb{Z} \to [0,1]$. Thus, in the long-run, aggregate (or average) income $Y = \int z_i d\Phi_I$ is constant, while individual income fluctuates. Let $Z_0 : I \to Z$ be a measurable function that assigns all individuals an initial income value. Also, let s_t denote the state of the economy in period t, a vector containing individual incomes and asset holdings of all agents.
Agents live forever and order consumption sequences according to the utility function

$$U = E_{s_0} \sum_{0}^{\infty} \beta^t u(c_{i,t}) \tag{1}$$

where E_{s_0} is the mathematical expectation conditional on s_0, $0 < \beta < 1$ discounts future utility, $c_{i,t}$ is consumption by agent i in period t, and $u : R^+ \to R$ is an increasing, strictly concave, twice-continuously differentiable function that satifies Inada conditions and is identical for all agents in the economy.

2.2 Asset markets with debt constraints

Agents engage in sequential trade of a complete set of state-contingent bonds. Individual endowment realisations are verifiable and contractable, but asset contracts are not completelely enforceable: at any point, individuals can default on their contractual payments at the price of eternal exclusion from financial markets. Thus the total amount an agent can borrow today against any income state tomorrow is bounded by the option to default

into financial autarky. There, consumption is forever equal to income. Given the markov structure of income, the value of default as a function of the vector of current income z can be written as

$$W(z) = \sum_{t=0}^{\infty} (\beta F)^t U(z) = (I - \beta F)^{-1} U(z) \tag{2}$$

Note that the monotonicity of F implies monotonicity of $W(z)$ (Dardanoni 1995).
I denote holdings of Arrow-Debreu securities paying off in state s_t by $a(s_t)$. In any state s_t, $V(z(s_t), a(s_t))$ is the contract value as a function of income $z(s_t)$ and current asset holdings $a(s_t)$. As in Alvarez and Jermann (2000), individual i's participation constraint for any state s_{t+1} tomorrow can be written as a portfolio constraint on the claims she can issue against s_{t+1} income.[1] This borrowing constraint is "not too tight" in the words of Alvarez and Jermann (2000) if it assures participation but does not constrain contracts otherwise

$$a_i(s_{t+1}) \geq A_i(s_{t+1}) = min\{\alpha(s_{t+1}) : V(z_i(s_{t+1}), \alpha(s_{t+1})) \geq W(z_i(s_{t+1}))\} \tag{3}$$

To focus on the interesting case of limited insurance, I make the following assumptions about the endowment process and preferences:

Assumption 1

$$W(z^1) > \sum_{0}^{\infty} \beta^t u(Y) \tag{4}$$

Assumption 2

$$\frac{u'(z^1)}{\beta u'(z^N)} < 1 \tag{5}$$

Assumption 1 assures that full insurance is not possible, since the autarky value at high income exceeds that of consuming average income in the economy forever. Assumption 2 implies that there is no positive net interest rate that would implement the autarky equilibrium, as the marginal rate of substitution between the highest and lowest income state is too low. Alvarez and Jermann (2000) show that this is sufficient to rule out autarky as an equilibrium.

[1] An alternative is to restrict choices directly, by requiring that the chosen consumption sequence fulfill participation constraints, as in Kehoe and Levine (1993).

2.3 The household's problem

Every period, households maximise their expected utility by choosing current consumption and assets subject to budget and participation constraints

$$\begin{aligned} V(z(s), a(s)) = max_{c,\{a(s')\}}\{u(c) + \beta E_s V(z', a(s'))\} \\ s.t.\ c + \sum_{s'} a(s')q(s') \leq a(s) + z(s) \\ a(s') \geq A(s') \\ A(s') = min\{\alpha(s') : V(z(s'), \alpha(s')) \geq W(z(s'))\} \end{aligned} \tag{6}$$

where c, a' are policy functions of the state variables $(z(s), a(s))$.

2.4 Definition of competitive equilibrium

The competitive equilibrium in this economy is a set of asset prices $q(s')$, a set of individual decision rules $c(z, a), a'(z, a)$ with associated value functions $V(z, a)$
such that

1. $V(z, a)$ are the households maximum value functions associated with the household problem given $q(s')$
2. $V(z, a)$ is attained by $c(z, a), a'(z, a)$
3. Markets for state-contingent assets clear
   $\int a_i(s')d\Phi_I = 0, \forall s'$

The competitive equilibrium is called "stationary" if the distribution of individual consumption is stationary through time.

3 Efficient allocations

Alvarez and Jermann (2000) show that a version of the first welfare theorem applies to this economy as long as interest rates are "high", in the sense that today's market value of total

future resources is finite.[2] This allows me to focus on participation-constrained efficient allocations, where the assumption of some risk sharing assures that the interest rate condition is met. More particularly, I exploit the results in Marcet and Marimon (2009), and focus on the solution to the participation-constrained social planner's problem.

3.1 The planner's problem

Marcet and Marimon (2009) show how the efficient allocation solves the following planner's problem. For a given measurable assignment of welfare weights to individuals $\mu_0 : I \to R^+$ in a linear social welfare function $\Omega = \int \mu_i \sum_0^\infty \beta^t u(c_{i,t}) d\Phi_I$ the problem of the planner is to distribute resources optimally subject to individuals' participation constraints and the aggregate resources of the economy

$$\begin{aligned} \mathbb{VV}(\mu_0, Z_0) = max_{\{c_i(s_t)\}} E_0 \int \mu_i \sum_{t=0}^{\infty} \beta^t u(c_i(s_t)) d\Phi_I \\ s.t. \int c_i(s_t) d\Phi_I \leq \int z_i(s_t) d\Phi_I, \ \forall s_t \\ V_i(s_t) \geq W(z_i(s_t)), \ \forall s_t, i \end{aligned} \tag{7}$$

where the planner's maximum value $\mathbb{VV}$ is a function of the initial measure of weights and income induced by μ_0, Z_0. I assume that the initial weighting function μ_0 is measurable and takes a finite number of finite, positive values $\mu_1, ..., \mu_k$ with $\Phi_I(i : \mu_{i,0} = \mu_k) > 0, \ for \ k = 1, ..., K$ and $\Phi_I(\{i : \mu_i \nsubseteq \{\mu_1, ..., \mu_K\}\}) = 0$.

Note that this problem is non-standard, because the participation-constraints in (7) introduce history dependence. Intuitively, the planner provides value to individuals who have attractive outside options by promising them high consumption today and in the future. But this requires him to honour promises made in the past, making the problem non-recursive. As a solution, this section applies a technique proposed by Marcet and Marimon (2009) that makes the problem recursive. Their results, however, do not apply to continuum economies in general, as they focus on an environment with a finite number of agents. But with a finite number of income values and a discrete initial distribution of planner weights, we can always replace integration over an infinity of indivuals i by

[2] An additional technical condition requires that for all i, there is a constant ζ_i such that for all z_t, $|u(c_{i,t}(s_t))| < \zeta_i(u'(c_{i,t}(s_t)))c_{i,t}(z_t))$. Note that this is a joint condition on utility and the equilibrium allocation. It is met in most relevant cases, for example if relative risk aversion is different from 1 at zero, or if consumption is uniformly bounded away from zero, which is the case in the setting of this paper.

summation over a countable number of sets of individuals that share all relevant characteristics. In particular, in any period t, we can split the uncountable set I into KN^t sets of individuals $I_{\mu_0,\{z\}}$ that share initial weight μ_k and income history $\{z_0, z_1, ..., z_t\}$. This ensures the countability of the planner's state space. A later section shows that this space remains, in fact, strictly finite.

Marcet and Marimon (2009) show how to capture the history dependence of the problem by an individual-specific summary variable. Particularly, they show that, denoting γ_i the Lagrange multiplier on i's participation constraint in the sequential problem (7), we can write the latter as

$$\mathbb{VV}(\Phi_{\mu,z}) = \min_{\gamma_{jl}\geq 0} \max_{\{c_{jl}\}} \sum_{\mu_j \in \bigvee^t, z_l \in Z} \Phi_{\mu,z}(\mu_j, z_l)[(\mu_j + \gamma_{jl})u(c_{jl}) - \gamma_{jl}W_z] + \beta E[\mathbb{VV}(\Phi_{\mu',z'})] \quad (8)$$

$$s.t. \sum_{\mu_j \in \bigvee^t, z_l \in Z} \Phi_{\mu,z}(\mu_j, z_l)[c_{jl} - z_l] \leq 0 \quad (9)$$

$$\mu_i' = \mu_i + \gamma_i \ \forall i \quad (10)$$

$$\Phi_{\mu,z} : \bigvee^t \times Z \to [0,1] \quad (11)$$

$$\bigvee^t = \{\mu : \Phi_I(i : \mu_{i,t} = \mu) > 0\} \quad (12)$$

where I write x_{jl} for the function $x(\mu_j, z_l)$. Note that the weights of individuals in the social welfare function are now updated every period to meet participation constraints, according to the law of motion (10). Intuitively, by increasing individual weights μ_i the planner allocates a higher than expected consumption path to individuals with binding participation constraints, to keep them "happy" with the contract. Policies c_{jl}, γ_{jl} are a function of planner weights at the beginning of the period and current income realisations only, so do not depend on past state variables. In other words, the time-varying individual weights now summarise history-dependence of the problem. Importantly, the cardinality of the set of individual planner weights with positive mass $\bigvee^t$ increases by a factor of at most N every period, and therefore remains countable. Equivalently, the integration across individuals along measure Φ_I is replaced by the weighted summation over (the Euclidean product of) the set of current income realisations Z and the time-varying set of planner weights with positive mass $\bigvee^t$, where the weights have discrete measure $\Phi_{\mu,z}$. With discrete $\bigvee^t$ and Z, the state space is finite and bounded, and thus compact, for all t. And Tychonoff's theorem ensures that it remains compact even for a countably infinite number of periods. With concave utility and finite resources, and in the absence of aggregate state variables entering the participation constraints, the constraint set is

therefore compact and convex. It is also non-empty since autarky is trivially feasible and incentive-compatible. Marcet and Marimon (2009) show how this is sufficient to ensure the equivalence of the sequential problem (7) and the transformed problem (8).[3] In particular, the planner's value function is single valued and, given continuously differentiable utility, differentiable. And finally, Inada conditions and concavity of the utility function imply that, to characterise the optimum, participation constraints and the first order conditions suffice.

3.2 Properties of efficient allocations

Although this paper is mainly concerned with the stationary joint distribution of consumption, income and wealth, the rest of this section shows two features of any equilibrium with limited commitment: first, there is asymmetry in insurance, as the planner insures consumers against drops in income, while accomodating rises in income with potentially strong consumption increases. Thomas et al (1998) show this in an environment with a finite number of agents, while I analyse the implications for the stationary joint distributions in a continuum economy. Relatedly, contracts feature "amnesia" (Kocherlakota 1996), as history dependence of individual consumption is cut off once participation constraints bind. Throughout, I denote as "continuation value" $V(\mu_t, z_t)$ the utility that an individual with current weight μ_t and income z_t can expect under the planners consumption allocation, as opposed to the autarky value $W(z_t)$ she gets from consuming her income stream from today onwards.

It is easy to see that the solution of the planner's problem defines an operator Γ that maps today's distribution of individual weights and current income into a distribution of weights and income tomorrow.[4] The next Lemma summarises some old and new results that characterise Γ.

Lemma 1 *The planner's decision rule* Γ *has the form*

$$\mu_{i,t+1} = max\{\mu_{t+1}(z_{i,t+1}), \mu_{i,t}\}$$

[3]In other words, the problem fulfills conditions A1 to A5 in Marcet and Marimon (2009). For further detail, see also the proof of uniqueness and existence in the Appendix.

[4]Or formally $\Gamma : (Z \times R_+^{KN^t}, \mathbb{Z} \times \mathbb{B}^{KN^t}) \to [0, 1]$, where B^n is the Borel algebra of the n-dimensional positive Euclidean space, and the cardinality of the set of welfare weights, equal to K in period 0, increases by N every period.

For every t, $\mu_t(z)$ is strictly increasing in z, and for every z, the sequence $\mu_t(z)$ increases strictly over time. Also, the set of individual planner weights with positive mass is strictly finite: $|\{\mu_j : \Phi_I(\{i : \mu_{i,t} = \mu_j\}) > 0\}| < \infty, \forall t$.

That individual weights increase when participation constraints bind but are constant otherwise is well-known from Marcet and Marimon (2009), and follows directly from the equivalence of γ and the Lagrange multipliers of the untransformed planner's problem. Also, since the outside option of autarky only depends on current income, planner weights of individuals with binding participation constraints $\mu_t(z_{i,t})$ are, for any t, equally a function only of their current income $z_{i,t}$. This lack of history dependence in consumption of constrained individuals is well-known as the "amnesia" property of consumption allocations with limited commitment (since Kocherlakota 1996).[5]
On the other hand, that $\mu_t(z_{i,t})$, the minimum planner weight that ensures participation of individuals with income $z_{i,t}$, is strictly increasing in both income and time has not been shown before. But this result is very useful for showing existence and uniqueness of a stationary solution to (8), and to compute it efficiently using first order conditions. It is proved in the appendix, along with its implication that $\bigvee^t$, the set of planner weights $\mu_{i,t}$, is not only countable but strictly finite.

Lemma 1 has immediate consequences for the dynamics of the joint distribution of consumption and income. To see this, note that, for λ the Lagrange multiplier associated with the ressource constraint (9), the planner's intratemporal optimality condition equates weighted marginal utilities across agents, $\lambda = (\mu_i + \gamma_i)U'(c_i)\forall i$. From this, relative consumption is monotonously increasing in planner weights

$$\frac{U'(c_{i,t})}{U'(c_{j,t})} = \frac{\mu_{j,t} + \gamma_{j,t}}{\mu_{i,t} + \gamma_{i,t}} \tag{13}$$

So the current distribution of planner weights maps monotonously into current consumption. There are thus N minimum participation-compatible consumption values $c^i_{0,t}, i = 1, ..., N$ that correspond to the minimum planner weights $\mu_t(z)$ and are increasing

[5] To see this formally, consider two agents i, j with different weights $\mu_{i,t} \neq \mu_{j,t}$ who receive a same income shock $z_{i,t+1} = z_{i,t+1}$ that implies autarky values higher than their continuation utility at current weights. With equal income today, they face the same conditional measures over future income realisations. So if $\mu_{i,t+1}$ is the minimum weight that meets i's participation constraint, it is also the minimum weight that meets j's participation constraint. And since continuation values $V(\mu_t, z_t)$ are strictly increasing in μ_t, the cutoff $\mu_t(z_{i,t})$ is unique.

in income. From this, it is easy to see that the highest income earners have highest consumption, while those with lowest consumption have necessarily the lowest income level. This lowest consumption level, since it solves the participation constraint at minimum income with equality, is easily seen to be constant through time, and equal to z^N. So there is a constant lower bound of consumption equal to minimum income.

Intratemporal optimality on the other hand requires growth rates of marginal utility to equal relative growth rates of planner weights, discounted and adjusted for changes in the planner's marginal value of resources λ

$$\frac{U'(c_i)}{U^{prime}(c_i')} = \frac{\mu_i' + \gamma_i'}{\mu_i'} \frac{\lambda_i}{\lambda_i'} \ \forall i \tag{14}$$

This immediately implies that all unconstrained agents, who have constant planner weights, share the same growth rate of marginal utility, equal to the change in the discounted marginal value of resources to the planner $\frac{\lambda'}{\lambda} = \beta R$. The result is a convenient law of motion for consumption of unconstrained agents as a function of equilibrium interest rate R

$$U'(c_i) = \beta R U'(c_i') \tag{15}$$

Equations (15) and (14) show two important characteristics of consumption transitions in limited commitment economies: discreteness and asymmetry. This is because, unless $R\beta = 1$ and insurance is perfect, all unconstrained agents share common, discrete falls in marginal utility over time, independent of their current level of income. Agents with binding participation constraints after a positive income shock, on the other hand, experience jumps in consumption to a level that is specific to their current income.

4 Existence and uniqueness of stationary equilibrium and its distributional characteristics

This section provides an analytical characterisation of the joint distribution of consumption and income. As in Krueger and Perri (2005), I concentrate on stationary consumption distributions.[6] The fact that stationarity of the consumption distribution implies a constant interest rate in the economy and vice versa conveniently means that we can index

[6]Note that limited commitment economies also admit non-stationary pareto-inefficient equilibria, where a path of decreasing interest rates confirms expectations of ever tighter borrowing limits, leading to convergence to autarky. See Bloise et al (2009).

different stationary distributions by the value of R.[7] But contrary to Krueger and Perri (2005), I prove the existence of a unique market clearing interest rate. To do this it turns out to be convenient to first characterise the stationary consumption allocation for a given R, and then to exploit its characteristics to show market-clearing at a particular unique value $R^\star$.

4.1 The stationary distribution of consumption and income

Proposition 1 *For $1 < R < \frac{1}{\beta}$ the interest rate in stationary equilibrium, the joint distribution of income and consumption $\Phi_C : \mathbb{C} \times \mathbb{Z} \longrightarrow [0,1]$ has the following features:*

1. *Φ_C is discrete, with positive mass at consumption values between minimum income and some upper bound c_0^1 smaller than the highest income level: $\mathbb{C} \subseteq [y_N, c_0^1]$, $c_0^1 < z^1$.*

2. *There are N minimum levels of consumption c_0^i, $i = 1, ..., N$ under which consumption of agents with income i never falls and where participation constraints at income z^i hold with equality. These threshold levels are constant through time and increasing in income $c_0^1 < c_0^2 < .... < c_0^N$. The lower bound of the distribution is minimum income $c_0^N = z^N$.*

3. *Every consumption threshold c_0^i is an upper bound to a geometric subdistribution of consumption Φ_C^i, with support $\{c_j^i\}$ recursively defined by the law of motion $U'(c_{j+1}^i) = (\beta R)^{-1} U'(c_j^i), j = 0, 1, 2, ...,$ and bounded below by z_N. Φ_C is thus a mixture of $N-1$ geometric distributions. The appendix contains an analytical expression for the frequencies in this distribution.*

4. *Individuals at the highest income level z^1 all have maximum consumption level c_0^1. The support of consumption conditional on income $z^i < z^1, i > 1$ is $[c_0^i, c_1^1]$. So the support of consumption narrows as income rises. For i.i.d. transitions (identical rows in F), this implies that the conditional variance of consumption falls monotonously in income.*

[7] To see this, look at any minimum participation compatible consumption value c_0^i and that corresponding to the first unconstrained transition away from it c_1^i. We have $R\beta = \frac{u'(c_0^i)}{u'(c_1^i)}$. Stationarity implies that this is a constant. The converse is proved by the construction of the stationary distribution in the appendix.

The proof of proposition 1 is by construction of the stationary distribution, and can be found in the appendix. The joint distribution of financial returns and income follows as a corollary.

Corollary 1 *The joint distribution of net financial returns and income* $\Phi_{y_{fin},z} : \mathbb{B}([c_0^1 - z^1, c_1^1 - z^N]) \times \mathbb{Z} \longrightarrow [0,1]$ *has the following features:*

- $\Phi_{y_{fin},z}$ *is discrete, and transfers are bounded above and below by* $c_1^1 - z^N, c_0^1 - z^1$ *respectively.*
- *Individuals at minimum income have positive financial returns* $y_{fin,0}^N \geq 0$*. All individuals at the highest income level* z^1*, and participation-constrained individuals at income* $z^i > z^N$ *have negative financial returns* $y_{fin,0}^i \leq 0$*, with strict inequality for* $i = 1$*.*
- *To the geometric consumption distribution with upper bound* c_0^i *corresponds a distribution that consists of a mass point at* $y_{fin,0}^i \leq 0$*, plus a support* $c_j^i - y_k, k = 1, ...i-1; j = 1, 2, ...$ *The frequency distribution follows from that of the joint distribution of consumption and income, which can be found in the appendix.*

Proposition 1 and its corollary show how the asymmetric nature of partial insurance under limited commitment affects the joint cross-sectional distribution: High income individuals have a narrow distribution of consumption, as their minimum participation-compatible consumption level is binding. They also have low financial returns, as they are making net contributions into the insurance scheme. Low income earners, on the other hand, receive net payments from insurance claims, but have a variety of consumption values that decline with the length of their low income spell.

4.2 A closed form example

A simplified version of the economy, with CRRA preferences $u = \frac{c^{1-\sigma}}{1-\sigma}$, two income values $\{z^h, z^l\}$ and transition matrix $F = [p, 1-p; 1-q, q]$, yields a closed form solution.

Proposition 2 *With* $N = 2$ *and CRRA preferences, and for* $1 < R < \frac{1}{\beta}$ *the interest rate in stationary equilibrium, denote the joint distribution of income and consumption*

$\Phi_C : \mathbb{C} \times \{z^l, z^h\} \longrightarrow [0,1]$. *The discrete support of consumption* $\mathbb{C}$ *is*

$$c_1 = f(p, q, m, W_h, W_l) \tag{16}$$

$$c_i = c_1(\beta R)^{\frac{i}{\sigma}}, 1 < i < m$$

$$c_m = z^l \tag{17}$$

$$for\ m = min\{x \in N : x > \frac{\sigma[ln(z^l) - ln(c_1)]}{ln(\beta R)}\} \tag{18}$$

The frequency mass function is geometric, given by

$$\Phi_C(c_1, z_h) = \frac{1-q}{2-q-p} \doteq \nu \tag{19}$$

$$\Phi_C(c_i, z_l|_{1<i<m}) = \nu(1-p)q^{i-1} \tag{20}$$

$$\Phi_C(c_m, z_l) = \nu\frac{(1-p)q^{m-1}}{1-q} \tag{21}$$

$$\Phi_C(\cdot, \cdot) = 0\ otherwise \tag{22}$$

Here, W_h, W_l are the autarky values at z_h, z_l given by

$$W_h = \frac{(1-\beta q)u(y_0 + \frac{1}{\nu}\epsilon) + \beta(1-p)u(y_0 - \frac{1}{1-\nu}\epsilon)}{1 - \beta(q+p) - \beta^2(1-(q+p))} \tag{23}$$

$$W_l = \frac{\beta(1-q)u(y_0 + \frac{1}{\nu}\epsilon) + (1-\beta p)u(y_0 - \frac{1}{1-\nu}\epsilon)}{1 - \beta(q+p) - \beta^2(1-(q+p))} \tag{24}$$

and $f() = \{\frac{(1-\sigma)(1-\beta q(\beta R)^{\frac{1-\sigma}{\sigma}})}{1+\beta(1-p-q)(\beta R)^{\frac{1-\sigma}{\sigma}} - (1-p)\beta^m q^{m-1}(\beta R)^{\frac{m(1-\sigma)}{\sigma}}}[\frac{1-\beta(p+q)-\beta^2(1-p-q)}{1-\beta q}W_h - (1-p)\beta^m q^{m-2}(qW_l - \frac{(1-q)W_h}{1-\beta q})]\}^{\frac{1}{1-\sigma}}$. Note that the frequency mass function Φ_C is the same with general, non-CRRA, preferences.

Proof

To obtain the discrete support of consumption $\mathbb{C}$, define c_m as the minimum participation-compatible consumption for an individual in the low income state z^l. As she cannot move further down in consumption, she is necessarily participation-constrained in both income states tomorrow, receiving values W_h and W_l respectively. Thus c_m is determined from her participation constraint as

$$W_l = U(c_m) + \beta[(1-q)W_h + qW_l]$$

which is solved by $c_m = z^l$ from the definition of W_l. So minimum consumption is equal to minimum income.

The strict monotonicity of the sequence $\mu_t(z^h)$ and the finiteness of initial weights together imply that for any $\mu_{i,0}$, we have $\mu_t(z^h) > \mu_{i,0}$ for some finite t. So in the stationary allocation, an individual in the high income state is always constrained, receiving minimum participation-compatible consumption c_1, whose value we need to determine. Tomorrow she either remains at high income, receiving W_h, or gets a negative income shock and moves down in consumption according to (15), which for CRRA preferences becomes $c_i' = (\beta R)^{\frac{1}{\sigma}} c_1$. Thus, the expected value of her consumption stream under the contract can be expressed as the sum of m lotteries with two outcomes: either, in case of a positive income shock z^h, she receives value W_h. Or, in case she moves to low income z^l, she gets current utility $\frac{[(\beta R)^{\frac{i}{\sigma}} c_1]^{1-\sigma}}{1-\sigma}$, $i = 1$ plus participation in the next lottery for $i = 2$, and so forth. If she has not received a positive shock after m-1 periods, her consumption cannot fall by another whole step without violating her participation-constraint at low income. So there is a final lottery between receiving W_h and W_l. This means c_1 is uniquely determined by her participation constraint

$$\begin{aligned} W_h &= \frac{c_1^{1-\sigma}}{1-\sigma} + p\beta W_h + (1-p)\beta \sum_{i=1}^{m-1} \{(\beta q)^{i-1} \frac{[(\beta R)^{\frac{i}{\sigma}} c_1]^{1-\sigma}}{1-\sigma} \\ &\quad + \beta^i q^{i-1}(1-q) W_h\} + (1-p)\beta^m q^{m-1} W_l \end{aligned} \tag{25}$$

To derive the mass function Φ_C, note that the stationary mass at c_1 is that at income state z^h, equal to the first entry of the normalised left eigenvector of transition matrix F associated with a unit eigenvalue $\nu = \frac{1-q}{2-q-p}$. $\Phi_C(c_2, z_l)$ is simply ν times transition probability to low income $(1-p)$, and $\Phi_C(c_i, z_l) = \nu(1-p)q^{i-1}$, $i = 2...m-1$ declines geometrically with survival probability q, the probability of remaining in low income state z^l. Finally, the lower bound c_m has mass $\Phi_C(c_m) = \Phi_C(c_{m-1})\frac{q}{1-q}$. ■

The next corollary summarises the shape of the distribution and derives some of its second moments. The proof is in the appendix.

Corollary 2 *With CRRA preferences and 2 income values, the following is true:*

1. *The covariance between income and consumption is positive. The covariances between income and both financial returns and wealth are negative.*

2. *The mean of consumption increases in income. Its conditional variance decreases.*

3. *If $\Phi_C(c_m, z_l) \approx 0$, the cross-sectional variance of log-consumption in stationary equilibrium is*

$$Var_c = \tau[\frac{\log(\beta R)}{\sigma}]^2 \tag{26}$$

where $\tau > 0$ is a function of transition probabilities only. If there is a non-negligible mass at the truncation point, $\Phi_C(c_m, z_l) > 0$, this is an upper bound for the cross-sectional variance of individual consumption.

With 2 income values, the asymmetric nature of insurance under limited commitment thus implies a geometric cross-sectional distribution of consumption. Negative income shocks lead to a sequence of equal small steps down the distribution, while positive income shocks lead to a variety of consumption responses. And insurance becomes more efficient at higher interest rates, as illustrated by the negative relationship between the cross-sectional variance of consumption and R in corollary 2.

This section extends previous characterisations of limited commitment economies with two income values in several ways. Krueger and Perri (2005), and similarly Krueger and Uhlig (2006), show for the i.i.d. case that the stationary efficient allocation under limited commitment is discrete with geometrically declining mass, for a given constant interest rate. For the CRRA case, I solve for the whole distribution including the support of consumption, wealth and financial income (see appendix) in closed form, analysing the more general case with persistent income. Moreover, the corollaries to proposition 2 characterise conditional and second moments of the distribution, including a closed form for the variance of log consumption in the case of negligible truncation, showing how lower interest rates are associated with higher consumption variance in stationary equilibrium.

4.3 Existence and uniqueness of a market-clearing interest rate

The previous sections characterised the equilibrium distribution of consumption and income for a given level of interest rates R. This section proves the existence of a unique stationary market-clearing interest rate $R^\star > 1$.

Proposition 3 *If agents have non-increasing relative risk aversion*

$$\frac{u'(c_1)c_1}{u''(c_1)} \geq \frac{u'(c_2)c_2}{u''(c_2)} \ \forall c_1 < c_2 \tag{27}$$

then there exists a unique stationary market-clearing interest rate $R^\star > 1$.

Proof The algorithm used in Proposition 1 defines a mapping $\odot$ from the interval $I_R = [\frac{u'(z^1)}{\beta u'(z^N)}, \frac{1}{\beta}[$ of interest rates to the space of stationary consumption distributions. By summing over the distribution and subtracting constant aggregate income Y, this yields excess demand as a function of interest rates $\overline{\odot} = \int \odot \, d\Phi_I - Y$. Note that this mapping is single-valued, as the algorithm has a unique solution for any $R \in I_R$, and that $\odot(R)$ coincides for $R > 1$ with the stationary solution to the planners problem given interest rate R. Note also that consumption equals income in autarky, so $\overline{\odot}(R_{aut}) = 0$ for $R_{aut} = \frac{u'(z^1)}{\beta u'(z^N)}$. The proof shows that $\overline{\odot}(R)$ is decreasing for $R_{aut} < R < 1$ and increasing for $1 < R < \frac{1}{\beta}$. This implies that for some $R^\star > 1$ excess demand is negative. Existence then follows from the fact that excess demand must be positive for $R = \frac{1}{\beta}$ as perfect insurance is unfeasible by assumption. Uniqueness follows from the monotonicity of $\overline{\odot}(R)$ for $R > 1$.

From Proposition 1, the consumption distribution Φ_C splits naturally into N subdistributions Φ_C^m bounded above by c_0^m, the minimum participation-compatible consumption at income $z^m, m = 1, ..., N$. For any m, consider Φ_C^m as a function of the interest rate R. For an individual who is constrained at income z^m we can write

$$V(c_0^m, z^m) - W(z^m) = \sum_{i=0}^{n} \beta^i [\pi_{i|m} u(c_i^m) - \sum_j \pi_{ij|m} u(z_{ij})] = 0 \tag{28}$$

where the last equality follows from the fact that the participation constraint is binding. Here, i is the index for unconstrained transitions of consumption starting from the constrained level c_0^m, $i = 0, 1, ..., n$. $z_{ij}, j = N, N-1, ...$ are the possible income states for an individual who has remained unconstrained for i periods, with associated conditional probabilities $\pi_{ij|m}$, while $\pi_{i|m} = \sum_j \pi_{ij|m}$ is the marginal probability that an individual at income z^m remains unconstrained for i periods, and $\pi_0 = 1$. Note that in (28), only unconstrained states appear, as continuation and autarky values cancel in the participation constraint for all constrained future states. Differentiating (28) totally with respect to c_i yields a condition for any participation-compatible perturbation to the planner's allocation

$$0 = \sum_{i=0}^{n} \beta^i \pi_{i|m} u'(c_i) dc_i = u'(c_0^m) \sum_{i=0}^{n} \pi_{i|m} R^{-i} dc_i \tag{29}$$

where the second equality follows from the law of motion (15). Since R^{-i} is a positive sequence, dc_i has to take both negative and positive values.

Differentiating the law of motion (15) totally yields a recursive definition of $\frac{dc_i}{dR}$

$$\frac{dc_i}{dR} = \frac{\frac{u''(c_{i-1})}{u'(c_{i-1})}}{\frac{u''(c_i)}{u'(c_i)}} \frac{dc_{i-1}}{dR} - \frac{u'(c_i)}{u''(c_i)} \frac{1}{R} \doteq \alpha_1 \frac{dc_{i-1}}{dR} + \alpha_2 \tag{30}$$

The constant term α_2 is strictly positive, while $0 < \alpha_1 \leq 1$ for any utility function satisfying non-increasing relative risk aversion. This implies that for any $\frac{dc_{i-1}}{dR} < 0$, $\frac{dc_i}{dR} > \frac{dc_{i-1}}{dR}$, while if $\frac{dc_{i-1}}{dR} > 0$, $\frac{dc_i}{dR} > 0$. In other words, the sequence dc_i crosses the zero line exactly once from below. The change of aggregate consumption by individuals on the mth subdistribution, denoted C^m, is therefore simply

$$\frac{dC^m}{dR} = \nu \sum_{i=0}^{m} \pi_i dc_i < (>) \nu \sum_{i=0}^{m} \pi_i R^{-i} dci = 0 \ for \ R < 1 \ (R > 1) \tag{31}$$

where the inequality (inverse inequality) follows from the fact that R^{-i} overweighs (underweighs) the latter, positive elements of the sequence dc_i when interest rates are below (above) 1. As this holds for all m, $\overline{\odot}$ is decreasing in interest rates at levels $R < 1$, reaching a minimum at $R = 1$, and rises with R from thereon. Since for $R = 1/\beta$ insurance is perfect, which is unfeasible by assumption 1, excess demand crosses the zero line exactly once at some $1 < R^* < \frac{1}{\beta}$. ■

5 The distribution of consumption and wealth compared to the data

This section looks at the stationary joint distribution of consumption, income and wealth, characterised theoretically in the previous section, for a calibrated version of the US economy. I compare these to the distributions in a standard self-insurance economy on the one hand, and in US micro-data on the other.

Previous studies on consumption insurance in calibrated economies usually have not looked at the shape of the implied joint distributions, but focused on particular moments of marginal distributions. This is true also for studies of limited commitment economies, such as Krueger and Perri (2006) who analyse changes in cross-sectional variances of income and consumption over time, or Cordoba (2008), who concentrates on variances and the upper tails of marginal distributions. Studies of the empirical distribution of consumption and income, on the other hand, have pointed out asymmetries. Battistin et al

(2007), for example, conclude that the marginal distribution of consumption is close to a log-normal, i.e. has significant right-hand skew. Dynan et al (2006) show that in PSID data, while consumption responds more strongly to negative income shocks, this asymmetry has fallen over time, which they take as evidence of declining liquidity constraints. Krueger and Perri (2008), on the other hand, show that in the Italian Household Survey the relation between nondurable consumption and income changes unexplained by a first stage regression on household characteristics is largely linear, with a slightly stronger response of consumption to positive income changes. This section looks at asymmetries in joint distributions both in theory and US micro-data.

5.1 A quantitative model calibrated to the US economy

This section briefly describes the Krueger and Perri (2006) calibration of a limited commitment economy with production. For the income process, the authors assume the log of post tax labour income plus transfers (LEA+) $log(z_t)$ to be the sum of a group specific component α_t and an idiosyncratic part y_t. The latter, in turn, is the sum of a persistent AR(1) process m_t, with persistence parameter ρ and variance σ_m^2, plus a completely transitory component ε_t which has mean zero and variance σ_ε^2.
The process for LEA+ is thus of the form

$$\begin{aligned} log(z_t) &= \alpha_t + y_t \\ y_t &= m_t + \varepsilon_t \\ m_t &= \rho m_{t-1} + \nu_t \\ \varepsilon &\sim N(0, \sigma_\varepsilon^2) \\ \nu_t &\sim N(0, \sigma_\nu^2) \end{aligned} \tag{32}$$

Using data from the Consumer Expenditure Survey (CEX), the authors first partial out the group-specific component α_t as a function of education and other variables, identifying the variance of the idiosyncratic part of income y_t, as well as (from the short panel dimension of the CEX) its first order autocorrelation. Setting $\rho = 0.09989$, the value estimated by Storesletten et al (2004), then allows the identification of σ_ν^2 and σ_ε^2. In this study, I use $\sigma_\nu^2 = 0.26$ and $\sigma_\varepsilon^2 = 0.12$, the estimate for the year 2003, the endpoint of the Krueger and Perri (2006) sample. I then use the standard Tauchen and Hussein (1999) method to approximate the resulting process using a 7-state Markov chain for m_t, and a binary process for ν_t. It is important to note that the resulting 14 state Markov

process does not fulfil the monotonicity assumption of the theory section, as transitions are identical across transitory shocks. The income process thus belongs to a more general class than that analysed in the previous sections.
For preferences, I choose a CRRA utility function with coefficient of relative risk aversion of 1 (log-preferences) and a discount factor of 0.96. In order capture the features of the US economy more accurately than in the simple theoretical model, I allow agents to save at the equilibrium interest rate after default, and introduce production in the economy. In particular, I assume that competitive firms hire capital and labour from households to operate a Cobb-Douglas technology

$$Y = AK^{\alpha}L^{1-\alpha} \tag{33}$$

and set the labour share α to 0.3. Again, the calibration follows Krueger and Perri (2006), who choose the depreciation rate of capital δ and total factor productvity A to target a capital-output ratio of 2.6 and an interest rate of 4 percent in their benchmark period. The corresponding values of A and δ are 0.9637 and 0.0754 respectively. The computational algorithm to solve for the stationary equilibrium first solves for the stationary equilibrium for a given interest rate, following the appendix that describes the recursions that derive the stationary consumption distribution in the general case.[8] I then use the bisection method to find the market clearing interest rate $R^{\star}$.

5.2 Joint distributions of c,y,w - Theory and non-parametric estimates from US micro-data

This section presents the joint distributions of consumption, wealth and income. In particular, I compare the distributions in Krueger and Perri's (2006) limited commitment economy to non-parametric estimates of their counterparts in US-microdata, as well as those in a simple self-insurance Ayagari economy. The latter has the same income process, technology and preferences described before, but agents can only save and borrow in uncontingent bonds subject to a borrowing limit equal to annual income. I calculate

[8] I amend this for the fact that, with purely transitory shocks ν_t, the monotonicity condition for F does not hold. So I need to reshuffle income states occasionally in order to have decreasing minimum-participation-compatible consumption values $c_0^1 > c_0^2 > ... > c_0^N$ during the algorithm. The solution is facilitated by the fact that, if this monotonicity condition holds, c_0^i can be found quickly using bisections on an interval $[z_i, c_0^{i+1}]$. This yields an algorithm that is extremely efficient when solving for the stationary consumption distribution.

the joint distributions by applying a simple histogramm density estimator to the exact theoretical distribution of the limited commitment model, and to a simulation of the Ayagari economy.[9] To compare the theoretical densities to the data, I then estimate bivariate kernel densities for US data on consumption and wealth, based on an optimal choice of the bandwith as in Botev et al (2009).

5.2.1 The distribution of consumption and income

Figures 3 and 4 use consumption and income data from the 2003 wave of the US Consumer Expenditure Survey (CEX) to confront their estimated joint density with that from the models. Particularly, I use the dataset constructed by Krueger and Perri (2006), and their definition of income and consumption. Their income measure corresponds to the CEX measure of after-tax labour earnings plus transfers (the sum of wages and salaries of all household members, plus a fixed fraction of self-employment farm and nonfarm income, minus reported federal, state, and local taxes (net of refunds) and Social Security contributions). Importantly, the consumption series includes an imputed measure of services from durables (for details see Krueger and Perri 2006). From both of these series I partial out the effect of a vector of observable individual characteristics, to control for ex-ante differences or predictable changes in life-time wealth.[10] Figure 2 and 3 show that the results from the theory continue to hold with the more general income process: the marginal distribution of consumption in the Krueger and Perri (2006) calibration, presented in figure 2 where equal colours correspond to individuals who were last constrained in the same income state, is a mixture of geometric subdistributions. And figure 3 shows that consumption rises on average with current income, but is highly heteroscedastic. In particular, the conditional variance of consumption declines as we move up the income distribution. The Ayagari economy, interestingly, also has some decline in conditional variances, although less so than the limited commitment economy. The data has a roughly homoscedastic, increasing shape of the conditional distribution.
Figure 4 presents the joint distribution of consumption and income growth. Its first strik-

[9]The histogramm density estimation for the Ayagari economy is based on an individual simulated income and consumption path of 100.000 periods, of which I discard the first 1000 for the estimation.

[10]Particularly, unless otherwise mentioned, I use residuals from a regression of income and consumption on a cubic in the household head's age, and dummies that equal 1 if the household head has a unversity degree, a college degree, a high school degree, is male, is black, is asian, or of some other non-white race. I concentrate on households where the head is between 16 and 64 years of age.

ing features are the important differences between the 2 model densities: in the limited commitment economy, as suggested by theory, income declines are perfectly shared, resulting in a floor to the distribution slightly below zero. Positive income shocks are followed by a variety of positive consumption responses, leading to a strong rise in the conditional variance of the distribution for larger shocks. The Ayagari model on the other hand has a much more homoscedastic shape around a roughly linear mean response of consumption to income growth. To compare these distributions to the data, figure 4 uses log-differences of the raw data, not the residuals from the first stage regression. The resulting estimate of the distribution shows neither the downward cap, nor the heteroscedasticity of the limited commitment model. Rather, the cloud character of the picture suggest important measurement error in the CEX data. The picture is practically unchanged if we use the residuals from the first stage regression.

Figure 2: The marginal distribution of consumption and its subdistributions

[Insert Figure 2 about here.]

Figure 3: The joint distribution of consumption and income

[Insert Figure 3 about here.]

Figure 4: The joint distribution of consumption and income growth

[Insert Figure 4 about here.]

5.2.2 The distribution of wealth and income

Figure 5 performs a similar exercise for the joint distribution of wealth and income, using the net worth variable of the 2004 wave of the Survey of Consumer Finances (SCF), and chopping off the upper 1 percent of all distributions to control for outliers and top-coding. In the limited commitment economy insurance lowers the financial wealth of high income households. So, even with the more general income process, the income rich have minimum wealth. Since individuals slowly deplete their wealth levels after a negative income shock, the income poor have a variety of positive wealth levels, including the highest in the

economy. In the Ayagari economy, on the other hand, the bufferstock nature of wealth leads on average to a positive relationship between income and wealth levels. But there is large variation around the mean, as individuals slowly build up, or draw down, their wealth after income changes. The mass of individuals at the borrowing constraint clearly rises as income falls. Comparing this to the SCF data, we see both an increase in mean wealth, as well as in its variance, as income, measured as salaries plus a proportion of business income, rises.

The SCF is a cross-section, so does not allow us to look at changes in wealth. But figure 6 compares the joint distributions of financial income and earned income in the model and the data. Again, the insurance mechanism leads to a strong negative correlation between income and financial returns in the limited commitment model, with the expected declining conditional variances. In the data, we find a positive relationship between financial and other income, as in the Ayagari model.

Figure 5: The cross-sectional distribution of wealth and income

[Insert Figure 5 about here.]

Figure 6: The cross-sectional distribution of financial income and income

[Insert Figure 6 about here.]

6 Conclusion

This study has looked at the equilibrium distribution of agents in an economy where limited commitment to contracts constrains risk-sharing. The theoretical contribution was to prove existence and uniquencess of a stationary equilibrium in a continuum limited commitment economy, and to provide an analytical characterisation of the distribution of consumption and income, including a closed form solution for an example with two income states and CRRA preferences. The theory showed how the asymmetric nature of insurance in the model, where negative shocks are shared but positive shocks lead to idiosyncratic consumption growth, implies declining conditional variances of wealth and consumption along the income distribution, and a negative relationship between wealth and income on average. The quantitative part of the paper looked at a limited commitment economy with capital and a more general income process, to compare the joint

equilibrium distributions, and their characteristic non-linearity and heteroscedasiticity, with non-parametric estimates of the counterparts in US micro data, and those in a simple Ayagari economy. The results showed that, even with a more realistic income process featuring both near-permanent and transitory shocks, the limited commitment economy still produces very asymmetric joint distributions: consumption growth has a floor slightly below zero, but an upward tail that becomes more important for stronger positive income shocks. And both the mean and variance of wealth fall with income. Importantly, both the data and the Ayagari model have less heteroscedastic distributions, and mean wealth that rises with income.

The approach of this paper, to focus on the shape of joint distributions in order to test economic models with heterogeneous agents against the empiricial evidence, provides plenty of room for further research. One direction would be to generalise the model economies analysed here, to see if their characteristics are robust. For example, Broer (2009c) shows that amending the calibration used in this paper to include some heterogeneity in discount factors can largely reconcile the model-impact of near-permanent income shocks on current consumption growth with the data. On the other hand, a more thorough description of the joint distributions in micro-data is needed. Here, the new dataset provided by Blundell et al (2008), who have imputed a series of non-durable consumption for the PSID on the basis of its food expenditure information and a consumption demand function estimated on CEX data, seems very promising. And finally, the equality of model distributions and data should be tested more rigorously, accounting appropriately for the important role of measurement error in the data.

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8 Appendix

8.1 Proof of Lemma 1

The planner's decision rule Γ *has the form*

$$\mu_{i,t+1} = max\{\mu_{t+1}(z_{i,t+1}), \mu_{i,t}\}$$

For every t, $\mu_t(z)$ *is strictly increasing in* z, *and for every* z, *the sequence* $\mu_t(z)$ *increases strictly over time. Also, the set of individual planner weights with positive mass is strictly finite:* $|\{\mu_j : \Phi_I(\{i : \mu_{i,t} = \mu_j\}) > 0\}| < \infty, \forall t$.

Proof

To prove the first statement write the difference between continuation values $V(\mu_i, z^i)$ and autarky values $W(z^i)$ as

$$\Delta_t(\mu, z^i) \doteq V(\mu, z^i) - W(z^i)$$

$$u(\mu) - z^i + F_i \sum_{s=1}^{\infty} \beta^s F^{s-1} max\{(\overline{u_{t+s}(\mu)} - u(z)), \overline{0}\} \tag{34}$$

where F_i is the ith row of F, $\overline{u(z)}$ the $Nx1$ vector of utilities from consuming income, $\overline{u_t(\mu)}$ the constant vector of utility from having planner weight μ in period t, and $\overline{0}$ the zero vector. Δ is thus the discounted sum of "utility in excess of autarky" across states where individuals are unconstrained, as in all constrained states autarky and continuation values cancel. It can be interpreted as a measure of insurance benefits promised to individual i. Note that $\Delta_t(\mu_t(z^i), z^i) = 0$ defines the minimum participation compatible planner weight $\mu_t(z^i)$.

I first show $\Delta(\mu, z^j)$ is strictly increasing in z^j, for all μ. For any μ

$$\begin{aligned}
\Delta_t(\mu, z^j) &= u_t(\mu) - u(z^j) + F_j \sum_{s=1}^{\infty} \beta^s F^{s-1} max\{(\overline{u_{t+s}(\mu)} - \overline{u(z)}), \overline{0}\} \\
&< u_t(\mu) - u(z^{j+1}) + F_j \sum_{s=1}^{\infty} \beta^s F^{s-1} max\{(\overline{u_{t+s}(\mu)} - \overline{u(z)}), \overline{0}\} \\
&\leq u_t(\mu) - u(z^{j+1}) + F_{j+1} \sum_{s=1}^{\infty} \beta^s F^{s-1} max\{(\overline{u_{t+s}(\mu)} - \overline{u(z)}), \overline{0}\} \qquad (35) \\
&= \Delta_t(\mu, z^{j+1}) \qquad (36)
\end{aligned}$$

where the second inequality follows from monotonicity of F, since the vector $max\{(u_{t+s}(\mu)-\overline{u(z)}),\overline{0}\}$ is decreasing in income values. Since $\Delta_t(\mu_t(z^j),z^j)=\Delta_t(\mu_t(z^{j+1}),z^{j+1})=0$ it follows that $\mu_t(z^j)>\mu_t(z^{j+1})$.

To see that the sequence $\mu_t(z)$ is <u>strictly</u> increasing for every income level z, note first that Assumption 1 implies a positive mass of agents with binding participation constraints every period, who experience an increase in their planner weights. Given constant resources, any constant $\mu_{i,t+1}=\mu_{i,t}$ then implies strictly declining consumption of individual i according to (13). Thus, since the autarky value $W(z^j)$ is constant through time, constant or declining cutoff values $\mu_{t+1}(z^j)\leq\mu_t(z^j)$ violate participation constraints. So $\mu_{t+1}(z^j)>\mu_t(z^j)$.

Finally, to see that the set of individual planner weights with positive mass is strictly finite, note first that the minimum participation compatible planner weights $\mu_t(z)$ lie in a finite interval defined by $1<\frac{\mu_t(z^1)}{\mu_t(z^N)}\leq\frac{z^1}{z^N}\ \forall t$, as otherwise participation constraints for some individuals are slack either today or at some point in the future. Since initial planner weights are strictly positive and finite, the ratio of maximum and minimum planner weights is bounded in all periods. As the sequence of $\mu_t(z^N)$ is strictly increasing there is an $\epsilon>0$ such that $\frac{\mu_{t+1}(z^N)}{\mu_t(z^N)}>1+\epsilon,\ \forall t$. But then the number of periods an individual can remain unconstrained is strictly bounded by $T=min(x\in\mathbb{N}:x>\frac{ln(z^1)-ln(z^N)}{ln(1+\epsilon)})$. Since in every period there are at most N new weights, the number of planner weights is bounded by N^T plus the number of initial weights K. ■

8.2 Proof of Proposition 1: The consumption distribution in the general case

For $1<R<\frac{1}{\beta}$ the interest rate in stationary equilibrium, the joint distribution of income and consumption $\Phi_C:\mathbb{C}\times\mathbb{Z}\longrightarrow[0,1]$ has the following features:

1. Φ_C is discrete, with positive mass at consumption values between minimum income and some upper bound c_0^1 smaller than the highest income level: $\mathbb{C}\subseteq[y_N,c_0^1],\ c_0^1<z^1$.

2. There are N minimum levels of consumption $c_0^i,\ i=1,...,N$ under which consumption of agents with income i never falls and where participation constraints at income z^i hold with equality. These threshold levels are increasing in income

$c_0^1 < c_0^2 < .... < c_0^N$. The lower bound of the distribution is minimum income $c_0^N = z^N$.

3. Every consumption threshold c_0^i is an upper bound to a geometric subdistribution of consumption Φ_C^i, with support $\{c_j^i\}$ recursively defined by the law of motion $U'(c_{j+1}^i = \beta R U'(c_j^i), j = 0, 1, 2, ...$, and bounded below by z_N. Φ_C is thus a mixture of $N-1$ geometric distributions $\Phi_{C,i}, i = 1, ..., N-1$. The appendix contains an analytical expression for the frequencies in this distribution.

4. Individuals at the higest income level z^1 all have maximum consumption level c_0^1. The support of consumption conditional on income $z^i, i > 1$ is $[c_0^i, c_1^1]$. So the support of consumption narrows as income rises. For i.i.d. transitions (identical rows in F), this implies that the conditional variance of consumption falls monotonously in income.

Proof

The proof is by construction of the stationary consumption distribution.

Ad 1-3: The support $\mathbb{C}$

I construct $\mathbb{C}$ "bottom-up", starting from its lower bound, which we know to be minimum income. Also, from Lemma 1, we know that minimum participation-compatible levels of consumption c_0^i increase in income z^i. Since c_0^i solves the participation constrained of individuals at income z^i with equality, this allows me to recursively determine c_0^i by substituting into the ith participation constraint the autarky values at incomes $z^j > z^i, j = i, i-1, ..., 1$ for future states with non-negative income shocks, and the consumption values given by the law of motion (15) for unconstrained states. Starting at $i = N-1$ and moving up income levels assures that this procedure can keep account of binding participation constraints as individuals move down in consumption from c_0^i to Z^N.

To see this in detail, denote as $c^i(c, R)$ the result of applying the law of motion for unconstrained transitions (15) i times starting from level c at interest rate R.

We know $c_0^N = Z^N$. Consider minimum participation-compatible consumption in the second lowest income state $N-1$. There, individuals receive c_0^{N-1} today, the value of which we want to determine. They face the "danger" of moving, with probability $f_{N-1,N}$, to state N, and thus down to $c^1(c_0^{N-1}, R)$ tomorrow. With probability $f_{N-1,i}$, however, they move to income $z^i > z^N$ receiving $W(i)$. So c_0^{N-1} is uniquely determined from the

participation constraint

$$W(N-1) =$$

$$U(c_0^{N-1}) + f_{N-1,N} \sum_{s=0}^{\infty} \beta^{s+1} f_{NN}^s max\{U(c^s(c_0^{N-1}, R), N), U(z^N)\} + \tag{37}$$

$$\beta \sum_{i=1}^{N-1} f_{N-1,i} W(i) + f_{N-1,N} \frac{\beta^2}{1 - f_{N,N}\beta} \sum_{1}^{N-1} f_{N,i} W(i) \tag{38}$$

Here, the second term on the right-hand side of the equation is the value from the declining consumption path starting at c_0^{N-1} and truncated at minimum level c_0^N, weighted by the probability to remain in income state N. The third term is the continuation value when not receiving a negative income shock tomorrow, the fourth from moving down in income tomorrow and then receiving positive income shocks at a later date. Note that the right hand side is increasing in c_0^{N-1} while the left hand side is constant. So the solution is unique.

3. Analogously, one can determine the other values c_0^i from repeated application of this algorithm.

The support of the consumption distribution $\mathbb{C}$ is simply the union of downward-sloping paths starting at minimum participation-compatible consumption c_0^i $\mathbb{C} = \cup_{i=1}^N \{max[c^j(c_0^i, R), z^N], j = 0, 1, 2, ...\}$. Note that the highest level of consumption c_0^1 is strictly lower than the highest income level z^h from assumption A2, which implies that there is at least one unconstrained transition of individuals at z^1 that receive a shock z^N, which happens with positive probability. With $c_0^1 \geq z^1$ the participation constraint would thus be slack, as continuation utilities under insurance are strictly greater than in autarky in at least 1 state of the world. This however cannot be optimal for the planner, so we have $c_0^1 < z^1$.

Ad 3: The frequency distribution on $\mathbb{C}$

I construct the frequency distribution "top-down". From Lemma 1, I know that all high income individuals are constrained, at the minimum participation-compatible consumption for individuals with the highest income c_0^1. Thus, its mass is equal to the stationary mass of individuals at z^1. The rest of the frequency distribution is then based on the transition probabilities as follows:

Define Φ_C^i to be the subdistribution of consumption that contains all indivduals that were last constrained at income z^i. Out of individuals with highest income last period, all but those that remain at z^1 move down in consumption to $c^1(c_0^1, R)$, according to the law of motion (15). Denoting the ith row of F as F_i, and defining the Matrix $\overline{F}_i$ as F

with the first i columns and rows replaced by zeros, and disregarding other thresholds for now, this would yield a geometric distribution on the downward sloping path from $c^1(c_0^1, R)$ equal to $\coprod_{1,n} = \nu F_1 \overline{F}_1^n$ on support given by $(c^n(c_0^1, R), z)$. However, since after $T_2 = \lfloor \frac{ln(U'(c_0^1)) - ln(U'(c_0^2))}{ln(\beta) ln(R)} \rfloor$ periods individuals at income 2 hit their participation constraint on the downward-sloping path $c^n(c_0^1, R)$, they drop out of this distribution, equivalent to $\overline{F}_1$ shrinking to $\overline{F}_2$. Equivalent reasoning for lower values of income yields the following vector valued sequence of joint frequencies on $c_j^1, z_k, j = 1, 2, ..., k = 1, ..., N$ [11]

$$\coprod_{1,n=\sum_{l=1}^{j-1} T_l + t_j} = \nu F_1 \ \Pi_{l=1}^{j-1} \overline{F}_l^{T_l} \ \overline{F}_j^{t_j}, j = 1, ...N, t_j = 1, ...T_j \tag{39}$$

for $T(i) = 1, T_{k>i} = \lfloor \frac{ln(U'(c_0^k)) - ln(U'(c_0^{k+1}))}{ln(\beta) ln(R)} \rfloor$ the integer number of unconstrained transitions between threshold levels of consumption c_0^k and c_0^{k+1}. The marginal subdistribution Φ_C^1 is simply the the row sum of the expression.

More generally, the joint subdistribution of income and consumption starting at consumption threshold c_0^i with support $c_j^i, z_k, j = 1, 2, ..., k = 1, ..., N$ has the vector valued sequence of frequencies

$$\coprod_{i,n=\sum_{l=i}^{j-1} T_l + t_j} = \nu_i \ F_i \Pi_{l=i}^{j-1} \overline{F}_l^{T_l} \ \overline{F}_j^{t_j}, j = i, ...N, t_j = 1, ...T_j \tag{40}$$

where $\nu_i = \Phi_Z(z^i) - \sum_{n=0}^{i-1} \sum_t \coprod_{n,t}(i, i)$ is the stationary mass of individuals at income level z^i minus those with income z^i and consumption above the treashold c_0^i.

Ad 4: The conditional distribution of consumption

The strictly positive entries of F ensure that the least upper bound of consumption by individuals at income $z^i, i = 2, ..., N$ is the first downward step from the threshold level for z^1, $c^1(c_0^1, R) = c_1^1$. The greatest lower bound of consumption for indivdiuals at income z^i is of course threshold value c_0^i, so the minimum interval covering the discrete support of consumption conditional on income $z^i, i > 1$ is $[c_1^1, c_0^i]$. Since c_0^i increases with income, the width of the interval decreases.

Monotonicity of transitions ensures that individuals at lower incomes are concentrated in lower parts of subdistributions, which can be shown to lead to conditional means that increase in income. Conditional variances on the other hand are non-monotononic. Assuming i.i.d. uncertainty, however, or identical rows in F, it is evident that $\Phi_{C|z^i}$ is simply $\Phi_{C|z^{i+1}}$ with a truncated tail, and the tail-mass moved to the truncation point. This implies conditional variances that decrease monotononously in income values.

[11] To keep notation concise I take $\Pi_{i=1}^0 x_i = 1, \Sigma_{i=1}^0 x_i = 0, \forall x_i$.

8.3 Proof of Corollary 2

With CRRA preferences and 2 income values, the following is true:

1. *The covariance between income and consumption is positive. The covariances between income and both financial returns and wealth are negative.*

2. *The mean of consumption increases in income. Its conditional variance decreases.*

3. *If $\Phi_C(c_m) \approx 0$, the cross-sectional variance of log-consumption in stationary equilibrium is*

$$Var_c = \tau[\frac{\log(\beta R)}{\sigma}]^2 \tag{41}$$

where $\tau > 0$ is a function of transition probabilities only. If there is a non-negligible mass at the truncation point, $\Phi_C(c_m) > 0$, this is an upper bound for the cross-sectional variance of individual consumption.

Proof Ad 1: The covariance of income and consumption is given by

$$E(c - \mu_c)(z - \overline{z}) \tag{42}$$

$$= \nu(c_1 - \mu_c)(z^h - \overline{z}) + \sum_2^m \nu(1-p)q^{i-1}(c_i - \mu_c)(z^l - \overline{z}) \tag{43}$$

$$= \nu(c_1 - \mu_c)[(z^h - \overline{z}) - (z^l - \overline{z})] \tag{44}$$

$$= \nu(c_1 - \mu_c)[z^h - z^l] > 0 \tag{45}$$

where μ_c is the mean of consumption, and the second equality imposes market clearing $\sum_2^m \nu(1-p)q^{i-1}(c_i - \overline{c}) = -\nu(c_1 - \overline{c})$.
To see the second statement, note that the joint distribution of financial returns and income is $\Phi_{y_{fin},z} : \mathbb{B}([c_1 - z^h, c_1 - z^l]) \times \{z^l, z^h\} \longrightarrow [0,1]$ is given by

$$\Phi(y_{fin,1}, z^h) = \frac{1-q}{2-q-p} \doteq \nu \tag{46}$$

$$\Phi(y_{fin,i}|_{1<i<m}, z^l) = \nu(1-p)q^{i-1} \tag{47}$$

$$\Phi(y_{fin,m}, z^l) = \nu\frac{(1-p)q^{m-1}}{1-q} \tag{48}$$

$$\Phi(\cdot,\cdot) = 0 \; otherwise \tag{49}$$

for

$$
\begin{aligned}
y_{fin,1} &= c_1 - z_h < 0 \\
y_{fin,i} &= z_l - c_1(\beta R)^{\frac{i}{\sigma}} > 0, 1 < i < m \\
y_{fin,i} &= 0
\end{aligned} \tag{50}
$$

where $\mathbb{B}(I)$ denotes the Borel sets on interval I. So high income agents have strictly negative financial income, while individuals with low income have non-negative financial income. This of course implies negative covariance, equal to

$$E(y_{fin} - \overline{y_{fin}})(z - \overline{z}) \tag{51}$$

$$\nu(c_1 - z^h)(z^h - \overline{z}) + \sum_2^m \nu(1-p)q^{i-1}(c_i - z^l)(z^l - \overline{z}) \tag{52}$$

$$\leq \nu(c_1 - z^h)(z^h - \overline{z}) < 0 \tag{53}$$

where the last line follows from $(c_i - z^l)(z^l - \overline{z}) \leq 0, \ \forall i > 1$ The joint distribution of financial wealth and income $\Phi_W : \mathbb{R} \times \{z^l, z^h\} \longrightarrow [0,1]$ has the same frequencies as $\Phi_{y_{fin},z}$, on a support defined by the recursion

$$
\begin{aligned}
A_m &= \frac{1-p}{R} A_1 \\
A_i &= y_{fin,i} + \frac{1-p}{R} A_1 + \frac{p}{R} A_{i+1} > A_{i+1}, 1 < i \leq m-1 \\
A_1 &= \frac{1}{1 - \frac{q}{R}} y_{fin,1} + \frac{1-q}{R} A_2 < 0
\end{aligned} \tag{54}
$$

The covariance of income and financial wealth is given by

$$E(A - \overline{A})(z - \overline{z}) \tag{55}$$

$$= \nu(A_1)(z^h - \overline{z}) + \sum_2^m \nu(1-p)q^{i-1}(A_i)(z^l - \overline{z}) \tag{56}$$

$$= (z^h - \overline{z})\nu(A_1) - (z^l - \overline{z})\nu(A_1) \tag{57}$$

$$= (z^h - z^l)\nu(A_1) \tag{58}$$

$$< 0 \tag{59}$$

where the third line exploits the fact that financial wealth sums to zero across individuals.
Ad 2: Both statements follow immediately from the fact that high-income individuals are located at a mass point on the upper bound of the consumption support.
Ad 3:

1. Denote the first entry of the normalised left eigenvector of transition matrix F associated with a unit eigenvalue as $\nu = \frac{(1-q)}{(2-q-p)}$, and the log of x as $\widehat{x}$.
2. The mean of $\log c$ is

$$\mu_c = \nu\{\widehat{c_h} + (1-p)\sum_{i=1}^{\infty}[\frac{\widehat{\beta R}}{\sigma}i + \widehat{c_h}]q^{i-1}\} \tag{60}$$

$$= \widehat{c_h} + \frac{1-p}{(1-q)(2-q-p)}\frac{\widehat{\beta R}}{\sigma} \tag{61}$$

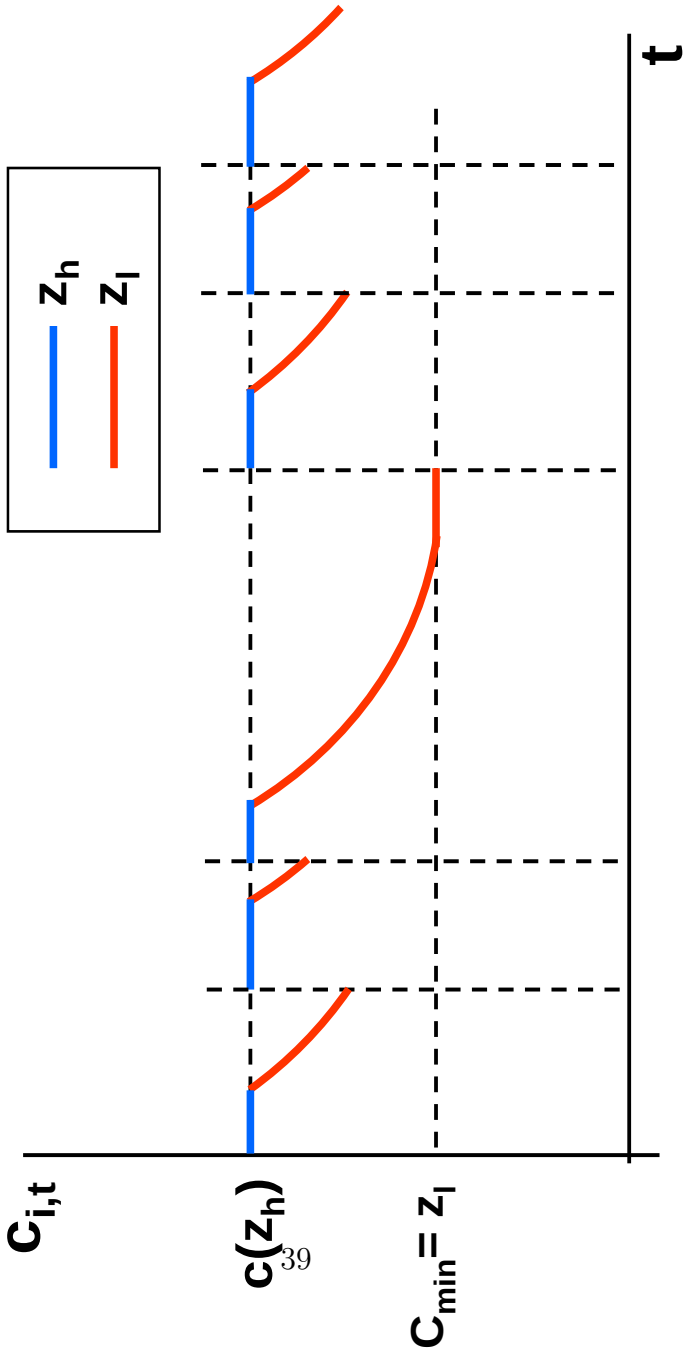
3. The variance is

$$\begin{aligned}
VAR_c = \quad & \nu\frac{(1-p)^2}{(1-q)^2(2-q-p)^2}[\frac{\widehat{\beta R}}{\sigma}]^2 && (62)\\
& +\nu(1-p)[\frac{\widehat{\beta R}}{\sigma}]^2\sum_{i=1}^{\infty}[i - \frac{(1-p)}{(1-q)(2-q-p)}]^2 q^{i-1} && (63)\\
= \quad & \nu[\frac{\widehat{\beta R}}{\sigma}]^2\{\frac{(1-p)^2}{(1-q)^2(2-q-p)^2} && (64)\\
& +(1-p)[\frac{(1+q)}{(1-q)^3} - 2\frac{(1-p)}{(1-q)^3(2-q-p)} + \frac{(1-p)^2}{(1-q)^3(2-q-p)^2}]\} && (65)\\
= \quad & \nu[\frac{\widehat{\beta R}}{\sigma}]^2[-\frac{(1-p)^2}{(1-q)^3(2-q-p)} + \frac{(1-p)(1+q)}{(1-q)^3}] && (66)\\
= \quad & [\frac{\widehat{\beta R}}{\sigma}]^2\frac{(1-p)(1+q(1-p-q))}{(1-q)^2(2-p-q)^2} && (67)
\end{aligned}$$

The more general result for the truncated case with $\Phi_C(c_m) > 0$ is not difficult, but algebraically messy, to compute. But note that the variance of an truncated geometric distribution is stricly lower, and that for the i.i.d. case $1-p = q$ both the mean and the variance reduce to those for an ordinary geometric distribution. ■

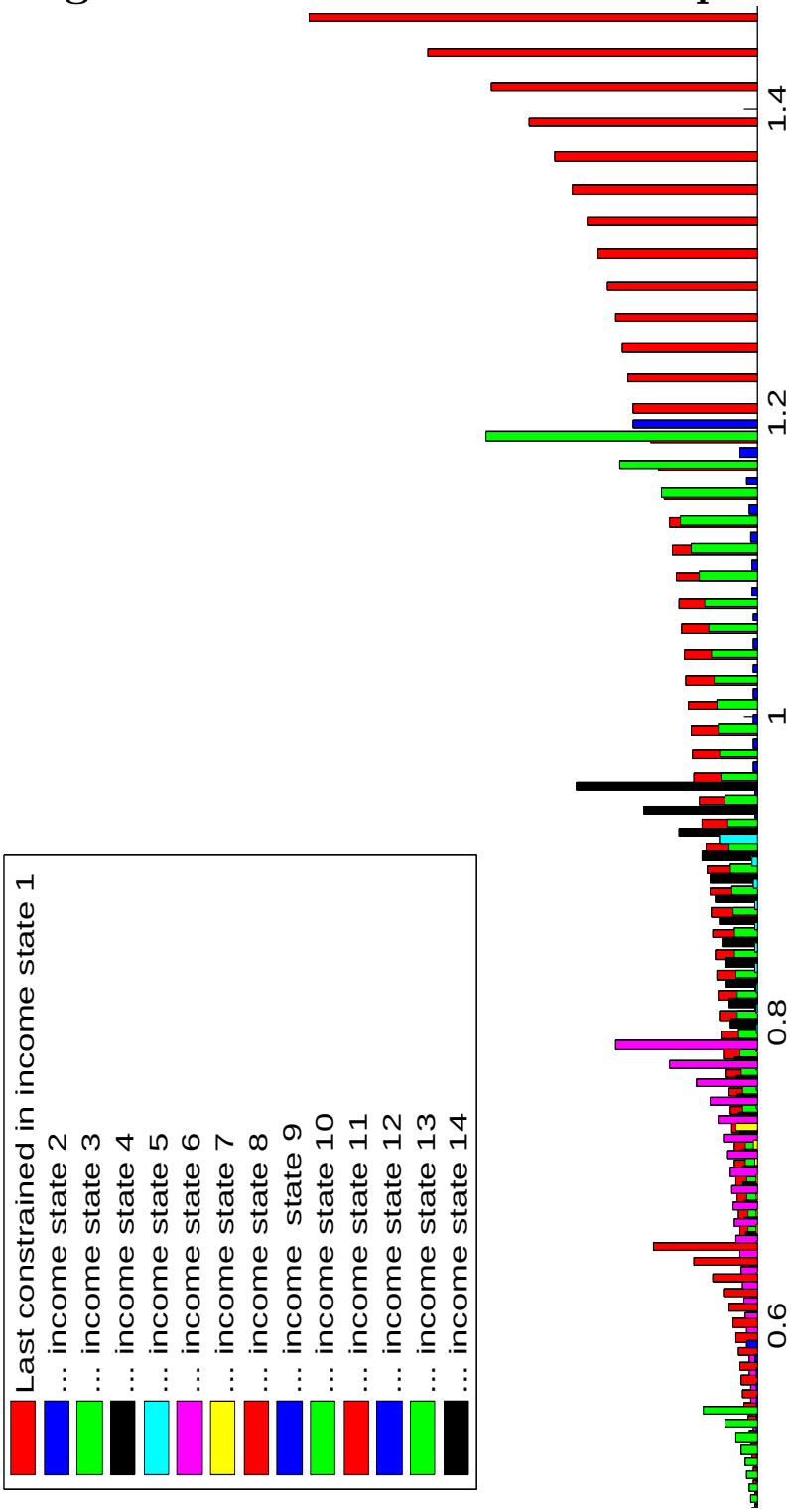
9 Tables and figures

Figure 1: The consumption path with two income values



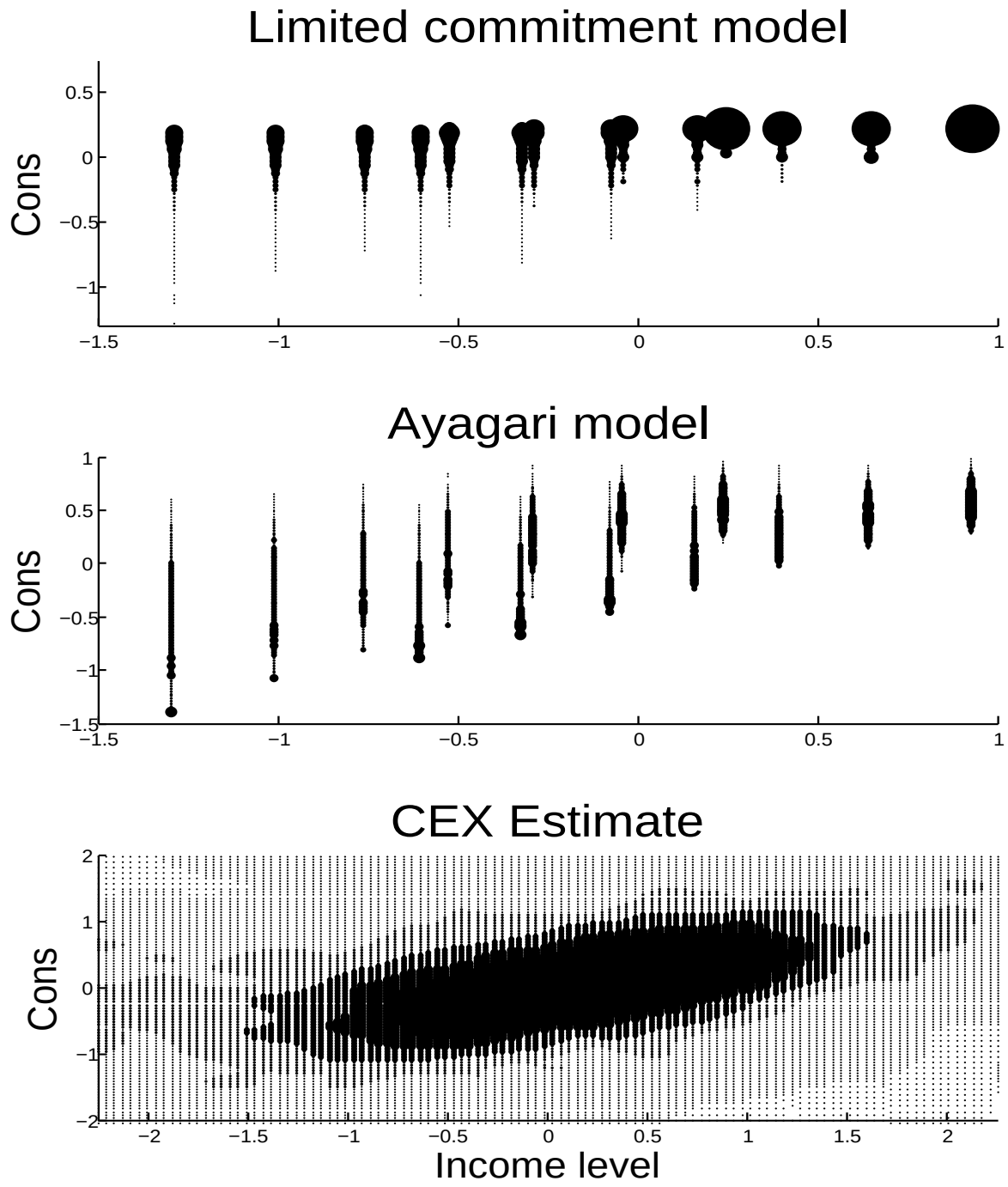
The figure shows an individual's consumption as she moves through periods of high (blue line)

Figure 2: Marginal distribution of consumption



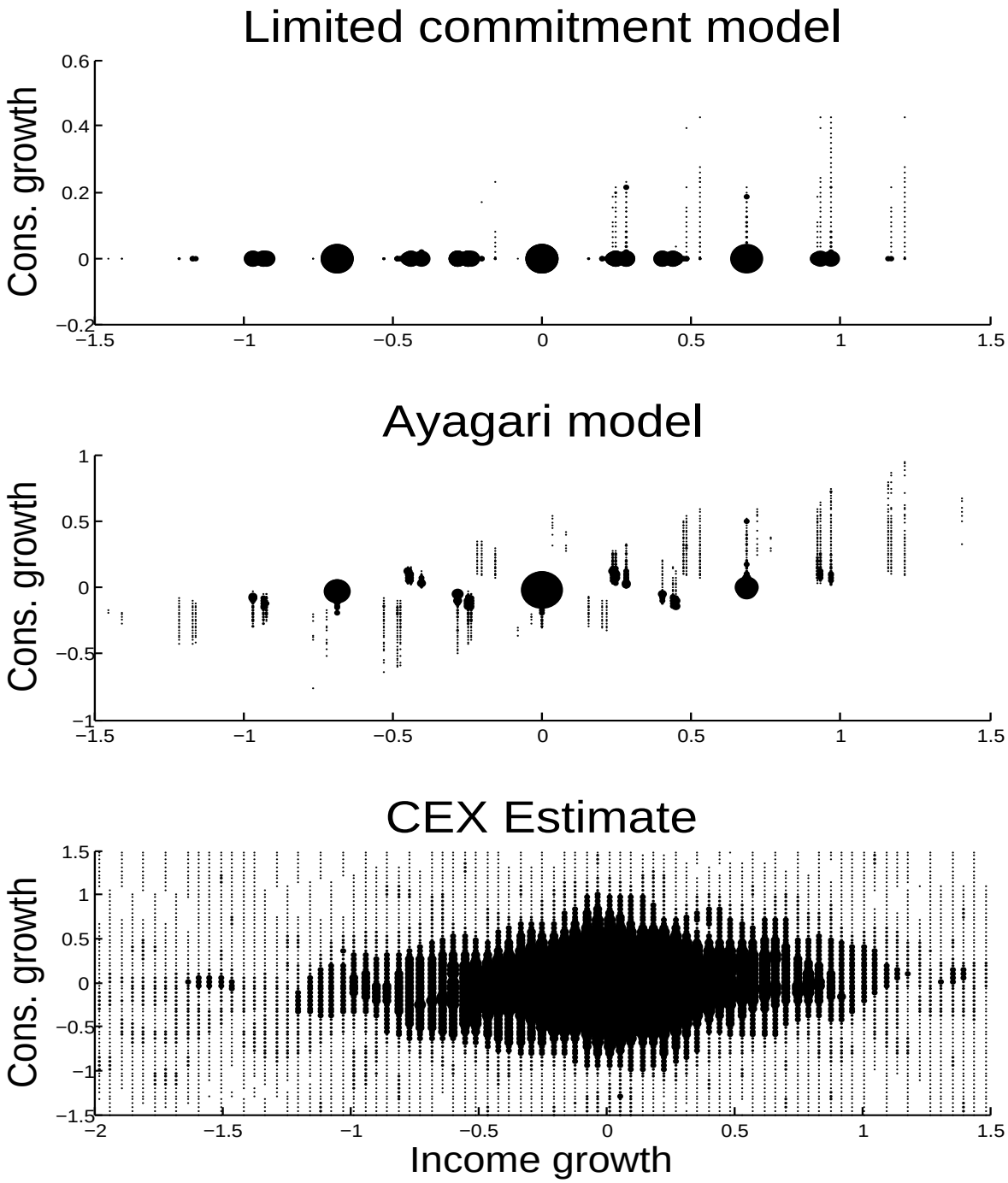
The figure shows the marginal distribution of consumption in the Krueger and Perri (2006) calibration. Equal colours denote individuals that were last constrained at equal income values and are thus located on the same geometric subdistribution of consumption.

Figure 3: Joint distribution of consumption and income



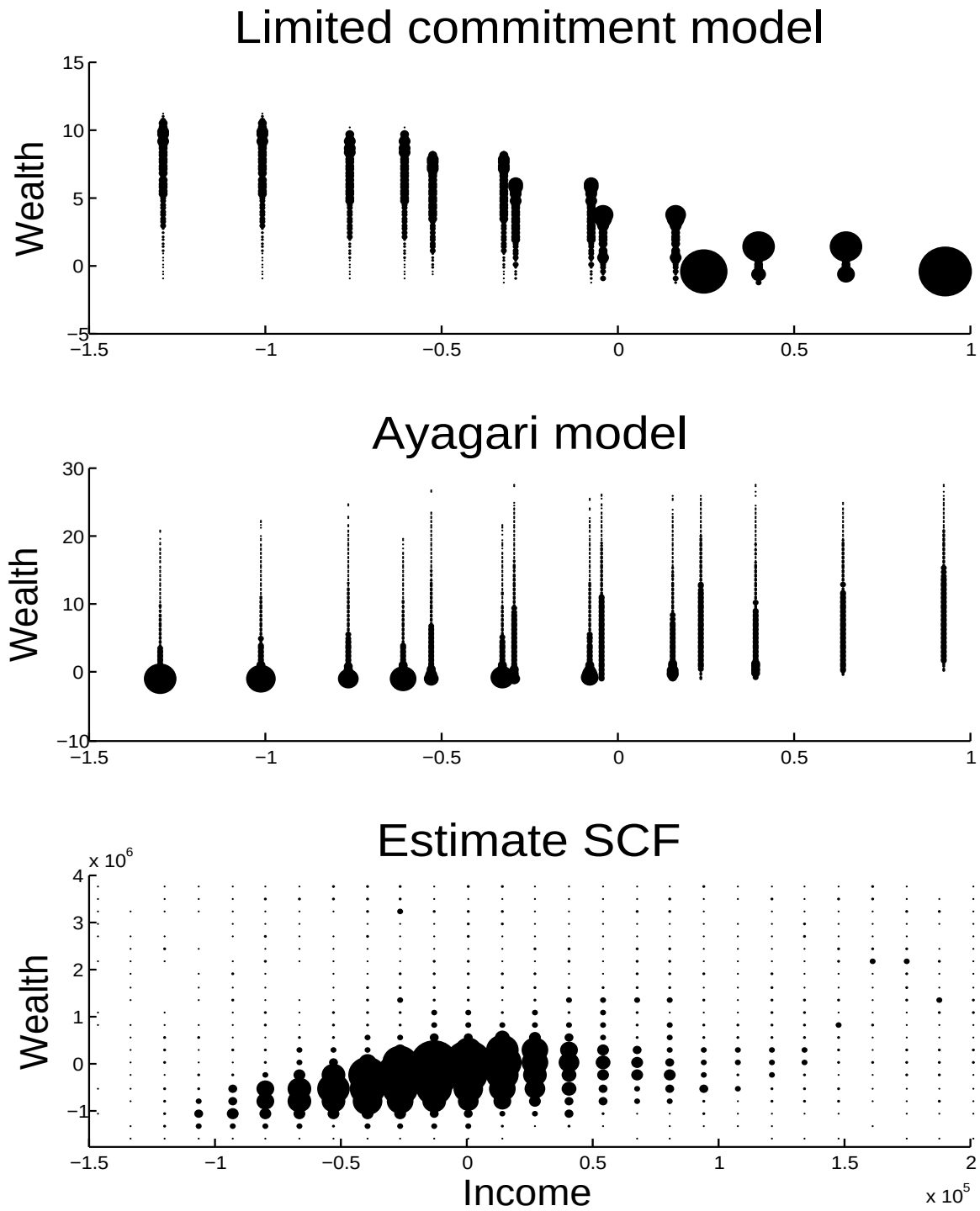
The figure shows the joint densities of consumption and income in the limited commitment economy, a simple Ayagari economy, and in CEX data. The size of dots is proportional to the frequency mass at that point. The kernel density estimate of the empirical distribution uses an optimal bandwith (Botev et al 2008), and is based on residuals from a first-stage regression of the variables on observable individual characteristics as described in the main text.

Figure 4: Joint distribution of consumption and income changes



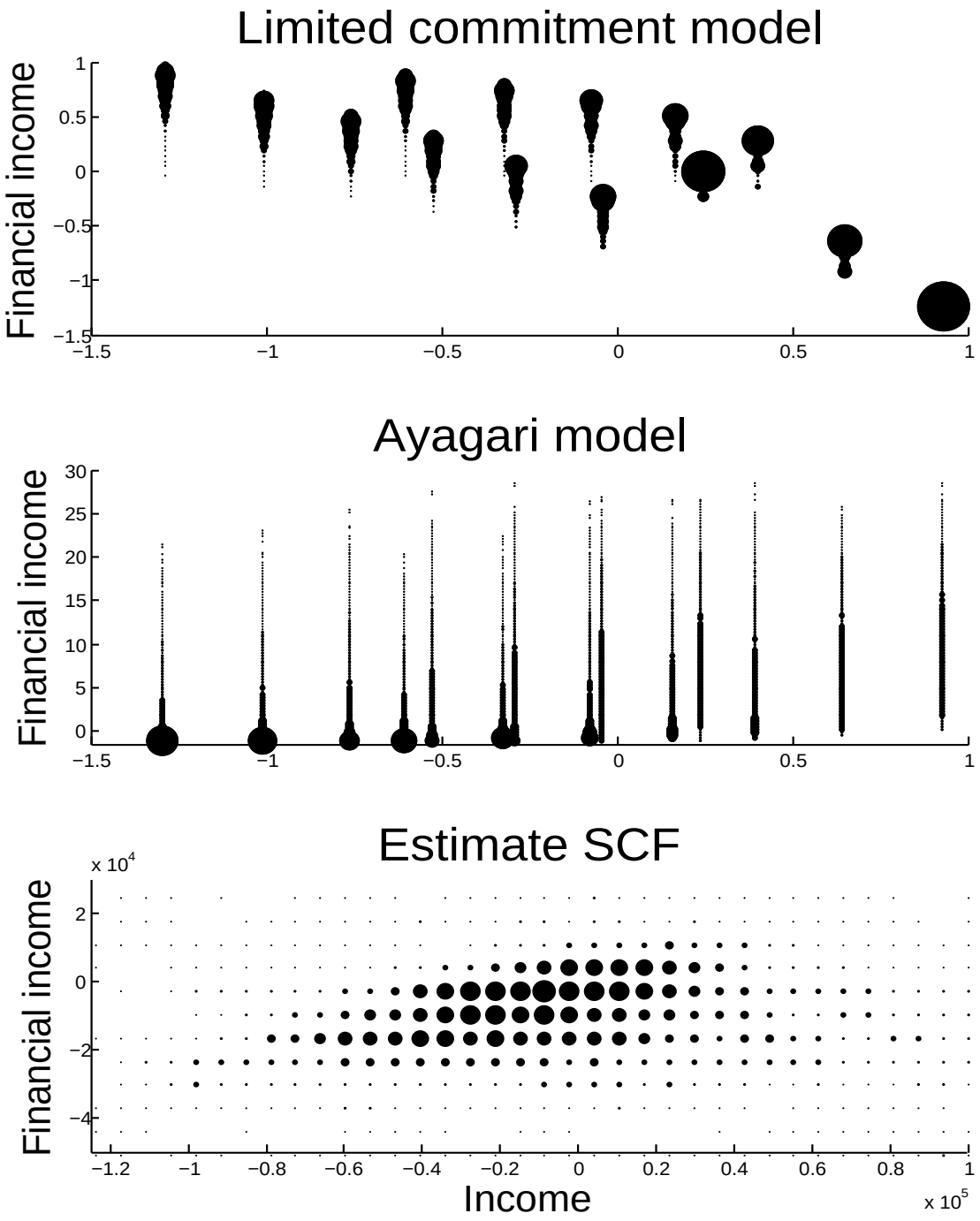
The figure shows the joint densities of consumption and income growth in the limited commitment economy, a simple Ayagari economy, and in CEX data. The size of dots is proportional to the frequency mass at that point. The kernel density estimate of the empirical distribution uses an optimal bandwith (Botev et al 2008), and is based on differences in the raw data.

Figure 5: Joint distribution of wealth and income



The figure shows the joint densities of wealth and income in the limited commitment economy, a simple Ayagari economy, and in SCF data. The size of dots is proportional to the frequency mass at that point. The kernel density estimate of the empirical distribution uses an optimal bandwith (Botev et al 2008), and is based on residuals from a first-stage regression of the variables on observable individual characteristics as described in the main text.

Figure 6: Joint distribution of net financial returns and income



The figure shows the joint densities of financial returns and income in the limited commitment economy, a simple Ayagari economy, and in SCF data. The size of dots is proportional to the frequency mass at that point. The kernel density estimate of the empirical distribution uses an optimal bandwith (Botev et al 2008), and is based on residuals from a first-stage regression of the variables on observable individual characteristics as described in the main text.