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EQUILIBRIUM CORPORATE FINANCE

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Equilibrium corporate finance *

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Abstract

We study a general equilibrium model with production where financial markets are incomplete. At a competitive equilibrium firms take their production and financial decisions so as to maximize their value. We show that shareholders unanimously support value maximization. Furthermore, competitive equilibria are constrained Pareto efficient. Finally the Modigliani-Miller theorem typically does not hold and the firms' corporate financing structure is determined at equilibrium. Such results extend to the case where informational asymmetries are present and contribute to determine the firms' capital structure.

Keywords: capital structure, competitive equilibria, incomplete markets, asymmetric information

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1 Introduction

We study a general equilibrium economy with incomplete markets, production and non-trivial corporate financing decisions. Corporate financing decisions are non-trivial because constraints in financial markets, e.g., borrowing constraints on the part of the agents, incomplete financial markets, asymmetric information between corporate investors and managers or between bond holders and equity holders, imply that the Modigliani-Miller (MM) theorem does not hold and production and financing decisions of firms cannot be separated.

In this class of economies, indeed because production and financing decisions of firms cannot be separated, corporate finance quantities like the capital structure and inside ownership levels depend on aggregate shocks as well as on idiosyncratic shocks. Also, corporate finance quantities are determined jointly with production decisions and cash flows, therefore affecting asset prices.

Various foundational issues, in particular regarding the specification of a proper objective function of the firm when markets are incomplete, have arguably hindered the study of the macroeconomic properties of these economies as well as the development of the integrated study of corporate finance with macroeconomics and asset pricing theory.

In this paper we hence concentrate on the foundational theoretical properties of these economies. To this end we restrict the analysis to a simple two-period economy along the lines of classical General Equilibrium models with Incomplete Markets (GEI). Bisin et al. (2009) extend the analysis to Bewley economies with production, the main workhorse of heterogeneous agents macroeconomics.[1]

We consider first the case where firms' equity cannot be sold short and show that *i) value maximization is unanimously supported by shareholders* as the firm's objective and *ii) competitive equilibria are constrained efficient*. Furthermore we show that *iii) the capital structure of firms at equilibrium is determinate* in a precise and specific sense. In particular, it typically varies with aggregate states, over the business cycle.

Our analysis and results extend to the case where firms can default on the debt they issue, as well as to the case where agents are allowed to sell stocks short. In the final sections of the paper we also introduce informational asymmetries between the decision maker in the firm (e.g., the manager) and equity holders or bondholders. This class of informational asymmetries provide the backbone of models of the capital structure and of

[1] See Heathcote et al. (2009) for a recent survey of Bewley models.

incentive compensation in corporate finance models. It is important therefore to extend our analysis to these models if we intend it as a foundation for equilibrium corporate finance. We show that unanimity of value maximization continues to obtain for both economies with moral hazard and with adverse selection. Constrained efficiency instead is preserved with moral hazard but not with adverse selection.[2] Also, the Modigliani-Miller theorem does not hold in general in the presence of asymmetric information as incentive issues further contribute to determine the firms' capital structure.

We first introduce the economy with riskless debt and no short sales and the definition of equilibrium, in Section 2. In this section, after showing that equilibria always exist, we also discuss and compare the equilibrium notion considered with the alternative ones adopted in the previous literature. In Section 3 we present our main results on unanimity, efficiency and firms' capital structure. In Section 4 we extend the analysis to account for risky debt and short sales. Finally in Section 5 we study economies with asymmetric information.

2 The economy

The economy lasts two periods, $t = 0, 1$ and at each date a single consumption good is available. The uncertainty is described by the fact that at $t = 1$ one state s out of the set $\mathcal{S} = \{1, ..., S\}$ realizes. We assume for simplicity that there is a single type of firm in the economy which produces the good at date 1 using as only input the amount k of the commodity invested in capital at time 0.[3] The output depends on k as well as another technology choice ϕ, affecting the stochastic structure of the output at date 1,[4] according to the function $f(k, \phi; s)$, defined for $k \in K, \phi \in \Phi$, and $s \in \mathcal{S}$. We assume that $f(k, \phi; s)$ is continuously differentiable, increasing in k and concave in k, ϕ; moreover, Φ, K are closed,

[2]Bisin and Gottardi (2006) identify an externality in pure exchange insurance economies with adverse selection which precludes constrained efficiency. In the adverse selection economies studied in this paper an analogous externality appears in production.

[3]It should be clear from the analysis which follows that our results hold unaltered if the firms' technology were described, more generally, by a production possibility set $Y \subset \mathbb{R}^{S+1}$.

[4]The parameter ϕ may describe, for instance, the loading on different factors affecting the firm's output. To illustrate this, consider the following instance of production function
$f(k, \phi; s) = [a(s) + \phi\epsilon(s)]\, k^{\alpha}$ where $\phi \in \{0, 1\}$ is the loading of the firm's cash-flow on the risk component given by $\epsilon(s)$. See also the example in Section 3.3.1.

compact[5] subsets of $\mathbb{R}_+$ and $0 \in K$.

In addition to firms, there are I types of consumers. Consumer $i = 1, .., I$ has an endowment of w_0^i units of the good at date 0 and $w^i(s)$ units at date 1 in each state $s \in \mathcal{S}$, thus the agent's endowment is also subject to the shock affecting the economy at $t = 1$. He is also endowed with θ_0^i units of stock of the representative firm. Consumer i has preferences over consumption in the two dates, represented by $\mathbb{E}u^i\left(c_0^i, c^i(s)\right)$, where $u^i\left(\cdot\right)$ is also continuously differentiable, increasing and concave.

There is a continuum of firms, of unit mass, as well as a continuum of consumers of each type i, which for simplicity is also set to have unit mass.

2.1 Competitive equilibrium

We examine the case where firms take both production and financial decisions, and their equity and debt are the only assets in the economy. Let the outstanding amount of equity be normalized to 1 (the initial distribution of equity among consumers satisfies $\sum_i \theta_0^i = 1$) and assume this is kept constant. Hence the choice of a firm's capital structure is only given by the decision concerning the amount B of bonds issued, which in turn also equals the firm's debt/equity ratio. The problem of the firm consists in the choice of its production plan k, ϕ and its financial structure B. To begin with, we assume all firms' debt is risk free.[6]

Firms are perfectly competitive and hence take prices as given. The notion of price taking behavior has no ambiguity when referred to the bond price p. For equity, however, the situation is more complex, since a firm's cash flow, and hence the return on equity, is $[f(k, \phi; s) - B]$ and varies with the firm's production and financing choices, k, ϕ, B. Thus equity is a different "product" for different choices of the firm. What should be its price when all this continuum of different "products" are not actually traded in the market? In this case the price is only a "conjecture", entertained by firms, as pointed out by Grossman and Hart (1979). This can be described by a map $q(k, \phi, B)$ specifying the market valuation of the firm's cash flow for any possible choice k, ϕ, B.[7]

When financial markets are complete, the present discounted valuation of any future

[5]The condition that the set of admissible values of k is bounded above is by no means essential and is only introduced for simplicity.

[6]We shall allow for the possibility that firms' default on their debt in Section 5.1.

[7]These price maps are also referred to as *price perceptions* (see Grossman and Hart (1979), Kihlstrom and Matthews (1990) and Magill and Quinzii (1998)).

payoff is uniquely determined by the price of the existing assets. This is no longer true when markets are incomplete, in which case the prices of the existing assets do not allow to determine unambiguously the value of any future cash flow. The specification of the price conjecture is thus more problematic in such case (see also the discussion in the next section). Still, firms operate on the basis of a given price conjecture $q(k, \phi, B)$ and choose their production and financing plans k, ϕ, B so as to maximize their value, as determined by such pricing map and the bond price[8]. The firm's problem is then:

$$V = \max_{k,\phi,B} -k + q(k, \phi, B) + pB \tag{1}$$

subject to the solvency constraint (ensuring that the bonds issued are risk free):

$$f(k, \phi; s) \geq B, \ \forall s \in \mathcal{S} \tag{2}$$

Let $\bar{k}, \bar{\phi}, \bar{B}$ denote the solutions to this problem.

At $t = 0$, each consumer i chooses his portfolio of equity and bonds, θ^i and b^i respectively, so as to maximize his utility, taking as given the price of bonds p and the price of equity q. In the present environment a consumer's long position in equity identifies a firm's equity holder, who may have a voice in the firm's decisions. It should then be treated as conceptually different from a short position in equity, which is not simply a negative holding of equity. To begin with, we rule out altogether the possibility of short sales and assume that agents cannot short-sell the firm equity nor its debt:

$$b^i \geq 0, \ \theta^i \geq 0, \ \forall i \tag{3}$$

The problem of agent i is then:

$$\max_{\theta^i, b^i, c^i} \mathbb{E} u^i(c_0^i, c^i(s)) \tag{4}$$

subject to (3) and

$$c_0^i \ = w_0^i \ + [-k \ + q \ + p \ B \ ] \, \theta_0^i - q \ \theta^i \ - p \ b^i \tag{5}$$

$$c^i(s) = w^i(s) + [f(k, \phi; s) - B \ ] \, \theta^i \ + b^i, \ \forall s \in \mathcal{S} \tag{6}$$

Let $\bar{\theta}^i, \bar{b}^i, \bar{c}_0^i, (\bar{c}^i(s))_{s \in \mathcal{S}}$ denote the solutions of this problem. In equilibrium, the following

[8] We will later show that such decision is unanimously supported by the firm's shareholders.

market clearing conditions for the assets must hold:[9]

$$\begin{array}{l} \sum_i b^i \ \leq B \\ \sum_i \theta^i \ \leq 1 \end{array} \tag{7}$$

In addition, the equity price map faced by firms must satisfy the following consistency condition:

Ci) $q(\bar{k}, \bar{\phi}, \bar{B}) = q$;

This condition requires that, in equilibrium, the price of equity conjectured by firms coincides with the price of equity, faced by consumers in the market: firms' conjectures are "correct" in equilibrium. We also restrict out of equilibrium conjectures by firms, requiring they satisfy:

Cii) $q(k, \phi, B) = \max_i \mathbb{E}\left[\overline{MRS}^i(s)(f(k, \phi; s) - B)\right]$, $\forall k, \phi, B$, where $\overline{MRS}^i(s)$ denotes the marginal rate of substitution between consumption at date 0 and at date 1 in state s for consumer i, evaluated at his equilibrium consumption level $\bar{c}^i$.

Condition Cii) says that for any k, ϕ, B the value of the equity price map $q(k, \phi, B)$ equals the highest marginal valuation - across all consumers in the economy - of the cash flow associated to k, ϕ, B. The consumers' marginal rates of substitutions $\overline{MRS}^i(s)$ used to determine the market valuation of the future cash flow of a firm are taken as given, unaffected by the firm's choice of k, ϕ, B. This is the sense in which, in our economy, firms are competitive: each firm is "small" relative to the mass of consumers and each consumers holds a negligible amount of shares of the firm.

To better understand the meaning of condition Cii), note that the consumers with the highest marginal valuation for the firm's cash flow when the firm chooses k, ϕ, B are those willing to pay the most for the firm's equity in that case and the only ones willing to buy equity - at the margin - at the price given by Cii). Under condition Ci), as we show in (8) below, such property is clearly satisfied for the firms' equilibrium choice $\bar{k}, \bar{\phi}, \bar{B}$. Condition Cii) requires that the same is true for any other possible choice k, ϕ, B: the value attributed to equity equals the maximum any consumer is willing to pay for it. Note that this would be the equilibrium price of equity of a firm who were to "deviate" from the equilibrium

[9] We state here the conditions for the case of symmetric equilibria, where all firms take the same production and financing decision, so that only one type of equity is available for trade to consumers. They can however be easily extended to the case of asymmetric equilibria as, for instance, the one considered in the example of Section 3.2.1.

choice $\bar{k}, \bar{\phi}, \bar{B}$ and choose k, ϕ, B instead: the supply of equity with cash flow corresponding to k, ϕ, B is negligible and, at such price, so is its demand. In this sense, we can say that condition Cii) imposes a consistency condition on the out of equilibrium values of the equity price map; that is, it corresponds to a "refinement" of the equilibrium map, somewhat analogous to backward induction. Summarizing,

Definition 1 *A competitive equilibrium of the economy is a collection* $\left(\bar{k}, \bar{\phi}, \bar{B}, \{\bar{c}^i, \bar{\theta}^i, \bar{b}^i\}_i, \bar{p}, \bar{q}, q(\cdot)\right)$ *such that: i)* $\bar{k}, \bar{\phi}, \bar{B}$ *solve the firm problem (1) s.t. (2) given* $\bar{p}, q(\cdot)$*; ii) for all* i*,* $\bar{c}^i, \bar{\theta}^i, \bar{b}^i$ *solve consumer* i*'s problem (4) s.t. (3), (5) and (6) for given* $\bar{p}, \bar{q}$*; iii) markets clear, (7) holds; iv) the equity price map* $q(\cdot)$ *is consistent, that is satisfies the consistency conditions Ci) and Cii).*

It readily follows from the consumers' first order conditions that in equilibrium the price of equity and the bond satisfy:

$$\begin{aligned} \bar{q} &= \max_i \mathbb{E}\left[\overline{MRS}^i(s)(f(\bar{k}, \bar{\phi}; s) - \bar{B})\right] \\ \bar{p} &= \max_i \mathbb{E}\left[\overline{MRS}^i(s)\right] \end{aligned} \tag{8}$$

as stated in consistency condition Ci).

Remark 1 *It is also of interest to point out that, when the price conjectures satisfy condition Cii), the model is equivalent to one where markets for all possible 'types' of equity are open (that is, equity corresponding to any possible value of* k, ϕ, B *is available for trade to consumers) and, in equilibrium all such markets - except the one corresponding to* $\bar{k}, \bar{\phi}, \bar{B}$ *- clear at zero trade.*[10]

To see this, suppose that consumers can trade any claim with payoff $[f(k, \phi; s) - B\ ]$*, at the price* $q(k, \phi, B)$*, for all* $(k, \phi) \in \Phi \times K$ *and* B *satisfying (2). The expressions of the budget constraints for type* i *consumers in (5) and (6) have then to be modified as follows:*

$$\begin{aligned} c_0^i &= w_0^i + \left[-\bar{k} + \bar{q} + p\ \bar{B}\ \right]\theta_0^i - \int_{\Phi \times K} \int_{\min_s f(k,\phi;s) \geq B} q(k, \phi, B)\ d\theta^i(k, \phi, B) - p\ b^i \\ c^i(s) &= w^i(s) + \int_{\Phi \times K} \int_{\min_s f(k,\phi;s) \geq B} [f(k, \phi; s) - B\ ]\, d\theta^i(k, \phi, B) + b^i,\ \forall s \in \mathcal{S} \end{aligned} \tag{9}$$

[10] An analogous specification of the price conjecture has been earlier considered by Makowski (1980), Makowski (1983), Makowski and Ostroy (1987) in a competitive equilibrium model with differentiated products, and by Allen and Gale (1991) and Pesendorfer (1995) in models of financial innovation. See also Geanakoplos (2004).

Similarly, to the market clearing conditions in (7) we should add: $\sum_i \theta^i\ (k,\phi,B) \leq 0$ *for all* $(k,\phi,B) \neq (\bar{k},\bar{\phi},\bar{B})$. *It is immediate to verify that, when condition Cii) holds, if* $\bar{c}^i, \bar{\theta}^i, \bar{b}^i$ *solves consumer* i*'s problem (4) subject to (3), (5) and (6), a solution to the problem of maximizing* i*'s utility subject to (9) obtains again at* $\bar{c}^i, \bar{b}^i$ *and* $\theta^i(\bar{k},\bar{\phi},\bar{B}) = \bar{\theta}^i,\ \theta^i(k,\phi,B) = 0$ *for all other* $(k,\phi,B) \neq (\bar{k},\bar{\phi},\bar{B})$. *This follows from the fact that the utility of all consumers is continuously differentiable and concave in the holdings of any type of equity and, when* $q(k,\phi,B)$ *satisfies condition Cii), their marginal utility of a trade in equity of any type* $(k,\phi,B) \neq (\bar{k},\bar{\phi},\bar{B})$, *evaluated at zero trade, is less or equal than its price.*
Hence the equilibrium allocation is unchanged if consumers are allowed to trade all possible types of equity at these prices. Note that this argument crucially relies on the no short sale condition; see also Hart (1979).

Definition 1 of a competitive equilibrium is stated for simplicity for the case of symmetric equilibria, where all firms choose the same production plan. When the equity price map satisfies the consistency conditions Ci) and Cii) the firms' choice problem is not convex. Asymmetric equilibria might therefore exist, in which different firms choose different production plans. The proof of existence of equilibria indeed requires that we allow for such asymmetric equilibria, so as to exploit the presence of a continuum of firms of the same type to convexify the firms' choice problem. A standard argument allows then to show that firms' aggregate supply is convex valued and hence that the existence of (possibly asymmetric) competitive equilibria holds. We relegate a sketch of the proof in the Appendix.

Proposition 1 *A competitive equilibrium always exist.*

2.2 Objective function of the firm

Starting with the initial contributions of Dreze (1974), Grossman and Hart (1979) and Duffie and Shafer (1986), a large literature has dealt with the question of what is the appropriate objective function of the firm when markets are incomplete.[11] The issue arises because, as mentioned above, firms' production decisions may affect the set of insurance possibilities available to consumers by trading in the asset markets.[12]

[11] See, e.g., Bonnisseau and Lachiri (2004), DeMarzo (1993), Dierker et al. (2002), Dreze et al. (2007), Kelsey and Milne (1996) and many others.

[12] It is only in rather special environments, as pointed out by Diamond (1967) (see also the more recent contribution by Carceles Poveda and Pirani (2009)), that the spanning condition holds and such issue does

If agents are allowed infinite short sales of the equity of firms, as in the standard GEI model, a *small* firm will possibly have a *large* effect on the economy by choosing a production plan with cash flows which, when traded as equity, change the asset span. It is clear that the price taking assumption appears hard to justify in such context, since changes in the firm's production plan have non-negligible effects on allocations and hence equilibrium prices. The GEI literature has struggled with this issue, trying sometimes to maintain a competitive equilibrium notion in an economic environment in which firms are potentially *large*.

In the environment considered in this paper, this problem does not arise since consumers face a constraint preventing short sales, (3). This guarantees that each firm's production plan has a negligible (infinitesimal) effect on the set of admissible trades and allocations available to consumers. As argued by Hart (1979) and Allen and Gale (1988), price taking behavior is justified in this case, when the number of firms is large. Evidently, for price taking behavior to be justified a no short sale constraint is more restrictive than necessary and a bound on short sales of equity would suffice. We will explore how to allow for short sales in Section 4.

When short sales are not allowed, while the decisions of a firm have a negligible effect on equilibrium allocations and market prices, still each firm's decision has a non-negligible impact on its present and future cash flows. Price taking cannot therefore mean that the price of its equity is taken as given by a firm, independently of its decisions. However, as argued in the previous section, the level of the equity price associated to out-of-equilibrium values of k, ϕ, B is not observed in the market. It is rather *conjectured* by the firm. In a competitive environment we require such conjecture to be consistent, as required by condition Cii) in the previous section. This notion of *consistency* of conjectures implicitly requires they are *competitive*, that is, determined by a given pricing kernel, independent of the firm's decisions.[13]

But which pricing kernel? Here lies the core of the problem with the definition of the objective function of the firm when markets are incomplete. When markets are incomplete, in fact, the marginal valuation of *out-of-equilibrium* production plans differs across agents of different types in equilibrium. In other words, equity holders may not be unanimous with respect to their preferred production plan for the firm. In addition, the set of the

not arise.

[13]Independence of the pricing kernel in our set-up is guaranteed by the fact that $\overline{MRS}^i(s)$ is evaluated at the equilibrium consumption level of type i, for each i.

firm's shareholders is itself endogenously determined in equilibrium. The problem with the definition of the objective function of the firm when markets are incomplete is therefore the problem of aggregating the marginal valuations for *out-of-equilibrium* production plans of the firm's (actual and potential) equity holders. The different equilibrium notions we find in the literature differ primarily in the specification of a consistency condition on $q(k, \phi, B)$, the price map which the firms adopt to aggregate agents' marginal valuations.

A minimal consistency condition on $q(k, \phi, B)$ is clearly given by condition i) in the previous section, which only requires the conjecture to be correct in correspondence to the firm's equilibrium choice. Duffie and Shafer (1986) indeed only impose such condition and considers as admissible any pricing kernel which satisfies it and induces prices with no arbitrage opportunities, that is lies in the same space where agents' marginal rates of substitution lie. They find then a rather large indeterminacy of the set of competitive equilibria.

Consider then the consistency condition proposed by Dreze (1974) in an important early contribution to this literature. Stated in our environment, his condition is:

$$q(k, \phi, B) = \mathbb{E} \sum_i \bar{\theta}^i \overline{MRS}^i(s) \left[f(k, \phi; s) - B \right], \ \forall k, B \tag{10}$$

It requires the price conjecture for any plan k, ϕ, B to equal - pro rata - the marginal valuation of the agents who in equilibrium are equity holders of the firm (that is, the agents who value the most the plan chosen by the firm in equilibrium and hence choose to buy equity). It does not however require that the firm's equity holders are those who value the most any possible plan of the firm. Intuitively, the choice of a plan which maximizes the firm's value with $q(k, \phi, B)$ as in (10) corresponds to a situation in which the firm's equity holders choose the plan which is optimal for them[14] without contemplating the possibility of selling the firm in the market, to allow the buyers of equity to operate the production plan they prefer. Equivalently, the value of equity for out of equilibrium production plans is determined using the - possibly incorrect - conjecture that the firms' equilibrium shareholders will still own the firm if it changes its production plan.

It is useful to compare our notion of equilibrium with that of Dreze (1974). Our consistency condition Cii) requires that each plan is evaluated according to the marginal valuation of the agent who values it the most. It is then easy to see that any allocation constituting an equilibrium according to Definition 1 is also a Dreze equilibrium: all shareholders have in

[14]It is in fact immediate to verify that the plan which maximizes the firm's value with $q(k, \phi, B)$ as in (10) is also the plan which maximizes the welfare of the given set of shareholders of the firm.

fact the same valuation for the firm's production plan and their marginal utility for any other possible plan is lower, hence a fortiori the chosen plan maximizes the weighted average of the shareholders' valuations. But the reverse implication is not true, i.e., a Dreze equilibrium is not in general an equilibrium according to our definition.

Grossman and Hart (1979) propose another consistency condition and hence a different equilibrium notion in a related environment. This condition (again, restated in our set-up) is:

$$q(k,\phi,B) = \mathbb{E}\sum_i \theta_0^i \overline{MRS}^i(s)\left[f(k,\phi;s) - B\right],\ \forall k, B$$

We can interpret such notion as describing a situation where the firm's plan is chosen by the initial equity holders (i.e., those with some predetermined stock holdings at the beginning of date 0) so as to maximize their welfare, again without contemplating the possibility of selling the equity to other consumers who value it more. Equivalently, the value of equity for out of equilibrium production plans is derived using the conjecture belief that the firm's initial shareholders stay in control of the firm also out of equilibrium.

To summarize, in our equilibrium notion the firm evaluates different production plans using possibly different marginal valuations (that is, possibly different pricing kernels, but all still consistent with the consumers' marginal rate of substitution at the equilibrium allocation). This is not the case of Dreze (1974) nor of Grossman and Hart (1979). This is a fundamental distinguishing feature of our equilibrium notion with respect to the many others proposed in the GEI literature, including those which have applied theoretical constructs from the theory of social choice and voting to model the control of equity holders over the firm's decisions; see for instance DeMarzo (1993), Boyarchenko (2004), Cres and Tvede (2005).

But *the proof is in the pudding*. Our equilibrium notion, besides being logically consistent as no *small* firm has *large* effects, also has some desirable properties: i) it delivers a Unanimity result and ii) it produces equilibria which satisfy a constrained version of the First Welfare Theorem.

3 Unanimity, efficiency, and Modigliani-Miller

We turn to state and prove our main results for the simplest benchmark economy just introduced, with riskless debt and no short sales.

3.1 Unanimity

In our setup equity holders unanimously support their firm's choice of the production and financial decisions which maximize its value (or profits), as in (1). This follows from the fact that, when the equity price map satisfies the consistency conditions Ci) and Cii), as we already noticed in Remark 1, the model is equivalent to one where a continuum of types of equity is available for trade to consumers, corresponding to any possible choice of k, ϕ, B the representative firm can make, at the price $q(k, \phi, B)$.[15]

Unanimity then holds by the same argument as the one used to establish this property for Arrow-Debreu economies. More formally, notice that we can always consider a situation where, in equilibrium, each consumer holds at most a negligible fraction of each firm. The effect on consumers' utility of alternative choices by a firm can then be evaluated using the agents' marginal utility. For any possible choice k, ϕ, B of a firm, the (marginal) utility of a type i agent if he holds the firm's equity, $\mathbb{E}\left[\overline{MRS}^i(s)\left(f(k,\phi;s)-B\right)\right]$, is always less or at most equal to his utility if he sells the firm's equity at the market price, given by $\max_i \mathbb{E}\left[\overline{MRS}^i(s)\left(f(k,\phi;s)-B\right)\right]$. Hence the firm's choice which maximizes the latter also maximizes the equity holder's utility.

Proposition 2 *At a competitive equilibrium, equity holders unanimously support the production and financial decisions $\bar{k}, \bar{\phi}, \bar{B}$ of the firms; that is, every agent i holding a positive initial amount θ_0^i of equity of the representative firm will be made - weakly - worse off by any other choice k', ϕ', B' of the firm.*

3.2 Efficiency

We show next that all competitive equilibria of the economy described exhibit desirable welfare properties. Evidently, since the hedging possibilities available to consumers are limited by the presence of the equity of firms and risk free bonds as the only assets, we cannot expect competitive equilibrium allocations to be fully Pareto efficient, but only to make the best possible use of the existing markets, that is to be constrained Pareto efficient in the sense of Diamond (1967).

[15] As already argued in Remark 1, this property depends on the fact that consumers face a no short sale condition. In Section 4, we will show that the unanimity, as well as the constrained efficiency, results extend to the case where limited short sales are allowed, provided an appropriate specification of the markets for selling short assets is considered.

To this end, we say a consumption allocation $(c^i)_{i=1}^2$ is *admissible* if:[16]

1. it is *feasible*: there exists a production plan k, ϕ of firms such that

$$\begin{aligned} \sum_i c_0^i \ + k &\leq \sum_i w_0^i \\ \sum_i c^i(s) &\leq \sum_i w^i(s) + f(k, \phi; s), \ \forall s \in \mathcal{S} \end{aligned} \tag{11}$$

2. it is *attainable with the existing asset structure*: there exists B and, for each consumer's type i, a pair θ^i, b^i such that:

$$c^i(s) = w^i(s) + [f(k, \phi; s) - B\ ]\,\theta^i \ + b^i, \ \forall s \in \mathcal{S} \tag{12}$$

Next we present the notion of *efficiency* restricted by the *admissibility* constraints:

Definition 2 *A competitive equilibrium allocation is constrained Pareto efficient if we cannot find another admissible allocation which is Pareto improving.*

The validity of the First Welfare Theorem with respect to such notion can then be established by an argument essentially analogous to the one used to establish the Pareto efficiency of competitive equilibria in Arrow-Debreu economies.[17]

Proposition 3 *Competitive equilibria are constrained Pareto efficient.*

3.2.1 Efficiency and asymmetric equilibria

Dierker et al. (2002) present an economy with the property that all Dreze equilibria are constrained inefficient. This appears to contradict the results in this paper. According to our equilibrium notion, in fact, all equilibria are constrained efficient, an equilibrium exist and any equilibrium is also a Dreze equilibrium. The apparent contradiction is due, however, to Dierker et al. (2002)'s restriction to symmetric equilibria. We will show that, in their economy, a unique competitive equilibrium exists which is asymmetric and constrained efficient. This equilibrium only is selected by our definition.

[16] To keep the notation simple we state here the definition of admissible allocations for symmetric allocations, as we did for competitive equilibria. Our analysis and the efficiency result hold however in the more general case where asymmetric allocations are allowed; see also the next section.

[17] See also Allen and Gale (1988) for a constrained efficiency result in a related environment.

Let $\mathcal{S} = \{s', s''\}$. There are two types of consumers, with type 2 having twice the mass of type 1, and (non VNM) preferences, respectively, $u^1(c_0^1, c^1(s'), c^1(s'')) = c^1(s')/\left(1 - (c_0^1)^{\frac{9}{10}}\right)^{\frac{10}{9}}$ and $u^2(c_0^2, c^2(s'), c^2(s'')) = c_0^2 + (c^2(s''))^{1/2}$, endowments $w_0^1 = .95$, $w_0^2 = 1$ and $w^1(s) = w^2(s) = 0$ for all $s \in \mathcal{S}$. The technology of the representative firm is described by $f(k, \phi; s) = \phi k$ for $s = s'$ and $(1 - \phi)k$ for $s = s''$, where $\phi \in \Phi = [2/3, 0.99]$. We abstract from the firms' financial decisions and set $B = 0$. The problem faced by firms in this environment is then $\max_{\phi,k} -k + q(k, \phi)$, where $q(k, \phi) = \max\left\{\frac{\partial u^1/\partial c^1(s')}{\partial u^1/\partial c_0^1}\phi k; \frac{\partial u^2/\partial c^2(s'')}{\partial u^2/\partial c_0^2}(1 - \phi)k\right\}$.

In this economy, Dierker et al. (2002) find a unique Dreze equilibrium where all firms choose a production plan with $\phi \approx 0.7$.[18]

According to our equilibrium concept, however, a symmetric equilibrium, where all firms choose the same value of k and ϕ, does not exist. Given the agents' endowments and preferences, both types of consumers buy equity in equilibrium. It is then easy to see that the firms' optimality condition with respect to ϕ can never hold for an interior value of ϕ nor for a corner solution.[19] On the other hand, an asymmetric equilibrium exists, where a fraction 1/3 of the firms choose $\phi^1 = 0.99$ and $k^1 = 0.3513$ and the remaining fraction chooses $\phi^2 = 2/3$ and $k^2 = 0.1667$, type 1 consumers hold only equity of the firms choosing ϕ^1, k^1 and type 2 consumers only equity of the other firms. At this allocation, we have $\frac{\partial u^1/\partial c^1(s')}{\partial u^1/\partial c_0^1} = 1.0101$, $\frac{\partial u^2/\partial c^2(s'')}{\partial u^2/\partial c_0^2} = 3$. Also, the marginal valuation of type 1 agents for the equity of firms choosing ϕ^2, k^2 is 0.1122, thus smaller than the market value of these firms' equity, equal to 0.1667, while the marginal valuation of type 2 agents for the equity of the firms choosing ϕ^1, k^1 is 0.0105, smaller than the market value of these firms' equity, equal to 0.3513. Therefore, at these values the firms' optimality conditions are satisfied. It can then be easily verified that this constitutes a competitive equilibrium according to our definition and that the equilibrium allocation is constrained efficient.

[18]The definition of Dreze equilibrium in Dierker et al. (2002) uses a specification of the firms' conjecture over their market value for out of equilibrium production plans that differs from the map $q(\phi, k)$ satisfying the consistency conditions imposed here in two important respects. The market value is computed i) by considering only the set of equilibrium shareholders rather than all consumers, and ii) by taking into account the effect of each plan on the marginal rate of substitution of shareholders rather than taking such rates as given.

[19]Consider for instance $\phi = 0.99$. To have an equilibrium at this value the marginal valuation of equity for both consumers must be the same at $\phi = 0.99$ and higher than at any other values of ϕ, but this second property clearly cannot hold for type 2 consumers.

3.3 Modigliani-Miller

In this section we study the properties of the firms' corporate finance and investment decisions at an equilibrium. To this end, it is convenient to introduce the notation I^e to denote the collection of all agents i such that $\bar{q} = \mathbb{E}\left[\overline{MRS}^i(s)\left(f(\bar{k},\bar{\phi};s) - \bar{B}\right)\right]$ that is, the collection of all agents that in equilibrium either hold equity or are indifferent between holding and not holding equity. We can similarly define the collection I^d of all agents i such that $\bar{p} = \mathbb{E}\overline{MRS}^i(s)$, that is, the collection of all agents that in equilibrium either hold bonds or are indifferent between holding and not holding bonds. With a slight abuse of language we denote the agents in I^e as equity holders and those in I^d bond holders.

The first order conditions are then different according to whether the no default constraint (2) binds or not. When it binds, we also need to take into account the possibility of joint changes in B, k and ϕ to identify the appropriate first order conditions. Letting $\underline{s}$ denote the lowest output state[20], we obtain the following characterization of the firms' optimality conditions:[21]

Proposition 4 *The optimal production and financing decisions of a firm are obtained:*
(i) either at an interior solution, $f(k,\phi;\underline{s}) > B$, where all equity holders are also bond holders (while the reverse may not be true: $I^e \subseteq I^d$):

$$\max_{i\in I^e} \mathbb{E}\overline{MRS}^i(s) = \min_{i\in I^e} \mathbb{E}\overline{MRS}^i(s) = p = \max_i \mathbb{E}\overline{MRS}^i(s) \tag{13}$$

and

$$\max_{i\in I^e} \mathbb{E}\left[\overline{MRS}^i(s) f_k(s)\right] = \min_{i\in I^e} \mathbb{E}\left[\overline{MRS}^i(s) f_k(s)\right] = 1; \tag{14}$$

(ii) or at a corner solution, $f(k,\phi;\underline{s}) = B$, where all equity holders have again the same marginal valuation for the bond, but such valuation may now be strictly less than its price p (hence no equity holder is a bond holder):

$$p \geq \max_{i\in I^e} \mathbb{E}\overline{MRS}^i(s) = \min_{i\in I^e} \mathbb{E}\overline{MRS}^i(s)\ , \tag{15}$$

$$1 \geq \max_{i\in I^e} \mathbb{E}\left[\overline{MRS}^i(s)\ f_k(s)\right] = \min_{i\in I^e} \left[\mathbb{E}\overline{MRS}^i(s)\ f_k(s)\right], \tag{16}$$

and

$$f_k(\underline{s})\left(p - \max_{i\in I^e} \mathbb{E}\overline{MRS}^i(s)\ \right) = 1 - \max_{i\in I^e} \mathbb{E}\left[\overline{MRS}^i(s)\ f_k(s)\right] \tag{17}$$

[20] This may clearly depend on k,ϕ but we omit to make it explicit for simplicity of the notation.

[21] We focus here on the conditions concerning the investment level k and capital structure B, ignoring those regarding ϕ, which are straighforward.

Thus in both cases all shareholders value equally the effect on the payoff of equity of an infinitesimal increase in the investment level k.[22] In addition, at an interior solution such value is always equal to the marginal cost of the investment. In contrast, at a corner solution this value may be strictly smaller. This happens whenever all equity holders value the bond less than p (that is, no equity holder is a bond holder), in which case the "gap" in the two expressions is exactly equal.

We can now study the implications of the above characterization of the firm's optimality conditions for the firm's optimal financing choice, described by B. Is such choice indeterminate? Equivalently, does the Modigliani-Miller irrelevance result hold in our setup? The answer clearly depends on whether the solution of the firm's problem obtains at a point where the no default constraint is slack or binds. We consider each of these two cases in turn.

When $f(k, \phi; \underline{s}) > B$ the value of the firm V is locally invariant with respect to any change in B. Furthermore, this invariance result extends to any admissible[23] change in B: all equity holders are in fact indifferent with respect to any admissible, discrete change ΔB, whether positive or negative. The other agents might not be indifferent, but the optimality of B, k, ϕ implies their valuation of the firm is always lower.

When the optimum obtains at a corner, $f(k, \phi; \underline{s}) = B$, either the same property still holds (V is invariant with respect to any admissible change in B), or V is strictly increasing in B. The latter property occurs when no equity holder is also a bond holder (in fact each shareholder would like to short the bond), in which case the firm's problem has a unique solution for B.

To sum up, except in the case in which no equity holder is also a bond holder, at a competitive equilibrium the value of the firm V is invariant with respect to any admissible change in B. It is important to note however that, while in such situation the capital structure is indeterminate for any individual firm, this does not mean that the capital structure of the economy, that is of all firms in the economy, is also indeterminate. In particular, the equilibrium is invariant only to changes in the aggregate stock of bonds in the economy ΔB such that all equity holders remain also bond holders and this imposes a lower bound on the aggregate value of ΔB consistent with the given equilibrium (given by $-\min_{i \in I^e} \bar{b}^i / \bar{\theta}^i$).

[22] The same is also true for the effect of an infinitesimal change in ϕ.

[23] An upper bound on the admissible levels of B is obviously given by the value at which the no default constraint binds, while the lower bound is 0.

We have thus established the following:

Proposition 5 *At a competitive equilibrium, the capital structure choice of each individual firm is indeterminate, except when the firm's no default constraint binds and no equity holder is also a bond holder (in which case there is a unique optimal level of B, at $f(\bar{k}, \bar{\phi}; \underline{s})$). On the other hand, the equilibrium capital structure of all firms in the economy is, at least partly, determinate: for any equilibrium value $\bar{B}$ only the values of the capital structure for all firms in the economy given by $\bar{B} + \Delta B$ such that $\Delta B \geq -\min_{i \in I^e} \bar{b}^i / \bar{\theta}^i$ are consistent with such equilibrium.*

Thus the Modigliani-Miller irrelevance result does not fully hold in equilibrium. The reason for this result is the presence of borrowing constraints, which restrict the set of equilibrium values of the capital structures to an interval.[24]

3.3.1 Capital structure and business cycles

It is useful to illustrate the properties of the equilibrium and the firms' production and financial decisions by considering a simple example, with two types of consumers, $I = 2$. Suppose both consumers have initial equity holdings $\theta_0 = .5$ and preferences described by $\mathbb{E}u^i(c_0^i, c^i(s)) = u(c_0^i) + \beta \mathbb{E}u(c^i(s))$, $i = 1, 2$; with $u = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma = 2$ and $\beta = .95$. The production technology exhibits two factors and multiplicative shocks affecting each of them: $f(k, \phi; s) = \phi a_1(s) k^\alpha + (1 - \phi) a_2(s) k^\alpha$, where $a_h(s)$ is the aggregate productivity shock affecting factor $h = 1, 2$ and $\phi \in \Phi = \{0, 1\}$ describes the choice of one of the two factors. We assume $\alpha = .75$. The structure of endowment and productivity shocks is reported in Table 1, for $\mathcal{S} = \{s_1, s_2, s_3\}$.

	s_1	s_2	s_3
w^1	2	3	3
w^2	3	5	8
a_1	2.0423	3.4286	4.9420
a_2	2.2464	3.4286	4.4930

Table 1: Example with risk free debt: stochastic structure.

[24] See Stiglitz (1969) for a first result along these lines.

We can think of s_1 as a recession state and s_3 as a boom. Consider a case in which at date 0 the state is the recession, i.e. $w_0^i = w^i(s_1)$ for all i, and $\pi(s_1) = .1$, $\pi(s_2) = .3$ and $\pi(s_3) = .6$, i.e., the persistence of the recession is relatively low.[25]

We find that in this case there is a unique equilibrium allocation where firms' factor loadings and investment are $\phi = 1$, $k = .20419$ while their capital structure is given by any level of B lying in the interval $[.30615, .62034]$.

In order to better illustrate the determinants of the firms' equilibrium capital structure, set $\phi = 1$ and treat parametrically the level of debt issued by each firm. For any given value B^{ex} of such debt we find the investment level k which maximizes firms' value, the individual consumption and portfolio holdings $\{c^i, \theta^i, b^i\}_{i=1}^2$ solving (4) and the prices $\{q, p\}$ such that markets clear and the consistency conditions for q hold. In Figure 1 we plot, as B^{ex} is varied from 0 to .62034, the values obtained for the consumers' asset holdings, on the first line, and their marginal valuations for the assets, on the second line. We can then use this figure to determine when we have an equilibrium, which happens when the optimality condition for the firms' financing decisions is satisfied. At $B^{ex} = 0$ the default constraint does not bind. From the top left panel we see that both consumers hold equity and from the lower right panel that consumer 2 has a higher marginal valuation for the bond than consumer 1. At $B^{ex} = 0$ any firm can so increase its value[26] by issuing debt, thus $B = 0$ is not an equilibrium value.

As B^{ex} is progressively increased from 0 to .30615, it remains true that consumer 2 has a higher marginal valuation for the bond. As for equity, the two consumers' valuation is still the same for values of B^{ex} less than .16421, while for values greater than .16421, agent 1's valuation becomes higher than 2's and hence only 1 holds equity. Thus for all values of B^{ex} from 0 to .30615 it is not true that all equity holders are also bond holders; since the default constraint never binds in this region, any firm can increase its value by issuing debt.

At $B^{ex} = .30615$, on the other hand, the two consumers have the same marginal valuation for the bond (bottom right panel) and only consumer 1 holds equity. Thus, all equity holders

[25] This implies that, if the firm loads everything on factor 1, i.e. $\phi = 1$, the correlations of the consumers' endowments with the firms' productivity shocks are .7279 for 1 and .9963 for 2. If the firm loads everything on factor 2 instead, i.e. $\phi = 0$, the correlations of the consumers' endowments with the firms' productivity shocks are .7662 for 1 and .9898 for 2.

[26] The firms' value is determined using the equity price map obtained, as stated in the consistency condition Cii) of Section 2.1, from the consumers' marginal rate of substitution at the equilibrium allocation associated to $B^{ex} = 0$.

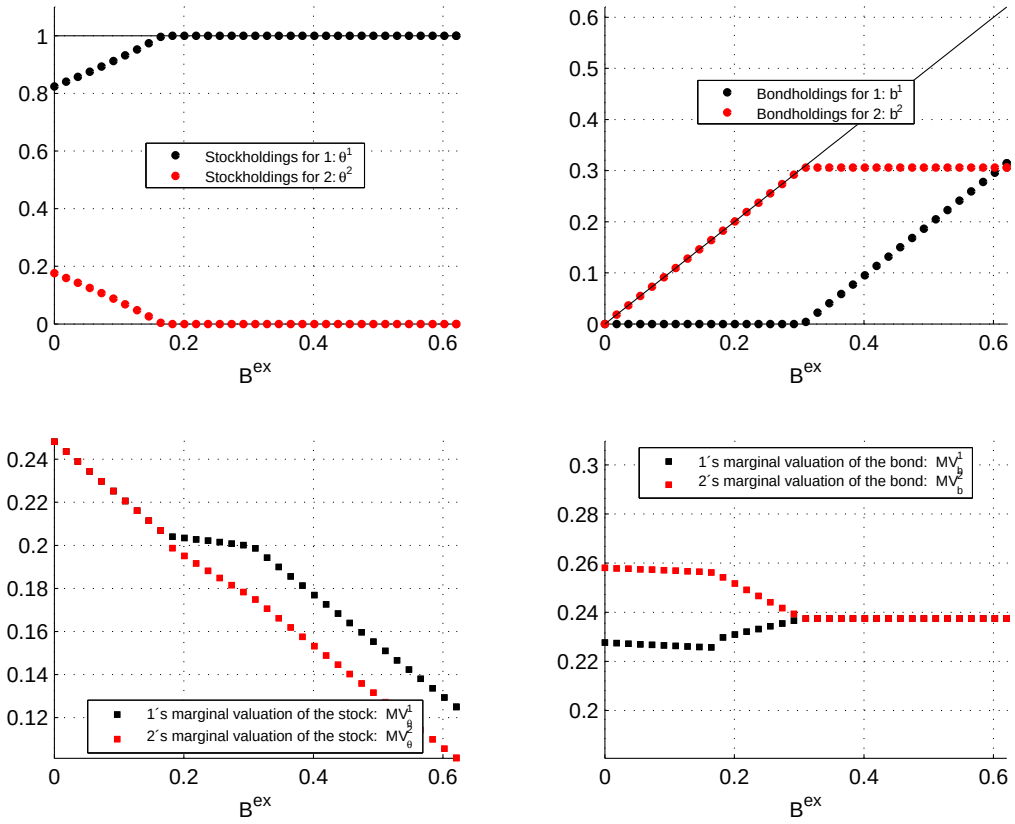

Figure 1: Parametric exercise: market clearing values, for given $B^{ex} \in [0, .62034]$, $\phi = 1$. i) First row: consumers' asset holdings. ii) Second row: consumers' willingness to pay for equity $\mathbb{E}MRS^i(s)\left[a_1(s)k^\alpha - B^{ex}\right]$ and bonds $\mathbb{E}MRS^i(s)$, $i = 1, 2$.

are also bond holders and the prices and allocations obtained when $B^{ex} = .30615$ (with $k = .20419$) constitute an equilibrium of our model. As B^{ex} is increased beyond .30615, up to its maximal level such that the no default condition is satisfied (.62034), the allocation and bond prices remain the same and still constitute an equilibrium. Values of $B^{ex} > .62034$ can only be sustained if the firm's investment k is increased so as to satisfy the no default constraint: we find however that this is never an equilibrium.

To sum up, the equilibrium consumption and investment levels are uniquely determined while the capital structure of all firms in the economy is only partly determinate, given by any $B \in [.30615, .62034]$. This is in accord with our findings in Proposition 5 for the case in which the default constraint does not bind (as it is here).

Figure 2 then shows that, also in accord with Proposition 5, the financial decision of each individual firm is indeterminate. It plots the value of an arbitrary firm, $-k+q(k,\phi,B)+pB$, for $\phi = 1$ and different levels of k and B: we see that the firm's maximal level is attained at $k = .20419$ and all $B \in [0, .62034]$.

We can also investigate how the equilibrium capital structure varies with the business cycle, that is, in this simple environment, how it varies with the aggregate state when firms' decisions are made and the persistence of the shocks. In the four columns of Table 2, we have reported the equilibrium values for investment, asset prices, firms' capital structure and consumers' portfolios for the cases where at date 0 the state is, respectively, a recession, as above, or a boom ($w_0^i = w^i(\bar{s_1})$ for all i), and where the persistence of the initial state is low (.1, as above) or high (.6). For the capital structure we report the lower bound of B in the equilibrium region. As we see from the table, when the persistence of the shocks is low, B increases from 1.068 to 1.6467 going from recession to boom. In a boom, with low persistence, consumers expect to face hard times in the future and firms' productivity to be low; hence they demand debt relatively more than equity because, in this situation, debt represents a better hedge than equity against expected low idiosyncratic shocks. On the other hand, when the persistence of the shock is high, B decreases from 2.2077 to 1.8957 from recession to boom.[27]

[27] There is a large body of literature about the cyclical properties of leverage, defined as $\frac{pB}{-k+q+pB}$ in our notation. There appears to be a consensus that leverage is counter cyclical; see Korajczyk and Levy (2003), and also Choe et al. (1993), Kashyap et al. (1993), Gertler and Gilchrist (1993), Gertler and Gilchrist (1994), Covas and Haan (2007), Levy and Hennessy (2007).

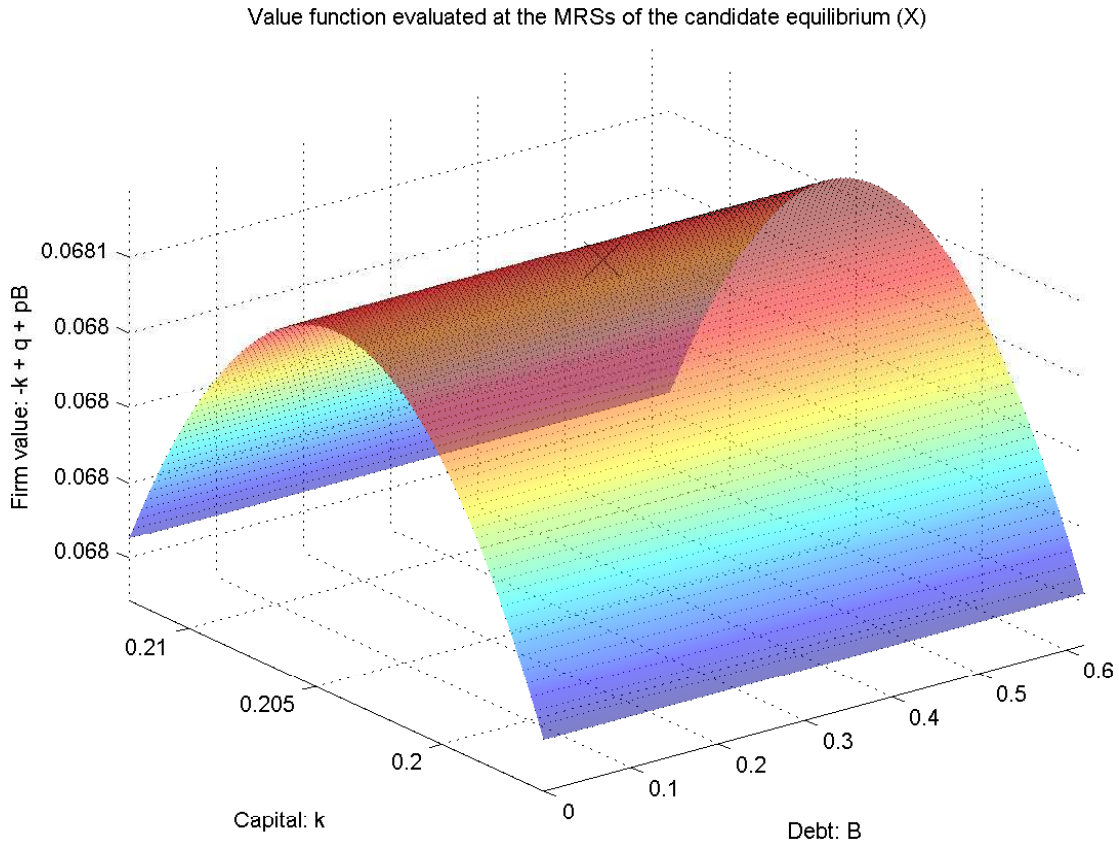

Figure 2: Value of an arbitrary firm, $-k+q(k,\phi,B)+pB$, as a function of k and B (for $\phi = 1$), where $q(k,\phi,B)$ is computed using the consumers' MRSs at the equilibrium allocation. The $\times$ in the plot represents the lower bound of the Modigliani-Miller region, i.e. $B = .30615$.

Initial state	s_1 (recession)	s_3 (boom)
Low persistence	$\pi(s_1) = .1, \pi(s_2) = .3$	$\pi(s_2) = .3, \pi(s_3) = .1$
ϕ	1	0
k	.20419	2.2502
B	.30615	1.8981
q	.19956	1.7652
p	.23744	.65074
$pB/(-k + pB + q)$	1.068	1.6467
θ^1; b^1	1; 0	.37086; 0
High persistence	$\pi(s_1) = .6, \pi(s_2) = .3$	$\pi(s_2) = .3, \pi(s_3) = .6$
ϕ	0	1
k	.36531	1.6612
B	.65132	2.4525
q	.21824	1.1652
p	.41275	.42801
$pB/(-k + pB + q)$	2.2077	1.8957
θ^1; b^1	1; 0	.45364; 0

Table 2: Equilibrium values for different specifications of the state at date 0 and different persistence of the initial state.

4 Intermediated short sales

We extend now the analysis to the case where consumers can sell short the firm's equity.[28] We have already observed that, in environments where firms' production decisions affect the returns on available assets, allowing for unlimited short sales of equity is inconsistent with the notion of competitive equilibrium (that is, with price-taking behavior), as in this case many agents can take large portfolio positions in a firm's equity. But a short position on equity is, both conceptually and in the practice of financial markets, different from a simple negative holding of equity. A short sale is not a simple sale; it is a loan contract with a promise to repay an amount equal to the future value of equity. In this sense, it is natural to model short sales as subject to frictions, because, e.g., of the possibility of default. Naturally we can allow such frictions to be arbitrarily small. In this case, the notion of competitive equilibrium is well-defined.

In this section, we consider a specific form of friction affecting short sales of equity and we show how the results of the previous section, including unanimity and constrained efficiency, extend to the case where short sales are allowed.

In order to model short sales, we introduce financial intermediaries, who can issue claims corresponding to both short and long positions (more generally, derivatives) on the firm's equity.[29] As before, equity is traded in the market at $t = 0$ at a price q and the outstanding amount of equity is normalized to 1. Intermediaries bear no cost to issue claims, but face the possibility of default on the short positions they issue (i.e., on the loans granted via the sale of such positions). We consider here for simplicity the case in which the default rate on such positions is exogenously given and equal to δ in every state. This is primarily for simplicity; the analysis and results of this section extend to situations where the default rate varies with the type i of a consumer and the portfolio held[30] by him.[31]

To protect themselves against the risk of default on the short positions issued, intermediaries have to hold an appropriate portfolio of claims (which acts then as a form of collateral against the

[28] We could allow for short sales of the bond as well, at only a notational cost.

[29] We could also allow intermediaries to issue different types of derivatives on the firm's equity, again at only notational cost.

[30] For this it suffices that both the type and the portfolio choice of each individual are observable so that the price of short positions in the derivative may depend on both, i.e. be type specific and nonlinear. We can then think of the map describing the default of rate of an individual of a given type and with a given portfolio as being endogenously determined in equilibrium as the result of the default choice of individuals, when they face, for instance, some penalty for defaulting (as in Dubey et al. (2005)) and default is chosen at the initial date.

[31] Any other cost of intermediation, as long as it is proportional to the amount intermediated, would give us the same results.

risk of their insolvency) and may charge a different price for long and short positions. The best hedge against default risk on short positions on equity is clearly equity itself and we focus so our attention here on the case where only equity is held to hedge consumers' default risk.

The self-financing constraint of the intermediary intermediating m units of the derivative on the firm's equity is then:

$$m \leq m(1-\delta) + \gamma \tag{18}$$

where m is the number of long (and short) positions issued and γ the amount of equity of the firm retained as collateral by the intermediary. Let q^+ (resp. q^-) be the price at which long (resp. short) positions in the derivative issued by the intermediary are traded. The intermediary chooses the amount of long and short positions in the derivative intermediated, $m \in R_+$, and the amount of equity held as collateral, $\gamma \in R_+$, so as to maximize its total revenue at date 0:

$$\max_{m,\gamma}(q^+ - q^-)m - q\gamma \tag{19}$$

subject to the self-financing constraint (18).

The intermediation technology is characterized by constant returns to scale. A solution to the intermediary's choice problem exists provided

$$q \geq \frac{q^+ - q^-}{\delta}$$

and is characterized by $\gamma = \delta m$ and $m > 0$ only if $q = \frac{q^+ - q^-}{\delta}$.

In this set-up derivatives are thus "backed" by equity in two ways: (i) the yield of each derivative is "pegged" to the yield of equity of the firm;[32] (ii) to issue any short position in the derivative, the intermediary has to hold - as a collateral against the risk of his customers' default - an appropriate amount of equity of the same firm to whose return the derivative is pegged.

Let $\lambda_+^i \in R_+$ denote consumer i's holdings of long positions in the derivative, and $\lambda_-^i \in R_+$ his holdings of short positions. The consumer's budget constraints in this set-up[33] are then as follows:

$$c_0^i = w_0^i + [-k + q + p\ B\ ]\,\theta_0^i - q\ \theta^i - p\ b^i - q^+\lambda_+^i + q^-\lambda_-^i \tag{20}$$

$$c^i(s) = w^i(s) + [f(k,\phi;s) - B\ ]\,(\theta^i + \lambda_+^i - \lambda_-^i(1-\delta)) + b^i,\ \forall s \in \mathcal{S} \tag{21}$$

The consumer's choice problem consists in maximizing his expected utility subject to the above constraints and $\left(\theta^i, b^i, \lambda_+^i, \lambda_-^i\right) \geq 0$.

[32]The role of equity as a benchmark to which the return on derivatives can be pegged can be justified on the basis of the fact that asset returns cannot be written as a direct function of future states of nature.

[33]In the expression of the date 1 budget constraint we take into account the fact that the consumer will default on a fraction δ of his short positions (equivalently, that he defaults with probability δ).

The asset market clearing conditions are now, for equity

$$\gamma + \sum_{i \in I} \theta^i = 1$$

and for the derivative security

$$\sum_{i \in I} \lambda_+^i = \sum_{i \in I} \lambda_-^i = m$$

The firm's choice problem is unchanged, still given by (1) subject to (2). However, the specification of the equity price map $q(k, \phi, B)$ has to be properly adjusted, to reflect the fact that now also intermediaries as well as consumers may demand equity in the market: $q(k, \phi, B)$ equals then the *maximal valuation, at the margin, among consumers and intermediaries, of the equity's cash flow* when the firm's decisions are given by k, ϕ, B:

$$q(k,\phi,B) = \max \left\{ \max_i \mathbb{E}\left[\overline{MRS}^i(s)\left(f(k,\phi;s) - B\right)\right], \frac{\max_i \mathbb{E}\left[\overline{MRS}^i(s)\left(f(k,\phi;s) - B\right)\right] - \min_i \mathbb{E}\left[\overline{MRS}^i(s)\left(f(k,\phi;s) - B\right)\right]}{\delta} \right\} \tag{22}$$

The second term on the right hand side of the above expression is the intermediaries' marginal valuation for equity and can be interpreted as the *value of intermediation*. Since an appropriate amount of equity, to be retained as collateral, is needed to issue the corresponding derivative claims (short and long positions on equity), the intermediary's willingness to pay for any type of equity is determined by the consumers' marginal valuation for the corresponding derivative claims which can be issued. Hence the above specification of the firms' equity price conjecture allows firms to take into account also the effects of their decisions on the value of intermediation.

A competitive equilibrium of the economy with short sales can then be defined from the above expressions along the same lines of Definition 1 in Section 2.1. Two possible situations can arise then in equilibrium:

1. $q = (q^+ - q^-)/\delta > q^+$, which is in turn equivalent to $q^+ > q^-/(1-\delta)$. In this case equity sells at a premium over the long positions on the derivative claim issued by the intermediary (because of its additional value as input in the intermediation technology). Thus all the amount of equity outstanding is purchased by the intermediary, who can bear the additional cost of equity thanks to the presence of a sufficiently high spread $q^+ - q^-$ between the cost of long and short positions on the derivative.

2. $q = q^+$. In this case there is a single price at which equity and long positions in the derivative can be traded. Consumers are then indifferent between buying long positions in equity and the derivative and some if not all the outstanding amount of equity is held by consumers.

When consumers hold all the outstanding amount of equity, intermediaries are non active at equilibrium and the bid ask spread $q^+ - q^-$ is sufficiently low (in particular, it is less or equal than δq).

For this economy unanimity holds exactly as in the economy with no short sales. Furthermore, we can again show that the First Welfare Theorem holds:

Proposition 6 *Competitive equilibria of the economy with intermediated short-sales are constrained Pareto efficient.*

The argument of the proof of such claim is essentially the same as the one for Proposition 3, and again relies on the fact that a competitive equilibrium of the model described above is equivalent to one where all markets, that is not only the markets for all possible types of equity (associated to any possible choice k, ϕ, B of firms), but also the markets for all types of corresponding derivatives are open for trade to consumers. In particular, for all $(k', \phi', B') \neq \left(\bar{k}, \bar{\phi}, \bar{B}\right)$ the buying price (that is, for long positions) and the selling prices are, respectively:

$$q^+(k', \phi', B') = \max_i \mathbb{E}\left[\overline{MRS}^i(s)\left(f(k', \phi'; s) - B'\right)\right]$$
$$q^-(k', \phi', B') = \min_i \mathbb{E}\left[\overline{MRS}^i(s)\left(f(k', \phi'; s) - B'\right)\right]$$

and at these prices both the market for long and short positions clear with a zero level of trade. This follows from the above specification of the consistency conditions imposed on the firms' price conjectures, hence the efficiency result.

Note that in the present economy with intermediated short sales consumers face no upper bound on their short sales of equity, but the presence of a bid ask spread still limits their hedging possibilities. It is interesting to compare our efficiency result with Theorem 5 in Allen and Gale (1991), where it is shown that the competitive equilibria of an economy with finite, exogenous bounds $\bar{K}$ on short sales are constrained inefficient.[34] In their set-up, long and short positions trade at the same price, i.e., the bid ask spread is zero, and firms cannot internalize the effect of their choices, at the margin, on the *value of intermediation*. The inefficiency result in Allen and Gale (1991) then follows from the fact that in equilibrium the expression of market value which firms maximize ignores the effect of their decisions on the value of the intermediated short sale positions taken by agents. In other words, a firm is restricted not to exploit the gains from trade arising from the demand for short positions in the firm's equity. In our economy with intermediated short

[34]Though firms' decisions in Allen and Gale (1991) concern primarily which securities to issue, their analysis could be easily reformulated in a set-up where firms have to choose their level of output and take financial decisions, as in this paper.

sales, instead, equity is an input in the intermediation process which allows short sales positions to be traded in the market. Hence the firm takes into account the value of its equity not only for the consumers but also for the intermediaries when making its production and financial decisions. The gains from trade due to intermediation are so exploited by firms.[35]

It is also useful to contrast our findings with the inefficiency result in Pesendorfer (1995). In Example 2 Pesendorfer (1995) shows that a competitive economy where financial intermediaries may introduce complementary innovations in the market may get stuck at an equilibrium in which no intermediary innovates, even though welfare would be higher if all innovations were traded in the market. The result in this example is related to similar findings obtained in competitive equilibrium models with differentiated goods; notably, Hart (1980) and Makowski (1980). In fact the inefficiency arising in the economy considered by Pesendorfer is conceptually similar to that of Allen and Gale (1991) just discussed: each intermediary is implicitly restricted not to trade with other intermediaries; equivalently, equilibrium prices for non-traded innovations are restricted not to include at the margin their effect on the value of intermediation. If instead prices for non-traded innovations were specified so as to equal the maximum between the consumers' and the intermediaries' marginal valuation, as in our equation (22), constrained efficiency would obtain at equilibrium.

5 Asymmetric information

We have shown that production and financing decisions of firms cannot be fully separated, along the lines of the Modigliani-Miller result, when markets are incomplete and short sales are either not allowed or are intermediated. Nonetheless, as we have seen, unanimity and constrained efficiency characterize competitive equilibria in these economies. In this section we will study economies in which an additional link between production and financing decisions is due to the presence of asymmetric information between debt holders, equity holders and the firm's management (the agents who manage the firm and choose its production plans).

[35] Another way to understand the difference between the present set-up and the one in Allen and Gale (1991) is by comparing the degree of completeness of the market in the two cases. Here, as argued above, the situation is effectively one where the markets for all possible derivative claims (corresponding to any plan k, ϕ, B) are open and clear at the equilibrium prices. Hence if no firms chooses a particular plan k', ϕ', B', the market for the associated derivatives is cleared at no trade, possibly with a large spread between the price for buying and selling positions. This is not the case in Allen and Gale (1991). To have an equilibrium in their set-up, where long and short positions are restricted to trade at the same price, the bound on short sales $\bar{K}$ must be 0 for the claims corresponding to values of k, ϕ, B different from those chosen by firms. Effectively, then, these markets are closed and an inefficiency may so arise.

In corporate finance models with such informational asymmetries have been studied for decades now, at least since the work of Jensen and Meckling (1976). In fact, these models are workhorses for much of corporate finance and, in particular, for the study of the determinants of firms' capital structure and managerial incentive compensation.[36] It is thus important to extend our analysis to allow for the consideration of these issues. At the same time, while this earlier work is typically cast in a partial equilibrium framework, a general equilibrium model allows to study the interaction between managerial incentive contracts, the equilibrium property of the firms' capital structure, and the general equilibrium effects of these agency problems, like the endogenous determination of aggregate risk in the economy and its implications for asset pricing.

Once again we shall mostly stress foundational issues, from the specification of the objective function of the firm to the analysis of the effects of its financial decisions and the efficiency properties of equilibria, rather than applications. This is necessary because, while general equilibrium theory has been extended to the study of economies with asymmetric information, from the seminal work of Prescott and Townsend (1984) to, e.g., the more recent work of Dubey et al. (2005), Bisin and Gottardi (1999) and Bisin and Gottardi (2006), most of this work concerns asymmetric information on the consumption side.[37] We shall consider two classes of models, where the asymmetric information concerning the firms' financing decisions is either of the moral hazard or the adverse selection type. We shall see that the moral hazard/adverse selection distinction is not important for unanimity, but it is for efficiency.

5.1 Unobservable risk composition - moral hazard

An implicit assumption in the analysis of the economy considered in the previous sections is that firms' production and financial decisions k, ϕ, B are observed by all the agents so that they can correctly anticipate, when they choose their trades in the asset markets at date 0, what the payoff in each state will be. This is in line with standard analysis of economies with traded equity or of the Modigliani Miller Theorem. Still it may appear rather demanding. In this section we consider then the case where the choice of ϕ, unlike that of k and B, is not observed by bond holders nor by equity holders in financial markets at time 0. In this environment, therefore, the characteristics of the agents who are in charge of the firm's production and technological decisions matter. We call these agents *managers* and we postulate that managers are endogenously chosen in equilibrium among the different types of consumers in the economy by the firms' equity holders.

[36] See Tirole (2006).

[37] Exceptions include Acharya and Bisin (2009), Magill and Quinzii (2002), Dreze et al. (2008), Zame (2007), Prescott and Townsend (2006).

An agent, if chosen as manager of a firm, will pick ϕ so as to maximize his utility, since the choice of ϕ is not observable. The choice of ϕ affects this agent's utility both because the agent may hold a portfolio whose return is affected by ϕ but also because the agent may incur some disutility cost associated to different choices of ϕ. Let this disutility costs be $v^i(\phi)$ for a type i consumer We will assume that the manager's portfolio is observable. In fact, without loss of generality, we assume that managers cannot trade their way out of the compensation package chosen by the equity holders.[38]

For simplicity, we continue to examine the case where the firm's equity and debt are the only assets in the economy, but we allow for the possibility that firms default on their debt in some states. Hence corporate debt is now a risky asset and its return $\min\left\{1, \frac{f(k,\phi;s)}{B}\right\}$ varies, like equity's, with the state as well as the firm's production, k, ϕ, and financial decisions, B.

The consumption side of the economy is as in Section 2: each consumer i is subject to endowment shocks w_0^i at date 0 and $w^i(s)$ at date 1 in state s and has an initial endowment of shares θ_0^i.

The equity holders of a firm must now choose the level of its physical capital k, its financial structure, described by B, as well as the type i of agent serving as its manager and his compensation package, so as to maximize the firm's market valuation. The manager's compensation package consists of a net payment x_0, in units of the consumption good at date 0, together with a portfolio of θ^m units of equity and b^m units of bonds and is chosen taking into account the manager's incentives (that is, the effect of the compensation on the manager's choice of ϕ). While ϕ is not observed by either equity holders or bond holders, ϕ is indirectly chosen by the firm's equity holders, provided the appropriate incentive compatibility constraints are satisfied. All agents in the economy can then anticipate the manager's choice of ϕ given his/her incentives, that is, given his/her type i and his/her compensation package.

Each firm is still perfectly competitive and hence takes prices as given. It evaluates the effects of alternative choices of k, B, ϕ on the market value of its equity, as in the previous sections, on the basis of a given price conjecture $q(k, B, \phi)$. In addition, the same is true now for the value of the firm's bonds, whose return is also risky in principle and varies with k, B, ϕ. The market valuation of the firms' debt for different production and financing choices is then also described by a given price conjecture, $p(k, \phi, B)$.

Let $W^i(\phi, k, B)$ denote the total cost of the compensation package for a manager of type i, which induces him to choose ϕ when the firm's production and financial decisions are given by k, B and the corresponding market value of equity and bonds is $q(k, \phi, B)$, $p(k, B, \phi)$. This cost is given by:

[38]See Acharya and Bisin (2009) and Bisin et al. (2008) for economies where much is made of the opposite assumption.

a) the payment x_0^i made to this agent at date 0,

b) plus the value of the portfolio $q(k,\phi,B)\left(\theta^{i,m}-\theta_0^i\right)+p(k,B,\phi)b^{i,m}$ attributed to him,

c) minus the amount of the dividends due to this agent on account of his initial endowment θ_0^i of equity, $\theta_0^i\left[-k+p(k,\phi,B)B-W^i(\phi,k,B)\right]$.

After collecting terms and simplifying, we obtain so the following expression for $W^i(\phi,k,B)$:

$$W^i(\phi,k,B)=\frac{\left\{x_0^i+q(k,B,\phi)\left(\theta^{i,m}-\theta_0^i\right)+p(k,B,\phi)b^{i,m}-\theta_0^i\left[p(k,B,\phi)B-k\right]\right\}}{1-\theta_0^i} \tag{23}$$

In order to analyze the firm's choice we proceed in two steps. We first state the optimal choice problem of a firm who has a hired as manager a type i consumer:

$$V^i=\max_{k,B,\phi,x_0^i,\theta^{i,m},b^{i,m}} -k+q(k,B,\phi)+p(k,B,\phi)B-W^i(\phi,k,B) \tag{24}$$

s.t. (23) and:

$$\mathbb{E}u^i\left(w_0^i\ +x_0^i,w^i(s)+\max\{f(k,\phi;s)-B,0\}\theta^{i,m}+\min\left\{1,\frac{f(k,\phi;s)}{B}\right\}b^{i,m}\right)-v^i(\phi)\geq$$

$$\mathbb{E}u^i\left(w_0^i+x_0^i,w^i(s)+\max\{f(k,\phi';s)-B,0\}\theta^{i,m}\ +\min\left\{1,\frac{f(k,\phi';s)}{B}\right\}b^{i,m}\right)-v^i(\phi'),\ \forall\phi'\in\Phi \tag{25}$$

$$\mathbb{E}u^i\left(w_0^i\ +x_0^i,w^i(s)+\max\{f(k,\phi;s)-B,0\}\theta^{i,m}\ +\min\left\{1,\frac{f(k,\phi;s)}{B}\right\}b^{i,m}\right)-v^i(\phi)\geq\bar{U}^i \tag{26}$$

The firm maximizes its value under constraints (25) and (26). The first is the incentive constraint of a type i manager which ensures that, given his compensation, he indeed chooses ϕ rather than any other $\phi'\in\Phi$. With this constraint the firm internalizes the effect of its choices of k and B on ϕ. The second is the participation constraint, where $\bar{U}^i$ is the reservation utility for a manager of type i, which is endogenously determined in equilibrium (see below). On the other hand, the no default constraint is no longer present.

Next, the type $\bar{\imath}\in I$ of agent to be hired as manager is chosen by selecting the type which maximizes the firm's value:

$$\max_{i\in I} V^i \tag{27}$$

for V^i indicating the solution of problem (24).

Each consumer of a given type j, if not hired as manager, has to choose his portfolio of equity and bonds, θ^j and b^j, taking as given the price of bonds p and the price of equity q, as well as the dividends paid on equity at the two dates and the bonds' yield, so as to maximize his utility.[39].

[39]We maintain here the assumption that agents cannot sell short the firm's equity nor its debt. No conceptual difficulty is involved in allowing for intermediated short sales as in Section 4.

The problem of such an agent is then:

$$\max_{\theta^j, b^j, c^j} \mathbb{E} u^j(c_0^j, c^j(s)) \tag{28}$$

subject to

$$c_0^j = w_0^j + \left[-k + q + pB - W^i\right] \theta_0^j - q\ \theta^j - p\ b^j \tag{29}$$

$$c^j(s) = w^j(s) + \max\{f(k,\phi;s) - B, 0\}\theta^j + \min\left\{1, \frac{f(k,\phi;s)}{B}\right\} b^j,\ \forall s \in \mathcal{S} \tag{30}$$

and

$$b^j \geq 0,\ \theta^j \geq 0,\ \forall j \tag{31}$$

Let once again $\bar{\theta}^j, \bar{b}^j, \bar{c}^j$ denote the solutions of this problem and $\bar{U}^j$ the corresponding level of the agent's expected utility. It represents the endogenous reservation utility for a type j agent if hired as a manager.

In equilibrium, the bond and equity price maps faced by the firms must satisfy the following consistency conditions:

Ci-mh) $p = p(\bar{k}, \bar{B}, \bar{\phi})$ and $q = q(\bar{k}, \bar{B}, \bar{\phi})$;

Cii-mh) $p(k, B, \phi) = \max_i \mathbb{E}\left[\overline{MRS}^i(s) \min\left\{1, \frac{f(k,\phi;s)}{B}\right\}\right]$ and
$q(k, B, \phi) = \max_i \mathbb{E}\left[\overline{MRS}^i(s) \max\left\{f(k,\phi;s) - B, 0\right\}\right]$ for all k, B, ϕ,

where $\overline{MRS}^i(s)$ denotes, as before, consumer i's marginal rate of substitution between consumption at date 0 and at date 1 in state s, evaluated at his equilibrium consumption choice $\bar{c}^i$. Condition Ci*-mh)* requires that in equilibrium the prices faced by consumers in the market equal the prices conjectured by the firms for their equilibrium choices. Condition Cii*-mh)* ensures that the firms' conjecture concerning the market value of the bond and equity for each possible k, B, ϕ equals the highest marginal valuation across all consumers for the return on these assets, evaluated at their equilibrium consumption choices. The specification of this price conjecture implicitly assumes that investors correctly anticipate the payoff distribution of bond and equity, given the observed levels of k and B as before and now, the inference over ϕ, using (25), from the information over the manager's compensation package as well as k, B.

In addition, the following market clearing conditions must hold:[40]

$$\sum_{i \neq \bar{\imath}} \bar{c}_0^i + \bar{k} + x_0^{\bar{\imath}} \leq \sum_{i \neq \bar{\imath}} w_0^i,$$

$$\sum_{i \neq \bar{\imath}} \bar{c}^i(s) + \max\{f(\bar{k}, \bar{\phi}; s) - \bar{B}, 0\}\theta^{\bar{\imath},m} + \min\left\{1, \frac{f(\bar{k}, \bar{\phi}; s)}{\bar{B}}\right\} b^{\bar{\imath},m} \leq \sum_{i \neq \bar{\imath}} w^i(s) + f(\bar{k}, \bar{\phi}; s),\ \forall s \in \mathcal{S} \tag{32}$$

[40]Recall that we have assumed for simplicity that the mass of agents of any given type i is equal to the mass of existing firms. This is obviously by no means essential.

Summarizing,

Definition 3 *A competitive equilibrium of an economy with moral hazard is a collection*

$$\left\{\left(\bar{k},\bar{B},\bar{\phi},\bar{\imath},x_0^{\bar{\imath}},\theta^{\bar{\imath},m},b^{\bar{\imath},m},\bar{W}\right),\left(\bar{c}^i,\bar{\theta}^i,\bar{b}^i,\bar{U}^i\right)_{i=1}^I,\bar{p},\bar{q},p(\cdot),q(\cdot)\right\}$$

such that: i) $\bar{k},\bar{B},\bar{\phi},\bar{\imath},x_0^{\bar{\imath}},\theta^{\bar{\imath},m}$ *and* $b^{\bar{\imath},m}$ *solve the firm problem (27) given* $p(\cdot),q(\cdot)$ *and* $\{\bar{U}^i\}_{i=1}^I$*; ii)* $p(\cdot),q(\cdot)$ *satisfy the consistency conditions Ci-mh) and Cii-mh), respectively; iii) for all i,* $\bar{c}^i,\bar{\theta}^i,\bar{b}^i$ *solve consumer i's problem (28) s.t. (29), (30) and (31) for given* $\bar{p},\bar{q},\bar{k},\bar{B},\bar{\phi}$ *and* $\bar{W}^{\bar{\imath}}=W^i(\bar{k},\bar{B},\bar{\phi})$*;* $\bar{U}^i=\mathbb{E}u^i(\bar{c}_0^i,\bar{c}^i(s))$ *and v) markets clear, (32).*

5.1.1 Unanimity and efficiency

In the economy with moral hazard just described each firm chooses the production and financing plan which maximizes its value. The firm takes fully into account the effects that its production and financing plan as well as its choice of management and associated compensation package have on its value and, in equilibrium, the model is equivalent to one where the markets for all types of equity and bonds are open. Consequently, by a very similar argument to the one developed in Section 3, equity holders' unanimity holds regarding the firm's production and financing decisions as well as the choice of management; that is the choice of k and B, as well as the decision over the manager and its compensation inducing the choice of ϕ.

Proposition 7 *At a competitive equilibrium of the economy with moral hazard, equity holders unanimously support the production and financial decisions of firms as well as the choice of management,* $\bar{k},\bar{B},\bar{\phi},\bar{\imath},x_0^{\bar{\imath}},\theta^{\bar{\imath},m},b^{\bar{\imath},m}$*; that is, every agent i holding a positive initial amount* θ_0^i *of equity of the representative firm will be made - weakly - worse off by any other admissible choice of a firm* $k',B',\phi',i',x_0^{i'},\theta^{i',m},b^{i',m}$ *which satisfies (25) and (26).*

We show next that all competitive equilibria of the economy described exhibit desirable welfare properties. Evidently, we cannot expect competitive equilibrium allocations to be fully Pareto efficient: first of all, the hedging possibilities available to consumers are limited by market incompleteness (equity and risky debt are the only assets traded). More importantly, the economy is characterized by the presence of moral hazard: the risk composition of the firms' cash-flow is chosen by the firms' managers and is not observable by the other agents (equity holders and bond holders). Given these constraints, equilibrium allocations are Pareto efficient, or constrained Pareto efficient in the sense of Diamond (1967) and Prescott and Townsend (1984).

More formally, a consumption allocation $(c^i)_{i=1}^I$ is *admissible* in the presence of moral hazard if:

1. it is *feasible*: there exists a production plan k and a risk composition choice ϕ of firms such that (11) holds;

2. it is *attainable with the existing asset structure*: that is, there exists B and, for each consumer's type i, a pair θ^i, b^i such that

$$c^i(s) = w^i(s) + \max\{0, f(k,\phi;s) - B\}\theta^i \ + \min\left\{1, \frac{f(k,\phi;s)}{B}\right\} b^i, \ \forall s \in \mathcal{S}; \tag{33}$$

3. It is *incentive compatible*: given the production plan k and the financing plan B, there exists $\bar{\imath}$ such that:

$$\mathbb{E}u^i(c_0^{\bar{\imath}}, w^{\bar{\imath}}(s) + \max\{f(k,\phi;s) - B, 0\}\theta^{\bar{\imath}} \ + \min\left\{1, \frac{f(k,\phi;s)}{B}\right\} b^{\bar{\imath}}) - v^{\bar{\imath}}(\phi) \geq$$
$$\mathbb{E}u^i(c_0^{\bar{\imath}}, w^{\bar{\imath}}(s) + \max\{f(k,\phi';s) - B, 0\}\theta^{\bar{\imath}} \ + \min\left\{1, \frac{f(k,\phi';s)}{B}\right\} b^{\bar{\imath}}) - v^{\bar{\imath}}(\phi'), \ \forall \phi' \in \Phi$$

Constrained Pareto optimality is now straightforwardly defined as in Definition 2, with respect to the stronger notion of admissibility described above.

The First Welfare theorem can then be established by an argument very similar to the one used earlier, for Proposition 3.

Proposition 8 *Competitive equilibria of the economy with moral hazard are constrained Pareto efficient.*

5.1.2 Capital structure with moral hazard

In equilibrium the financing plans of the firm are determined both by the demand of investors and by managers' incentives. As in the economy considered in Section 3.3, investors' demand for bonds and equity gives the firm the incentive to leverage its position and finance production also with bonds. With riskless debt, as we noted in Section 3.3, this implies a lower bound on the quantity of corporate bonds issued by firms in equilibrium (while the upper bound is just given by feasibility, that is the no default constraint). When the firms' debt is risky, since the return on equity is a nonlinear function of B, both the aggregate and the individual firm's level of B are more precisely determined in equilibrium.[41]

[41] If risky debt is allowed in the setup of Section 3.3 (with no moral hazard), an optimal choice for the firms obtains when all equity holders have the same valuation - and the same as bond holders - for bonds' payoff in the no default states. Differently from the case where debt is riskless this does not imply that all equity holders are also bond holders, since there is a second component of bonds' payoff, in the default states. Moreover, all bond holders have the same valuation for each of the two components of bonds' payoffs,

In the presence of moral hazard the capital structure of the firm, together with the portfolio composition of its manager, also plays a role in determining the unobservable choice of ϕ and hence the returns on the firm's bonds and equity. This fact can be used to align the manager's incentives with those of the firm's equity holders and further contributes to determine the firm's capital structure. For instance, a manager of a leveraged firm with a large amount of the firm's equity in his portfolio has the incentive to choose values of ϕ that induce a higher loading on riskier factors. This is because in this economy debt is risky and equity holders primarily benefit from the upside risk. Bond holders will therefore pay a premium for corporate bonds of less leveraged firms, whose managers also hold a larger proportion of debt than equity.

Thus both the capital structure and the portfolio composition of its manager can be used to enhance his incentives and hence to increase a firm's value. As a consequence, the Modigliani-Miller's irrelevance region not only of aggregate but also of individual firms' financial decisions is even further reduced in the presence of moral hazard. We illustrate these issues by means of the following example.

An example Consider the same specification of the economy considered in the example of Section 3.3.1 except for the fact that the structure of endowment and productivity shocks is now as reported in the following[42] Table 3. As before, at date 0 the state is s_1, $w_0^i = w^i(s_1)$ for all i, and

	s_1	s_2	s_3
w^1	1	2	1
w^2	1	1	2
a_1	.1053	1.2857	2
a_2	.1580	1.2857	1.7

Table 3: Example with risky debt: stochastic structure.

$\pi(s_1) = .1$, $\pi(s_2) = .3$ and $\pi(s_3) = .6$. The disutility cost for implementing $\phi = 1$ is $v^i(1) = .0154$ for all i; on the other hand, $v^i(0) = 0$ for all i.

taken separately. If in equilibrium default occurs in some states, the firms' aggregate capital structure is fully determinate, while individual capital structure is only partially determinate (the optimum is given by an interval of values of B). Here we omit the formal statement of the firms' optimality conditions with risky debt and leave it to a technical appendix available online.

[42]These values imply that now, if the firm loads everything on factor 1, i.e. $\phi = 1$, the correlations of the consumers' endowments with the firms' productivity shocks are $-.3441$ for 1 and .8370 for 2. If it loads everything on factor 2, $\phi = 0$, the correlations are instead $-.1932$ for 1 and .7413 for 2.

	Moral Hazard	No Moral Hazard
ϕ	1	1
$\bar{\imath}$	1	1
k	.06553	.061875
B	.040425	.035792
$x_0^{\bar{\imath}}$	-.05815	-.05654
θ^1; b^1	1; 0	1; 0
q	.081488	.0802566
p	.425043	.4323179
W	.013536	.0135780
$pB/(-k+q+pB-W)$	.8764	.7631
$U^1\ (manager)$	-1.77393	-1.774696
$U^1\ (non-manager)$	-1.77393	-1.774696
U^2	-1.654437	-1.654280

Table 4: Equilibrium values with risky debt and with or without moral hazard.

In addition, firms can issue risky debt. In Table 4 we report the equilibrium values respectively for the case in which there is moral hazard (the choice of ϕ is not observable, hence both the manager's incentive and participation constraints must be satisfied) and the case in which there is no moral hazard (the choice of ϕ is observable, hence only the manager's participation constraint must be satisfied): We see that in both cases the equilibrium choice of the loading factor is $\phi = 1$, the type 1 agent is hired as manager and the compensation package awards him all the shares of the firm and no bonds. Also in both cases the participation constraint binds: the utility of a type 1 agent if hired as a manager, $U^1\ (manager)$, is in fact equal to his utility if he is not a manager and free to trade in the markets, and default occurs only in state s_1 [43]. But when there is moral hazard the incentive constraint binds and the equilibrium investment and consumption levels in the two cases are different: we see in fact that the equilibrium levels of k and B are higher with moral hazard, so that the dividends awarded to the manager in states s_2 and s_3 are higher ($.1261 > .1237$; $.2186 > .2123$) and so is his total expected utility ($-1.77393 > -1.774696$). Finally, and importantly, in both cases the capital structure not only in the aggregate but also

[43] With moral hazard: $[.1053, 1.2857, 2].06553^{.75} - .040425 = [-.0268, .1261, .2186]$. Without moral hazard: $[.1053, 1.2857, 2].061875^{.75} - .035792 = [-.0227, .1237, .2123]$.

of each individual firms is uniquely determined (with moral hazard, $B = .040425$; without moral hazard, $B = .035792$).

5.2 Unobservable manager's quality - adverse selection

Consider next an environment where the technology of an arbitrary firm is still described by the production function $f(k, \phi; s)$, but ϕ represents the quality of the agent hired as manager of the firm, which affects the stochastic structure of the firm's future output. Thus $\phi \in \Phi$ is not as in the previous section an unobservable choice of the firm's manager, but a privately observed characteristic of each agent in the economy which affects the productivity of the firm if the agent is hired as manager of the firm. We also assume managers receive some benefits from control, given by ς_ϕ, in units of the consumption good, which are diverted from the firm's output at time 1.

The problem of the equity holders of the firm is again that of choosing the production plan k and the financial structure B, as well as the type of agent serving as manager, were the type is now given by an observable component i and a second, unobservable component, the quality ϕ, together with the associated manager's compensation package. The manager's compensation package consists of an amount x_0 of the consumption good at date 0, θ^m units of equity and b^m of bonds. Since agents know also the quality component ϕ of their type at the beginning of date 0, before they may be hired as managers, this economy is one of adverse selection.

For simplicity we restrict here our attention on the case where Φ is a finite set. Let χ^i_ϕ denote the mass of agents of type i and quality ϕ. To ensure that firms are never rationed in equilibrium in their demand of managers we need to appropriately redefine the size of the mass of firms in the economy and set it here at a level smaller than χ^i_ϕ for all i, ϕ. Furthermore, we assume that the firms' technology is such that some production and financing levels and a compensation package can always be found so as to separate managers of different unobservable qualities. This is guaranteed by the following (stricter than necessary) *single crossing property* assumption:

Assumption 1 *The firms' technology is such that, for any tuple $v = (x_0, b, \theta, B, k) \in \mathbb{R} \times \mathbb{R}^4_+$ the vectors*

$$D_v \mathbb{E}u^i \left(w^i_0 + x_0, w^i(s) + \varsigma_\phi + \max\{0, f(k, \phi; s) - \varsigma_\phi - B\}\theta + \min\left\{1, \frac{f(k, \phi; s) - \varsigma_\phi}{B}\right\} b \right), \ \phi \in \Phi$$

are linearly independent.

Let $W^i(\phi, k, B) = x^i_0 + \frac{q(k,B,\phi)\left(\theta^{i,m} - \theta^i_0\right) + p(k,B,\phi) b^{i,m} - \theta^i_0 \left[p(k,B,\phi)B - k - x^i_0\right]}{1 - \theta^i_0}$ denote the total cost of the compensation package $x^i_0, \theta^{i,m}, b^{i,m}$ for a manager of type i and quality ϕ, when the firms' decisions are given by k, B and the corresponding market value of equity and bonds is $q(k, B, \phi)$,

$p(k, B, \phi)$. Hence the value maximization problem of a firm who is hiring as manager an agent of type i and unobservable quality ϕ takes the following form:

$$V^i(\phi) = \max_{k,B,x_0^i,\theta^{i,m},b^{i,m}} -k + q(k, B, \phi) + p(k, B, \phi)B - W^i(\phi, k, B; q, p)$$

$s.t.$

$$\bar{U}^i \geq \mathbb{E}u^i\Bigg(w_0^i + x_0^i, w^i(s) + \varsigma_{\phi'} + \max\{0, f(k, \phi'; s) - \varsigma_{\phi'} - B\}\theta^{i,m} + \\ + \min\left\{1, \frac{f(k, \phi'; s) - \varsigma_{\phi'}}{B}\right\} b^{i,m}\Bigg), \ \forall \phi' \neq \phi \tag{34}$$

$$\bar{U}^i \leq \mathbb{E}u^i\left(w_0^i + x_0^i, w^i(s) + \varsigma_{\phi} + \max\{0, f(k, \phi; s) - \varsigma_{\phi} - B\}\theta^{i,m} + \min\left\{1, \frac{f(k, \phi; s) - \varsigma_{\phi}}{B}\right\} b^{i,m}\right) \tag{35}$$

Constraint (34) is the incentive compatibility constraint, which requires that a firm choosing a manager of type i and quality ϕ will set a compensation package which agents of the same type i but different quality $\phi' \neq \phi$ will not accept. This is because their reservation utility $\bar{U}^i$, describing as before the utility they can get by not being a manager and trading in the existing markets, is higher. Constraint (35) is then the participation constraint, which requires instead that an agent of type i and quality ϕ indeed prefers being hired as manager and receiving the proposed compensation package than receiving his reservation utility, $\bar{U}^i$.

The single crossing property guarantees that, for any i, there always exists a compensation scheme such that constraints (34) and (35) are satisfied non-trivially: only agents of quality ϕ become managers and all agents of quality $\phi' \neq \phi$ prefer not to. However firms may also choose a production and a financing plan k, B and a compensation package such that a non singleton subset $\Phi' \subseteq \Phi$ of quality types prefer being hired as managers. The specification of the program yielding the maximal value $V^i(\Phi')$ of the firm in this case is analogous to the one above.

In equilibrium firms choose the type i and quality ϕ (or alternatively, sets of qualities Φ') of the agent to be hired as manager which maximize their value:

$$\max_{i,\phi} V^i(\phi)$$

If at equilibrium the optimal choice of the firm is to hire a single quality type $\bar{\phi}$ as manager, we call the equilibrium separating, following Rothschild and Stiglitz (1976). On the other hand, if the optimal choice is to hire a nonsingleton set $\Phi' \subseteq \Phi$ of quality types, we say the equilibrium is (partially) pooling, where agents of different quality become managers.

By a similar argument as in Bisin and Gottardi (2006), we can show that competitive equilibrium are necessarily separating and moreover that, differently from the economy with moral

hazard, equilibrium allocations are not in general constrained Pareto efficient, in the sense of Diamond (1967) and Prescott and Townsend (1984). On the other hand, unanimity still holds in this environment.

6 Conclusion

In the presence of financial frictions, such as incomplete markets and/or borrowing restrictions and informational asymmetries between managers and equity holders or bond holders, production decisions are not necessarily separated from financing decisions. Corporate financing decisions, in these economies, are therefore not indeterminate and one can investigate their interaction with the properties of the equilibrium allocation and prices. The conceptual problems usually associated with modelling firm decisions when markets are incomplete or with asymmetric information can be overcome with *appropriate,* and *natural* modeling choices. We conclude therefore that the economies we study in this paper are an appropriate foundation for macroeconomics and finance in production economies when financial markets are incomplete and/or information is asymmetric.

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7 Appendix - Not for Publication

We collect here most proofs.

7.1 Proof of Proposition 1

We only provide here an outline of the main steps. Since short sales are not allowed, the consumers' budget set is non empty, compact and convex for all[44] $(\phi, k) \in \Phi \times K$, all $B \geq 0$ satisfying (2), and all $(p, q) \gg 0$. Under the assumptions made on individual preferences, consumers' net demand functions (for bonds, equity and the consumption good) are then well behaved. Using condition Cii), the pricing map $q(\phi, k, B)$ in the firm's problem (1) can be written as a function of the agents' consumption $(c_0^i, (c^i(s)_{s\in\mathcal{S}})$. The convex hull of the correspondence describing the firms' net supply of bonds and of the consumption good as well as their choice of the other technology parameter ϕ, is then also well behaved, for all $p \geq 0$ and $c_0^i \in (0, \max\{\sum_i w_0^i\}]$, $c^i(s) \in (0, \max \sum_i w^i(s)]$ $\forall s \in \mathcal{S}$. By a standard fixed point argument there exists so a value of $\bar{\phi}, \bar{k}, \bar{B}, \bar{p}, \bar{q}, (\bar{c}_0^i, (\bar{c}^i(s)_{s\in\mathcal{S}})_{i=1}^I$ such that: (a) $\bar{q}$ equals the value of the price map specified in condition Cii) evaluated at $\bar{\phi}, \bar{k}, \bar{p}$ and $(\bar{c}_0^i, (\bar{c}^i(s)_{s\in\mathcal{S}})_{i=1}^I$, (b) $\bar{\phi}, \bar{k}, \bar{B}$ belong to the convex hull of the firms' optimal choice correspondence when $p = \bar{p}$ and the terms $\overline{MRS}^i$ appearing in the equity price map specified in condition Cii) are evaluated at $(\bar{c}_0^i, (\bar{c}^i(s)_{s\in\mathcal{S}})_{i=1}^I$, (c) for each $i = 1, .., I$, $(\bar{c}_0^i, (\bar{c}^i(s)_{s\in\mathcal{S}})$ is a solution of the choice problem of type i consumers at $\bar{q}, \bar{p}$, (d) the market clearing conditions hold. Finally, by Caratheodory's Theorem, $\bar{\phi}, \bar{k}, \bar{B}$ can be written as a convex combination of finitely many points belonging to the firms' optimal choice correspondence.■

7.2 Proof of Proposition 3

Suppose $(\hat{c}^i)_{i=1}^I$ is admissible and Pareto dominates the competitive equilibrium allocation $(\bar{c}^i)_{i=1}^I$. By the definition of admissibility a collection $\hat{k}, \hat{\phi}, \hat{B}$ and $\left(\hat{\theta}^i, \hat{b}^i\right)_{i=1}^I$ exists such that (11) and (12) are satisfied. Since $\bar{c}^i$ is the optimal choice of a type i consumer at the equilibrium prices $\bar{q}, \bar{p}$ and, as argued in Remark 1, the consumer's choice problem is analogous to one where any possible type of equity is available for trade, at a price $q(k, \phi, B)$ satisfying the consistency condition Cii), we get

$$\hat{c}_0^i + \hat{q}\hat{\theta}^i \quad + \bar{p}\ \hat{b}^i - w_0^i \geq \bar{c}_0^i + \bar{q}\ \bar{\theta}^i + \bar{p}\ \bar{b}^i - w_0^i\ ,$$

[44] Strictly speaking, the nonemptiness of the budget set is ensured for all $k \in K$ provided the maximal element of $k \in K$, $k_{\max}$, is such that $w_0^i \geq \theta_0^i k_{\max}$ for all i.

where $\hat{q} = \max_i \mathbb{E}\overline{MRS}^i(s)\ \left[f(\hat{k},\hat{\phi};s) - \hat{B}\right]$. Or, equivalently,

$$\left[-\hat{k}\ +\hat{q}\ +\bar{p}\ \hat{B}\ \right]\theta_0^i + \tau^i \geq \left[-\bar{k}\ +\bar{q}\ +\bar{p}\ \bar{B}\ \right]\theta_0^i, \tag{36}$$

for $\tau^i \equiv \hat{c}_0^i + \hat{q}\hat{\theta}^i\ \ + p\ \hat{b}^i - \left[-\hat{k}\ +\hat{q}\ +\bar{p}\ \hat{B}\ \right]\theta_0^i - w_0^i$. Since (36) holds for all i, strictly for some i, summing over i yields:

$$\left[-\hat{k}\ +\hat{q}\ +\bar{p}\ \hat{B}\ \right] + \sum_i \tau^i > \left[-\bar{k}\ +\bar{q}\ +\bar{p}\ \bar{B}\ \right] \tag{37}$$

The fact that $\bar{k}, \bar{\phi}, \bar{B}$ solves the firms' optimization problem (1) in turn implies that:

$$-\bar{k}\ +\bar{q}\ +\bar{p}\ \bar{B} \geq -\hat{k}\ +\hat{q}\ +\bar{p}\ \hat{B},$$

which, together with (37), yields:

$$\sum_i \tau^i > 0,$$

or equivalently:

$$\sum_i \hat{c}_0^i + \hat{k} > \sum_i w_0^i,$$

a contradiction to (11) at date 0. ■

7.3 Proof of Proposition 4

Note first that

$$q(k,\phi,B+dB) = \max_i \mathbb{E}\overline{MRS}^i(s)\ \left[f(k,\phi;s) - B - dB\right].$$

Since for all $i \notin I^e$, $\mathbb{E}\overline{MRS}^i(s)\ [f(k,\phi;s) - B] < q(k,\phi,B)$, the max in the above expression is attained for some $i \in I^e$ and hence

$$q(k,\phi,B+dB) = q(k,\phi,B) + \max_{i\in I^e} \mathbb{E}\overline{MRS}^i(s)\ \left[-dB\right].$$

The right and left derivative of $q(k,\phi,B)$ with respect to B are then given by:

$$\frac{\partial q}{\partial B}_+ = -\min_{i\in I^e} \mathbb{E}\overline{MRS}^i(s)\ ;\ \ \frac{\partial q}{\partial B}_- = -\max_{i\in I^e} \mathbb{E}\overline{MRS}^i(s) \tag{38}$$

and may differ. Similarly the derivatives with respect to k are:

$$\frac{\partial q}{\partial k}_+ = \max_{i\in I^e} \mathbb{E}\left[\overline{MRS}^i(s)\ f_k(s)\right];\ \frac{\partial q}{\partial k}_- = \min_{i\in I^e} \mathbb{E}\left[\overline{MRS}^i(s)\ f_k(s)\right] \tag{39}$$

where f_k denotes the derivative of f with respect to k.

The first order conditions are then different according to whether the no default constraint (2) binds or not. Recalling that $\underline{s}$ denotes the lowest output state they are given by:

i. $f(k,\phi;\underline{s}) > B$ and

$$\begin{array}{rcl} \dfrac{\partial V}{\partial B_+} & = & \dfrac{\partial q}{\partial B_+} + p \leq 0, \ \dfrac{\partial V}{\partial k_+} = \dfrac{\partial q}{\partial k_+} - 1 \leq 0, \\ \dfrac{\partial V}{\partial B_-} & = & \dfrac{\partial q}{\partial B_-} + p \geq 0, \ \dfrac{\partial V}{\partial k_-} = \dfrac{\partial q}{\partial k_-} - 1 \geq 0; \end{array} \tag{40}$$

Since (38) implies that $\frac{\partial q}{\partial B}_+ \geq \frac{\partial q}{\partial B}_-$, the above conditions (with respect to B) are equivalent to:

$$\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p = \frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p = 0,$$

that is:

$$\max_{i \in I^e} \mathbb{E}\overline{MRS}^i(s) \ = \min_{i \in I^e} \mathbb{E}\overline{MRS}^i(s) \ = p = \max_i \mathbb{E}\overline{MRS}^i(s)$$

or (13) holds. Similarly, from (39) we see that $\frac{\partial q}{\partial k}_+ \geq \frac{\partial q}{\partial k}_-$, the above conditions (with respect to k) are equivalent to:

$$\frac{\partial q}{\partial k_+} - 1 = \frac{\partial q}{\partial k_-} - 1 = 0,$$

that is,

$$\max_{i \in I^e} \mathbb{E}\left[\overline{MRS}^i(s) \ f_k(s)\right] = \min_{i \in I^e} \mathbb{E}\left[\overline{MRS}^i(s) \ f_k(s)\right] = 1$$

or (14) holds.

ii. $f(k,\phi;\underline{s}) = B$ and

$$\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p \geq 0, \ \frac{\partial V}{\partial k_+} = \frac{\partial q}{\partial k_+} - 1 \leq 0. \tag{41}$$

This condition can be equivalently written as

$$p = \max_i \mathbb{E}\overline{MRS}^i(s) \ \geq \max_{i \in I^e} \mathbb{E}\overline{MRS}^i(s) \ . \tag{42}$$

and

$$1 \geq \max_{i \in I^e} \mathbb{E}\left[\overline{MRS}^i(s) \ f_k(s)\right]. \tag{43}$$

Note that (42) is always satisfied. In particular, it holds as equality when at least one equity holder is also a bond holder, or $I^e \cap I^d \neq \varnothing$, and as a strict inequality when no equity holder is also a bond holder, or all equity holders would like to short the risk free asset.

To verify whether a solution indeed obtains at $f(k,\phi;\underline{s}) = B$ when (43) holds, we need to

consider also the optimality with respect to joint changes[45] in k and B or[46]:

$$\frac{\partial V}{\partial B_+}dB + \frac{\partial V}{\partial k_+}dk = \left(\frac{\partial q}{\partial B_+} + p\right)dB + \left(\frac{\partial q}{\partial k_+} - 1\right)dk \leq 0 \text{ for } dB = f_k(\underline{s})dk > 0$$
$$\frac{\partial V}{\partial B_-}dB + \frac{\partial V}{\partial k_-}dk = \left(\frac{\partial q}{\partial B_-} + p\right)dB + \left(\frac{\partial q}{\partial k_-} - 1\right)dk \geq 0 \text{ for } dB = f_k(\underline{s})dk < 0$$

Using again (38),(39) to substitute for the derivatives of q w.r.t. B and k into the first of the above expressions yields:

$$\left[\begin{array}{c} f_k(\underline{s})\left(-\min_{i\in I^e}\mathbb{E}\overline{MRS}^i(s) \ + \max_i \mathbb{E}\overline{MRS}^i(s)\ \right) + \\ \max_{i\in I^e}\mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right) - 1 \end{array}\right] \leq 0,$$

or

$$\begin{aligned} & f_k(\underline{s})\left(-\min_{i\in I^e}\mathbb{E}\overline{MRS}^i(s) \ + \max_i \mathbb{E}\overline{MRS}^i(s)\ \right) \\ \leq \ & 1 - \max_{i\in I^e}\mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right), \end{aligned} \tag{44}$$

where the term on the r.h.s. is always nonnegative by (43) and the one on the l.h.s. is obviously always nonnegative. Proceeding similarly with the second expression above, we get:

$$\left[\begin{array}{c} f_k(\underline{s})\left(-\max_{i\in I^e}\mathbb{E}\overline{MRS}^i(s_1) \ + \max_i \mathbb{E}\overline{MRS}^i(s)\ \right) \\ + \min_{i\in I^e} E_{s_0}\left(\overline{MRS}^i(s_1)\ f_k(s)\right) - 1 \end{array}\right] \geq 0,$$

or

$$\begin{aligned} & 1 - \min_{i\in I^e}\mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right) \\ \leq \ & f_k(\underline{s})\left(-\max_{i\in I^e}\mathbb{E}\overline{MRS}^i(s) \ + \max_i \mathbb{E}\overline{MRS}^i(s)\right), \end{aligned} \tag{45}$$

and again both terms are non negative.

Putting (44) and (45) together yields

$$\begin{aligned} 1 - \max_{i\in I^e}\mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right) \ & \geq \\ f_k(\underline{s})\left(\begin{array}{c} -\min_{i\in I^e}\mathbb{E}\overline{MRS}^i(s) + \\ \max_i \mathbb{E}\overline{MRS}^i(s) \end{array}\right) \ & \geq \\ f_k(\underline{s})\left(\begin{array}{c} -\max_{i\in I^e}\mathbb{E}\overline{MRS}^i(s) + \\ \max_i \mathbb{E}\overline{MRS}^i(s) \end{array}\right) \ & \geq \end{aligned}$$

[45]This is obviously not necessary when the first order conditions are satisfied at an interior solution, that is when (13) and (14) hold.

[46]Without loss of generality, we can limit our attention to changes in B and k such that the no default constraint still binds, or $f_k(\underline{s})dk \geq dB$ holds as equality.

$$\geq 1 - \min_{i \in I^e} \mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right),$$

where the second inequality follows from the fact that

$$-\min_{i \in I^e} \mathbb{E}\overline{MRS}^i(s)\ \geq -\max_{i \in I^e} \mathbb{E}\overline{MRS}^i(s)$$

Since, by the same argument,

$$-\min_{i \in I^e} \mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right) \geq -\max_{i \in I^e} \mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right),$$

the above condition can only hold as equality:

$$\begin{aligned} 1 - \max_{i \in I^e} \mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right) &= \\ f_k(\underline{s})\left(-\min_{i \in I^e} \mathbb{E}\overline{MRS}^i(s)\ + \max_i \mathbb{E}\overline{MRS}^i(s)\ \right) &= \\ f_k(\underline{s})\left(-\max_{i \in I^e} \mathbb{E}\overline{MRS}^i(s)\ + \max_i \mathbb{E}\overline{MRS}^i(s)\ \right) &= \\ = 1 - \min_{i \in I^e} \mathbb{E}\left(\overline{MRS}^i(s)\ f_k(s)\right) & \end{aligned} \tag{46}$$

This implies that (15), (16), (17) hold, thus completing the proof.■