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RECURSIVE CONTRACTS

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We obtain a recursive formulation for a general class of contracting problems involving incentive constraints. These constraints make the corresponding maximization *sup* problems non-recursive. Our approach consists of studying a recursive Lagrangian. Under standard general conditions, there is a recursive *saddle-point* (*infsup*) functional equation (analogous to a Bellman equation) that characterizes the recursive solution to the planner's problem and *forward-looking* constraints. Our approach has been applied to a large class of dynamic contractual problems, such as contracts with limited enforcement, optimal policy design with implementability constraints, and dynamic political economy models.

Keywords:

Recursive methods, dynamic optimization, Ramsey equilibrium, time inconsistency, limited participation, contract default, saddle-points, Lagrangian multipliers.

JEL numbers:
C61, C63, D58, E27

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1 Introduction

Recursive methods have become a basic tool for the study of dynamic economic models. For example, Stokey et al. (1989) and Ljungqvist and Sargent (2004) describe a large number of macroeconomic models that can be analysed using recursive methods. A main advantage of this approach is that it characterizes optimal decisions – at any time t – as time-invariant functions of a small set of state variables. In engineering systems, knowledge of the available technology and of the current state is enough to decide the optimal control, since current returns and the feasible set depend only on past and current predetermined variables. In this case the value of future states is assessed by the value function and, under standard dynamic programming assumptions, the Bellman equation is satisfied and a standard recursive formulation is obtained.

However, one key assumption to obtain the Bellman equation is that future choices do not constrain the set of today's feasible choices. Unfortunately, this assumption does not hold in many interesting economic problems. For example, in contracting problems where agents are subject to intertemporal participation, or other intertemporal incentive constraints, the future development of the contract determines the feasible action today. Similarly, in models of optimal policy design agents' reactions to government policies are taken as constraints and, therefore, future actions limit the set of current feasible actions available to the government. Many dynamic games – for example, dynamic political-economy models – share the same feature that an agent's current feasible actions depend on functions of future actions.

In general, in the presence of forward-looking constraints – as in rational expectations models where agents commit to contracts subject to incentive constraints (e.g. commitment may be limited) – optimal plans, or contracts, do not satisfy the Bellman equation and the solution is not recursive in the standard sense. In this paper we provide an integrated approach for a recursive formulation of a large class of dynamic models with forward-looking constraints by reformulating them as equivalent recursive saddle-point problems.

Our approach has a wide range of applications. In fact, it has already proved to be useful in the study of very many models[1]. Just to mention a few examples: growth and business cycles with possible default (Marcet and Marimon (1992), Kehoe and Perri (2002), Cooley, *et al.* (2004)); social insurance (Attanasio and Rios-Rull (2000)); optimal fiscal and monetary policy design with incomplete markets (Aiyagari, Marcet, Sargent and Seppälä (2002), Svensson and Williams (2008)), and political-economy models (Acemoglu, Golosov and Tsyvinskii (2011)). For brevity, however, we do not present further applications here and limit the presentation of the theory to the case of full information.

We build on traditional tools of economic analysis such as duality theory of optimization, fixed point theory, and dynamic programming. We proceed in three steps. We first study the planner's problem with incentive constraints (**PP**) as an infinite-dimensional maximization problem, and we embed this

[1]As we write this version google scholar reports that the working paper has been cited 290 times. Many of these citations are applications of the method.

problem in a more general class of planner's problems ($\mathbf{PP}_\mu$); these problems are parameterized by the weight (μ) of a (Benthamite) social welfare function, which accounts for the functions appearing in the constraints with future controls (*forward-looking constraints*). The objective function of $\mathbf{PP}_\mu$ is similar to Pareto-optimal problems where μ is the vector of weights given to the different infinitely-lived agents in the economy.

Second, we consider the Lagrangean which incorporates the *forward-looking constraints* of the first period, which defines our starting *saddle-point planner's problem* ($\mathbf{SPP}_\mu$) and we prove a duality result between this saddle-point problem and the planner's problem ($\mathbf{PP}_\mu$). This construction helps to characterize the 'non-recursivity problem'and provides a key step towards its resolution.

As is well known, the solution of dynamic models with forward-looking constraints is, in general, time-inconsistent, in the following sense: if at some period $t > 0$ the agent solves $\mathbf{PP}_\mu$ for the whole future path given the state variables found at t, the agent will not choose the path that he had chosen in period zero (unless, of course, the forward-looking constraints are not binding, up to period t). This 'non-recursivity problem'is at the root of the difficulties in expressing the optimal solution with a time-invariant policy function.

A key insight of our approach is to show that there is a *modified problem* $\mathbf{PP}_{\mu'}$ such that if the agent reoptimizes this problem at $t = 1$ for a certain μ', the solution from period $t = 1$ onwards is the same that had been prescribed by $\mathbf{PP}_\mu$ from the standpoint of period zero. The key is to choose the vector of weights μ' appropriately. We show that the appropriate μ' is given by the Lagrange multipliers of $\mathbf{SPP}_\mu$ in period zero. This procedure of sequentially connecting *saddle-point problems* is well defined and it is recursive when solutions are unique. The problem $\mathbf{PP}_{\mu'}$ can be thought of as the 'continuation problem'that needs to be solved each period in order to implement the constrained-efficient solution. This supports our claim that the recursive formulation is obtained by introducing the vector μ, summarizing the evolution of the Lagrange multipliers, as co-state variable in a time-invariant policy function. As a result, with our method it is easy to guarantee existence of the solution to $\mathbf{PP}_{\mu'}$ for *any* $\mu' \geq 0$, making the practical implementation of this method no more complicated than standard dynamic programming problems.

Third, we extend dynamic programming theory to show that the sequence of *modified saddle-point problems* ($\mathbf{SPP}_{\mu_t}$) satisfies a *saddle-point functional equation* (**SPFE**; a *saddle-point Bellman equation*) and, conversely, that policies obtained from solving the *saddle-point functional equation* (**SPFE**) provide a solution to the original $\mathbf{SPP}_\mu$ and, therefore, to the $\mathbf{PP}_\mu$ problem. This latter sufficiency result is very general; in particular, it does not rely on convexity assumptions. This is important because incentive constraints do not have a convex structure in many applications. However, this result is limited in that we assume (local) uniqueness of solutions. We discuss the role this assumption plays and, in particular, we show how our approach, and results, do not depend on this assumption.

In addition, we also show how standard dynamic programming results, based on a *contraction mapping theorem*, generalize to our *saddle-point functional* *equation* (**SPFE**). An immediate consequence of these results is that one can use standard computational techniques that have been used to solve dynamic programming problems – such as the solution of first-order-conditions for a given recursive structure of the policy function, or value function iteration – to solve dynamic saddle-point problems. Not only the computational techniques needed but also our assumptions are standard in dynamic economic models.

Our approach is related to other existing approaches that study dynamic models with expectations constraints, in particular to the pioneering works of Abreu, Pearce and Stacchetti (1990), Green (1987) and Thomas and Worrall (1988), and the applications that have followed. We briefly discuss how these, and other, works relate to ours in Section 6, after presenting the main body of the theory in Sections 4 and 5. Section 2 provides a basic introduction to our approach and Section 3 a couple of canonical examples (most proofs are contained in the Appendix).

2 Formulating contracts as recursive saddle-point problems

In this section we give an outline of our approach, leaving the technical details and proofs to sections 4 and 5. Our interest is in solving problems that have the following representation:

PP

$$\sup_{\{a_t,x_t\}} \quad \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t r(x_t,\, a_t,\, s_t), \tag{1}$$

$$\text{s.t. } x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}), \;\; p(x_t, a_t,\, s_t) \geq 0, \;\; t \geq 0, \tag{2}$$

$$\mathrm{E}_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) + h_1^j(x_t,\, a_t,\, s_t) \geq 0, \; j = 1, ...l, \;\; t \geq 0, \tag{3}$$

$$x_0 = x, \; s_0 = s,$$

and a_t is measurable with respect to $(\ldots, s_{t-1}, s_t)$,

where r, ℓ, p, h_0, h_1 are known functions, β, x, s known constants, $\{s_t\}_{t=0}^{\infty}$ an exogenous stochastic Markov process, $N_j = \infty$ for $j = 0, ..., k$, and $N_j = 0$ for $j = k+1, ..., l$.

Standard dynamic programming methods only consider constraints of form (2) (see, for example, Stokey, *et al.* (1989) and Cooley, (1995)). Constraints of form (3) are not a special case of (2), since they involve expected values of future variables[2]. We know from Kydland and Prescott (1977) that, under these constraints, the usual Bellman equation is not satisfied, the solution is *not,* in general, of the form $a_t = f(x_t, s_t)$ for all t, and the whole history of

[2]One might think that expressing (3) in the form $v(x_t, s_t) - \psi(x_t, s_t) \geq 0$, where v is the discounted sum $\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n h_0(x_{t+n},\, a_{t+n},\, s_{t+n}), x_t, a_t, s_t)$ and $\psi = h_1 - h_0$ converts (3) into (2). But this does not solve the problem since v is not known a priori.

past shocks s_t can matter for today's optimal decision. By letting $N_j = \infty$ **PP** covers a large class of problems where discounted present values enter the implementability constraint. For example, long term contracts with *intertemporal participation constraints* take this form.[3] Alternatively, by letting $N_j = 0$ **PP** covers problems where intertemporal reactions of agents must be taken into account. For example, dynamic Ramsey problems, where the government chooses policy variables subject to optimal dynamic behavior by the agents in the economy, have this form[4]. Even though we focus on the two canonical cases $N_j = \infty$ and $N_j = 0$, intermediate cases can be easily incorporated. It is then without loss of generality that we let $N_j = \infty$, for $j = 0, ..., k$, and $N_j = 0$ for $j = k+1, ..., l$.

A first step of our approach is to consider a more general class of problems, parameterized by μ:

$$\mathbf{PP}_\mu \qquad \sup_{\{a_t, x_t\}} \mathrm{E}_0 \sum_{j=0}^{l} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t,\, a_t,\, s_t)$$

$$\text{s.t. } x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}), \;\; p(x_t, a_t,\, s_t) \geq 0, \tag{4}$$

$$\mathrm{E}_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) + h_1^j(x_t,\, a_t,\, s_t) \geq 0, \;\; t \geq 0, \tag{5}$$

$$x_0 = x, \;\; s_0 = s, \tag{6}$$

and a_t is measurable with respect to $(\ldots, s_{t-1}, s_t)$.

The main difference with **PP** is that in $\mathbf{PP}_\mu$ we have incorporated the h_0^j functions of the *forward-looking* constraints (3) into the objective function. Also, the superindex j now starts from $j = 0$, with h_0^0,, to account for the reward function of the original problem. More precisely, if we let $h_0^0 = r$, we set $\mu = (1, 0, ..., 0)$ and we choose a very large h_1^0 to guarantee that (5) is never binding for $j = 0$, $\mathbf{PP}_\mu$ is the original **PP**. Furthermore, it should also be noticed that the value function of this problem, when well defined – say, $V_\mu(x,\, s)$ – is homogeneous of degree one in μ; a property that our approach exploits (and the reason for collecting the original return function r of **PP** in the objective function, together with the forward-looking elements of the constraints).

Notice that $\mathbf{PP}_\mu$ is an infinite-dimensional maximization problem which, under relatively standard assumptions, is guaranteed to have a solution *for arbitrary* $\mu \geq 0$. The solution is a *plan*[5] $\boldsymbol{a} \equiv \{a_t\}_{t=0}$, where $a_t(\ldots, s_{t-1}, s_t)$ is a state-contingent action (Proposition 1).

[3]Combining (2) and (3) accounts for a broad class of constraints. For example, a nonlinear participation constraint of the form $g(\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n h(x_{t+n},\, a_{t+n},\, s_{t+n}), x_t, a_t, s_t) \geq 0$ can easily be incorporated in our framework with one constraint of form (2), $g(w_t, x_t, a_t, s_t) \geq 0$ (with control variables (w_t, a_t)), and one of form (3), $\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n h(x_{t+n},\, a_{t+n},\, s_{t+n}) = w_t$.

[4]See Section 6 for references to related work using constraints of the form $N_j = \infty$ and $N_j = 0$.

[5]We use bold notation to denote sequences of measurable functions.

An intermediate step in our approach is to transform program $\mathbf{PP}_\mu$ into a saddle-point problem in the following way. Consider writing the Lagrangean for $\mathbf{PP}_\mu$ when a Lagrange multiplier $\gamma \in R^{l+1}$ is attached to the *forward-looking* constraints *only* in period $t = 0$ and the remaining constraints for $t > 0$ are left as constraints. The Lagrangean then is

$$\begin{aligned} L(\mathbf{a}, \boldsymbol{\gamma}; \mu) &= \mathrm{E}_0 \sum_{j=0}^{l} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t,\, a_t,\, s_t) + \\ & \sum_{j=0}^{l} \gamma^j \left(\mathrm{E}_0 \sum_{t=1}^{N_j+1} \beta^t h_0^j(x_t,\, a_t,\, s_t) + h_1^j(x_0,\, a_0,\, s_0) \right). \end{aligned}$$

If we find a saddle-point of L subject to (4) for all $t \geq 0$ and (5) for all $t \geq 1$, the usual equivalences between this saddle-point and the optimal allocation of $\mathbf{PP}_\mu$ can be exploited. Using simple algebra it is easy to show that L can be rewritten as the objective function in the following saddle point problem[6]:

$$\begin{aligned} \mathbf{SPP}_\mu \qquad & \inf_{\gamma \in R_+^{l+1}} \sup_{\{a_t, x_t\}} \mu h_0(x_0,\, a_0,\, s_0) + \gamma h_1(x_0,\, a_0,\, s_0) \\ & + \beta\, \mathrm{E}_0 \sum_{j=0}^{l} \varphi^j(\mu,\, \gamma) \sum_{t=0}^{N_j} \beta^t\; h_0^j(x_{t+1},\, a_{t+1},\, s_{t+1}) \qquad (7) \end{aligned}$$

$$\begin{aligned} \text{s.t. } & x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}),\;\; p(x_t, a_t,\, s_t) \geq 0, \quad t \geq 0, \\ & \mathrm{E}_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) + h_1^j(x_t,\, a_t,\, s_t) \geq 0,\;\; j = 0, ..., l,\;\; t \geq 1, \end{aligned}$$

for initial conditions $x_0 = 0$ and $s_0 = s$, where φ is defined as

$$\begin{aligned} \varphi^j(\mu,\, \gamma) \;\; & \equiv \mu^j + \gamma^j \quad \text{if } N_j = \infty, \text{ i.e. } j = 0, ..., k \\ & \equiv \gamma^j \qquad\quad\; \text{if } N_j = 0, \text{ i.e. } j = k+1, ..., l. \end{aligned}$$

The usefulness of $\mathbf{SPP}_\mu$ comes from the fact that its objective function has a very special form: the term inside the expectation in (7) is precisely the objective function of $\mathbf{PP}_{\varphi(\mu,\,\gamma^*)}$ given the states (x_1^*, s_1). This will allow us to show that if $(\{a_t^*\}_{t=0}^{\infty}, \gamma^*)$ solves $\mathbf{SPP}_\mu$ for initial conditions (x, s) then $\{a_{t+1}^*\}_{t=0}^{\infty}$ solves $\mathbf{PP}_{\varphi(\mu,\,\gamma^*)}$ given initial conditions (x_1^*, s_1), where $x_1^* = \ell(x_0,\, a_0^*,\, s_1)$. That is, the continuation problem that needs to be solved in the next period is precisely a planner problem where the weights have been shifted according to φ.

We show that, under fairly general conditions, solutions to $\mathbf{PP}_\mu$ are solutions to $\mathbf{SPP}_\mu$ (Theorem 1), and viceversa (Theorem 2). Also, the usual slackness conditions will guarantee that if $(\{a_t^*\}, \gamma^*)$ solves $\mathbf{SPP}_\mu$, then

$$\mathrm{E}_0 \sum_{j=0}^{l} \gamma^{j*} \left[\sum_{t=1}^{N_j+1} \beta^t h_0^j(x_t^*,\, a_t^*,\, s_t) + h_1^j(x,\, a_0^*,\, s) \right] = 0, \qquad (8)$$

[6]We use the notation $\mu h_0(x, a, s) \equiv \sum_{j=0}^{l} \mu^j h_0^j(x, a, s)$.

so that the values achieved by the objective functions of $\mathbf{SPP}_\mu$ and $\mathbf{PP}_\mu$ coincide.

If $\mathbf{PP}_\mu$ were a standard dynamic programming problem (i.e. without (5)), then the following Bellman equation would be satisfied:

$$\begin{aligned} V_\mu(x,s) &= \sup_a \{\mu h_0(x,\,a,\,s) + \beta\,\mathrm{E}\,[V_\mu(\ell(x,\,a,\,s'),s') \mid s]\} \\ & \quad \text{s.t.} \;\; p(x,a,\,s) \geq 0. \end{aligned} \tag{9}$$

The reason that the Bellman equation holds is that in standard dynamic programming if $\mathbf{PP}_\mu$ is reoptimized at period $t=1$ given initial conditions (x_1^*, s_1), the reoptimization simply confirms the choice that had been previously made for $t \geq 1$. However, with forward-looking constraints (5) this Bellman equation is *not* satisfied, the reason being that if the problem is reoptimized at $t=1$ the choice will violate the forward-looking constraint of period $t=0$, if (5) is binding at $t=0$. A central element of our approach is that, as suggested by the objective function of $\mathbf{SPP}_\mu$, if the solution is reoptimized in period $t=1$ with the new weights $\mu' = \varphi(\mu,\,\gamma^*)$ – that is, if in period one the reoptimization is for the problem $\mathbf{PP}_{\mu'}$, – the result confirms the solution of the original problem $\mathbf{PP}_\mu$. This allows the construction of a recursive formulation of our original $\mathbf{PP}_\mu$ problem where the value function with modified weights is included in the right-hand side of a functional 'Bellman-like'equation to capture the terms in (7). More specifically, we show that under fairly general assumptions solutions to $\mathbf{SPP}_\mu$ obey a saddle-point functional equation (**SPFE**). More specifically, we look for functions W that satisfy the following:

SPFE

$$\begin{aligned} W(x,\,\mu,\,s) = \inf_{\gamma\geq 0} \sup_a \{&\mu h_0(x,\,a,\,s) + \gamma h_1(x,\,a,\,s) + \beta\,\mathrm{E}\,[W(x',\,\mu',\,s')|\,s]\} \\ &\text{s.t. } x' = \ell(x,\,a,\,s'),\; p(x,a,\,s) \geq 0 \\ &\text{and } \mu' = \varphi(\mu,\,\gamma), \end{aligned}$$

and we show that this holds for $W(x,\,\mu,\,s) = V_\mu(x,s)$ (Theorem 3).

We only consider problems where the infsup problem has unique optimal choices.[7] In this case optimal allocations and multipliers are uniquely determined and there is *a policy function* ψ, i.e. $(\,a^*,\,\gamma^*) = \psi(x,\mu,s)$, associated with a value function W satisfying **SPFE**. Finally we show that the following recursive formulation

$$\begin{aligned} (\,a_t^*,\,\gamma_t^*) &= \psi(x_t^*, \mu_t^*, s_t) \\ \mu_{t+1}^* &= \varphi(\mu_t^*,\,\gamma_t^*) \end{aligned}$$

gives the optimal policy we are seeking. More precisely, we first show that, given $\{a_t^*, \gamma_t^*\}$ generated by ψ for initial conditions (x,μ,s), $(\{a_t^*\}_{t=0}^\infty, \gamma_0^*)$ solves $\mathbf{SPP}_\mu$ in state $(x,\,s)$, that $(\{a_t^*\}_{t=1}^\infty, \gamma_1^*)$ solves $\mathbf{SPP}_{\mu_1^*}$ in state $(x_1^*,\,s_1)$, etc. (Theorem 4). As a result, the path $\{a_t^*\}$ for $\mu_0 = (1,0,...0)$ is a solution to **PP.**

[7]We discuss this issue in more detail in Sections 4 and 5.

In this sense, the modified problem $\mathbf{PP}_\mu$ is the 'correct continuation problem' to the planners' problem: if μ is properly updated, the solution can be found each period by re-optimizing $\mathbf{PP}_{\mu_t^*}$.

2.1 The Principle of Optimality with *forward-looking* constraints

We now briefly discuss in which sense our central result is not a simple restatement of the standard dynamic programming *principle of optimality* to our saddle-point formulation[8]. This principle says (when there are no *forward-looking* constraints)[9]: if V_μ satisfies the Bellman equation (9), evaluated at (x, s)), it is the (*sup*) value of $\mathbf{PP}_\mu$, when the initial state is $(x,\, s)$, and a sequence $\{a_t^*\}_{t=1}^{\infty}$ solves $\mathbf{PP}_\mu$ if and only if it satisfies:

$$\begin{aligned} V_\mu(x_t^*, s_t) &= \mu h_0(x_t^*,\, a_t^*,\, s) + \beta \mathrm{E}\left[V_\mu(x_{t+1}^*, s_{t+1}) |\, s_t\right] \\ x_{t+1}^* &= \ell(x_t^*,\, a_t^*,\, s_{t+1}),\; x_t^* = x. \end{aligned}$$

In our context, under standard assumptions it is true that if $\{a_t^*\}_{t=1}^{\infty}$ solves $\mathbf{PP}_\mu$ when the initial state is (x, s) and attains the value $V_\mu(x, s)$, then $W(x,\, \mu,\, s) \equiv V_\mu(x, s)$ and

$$W(x_t^*, \mu_t^*, s_t) = \mu_t^* h_0(x_t^*, a_t^*, s_t) + \gamma_t^* h_1(x_t^*, a_t^*, s_t) + \beta\ \mathrm{E}\left[W(x_{t+1}^*, \mu_{t+1}^*, s_{t+1}) |\, s_t\right] \tag{10}$$

$$\begin{aligned} &x_{t+1}^* = \ell(x_t^*,\, a_t^*,\, s_{t+1}),\; x_t^* = x \\ &\mu_{t+1}^* = \varphi(\mu_t^*,\, \gamma_t^*),\; \mu_0^* = (1, 0, ..., 0), \end{aligned}$$

where $\{\gamma_t^*\}_{t=1}^{\infty}$ is the sequence of Lagrange multipliers associated with the sequence of $\mathbf{SPP}_\mu$ (Theorems 1 and 3).

However, the converse, sufficiency, theorem that if $W(x,\, \mu,\, s)$ satisfies **SPFE** and $(\{a_t^*\}_{t=1}^{\infty}, \{\gamma_t^*\}_{t=1}^{\infty})$ satisfies (10) then $V_\mu(x, s) \equiv W(x,\, \mu,\, s)$ is the value of $\mathbf{PP}_\mu$ and $\{a_t^*\}_{t=1}^{\infty}$ solves $\mathbf{PP}_\mu$ is only true if, *in addition*, $W(x,\, \mu,\, s) = \mu\omega(x,\, \mu,\, s)$ and

$$\omega_j(x_t^*,\, \mu_t^*,\, s_t) = h_0^j(x_t^*,\, a_t^*,\, s_t) + \beta\ \mathrm{E}\left[\omega_j(x_{t+1}^*,\, \mu_{t+1}^*,\, s_{t+1}) |\, s_t\right], \quad \text{if} \quad j = 0, ..., k, \quad \text{and} \tag{11}$$

$$\omega_j(x_t^*,\, \mu_t^*,\, s_t) = h_0^j(x_t^*,\, \mu_t^*,\, s_t) \quad \text{if} \quad j = k+1, ..., l. \tag{12}$$

These recursive equations for the *forward-looking* constraints are needed to guarantee that these constraints are also satisfied in the original $\mathbf{PP}_\mu$. Notice that if $W(x,\, \mu,\, s)$ is differentiable in μ then, by *Euler's Theorem*, $W(x,\, \mu,\, s) = \mu\omega(x,\, \mu,\, s)$ (where $\omega_j \equiv \partial_{\mu^j} W$) and equations (11) and (12) follow from the

[8]This subsection clarifies our approach and what is new with respect to our previous work; it can be skipped by the reader only interested in how our approach works and in our main results.

[9]See, for example, Stokey *et al* (1989).

Envelope Theorem. We show, and use, the fact that if $\{a_t^*\}_{t=1}^{\infty}$ is unique (at least locally unique) then $W(x, \mu, s)$ is differentiable in μ and, therefore, we recover the *Principle of Optimality* for our saddle-point formulation (Theorems 4 and 5), without having to impose equations (11) and (12) as 'promise-keeping'constraints[10].

Before we turn to these results in Sections 4 and 5, in the next Section we show how our approach is implemented in a couple of canonical examples.

3 Two Examples

In this Section we illustrate our approach with two examples. In the first, there are only *intertemporal participation constraints*, so it is a case when $N_j = \infty$ (i.e. $k = l$); in the second, there are only *intertemporal one-period (Euler) constraints* and hence it is a case with $N_j = 0$ (i.e. $k = 0$). The first is similar to the model studied in Marcet and Marimon (1992), Kocherlakota (1996), Kehoe and Perri (2002), among others, and it is canonical of models with intertemporal default constraints; the second is based on the model studied by Aiyagari et al. (2002) and it is a canonical model with Euler constraints, as in Ramsey equilibria of optimal fiscal and monetary policy.

3.1 Intertemporal participation constraints.

We consider as an example a model of a partnership, where several agents can share their individual risks and jointly invest in a project which can not be undertaken by single (or subgroups of) agents. Formally, there is a single good and J infinitely-lived consumers. The preferences of agent j are represented by $E_0 \sum_{t=0}^{\infty} \beta^t \; u(c_t^j)$; u is assumed to be bounded, strictly concave and monotone, with $u(0) = 0$; c represents individual consumption. Agent j receives an endowment of consumption good y_t^j at time t and, given a realization of the vector y_t, agent j has an outside option that delivers total utility $v_j^a(y_t)$ if he leaves the contract in period t, where v_j^a is some known function. It is often assumed that the outside option is the autarkic solution: $v_j^a(y_t) = \mathrm{E}\left[\sum_{n=0}^{\infty} \beta^n \;\; u(y_{t+n}^j) \mid y_t^j\right]$, which implicitly assumes that if agent j defaults in period t he is permanently excluded from the partnership and he has no furhter claims on its production or capital in, or after, period t.

Total production is given by $F(k, \theta)$, and it can be split into consumption c and investment i. The stock of capital k depreciates at the rate δ. The joint process $\{\theta_t, y_t\}_{t=0}^{\infty}$ is assumed to be Markovian and the initial conditions (k_0, θ_0, y_0)

[10] In our previous work (Marcet and Marimon (1998, 1999)) we assumed uniqueness of solutions and used the fact that the contraction mapping theorem guarantees the uniqueness of the value function. Messner and Pavoni (2004) showed how the *principle of optimality* could fail in our context when solutions are not unique, the missing element being the recursivity of the forward-looking constraints. The above statement of the *principle of optimality* for problems with forward-llooking constraints and, correspondingly, our sufficiency theorems address this issue, which we further discuss in Section 6.

are given. The planner looks for pareto-optimal allocations that ensure that no agent ever leaves the contract. Letting $Y_t = \sum_{j=1}^{J} y_t^j > 0$, the planner's problem takes the form:

$$\textbf{PP} \qquad \max_{\{c_t, i_t\}} \ \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \ \sum_{j=1}^{J} \alpha^j \ u(c_t^j)$$
$$\text{s.t. } k_{t+1} = (1-\delta)k_t + i_t,$$
$$F(k_t, \theta_t) + Y_t - \left(\sum_{j=1}^{J} c_t^j + i_t \right) \geq 0, \text{ and}$$
$$\mathrm{E}_t \sum_{n=0}^{\infty} \beta^n \ u(c_{t+n}^j) \geq v_j^a(y_t) \qquad \text{for all } j, t \geq 0.$$

It is easy to map this planner's problem into our **PP** formulation. Let $s \equiv (\theta, y)$; $x \equiv k$; $a \equiv (i, c)$; $\ell(x,\, a,\, s) \equiv (1-\delta)k + i$; $p(x,\, a,\, s) \equiv F(k, \theta) + \sum_{j \in J} y^j - \left(\sum_{j \in J} c^j + i \right)$; $r(x,\, a,\, s) \equiv \sum_{j=1}^{J} \alpha^j \ u(c^j)$; $h_0^j(x,\, a,\, s) \equiv u(c^j)$; $h_1^j(x,\, a,\, s) \equiv u(c^j) - v_j^a(y_t)$, $j = 1, ..., J$. Problems $\mathbf{PP}_\mu$ and $\mathbf{SPP}_\mu$ are obtained mechanically by insuring that (5) is not binding for $j = 0$.

Finally we obtain the recursive formulation that we are seeking. **SPFE** takes the form[11]

$$W(k,\, \mu, y, \theta) = \inf_{\gamma \geq 0} \sup_{c,i} \{ \sum_{j=1}^{J} (\mu^0 \alpha^j + \mu^j) \ u(c^j) + \gamma^j \left(u(c^j) - v_j^a(y) \right)$$
$$+ \beta \ \mathrm{E} \left[W(k',\, \mu', y', \theta') \,|y, \theta \right] \}$$
$$\text{s.t. } k' = (1-\delta)k + i, \quad F(k, \theta) + \sum_{j=1}^{J} y^j - \left(\sum_{j=1}^{J} c^j + i \right) \geq 0$$
$$\text{and } \mu' = \mu + \gamma.$$

We know that $W(k,\, \mu, y, \theta) = V_\mu(k, y, \theta)$ solves this functional equation. Letting ψ be the policy function associated with it, solutions to **PP** satisfy

$$(c_t^*, i_t^*, \gamma_t^*) \;=\; \psi(k_t^*,\, \mu_t^*,\, \theta_t,\, y_t) \text{ and}$$
$$\mu_{t+1}^* \;=\; \mu_t^* + \gamma_t^*,$$

with initial conditions $(k_0, \mu_0,\, \theta_0,\, y_0)$, where $\mu_0 = (1, 0, ..., 0)$.

The planner would obtain the full commitment solution (subject to intertemporal participation constraints) from period t onwards if in period $t = 1$ he solved $\mathbf{PP}_{\mu_1^*}$ given initial conditions $(k_1^*, \theta_1,\, y_1)$, provided that the weights α of the agents were adjusted according to μ_1^*. Co-state variables μ_t^* become the additional weight that the planner should assign to each agent above the initial

[11] Here we incorporate the knowledge that $\gamma^{0*} = 0$.

weight α^j if the planner reoptimizes in period one. The variable μ_t is all that needs to be remembered from the past.

This recursive formulation allows easy computation of solutions using either first-order conditions or value function iteration. It also helps in characterizing the solution to the optimal problem: the weights μ_t^* evolve according to whether or not their participation constraints are binding. Every time that the participation constraint for an agent is binding, his weight is increased permanently by the amount of the corresponding Lagrange multiplier. An agent is induced not to default by increasing his consumption permanently, not only in the period where he is tempted to default, but smoothly over time.

Due to these changing weights, relative marginal utilities across agents are not constant when participation constraints are binding, since the first-order-conditions imply

$$\frac{u'(c_t^i)}{u'(c_t^j)} = \frac{\alpha^j + \mu_{t+1}^j}{\alpha^i + \mu_{t+1}^i} \quad \text{, for all } i, j \text{ and } t.$$

It follows that individual paths of consumption depend on individual histories (in particular, on past 'temptations to default') and not just on the initial wealth distribution and the aggregate consumption path, as in the Arrow-Debreu competitive allocations. This dependence on the past is completely summarized by μ_t (and, by homogeneity, the weights $\alpha^j + \mu_{t+1}^j$ can be normalized to add up to one). This also shows that if enforcement constraints are never binding (e.g. punishments are severe enough) then $\mu_t = \mu_0$ and we recover the "constancy of the marginal utility of expenditure", and the "constant proportionality between individual consumptions," given by $u'(c_t^i)/u'(c_t^j) = \alpha^j / \alpha^i$. In other words, the evolution of the co-state variables can also be interpreted as the evolution of the distribution of wealth. If intertemporal participation constraints are binding infinitely often there may be a non-degenerate distribution of consumption in the long-run; in contrast with an economy where intertemporal participation constraints cease to be binding, as in an economy with full enforcement.[12]

The evolution of the weights μ also helps to characterize the decision for capital: the intertemporal Euler equation of $\mathbf{SPP}_\mu$ is given by:

$$\mu_{t+1}^i u'(c_t^i) = \beta \; \mathrm{E}_t \left[\mu_{t+2}^i u'(c_{t+1}^i) \left(F_{k_{t+1}} + (1 - \delta) \right) \right].$$

That is, the 'stochastic discount factor' $\beta u'(c_{t+1}^i)/u'(c_t^i)$ is distorted by $\left(1 + \gamma_{t+1}^i / \mu_{t+1}^i\right)$, a distortion which does not vanish unless the non-negative process $\left\{\gamma_t^i\right\}$ converges to zero.

[12] See, for example, Broer (2009) for a characterization of the non-degenerate stationary distribution of consumption, in a similar model with a finite number of types and a continuum of agents of each type.

3.2 Intertemporal one-period constraints: a Ramsey problem

We present an abridged version of the optimal taxation problem studied by Aiyagari at al. (2002). A representative consumer solves

$$
\begin{aligned}
& \max_{\{c_t, e_t, b_{t+1}\}} E_0 \sum_{t=0}^{\infty} \delta^t \quad [u(c_t) + v(e_t)] \\
\text{s.t.} \quad & c_t + b_{t+1} p_t^b = e_t(1-\tau_t) + b_t \text{ and} \\
& b_{t+1} \geq \underline{B}, \text{ for a given } b_0,
\end{aligned}
$$

where c is consumption and e is effort (e.g. hours worked). The government must finance exogenous random expenditures g by issuing debt and collecting taxes. Feasible allocations satisfy $c_t + g_t = e_t$. The budget of the government mirrors the budget of the representative agent. For convenience we assume the government can not get too much in debt due to a constraint $b_{t+1} \geq \underline{B}$. In a competitive equilibrium, the following intertemporal and intratemporal equations must be satisfied (provided $b_{t+1} > \underline{B}$):

$$
\begin{aligned}
p_t^b u'(c_t) &= \beta \, E_t u'(c_{t+1}) \\
-\frac{v'(e_t)}{u'(c_t)} &= 1 - \tau_t.
\end{aligned}
$$

In a Ramsey equilibrium the government chooses sequences of taxes and debt that maximize the utility of the consumer subject to the allocations being a competitive equilibrium allocations. Substituting the above equilibrium equations into the budget constraint of the consumer, the Ramsey equilibrium can be found by solving

$$
\begin{aligned}
\mathbf{PP} \quad & \max_{\{c_t, b_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \; [u(c_t) + v(e_t)] \\
\text{s.t.} \quad & E_t [\beta b_{t+1} u'(c_{t+1})] = u'(c_t)(b_t - c_t) - e_t v'(e_t) \qquad (13) \\
& b_{t+1} \geq \underline{B}, \text{ for a given } b_0,
\end{aligned}
$$

where $e_t = c_t + g_t$ is left implicit. This problem can be represented as a special case of **PP** by letting $s \equiv g$; $x \equiv b$; $a \equiv (c, b')$; $p(x, a, s) \equiv b' - \underline{B}$; $\ell(x, a, s') \equiv b'$; $r(x, a, s) \equiv u(c) + v(e)$, $h_0^1(x, a, s) \equiv \beta b u'(c)$; $h_1^1(x, a, s) \equiv u'(c)(b-c) - e v'(e)$. Problems $\mathbf{PP}_\mu$ and $\mathbf{SPP}_\mu$ are then easily defined.

Finally, we obtain the recursive formulation that we are seeking. In its original notation, **SPFE** takes the form[13]

[13] Here we incorporate the knowledge that $\gamma^{0*} = 0$.

$$W(b,\,\mu,g) = \inf_{\gamma^1 \geq 0} \sup_{c,i} \{\mu^0\,[u(c_t)+v(e_t)] + \mu^1 b u'(c)$$
$$+\,\gamma^1\,[u'(c)(b-c) - e v'(e)] \qquad + \beta\,\mathrm{E}\,[W(b',\,\mu',\,g')\,|g\,]\}$$
$$\text{s.t.} \qquad b' \geq \underline{B},\ \mu^{0\prime} = \mu^0,\ \mu^{1\prime} = \gamma^1.$$

Letting ψ be the policy function associated with this functional equation, efficient allocations satisfy

$$(c_t^*, b_{t+1}^*, \gamma_t^{1*}) \,=\psi(b_t^*,\,\mu_t^*,\,g_t)$$

for $\mu_{t+1}^* = (1, \gamma_t^{1*})$ with initial conditions $(b_0, \mu_0,\, g_0)$, where $\mu_0 = (1,0)$.

It is clear that in this case the only element of μ that matters is the Lagrange multiplier γ_{t-1}. The planner would obtain the full commitment solution from period t onwards if in period $t=1$ he solved $\mathbf{PP}_{\mu_1^*}$ given initial conditions (b_1^*, g_1),; that is, if the objective function were modified to include the term $\gamma_0^1 b_1\ u'(c_1)$ in addition to the consumer's discounted utility from $t=1$ onwards. This term captures the commitment to enforcing the Euler equation (13) at $t=0$.

This recursive formulation allows easy computation of solutions using either first-order-conditions or value function iteration. It also helps charactere the solution to the optimal problem. The first-order conditions of the Ramsey problem imply that solutions satisfy

$$\mathrm{E}_t\left[(\gamma_t^1 - \gamma_{t+1}^1)u'(c_{t+1})\right] = 0. \tag{14}$$

As discussed in Aiyagari et al. (2002), with incomplete markets, this implies that $\left\{\gamma_{1,t}^*\right\}$ is a non-negative submartingale. Lagrange multipliers modify the weight given to debt relative to the complete markets case. The optimal policy can now be understood as forcing the planner in each period to modify the deadweight loss of taxation with weight γ_{t-1}^*.

4 The relationship between $\mathbf{PP}_\mu$, $\mathbf{SPP}_\mu$, and SPFE

This section proves the relationships between the initial maximization problem $\mathbf{PP}_\mu$, the saddle-point problem $\mathbf{SPP}_\mu$ and the saddle-point functional equation **SPFE** discussed in the previous Sections. We first describe the basic structure of the problems being considered.

4.1 Basic Structure

There exists an exogenous stochastic process $\{s_t\}_{t=0}^\infty$, $s_t \in S$, defined on the probability space $(S_\infty, \mathcal{S}, P)$. As usual, s^t denotes a history $(s_0, ..., s_t) \in S_t$ and $\mathcal{S}_t$ the σ-algebra of events of s^t; while $\{s_t\}_{t=0}^\infty \in S_\infty$, with $\mathcal{S}$ the corresponding σ-algebra. An action in period t, history s^t, is denoted by $a_t(s^t)$, where $a_t(s^t) \in$

$A \subset R^m$; when there is no confusion, it is simply denoted by a_t. Given s_t and the endogenous state $x_t \in X \subset R^n$, an action a_t is feasible if $p(x_t, a_t, s_t) \geq 0$. If the latter feasibility condition is satisfied, the endogenous state evolves according to $x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1})$. Plans, $\boldsymbol{a} = \{a_t\}_{t=0}^{\infty}$, are elements of $\mathcal{A} = \{\boldsymbol{a} : \forall t \geq 0,\, a_t : S_t \to A \text{ and } a_t \in \mathcal{L}_{\infty}^{m}(S_t, \mathcal{S}_t, P),\ \}$, where $\mathcal{L}_{\infty}^{m}(S_t, \mathcal{S}_t, P)$ denotes the space of m-valued, essentially bounded, $\mathcal{S}_t$-measurable functions. The corresponding endogenous state variables are elements of $\mathcal{X} = \{\boldsymbol{x} : \forall t \geq 0,\, x_t \in \mathcal{L}_{\infty}^{n}(S_t, \mathcal{S}_t, P)\}$.

Given initial conditions (x, s), a plan $\boldsymbol{a} \in \mathcal{A}$ and the corresponding $\boldsymbol{x} \in \mathcal{X}$, the evaluation of the plan in $\mathbf{PP}_\mu$ is given by

$$f_{(x,\mu.s)}(\boldsymbol{a}) = \mathrm{E}_0 \sum_{j=0}^{k} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t,\, a_t,\, s_t).$$

We can describe the forward-looking constraints by defining $g : \mathcal{A} \to \mathcal{L}_{\infty}^{k+1}$ coordinatewise as

$$g(\boldsymbol{a})_t^{\,j} = \mathrm{E}_t \left[\sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},\, a_{t+n},\, s_{t+n}) \right] + h_1^j(x_t,\, a_t,\, s_t).$$

Given initial conditions (x, s), the corresponding feasible set of plans is then

$$\begin{aligned} \mathcal{B}(x,s) \;=\; & \{\boldsymbol{a} \in \mathcal{A} :\; p(x_t, a_t,\, s_t) \geq 0,\; g(\boldsymbol{a})_t \geq 0,\, \boldsymbol{x} \in \mathcal{X}\ , \\ & x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}) \text{ for all } t \geq 0, \text{ given } (x_0, s_0) = (x, s)\}\,. \end{aligned}$$

Then $\mathbf{PP}_\mu$ can be written in compact form as

$$\mathbf{PP}_\mu \qquad \sup_{\boldsymbol{a} \in \mathcal{B}(x,s)} f_{(x,\mu.s)}(\boldsymbol{a}).$$

We denote solutions to this problem as $\boldsymbol{a}^*$ and the corresponding sequence of state variables $\boldsymbol{x}^*$. When the solution exists we define the value function of $\mathbf{PP}_\mu$ as

$$V_\mu(x, s) = f_{(x,\mu.s)}(\boldsymbol{a}^*) \tag{15}$$

Similarly, we can also write $\mathbf{SPP}_\mu$ in a compact form, by defining

$$\begin{aligned} \mathcal{B}'(x,s) \;=\; & \{\boldsymbol{a} \in \mathcal{A} : p(x_t, a_t,\, s_t) \geq 0,\, g(\boldsymbol{a})_{t+1} \geq 0;\; \boldsymbol{x} \in \mathcal{X} \\ & x_{t+1} = \ell(x_t,\, a_t,\, s_{t+1}) \text{ for all } t \geq 0, \text{ given } (x_0, s_0) = (x, s)\}\,. \end{aligned}$$

$$\mathbf{SPP}_\mu \qquad \inf_{\gamma \in R_+^l} \sup_{\boldsymbol{a} \in \mathcal{B}'(x,s)} \left\{ f_{(x,\mu,.s)}(\boldsymbol{a}) + \gamma g(\boldsymbol{a})_0 \right\}.$$

Note that $\mathcal{B}'$ only differs from $\mathcal{B}$ in that the forward-looking constraints in period zero $g(\boldsymbol{a})_0 \geq 0$ are not included as a condition in the set $\mathcal{B}'$, but instead these constraints form part of the objective function of $\mathbf{SPP}_\mu$.

4.2 Assumptions and existence of solutions to $\mathbf{PP}_\mu$

We consider the following set of assumptions:

A1. s_t takes values on a set $S \subset R^K$. $\{s_t\}_{t=0}^{\infty}$ is a Markovian stochastic process defined on the probability space $(S_\infty, \mathcal{S}, P)$.

A2. **(a)** $X \subset R^n$ and A is a closed subset of R^m. **(b)** The functions $p : X \times A \times S \to R$ and $\ell : X \times A \times S \to X$ are measurable and continuous.

A3. Given (x, s), there exist constants $B > 0$ and $\varphi \in (0, \beta^{-1})$, such that if $p(x, a, s) \geq 0$ and $x' = \ell(x, a, s')$, then $\|a\| \leq B \|x\|$ and $\|x'\| \leq \varphi \|x\|$.

A4. The functions $h_i^j(\cdot, \cdot, s)$, $i = 0, 1$, $j = 0, ..., l$, are continuous and uniformly bounded, and $\beta \in (0, 1)$.

A5. The function $\ell(\cdot, \cdot, s)$ is linear and the function $p(\cdot, \cdot, s)$ is concave. X and A are convex sets.

A6. The functions $h_i^j(\cdot, \cdot, s)$, $i = 0, 1$, $j = 0, ..., l$, are concave.

A6s. In addition to **A6**, the functions $h_0^j(x, \cdot, s), j = 0, ..., l$, are strictly concave.

A7. For all (x, s), there exists a program $\{\widetilde{a}_n\}_{n=0}^{\infty}$, with initial conditions (x, s), which satisfies the inequality constraints (4) and (5) with strict inequality.

Assumptions **A1** and **A2** are part of our basic structure, described in the previous sub-section. These assumptions, together with **A3-A4,** are standard and we treat them as our basic assumptions. Assumptions **A5-A7** are often made but they are not satisfied in some interesting models; however, these assumptions are only used in some of the results below. For example, the concavity assumptions **A5-A6** are not needed for many results, and assumption **A7** is a standard interiority assumption, only needed to guarantee the existence of Lagrange multipliers.

The following proposition gives sufficient conditions for a maximum to exist for any μ. The aim is not to have the most general existence theorem[14], but to stress that one can find fairly general conditions under which $\mathbf{PP}_\mu$ has a solution for any μ, which will be crucial in the discussion of how our approach compares with that of Abreu, Pearce and Stachetti, since this ensures that the continuation problem (namely $\mathbf{PP}_{\varphi(\mu,\gamma)}$) is well defined for any γ.

Proposition 1. Assume **A1-A6** and that the set of possible exogenous states S is countable. Fix $(x, \mu, s) \in X \times R_+^{l+1} \times S$. Assume there exists a feasible plan $\widetilde{\boldsymbol{a}} \in \mathcal{B}(x, s)$ such that $f_{(\mu,x,s)}(\widetilde{\boldsymbol{a}}) > -\infty$. Then there exists a program $\boldsymbol{a}^*$ which solves $\mathbf{PP}_\mu$ with initial conditions $x_0 = x$, $s_0 = s$.

[14]For example, not requiring **A6** or the countability of S, which will require additional assumptions.

Furthermore, if **A6s** is also satisfied then the solution is (almost surely) unique.

Proof: See Appendix.

4.3 The relationship between $\mathbf{PP}_\mu$ and $\mathbf{SPP}_\mu$

The following result says that a solution to the maximum problem is also a solution to the saddle point problem. It follows from the standard theory of constrained optimization in linear vector spaces (see, for example, Luenberger (1969, Section 8.3, Theorem 1 and Corollary 1). As in the standard theory, convexity and concavity assumptions (**A5 to A6**), as well as an interiority assumption (**A7**) are necessary to obtain the result.

Theorem 1 ($\mathbf{PP}_\mu \Longrightarrow \mathbf{SPP}_\mu$). Assume **A1-A7** and fix $\mu \in R_+^{l+1}$. Let $\boldsymbol{a}^*$ be a solution to $\mathbf{PP}_\mu$ with initial conditions (x, s). There exists a $\gamma^* \in R_+^l$ such that $(\boldsymbol{a}^*, \gamma^*)$ is a solution to $\mathbf{SPP}_\mu$ with initial conditions (x, s).

Furthermore, the value of $\mathbf{SPP}_\mu$ is the same as the value of $\mathbf{PP}_\mu$. more precisely:

$$V_\mu(x, s) = f_{(x,\mu,.s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \tag{16}$$

Proof: This is an immediate application of Theorem 1 (8.3) in Luenberger (1969), p. 217.

The following is a theorem on the sufficiency of a saddle point for a maximum.

Theorem 2 ($\mathbf{SPP}_\mu \Longrightarrow \mathbf{PP}_\mu$). Given any $(x,\, \mu,\, s) \in X \times R_+^{l+1} \times S$, let $(\boldsymbol{a}^*,\, \gamma^*)$ be a solution to $\mathbf{SPP}_\mu$ for initial conditions (x, s). Then $\boldsymbol{a}^*$ is a solution to $\mathbf{PP}_\mu$ for initial conditions $(x,\, s)$.

Furthermore, the value of the two programs is the same and (16) holds.

Notice that Theorem 2 is a sufficiency theorem 'almost free of assumptions.' All that is needed is the basic structure of section 4.1 defining the corresponding infinite-dimensional optimization and saddle-point problems *together with* the assumption that a solution to $\mathbf{SPP}_\mu$ exists. Once these conditions are satisfied assumptions **A2** to **A7** are not needed.

Proof: The following proof is an adaptation, to $\mathbf{SPP}_\mu$, of a sufficiency theorem for Lagrangian saddle points (see, for example, Luenberger (1969), Theorem 8.4.2, p.221).

If $(\boldsymbol{a}^*,\, \gamma^*)$ solves $\mathbf{SPP}_\mu$, minimality of γ^* implies that, for every $\gamma \geq 0$,

$$(\gamma^* + \gamma)\, g(\boldsymbol{a}^*)_0 \geq \gamma^* g(\boldsymbol{a}^*)_0;$$

therefore, $g(\boldsymbol{a}^*)_0 \geq 0$, but since $\boldsymbol{a}^* \in \mathcal{B}'(x, s)$, it follows that $\boldsymbol{a}^* \in \mathcal{B}(x, s)$; i.e. $\boldsymbol{a}^*$ is a feasible program for $\mathbf{PP}_\mu$. Furthermore, the minimality of γ^* implies that

$$\gamma^* g(\boldsymbol{a}^*)_0 \leq 0 g(\boldsymbol{a}^*)_0 = 0,$$

but since $\gamma^* \geq 0$ and $g(\boldsymbol{a}^*)_0 \geq 0$, it follows that $\gamma^* g(\boldsymbol{a}^*)_0 = 0$. Now, suppose there exists $\tilde{\boldsymbol{a}} \in \mathcal{B}(x,s)$ satisfying $f_{(x,\mu,s)}(\tilde{\boldsymbol{a}}) > f_{(x,\mu,s)}(\boldsymbol{a}^*)$. Then, since $\gamma^* g(\tilde{\boldsymbol{a}})_0 \geq 0$, it must be that

$$f_{(x,\mu,s)}(\tilde{\boldsymbol{a}}) + \gamma^* g(\tilde{\boldsymbol{a}})_0 > f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0,$$

which contradicts the maximality of $\boldsymbol{a}^*$ for $\mathbf{SPP}_\mu$.

Finally, using $\gamma^* g(\boldsymbol{a}^*)_0 = 0$, we have $f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 = V_\mu(x,s)$ ■

4.4 The relationship between $\mathbf{SPP}_\mu$ and SPFE

Recall that a function $W : X \times R_+^{l+1} \times S \to R$ satisfies **SPFE** at $(x,\ \mu,\ s)$ when

$$W(x,\ \mu,\ s) = \min_{\gamma \geq 0} \max_{a \in \mathcal{A}} \left\{ \mu h_0(x,\ a,\ s) + \gamma h_1(x,\ a,\ s) + \beta \ \mathrm{E}\left[W(x',\ \mu',\ s') |\ s \right] \right\} \tag{17}$$

$$\text{s.t. } x' = \ell(x,\ a,\ s), \quad p(x, a,\ s) \geq 0 \tag{18}$$

$$\text{and } \mu' = \varphi(\mu,\ \gamma). \tag{19}$$

In (17) we substituted inf sup with min max, implicitly assuming that a solution to the saddle point problem exists, in which case the value $W(x,\ \mu,\ s)$ is uniquely determined[15]. In other words, the right-hand side of **SPFE** is well defined for all (x, μ, s) and W for which a saddle point exists.

We say that W satisfies **SPFE** if it satisfies **SPFE** in *any possible state* $(x, \mu, s) \in X \times R_+^{l+1} \times S$. Given W, we define the *saddle-point policy correspondence* (*SP policy correspondence*) $\Psi : X \times R_+^{l+1} \times S \to A \times R_+^{l+1}$ by

$$\begin{aligned}
&\Psi_W(x,\ \mu,\ s) = \\
&\Big\{ (\,a^*, \gamma^*) : a^* \in \arg \max_{a \in \mathcal{A},\ x' \in X} \mu h_0(x,\ a,\ s) + \gamma^* h_1(x,\ a,\ s) + \beta \ \mathrm{E}\left[W(x',\ \mu^{*\prime},\ s') |\ s \right] \\
&\text{for } \mu^{*\prime} = \varphi(\mu,\ \gamma^*) \text{ and } (18); \\
&\gamma^* \in \arg \min_{\gamma \geq 0} \mu h_0(x,\ a^*,\ s) + \gamma h_1(x,\ a^*,\ s) + \beta \ \mathrm{E}\left[W(x^{*\prime},\ \mu',\ s') |\ s \right], \\
&\text{for } x^{*\prime} = \ell(x,\ a^*,\ s) \text{ and } (19) \Big\}.
\end{aligned}$$

If Ψ_W is single valued, we denote it by ψ_W, and we call it a *saddle-point policy function* (*SP policy function*).

We define the function $W^*(x, \mu, s) \equiv V_\mu(x, s)$. The following theorem says that W^* satisfies **SPFE.**

Theorem 3 ($\mathbf{SPP}_\mu \Longrightarrow$ SPFE). Assume that $\mathbf{SPP}_\mu$ has a solution for any $(x, \mu, s) \in X \times R_+^{l+1} \times S$. Then W^* satisfies **SPFE**. Furthermore, letting $(\boldsymbol{a}^*,\ \gamma^*)$ be a solution to $\mathbf{SPP}_\mu$ at (x, s), we have $(a_0^*, \gamma^*) \in \Psi_{W^*}(x,\ \mu,\ s)$.

[15]See Lemma 3A in Appendix B.

As in Theorem 2, Theorem 3 is also a theorem 'almost free of assumptions,' once the underlying structure and the existence of a well-defined solution to $\mathbf{SPP}_\mu$ at all possible (x, μ, s) is assumed.

Proof: By theorem 2, we have that whenever $\mathbf{SPP}_\mu$ has a solution W^* is well defined. Then, we first prove that, for any given (x, μ, s), if $(\boldsymbol{a}^*, \gamma^*)$ solves $\mathbf{SPP}_\mu$ at (x, s) the following recursive equation is satisfied:

$$W^*(x, \mu, s) = \mu h_0(x, a_0^*, s) + \gamma^* h_1(x, a_0^*, s) + \beta \,\mathrm{E}\left[W^*(x_1^*, \varphi(\mu, \gamma^*), s') | \, s\right] \tag{20}$$

To prove $\leq$ in (20) we write

$$\begin{aligned} W^*(x, \mu, s) &= f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\ &= \mu h_0(x, a_0^*, s) + \gamma^* h_1(x, a_0^*, s) \\ &\quad + \beta \mathrm{E}\left[\sum_{j=0}^{l} \varphi^j(\mu, \gamma^*) \sum_{t=0}^{N_j} \beta^t \; h_0^j(x_{t+1}^*, a_{t+1}^*, s_{t+1}) \mid s\right] \\ &= \mu h_0(x, a_0^*, s) + \gamma^* h_1(x, a_0^*, s) + \beta \mathrm{E}\left[f_{(x_1^*, \varphi(\mu, \gamma^*), s_1)}(\sigma \boldsymbol{a}^*) | \, s\right] \\ &\leq \mu h_0(x, a_0^*, s) + \gamma^* h_1(x, a_0^*, s) + \beta \mathrm{E}\left[V_{\varphi(\mu, \gamma^*)}(x_1^*, s_1) | \, s\right] \\ &= \mu h_0(x, a_0^*, s) + \gamma^* h_1(x, a_0^*, s) + \beta \mathrm{E}\left[W^*(x_1^*, \varphi(\mu, \gamma^*, s)) | \, s\right], \end{aligned}$$

where $\sigma \boldsymbol{a}^*$ is the original optimal sequence shifted one period; formally, letting the shift operator $\sigma : S^{t+1} \to S^t$ be given by $\sigma(s^t) = (s_1, s_2 ..., s_t)$, we define the $\mathcal{S}_{t+1}$-measurable function σa_t^* as $\sigma a_t^*(s) \equiv a_{t+1}^*(s)$. The first equality follows from the definition of W^*, and because Theorem 2 guarantees (16) the second equality follows from the definition of f, g and simple algebra. The third equality follows from the definitions of f, φ, and $\boldsymbol{a}^*$. The weak inequality follows from the fact that $\boldsymbol{a}^*$ is a feasible solution to the problem $\mathbf{PP}_{\varphi(\mu, \gamma^*)}$ with initial conditions (x_1^*, s_1) and that this program achieves $V_{\varphi(\mu, \gamma^*)}(x_1^*, s_1)$ at its maximum. The last equality follows from Theorem 2 and (16).

To show $\geq$ in (20) we construct a sequence $\boldsymbol{a}^+$ that consists of the optimal choice for $\mathbf{SPP}_\mu$ for initial conditions (x, s) in the initial period, but subsequently is followed by the optimal choices for $\mathbf{PP}_{\varphi(\mu, \gamma^*(x,\mu,s))}$ for initial conditions (x_1^*, s_1). To define $\boldsymbol{a}^+$ formally, we explicitly denote by $(\boldsymbol{a}^*(x, \mu, s), \gamma^*(x, \mu, s))$ a solution to $\mathbf{SPP}_\mu$ for given initial conditions (x, s) and we let

$$\begin{aligned} a_0^+(x, \mu, s) &= a_0^*(x, \mu, s) \text{ and} \\ a_t^+(x, \mu, s) &= \sigma a_{t-1}^*(x_1^*, \varphi(\mu, \gamma^*(x, \mu, s), s), \; s_1) \end{aligned}$$

for all (x, μ, s) and $t \geq 1$. Also, we let $\boldsymbol{x}^+$ be the corresponding sequence of state variables.

In what follows, we again simplify notation and go back to denoting $a_t^*(x,\mu,s)$ by a_t^*, $\gamma^*(x,\mu,s)$ by γ^*, and $a_t^+(x,\mu,s)(s^t)$ by a_t^+. Then, we have:

$$\begin{aligned}
W^*(x,\mu,s) &= f_{(x,\mu,s)}(\boldsymbol{a}^*)+\gamma^* g(\boldsymbol{a}^*)_0 \\
&\geq f_{(x,\mu,s)}(\boldsymbol{a}^+)+\gamma^* g(\boldsymbol{a}^+)_0 \\
&= \mu h_0(x,\, a_0^+,\, s)+\gamma^* h_1(x,\, a_0^+,\, s) \\
&\quad +\beta \mathrm{E}\left[\sum_{j=0}^{l}\varphi(\mu,\,\gamma^*)^j\sum_{t=0}^{N_j}\beta^t\ h_0^j(x_{t+1}^+,\, a_{t+1}^+,\, s_{t+1})\mid s\right] \\
&= \mu h_0(x,\, a_0^*,\, s)+\gamma^* h_1(x,\, a_0^*,\, s) \\
&\quad +\beta \mathrm{E}\left[W^*(x_1^*,\varphi\left(\mu,\,\gamma^*\right),s_1)\right|s],
\end{aligned}$$

where the first equality has been argued before, the first inequality follows from the fact that $\boldsymbol{a}^+(x,\mu,s)$ is a feasible allocation in $\mathbf{SPP}_\mu$ for initial conditions (x,s) but that $\boldsymbol{a}^*(x,\mu,s)$ is a solution to the max part to $\mathbf{SPP}_\mu$ at (x,s). The second equality just applies the definition of f and g, and the last equality follows because $\boldsymbol{a}^+$ is optimal for $\mathbf{PP}_{\varphi(\mu,\,\gamma^*(x,\mu),s)}$ given initial conditions (x_1^*,s_1) from period 1 onwards and because Theorem 2 ensures that (16) holds.

Notice that for this step of the proof it is crucial that we use $\mathbf{SPP}_\mu$ in order to obtain a recursive formulation. The first inequality above only works because we are considering a saddle point problem. Indeed, the $\boldsymbol{a}^+$ sequence (which reoptimizes in period $t=1$) is feasible for $\mathbf{SPP}_\mu$ because this problem does not impose the forward looking constraints in $t=0$. The sequence $\boldsymbol{a}^+$ would not be feasible in the original problem $\mathbf{PP}_\mu$, because by reoptimizing at period $t=1$ the forward-looking constraints at $t=0$ would be typically violated.

This ends the proof of (20).

To show that W^* satisfies **SPFE** we now prove that the right-hand side of SPFE is well defined at W^* and that (a_0^*,γ^*) is a saddle point of the right-hand side of (17) or, formally, that $(a_0^*,\gamma^*)\in\Psi_{W^*}(x,\,\mu,\,s)$.

We first prove that a_0^* solves the max part of the right-hand side of SPFE. Given any $\widetilde{a}\in A$, $p(x,\widetilde{a},s)\geq 0$, letting $\widetilde{a}_t^*(s^t)\equiv a_{t-1}^*(\ell(x,\,\widetilde{a},\,s'),\,\varphi(\mu,\,\gamma^*),\ s')(\sigma(s^t))$ for $t\geq 1$, the definition of $\widetilde{a}_t^*$, and (16) give the following first equality:

$$\begin{aligned}
&\mu h_0(x,\,\widetilde{a},\,s)+\gamma^* h_1(x,\,\widetilde{a},\,s)+\beta\mathrm{E}\left[W^*(\ell(x,\,\widetilde{a},\,s'),\,\varphi(\mu,\,\gamma^*),\ s')\right|s] \\
=\ &\mu h_0(x,\,\widetilde{a},\,s)+\gamma^* h_1(x,\,\widetilde{a},\,s) \\
&+\beta\mathrm{E}\left[\sum_{j=0}^{l}\varphi(\mu,\,\gamma^*)^j\sum_{t=0}^{N_j}\beta^t\ h_0^j(\widetilde{x}_{t+1}^*,\,\widetilde{a}_{t+1}^*,\,s_{t+1})\mid s\right] \\
=\ &f_{(x,\mu,s)}(\widetilde{\boldsymbol{a}}^*)+\ \gamma^* g(\widetilde{\boldsymbol{a}}^*)_0 \\
\leq\ &f_{(x,\mu,s)}(\boldsymbol{a}^*)+\ \gamma^* g(\boldsymbol{a}^*)_0 \\
=\ &W^*(x,\mu,s).
\end{aligned}$$

The second equality follows by definition, and the inequality holds because $(\boldsymbol{a}^*, \gamma^*)$ solves the max part of $\mathbf{SPP}_\mu$, while the third equality follows from (16). Now we can combine this with (20) to obtain that for all feasible $\widetilde{a} \in A$

$$\begin{aligned} & \mu h_0(x,\, \widetilde{a},\, s) + \gamma^* h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W^*(x_1^*,\, \varphi(\mu,\, \gamma^*),\; s')\middle|\, s\right] \\ \leq\; & \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta \mathrm{E}\left[W^*(x_1^*,\, \varphi(\mu,\, \gamma^*),\; s')\middle|\, s\right], \end{aligned}$$

implying that a_0^* solves the max part of the right-hand side of the **SPFE**.

A similar argument shows that γ^* solves the min part. For any $\widetilde{\gamma} \in R_+^{l+1}$ now let

$$\begin{aligned} & \mu h_0(x,\, a_0^*,\, s) + \widetilde{\gamma} h_1(x,\, a_0^*,\, s) + \beta \mathrm{E}\left[W^*(x_1^*,\, \varphi(\mu,\, \widetilde{\gamma}),\; s')\middle|\, s\right] \\ \geq\; & \mu h_0(x,\, a_0^*,\, s) + \widetilde{\gamma} h_1(x,\, a_0^*,\, s) + \\ & + \beta \mathrm{E}\left[\sum_{j=0}^{l} \varphi(\mu,\, \widetilde{\gamma})^j \sum_{t=0}^{N_j} \beta^t\; h_0^j(x_{t+1}^*,\, a_{t+1}^*,\, s_{t+1}) \mid s\right] \\ =\; & f_{(x,\mu,s)}(\boldsymbol{a}^*) + \widetilde{\gamma}\; g(\boldsymbol{a}^*)_0 \\ \geq\; & f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^*\; g(\boldsymbol{a}^*)_0 \\ =\; & W^*(x, \mu, s) \\ =\; & \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta \mathrm{E}\left[W^*(\ell(x,\, a_0^*,\, s'),\, \varphi(\mu,\, \gamma^*),\; s')\middle|\, s\right], \end{aligned}$$

where the inequality follows from the facts that shifting the policies one period back, the plan $\boldsymbol{a}^*$ is a feasible plan for the $\mathbf{PP}_{\varphi(\mu,\,\widetilde{\gamma})}$ problem with initial conditions (x_1^*, s') and that $W^*(x_1^*,\, \varphi(\mu,\, \widetilde{\gamma}),\; s')$ is the optimal value of $\mathbf{PP}_{\varphi(\mu,\,\widetilde{\gamma})}$. The second inequality follows because $(\boldsymbol{a}^*, \gamma^*)$ is a saddle point of $\mathbf{SPP}_\mu$ and the equalities follow from definitions, Theorem 2 and (20).

Therefore, γ^* solves the min part of the right side of **SPFE**.

Therefore (a_0^*, γ^*) is a saddle point of the right-hand side of **SPFE.** This implies the first equality in

$$\begin{aligned} & \min_{\gamma \geq 0} \max_{a \in \mathcal{A}} \left\{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta\; \mathrm{E}\left[W^*(x',\, \mu',\, s')\middle|\, s\right]\right\} \\ =\; & \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) + \beta\; \mathrm{E}\left[W^*(x_1^*,\, \varphi(\mu,\, \gamma^*),\, s')\middle|\, s\right] \\ =\; & W^*(x, \mu,\, s), \end{aligned}$$

and the second equality comes, again, from (20). This proves that W^* satisfies **SPFE.**■

The argument used in the proof of Theorem 3 can be iterated a finite number of times to show the underlying recursive structure of the $\mathbf{PP}_\mu$ formulation. If

$\mathbf{PP}_\mu$ has a unique solution $\{a_t^*\}_{t=0}^{\infty}$ at (x,s), then by Theorem 1 there is a $\mathbf{SPP}_\mu$ at (x,s) with solution $(\{a_t^*\}_{t=0}^{\infty}, \gamma^*)$, which in turn defines a $\mathbf{PP}_{\varphi(\mu,\,\gamma^*)}$ problem. As has been seen in the proof of Theorem 3, $\{a_t^*\}_{t=1}^{\infty}$ solves $\mathbf{PP}_{\varphi(\mu,\,\gamma^*)}$ at $(\ell(x,\, a_0^*,\, s), s_1)$ and by Theorem 1 there is a γ_1^* such that $(\{a_t^*\}_{t=1}^{\infty}, \gamma_1^*)$ solves $\mathbf{SPP}_{\varphi(\mu,\,\gamma^*)}$ at $(\ell(x,\, a_0^*,\, s), s_1)$. In turn, $\{a_t^*\}_{t=2}^{\infty}$ solves $\mathbf{PP}_{\varphi^{(2)}(\mu,\,\gamma^*)}$ at $(\ell^{(2)}(x,\, a_0^*,\, s), s_1)$, where $\varphi^{(2)}(\mu,\, \gamma^*) \equiv \varphi(\varphi(\mu,\, \gamma^*),\, \gamma_1^*,\, s_1)$ and $\ell^{(2)}(x,\, a_0^*,\, s) \equiv \ell(\ell(x,\, a_0^*,\, s),\, a_1^*,\, s_1)$. Similarly, let $\varphi^{(n+1)}(\mu,\, \gamma^*) \equiv \varphi(\varphi^{(n)}(\mu,\, \gamma^*),\, \gamma_n^*,\, s_n)$. Then by recursively applying the argument of the proof of Theorem 3 we obtain the following result.

Corollary 3.1. (Recursivity of $\mathbf{PP}_\mu$). If $\mathbf{PP}_\mu$ satisfies the assumptions of Theorem 1 and has a unique solution $\{a_t^*\}_{t=0}^{\infty}$ at (x,s), then, for any (t, x_t^*, s_t), $\left\{a_{t+j}^*\right\}_{j=0}^{\infty}$ is the solution to $\mathbf{PP}_{\varphi^{(t)}(\mu,\,\gamma^*)}$ at (x_t^*, s_t), where γ^* is the minimizer of $\mathbf{SPP}_\mu$ at (x,s).

The value function has some interesting properties that we would like to emphasize. First, notice that

$$\begin{aligned} W^*(x,\mu,s) &= f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\ &= f_{(x,\mu,s)}(\boldsymbol{a}^*) \\ &= \mathrm{E}_0 \sum_{j=0}^{l} \sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t^*,\, a_t^*,\, s_t). \end{aligned}$$

Therefore, if $\{a_t^*\}_{t=0}^{\infty}$ at (x,s) is uniquely defined, then W^* has a unique representation

$$\begin{aligned} W^*(x,\mu,s) &= \sum_{j=0}^{l} \mu^j \omega_j^*(x,\mu,s) \\ &= \mu\omega^*(x,\mu,s), \end{aligned}$$

where, for $j = 0, ...k, \omega_j(x,\mu,s) \equiv \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t h_0^j(x_t^*,\, a_t^*,\, s_t)$, and, for $j = k+1, ...l, \omega_j(x,\mu,s) \equiv h_0^j(x_0^*,\, a_0^*,\, s_0)$. Similarly, the value function of $\mathbf{SPP}_{\varphi(\mu,\,\gamma^*)}$ at (x_1^*, s_1), $x_1^* = \ell(x,\, a_0^*,\, s)$, satisfies

$$W^*(x_1^*,\, \varphi(\mu,\, \gamma^*),\, s_1) \equiv \varphi(\mu,\, \gamma^*)\omega^*(x_1^*,\, \varphi(\mu,\, \gamma^*),\, s_1).$$

This representation not only has an interesting economic meaning – for example, as a 'social welfare function,' with varying weights, in problems with intertemporal participation constraints – but is also very convenient analytically. In particular, this reprensentation shows[16] that W^* is *convex and homogenous of degree one* in μ, with $W^*(x,0,s) = 0$, for all (x,s)[17]. In addition, the following Corollary to Theorem 3 also shows that W^* satisfies what we call the

[16] See Lemma 2A in Appendix B.

[17] A function which is convex, homogeneous of degree one and finite at 0, is also called a *sublinear function* (see Rockafellar, 1981, p.29).

saddle-point inequality property **SPI**. Lemmas 1 and 2 below show how these properties are extended to general W functions satisfying **SPFE**.

A function $W(x,\mu,s) = \sum_{j=0}^{l} \mu^j \omega_j(x,\mu,s)$ satisfies the *saddle-point inequality property* **SPI** at $(x,\,\mu,\,s)$ if and only if there exist (a^*,γ^*) satisfying

$$\mu h_0(x,\,a^*,\,s) + \widetilde{\gamma} h_1(x,\,a^*,\,s) + \beta\,\mathrm{E}\left[\varphi(\mu,\,\widetilde{\gamma})\omega(x^{*\prime},\,\varphi(\mu,\,\gamma^*),\,s')|\,s\right]$$

$$\geq \mu h_0(x,\,a^*,\,s) + \gamma^* h_1(x,\,a^*,\,s) + \beta\,\mathrm{E}\left[\varphi(\mu,\,\gamma^*)\omega(x^{*\prime},\,\varphi(\mu,\,\gamma^*),\,s')|\,s\right]. \quad (21)$$

$$\geq \mu h_0(x,\,\widetilde{a},\,s) + \gamma^* h_1(x,\,\widetilde{a},\,s) + \beta\,\mathrm{E}\left[\varphi(\mu,\,\gamma^*)\omega(\widetilde{x}',\,\varphi(\mu,\,\gamma^*),\,s')|\,s\right], \quad (22)$$

for any $\widetilde{\gamma} \in R_+^{l+1}$ and $(\widetilde{a},\widetilde{x}')$ satisfying the technological constraints at $(x,\;s)$; that is, in **SPI** the multiplier minimization is taken in relation to the optimal continuation values.

Corollary 3.2. (SPP$_\mu$ $\Longrightarrow$SPI). Let $W^*(x,\mu,s) \equiv V_\mu(x,s)$ be the value of **SPP**$_\mu$ at (x,s), for an arbitrary $(x,\,\mu,\,s)$. Then $W^*(x,\mu,s) = \sum_{j=0}^{l} \mu^j \omega_j^*(x,\mu,s)$ satisfies **SPI**.

Proof: We only need to show that (21) is satisfied, but this is immediate from the following identities:

$$\begin{aligned} f_{(x,\mu,s)}(\boldsymbol{a}^*) &= \mu h_0(x,\,a_0^*,\,s) + \beta\,\mathrm{E}\left[\sum_{j=0}^{k} \mu^j \omega_j^*(x_1^*,\,\varphi(\mu,\,\gamma^*),\,s_1)|\,s\right] \\ \gamma g(\boldsymbol{a}^*)_0 &= \gamma\left[h_1(x,\,a_0^*,\,s) + \beta\,\mathrm{E}\left[\omega^*(x_1^*,\,\varphi(\mu,\,\gamma^*),\,s_1)|\,s\right]\right], \end{aligned}$$

and the definition of **SPP**$_\mu$ at (x,s); that is, for any $\widetilde{\gamma} \in R_+^{l+1}$,

$$\begin{aligned} & \mu h_0(x,\,a^*,\,s) + \widetilde{\gamma} h_1(x,\,a^*,\,s) + \beta\,\mathrm{E}\left[\varphi(\mu,\,\widetilde{\gamma})\omega^*(x^{*\prime},\,\varphi(\mu,\,\gamma^*),\,s')|\,s\right] \\ = \;& f_{(x,\mu,s)}(\boldsymbol{a}^*) + \widetilde{\gamma} g(\boldsymbol{a}^*)_0 \\ \geq \;& f_{(x,\mu,s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\ = \;& \mu h_0(x,\,a^*,\,s) + \gamma^* h_1(x,\,a^*,\,s) + \beta\,\mathrm{E}\left[\varphi(\mu,\,\gamma^*)\omega^*(x^{*\prime},\,\varphi(\mu,\,\gamma^*),\,s')|\,s\right]. \end{aligned}$$

■

We now show that, under fairly general conditions, programs satisfying **SPFE** are solutions to **SPP**$_\mu$ at (x,s). More formally,

Theorem 4 (SPFE $\Longrightarrow$ SPP$_\mu$) Assume W, satisfying **SPFE**, is continuous in (x,μ) and convex and homogeneous of degree one in μ. If the *SP policy correspondence* Ψ_W associated with W generates a solution $(\mathbf{a}^*,\boldsymbol{\gamma}^*)_{(x,\mu,s)}$, where $(\mathbf{a}^*)_{(x,\mu,s)}$ is uniquely determined, then $(\mathbf{a}^*,\boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is also a solution to **SPP**$_\mu$ at (x,s).

Notice that the assumptions on W are very general. In particular, if $W(x,\mu,s)$ is the value function of **SPP**$_\mu$ at (x,s) (i.e. $W(x,\mu,s) \equiv V_\mu(x,s)$) then (as Lemma 2A in Appendix B shows) it is convex and homogeneous of degree one in μ and, if **A2 - A5** are satisfied it is continuous and bounded in (x, μ). The only 'stringent condition' is that $(\mathbf{a}^*)_{(x,\mu,s)}$ must be uniquely determined, which is the case when W is concave in x and **A6s** is satisfied (see Corollary 4.1.).

Before proving these results, we show that, as we have seen for W^*, convex and homogeneous functions W satisfying **SPFE** have some interesting properties, which are used in the proof of Theorem 4. First, without loss of generality (see **F2** and **F3** in Appendix C), we can express the *recursive equation* (17) in the form

$$\begin{aligned}\mu\omega^d(x, \mu, s) &= \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) \\ &\quad +\beta \ \mathrm{E}\left[\varphi(\mu,\, \gamma)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s')|\, s\right], \qquad (23)\end{aligned}$$

where $\mu\omega^d(x, \mu, s) = W(x, \mu, s)$, and the vectors ω^d and $\omega^{d'}$are (partial) directional derivatives in μ of $W(x, \mu, s)$ and $W(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s')$, respectively. Therefore, the **SPFE** *saddle-point inequalities* take the form

$$\mu h_0(x,\, a^*,\, s) + \widetilde{\gamma} h_1(x,\, a^*,\, s) + \beta \ \mathrm{E}\left[\varphi(\mu,\, \widetilde{\gamma})\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \widetilde{\gamma}),\, s')|\, s\right]$$

$$\geq \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \ \mathrm{E}\left[\varphi(\mu,\, \gamma^*)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s')|\, s\right] \quad (24)$$

$$\geq \mu h_0(x,\, \widetilde{a},\, s) + \gamma^* h_1(x,\, \widetilde{a},\, s) + \beta \ \mathrm{E}\left[\varphi(\mu,\, \gamma^*)\omega^{d'}(\widetilde{x}',\, \varphi(\mu,\, \gamma^*),\, s')|\, s\right], \quad (25)$$

for any $\widetilde{\gamma} \in R_+^{l+1}$ and $(\widetilde{a}, \widetilde{x}')$ satisfying the technological constraints at $(x,\ s)$.

Second, as we show in Lemma 1, there is an equivalence between this **SPFE** property and the *saddle-point inequality property,* **SPI**, which substitutes (24) with

$$\mu h_0(x,\, a^*,\, s) + \widetilde{\gamma} h_1(x,\, a^*,\, s) + \beta \ \mathrm{E}\left[\varphi(\mu,\, \widetilde{\gamma})\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s')|\, s\right]$$

$$\geq \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \ \mathrm{E}\left[\varphi(\mu,\, \gamma^*)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s')|\, s\right]. \quad (26)$$

Third, as we show in Lemma 2, if in addition $(\mathbf{a}^*, \boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is uniquely determined, then W is differentiable in μ. Alternatively, if W is not differentiable in μ', then different choices of $\omega^{d'}$ can result in different solutions and the union of all these different solutions are the solutions to the saddle point problem, given by (26) and (25).

Lemma 1 (SPI $\Longleftrightarrow$ SPFE). If $W(x, \cdot, s)$ is convex and homogeneous of degree one, then (24) is satisfied if and only if (26) is satisfied. Furthermore, the inequality (26) is satisfied if and only if the following conditions are satisfied, for $j = 0, ..., l,$:

$$h_1^j(x,\, a_0^*,\, s) + \beta \ \mathrm{E}\left[\omega_j^{d'}(x_1^*,\, \mu_1^*,\, s_1)|\, s\right] \ \geq \ 0 \qquad (27)$$

$$\gamma^{*j}\left[h_1^j(x,\, a_0^*,\, s) + \beta \ \mathrm{E}\left[\omega_j^{d'}(x_1^*,\, \mu_1^*,\, s_1)|\, s\right]\right] \ = \ 0. \qquad (28)$$

Proof of Lemma 1: That **SPI** $\Longrightarrow$ **SPFE** follows from **F4** (see Appendix C). With respect to $W(x^{*\prime}, \varphi(\mu, \gamma), s')$, **F4** takes the form:

$$\varphi(\mu, \widetilde{\gamma})\omega^{d'}(x^{*\prime}, \varphi(\mu, \widetilde{\gamma}), s') \geq \varphi(\mu, \widetilde{\gamma})\omega^{d'}(x^{*\prime}, \varphi(\mu, \gamma^*), s').$$

Therefore, (26) together with this latter inequality results in the following inequalities, which show that (24) is satisfied whenever (26) is satisfied:

$$\begin{aligned} & \mu h_0(x, a^*, s) + \widetilde{\gamma} h_1(x, a^*, s) + \beta \operatorname{E}\left[\varphi(\mu, \widetilde{\gamma})\omega^{d'}(x^{*\prime}, \varphi(\mu, \widetilde{\gamma}), s') | \, s\right] \\ \geq \; & \mu h_0(x, a^*, s) + \widetilde{\gamma} h_1(x, a^*, s) + \beta \operatorname{E}\left[\varphi(\mu, \widetilde{\gamma})\omega^{d'}(x^{*\prime}, \varphi(\mu, \gamma^*), s') | \, s\right] \\ \geq \; & \mu h_0(x, a^*, s) + \gamma^* h_1(x, a^*, s) + \beta \operatorname{E}\left[\varphi(\mu, \gamma^*)\omega^{d'}(x^{*\prime}, \varphi(\mu, \gamma^*), s') | \, s\right]. \end{aligned}$$

To see that **SPFE** $\Longrightarrow$ **SPI,** let

$$G_{(x, a^*, s)}(\gamma, \mu) \equiv \mu h_0(x, a^*, s) + \gamma h_1(x, a^*, s) + \beta \operatorname{E}\left[\varphi(\mu, \gamma)\omega^{d'}(x^{*\prime}, \varphi(\mu, \gamma), s') | \, s\right],$$

and

$$F_{(x, a^*, s)}(\gamma, \mu) \equiv \mu h_0(x, a^*, s) + \gamma h_1(x, a^*, s) + \beta \operatorname{E}\left[\varphi(\mu, \gamma)\omega^{d'}(x^{*\prime}, \varphi(\mu, \gamma^*), s') | \, s\right].$$

Then, (24) reduces to $G_{(x, a^*, s)}(\gamma, \mu) \geq G_{(x, a^*, s)}(\gamma^*, \mu)$ and (26) to $F_{(x, a^*, s)}(\gamma, \mu) \geq F_{(x, a^*, s)}(\gamma^*, \mu)$. Since $G_{(x, a^*, s)}(\gamma^*, \mu) = F_{(x, a^*, s)}(\gamma^*, \mu)$, the above inequalities show that, if $f_{(x, a^*, s)}(\gamma^*, \mu) \in \partial_\gamma F_{(x, a^*, s)}(\gamma^*, \mu)$ for all $\gamma \geq 0$, then

$$\begin{aligned} G_{(x, a^*, s)}(\gamma, \mu) - G_{(x, a^*, s)}(\gamma^*, \mu) \quad \geq \quad & F_{(x, a^*, s)}(\gamma, \mu) - F_{(x, a^*, s)}(\gamma^*, \mu) \\ & (\gamma - \gamma^*) \, f_{(x, a^*, s)}(\gamma^*, \mu); \end{aligned}$$

that is, $f_{(x, a^*, s)}(\gamma^*, \mu) \in \partial_\gamma G_{(x, a^*, s)}(\gamma^*, \mu)$.

Now let $g_{(x, a^*, s)}(\gamma^*, \mu)$ be an extreme point of $\partial_\gamma G_{(x, a^*, s)}(\gamma^*, \mu)$. Since $G_{(x, a^*, s)}(\gamma, \mu)$ is homogenous of degree one in γ, it follows by **F2** (Appendix C) that there exists $\gamma_k \longrightarrow \gamma^*$, with G differentiable at γ_k and $\nabla G_{(x, a^*, s)}(y_k, \mu) \longrightarrow g_{(x, a^*, s)}(\gamma^*, \mu)$. By homogeneity of degree zero of $\omega^{d'}(x^{*\prime}, \mu', s')$ with respect to μ',

$$\nabla G_{(x, a^*, s)}(\gamma_k, \mu) = h_1(x, a^*(x, \mu, s), s) + \beta \operatorname{E}\left[\omega^{d'}(x^{*\prime}(x, \mu, s), \varphi(\mu, \gamma_k), s') | \, s\right].$$

Given the differentiability of $\nabla G_{(x, a^*, s)}(y_k, \mu)$ at γ_k, the continuity[18] of φ and $\omega^{d'}$ implies that

$$g_{(x, a^*, s)}(\gamma^*, \mu) = h_1(x, a^*(x, \mu, s), s) + \beta \operatorname{E}\left[\omega^{d'}(x^{*\prime}(x, \mu, s), \varphi(\mu, \gamma^*), s') | \, s\right],$$

[18] The continuity of $\omega^{d'}$ is given, for example, by Theorem 4F (& Corollary 4G) in Rockafellar (1981).

and, therefore, $g_{(x,\, a^*,\, s)}(\gamma^*, \mu) \in \partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$ – in fact, it is also an extreme point of $\partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu)$. This shows that $\partial_\gamma F_{(x,\, a^*,\, s)}(\gamma^*, \mu) = \partial_\gamma G_{(x,\, a^*,\, s)}(\gamma^*, \mu)$, which, in turn, implies the equivalence between (24) and (26).

Finally, the proof of the Kuhn-Tucker conditions is standard. First, the necessity of (27) follows from the fact that $\gamma^* \geq 0$ is finite, which will not be the case if, for some $j = 0, ..., l$,

$$h_1^j(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\omega_j^{d'}(x_1^*,\, \mu_1^*,\, s_1) |\, s\right] < 0.$$

To see the necessity of (28), let $\gamma_{(i)}^{*j} = \gamma^{*j}$, if $j \neq i$, and $\gamma_{(i)}^{*i} = 0$. Then (26) results in:

$$\mu h_0(x,\, a^*,\, s) + \gamma_{(i)}^* h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \gamma_{(i)}^*)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s') |\, s\right]$$

$$\geq \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\varphi(\mu,\, \gamma^*)\omega^{d'}(x^{*\prime},\, \varphi(\mu,\, \gamma^*),\, s') |\, s\right],$$

which, together with (27), implies that

$$0 \geq \gamma^{*j}\left[h_1^j(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\omega_j^{d'}(x_1^*,\, \mu_1^*,\, s_1) |\, s\right]\right] \geq 0.$$

To see that (27) and (28) imply (26), suppose they are satisfied and there exists a $\widetilde{\gamma} \geq 0$ for which (26) is not, then it must be that

$$\begin{aligned} & \widetilde{\gamma}\left[h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\omega^{d'}(x_1^*,\, \mu_1^*,\, s_1) |\, s\right]\right] \\ < \ & \gamma^*\left[h_1(x,\, a_0^*,\, s) + \beta\ \mathrm{E}\left[\omega^{d'}(x_1^*,\, \mu_1^*,\, s_1) |\, s\right]\right] = 0, \end{aligned}$$

which contradicts (27)■

Lemma 2. If $(\mathbf{a}^*, \boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is generated by $\Psi_W(x,\, \mu,\, s)$ and $(\mathbf{a}^*)_{(x,\mu,s)}$ is uniquely defined, then $W(x_t^*, \mu_t^*, s_t)$ is differentiable with respect to μ_t^*, for every (x_t^*, μ_t^*, s_t), with (x_t^*, μ_t^*) realized by[19] $(\mathbf{a}^*, \boldsymbol{\gamma}^*)_{(x,\mu,s)}$.

Proof of Lemma 2: By (28) the recursive equation (23) simplifies to

$$\mu\omega^d(x, \mu, s) = \mu h_0(x,\, a^*,\, s) + \beta\ \mathrm{E}\left[\sum_{j=0}^{k} \mu^j \omega_j^{d'}(x^{*\prime},\, \mu + \gamma^*),\, s') |\, s\right].$$

Assume, for the moment, that $(\mathbf{a}^*, \boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is uniquely determined. By recursive iteration, it follows that

[19]That is, $(x_0^*, \mu_0^*) \equiv (x, \mu)$, $x_{t+1}^* = \ell(x_t^*, a_t^*, s_{t+1})$ and $\mu_{t+1}^* = \varphi(\mu_t^*, \gamma_t^*)$.

$$\begin{aligned}\mu\omega^d(x_0,\mu_0,s_0) &= \mu h_0(x_0,\, a_0^*,\, s_0) \\ &\quad +\beta\ \mathrm{E}_0\left[\sum_{j=0}^{k}\mu^j\left(h_0^j(x_1^*,\, a_1^*,\, s_1)+\beta\omega_j^{d''}(x_2^*,\, \mu+\gamma_0^*+\gamma_1^*),\, s_2)\right)\mid s_0\right] \\ &= \mu h_0(x_0,\, a_0^*,\, s_0)+\beta\ \mathrm{E}_0\left[\sum_{j=0}^{k}\mu^j\sum_{t=1}^{\infty}\beta^t h_0^j(x_t^*,\, a_t^*,\, s_t)\mid s_0\right].\end{aligned}$$

Therefore, the uniqueness of $(\mathbf{a}^*,\boldsymbol{\gamma}^*)_{(x,\mu,s)}$ implies: *i)* $\omega^d(x,\, \mu,\, s)$ is uniquely defined: $\omega^d(x,\, \mu,\, s)=\omega(x,\, \mu,\, s)\equiv\nabla_\mu W(x,\, \mu,\, s)$, which, in turn, implies that $W(x,\cdot,s)$ is differentiable; and *ii)* $\omega_j(x,\, \mu,\, s)=\mathrm{E}_0\sum_{t=0}^{\infty}\beta^t h_0^j(x_t^*,\ a^*(x_t^*,\, \mu_t^*,\, s_t),\, s_t)$, for $j=0,\ldots k$ (with $(x_0^*,\, \mu_0^*,\, s_0)\equiv(x,\, \mu,\, s)$, $x_{t+1}^*=\ell(x_t^*,\, a^*(x_t^*,\, \mu_t^*,\, s_t),\, s_t)$, and $\mu_{t+1}^*=\mu^*+\gamma^*(x_t^*,\, \mu_t^*,\, s_t))$, and $\omega_j(x,\, \mu,\, s)=h_0^j(x,\, a^*(x,\, \mu,\, s),\, s)$, for $j=k+1\ldots l$

Given $(\mathbf{a}^*)_{(x,\mu,s)}$, suppose now $\left(\mathbf{a}^*,\widetilde{\boldsymbol{\gamma}}^*\right)_{(x,\mu,s)}$ is also generated by $\Psi_W(x,\, \mu,\, s)$. Both saddle-point paths must have the same value (see Lemma 3A in Appendix B). In particular, following the same recursive argument,

$$\begin{aligned}\mu\omega^d(x_0,\mu_0,s_0) &= \mu h_0(x_0,\, a_0^*,\, s_0)+\beta\ \mathrm{E}\left[\sum_{j=0}^{k}\mu^j\omega_j^{d'}(x^{*\prime},\, \mu+\widetilde{\gamma}^*),\, s')\mid s\right] \\ &= \mu h_0(x_0,\, a_0^*,\, s_0)+\beta\ \mathrm{E}_0\left[\sum_{j=0}^{k}\mu^j\sum_{t=1}^{\infty}\beta^t h_0^j(x_t^*,\, a_t^*,\, s_t)\mid s_0\right],\end{aligned}$$

which proves the differentiablity of W with respect to μ, even when $(\boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is not uniquely determined (i.e. there may be kinks in the Pareto frontier)■

An immediate, and important, consequence of Lemma 2 is the following result:

Corollary: If $(\mathbf{a}^*)_{(x,\mu,s)}$ is uniquely defined by $\Psi_W(x,\, \mu,\, s)$, from any initial condition (x,μ,s), then the following (recursive) equations are satisfied:

$$\omega_j(x,\, \mu,\, s)=h_0^j(x,\, a^*(x,\, \mu,\, s),\, s)+\beta\ \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')\mid s\right],$$
$$\text{if} \quad j=0,...,k, \quad \text{and} \tag{29}$$
$$\omega_j(x,\, \mu,\, s)=h_0^j(x,\, a^*(x,\, \mu,\, s),\, s) \quad \text{if} \quad j=k+1,...,l. \tag{30}$$

Furthermore, $(\mathbf{a}^*)_{(x,\mu,s)}$ is uniquely defined by $\Psi_W(x,\, \mu,\, s)$ whenever $W(\cdot,\, \mu,\, s)$ is concave and **A6s** is satisfied.

Notice that, in proving Lemma 2, the uniqueness of the solution paths has implied the uniqueness of the value function decomposition: $W = \mu\omega$. This unique decomposition has implied the recursive equations (29) and (30). Uniqueness of the value function decomposition is equivalent to the differentiability of the value function. In fact, once it has been established that the value function is differentiable, one can obtain equations (29) and (29) as a simple application of the *Envelope Theorem*. For example, equation (29) is just[20]:

$$\partial_j W(x,\,\mu,\,s) = h_0^j(x,\,a^*(x,\,\mu,\,s),\,s) + \beta\,\mathrm{E}\left[\partial_j W(x^{*\prime}(x,\,\mu,\,s),\,\mu^{*\prime}(x,\,\mu,\,s),\,s')|\,s\right].$$

We now turn to the proof of Theorem 4, where the recursive equations (29) and (30) play a key role.

Proof (Theorem 4): By Lemma 2, there is a unique representation $W(x,\,\mu,\,s) = \mu\omega(x,\,\mu,\,s)$. To see that solutions of **SPFE** satisfy the participation constraints of $\mathbf{SPP}_\mu$, we use the first-order-conditions (27) and (28), as well as the recursive equations of the *forward-looking* constraints (29) and (30) of the previous Corollary. As in the proof of Lemma 2, equation (29) can be iterated to obtain

$$\omega_j(x,\,\mu,\,s) = \mathrm{E}_0\left[\sum_{t=0}^{\infty}\beta^t h_0^j(x_t^*,\,a_t^*,\,s_t)|\,s\right],\ \text{if } j = 0,...,k. \tag{31}$$

Following the same steps for any $t > 0$ and state $(x_t^*,\,\mu_t^*,\,s_t)$, equation (30) and (31) together with the inequality (27) show that the intertemporal participation constraints in $\mathbf{PP}_\mu$ – and therefore in $\mathbf{SPP}_\mu$ – are satisfied; that is,

$$\mathrm{E}_t\sum_{n=1}^{N_j+1}\beta^n\,h_0^j(x_{t+n}^*,\,a_{t+n}^*,\,s_{t+n}) + h_1^j(x_t^*,\,a_t^*,\,s_t) \geq 0,\ ;\ t \geq 0, \quad j = 0,...,l. \tag{32}$$

Now, to see that solutions of **SPFE** are, in fact, solutions of $\mathbf{SPP}_\mu$ we argue by contradiction. Suppose there exist a program $\{\widetilde{a}_t\}_{t=0}^{\infty}$, and $\{\widetilde{x}_t\}_{t=0}^{\infty}$, $\widetilde{x}_0 = x, \widetilde{x}_{t+1} = \ell(\widetilde{x}_t,\,\widetilde{a}_t,\,s_{t+1})$ satisfying the constraints of $\mathbf{SPP}_\mu$ with initial condition (x, s) and such that

$$\begin{aligned}&\mu h_0(x,\,\widetilde{a}_0,\,s) + \gamma^* h_1(x,\,\widetilde{a}_0,\,s)\\ &+\beta\mathrm{E}\left[\sum_{j=0}^{k}\left(\mu^j+\gamma^{*j}\right)\sum_{n=1}^{\infty}\beta^t\,h_0^j(\widetilde{x}_t,\,\widetilde{a}_t,\,s_t) + \sum_{j=k+1}^{l}\gamma^{*j}h_0^j(\widetilde{x}_1,\,\widetilde{a}_1,\,s_1)|\,s\right]\\ &>\ \mu h_0(x,\,a_0^*,\,s) + \gamma^* h_1(x,\,a_0^*,\,s)\\ &+\beta\mathrm{E}\left[\sum_{j=0}^{k}\left(\mu^j+\gamma^{*j}\right)\sum_{n=1}^{\infty}\beta^t\,h_0^j(x_t^*,\,a_t^*,\,s_t) + \sum_{j=k+1}^{l}\gamma^{*j}h_0^j(x_1^*,\,a_1^*,\,s_1)|\,s\right].\end{aligned} \tag{33}$$

[20]We use the standard notation $\partial_j W(x,\,\mu,\,s) \equiv \frac{\partial W(x,\mu,s)}{\partial \mu_i}$, and also $\omega_j(x,\,\mu,\,s) \equiv \partial_j W(x,\,\mu,\,s)$.

The following string of equalities and inequalities, which we explain at the end, contradict this inequality:

$$
\begin{aligned}
&\mu h_0(x,\, a_0^*,\, s) + \gamma_0^* h_1(x,\, a_0^*,\, s) \\
&+ \beta \mathrm{E}\left[\sum_{j=0}^{k} \left(\mu^j + \gamma^{*j}\right) \sum_{n=1}^{\infty} \beta^t \; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l} \gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1) |\, s \right] \\
&= \mu h_0(x,\, a_0^*,\, s) + \gamma_0^* h_1(x,\, a_0^*,\, s) + \beta \, \mathrm{E}\left[\mu_1^* \omega(x_1^*,\, \mu_1^*,\, s_1) |\, s\right] && (34) \\
&\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) + \beta \, \mathrm{E}\left[\mu_1^* \omega(\widetilde{x}_1,\, \mu_1^*,\, s_1) |\, s\right] && (35) \\
&= \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) \\
&+ \beta \, \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, a^*(\widetilde{x}_1,\, \mu_1^*, s_1),\, s_1) + \gamma^*(\widetilde{x}_1,\, \mu_1^*, s_1) h_1(\widetilde{x}_1,\, a^*(\widetilde{x}_1,\, \mu_1^*, s_1),\, s_1)\right. && (36) \\
&+ \left. \beta \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \omega(x^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1),\, \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1), s_2) |\, s\right] \\
&\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) \\
&+ \beta \, \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \gamma^*(\widetilde{x}_1,\, \mu_1^*, s_1) h_1(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1)\right. && (37) \\
&+ \left. \beta \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \omega(\widetilde{x}_2,\, \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1), s_2) |\, s\right] \\
&\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) \\
&+ \beta \, \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \gamma^*(\widetilde{x}_1,\, \mu_1^*, s_1) h_1(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \beta \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \omega(\widetilde{x}_2,\; \mu_1^*, s_2) |\, s\right] && (38) \\
&\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) \\
&+ \beta \, \mathrm{E}\left[\mu_1^* \left[h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \beta \omega(\widetilde{x}_2,\; \mu_1^*, s_2)\right] |\, s\right] && (39) \\
&= \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) + \beta \, \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) |\, s\right] \\
&+ \beta^2 \, \mathrm{E}\left[\mu_1^* h_0(\widetilde{x}_2,\, a^*(\widetilde{x}_2,\, \mu_1^*, s_2),\, s_2) + \gamma^*(\widetilde{x}_2,\, \mu_1^*, s_2) h_1(\widetilde{x}_2,\, a^*(\widetilde{x}_2,\, \mu_1^*, s_2),\, s_2)\right. && (40) \\
&+ \left. \beta \mu^{*\prime}(\widetilde{x}_2,\, \mu_1^*, s_2) \omega(x^{*\prime}(\widetilde{x}_2,\, \mu_1^*, s_2),\, \mu^{*\prime}(\widetilde{x}_2,\, \mu_1^*, s_2), s_2) |\, s\right] \\
&\geq \mu h_0(x,\, \widetilde{a}_0,\, s) + \gamma_0^* h_1(x,\, \widetilde{a}_0,\, s) \\
&+ \beta \mathrm{E}\left[\sum_{j=0}^{k} \left(\mu^j + \gamma^{*j}\right) \sum_{t=1}^{\infty} \beta^t \; h_0^j(\widetilde{x}_t^{;} \widetilde{a}_t,\, s_t) + \sum_{j=k+1}^{l} \gamma^{*j} h_0^j(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) |\, s \right]. && (41)
\end{aligned}
$$

Notice that the first equality (34) is just uses the value function decomposition, the other two equalities (36) and (40) are simple expansions of the saddle-point value paths (i.e., of (23)) and in these expansions equations (29) and (30) play a key role. Inequalities (35) and (37) follow from the maximality property of **SPFE.** Inequalities (38) and (39) require explanation. Inequality (38) follows from one of the properties of convex and homogeneous of degree one functions (i.e. **F4**: $\widehat{\mu}\omega(\widehat{\mu}) \geq \widehat{\mu}\omega(\mu)$, see Appendix), given that (38) is simply $\mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1)\omega(\widetilde{x}_2,\, \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1) \geq \mu^{*\prime}(\widetilde{x}_1,\, \mu_1^*, s_1)\omega(\widetilde{x}_2,\; \mu_1^*, s_2)$. Inequality (39) follows from applying the slackness inequality (27), as well as equations (30) and (31) to the plan generated by **SPFE** in state $(\widetilde{x}_2,\; \mu_1^*, s_2)$ (i.e. to $\{a_t^*(\widetilde{x}_2,\; \mu_1^*, s_2)\}_{t=2}^{\infty}$); these inequalities are needed to show that this plan satisfies the corresponding **SPP** constraints (32); that is, $\left[h_1^j(\widetilde{x}_1,\, \widetilde{a}_1,\, s_1) + \beta\omega_j(\widetilde{x}_2,\; \mu_1^*, s_2)\right] \geq 0$, $j = 0, ..., l$. Finally, since the equality (40) is simply the equality (36) after one iteration, repeated iterations result in the last inequality (41), which contradicts (33).

It only remains to be shown that the inf part of **SPP** is also satisfied. Reasoning again by contradiction, suppose there exist a $\widetilde{\gamma} \geq 0$ such that

$$\begin{aligned}
& \mu h_0(x,\, a_0^*,\, s) + \widetilde{\gamma} h_1(x,\, a_0^*,\, s) \\
& + \beta \mathrm{E}\left[\sum_{j=0}^{k}\left(\mu^j + \widetilde{\gamma}^j\right)\sum_{n=1}^{\infty}\beta^t\; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l}\widetilde{\gamma}^j h_0^j(x_1^*,\, a_1^*,\, s_1)|\, s\right] \\
< \;& \mu h_0(x,\, a_0^*,\, s) + \gamma^* h_1(x,\, a_0^*,\, s) \\
& + \beta \mathrm{E}\left[\sum_{j=0}^{k}\left(\mu^j + \gamma^{*j}\right)\sum_{n=1}^{\infty}\beta^t\; h_0^j(x_t^*,\, a_t^*,\, s_t) + \sum_{j=k+1}^{l}\gamma^{*j} h_0^j(x_1^*,\, a_1^*,\, s_1)|\, s\right].
\end{aligned} \tag{42}$$

Using the value function decomposition representation, this inequality can also be expressed as

$$\begin{aligned}
& \widetilde{\gamma}\left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta\; \mathrm{E}\left[\omega(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')|\, s\right]\right] \\
< \;\; & \gamma^*(x,\, \mu,\, s)\left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta\; \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')|\, s\right]\right],
\end{aligned}$$

but the first-order-conditions (27) and (28) require that (26) is satisfied, i.e.

$$\begin{aligned}
& \widetilde{\gamma}\left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta\; \mathrm{E}\left[\omega(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')|\, s\right]\right] \\
\geq \;\; & \gamma^*(x,\, \mu,\, s)\left[h_1(x,\, a^*(x,\, \mu,\, s),\, s) + \beta\; \mathrm{E}\left[\omega_j(x^{*\prime}(x,\, \mu,\, s),\, \mu^{*\prime}(x,\, \mu,\, s),\, s')|\, s\right]\right] = 0,
\end{aligned}$$

which contradicts (42)■

The Corollary to Lemma 2 implies the following Corollary to Theorem 4:

Corollary 4.1. Assume W, satisfying **SPFE,** is continuous in (x, μ), convex and homogeneous of degree one in μ, concave in x and that **A6s** is satisfied**.** If $(\mathbf{a}^*, \boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is generated by $\Psi_W(x,\, \mu,\, s)$ then $(\mathbf{a}^*, \boldsymbol{\gamma}^*)_{(x,\mu,s)}$ is also a solution to $\mathbf{SPP}_\mu$ at (x, s).

5 DSPP and the contraction mapping theorem

In this Section we show how our main results – Theorems 3 and 4 – can also be obtained by applying the *Contraction Mapping Theorem* to the *Dynamic Saddle-Point Problem,* corresponding to **SPFE**. This Section provides more general sufficient conditions for obtaining a solution to the original problem $\mathbf{PP}_\mu$ starting from **SPFE**. While these conditions are satisfied whenever the conditions of Theorem 4 are satisfied, they help to better understand the passage **SPFE**→$\mathbf{PP}_\mu$ and, in particular, they show how the standard method of value function iteration extends to our *saddle-point problems* and, therefore, that computing solutions to our original $\mathbf{PP}_\mu$ does not require special computational techniques. Furthermore, it also shows the interest of using the $W = \mu\omega$ representation in computing recursive contracts (i.e. taking ω as the starting vector valued function) and how, in contrast with the 'promise keeping' approach to solving contractual problems, 'promised values' are not part of the constraints, but an outcome of the recursive contract[21].

We first define some spaces of "value" functions:

$$\begin{aligned}\mathcal{M}_b = &\{W : X \times \mathcal{R}_+^{l+1} \times S \to \mathcal{R} \\ &\quad i) W(\cdot,\,\cdot,\,s) \text{ is continuous, and } W(\cdot,\,\mu,\,s) \text{ bounded, when } \|\mu\| \leq 1, \\ &\quad ii) W(x,\,\cdot,\,s) \text{ is convex and homogeneous of degree one}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{bc} = &\{W \in \mathcal{M}_b \text{ and} \\ &\quad iii) W(\cdot,\,\mu,\,s) \text{ is concave}\}.\end{aligned}$$

$\mathcal{M}_b$ is a space of continuous, bounded functions (in x), and convex and homogenous of degree one (in μ)[22], while $\mathcal{M}_{bc}$ is the subspace of concave functions (in x). Both spaces are normed vector spaces with the norm

$$\|W\| = \sup\left\{|W(x, \mu, s)| : \|\mu\| \leq 1, x \in X, s \in S\right\}.$$

We show in Appendix D (Lemma 6A) that they are complete metric spaces; therefore, suitable spaces for the *Contraction Mapping Theorem.*

Since, whenever W satisfies (ii) it can be represented as $W(x,\,\mu,\,s) = \mu\omega(x,\,\cdot,\,s)$ (see Lemma 4A), it is convenient to define the corresponding spaces of the functions:

$$\begin{aligned}M_b = &\{\omega : X \times \mathcal{R}_+^{l+1} \times S \to \mathcal{R}^{l+1} \text{ s.t., for } j = 0, ..., l, \\ &\quad i) \omega_j(\cdot,\,\cdot,\,s) \text{ is continuous, and } \omega_j(\cdot,\,\mu,\,s) \text{ bounded, when } \|\mu\| \leq 1 \\ &\quad ii) \omega_j(x,\,\cdot,\,s) \text{ is convex and homogeneous of degree zero}\}\end{aligned}$$

and

$$\begin{aligned}M_{bc} = &\{\omega \in M_b \text{ s.t., for } j = 0, ..., l, \\ &\quad iii) \omega_j(\cdot,\,\mu,\,s) \text{ is concave}\}.\end{aligned}$$

Notice that $\omega \in M$ uniquely defines a function $W \in \mathcal{M}$, given by $W \equiv \mu\omega$, but $W \in \mathcal{M}$ does not uniquely define a $\mathcal{R}^{l+1}$ valued function $\omega \in M$; it does, however, when, in addition, W is differentiable in μ (see Appendix C)[23].

[21] We further discuss the 'promise keeping' approach in Section 6.

[22] Without loss of generality, we could also require that $W(x, 0, s) < \infty$ and then replace (ii) with $W(x,\,\cdot,\,s)$ is *sublinear* (see footnote 12).

[23] M denotes either M_b or M_{bc}.

As we have seen in Section 4[24], when $W^*(x,\mu,s) = V_\mu(x,s)$ is the value of $\mathbf{SPP}_\mu$, with initial conditions (x,s), then $W^*(x,\mu,s) = \sum_{j=0}^{l} \mu^j \omega_j^*(x,\mu,s)$ with $W^* \in \mathcal{M}_b$, whenever **A2** - **A4** are satisfied (and $W^* \in \mathcal{M}_{bc}$ if in addition **A5** - **A6** are satisfied); furthermore, $\omega^* \in M$ is unique whenever $(\mathbf{a}^*)_{(x,\mu,s)}$ is uniquely defined.

Given a function $\omega \in M$, and an initial condition (x,μ,s), we can define the following *Dynamic Saddle Point Problem*:

DSPP

$$\begin{aligned} &\inf_{\gamma \geq 0} \sup_{a} \{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta \, \mathrm{E} \left[\mu' \omega(x',\, \mu',\, s') |\, s \right] \} \\ &\text{s.t. } x' = \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0 \\ &\text{and } \mu' = \varphi(\mu,\, \gamma). \end{aligned}$$

To guarantee that this problem has well-defined solutions we make an interiority assumption:

A7b. For any $(x,s) \in X \times S$, there exists an $\widetilde{a} \in A$, satisfying $p(x,a,\,s) > 0$, such that, for any $\mu' \in R_+^{l+1}, \|\mu'\| < +\infty$, and $j = 0,...,l$, $h_1^j(x,\widetilde{a},s) + \beta \mathrm{E} \left[\omega^j(\ell(x,\, \widetilde{a},\, s'), \mu', s') |\, s \right] > 0$.

Notice that **A7b** is satisfied, whenever **A7** is satisfied and $\mu'\omega(\ell(x,\, \widetilde{a},\, s'), \mu', s')$ is the value function of $\mathbf{SPP}_{(\ell(x,\widetilde{a},\,s'),\mu',s')}$. In general, **A7b** is not a restrictive assumption in the class of possible value functions if the original problem has interior solutions. Nevertheless, an assumption, such as **A7**b is needed when one takes $\mathbf{DSPP}_{(x,\mu,s)}$ as the starting problem. This is a relatively standard $\min\max$ problem, except for the dependency of ω on $\varphi(\mu,\, \gamma)$. The following proposition shows that it has a solution. Obviously, solutions to $\mathbf{DSPP}_{(x,\mu,s)}$ satisfy **SPFE.** . An immediate consequence of **A7b**, is the following lemma:

Lemma 3. Assume **A4** and **A7b** and let $\omega \in M_b$. There exists a $B > 0$ such that if $(a^*(x,\mu,s), \gamma^*(x,\mu,s))$ is a solution to **DSPP** at (x,μ,s), then $\|\gamma^*(x,\mu,s)\| \leq B \, \|\mu\|$.

Proof: Denote by $(a^*, \gamma^*) \equiv (a^*(x,\mu,s), \gamma^*(x,\mu,s))$ the solution to **DSPP** at

[24] See also Lemma 2A in Appendix B.

(x, μ, s), and let $\widetilde{a}$ be the interior solution of **A7b**. Then

$$\begin{aligned}
& \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \text{ E}\left[\mu' \omega(\ell(x,\, a^*, s),\, \varphi(\mu,\, \gamma^*),\, s') | \, s\right] \\
= \;& \mu h_0(x,\, a^*,\, s) + \beta \text{ E}\left[\sum_{j=0}^{k} \mu^j \omega_j(\ell(x,\, a^*, s),\, \varphi(\mu,\, \gamma^*),\, s') | \, s\right] \\
+ \;& \gamma^* \left[h_1(x,\, a^*,\, s) + \beta \text{ E}\left[\omega(\ell(x,\, a^*, s),\, \varphi(\mu,\, \gamma^*),\, s') | \, s\right]\right] \\
= \;& \mu h_0(x,\, a^*,\, s) + \beta \text{ E}\left[\sum_{j=0}^{k} \mu^j \omega_j(\ell(x,\, a^*, s),\, \varphi(\mu,\, \gamma^*),\, s') | \, s\right] \\
\geq \;& \mu h_0(x,\, \widetilde{a},\, s) + \beta \text{ E}\left[\sum_{j=0}^{k} \mu^j \omega_j(\ell(x,\, \widetilde{a},\, s),\, \varphi(\mu,\, \gamma^*),\, s') | \, s\right] \\
+ \;& \gamma^* \left[h_1(x,\, \widetilde{a},\, s) + \beta \text{ E}\left[\omega(\ell(x,\, \widetilde{a}, s),\, \varphi(\mu,\, \gamma^*),\, s') | \, s\right]\right].
\end{aligned}$$

By assumption,

$$(\mu / \|\mu\|) h_0(x,\, a^*,\, s) + \beta \text{ E}\left[\sum_{j=0}^{k} (\mu^j / \|\mu\|) \omega_j(\ell(x,\, a^*, s),\, \varphi((\mu / \|\mu\|),\, (\gamma^* / \|\mu\|)),\, s') | \, s\right]$$

is uniformly bounded (**A4** and $\omega \in M_b$ imply that there is uniform bound for the max value), while if $(\gamma^{*j} / \|\mu\|) > 0$ then

$$(\gamma^{*j} / \|\mu\|) \left[h_1^j(x,\, \widetilde{a},\, s) + \beta \text{E}\left[\omega_j(\ell(x,\, \widetilde{a}, s),\, \varphi((\mu / \|\mu\|),\, (\gamma^* / \|\mu\|)),\, s') | \, s\right]\right] > 0.$$

Therefore, there must be a $B > 0$ such that $\|\gamma^*\| \leq B \|\mu\|$■

Proposition 2. Let $\omega \in M_{bc}$ and assume **A1-A6** and **A7b.** There exists (a^*, γ^*) that solves $\mathbf{DSPP}_{(x,\mu,s)}$. Furthermore if **A6s** is assumed, then $a^*(x, \mu, s)$ is uniquely determined.

Proof: This is a relatively standard proof of existence of an equilibrium, based on a fixed point argument; see Appendix D.

The following Corollary to Theorem 3, is a simple restatement of the theorem in terms of the *Dynamic Saddle Point Problem*:

Corollary 3.3. **($\mathbf{SPP}_\mu(x, s) \Longrightarrow \mathbf{DSPP}_{(x,\mu,s)}$).** Assume that $\mathbf{SPP}_\mu$ at (x, s) has a solution $(\boldsymbol{a}^*,\, \gamma^*)$ with value $V_\mu(x, s) = \sum_{j=0}^{l} \mu^j \omega_j^*(x, \mu, s)$. Then (a_0^*, γ^*) solves $\mathbf{DSPP}_{(x,\mu,s)}$.

When $\mathbf{DSPP}_{(x,\mu,s)}$ has a solution, it defines a **SPFE** operator $T^* : \mathcal{M} \longrightarrow \mathcal{M}$ given by

$$(T^*W)(x,\mu,s) = \min_{\gamma\geq 0}\max_{a}\{\mu h_0(x,\,a,\,s) + \gamma h_1(x,\,a,\,s) + \beta\,\mathrm{E}\,[W(x',\,\mu',\,s')|\,s]\}$$

s.t. $x' = \ell(x,\,a,\,s),\; p(x,a,\,s) \geq 0$
and $\mu' = \varphi(\mu,\,\gamma)$,

When $W \equiv \mu\omega$, with $\omega \in M$, and $\mathbf{DSPP}_{(x,\mu,s)}$ uniquely defines the values $h_0^j(x,\,a^*(x,\,\mu,\,s),\,s)$, $j = 0,...,l$, then T^* defines a mapping, $T : M \longrightarrow M$, given by

$$(T\omega_j)(x,\,\mu,\,s) = h_0^j(x,\,a^*(x,\,\mu,\,s),\,s) + \beta\,\mathrm{E}\,[\omega_j(x^{*\prime}(x,\,\mu,\,s),\,\mu^{*\prime}(x,\,\mu,\,s),\,s')|\,s]\,, \tag{43}$$

if $j = 0,...,k$, and

$$(T\omega_j)(x,\,\mu,\,s) = h_0^j(x,\,a^*(x,\,\mu,\,s),\,s),\text{ if } j = k+1,...,l. \tag{44}$$

Two remarks are in order. First, as already said, notice that $(T\omega_j)$ corresponds to the 'promise keeping' approach to solving contractual problems but in our approach $(T\omega_j)$ is not a constraint: it is an outcome. Second, a fixed point of T^* does not imply a fixed point of T when 'the planner' is indifferent to T reallocations (e.g. $\mu_i = \mu_j$, $\mu_i(T\omega_i) + \mu_j(T\omega_j) =$ constant) resulting in multiple (indeterminate) continuation values (for i and j)[25]

Proposition 3. Assume **DSPP** has a solution for any $\omega \in M$ and (x,μ,s). Then $T^* : \mathcal{M} \to \mathcal{M}$ is a well-defined contraction mapping. Let $W^* = T^*(W^*)$ and $W^* = \mu\omega^*$. If in addition the solutions $a^*(x,\,\mu,\,s)$ to **DSPP** are unique, then $\omega^* = T(\omega^*)$ is unique.

Proof: The first part follows from showing that Blackwell's sufficiency conditions for a contraction are satisfied for T^* (see Lemmas 7A to 10A in Appendix D); the second part from the definition of T.

Our last Theorem, Theorem 5, wraps up our sufficiency results and is, in fact, a Corollary to Theorem 4. It shows how, starting from a *Dynamic Saddle-Point Problem* and a corresponding well defined *Contraction Mapping* resulting in a unique value function, one obtains the solution to our original problem $\mathbf{PP}_\mu$. The previous Propositions 2 and 3 provide conditions guaranteeing that the assumptions of Theorem 5, regarding T, are satisfied.

Theorem 5 ($\mathbf{DSPP}_{(x,\mu,s)} \Longrightarrow \mathbf{SPP}_\mu(x,s)$). Assume $T : M \to M$ has a unique fixed point ω^*. Then the *value function* $W^*(\mu,x,s) = \mu\omega^*(x,\mu,s)$ is the value of $\mathbf{SPP}_\mu$ at $(x,\,s)$ and the solutions of **DSPP** define a *saddle-point correspondence*Ψ, such that if $(\mathbf{a}^*,\boldsymbol{\gamma}^*)$ is generated by Ψ from $(x,\,\mu,\,s)$, then $(\mathbf{a}^*,\boldsymbol{\gamma}^*)$ solves $\mathbf{SPP}_\mu$ at $(x,\,s)$ and $\mathbf{a}^*$ is the unique solution to $\mathbf{PP}_\mu$ at $(x,\,s)$.

[25] Messner and Pavoni's 'counterexample' is one of these cases of 'flats in the Pareto frontier', when one only considers the T^* map and not the T map; see Marimon, Messner and Pavoni (2010).

Proof: By assumption **SPFE** is satisfied. The proof of Theorem 4 is based on having a unique representation $W^*(\mu, x, s) = \mu\omega^*(x, \mu, s)$ which, in that proof, is given by Lemma 2. In Theorem 5 such a unique representation is assumed, which implicitly also means assuming that the values $h_0^j(x,\, a^*(x,\, \mu,\, s),\, s)\, j = 0, ..., l$ are uniquely determined, which in fact is all that is needed in the proof of Theorem 4.

6 Related work

Precedents of our approach can be found in Epple, Hansen and Roberds (1985), Sargent (1987) and Levine and Currie (1987), who introduced Lagrange multipliers as co-state variables in linear-quadratic Ramsey problems. Similarly, recent studies of optimal monetary policy in sticky price models have included Lagrange multipliers as co-states. Often, the reason given for including these past multipliers as co-states is the observation that past multipliers appear in the first-order-conditions of the Ramsey problem. Our work provides a formal proof that, with standard assumptions, co-state past multipliers deliver the optimal solution in a general framework, encompassing a larger class of models with *forward-looking* constraints.

The pioneer work of Abreu, *et al.* (1990) – APS, from now on, – characterizing sub-game perfect equilibria, shows that past histories can be summarized in terms of promised utilities. Earlier related work was by Green (1987) and Thomas and Worral (1988). This approach has been widely used in macroeconomics[26]. Some applications are by Kocherlakota (1995) in a model with participation constraints, and Cronshaw and Luenberger (1994) in a dynamic game. Also, as in the earlier work, Kydland and Prescott (1980), Chang (1998) and Phelan and Stacchetti (1999) study Ramsey equilibria using promised *marginal* utility as a state variable, and they note the analogy of their approach with APS's.

Both APS and our approach have in common that starting from non-recursive problems allow optimal solutions to be obtained (obviously, the same solutions) where *forward-looking* constraints have a recursive structure. In relatively simple problems (e.g., convex problems of full information and low dimensionality, in terms of state variables and number of forward-looking constraints) the two approaches can be seen as mirrors of each other. Nevertheless, there is a conceptual difference which sets these two approaches further apart as more complex problems are analyzed: our state (including the co-state μ) is predetermined, while promised-utilities – as co-state variables – are not; furthermore, in the APS approach in taking future promised-utilities as choice variables, the recursive structure of *forward-looking* constraints must be taken as 'promise keeping' constraints, while in our approach we obtain this recursive structure as a result (see, Subsection 2.1)[27].

[26] Ljungqvist and Sargent (2000) provides an excellent introduction, and reference, to most of this recent work.

[27] We only provide a summary discussion of the contrast between the two approaches. See

As is well known, promised utilities in the APS approach have to be restricted to lie in a set where the continuation problem is well defined; otherwise algorithmic computations break down. The set of feasible promised utilities is not known beforehand. It can only be characterized numerically, often leading to very complicated calculations. Whenever there are several natural state variables the set of feasible promised utilities is a function of the natural state variables and the problem of finding the set of feasible utilities is daunting. Considerable progress has been made either by improving algorithms or by redefining the problem at hand[28], but the issue of constraining promised utilities (or marginal utilities in a Ramsey problem) is always present.

One key advantage of our approach is that the continuation problem is given by $\mathbf{PP}_{\mu'}$, and it is easy to find standard assumptions guaranteeing that this problem has a solution for any co-state $\mu' \geq 0$; in fact, the set of feasible co-states is known beforehand: it is simply the positive orthant. The difficulties associated with computing a set of co-states for which the continuation problem is well defined are absent.

A second advantage lies in the dimensionality of the decision vector. In the APS approach the planner has to decide at t the utility promised at all possible states in $t+1$. If the underlying exogenous state variable can take N_s possible values the planner has to decide on at least N_s controls at t. Most applications of APS constrain themselves to assuming that the exogenous variable is binomial (say, it can be 'high'or 'low'), but if exogenous variables can take many values a high-dimensional decision vector has to be solved for. Again, there are ways of dealing with this, but it is no doubt an added difficulty. By contrast, in our approach the dimensionality of the decision variable is independent of N_s.

An additional issue is that the initial conditions for the state variables in our approach are given from the outset, namely $\mu^0 = 1$ and $\mu^j = 0$ for $j \geq 1$, while in the promised utility approach the promised-utility in the first period has to be solved for separately. It is well known that to do thisis necessary that the Pareto frontier is downward slopping; otherwise the computations can become very cumbersome.[29]

Finally, an interesting – but not exclusive – feature of our approach is that the evolution of μ often helps to directly characterize the behavior of the model. For example, in models with participation constraints the μ's allow to interpret the behavior of the model as changing the Pareto weights sequentially depending on how binding the participation constraints become. In Ramsey type models the behavior of the μ's is associated with the commitment technology and the role that budget constraints play in the objective function of the planner. We have discussed these interpretations in Section 3. Also, our approach facili-

Marcet (2008) for more details.

[28]See, for example, Abraham and Pavoni (2005) or Judd, Yeltekin and Conklin (2003).

[29]Even with two agents and a downward slopping Pareto frontier, as in Kocherlakota (1996), one may be interested in finding the efficient allocation that *ex-ante* gives the same utility to both agents. While this is trivial with our approach (just give the same initial weights in the **PP** problem), it becomes very tricky with the APS approach since the 'right promise' must be made to determine the initial conditions.

tates the identification of cases where despite the presence of forward-looking constraints the co-state variables do not need to be introduced in the model.[30]

The APS approach does, however, have some strengths over our approach. For example, it allows for the characterization of all feasible paths (not only the constrained-efficient) and it naturally applies to models with private information or models with multiple solutions. However, these initial advantages are also being overcome. Sleet and Yeltekin (2010) and Mele (2010) have extended our approach to address moral hazard problems. In problems with multiple (locally unique) solutions, it is also possible to find other feasible paths using our approach by changing the objective function. However, as the example of Messner and Pavoni (2004) shows, there are problems where optimal paths are bound to have a continuum of solutions (i.e. when the constrained-efficient Pareto frontier has flats). We have maintained the assumption of (local) uniqueness in this paper; nevertheless, Marimon, Messner and Pavoni (2011) have recently shown that there is a natural extension of our approach to solve problems with non-uniqueness [31].

Many applications of our approach can be found in the literature, although it is beyond the scope of this paper to discuss them in detail. This seems to be testimony to the convenience of using our approach, especially in the presence of intertemporal participation constraints with natural state variables such as capital (as in Subsection 3.1) or first-order Euler equation constraints with bonds as natural state (as in Subsection 3.2).

Perhaps most interesting is that the approach here can be used as an intermediate step in solving models that go beyond the pure formulation of **PP**. For example, a second generation of models considers *endogenous* participation constraints, as in the non-market exclusion models of Cooley et al. (2004), Marimon and Quadrini (2011), and Ferrero and Marcet (2005). In these models the functions h that appear in the incentive constraints are endogenous; they depend on the optimal or equilibrium solution, and the approach of this paper is often used as an intermediate step, defining the underlying contracts. This allows the study of problems where the outside option is determined in equilibrum as in models of debt renegotiation and long-term contracts. Furthermore, the work of Debortoli and Nunes (2010) extends our approach to study models of partial commitment and political economy, and the work of Marimon and Rendhal (2011) extends it to study models where agents can behave strategically with respect to their participation constraints, as in dynamic bargaining problems with endogenous separations.

[30] This can be the case, for example, in Cooley, Quadrini and Marimon (2004) and Anagnostopoulos, Cárceles-Poveda and Marcet (2011).

[31] Marimon, Messner and Pavoni (2011) show how the results presented here can be applied when the co-state is extended with the 'last non-negative multiplier'. Cole and Kubler (2010) also provide a solution for the non-uniqueness case. Their approach involves a mix of our approach and of the APS approach; they provide a solution for the two-agent case with intertemporal participation constraints.

7 Concluding remarks

We have shown that a large class of problems with implementability constraints can be analysed using an equivalent recursive saddle-point problem. This saddle-point problem obeys a saddle point functional equation, which is a version of the Bellman equation. This approach works for a very large class of models with incentive constraints: intertemporal enforcement constraint, intertemporal Euler equations in optimal policy and regulation design, etc. This means that a unified framework can be provided to analyse all these models. The key feature of our approach is that instead of having to write optimal contracts as history-dependent contracts one can write them as a stationary function of the standard state variables together with additional co-state variables. These co-state variables are – recursively – obtained from the Lagrange multipliers associated with the intertemporal incentive constraints, starting from pre-specified initial conditions. This simple representation also provides economic insight into the analysis of various contractual problems; for example, with intertemporal participation constraints it shows how the (Bethamite) social planner changes the weights assigned to different agents in order to keep them within the social contract; in Ramsey optimal problems it shows the cost of commitment to the benevolent government.

We have provided here the first complete account of the basic theory of *recursive contracts*. Nevertheless, we had already expounded most of the elements of the theory in our previous work (in particular, Marcet and Marimon (1988 & 1999)), which has allowed others to built on it. Many applications are already found in the literature, showing the convenience of our approach, especially when natural state variables are present. Useful extensions are already available encompassing a larger set of problems than the ones considered here.

APPENDIX

Appendix A (Proof of Proposition 1)

The proof of Proposition 1 relies on the following result:

Lemma 1A. Assume **A1-A6** and that S is countable**,** then

i) $\mathcal{B}(x,s)$ is non-empty, convex, bounded and $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed; therefore it is $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ compact;

ii) Given $d \in R$, the set $\left\{\boldsymbol{a} \in \mathcal{A} : f_{(x,\mu,s)}(\boldsymbol{a}) \geq d\right\}$ is convex and $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed.

The proof of Lemma 1A builds on three theorems. First, the *Urysohn metrization theorem* stating that regular topological spaces with a countable base are metrizable[32]. Second, the *Mackey-Arens theorem* stating that different topologies consistent with the same duality share the same closed convex sets; in our case, the duality is $(\mathcal{L}_\infty,\, \mathcal{L}_1)$ and the weakest and the strongest topology consistent with such duality; namely, the *weak-star*, $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ and the Mackey $\tau(\mathcal{L}_\infty,\, \mathcal{L}_1)$. Third, the *Alaoglu theorem* stating that norm bounded $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed subsets of $\mathcal{L}_\infty$ are $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ compact[33].

Proof:

Assumptions **A2**, and **A4** - **A6** imply that $\mathcal{B}(x,s)$ is convex, and closed under pointwise convergence. Since, by assumption S is countable, *Urysohn metrization theorem* guarantees that $\mathcal{B}(x,s)$ is, in fact, $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed. Assumptions **A3** and **A4** imply that $\mathcal{B}(x,s)$ is bounded in the $\|\cdot\|_\infty^\beta$ norm as needed for compactness, according to the *Alaoglu theorem.*

Assumptions **A5** and **A6** imply that $\mathcal{B}(x,s)$ and the upper contour sets

$$\{\boldsymbol{a} \in \mathcal{A} : f_\mu(\boldsymbol{a}) \geq d\}\,,$$

are convex and, together with the previous assumptions (**A2**, and **A4**), Mackey closed and, therefore, $\sigma(\mathcal{L}_\infty,\, \mathcal{L}_1)$ closed[34]■

Proof of Proposition 1: As in Bewley (1991), the central element of the proof follows from the *Hausdorff maximal principle* and an application of *the*

[32]See Dunford and Schwartz (1957) p. 24. in our case the metric we use is given by

$$\rho_\infty^\beta(\boldsymbol{a},\boldsymbol{b}) = \sum_{n=0}^{\infty} \beta^n \sup_{s^n \in S^n} \mid a_n(s) - b_n(s) \mid .$$

[33]See Schaefer (1966) p. 130 and p. 84, respectively.

[34]See Bewley (1972) for a proof of the Mackey continuity expected utility, without assuming S to be countable.

finite intersection property[35]. Let $\mathcal{P}_d = \{\boldsymbol{a} \in \mathcal{B}(x,s) : f_{(x,\mu.s)}(\boldsymbol{a}) \geq d\}$. Then by Lemma 1A , $\mathcal{P}_d$ is $\sigma(\mathcal{L}_\infty, \mathcal{L}_1)$ closed. By the interiority assumption of Proposition 1, for d low enough, it is non-empty. In fact, we can consider the family of sets $\{\mathcal{P}_d : d \in D\}$ for which $\mathcal{P}_d \neq \varnothing$, where $D \subset R$. The sets $\mathcal{P}_d$ are ordered by inclusion; in fact, if $d' > d$ then $\mathcal{P}_{d'} \subset \mathcal{P}_d$ and every finite collection of them has a non-empty intersection (i.e. $\{\mathcal{P}_d : d \in D\}$ satisfies *the finite intersection property*), but then by compactness of $\mathcal{B}(x,s)$ any family of subsets of $\{\mathcal{P}_d : d \in D\}$ – say, $\{\mathcal{P}_d : d \in B \subset D\}$ – has a non-empty intersection and, by inclusion, there is $\mathcal{P}_{\widehat{d}} = \cap\{\mathcal{P}_d : d \in B \subset D\} \neq \varnothing$. In particular, there is $\mathcal{P}_{d^*} = \cap\{\mathcal{P}_d : d \in D\} \neq \varnothing$ which – as the *the minimal principle* states – is a minimal member of the family $\{\mathcal{P}_d : d \in D\}$. It follows that if $\boldsymbol{a}^* \in \mathcal{P}_{d^*}$ then $f_{(x,\mu.s)}(\boldsymbol{a}^*) \geq f_{(x,\mu.s)}(\boldsymbol{a})$ for any $\boldsymbol{a} \in \mathcal{B}(x,s)$. Furthermore, if strictly concavity is assumed then $\mathcal{P}_{d^*}$ must be a singleton; otherwise convex combinations of elements of $\mathcal{P}_{d^*}$ will form a proper closed subset of $\mathcal{P}_{d^*}$ contradicting its minimality■

Appendix B (Some Properties of W^*)

Lemma 2A. Let $W^*(x,\mu,s) \equiv V_\mu(x,s)$ be the value of $\mathbf{SPP}_\mu$ at (x,s), for an arbitrary $(x,\, \mu,\, s)$. Then

i) $W^*(x,\cdot,s)$ is convex and homogeneous of degree one;

ii) if **A2**- **A4** are satisfied $W^*(\cdot,\mu,s)$ is continuous and uniformly bounded; and

iii) if **A5** and **A6** are satisfied $W^*(\cdot,\mu,s)$ is concave.

Proof: *i)* follows from the fact that, for any $\lambda > 0$, $f_{(x,\lambda\mu.s)}(\boldsymbol{a}) = \lambda f_{(x,\mu.s)}(\boldsymbol{a})$. To see this, let $(\gamma^*, \boldsymbol{a}^*)$ satisfy **SFPE**, i.e.

$$\begin{aligned} & f_{(x,\mu.s)}(\boldsymbol{a}^*) + \gamma g(\boldsymbol{a}^*)_0 \\ \geq\ & f_{(x,\mu.s)}(\boldsymbol{a}^*) + \gamma^* g(\boldsymbol{a}^*)_0 \\ \geq\ & f_{(x,\mu.s)}(\boldsymbol{a}) + \gamma^* g(\boldsymbol{a})_0, \end{aligned}$$

for any $\gamma \in R_+^{l+1}$ and $\boldsymbol{a} \in \mathcal{B}'(x,s)$. Then $(\lambda\gamma^*, \boldsymbol{a}^*)$ satisfies

$$\begin{aligned} & f_{(x,\lambda\mu.s)}(\boldsymbol{a}^*) + \gamma g(\boldsymbol{a}^*)_0 \\ \geq\ & f_{(x,\lambda\mu.s)}(\boldsymbol{a}^*) + \lambda\gamma^* g(\boldsymbol{a}^*)_0 \\ \geq\ & f_{(x,\lambda\mu.s)}(\boldsymbol{a}) + \lambda\gamma^* g(\boldsymbol{a})_0, \end{aligned}$$

for any $\gamma \in R_+^{l+1}$ and $\boldsymbol{a} \in \mathcal{B}'(x,s)$. *ii)* and *iii)* are straightforward; in particular, *ii)* follows from applying the Theorem of the Maximum (Stokey, Lucas and Prescott,1989, Theorem 3.6) and *iii)* follows from the fact that the constraint sets are convex and the objective function concave.■

[35] See, Kelley (1955) p. 33-34. for the Hausdorff principle and the Minimal principle, and p. 136 for the theorem stating that "a set is compact if and only if every family of closed sets which has the finite intersection property has a non-void intersection."

Lemma 3A: If the inf sup problem **SPFE** at $(x,\, \mu,\, s)$, has a solution then the value of this solution is unique.

Proof: It is a standard argument: consider two solutions to the right-hand side of **SPFE** at $(x,\, \mu,\, s)$, $(\widetilde{a}, \widetilde{\gamma})$ and $(\widehat{a}, \widehat{\gamma})$. Then repeated application of the saddle-point condition implies:

$$\begin{aligned}
& \mu h_0(x,\, \widetilde{a},\, s) + \widetilde{\gamma} h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W^*(\ell(x,\, \widetilde{a},\, s'), \varphi(\mu,\, \widetilde{\gamma}),\, s') |\, s\right] \\
\geq\ & \mu h_0(x,\, \widehat{a},\, s) + \widetilde{\gamma} h_1(x,\, \widehat{a},\, s) + \beta \mathrm{E}\left[W^*(\ell(x,\, \widehat{a},\, s'), \varphi(\mu,\, \widetilde{\gamma}),\, s') |\, s\right] \\
\geq\ & \mu h_0(x,\, \widehat{a},\, s) + \widehat{\gamma} h_1(x,\, \widehat{a},\, s) + \beta \mathrm{E}\left[W^*(\ell(x,\, \widehat{a},\, s'), \varphi(\mu,\, \widehat{\gamma}),\, s') |\, s\right] \\
\geq\ & \mu h_0(x,\, \widetilde{a},\, s) + \widehat{\gamma} h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W^*(\ell(x,\, \widetilde{a},\, s'), \varphi(\mu,\, \widehat{\gamma}),\, s') |\, s\right] \\
\geq\ & \mu h_0(x,\, \widetilde{a},\, s) + \widetilde{\gamma} h_1(x,\, \widetilde{a},\, s) + \beta \mathrm{E}\left[W^*(\ell(x,\, \widetilde{a},\, s'), \varphi(\mu,\, \widetilde{\gamma}),\, s') |\, s\right].
\end{aligned}$$

Therefore the value of the objective at both $(\widetilde{a}, \widetilde{\gamma})$ and $(\widehat{a}, \widehat{\gamma})$ coincides ■

Appendix C (Some Properties of convex homogeneous functions)

To simplify the exposition of these properties let $F : R^m_+ \to R$ be continuous, convex and homogeneous of degree one. The *subgradient set* of F at y, denoted $\partial F(y)$, is given by

$$\partial F(y) = \left\{ z \in R^m \mid F(y') \geq F(y) + (y' - y)z \text{ for all } y' \in R^m_+ \right\}.$$

The following **facts**, regarding F, are used in proving Lemmas 1 and 2:

F1. If F is convex, then it is differentiable at y if, and only if, $\partial F(y)$ consists of a single vector; i.e. $\partial F(y) = \{\nabla F(y)\}$, where $\nabla F(y)$ is called the *gradient* of F at y.

F2. If F is convex and finite in a neighborhood of y, then $\partial F(y)$ is the convex hull of the compact set

$$\{ z \in R^m \mid \exists\, y_k \longrightarrow y \text{ with } F \text{ differentiable at } y_k \text{ and } \nabla F(y_k) \longrightarrow z \}.$$

F3. Lemma 4A (Euler's formula). If F is convex and homogeneous of degree one and $z \in \partial F(y)$ then $F(y) = yz$.

F4. Lemma 5A. If F is convex and homogeneous of degree one, for any pair $\left(f, \widehat{f}\right)$, if $f^d(y) \in \partial F(y)$ and $f^d(\widehat{y}) \in \partial F(\widehat{y})$, then $\widehat{y} f^d(\widehat{y}) \geq \widehat{y} f^d(y)$.

F1 is a basic result on the differentiability of convex functions (see, Rockafellar, 1981, Theorem 4F, or 1970, Theorem 25.1). **F2** is a very convenient characterization of the subgradient set of a convex function (see Rockafellar, 1981, Theorem 4D, or 1970, Theorem 25.6). We now provide a proof of the last two facts.

Proof of Lemma 4A: Let $z \in \partial F(y)$. Then for any $\lambda > 0$, $F(\lambda y) - F(y) \geq (\lambda y - y)z$, and, by homogeneity of degree one: $(\lambda - 1)F(y) \geq (\lambda - 1)yz$. If $\lambda > 1$ this weak inequality results in $F(y) \geq yz$; while if $\lambda \in (0,1)$ in $F(y) \leq yz$■[36]

Proof of Lemma 5A: To see **F4** notice that if $f^d(y) \in \partial F(y)$ and $f^d(\widehat{y}) \in \partial F(\widehat{y})$, by convexity, homogeneity of degree one, and Euler's formula: $F(\widehat{y}) = \widehat{y} f^d(\widehat{y})$ and

$$\begin{aligned} F(\widehat{y}) &\geq F(y) + (\widehat{y} - y) f^d(y) \\ &= y f^d(y) + \widehat{y} f^d(y) - y f^d(y) = \widehat{y} f^d(y). \end{aligned}$$

Appendix D (Proof of Propositions 2 and 3)

Proof of Proposition 2: Given the assumptions of Proposition 2, for any (x, μ, s), and $\gamma \in \mathcal{R}_+^{l+1}$, let

$$F_{(x,\mu,s)}(\gamma) = \arg\sup_a \left\{ \mu h_0(x,\, a,\, s) + \beta \operatorname{E}\left[\sum_{j=0}^{k} \mu^j \omega_j(x',\, \mu',\, s') |\, s \right] \right\}$$
$$\text{s.t. } x' = \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0$$
$$\text{and } \mu' = \varphi(\mu,\, \gamma),$$
$$\text{and } h_1^j(x, a, s) + \beta \operatorname{E}\left[\omega^j(x', \mu', s') |\, s \right] \geq 0,\ j = 0, ..., l. \tag{45}$$

Since this is a standard maximization problem of a continuous function on a compact set, there is a solution $a^*(x, \mu, s; \gamma) \in F_{(x,\mu,s)}(\gamma)$. Furthermore, given that the constraint set is convex and has a non-empty interior (by **A2** and **A7b**), there is an associated multiplier vector; let $\gamma^{*j}(x, \mu, s; \gamma)$ be the multiplier corresponding to (45) for j. In particular, by Lemma 3, $\gamma^*(x, \mu, s; \gamma) \in G_{(x,\mu,s)}(a^*)$, where

$$G_{(x,\mu,s)}(a) =$$
$$\arg \inf_{\{\gamma \geq 0 : \|\gamma\| \leq B \|\mu\|\}} \{\mu h_0(x,\, a,\, s) + \gamma h_1(x,\, a,\, s) + \beta \operatorname{E}[\varphi(\mu,\, \gamma) \omega(x',\, \varphi(\mu,\, \gamma),\, s') |\, s]\}$$
$$\text{s.t. } x' = \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0.$$

The rest of the proof is a trivial application of the *Theorem of the Maximum* (e.g. Stokey et al. (1989), p. 62) and of *Kakutani's Fixed Point Theorem* (e.g. Mas-Colell et al. (1995), p.953). First notice that $\mu h_0(x,\, a,\, s) + \beta \operatorname{E}\left[\sum_{j=0}^{k} \mu^j \omega_j(x',\, \mu',\, s') |\, s \right]$ is continuous in a and. By **A2**, **A4** and the definition of M_{bc}, $G^*_{(x,\mu,s)}(\cdot)$ is also continuous. Second, let $A(x, s) \equiv \{a \in A : p(x, a,\, s) \geq 0\}$, by **A2** and **A3** $A(x; s)$ is compact and by **A5** is convex, while $\Psi(\mu) \equiv \{\gamma \geq 0 : \|\gamma\| \leq B \|\mu\|\}$ is trivially compact and convex. Therefore, $F_{(x,\mu,s)} : \Psi(\mu) \to A(x; s)$ and $G^*_{(x,\mu,s)} : A(x; s) \to$

[36] Notice that this does not imply that F is linear, which requires that $F(-y) = -F(y)$.

$\Psi(\mu)$ are upper-hemicontinuous, non-empty and convex-valued correspondences, jointly mapping a convex and compact set $\Psi \times A(x;s)$ onto itself. By *Kakutani's Fixed Point Theorem* there is a fixed point (a^*, γ^*) which is a solution to $\mathbf{DSPP}_{(x,\mu,s)}$. Furthermore, $F_{(x,\mu,s)}(\cdot)$ is a continuous function, when **A6s** is assumed■

Lemma 6A. $\mathcal{M}$ is a nonempty complete metric space.

Proof: That it is non-empty is trivial. Except for the homogeneity property, that every Cauchy sequence $\{W^n\} \in \mathcal{M}_{bc}$ converges to $W \in \mathcal{M}_{bc}$ satisfying *i*), *iii*), and the convexity property *ii*), follows from standard arguments (see, for example, Stokey, et al. (1989), Theorem 3.1 and Lemma 9.5). To see that the homogeneity property is also satisfied, for any $(x,\, \mu,\, s)$ and $\lambda > 0$,

$$\begin{aligned} & |W(x,\, \lambda\mu,\, s) - \lambda W(x,\, \mu,\, s)| \\ = \;& |W(x,\, \lambda\mu,\, s) - W^n(x,\, \lambda\mu,\, s) + \lambda W^n(x,\, \mu,\, s) - \lambda W(x,\, \mu,\, s)| \\ \leq \;& |W(x,\, \lambda\mu,\, s) - W^n(x,\, \lambda\mu,\, s)| + \lambda |W^n(x,\, \mu,\, s) - W(x,\, \mu,\, s)| \\ \to \;& 0 \end{aligned}$$

■

Lemma 7A. Assume **A2** - **A6** and **A7b**. The operator T^* maps $\mathcal{M}_{bc}$ onto itself.

Proof: First, notice that by Proposition 2, given $W \in \mathcal{M}_{bc}$, T^*W is well defined. The correspondences $\Gamma : X \to X$ and $\Phi : \mathcal{R}_+^{l+1} \to \mathcal{R}_+^{l+1}$ defined by $\Gamma(x)_{(\mu,s)} \equiv \{x' \in X : x' = \ell(x,\, a,\, s),\ p(x, a,\, s) \geq 0,\ \text{for some } a \in A\}$ and

$$\Phi(\mu)_{(x,s)} \equiv \left\{\mu' \in \mathcal{R}_+^{l+1} : \mu' = \varphi(\mu,\, \gamma),\ \text{for some } \gamma \in \mathcal{R}_+^{l+1}\right\}$$

are continuous and compact-valued (by **A2**, **A3** and **A5,** and the definition of φ, respectively) and, as in the proof of Proposition 2, by continuity of the objective function, it follows that $T^*W(\cdot,\, \cdot,\, s)$ is continuous, and given **A3** and **A4**, and the boundedness condition on W, it follows that T^*W also satisfies (i).To see that the homogeneity properties are satisfied, let $(a^*,\, \gamma^*)$ satisfy

$$(T^*W)(x,\, \mu,\, s) = \mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \mathrm{E} W(x^*,\, \mu^*,\, s').$$

Then, for any $\lambda > 0$

$$\lambda(T^*W)(x,\, \mu,\, s) = \lambda[\mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \mathrm{E} W(x^*,\, \mu^*,\, s')].$$

Furthermore,

$$\begin{aligned} & \lambda\mu h_0(x,\, a^*,\, s) + \lambda\gamma^* h_1(x,\, a^*,\, s) + \beta \mathrm{E} W(x^*,\, \lambda\mu^{*'},\, s') \\ = \;& \lambda\left[\mu h_0(x,\, a^*,\, s) + \gamma^* h_1(x,\, a^*,\, s) + \beta \mathrm{E} W(x^*,\, \mu^{*'},\, s')\right]. \end{aligned}$$

To see that **SPFE** is satisfied, let $\gamma \geq 0$, $\mu' = \varphi(\lambda\mu, \gamma)$, $a \in A(x,s)$ and $x' = \ell(x, a, s')$. Then

$$\begin{aligned} & \lambda\mu h_0(x, a^*, s) + \gamma h_1(x, a^*, s) + \beta \mathrm{E} W(x^*, \mu', s') \\ = \;& \lambda \left[\mu h_0(x, a^*, s) + \gamma\lambda^{-1} h_1(x, a^*, s) + \beta \mathrm{E} W(x^*, \mu'\lambda^{-1}, s')\right] \\ \geq \;& \lambda \left[\mu h_0(x, a^*, s) + \gamma^* h_1(x, a^*, s) + \beta \mathrm{E} W(x^*, \mu^{*'}, s')\right] \\ \geq \;& \lambda \left[\mu h_0(x, a, s) + \gamma^* h_1(x, a, s) + \beta \mathrm{E} W(x', \mu^{*'}, s')\right]. \end{aligned}$$

It follows that

$$\begin{aligned} (T^*W)(x, \lambda\mu, s) &= \lambda\mu h_0(x, a^*, s) + \lambda\gamma^* h_1(x, a^*, s) + \beta \mathrm{E} W(x^*, \lambda\mu^{*'}, s') \\ &= \lambda(T^*W)(x, \mu, s). \end{aligned}$$

Finally, since $W \in \mathcal{M}_{bc}$, it is straightforward to show that TW is concave in x (by **A5** and **A6),** and convex in μ■

Lemma 7A (monotonicity) Let $\widehat{W} \in \mathcal{M}$ and $\widetilde{W} \in \mathcal{M}$ be such that $\widehat{W} \leq \widetilde{W}$, then $(T^*\widehat{W}) \leq (T^*\widetilde{W})$.

Proof Fix (μ, x, s). Then for any μ' satisfying $\mu' = \varphi(\mu, \gamma) \geq 0$, for a given $\gamma \geq 0$,

$$\begin{aligned} & \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E}\widehat{W}(\ell(x, a, s), \mu', s')\} \\ \leq \;& \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E}\widetilde{W}(\ell(x, a, s), \mu', s')\}. \end{aligned}$$

It follows that

$$\begin{aligned} & \min_{\gamma \geq 0} \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E}\widehat{W}(\ell(x, a, s), \varphi(\mu, \gamma), s')\} \\ \leq \;& \min_{\gamma \geq 0} \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E}\widetilde{W}(\ell(x, a, s), \varphi(\mu, \gamma), s')\}. \end{aligned}$$

■

Notice that if $W \in \mathcal{M}_{bc}$ and $r \in \mathcal{R}$, $(W + r)(x, \mu, s) = \mu(\omega + r)(x, \mu, s) = \mu\omega(x, \mu, s) + r\|\mu\|$.

Lemma 8A (discounting) Assume **A4** and **A7b**. For any $W \in \mathcal{M}_b$, and $r \in \mathcal{R}_+$, $T^*(W + r) \leq T^*W + \beta r$.

Proof First notice that, for any (x, μ, s) and $\gamma \geq 0$,

$$\begin{aligned} & \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E}(W + r)(\ell(x, a, s), \varphi(\mu, \gamma), s')\} \\ = \;& \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E} W(\ell(x, a, s), \varphi(\mu, \gamma), s') + \beta r \|\varphi(\mu, \gamma)\|\} \\ = \;& \max_{a \in A(x,s)} \{\mu h_0(x, a, s) + \gamma h_1(x, a, s) + \beta \mathrm{E} W(\ell(x, a, s), \varphi(\mu, \gamma), s')\} + \beta r \|\varphi(\mu, \gamma)\|. \end{aligned}$$

Now, given $(x,\, \mu,\, s)$ and $a \in A(x,s)$, denote by $\gamma^+(a)$ the solution to the following problem:

$$\begin{aligned}
&\min_{\{\gamma\geq 0:\|\gamma\|\leq B\|\mu\|\}} \left\{ \begin{array}{c} \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j (\omega_j + r)(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma),\, s') \\ + \sum_{j=0}^{l} \gamma^j \left[h_1^j(x,\, a,\, s) + \beta \mathrm{E} (\omega_j + r)(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma),\, s') \right] \end{array} \right\} \\
= \; & \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^+(a)),\, s') \\
+ \; & \gamma^+(a) \left[h_1(x,\, a,\, s) + \beta \mathrm{E} \omega(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^+(a)),\, s') \right] + \beta r \left\| \varphi(\mu,\, \gamma^+(a)) \right\|
\end{aligned}$$

and let $\gamma^*(a)$ be the solution to

$$\begin{aligned}
&\min_{\{\gamma\geq 0:\|\gamma\|\leq B\|\mu\|\}} \left\{ \begin{array}{c} \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma),\, s') \\ \sum_{j=0}^{l} \gamma^j \left[h_1^j(x,\, a,\, s) + \beta \mathrm{E} \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma),\, s') \right] \end{array} \right\} \\
= \; & \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^*(a)),\, s') \\
+ \; & \gamma^*(a) \left[h_1(x,\, a,\, s) + \beta \mathrm{E} \omega(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^*(a)),\, s') \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^+(a)),\, s') \\
+ \; & \gamma^+(a) \left[h_1(x,\, a,\, s) + \beta \mathrm{E} \omega(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^+(a)),\, s') \right] + \beta r \left\| \varphi(\mu,\, \gamma^+(a)) \right\| \\
\leq \; & \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^*(a)),\, s') \\
+ \; & \gamma^*(a) \left[h_1(x,\, a,\, s) + \beta \mathrm{E} \omega(\ell(x,\, a,\, s), \varphi(\mu,\, \gamma^*(a)),\, s') \right] + \beta r \left\| \varphi(\mu,\, \gamma^*(a)) \right\|.
\end{aligned}$$

Piecing things together (denoting, as usual, $a^* \equiv a^*(x, \mu,\, s)$ and $\gamma^* \equiv \gamma^*(x, \mu,\, s)$); that is, $\gamma^* \equiv \gamma^*(a^*)$), we have:

$$\begin{aligned}
& T^*(W\, +\, r)(x, \mu,\, s) \\
= \; & \min_{\gamma\geq 0} \max_{a\in A(x,s)} \left\{ \begin{array}{c} \mu h_0(x,\, a,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma),\, s') \\ + \sum_{j=0}^{l} \gamma^j \left[h_1^j(x,\, a,\, s) + \beta \mathrm{E} \omega_j (\ell(x,\, a,\, s), \varphi(\mu,\, \gamma),\, s') \right] \end{array} \right\} \\
\leq \; & \mu h_0(x,\, a^*,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a^*,\, s), \varphi(\mu,\, \gamma^*,\, s') \\
+ \; & \gamma^* \left[h_1(x,\, a^*,\, s) + \beta \mathrm{E} \omega(\ell(x,\, a^*,\, s), \varphi(\mu,\, \gamma^*),\, s') \right] + \beta r \left\| \varphi(\mu,\, \gamma^*) \right\| \\
= \; & \mu h_0(x,\, a^*,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a^*,\, s), \varphi(\mu,\, \gamma^*,\, s') + \beta r \left\| \varphi(\mu,\, \gamma^*) \right\| \\
\leq \; & \mu h_0(x,\, a^*,\, s) + \beta \mathrm{E} \sum_{j=0}^{k} \mu^j \omega_j (\ell(x,\, a^*,\, s), \varphi(\mu,\, \gamma^*,\, s') + \beta r (1 + B) \left\| \mu \right\|.
\end{aligned}$$

By homogeneity, without loss of generality we can choose an arbitrary $\mu \neq 0$ such that: $\|\mu\| \leq (1+B)^{-1}$. The above inequalities show that $T^*(W + r) \leq T^*(W) + \beta r$■

Lemma 9A (Contraction property): The argument is the standard Blackwell's argument. We show that the contraction property is satisfied. Let $W, \widehat{W} \in \mathcal{M}_{bc}$.

Notice that $W \leq \widehat{W} + \left\| W - \widehat{W} \right\|$. Then, using the results of Lemmas 7A and 8A,

$$T^*W \leq T^*(\widehat{W} + \left\| W - \widehat{W} \right\|) \leq T^*(\widehat{W}) + +\beta \left\| W - \widehat{W} \right\|).$$

Reversing the roles of W and $\widehat{W}$, we obtain that

$$\left\| T^*W - T^*\widehat{W} \right\| \leq \beta \left\| W - \widehat{W} \right\|.$$

Therefore, T^* is a contraction mapping■

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