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SUNSPOTS AND MULTIPLICITY

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Sunspots and Multiplicity

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Abstract

This paper proves that a multiplicity of certainty equilibria is not necessary for the existence of sunspot effects in two-period general equilibrium models with incomplete markets. Sunspot effects are present, by definition, when real economic variables differ across realizations of extrinsic uncertainty. For the class of models delineated above and further restricted to numeraire assets whose payouts are identical across such realizations, the literature has remained silent on whether a multiplicity of certainty equilibria is necessary for sunspot effects. First, I prove that such a multiplicity is not necessary for sunspot effects in this particular class of models. Second, I prove that, over an entire set of economies commonly considered in sunspot examples, an equilibrium with sunspot effects can never be characterized as a randomization over multiple certainty equilibria.

Keywords

sunspots; extrinsic uncertainty; incomplete markets; randomization

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1 Introduction

The idea of sunspots has attracted considerable attention from economists, because sunspots are a formal representation of seemingly irrational behavior exhibited in financial markets. In these markets, traders respond to irrelevant information that has no bearing on "fundamentals." These responses, if adopted as the "market psychology," result in self-fulfilling optimal actions by traders. In terms of economic modeling, sunspots are realizations of extrinsic uncertainty, i.e., uncertainty that does not impact the fundamentals (household endowments and preferences and asset payouts) of the economy. As was shown thirty years ago (Cass and Shell, 1983; Balasko, 1983; Azariadis, 1981), even when the tenet of rational expectations is maintained, sunspots can affect the real equilibrium variables.

Since the introduction of sunspots thirty years ago, the origin of sunspot effects has been a source of confusion for economists. Initially, one of the leading explanations for sunspot effects is that they occur exclusively in economies with multiple certainty equilibria, in which sunspots serve to coordinate the beliefs of agents on one vector of certainty equilibrium prices.[1] A currently held belief is that even if a multiplicity of certainty equilibria is not necessary, it is sufficient for sunspot effects. The overall aim of this paper is to clear up these fallacies and to allow the theory to reveal that the origin of sunspot effects in dynamic models has no connection with the multiplicity of certainty equilibria.

In terms of dynamic models, this paper considers one particular class of such models: two-period general equilibrium models with incomplete markets. The financial element of the models is numeraire assets, whose payouts in the final period are identical across realizations of extrinsic uncertainty. The models do not contain intrinsic uncertainty, implying that the only independent numeraire asset is a risk-free bond whose payouts are equal in all realizations of extrinsic uncertainty and normalized to one.

For this particular class of models, I show in this paper that a multiplicity of certainty equilibria is neither necessary nor sufficient for sunspot effects.[2] The fact that multiple certainty equilibria are not necessary for sunspot effects has been demonstrated over other major classes of dynamic models of sunspots.[3] Showing that a statement is neither necessary nor sufficient for another requires only two well-chosen economies to serve as counter-examples. What I ultimately demonstrate is a general result, namely that both implications are false over an entire set of economies commonly considered in the literature.

Within the class of two-period financial models, this paper considers numeraire assets, which have far different implications for sunspot equilibria than the two other asset types: nominal assets and real assets. With nominal assets, Cass (1992) proves that sunspot equilibria are generically indeterminate when there are fewer assets than states of extrinsic uncertainty. This result only requires the reader to count. In economies with such market incompleteness, equilibria are generically indeterminate, but the set of equilibria without sunspot effects (namely those with identical real variables for all realizations of extrinsic uncertainty) is generically finite. With real assets, Gottardi and Kajii (1999) prove that for a generic subset of economies sunspot effects occur without a

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[1] Such an explanation can perhaps be traced to the seminal Cass and Shell (1983) paper. Most readers have tended to focus on the canonical example in the body of the paper, in which the multiplicity of certainty equilibria appears to be necessary for sunspot effects, rather than on the example in the appendix.

[2] The results are valid for two-period models without consumption in the initial period. The details can be fleshed out by the reader using the results in this paper. Additionally, the results extend to models with any finite number of periods.

[3] See Cass and Shell (1983) for 2-period models of restricted participation, Azariadis and Guesnerie (1986) for OLG models of production, Peck and Shell (1991) for markets with information asymmetries, and Garratt et al. (2004) for nonconvex exchange economies.

multiplicity of certainty equilibria. In their model, the asset yields are identical across realizations of extrinsic uncertainty, but the asset payouts, as functions of endogenous commodity prices, are not. Thus, their result states that a "potential multiplicity" of certainty equilibria is necessary for sunspot effects. With numeraire assets, the sunspot equilibria are not generically indeterminate (as is the case for nominal assets) and the asset payouts are parameters that are independent of endogenous prices (in contrast to the case for real assets).

Within the narrower class of two-period financial models with numeraire assets, the idea of Hens (2000) is to provide an example of an economy in which sunspot effects occur even though a unique spot market equilibrium exists for all distributions of ex-post final period endowments.[4] This uniqueness is guaranteed by assuming Cobb-Douglas preferences in the final period. The idea of Hens (2000) complements Gottardi and Kajii (1999), whose result requires that for some distribution of ex-post final period endowments, reached via ex-ante asset trade in the initial period, multiple spot market equilibria exist. However, to make this contribution, Hens (2000) must allow the asset payouts to depend upon the realizations of extrinsic uncertainty.

My first contribution is to complete the triangle begun by Gottardi and Kajii (1999) and Hens (2000) by providing an example with the concurrence of a unique certainty equilibrium and an equilibrium with sunspot effects. This example contains two elements: (i) numeraire assets whose payouts are equal for all realizations of extrinsic uncertainty and (ii) multiple spot market equilibria for some distribution of ex-post final period endowments. Element (i) distinguishes the contribution from Gottardi and Kajii (1999) and both elements distinguish it from the idea of Hens (2000). The example verifies that a multiplicity of certainty equilibria is not necessary for sunspot effects.

To highlight one important aspect of the preceding literature review, a necessary feature for the presence of sunspot effects is that for some set of ex-post final period endowments, multiple spot market equilibria exist (termed "ex-post multiplicity"). Without such an ex-post multiplicity, sunspot effects cannot occur unless the asset payouts differ across realizations of extrinsic uncertainty (as in Hens, 2000). In the dynamic setting of this model, an equilibrium includes all household decisions, including those ex-ante decisions made before the uncertainty is revealed. The multiplicity of such equilibria (termed "ex-ante multiplicity") is the property that is disproved in this paper and it is this ex-ante multiplicity that is referred to simply as "multiplicity" in the sequel. In fact, the first contribution of this paper can be restated as a verification that ex-ante multiplicity is not necessary for sunspot effects, even though ex-post multiplicity is.

My second contribution is to prove that the result stating that a multiplicity of certainty equilibria is not necessary for sunspot effects holds in many economies beyond the one selected for my example. In particular, I restrict attention to a set of economies within the previously described class of models that is commonly used for sunspot examples in the literature. For an economy within this set in which an equilibrium exists with sunspot effects, there may exist multiple certainty equilibria; I do not impose assumptions to rule this out. Even so, I prove that no equilibrium with sunspot effects can be characterized as a randomization over multiple certainty equilibria. As I later define, a sunspot equilibrium is a randomization over certainty equilibria if the commodity prices in each state of extrinsic uncertainty are identical to the final period commodity prices for

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[4]An error is present in the example provided by Hens (2000), so sunspots do not have effects in that example as claimed by the author. This led to 3 further papers attempting to clarify the issue. All these papers consider models without initial consumption and with sunspot-contingent numeraire assets. The error in Hens (2000) is initially pointed out by Barnett and Fisher (2002), but in their example sunspots still do not have effects because they assume risk neutrality. Ultimately, Hens et al. (2005), building from an insight that Hens and Pilgrim (2004) introduce, provide an example (their Proposition 2), with more than two households, in which sunspots have effects even though a unique spot market equilibrium exists for all distributions of ex-post final period endowments. It is this final statement that I refer to as the "idea of Hens (2000)."

some certainty equilibrium. Randomization is the mathematical equivalent of sunspots serving as a device to coordinate beliefs on one of the certainty equilibrium price vectors.[5] In summary, the result shows in general that no logical connection exists between a multiplicity of certainty equilibria and the presence of sunspot effects.

The paper is organized as follows. Section 2 introduces the model both with and without extrinsic uncertainty. Section 3 provides an example of an economy to prove that the multiplicity of certainty equilibria is not necessary for sunspot effects. Section 4 asserts that an equilibrium with sunspot effects need not be a randomization over certainty equilibria. Section 5 offers concluding remarks and Section 6 contains the proofs.

2 The Model

2.1 Sunspot equilibrium

I consider a general equilibrium model with two time periods and extrinsic uncertainty in the final period. The extrinsic uncertainty is modeled as a finite number of states $s \in \mathcal{S} = \{1, ..., S\}$ that can be realized in the final period. By convention, the initial period is state $s = 0$. In all states, a finite number of households $h \in \mathcal{H} = \{1, ..., H\}$ trade and consume a finite number of physical commodities $l \in \{1, .., L\}$. The model is a financial model, because assets may be included to allow households to transfer wealth between states.

For this paper, the uncertainty is only extrinsic. Extrinsic uncertainty means that the fundamentals (household endowments and utility functions and asset payouts) in the states $s \in \mathcal{S}$ are identical. A sunspot, by definition, is the realization of extrinsic uncertainty. Sunspot effects occur if real economic variables differ across sunspots.

Let h be any household. The consumption of a bundle of L physical commodities by h in state $s = 0$ is $x^h(0)$ and in state $s \in \mathcal{S}$ is $x^h(s)$. Consumption over all states is defined by the vector $x^h = \left(x^h(0), \left(x^h(s) \right)_{s \in \mathcal{S}} \right)$. The consumption set is defined by $X^h = \mathbb{R}_+^{L(S+1)}$ and the utility function by $u^h : X^h \to \mathbb{R}$. The utility function is given by the expected utility form:

$$ u^h(x^h) = v_0^h \left(x^h(0) \right) + \beta^h \left(\sum_{s \in \mathcal{S}} \pi(s) v_s^h(x^h(s)) \right), $$

where $v_s^h : \mathbb{R}_+^L \to \mathbb{R} \ \forall s \in \mathcal{S}$, $v_0^h : \mathbb{R}_+^L \to \mathbb{R}$, $\pi(s) > 0 \ \forall s \in \mathcal{S}$, and $\sum_{s \in \mathcal{S}} \pi(s) = 1$.

The standard assumptions for existence are:

1. v_s^h is C^0, concave, and locally non-satiated $\forall s \in \mathcal{S}$.

2. v_0^h is C^0, concave, and locally non-satiated.

The assumptions for extrinsic uncertainty are given by:

3. $v_s^h = v^h \ \ \forall s \in \mathcal{S}$.

4. v^h is strictly concave.

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[5] Randomization, both the colloquial term and the technical definition given in this paper, is the intuition for sunspot effects from both: (i) Cass and Shell (1983) for their static model under Observation 1 ("a sunspot equilibrium is constructed as a lottery over certainty equilibria," pg. 213) and (ii) Mas-Colell (1992) for a static model ("sunspots can matter only if they induce randomness," pg. 469).

The household endowments are defined by $e^h = \left(e^h(0), \left(e^h(s) \right)_{s \in \mathcal{S}} \right)$ and are assumed to be strictly positive: $e^h >> 0$.[6] The spot commodity prices are defined by $p = \left(p(0), (p(s))_{s \in \mathcal{S}} \right)$, where $p(0) \in \mathbb{R}_+^L \backslash \{0\}$ and $p(s) \in \mathbb{R}_+^L \backslash \{0\}$ $\forall s \in \mathcal{S}$ by Assumptions 1 and 2.

With $J < S$ numeraire assets, markets are incomplete with respect to the extrinsic uncertainty. The payout of asset j in state $s \in \mathcal{S}$ in terms of the commodity $l = L$ is given by $r_j(s)$. In terms of the real economic variables, it is innocuous to assume that the payout matrix $R = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & : & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}$ has full column rank.

The remaining assumptions for extrinsic uncertainty are:

5. $e^h(s) = e^h(1)$ $\forall s \in \mathcal{S}$.

6. $r_j(s) = r_j(1)$ $\forall s \in \mathcal{S}$.

Assumption 6 together with R full column rank implies that only one numeraire asset exists ($J = 1$) and that it has a risk-free payout normalized to 1 in all states. The asset choice by household h is denoted by $z^h \in \mathbb{R}$ and the price of the asset is $q \in \mathbb{R}_+$.

I define a *financial equilibrium with extrinsic uncertainty*, referred to as a *sunspot equilibrium*, as follows:

Definition 1 *A financial equilibrium with extrinsic uncertainty is a vector $\left(\left(x^h, z^h \right)_{h \in \mathcal{H}}, p, q \right)$ such that*

1. *given* (p, q), $\forall h \in \mathcal{H}$:

$$
\begin{aligned}
\left(x^h, z^h \right) \in \quad & \arg\max \quad u^h \left(x^h \right) \\
& \text{subj. to} \quad p(0) \left(x^h(0) - e^h(0) \right) + q z^h \leq 0. \\
& \qquad\qquad \left(p(s) \left(x^h(s) - e^h(1) \right) - z^h \leq 0 \right)_{s \in \mathcal{S}}.
\end{aligned}
$$

2. *markets clear:*

$$
\begin{aligned}
\sum_{h \in \mathcal{H}} x_l^h(0) - e_l^h(0) &= 0 \quad \forall l. \\
\sum_{h \in \mathcal{H}} x_l^h(s) - e_l^h(1) &= 0 \quad \forall l, \forall s \in \mathcal{S}. \\
\sum_{h \in \mathcal{H}} z^h &= 0.
\end{aligned}
$$

Given Assumptions 1-6 and $e^h >> 0$ $\forall h \in \mathcal{H}$, a *sunspot equilibrium* always exists.

Sunspots, i.e., the realizations of extrinsic uncertainty, have effects when they lead to different consumptions choices. If sunspots have effects, they are said to matter.

Definition 2 *Sunspots have effects iff $\exists h$ s.t. $x^h(s) \neq x^h(s')$ for some $s, s' \in \mathcal{S}$.*

2.2 Certainty equilibrium

A *financial equilibrium without extrinsic uncertainty* (a *certainty equilibrium*, for short) is defined by setting $S = 1$. The consumption set is $\tilde{X}^h = \mathbb{R}_+^{2L}$, consumption over all states is defined by a

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[6] By convention, for $x, y \in \mathbb{R}^m$, (i) $x >> y$ iff $x_i > y_i$ $\forall i$, (ii) $x \geq y$ iff $x_i \geq y_i$ $\forall i$, and (iii) $x > y$ iff $x \geq y$ and $x \neq y$.

vector $\tilde{x}^h = \left(\tilde{x}^h(0), \tilde{x}^h(1)\right)$, endowments are defined by $e^h = \left(e^h(0), e^h(1)\right) >> 0$, and the expected utility function $\tilde{u}^h : \tilde{X}^h \to \mathbb{R}$ is defined as:

$$\tilde{u}^h(\tilde{x}^h) = v_0^h\left(\tilde{x}^h(0)\right) + \beta^h v^h(\tilde{x}^h(1)),$$

where $v^h : \mathbb{R}_+^L \to \mathbb{R}$ satisfies Assumptions 1 and 4 and $v_0^h : \mathbb{R}_+^L \to \mathbb{R}$ satisfies Assumption 2. The functions v^h and v_0^h remain unchanged, but I use the notation $\tilde{u}^h$, as the dimension of the domain for this function is strictly smaller when compared to its counterparts u^h.

The spot commodity prices are defined by $\tilde{p} = (\tilde{p}(0), \tilde{p}(1))$, where $\tilde{p}(0), \tilde{p}(1) \in \mathbb{R}_+^L \backslash \{0\}$ by Assumptions 1 and 2. The household asset choice is denoted by $\tilde{z}^h \in \mathbb{R}$ and the asset price by $\tilde{q} \in \mathbb{R}_+$.

Definition 3 *A financial equilibrium without extrinsic uncertainty is a vector $\left(\left(\tilde{x}^h, \tilde{z}^h\right)_{h \in \mathcal{H}}, \tilde{p}, \tilde{q}\right)$ such that*

1. given $(\tilde{p}, \tilde{q})$, $\forall h \in \mathcal{H}$:

$$
\begin{aligned}
\left(\tilde{x}^h, \tilde{z}^h\right) \in \quad &\arg\max \quad \tilde{u}^h\left(\tilde{x}^h\right) \\
&\text{subj. to} \quad \tilde{p}(0)\left(\tilde{x}^h(0) - e^h(0)\right) + \tilde{q}\tilde{z}^h \leq 0. \\
&\qquad\qquad \tilde{p}(1)\left(\tilde{x}^h(1) - e^h(1)\right) - \tilde{z}^h \leq 0.
\end{aligned}
$$

2. markets clear:

$$
\begin{aligned}
\sum\nolimits_{h \in \mathcal{H}} \tilde{x}_l^h(0) - e_l^h(0) &= 0 \quad \forall l. \\
\sum\nolimits_{h \in \mathcal{H}} \tilde{x}_l^h(1) - e_l^h(1) &= 0 \quad \forall l. \\
\sum\nolimits_{h \in \mathcal{H}} \tilde{z}^h &= 0.
\end{aligned}
$$

3 The Example

3.1 Multiplicity is not necessary for sunspot effects

For the economy specified below, the following subsections show that (i) a *sunspot equilibrium* exists in which sunspots matter (Subsection 3.1.1) and (ii) a unique *certainty equilibrium* exists (Subsection 3.1.2). Taken together, this verifies that a multiplicity of *certainty equilibria* is not necessary for sunspot effects.

3.1.1 Sunspot equilibrium

Consider the following example with two households ($H = 2$), two states of extrinsic uncertainty ($S = 2$), and two physical commodities traded in each state ($L = 2$). The endowments for the households are given by:

	$e_1^h(0)$	$e_2^h(0)$	$e_1^h(1)$	$e_2^h(1)$
$h = 1$	16	32	47	2
$h = 2$	32	16	1	46

Table I: Endowments

Recall from Assumption 5 that $e^h(s) = e^h(1) \; \forall s \in \mathcal{S}$. The utility functions $u^h : \mathbb{R}_+^6 \to \mathbb{R}$ are defined by

$$u^h(x^h) = v_0^h\left(x^h(0)\right) + \beta^h\left(\sum\nolimits_{s \in \mathcal{S}} \frac{1}{2}v^h(x^h(s))\right),$$

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where the functions $v^h : \mathbb{R}_+^2 \to \mathbb{R}$ and $v_0^h : \mathbb{R}_+^2 \to \mathbb{R}$ are given by:

$$v^1\left(x^1(s)\right) = \left(\tfrac{1}{8}\right)^{-3} \frac{\left(x_1^1(s)\right)^{-2}}{-2} + \left(\tfrac{7}{8}\right)^{-3} \frac{\left(x_2^1(s)\right)^{-2}}{-2} \quad \forall s \in \mathcal{S}.$$
$$v^2\left(x^2(s)\right) = \left(\tfrac{7}{8}\right)^{-3} \frac{\left(x_1^2(s)\right)^{-2}}{-2} + \left(\tfrac{1}{8}\right)^{-3} \frac{\left(x_2^2(s)\right)^{-2}}{-2} \quad \forall s \in \mathcal{S}.^{[7]}$$

$$v_0^h\left(x^h(0)\right) = \alpha \log\left(x_1^h(0)\right) + (1-\alpha) \log\left(x_2^h(0)\right) \quad \forall h \in \mathcal{H}.$$

Notice that the initial period utility functions v_0^h are identical across households. The parameters $\left(\alpha, \beta^1, \beta^2\right)$ are backward engineered so that the equilibrium asset choice is an integer. The values for these parameters are given by:

$$\alpha = 0.6.$$
$$\left(\beta^1, \beta^2\right) = (0.184, 0.906).$$

As the utility functions are differentiable, concave, and strictly increasing, then (i) the first order conditions with respect to the variables x^h and z^h are necessary and sufficient conditions for an optimal solution to the household problem given in Definition 1, and (ii) the budget constraints in the household problem hold with equality. Solving this system of equations together with the market-clearing conditions yields the following *sunspot equilibrium:*

initial period	$\frac{p_1(0)}{p_2(0)} = 1.5$	$q = 0.50$
	$x^1(0) = (22.6, 22.6)$	$z^1 = -1$
	$x^2(0) = (25.4, 25.4)$	$z^2 = 1$

final period	$\frac{p_1(1)}{p_2(1)} = \frac{1}{8}$	$\frac{p_1(2)}{p_2(2)} = 8$
	$x^1(1) = (35, 2.5)$	$x^1(2) = (45.5, 13)$
	$x^2(1) = (13, 45.5)$	$x^2(2) = (2.5, 35)$

Table II: Sunspot equilibrium in which sunspots matter

In this example, the sunspots matter since $x^h(1) \neq x^h(2) \ \forall h$.

For the purpose of future welfare analysis, the utility of both households for the *sunspot equilibrium* in Table II is given by:

$u^1(x^1) = 3.076$	$u^2(x^2) = 3.028$

Table III: Utility values when sunspots matter

3.1.2 Unique certainty equilibrium

For the economy previously specified, a unique *certainty equilibrium* exists:

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[7]The inspiration for the functional forms $\left(v^h\right)_{h \in \mathcal{H}}$ and endowments $\left(e^h(1)\right)_{h \in \mathcal{H}}$ leading to the necessary ex-post multiplicity is Kubler and Schmedders (2010).

initial period	$\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)} = 1.5$	$\tilde{q} = 0.337$
	$\tilde{x}^1(0) = (22.52, 22.52)$	$\tilde{z}^1 = -0.857$
	$\tilde{x}^2(0) = (25.48, 25.48)$	$\tilde{z}^2 = 0.857$

final period	$\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)} = 0.280$
	$\tilde{x}^1(1) = (38.30, 3.58)$
	$\tilde{x}^2(1) = (9.70, 44.42)$

Table IV: Certainty equilibrium

It is straightforward to verify that the above variables satisfy the conditions characterizing *certainty equilibria.*[8] To verify that they are the only ones to satisfy the conditions characterizing *certainty equilibria*, I reduce the system of equilibrium equations to a system of two equations and two unknowns. Then, I confirm that this system has only one solution. The system of two equilibrium equations is constructed in two steps.

I. Final period first order conditions, budget constraints, and market clearing.

   Taking $\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}$ and $\tilde{z}^1$ as given, the first order conditions with respect to $\tilde{x}^h(1)$ $\forall h$ yield:

   $$\left(\frac{1}{8}\right)^{-3} \left(\tilde{x}_1^1(1)\right)^{-3} = \left(\frac{7}{8}\right)^{-3} \left(\tilde{x}_2^1(1)\right)^{-3} \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right).$$
   $$\left(\frac{7}{8}\right)^{-3} \left(\tilde{x}_1^2(1)\right)^{-3} = \left(\frac{1}{8}\right)^{-3} \left(\tilde{x}_2^2(1)\right)^{-3} \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right).$$

   Using the budget constraints $\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) \tilde{x}_1^h(1) + \tilde{x}_2^h(1) = \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) e_1^h(1) + e_2^h(1) + \tilde{z}^h$, the demand functions for commodity $l = 2$ are given by:

   $$\tilde{x}_2^1(1) = \frac{47\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) + 2 + \tilde{z}^1}{7\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)^{2/3} + 1}.$$
   $$\tilde{x}_2^2(1) = \frac{\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) + 46 - \tilde{z}^1}{\frac{1}{7}\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)^{2/3} + 1}.$$

   Defining $\rho = \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)^{1/3}$, the market clearing condition $\tilde{x}_2^1(1) + \tilde{x}_2^2(1) = 48$ reduces to a polynomial of degree 3:

   $$\frac{96}{7}\rho^3 - 48\rho^2 + 48\rho - \frac{144}{7} - \frac{48}{7}\tilde{z}^1 = 0. \tag{1}$$

   Eq. 1 is the first of two equations characterizing *certainty equilibria.*

II. First order conditions with respect to asset choice; Initial period first order conditions, budget constraints, and market clearing.

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[8]To obtain these equilibrium conditions, I take advantage of the fact that the utility functions are differentiable, concave, and strictly increasing. The first order conditions are then necessary and sufficient conditions for an optimal solution to the household problem given in Definition 3.

Let $\tilde{\lambda}^h(0)$ be the Lagrange multiplier associated with the budget constraint in the initial period and $\tilde{\lambda}^h(1)$ be the Lagrange multiplier associated with the budget constraint in the final period. The first order conditions with respect to the asset choice $\forall h$ yield:

$$\tilde{q} = \frac{\tilde{\lambda}^1(1)}{\tilde{\lambda}^1(0)} = \frac{\tilde{\lambda}^2(1)}{\tilde{\lambda}^2(0)}. \tag{2}$$

Given the variables $(\rho, \tilde{z}^1)$, the values $(\tilde{\lambda}^1(1), \tilde{\lambda}^2(1))$ can be calculated from Step I. Rearrange Eq. 2 to yield

$$\frac{\tilde{\lambda}^1(1)}{\tilde{\lambda}^2(1)} = \frac{\tilde{\lambda}^1(0)}{\tilde{\lambda}^2(0)}. \tag{3}$$

The Cobb-Douglas utility functions $\left(v_0^h\right)_{h \in \mathcal{H}}$ simplify the following calculations. The left-hand side of Eq. 3 is known, so the ratio $\frac{\tilde{\lambda}^1(0)}{\tilde{\lambda}^2(0)}$, together with the market clearing condition $\tilde{x}_2^1(0) + \tilde{x}_2^2(0) = 48$, uniquely determines $\left(\tilde{x}_2^1(0), \tilde{x}_2^2(0)\right)$ as functions of the variables $(\rho, \tilde{z}^1)$. From Eq. 2, $\tilde{q}$ is then determined as a function of $(\rho, \tilde{z}^1)$.

The first order conditions with respect to $\tilde{x}^h(0)$ $\forall h$, together with the market clearing condition $\tilde{x}_1^1(0) + \tilde{x}_1^2(0) = 48$, uniquely determine $\left(\tilde{x}_1^1(0), \tilde{x}_1^2(0)\right)$ and $\left(\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)}\right)$ as functions of the variables $(\rho, \tilde{z}^1)$. The final equation to consider is the initial period budget constraint for household $h = 1$: [9]

$$\left(\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)}\right) \tilde{x}_1^1(0) + \tilde{x}_2^1(0) + \tilde{q}\tilde{z}^1 - \left(\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)}\right) e_1^1(0) - e_2^1(0) = 0. \tag{4}$$

All variables in Eq. 4 are functions of $(\rho, \tilde{z}^1)$.

The two equations, Eq. 1 and Eq. 4, in two unknowns $(\rho, \tilde{z}^1)$, suffice to characterize the set of *certainty equilibria*. That is, once variables $(\rho, \tilde{z}^1)$ consistent with Eq. 1 and Eq. 4 are found, simple linear equations are used to compute all remaining equilibrium variables: $\left(\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)}, \tilde{q}, \tilde{x}^1(0), \tilde{x}^2(0), \frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}, \tilde{x}^1(1), \tilde{x}^2(1)\right)$. Thus, for multiple *certainty equilibria* to exist, there must exist multiple solutions to the system: Eq. 1 and Eq. 4.

In particular, as Eq. 4 is a linear equation, for multiple *certainty equilibria* to exist, given the equilibrium variable $\tilde{z}^1$, there must exist multiple values for ρ that satisfy Eq. 1. Consider the cubic polynomial Eq. 1 as a function $f(\rho, \tilde{z}^1) = 0$. The function has two local extrema: a local maximum $f(\rho_1)$ and a local minimum $f(\rho_2)$ with $\rho_1 < \rho_2$. Both ρ_1 and ρ_2 are independent of $\tilde{z}^1$, so define $\tilde{z}_i^1$ such that $f(\rho_i, \tilde{z}_i^1) = 0$ for $i = 1, 2$. The highest possible ρ such that multiplicity occurs in Eq. 1 is $\rho = \bar{\rho}$ such that $f(\bar{\rho}, \tilde{z}_1^1) = 0$. Likewise, the lowest possible ρ such that multiplicity occurs in Eq. 1 is $\rho = \underline{\rho}$ such that $f(\underline{\rho}, \tilde{z}_2^1) = 0$. The calculated values provide a search window $\rho \in [\underline{\rho}, \bar{\rho}] = [0.285, 2.049]$.

In the interval $\rho \in [\underline{\rho}, \bar{\rho}] = [0.285, 2.049]$, use Eq. 1 to compute $\tilde{z}^1$. Consider Eq. 4 as a function $g(\rho) = 0$. Figure 1 graphs the function g over the interval $\rho \in [\underline{\rho}, \bar{\rho}] = [0.285, 2.049]$.

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[9] The initial period budget constraint for the other household, $h = 2$, is redundant given that the market clearing conditions for both commodities have been used.
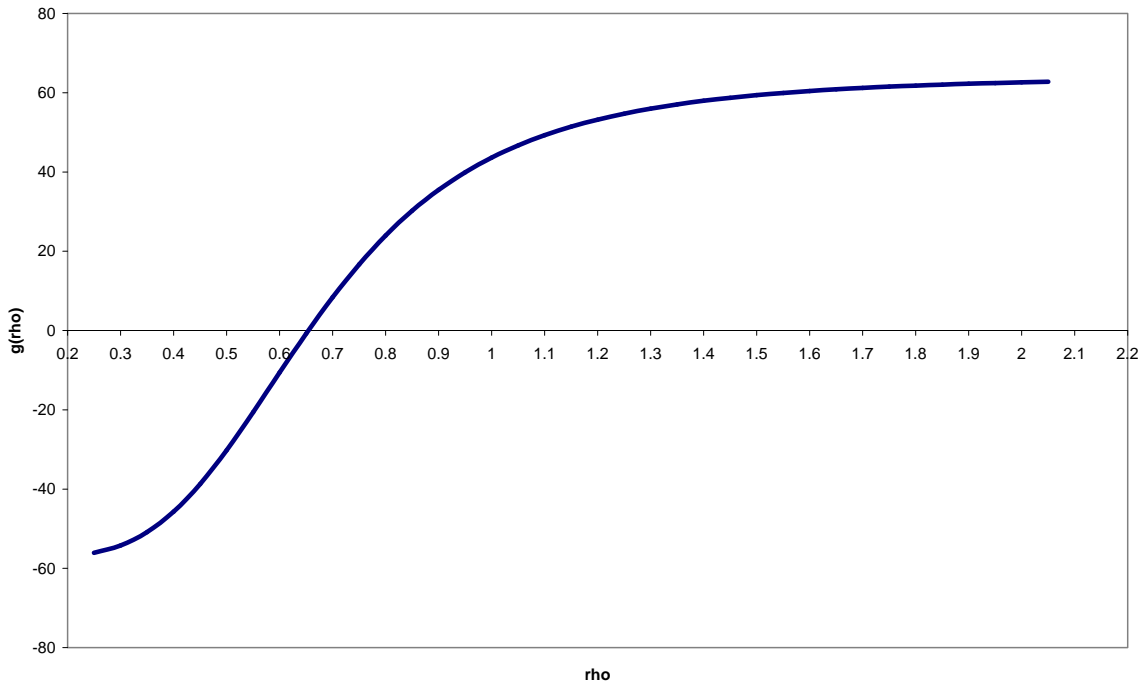
Figure 1: Unique solution to Eqs. 1 and 4

The function has a unique solution at the value $\rho = 0.654$. This verifies that the only solution to the system Eqs. 1 and 4 leads to the *certainty equilibrium* previously given in Table IV.

3.2 Welfare

For the economy previously specified, the following subsections show that (i) the *sunspot equilibrium* in which sunspots matter is constrained Pareto inefficient (Subsection 3.2.1) and (ii) there is no Pareto ranking between the *sunspot equilibrium* in which sunspots do not matter and the previously specified *sunspot equilibrium* in which sunspots matter (Subsection 3.2.2).

3.2.1 Constrained inefficiency when sunspots matter

The *sunspot equilibrium* allocation in Table II is Pareto inefficient. This follows from the strict concavity of v^h (Assumption 4) and the result that sunspots matter. More disturbing is that the equilibrium allocation in Table II is constrained Pareto inefficient. The natural interpretation of constrained Pareto inefficiency is that a planner can fix the asset choices (subject to market clearing) and asset price, allow households to optimize on all spot markets (subject to spot budget constraints and market clearing), and the resulting allocation Pareto dominates the equilibrium allocation in Table II.

Such a Pareto improvement is achieved by the planner fixing the asset choices and asset price as $z^1 = -1$, $z^2 = 1$, and $q = 0.50$. Then, a resulting constrained feasible allocation is given by:

initial period	$\frac{p_1(0)}{p_2(0)} = 1.5$	
	$x^1(0) = (22.6, 22.6)$	
	$x^2(0) = (25.4, 25.4)$	

final period	$\frac{p_1(1)}{p_2(1)} = 1$	$\frac{p_1(2)}{p_2(2)} = 1$
	$x^1(1) = (42, 6)$	$x^1(2) = (42, 6)$
	$x^2(1) = (6, 42)$	$x^2(2) = (6, 42)$

Table V: Constrained feasible allocation

The utility of both households for the constrained feasible allocation in Table V is given by:

$u^1(x^1) = 3.087$	$u^2(x^2) = 3.084$

Table VI: Utility values for constrained feasible allocation

Comparing Tables III and VI, the specified contrained feasible allocation Pareto dominates the *sunspot equilibrium* allocation. Thus, the *sunspot equilibrium* allocation is constrained Pareto inefficient. In sum, sunspots matter because they have welfare implications.

3.2.2 Possible Pareto ranking?

Using the *certainty equilibrium* in Table IV, it is straightforward to construct a *sunspot equilibrium* in which sunspots do not matter, i.e., an equilibrium in which the consumption choices are independent of the sunspots. In fact, this is the only *sunspot equilibrium* in which sunspots do not matter given the uniqueness of the c*ertainty equilibria*. To construct this equilibrium, the final period commodity prices for both states of extrinsic uncertainty are set equal to the final period commodity prices of the *certainty equilibrium* in Table IV.

initial period	$\frac{p_1(0)}{p_2(0)} = 1.5$	$q = 0.337$
	$x^1(0) = (22.52, 22.52)$	$z^1 = -0.857$
	$x^2(0) = (25.48, 25.48)$	$z^2 = 0.857$

final period	$\frac{p_1(1)}{p_2(1)} = 0.280$	$\frac{p_1(2)}{p_2(2)} = 0.280$
	$x^1(1) = (38.30, 3.58)$	$x^1(2) = (38.30, 3.58)$
	$x^2(1) = (9.70, 44.42)$	$x^2(2) = (9.70, 44.42)$

Table VII: Sunspot equilibrium in which sunspots do not matter

As all *certainty equilibria* are Pareto efficient (complete markets), then any equilibrium allocation without sunspot effects is necessarily Pareto efficient. The utility of both households for the *sunspot equilibrium* in Table VII is given by:

$u^1(x^1) = 3.072$	$u^2(x^2) = 3.113$

Table VIII: Utility values when sunspots do not matter

Recall that the *sunspot equilibrium* in Table II (in which sunspots matter) is constrained Pareto inefficient, whereas the *sunspot equilibrium* in Table VII is Pareto efficient. Comparing the utility values in Tables III and VIII, there is no Pareto ranking between the two *sunspot equilibria*, although the utilitarian welfare is higher for the Pareto efficient equilibrium allocation in Table VII.

4 Sunspot Effects without Randomization

Consider an economy with N distinct *certainty equilibria* for $N \geq 2$. The set of N *certainty equilibria* is given by: $\left\{ \left((\tilde{x}_n^h, \tilde{z}_n^h)_{h \in \mathcal{H}}, \tilde{p}_n, \tilde{q}_n \right)_{n=1,...,N} \right\}$. I define what it means for a *sunspot equilibrium* to be a randomization over these N *certainty equilibria*. Consider an environment with S states of extrinsic uncertainty $s \in \mathcal{S} = \{1, ..., S\}$ in the final period. Let $\omega : \mathcal{S} \to \{1, ..., N\}$ be any function, which maps each state of extrinsic uncertainty onto one of the possible *certainty equilibria*.

Definition 4 *Consider a set of certainty equilibria* $\left\{ \left((\tilde{x}_n^h, \tilde{z}_n^h)_{h \in \mathcal{H}}, \tilde{p}_n, \tilde{q}_n \right)_{n=1,..,N} \right\}$ *and a function* $\omega : \mathcal{S} \to \{1, ..., N\}$ *such that:*

$$\tilde{p}_n(1) \cdot \left(e^h(1) - \tilde{x}_n^h(1) \right) = \tilde{p}(1) \cdot \left(e^h(1) - \tilde{x}^h(1) \right) \quad \forall n \in \omega(\mathcal{S}) \text{ and } \forall h \in \mathcal{H}.$$

A sunspot equilibrium $\left((x^h, z^h)_{h \in \mathcal{H}}, p, q \right)$ *is a randomization over the certainty equilibria iff:*

$$p(s) = \tilde{p}_{\omega(s)}(1) \quad \forall s \in \mathcal{S}.$$

I am only interested in *sunspot equilibria* in which sunspots matter. If $\exists n$ such that $\omega(s) = n$ $\forall s \in \mathcal{S}$, the strict concavity of v^h (Assumption 4) guarantees that sunspots do not matter in the proposed *sunspot equilibrium*. Thus, I focus attention on functions ω such that $\#\omega(\mathcal{S}) > 1$, where $\#\omega(\mathcal{S})$ is the number of elements in the set $\omega(\mathcal{S})$.

The definition of *randomization* is mathematically equivalent to the statement that sunspots serve as a coordinating device.[10] That is, each state of extrinsic uncertainty $s \in \mathcal{S}$ must have the same final period commodity prices as one of the N *certainty equilibria*.

The following assumptions are required for Lemma 1:

7. v^h is C^1, differentiably strictly increasing[11], and additively separable across all commodities $\forall h \in \mathcal{H}$.

8. v_0^h is C^1, differentiably strictly increasing, strictly concave, and additively separable across all commodities $\forall h \in \mathcal{H}$.

Using Assumptions 7 and 8, the commodity prices in this section are normalized as $p(s) = \left(\left(\frac{p_l(s)}{p_L(s)} \right)_{l < L}, 1 \right)$ and $\tilde{p}(s) = \left(\left(\frac{\tilde{p}_l(s)}{\tilde{p}_L(s)} \right)_{l < L}, 1 \right)$.

Lemma 1 *For a set of certainty equilibria* $\left\{ \left((\tilde{x}_n^h, \tilde{z}_n^h)_{h \in \mathcal{H}}, \tilde{p}_n, \tilde{q}_n \right)_{n=1,..,N} \right\}$ *and a function* $\omega : \mathcal{S} \to \{1, ..., N\}$ *such that* $\#\omega(\mathcal{S}) > 1$ *and*

$$\tilde{p}_n(1) \cdot \left(e^h(1) - \tilde{x}_n^h(1) \right) = \tilde{p}(1) \cdot \left(e^h(1) - \tilde{x}^h(1) \right) \quad \forall n \in \omega(\mathcal{S}) \text{ and } \forall h \in \mathcal{H},$$

Assumptions 1-8 imply that:

i. $\left(\tilde{x}_{n,L}^h(0) \right)_{h \in \mathcal{H}} \neq \left(\tilde{x}_{m,L}^h(0) \right)_{h \in \mathcal{H}}$ $\forall n, m \in \omega(\mathcal{S})$ *with* $m \neq n$.

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[10]See Footnote 5.

[11]A function $f : X \to \mathbb{R}$ for $X \subseteq \mathbb{R}^m$ is differentiably strictly increasing if $Df(x) >> 0$ $\forall x \in X$.

ii. $\left(\tilde{q}_n D_L v_0^h \left(\tilde{x}_n^h(0)\right)\right)_{h \in \mathcal{H}} \neq \left(\tilde{q}_m D_L v_0^h \left(\tilde{x}_m^h(0)\right)\right)_{h \in \mathcal{H}}$ $\forall n, m \in \omega(\mathcal{S})$ *with* $m \neq n$.[12]

Proof. See Section 6. ∎

The lemma implies that a *sunspot equilibrium* as a *randomization* over the set of *certainty equilibria* can never be of the form:

$$\left(\left(x^h(0), z^h\right)_{h \in \mathcal{H}}, p(0), q\right) = \left(\left(\tilde{x}_n^h(0), \tilde{z}_n^h\right)_{h \in \mathcal{H}}, \tilde{p}_n(0), \tilde{q}_n\right) \quad \forall n \in \omega(\mathcal{S}).$$

If a *sunspot equilibrium* were to be a *randomization* over the set of *certainty equilibria*, the following randomization condition (RC) must hold (determined from first order conditions; see Section 6.2):

$$q D_L v_0^h \left(x^h(0)\right) = \sum_n \frac{\#\omega^{-1}(n)}{S} \left(\tilde{q}_n D_L v_0^h \left(\tilde{x}_n^h(0)\right)\right) \quad \forall h \in \mathcal{H}. \tag{RC}$$

From Lemma 1(ii), the summands on the right-hand side of Eq. RC are not equal.

The analysis from this point forward restricts attention to functions $\left(v_0^h\right)_{h \in \mathcal{H}}$ of the homogeneous Cobb-Douglas form. That is, I make the following assumption:

9. v_0^h is of the form $v_0^h \left(x^h(0)\right) = \sum_l \alpha_l \log \left(x_l^h(0)\right)$ $\forall h \in \mathcal{H}$, where $\alpha_l > 0$ $\forall l$.

As evidenced by the example in Section 2, Assumption 9 does not inhibit the occurrence of sunspot effects. I consider this specific utility form in order to show that Eq. RC does not hold for all possible utility parameters and endowments. With a more general utility function, Eq. RC may hold for some knife-edge cases of utility parameters and endowments. Future research can analyze the precise nature in which Eq. RC fails to hold over a generic subset of parameters, but that technical argument distracts from the main point: in models with dynamic household choice and strictly concave preferences, sunspot effects and the multiplicity of *certainty equilibria* are outcomes that are coincidental at best and neither is an implication of the other.

For Theorem 1, I consider a set of economies not just satisfying Assumption 9, but also with two households, two states of extrinsic uncertainty, and two commodities per state ($H = S = L = 2$). For economies within this set, the Edgeworth box can be used to provide intuition for the result.

Theorem 1 *For any economy satisfying Assumptions 1-9 with $H = S = L = 2$ and certainty equilibria* $\left\{\left(\left(\tilde{x}_n^h, \tilde{z}_n^h\right)_{h \in \mathcal{H}}, \tilde{p}_n, \tilde{q}_n\right)_{n=1,..,N}\right\}$, *a sunspot equilibrium* $\left(\left(x^h, z^h\right)_{h \in \mathcal{H}}, p, q\right)$ *in which sunspots matter can never be a randomization over the certainty equilibria.*

Proof. See Section 6. ∎

The logic of this statement is as follows. Consider any economy within the specified set of economies in which a *sunspot equilibrium* exists and the sunspots matter. For this economy, there may exist multiple *certainty equilibria*; I do not make assumptions to rule out this possibility. Even so, Theorem 1 asserts that the *sunspot equilibrium* is not related to the set of *certainty equilibria* using any misguided notion we may have about the nature of sunspots. The misguided notion to which I refer specifies that sunspots serve to coordinate beliefs on one of the possible vectors of *certainty equilibrium* prices. This is precisely the definition of *randomization* that I have posited, and given this definition, Theorem 1 declares that a *sunspot equilibrium* can never be a *randomization* over the *certainty equilibria*.

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[12] The convention is that $D_L f : \mathbb{R}^L \to \mathbb{R}$ denotes the derivative of the mapping f with respect to its Lth element.

5 Concluding Remarks

The presence of sunspot effects is independent of the number of underlying *certainty equilibria.* The first contribution of this paper has been to complement the literature by proving that a multiplicity of *certainty equilibria* is not a necessary condition for sunspot effects in a two-period model with numeraire assets whose payouts are independent of sunspots.

The second contribution is to analyze the necessity of multiple *certainty equilibria* for sunspot effects in a broader class of economies. An intuitive explanation of sunspot effects is that they arise as a *randomization* over *certainty equilibria* and sunspots do nothing more than coordinate beliefs. Contrary to this intuition, this paper's results have shown that *sunspot equilibria* need not be characterized as a *randomization* over multiple *certainty equilibria.*

6 The Proofs

6.1 Proof of Lemma 1

Fix a function $\omega : \mathcal{S} \to \{1, ..., N\}$ with $\#\omega(\mathcal{S}) > 1$ and consider any $n, m \in \omega(\mathcal{S})$ such that $m \neq n$. The equilibria $\left(\left(\tilde{x}_n^h, \tilde{z}_n^h \right)_{h \in \mathcal{H}}, \tilde{p}_n, \tilde{q}_n \right)$ and $\left(\left(\tilde{x}_m^h, \tilde{z}_m^h \right)_{h \in \mathcal{H}}, \tilde{p}_m, \tilde{q}_m \right)$ are distinct. From Assumptions 7 and 8 (notably strict concavity),

$$\left(\tilde{x}_n^h \right)_{h \in \mathcal{H}} \neq \left(\tilde{x}_m^h \right)_{h \in \mathcal{H}}. \tag{5}$$

To prove part (i) of the lemma, suppose for contradiction that $\left(\tilde{x}_{n,L}^h(0) \right)_{h \in \mathcal{H}} = \left(\tilde{x}_{m,L}^h(0) \right)_{h \in \mathcal{H}}$. For the remaining commodities $l < L$ in the initial period, it must be the case that $\left(\tilde{x}_{n,l}^h(0) \right)_{h \in \mathcal{H}} = \left(\tilde{x}_{m,l}^h(0) \right)_{h \in \mathcal{H}}$. If it were otherwise, say $\tilde{x}_{n,l}^h(0) > \tilde{x}_{m,l}^h(0)$ without loss of generality for some (h, l), the first order conditions for household h with respect to commodities l and L, together with Assumption 8, dictate that $\tilde{p}_{n,l}(0) < \tilde{p}_{m,l}(0)$. The price inequality implies that $\tilde{x}_{n,l}^{h'}(0) > \tilde{x}_{m,l}^{h'}(0)$ $\forall h' \in \mathcal{H}$, a violation of market clearing.

Thus, using Eq. 5, $\left(\tilde{x}_n^h(1) \right)_{h \in \mathcal{H}} \neq \left(\tilde{x}_m^h(1) \right)_{h \in \mathcal{H}}$. There are two cases to consider: (A) $\left(\tilde{x}_{n,L}^h(1) \right)_{h \in \mathcal{H}} = \left(\tilde{x}_{m,L}^h(1) \right)_{h \in \mathcal{H}}$ and (B) $\left(\tilde{x}_{n,L}^h(1) \right)_{h \in \mathcal{H}} \neq \left(\tilde{x}_{m,L}^h(1) \right)_{h \in \mathcal{H}}$. For Case (A), using the exact same logic as in the preceding paragraph, we arrive at $\left(\tilde{x}_{n,l}^h(1) \right)_{h \in \mathcal{H}} = \left(\tilde{x}_{m,l}^h(1) \right)_{h \in \mathcal{H}}$ $\forall l < L$, which cannot hold. Thus, we are left to consider Case (B). In Case (B), there exists a household h' such that $\tilde{x}_{n,L}^{h'}(1) > \tilde{x}_{m,L}^{h'}(1)$, without loss of generality. Denote $\tilde{\lambda}_n^{h'}(0)$ as the Lagrange multiplier associated with the initial period and $\tilde{\lambda}_n^{h'}(1)$ as the Lagrange multiplier associated with the final period (for *certainty equilibrium n*). By Assumptions 4 and 7, $\tilde{\lambda}_n^{h'}(1) = \beta^{h'} D_L v^{h'} \left(\tilde{x}_n^{h'}(1) \right)$ is a strictly decreasing function of $\tilde{x}_{n,L}^{h'}(1)$. Thus, $\tilde{\lambda}_n^{h'}(1) < \tilde{\lambda}_m^{h'}(1)$ for this household h'. The first order conditions with respect to the asset choice $\tilde{z}^h$ for all households $h \in \mathcal{H}$ are given by:

$$\tilde{q}_n = \frac{\tilde{\lambda}_n^h(1)}{\tilde{\lambda}_n^h(0)} \text{ and } \tilde{q}_m = \frac{\tilde{\lambda}_m^h(1)}{\tilde{\lambda}_m^h(0)}. \tag{6}$$

From the initial supposition $\left(\tilde{x}_{n,L}^h(0) \right)_{h \in \mathcal{H}} = \left(\tilde{x}_{m,L}^h(0) \right)_{h \in \mathcal{H}}$, $\tilde{\lambda}_n^h(0) = \tilde{\lambda}_m^h(0)$ $\forall h \in \mathcal{H}$. Thus, using

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Eq. 6 for household h', $\tilde{q}_n < \tilde{q}_m$. Using Eq. 6 for all households $h \in \mathcal{H}$ implies $\tilde{\lambda}_n^h(1) < \tilde{\lambda}_m^h(1)$ $\forall h \in \mathcal{H}$. Again, using Assumptions 4 and 7 implies that $\tilde{x}_{n,L}^h(1) > \tilde{x}_{m,L}^h(1)$ $\forall h \in \mathcal{H}$, a contradiction of market clearing. Thus, Case (B) results in a contradiction, as did Case (A), completing the proof of part (i).

To prove part (ii) of the lemma, suppose for contradiction that $\tilde{q}_n D_L v_0^h\left(\tilde{x}_n^h(0)\right) = \tilde{q}_m D_L v_0^h\left(\tilde{x}_m^h(0)\right)$ $\forall h \in \mathcal{H}$. As $D_L v_0^h$ is a strictly decreasing function of $\tilde{x}_{n,L}^h(0)$, if $\tilde{q}_n = \tilde{q}_m$, then $\tilde{x}_{n,L}^h(0) = \tilde{x}_{m,L}^h(0)$ $\forall h \in \mathcal{H}$, a contradiction of part (i) of this lemma. If $\tilde{q}_n > \tilde{q}_m$, without loss of generality, then $\tilde{x}_{n,L}^h(0) > \tilde{x}_{m,L}^h(0)$ $\forall h \in \mathcal{H}$, a contradiction of market clearing. This completes the proof of part (ii).

6.2 Proof of Theorem 1

Suppose, without loss of generality, that $\omega(1) = 1$ and $\omega(2) = 2$. For the economy without extrinsic uncertainty, the first order conditions with respect to the asset choice and the numeraire commodities yield the following equilibrium equations for the asset prices:

$$\tilde{q}_n = \frac{\beta^h D_2 v^h\left(\tilde{x}_n^h(1)\right)}{D_2 v_0^h\left(\tilde{x}_n^h(0)\right)} \quad \forall n. \tag{7}$$

For the economy with extrinsic uncertainty, the first order conditions with respect to asset choice and the numeraire commodities yield the following equilibrium equation for the asset price:

$$q = \frac{\beta^h\left(\sum_{s \in \mathcal{S}} \frac{1}{2} D_2 v^h\left(x^h(s)\right)\right)}{D_2 v_0^h\left(x^h(0)\right)}. \tag{8}$$

By the definition of *randomization*, $p(s) = \tilde{p}_{\omega(s)}(1)$ $\forall s \in \mathcal{S}$. From the strict concavity of v^h (Assumption 4), $x^h(s) = \tilde{x}_{\omega(s)}^h(1)$ $\forall s \in \mathcal{S}$ and $\forall h$. This fact, using Eqs. 7 and 8, allows the proposed *sunspot equilibrium* asset price to be written as:

$$q = \frac{\sum_n \frac{1}{2}\tilde{q}_n D_2 v_0^h\left(\tilde{x}_n^h(0)\right)}{D_2 v_0^h\left(x^h(0)\right)}, \tag{9}$$

or equivalently:

$$\frac{q}{x_2^h(0)} = \frac{1}{2}\frac{\tilde{q}_1}{\tilde{x}_{1,2}^h(0)} + \frac{1}{2}\frac{\tilde{q}_2}{\tilde{x}_{2,2}^h(0)}. \tag{10}$$

Eq. 10 holds for both households $h = 1, 2$.

Define $\sigma = \frac{\tilde{q}_1}{\tilde{q}_1 + \tilde{q}_2}$. Eq. 9 then yields:

$$\frac{q}{\frac{1}{2}\tilde{q}_1 + \frac{1}{2}\tilde{q}_2} = \frac{\sigma D_2 v_0^h\left(\tilde{x}_1^h(0)\right) + (1 - \sigma) D_2 v_0^h\left(\tilde{x}_2^h(0)\right)}{D_2 v_0^h\left(x^h(0)\right)}.$$

By the strict convexity of $D_2 v_0^h$,

$$\frac{q}{\frac{1}{2}\tilde{q}_1 + \frac{1}{2}\tilde{q}_2} \geq \frac{D_2 v_0^h\left(\sigma \tilde{x}_1^h(0) + (1 - \sigma)\tilde{x}_2^h(0)\right)}{D_2 v_0^h\left(x^h(0)\right)}, \tag{11}$$

with strict inequality if $\tilde{x}_{1,2}^h(0) \neq \tilde{x}_{2,2}^h(0)$.

Supposing that $q \leq \frac{1}{2}\tilde{q}_1 + \frac{1}{2}\tilde{q}_2$, then Eq. 11 implies $D_2 v_0^h\left(\sigma \tilde{x}_1^h(0) + (1 - \sigma)\tilde{x}_2^h(0)\right) \leq D_2 v_0^h\left(x^h(0)\right)$

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for both households $h = 1, 2$, with strict inequality for one from Lemma 1(i). Thus, $\sigma \tilde{x}_{1,2}^h(0) + (1 - \sigma)\tilde{x}_{2,2}^h(0) \geq x_2^h(0)$ for both households $h = 1, 2$, with strict inequality for one. This violates market clearing and verifies that:

$$q > \frac{1}{2}\tilde{q}_1 + \frac{1}{2}\tilde{q}_2. \tag{12}$$

Given the definition of *randomization*, $z^h = \tilde{z}_1^h = \tilde{z}_2^h$ for both households $h = 1, 2$. Fixing the variables $\left(\tilde{q}_1, \tilde{q}_2, \tilde{z}^1\right)$, Figure 2 illustrates how the initial period consumption is determined in equilibrium.

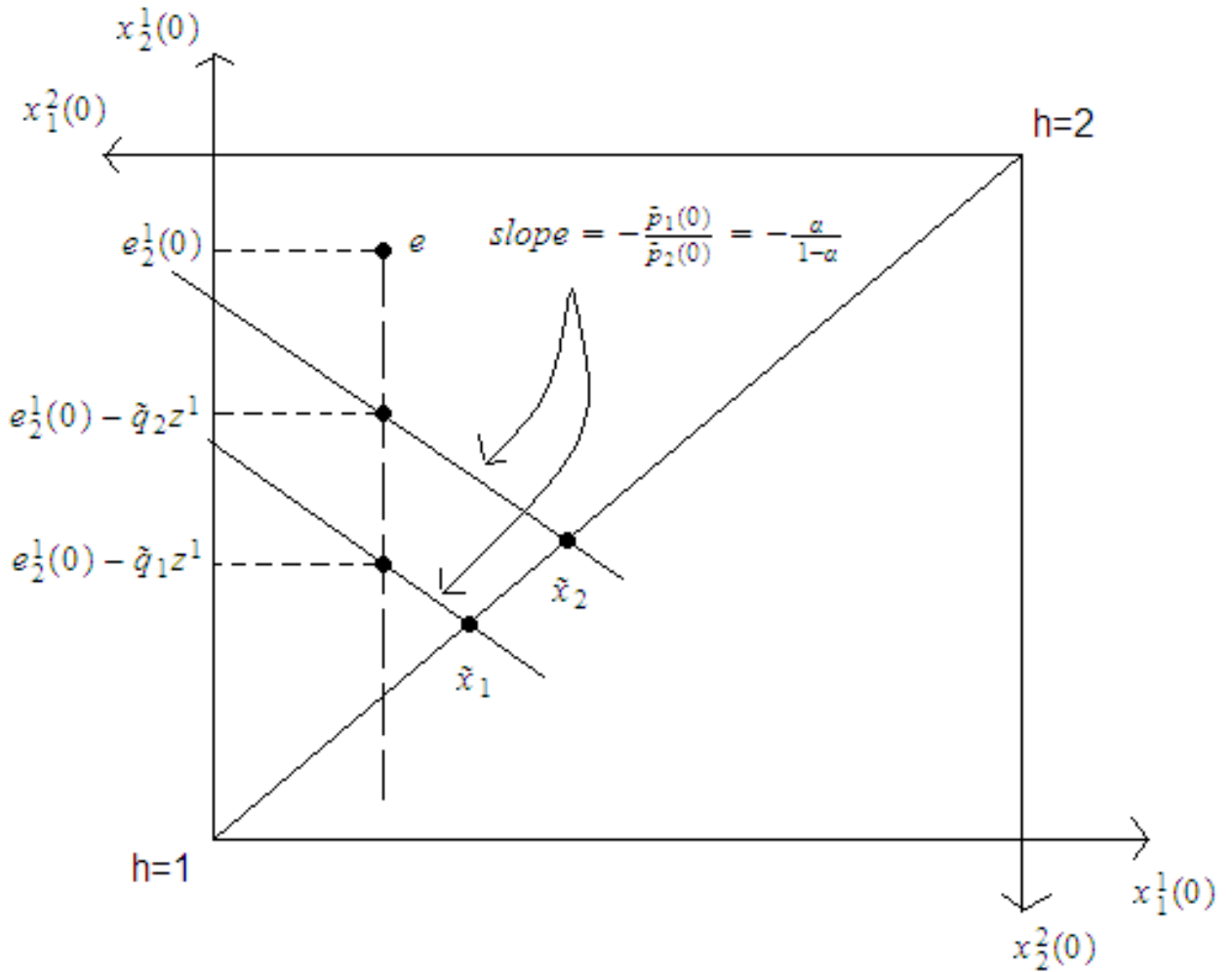Figure 2: Initial period Edgeworth box and multiple certainty equilibria

Using the figure, Lemma 1(i) requires $\tilde{q}_1 \neq \tilde{q}_2$. Otherwise, when $\tilde{q}_1 = \tilde{q}_2$, then $\tilde{x}_{1,2}^h(0) = \tilde{x}_{2,2}^h(0)$ for both households $h = 1, 2$, contradicting the result from Lemma 1(i).

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Suppose that $q = \theta \tilde{q}_1 + (1 - \theta) \tilde{q}_2$. The geometric argument is made using a figure in which $\theta \in [0, 1]$, but the principle applies for any $\theta \in \mathbb{R}$. Figure 3 illustrates how the initial period consumption in the *sunspot equilibrium* is determined relative to the initial period consumption in the two *certainty equilibria*. The figure magnifies the region of the Edgeworth box needed for the argument.
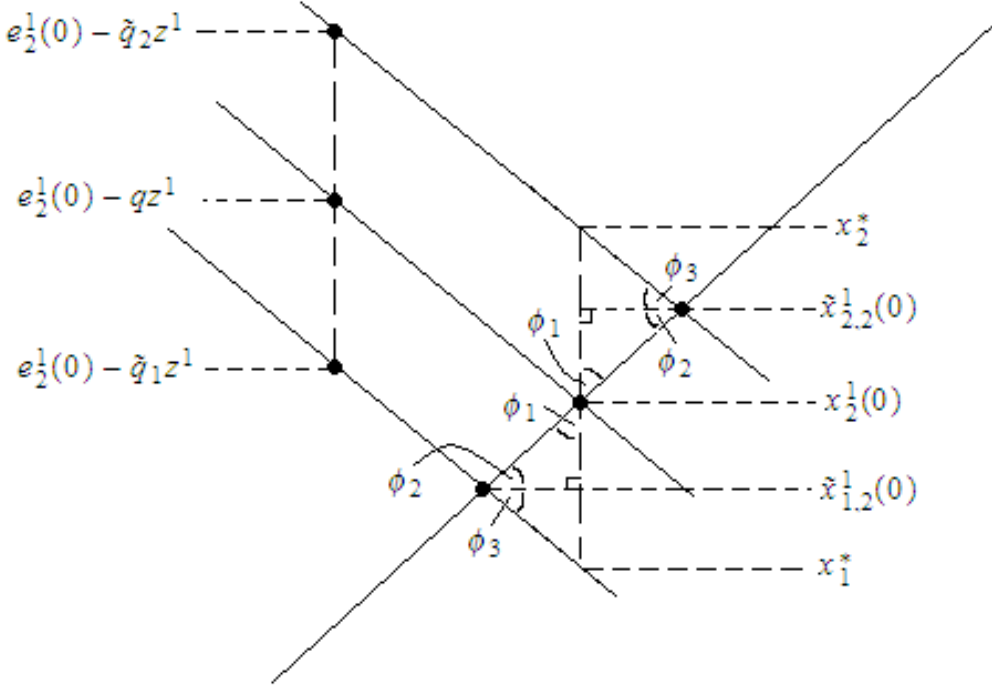

Figure 3: Initial period Edgeworth box with sunspot equilibrium variables

Consider the figure as we walk through the geometry. The angles ϕ_1 are identical, implying that the angles ϕ_2 are identical. As the budget lines are parallel, the angles ϕ_3 are identical. As $q = \theta \tilde{q}_1 + (1 - \theta) \tilde{q}_2$, then $x_2^1(0) = \theta x_1^* + (1 - \theta) x_2^*$. Using the equality among the angles ϕ_2 and ϕ_3, then

$$\frac{x_2^* - \tilde{x}_{2,2}^1(0)}{\tilde{x}_{2,2}^1(0) - x_2^1(0)} = \frac{\tilde{x}_{1,2}^1(0) - x_1^*}{x_2^1(0) - \tilde{x}_{1,2}^1(0)}. \tag{13}$$

This Eq. 13, together with $x_2^1(0) = \theta x_1^* + (1 - \theta) x_2^*$ and a small amount of algebra, implies $x_2^1(0) = \theta \tilde{x}_{1,2}^1(0) + (1 - \theta) \tilde{x}_{2,2}^1(0)$. Thus,

$$x_2^h(0) = \theta \tilde{x}_{1,2}^h(0) + (1 - \theta) \tilde{x}_{2,2}^h(0) \tag{14}$$

for both households $h = 1, 2$.

If $\frac{\tilde{q}_1}{\tilde{x}_{1,2}^h(0)} = \frac{\tilde{q}_2}{\tilde{x}_{2,2}^h(0)}$ for some household h, then using Eq. 6 and Assumption 7, $\tilde{x}_{1,2}^h(1) = \tilde{x}_{2,2}^h(1)$. By market clearing, $\tilde{x}_{1,2}^{h'}(1) = \tilde{x}_{2,2}^{h'}(1)$ for $h' \neq h$. Using Eq. 6 and Assumption 7 again, $\frac{\tilde{q}_1}{\tilde{x}_{1,2}^{h'}(0)} = $

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$\frac{\tilde{q}_2}{\tilde{x}_{2,2}^{h'}(0)}$. As this violates Lemma 1(ii), then $\frac{\tilde{q}_1}{\tilde{x}_{1,2}^{h}(0)} \neq \frac{\tilde{q}_2}{\tilde{x}_{2,2}^{h}(0)}$ for both households $h = 1, 2$.

For any household $h = 1, 2$, Eq. 10, the initial specification $q = \theta\tilde{q}_1 + (1 - \theta)\tilde{q}_2$, and Eq. 14 imply that:

$$\frac{\tilde{q}_1}{\tilde{x}_{1,2}^{h}(0)} \left(\theta\tilde{x}_{1,2}^{h}(0) - (1 - \theta)\tilde{x}_{2,2}^{h}(0) \right) = \frac{\tilde{q}_2}{\tilde{x}_{2,2}^{h}(0)} \left(\theta\tilde{x}_{1,2}^{h}(0) - (1 - \theta)\tilde{x}_{2,2}^{h}(0) \right). \qquad (15)$$

Given $\frac{\tilde{q}_1}{\tilde{x}_{1,2}^{h}(0)} \neq \frac{\tilde{q}_2}{\tilde{x}_{2,2}^{h}(0)}$ for both households $h = 1, 2$, Eq. 15 is only satisfied for both households if $\theta\tilde{x}_{1,2}^{h}(0) = (1 - \theta)\tilde{x}_{2,2}^{h}(0)$ for $h = 1, 2$. The market clearing condition $\tilde{x}_{1,2}^{1}(0) + \tilde{x}_{1,2}^{2}(0) = \tilde{x}_{2,2}^{1}(0) + \tilde{x}_{2,2}^{2}(0)$ implies that $\theta = \frac{1}{2}$ is the only possible value. However, this is inconsistent with our initial specification $q = \theta\tilde{q}_1 + (1 - \theta)\tilde{q}_2$ and Eq. 12. This completes the proof.

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