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DISTRIBUTIONS: THE CASE OF LUMPY INVESTMENTS

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# Detecting Propagation Effects by Observing Aggregate Distributions: The Case of Lumpy Investments

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## Abstract

By using an extensive panel data set of Italian firms, we show empirically that the fraction of firms that engage in a lumpy investment follows a non-normal, double-exponential distribution across region-year. We propose a simple sectoral model that generates the double-exponential distribution that arises from the complementarity of the firms' lumpy investments within a region. We calibrate the degree of complementarity by estimating an individual firm's behavior with

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the firm-level data. Simulations show that the degree of complementarity estimated at the firm level is consistent with the double-exponential fluctuations observed at the aggregate level.

**Keywords:** Interaction models, strategic complementarity, propagation effect, non-Gaussian fluctuations, double-exponential distribution

**JEL codes:** L16, E22

## 1 Introduction

Despite the presence of significant aggregate fluctuations in outputs in the short run, it is often difficult to identify the underlying shocks. While there exists much literature on the estimation of unobserved aggregate shocks using business cycle models, it is puzzling that we do not observe corresponding large shocks on the fundamentals reported by practitioners in normal times. This issue would be resolved if idiosyncratic shocks are the source of aggregate fluctuations. However, this explanation leads to another question as to why the law of large numbers does not cancel out the effects of idiosyncratic shocks.

This issue has led to a substantial amount of literature being devoted to the frictions that prevent the law of large numbers from averaging out the idiosyncratic shocks. For example, Jovanovic [20] presented several models that generate aggregate risk from idiosyncratic shocks through a propagation mechanism due to the strategic complementarity of the agents' actions. Moreover, Horvath [19] investigated the postponement of the effect of the law of large numbers in a sectoral model with a sparse input-output matrix. Recently, Gabaix [15] elucidated the role played by the idiosyncratic shocks on large firms, and Carvalho [8] extended Horvath [19] to a richer network structure of the input-output relations. Nirei [25] constructed a model with discrete actions

and strategic complementarity that generates a non-normal distribution for aggregate fluctuations with non-negligible variance even when the number of players is large.

This paper shows that the non-normal distribution well replicates the empirically observed distribution of aggregate investments. By using rich Italian firm-level data, we demonstrate that the aggregate fluctuations in the fraction of firms that engage in large investments in a region-year follow a non-normal, double-exponential distribution. We propose a model that includes a firm's lumpy investments and complementarity. We will analytically show that the model generates a double-exponential distribution. By calibrating the model with our estimate for the degree of complementarity obtained from the firm-level data, we show that the simulated distribution of the fraction of investing firms quantitatively replicates the empirically observed distribution.

Our model includes the complementarity and lumpiness of the firms' investments. The model is a simple version of Long and Plosser's [22] sectoral business cycle model, in which the firms' investments are complements in the sense that an investment by one firm induces the investment of another. The complementarity is a necessary ingredient of the model for generating the double-exponential distribution because if the firms' investments are independent, aggregate investments follow a normal distribution due to the central limit theorem.

We introduce lumpy investments to the Long-Plosser model. The lumpiness may come from the indivisibility of capital, or it may be a result of the optimization behavior when capital adjustments incur fixed costs, as we show in a supplementary model. In the class of sectoral models we investigate, we show that the lumpy investments always generate a double-exponential distribution, while the divisible investments necessarily lead to a normal distribution. The reason for the normal distribution is that when the investments are smooth, a firm's response to the other firms' investments is

locally linear. In this case, even though the firms' investments are correlated through complementarity, the idiosyncratic shocks to the firms are aggregated linearly, and the central limit theorem holds for the aggregate investments. Thus, both lumpiness (non-linearity) and complementarity are necessary for our model to generate a non-normal distribution.

To see why the propagation of lumpy investments leads to a double-exponential distribution, let us consider the simplest example of propagation effects where an agent acts with a certain probability, given that its neighbor acts. Then, the number of agents who act follows a geometric distribution, which has an exponential tail. The shape of the distribution is affected by the precise structure of the agents' interaction. For example, in Banerjee's [2] herd behavior model, an agent's action is affected by the actions taken by all the agents who acted before. Banerjee showed that, in this case, it is possible that all agents take the same action. In our model, the propagation effect does not degenerate to such a deterministic cascade, and it generally leads to a stochastic cascade with an exponential tail.

An obvious explanation for our empirical observation without resorting to the firms' interactions would be to consider an exogenous aggregate shock that follows a double-exponential distribution. However, this does not explain why the exogenous aggregate shock follows a double-exponential distribution. In reality, the aggregate investments are affected by many independent exogenous factors, and thus, the central limit theorem predicts that the aggregate shock should follow a normal distribution. Conversely, it is reasonable to have a set of unspecified exogenous factors as the main drivers of the aggregate fluctuations only when we observe a normal distribution for the aggregate fluctuations. Thus, we argue that the non-normal distribution of aggregate investments is an indicator of an interaction model, instead of a model with exogenous aggregate



shocks.

This paper has things in common with several threads of previous literature. Broadly, this study belongs to the research agenda on interacting agents, which was set forth by such authors as Brock and Durlauf [4] and Glaeser, Sacerdote, and Scheinkman [16]. The main objective of interacting agent models is to study the mechanisms by which the interactions of micro-level agents give rise to macro-level fluctuations. However, as Manski [24] pointed out, the interaction parameters of such models are often difficult to identify econometrically. Thus, the literature focuses not only on model building but also on developing a framework that facilitates empirical identification. In the context of information spillovers, Guiso and Schivardi [18] tackled the problem by using the structure of the firms' network. In the context of sectoral comovements over the business cycle, Shea [27, 28] and Bartelsman, Caballero, and Lyons [3] proposed to use the heterogeneity of the input-output matrix to estimate the interaction of sectoral activity. This paper follows the approach taken by Bartelsman et al. [3].

Long and Plosser [22] explained the sectoral comovements through sectoral complementarity. However, the complementarity in their model generates too small fluctuations, as pointed out by Lucas [23] and Dupor [13]. It fails to generate large fluctuations because the firms' decisions are locally linear in the aggregate output, and thus, the law of large numbers holds even though the model exhibits complementarity. Our paper differs from Long and Plosser [22] in that we introduce the firms' lumpy investment, which is one of the most commonly observed non-linear behaviors at the firm level. We show that the combination of non-linearity and complementarity generates quantitatively large fluctuations.

The lumpy investment behavior was empirically established by Doms and Dunne [12] and has attracted much attention from business cycle researchers since then. On

one hand, the empirical studies (Caballero and Engel [6]; Cooper, Haltiwanger, and Power [11]) suggested that the firms' lumpy investments contribute to the fluctuations of the aggregate investments through the variations at the extensive margin (i.e., firms at the threshold level of capital for lumpy adjustments). On the other hand, the theoretical research (Caplin and Spulber [7]; Caballero and Engel [5]) showed that the deviations of the extensive margin from its steady state level can quickly disappear in an aggregate economy with the firms' lumpy investments, the so-called (S,s) economy. This implies that complementarity and lumpiness (non-linearity) do not generate the aggregate fluctuations from the idiosyncratic shocks in an (S,s) economy with a continuum of firms. It is extensively debated whether the (S,s) economy gives rise to the business cycle fluctuation under realistic calibration in the framework of the continuum-of-firms models (Thomas [31]; Gourio and Kashyap [17]; Bachmann, Caballero, and Engel [1]).

In this context, we show that the idiosyncratic shocks can generate the aggregate fluctuations if there exist a large but finite number of firms, instead of a continuum of firms. In this case, the firms' capital will be distributed almost uniformly within an inaction band but with some small disturbances. If the firms' actions are independent of each other, this disturbance caused by the finite population alone generates little aggregate fluctuation due to the law of large numbers. If the firms' actions are correlated due to complementarity, the small aggregate fluctuations caused by the idiosyncratic shocks can be propagated. Suppose that a firm at the threshold level of capital decides to invest; this results in a discrete investment by the firm that is positioned closest to the threshold; then, the investment of the second firm can result in an investment by the next closest firm, and so on. This chain reaction stops at the point where there are no other firms positioned close enough to the threshold. The mechanism is similar to

a domino game: the line of falling tiles stops where two adjacent tiles are standing too far apart. Since the point where this domino effect stops can change by much through a slight change in the position of the tiles, the total number of falling tiles can vary considerably depending on the configuration of the initial points of the tiles. Similarly, when a finite number of firms interact with each other, the slight disturbance in the cross-section distribution of the position becomes an important source of aggregate fluctuations. Of course, we do not actually observe such a sequential effect in our static equilibrium model, but the domino effect provides us with a handy interpretation of the large shifts in equilibrium caused by the idiosyncratic shocks.

The rest of the paper is organized as follows. Section 2 describes the data and presents the evidence that the fraction of firms engaging in large investments follows a double-exponential distribution rather than a normal distribution. Section 3 presents a sectoral model with the lumpy investments. In Section 4, we estimate the structural decision rule on the firms' lumpy investments using micro-level data, and obtain an estimate for the degree of complementarity. We show our main results in Section 5. First, we numerically simulate the calibrated model with the estimate of the degree of complementarity, and show that the simulated distribution provides a good approximation of the double-exponential distribution. Second, we analytically derive the exponential tail of the aggregate fluctuations in our model. The model predicts that the slope of the double-exponential distribution is determined by the degree of complementarity among the firms' lumpy investments. Third, we show that our model without the lumpy investments generates a normal distribution. Section 6 concludes.

## 2 Aggregate fluctuations of the lumpy investments

### 2.1 Evidence for the lumpy investments in the Italian firm-level data

Figure 1 depicts the histograms of the investment-capital ratios of Italian firms of different sizes. To obtain the investment-capital ratios, we use the longitudinal data of Italian firms drawn from the Company Accounts Data Service (CADS). The data set consists of an unbalanced panel of over 30,000 firms from 20 regions, 23 industries, and 15 years from 1982 to 1996.<sup>1</sup> Both investment and capital are in real terms. Real capital is constructed using the perpetual inventory method by Cingano and Schivardi [9], in which the deflators and the depreciation rates of capital were obtained from the National Accounts data of the Italian National Institute for Statistics. We focus on the gross fixed investments of continuing firms, and thus, drop new entrants from our

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<sup>1</sup>The data set we use was compiled by Guiso and Schivardi [18]. The firms in the data set consist of the borrowers of leading banks in Italy, which may result in a selection bias. However, comparing the summary statistics of the data set with census data, Guiso and Schivardi [18] found that the sample is sufficiently representative despite the bias. We convert the original 4-digit industry code in our data set to a 2-digit industry code that is consistent with the OECD's industry classification. The following industries are included: Mining and quarrying; Food, beverages and tobacco; Textiles, apparel and leather; Wood products and furniture; Paper, paper products and printing; Industrial chemicals; Drugs and medicines; Rubber and plastic products; Iron and steel; Non-ferrous metals; Metal products; Non-electrical machinery; Electrical apparatus; Radio, TV and communication equipment; Shipbuilding and repairing; Other transport; Motor vehicles; Aircraft; Professional goods; Other manufacturing; Electricity, gas and water. The original data set contains 306,364 observations. We drop observations with missing industry or region codes and observations in the agricultural sector. We also drop observations of new entry, which include observations in the initial year 1982 and those with an investment-capital ratio greater than 1. As a result, we are left with 205,513 observations.

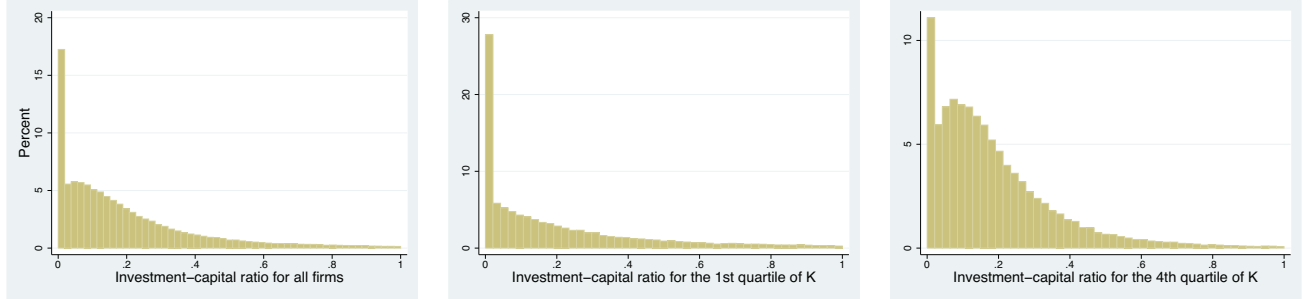


Figure 1: Histograms of the firms’ investment-capital ratios for different firm sizes in terms of their capital. The vertical axis shows the relative frequency of the observations in percent.

sample by deleting the observations that appear for the first time for a firm.

The large skewness and long right tail in the distribution of investment rates seen in Figure 1 has been widely observed in other studies as well (e.g., Doms and Dunne [12] and Cooper et al. [11]). Moreover, Doms and Dunne [12] found a characteristic lumpiness in the investment behavior: there were active capital adjustments in a short amount of time and almost no adjustments in the other periods. In order to concentrate on the lumpy investments behavior, we divide the firms’ investment behavior at any particular point in time into one of two categories: lumpy investments and inaction. That is, we convert the investment-capital ratio into a binary variable,  $d(i, t)$ , which takes 1 if the investment-capital ratio is greater than  $\bar{d}$  and 0 otherwise.

We set the threshold value for the lumpy investments as  $\bar{d} = 0.2$ , which is comparable to the threshold used in the literature on lumpy investments (e.g., Cooper et al. [11]). The results obtained in this paper are robust to the threshold value. In our data, 36% of the observations are classified as lumpy investments when using this threshold, and the sum of the lumpy investments on average accounts for 50% of the total fixed

investment in each year. This implies that the lumpy investments are an important factor in affecting the aggregate investments.

Figure 1 indicates that the distributions of the investment-capital ratios are similar across different sizes of the firm’s capital. For all the histograms, the relative frequency shows a slow decline from the level of about 6 percent when the investment-capital ratio is just above 0. The average investment-capital ratio for all observations is 0.194, while that for the first quartile and the last quartile in terms of size is 0.203 and 0.187, respectively. Moreover, the fraction of firms with the lumpy investments does not vary much across the firm sizes: it is 0.358 for all observations, and 0.367 for the first and 0.356 for the last quartile. Thus, the lumpy investment behavior as defined here seems homogeneous across the firm sizes.

## 2.2 Fluctuations of the fraction of investing firms

In order to capture the effect of the firms’ lumpy investments at an aggregate level, we consider the *fraction of investing firms*. The fraction of investing firms is defined as the number of firms that engage in a lumpy investment divided by the total number of firms in a region for a year. Let  $G_r$  denote the set of firms that belong to the same region  $r$ . Using the binary investment variable  $d(i, t)$ , we define the fraction of investing firms for each group as  $X(r, t) = \sum_{i \in G_r} d(i, t) / \#G_r$ .<sup>2</sup>

In this paper, we are interested in the variation of  $X(r, t)$  across region-year that is not explained by the exogenous aggregate or common shocks, such as national-level

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<sup>2</sup>The number of observations in  $G_r$  may change over time. The time subscript  $t$  is suppressed here for notational simplicity. We omit the regions that have a small number of firms, because the behavior of  $X$  can be overly volatile for those regions. The threshold for the small region is set as the fifth percentile of the size of the region group, which is 21.

shocks or region-specific fixed effects. To capture this unexplained variation in  $X(r, t)$ , we form a new fraction variable  $\tilde{X}(r, t)$  using the residual of the linear regression of  $X(r, t)$  on the year-dummy and region-dummy variables.<sup>3</sup>

Figure 2 shows the kernel density estimate of the distribution of  $\tilde{X}(r, t)$ . A normal density with the same variance as the sample is plotted alongside. We observe that the distribution exhibits a symmetric and leptokurtic pattern. The standard deviation of  $\tilde{X}(r, t)$  accounts for more than half of the standard deviation of  $X(r, t)$ . Thus, there is a considerable amount of variation in the fraction of investing firms that is not explained by common shocks at the national level or by any fixed effects.

### 2.3 Normality tests for the fluctuations of the fraction of investing firms

Our goal is to characterize and explain the distribution of the fraction of investing firms  $\tilde{X}$ . We first fit parametric distributions to the empirical distribution. Our null

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<sup>3</sup>We exclude the effect of common shocks on  $X(r, t)$  by subtracting the mean of  $X(r, t)$ , even though the investment decision is discrete. When the investment decision is binary as in our case, a common shock  $z$  affects the fraction  $X$  through the extensive margin, and thus, the impact of  $z$  is determined by the density of firms around the threshold of their binary decision. Consider a case where  $z$  affects the threshold of the firms' state (capital) through function  $x^*(z)$  where the firms obey a one-sided (S,s) policy in which they invest if their capital  $x_i$  is smaller than  $x^*$ . Then, the fraction of investing firms is written as  $\sum_{i \in G: x_i < x^*(z)} d(i) / \#G$ . If  $x_i$  follows the distribution function  $F$ , then, as the number of firms  $\#G$  tends to infinity, this fraction converges to  $F(x^*(z))$ , which is unbiasedly estimated by the average of  $X(r, t)$  across  $G_r$ . An underlying premise here is that each group has achieved the stationary state in which the distribution  $F$  of the state  $x_i$  is common across  $G_r$ . In a group with a finite number of firms, the actual realization of the fraction of investing firms can differ from  $F(x^*(z))$ . However, the error is as small as of order  $z^2$ .

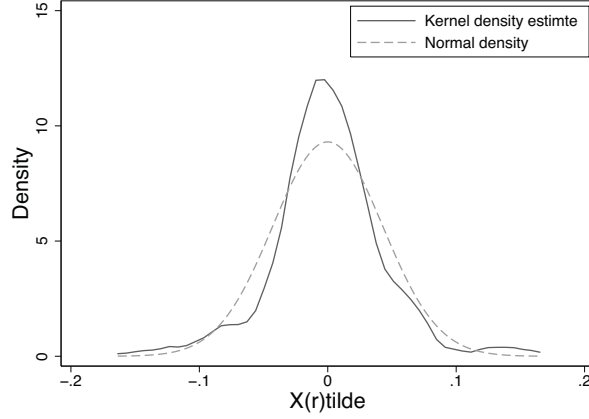


Figure 2: Kernel density estimate of  $\tilde{X}(r, t)$ .  $\tilde{X}$  is the fraction of investing firms  $X$  excluding the year effects and region-specific fixed effects. Width parameter is set as 0.009.

hypothesis is that  $\tilde{X}$  follows the normal distribution. One alternative hypothesis is that  $\tilde{X}$  follows the double-exponential distribution, in which  $|\tilde{X}|$  conditional on  $\tilde{X} > 0$  follows an exponential distribution with mean  $\lambda_+$  and  $|\tilde{X}|$  conditional on  $\tilde{X} < 0$  follows an exponential distribution with mean  $\lambda_-$ . In this case, the density function has two parts as  $\Pr(\tilde{X} = \tilde{x} | \tilde{x} > 0) = e^{-(1/\lambda_+)\tilde{x}} / (2\lambda_+)$  and  $\Pr(\tilde{X} = \tilde{x} | \tilde{x} < 0) = e^{-(1/\lambda_-)(-\tilde{x})} / (2\lambda_-)$ . The other alternative hypothesis is that  $\tilde{X}$  follows the Laplace distribution, in which  $|\tilde{X}|$  follows an exponential distribution with mean  $\lambda$ :  $\Pr(\tilde{X} = \tilde{x}) = e^{-(1/\lambda)|\tilde{x}|} / (2\lambda)$ . The Laplace distribution is a special case of the double-exponential distribution, with restriction  $\lambda_+ = \lambda_-$ .

We start from conventional normality tests based on the third and fourth moments. Table 1 shows those moments of  $\tilde{X}$ . The large kurtosis indicates that  $\tilde{X}$  is leptokurtic. The normality tests such as the skewness-kurtosis test, the Shapiro-Wilk test, and the



Variable	Obs.	Mean	Std. Dev.	Skewness	Kurtosis	Min	Max	Median
$\#G_r$	268	766.2	1094	2.363	8.599	21	5502	291
$X(r, t)$	268	0.326	0.078	0.107	3.618	0.081	0.579	0.325
$\tilde{X}(r, t)$	268	0	0.043	0.069	5.162	-0.154	0.156	-0.001

Table 1: Summary statistics of the size of region groups,  $\#G_r$ , the fraction of investing firms,  $X(r, t)$ , and the fraction of investing firms excluding the common shocks and fixed effects,  $\tilde{X}(r, t)$ .

Shapiro-Francia test overwhelmingly reject the normality hypothesis for  $\tilde{X}(r, t)$ . Note that the kurtosis of  $X$  is smaller than that of  $\tilde{X}$ . This suggests that the yearly common shocks and fixed effects “round” the distribution of aggregate fluctuations toward a normal distribution much like the addition of many random noises will decrease the kurtosis.

Next, we conduct another normality test that is based only on kurtosis and not on skewness. The concentration of the fourth moment is useful for testing the hypotheses among a class of symmetric distributions, in our case, the Laplacian and the Gaussian. The true kurtosis is equal to 3 under the Gaussian hypothesis and 6 under the Laplacian hypothesis. Consider the method of moment estimator  $(\sum_{i=1}^N x_i^4/N)/(\sum_{i=1}^N x_i^2/N)^2$ . Its asymptotic variance is  $V(x_i^4)/\sigma^2 N$ , which is equal to  $96/N$  under the Gaussian hypothesis. Thus, the sample kurtosis in Table 1 rejects the Gaussian hypothesis at the 1% significance level, although the estimator is not efficient. In contrast, the asymptotic variance of the method of moment estimator is  $(8! - 4!^2)/2^8 N = 155.25/N$  under the Laplacian hypothesis. Thus, the Laplacian hypothesis is not rejected for  $\tilde{X}(r, t)$  at the 5% significance level by the method of moment test. As such, the

	Maximum likelihood estimates				Log Likelihood	Vuong's statistic	
Exponential	$\lambda_+$	0.0310 (0.0027)	$\lambda_-$	0.0306 (0.0026)	479.1	2.44	2.96
Laplace	$\lambda$	0.0308 (0.0019)		465.9	0.22	Null	
Normal	$\mu$	0.000 (0.0026)	$\sigma$	0.0429 (0.0002)	464.3	Null	

Table 2: Parameter estimates for various distributions. Standard errors are in parentheses.

normality test based on kurtosis rejects the Gaussian hypothesis while failing to reject the Laplacian hypothesis.

Table 2 shows the maximum likelihood estimates for each parametrization. The estimated mean of the exponential distribution is 0.031 for the positive side and 0.0306 for the negative side. The estimated  $\lambda$  for the Laplacian distribution is 0.0308, which is the middle value for the exponential slopes for the positive and negative sides. We test the normality hypothesis using a likelihood-based test. Let  $L(i; H)$  denote the likelihood of a sample point  $i$  under the hypothesis  $H$ . Defining the log likelihood ratio for each  $i$  as  $l_i = \log L(i; H_1) - \log L(i; H_0)$ , Vuong [33] showed that the statistic  $V \equiv \sqrt{N} \bar{l}_i / \text{Std}(l_i)$  follows a standard normal distribution if the hypotheses  $H_0$  and  $H_1$  are equivalent. Thus, if  $V$  as computed for  $H_1$  against the null  $H_0$  is greater than 1.96, then the null is rejected in favor of the alternative at the 5% significance level. Vuong's statistics are reported in Table 2 for the exponential and the Laplacian hypotheses against the Gaussian null hypothesis and for the exponential against the Laplacian. The estimates show that the exponential hypothesis is favored against the Gaussian and the Laplacian hypotheses at the 5% significance.

To sum this section up, we constructed the fraction of lumpy-investing firms at the

regional level, and formed the distribution of the fractions that are not explained by national-level shocks or regional fixed effects. We showed that the empirical distribution of the fractions favors the model that generates an exponential pattern over those that generate normal distributions. In the following sections, we analyze the mechanism that generates the exponential pattern of the fluctuations.

### 3 Model of investment fluctuations

#### 3.1 Model and equilibrium

This section presents a simple model of investment fluctuations that generates an exponential tail for the distribution of  $X$  as observed in the previous section. One obvious hypothesis for the exponential distribution is that there is an exogenous common shock that follows an exponential distribution. However, economies usually have many exogenous independent factors, which will constitute a normally distributed common shock by the central limit theorem rather than an exponentially distributed one. Therefore, we pursue a model with complementarity and non-linearity (lumpiness) in which the aggregate fluctuations are not driven by the exogenous common shocks.

Consider that there are  $N$  firms, each of which monopolistically supplies a differentiated good subject to the following production function:

$$y_i = A_i n_i^\alpha \tag{1}$$

for  $i = 1, 2, \dots, N$ .  $A_i$  denotes the exogenous productivity, and  $n_i$  denotes the inputs. The returns-to-scale,  $\alpha$ , is less than one. We assume that the operation scale,  $n_i$ , of the firm is constrained to the following discrete set:

$$n_i \in \{1, \lambda_i^{\pm 1}, \lambda_i^{\pm 2}, \dots\}, \tag{2}$$

where  $\lambda_i > 1$  is an exogenous parameter for lumpiness. The constraint assumes that operations are subject to indivisibility in production units. For example, we can imagine the case where the firms must choose an integer for the number of plants they operate. Besides the indivisibility of capital, the fixed costs involved in adjusting capital are another important source of the discreteness captured by (2). In Appendix A, we present an extension of the model where the firm's state variable is capital and capital adjustment requires a fixed cost. In this case, the firm adjusts its capital only if the adjustment increases the value of the firm by more than the fixed cost. When the firm decides to adjust its capital, it does so through a lumpy investment, and the extent of the lumpiness is determined by the magnitude of the fixed cost. Thus, the fixed cost of adjustments can provide a micro-foundation for the discreteness constraint.

Input  $n_i$  is a composite good produced by using all the goods in a CES manner:

$$n_i = \left( \sum_{j=1}^N \chi_{i,j}^{1/\sigma} z_{i,j}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \quad (3)$$

where  $\sigma > 1$  and  $\chi_{i,j}$  denotes the input weights for industry  $i$ . Next, we define the aggregate index of the input prices for industry  $i$  as a weighted sum:

$$P_i \equiv \left( \sum_{j=1}^N \chi_{i,j} p_j^{1-\sigma} \right)^{1/(1-\sigma)}. \quad (4)$$

Then, the derived demand for good  $j$  is given in an isoelastic form as  $z_{i,j}^* = (p_j/P_i)^{-\sigma} \chi_{i,j} n_i$ . Thus, the derived demand  $z_{i,j}^*$  changes proportionally to the weight  $\chi_{i,j}$  when the input  $n_i$  varies. The minimized cost satisfies  $\sum_j p_j z_{i,j}^* = P_i n_i$ .

We assume that the firm is owned by a representative household. The household has a preference  $U(C)$  which is increasing in a composite consumption good  $C = \left( \sum_{j=1}^N \chi_{C,j}^{1/\sigma} z_{C,j}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}$ . The income of the representative household is derived only from the aggregate profit  $\sum_j \pi_j$ . We can incorporate labor in this model without al-

tering our results, but we refrain from doing so in order to avoid complicating the notation. We define the aggregate index of consumer prices as  $P_C \equiv (\sum_{j=1}^N \chi_{C,j} p_j^{1-\sigma})^{1/(1-\sigma)}$ , and normalize it to 1. Then, the derived demand for  $z_{C,j}$  is  $z_{C,j}^* = p_j^{-\sigma} \chi_{C,j} C$ . The optimal expenditure satisfies  $\sum_j p_j z_{C,j}^* = C = \sum_j \pi_j$ .

The market clearing condition for good  $i$  is  $y_i = \sum_j z_{j,i} + z_{C,i}$ . In this setup, the total demand function becomes isoelastic:

$$y_i = p_i^{-\sigma} Y_i, \quad (5)$$

where  $Y_i$  is the aggregate demand factor for good  $i$  defined as follows:

$$Y_i \equiv \sum_j P_j^\sigma n_j \chi_{j,i} + C \chi_{C,i}. \quad (6)$$

The aggregate output is defined as  $Y \equiv \sum_{i=1}^N p_i y_i = \sum_{i=1}^N y_i^{1-1/\sigma} Y_i^{1/\sigma}$ .

The equilibrium is defined as a pair of allocation  $(y_i, n_i, z_{i,j}, C, z_{C,j})$  and prices  $(p_i)$  such that  $(y_i, n_i, z_{i,j}, p_i)$  maximizes monopolist  $i$ 's profit  $\pi_i = p_i y_i - \sum_j p_j z_{i,j}$  subject to (1,2,3,5),  $(C, z_{C,j})$  maximizes  $U(C)$  subject to  $C = \sum_i \pi_i$ , and the markets clear for all  $N$  goods.

Since the economy has non-convex costs, multiple equilibria may exist for some productivity profiles  $(A_i)$ . We select an equilibrium that is close to its smoothly-adjusting counterpart in the following sense. The smoothly-adjusting equilibrium is defined by the equilibrium of our model without indivisibility constraint (2). The smoothly-adjusting equilibrium exists uniquely. Then, under the smoothly-adjusting equilibrium level of demand  $(Y_i)$  and input prices  $(P_i)$ , we compute the net number of firms that adjust their capital upward if the indivisibility constraint is imposed. If the net number of upwardly adjusting firms is positive, we select an equilibrium with the smallest aggregate output that is larger than that of the smoothly-adjusting

equilibrium. If the net number is negative, we select an equilibrium with the largest aggregate output that is smaller than that of the smoothly-adjusting equilibrium.<sup>4</sup> By this equilibrium selection, we exclude an equilibrium that is far from the smoothly-adjusting equilibrium in the case when a nearer equilibrium exists. Thus, we exclude the fluctuations that occur purely from information coordination such as in sun-spot equilibria.

A note on the general equilibrium properties of this model is in order. In this model, the representative household delegates the investment decision to the firms. Since each firm's operation faces non-convex costs due to the indivisibility constraint, there is a possibility that the equilibrium outcome differs from the first-best that is achieved if the representative household directly chooses the allocation  $(n_i)$ . In this sense, ours is a model of coordination failure in the capital market. In order to truly represent the coordination failure in the capital market, however, we need a dynamic model in which the intermediate input is replaced with investments. Appendix A shows the dynamic extension of our model. Since this paper is concerned with a regional economy, we can assume that the discount factor of the firms is exogenously determined as in a small open economy. Then, our results will hold in a dynamic setup. Our results may change in the dynamic general equilibrium of a closed economy, since the opportunity cost of investments is determined by the marginal rate of intertemporal substitution, which is affected by the investments. We defer the analysis of this closed economy to a separate paper (Nirei [26]).

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<sup>4</sup>Alternatively, we can select an equilibrium with an aggregate output closest to that of the smoothly-adjusting equilibrium. This does not change our results much (see Nirei [26] for details). This alternative selection has a simpler intuition, but its analysis becomes complicated.

### 3.2 Firms' threshold rule

At the optimal point of the firms' cost minimization, the cost of inputs satisfies  $\sum_j p_j z_{i,j} = P_i n_i$ . Then, the monopolist's problem is written as  $\max_{n_i} p_i y_i - P_i n_i$  subject to (1,2,5). Because of the indivisibility constraint (2), the firms' optimal behavior follows a threshold rule, in which the firms adjust their input only if it is outside the "inaction band"  $[n_i^*, \lambda_i n_i^*]$ . The optimal threshold  $n_i^*$  is determined by the condition that the profits must be equalized at the lower and upper bounds of the inaction band. Thus, the threshold is solved as

$$n_i^* = \left( \frac{\lambda_i^{\alpha(1-1/\sigma)} - 1}{\lambda_i - 1} A_i^{1-1/\sigma} Y_i^{1/\sigma} P_i^{-1} \right)^{\frac{1}{1-\alpha(1-1/\sigma)}}. \quad (7)$$

Using this threshold, the equilibrium is determined as a profile of  $n_i$  that satisfies  $n_i^* \leq n_i < \lambda_i n_i^*$  and the indivisibility constraint (2).

We can further characterize the threshold rule by considering the homogeneous case when the input weight  $\chi_{i,j}$  is the same across  $i$ :  $\chi_{i,j} = \chi_j$  for all  $i$  including  $\chi_{C,j}$ . The case of heterogeneous weights is investigated in the next section through numerical simulations. The homogeneous weight implies that the aggregate output can be simplified as  $Y = (\sum_i \chi_i^{1/\sigma} y_i^{1-1/\sigma})^{\sigma/(\sigma-1)}$  and that the input prices ( $P_i$ ) are equal to the price of consumption good  $P_C$ , which is normalized to one. The optimal threshold  $n_i^*$  is then given as the function of input profile ( $n_j$ ) and productivities ( $A_j$ ):

$$n_i^* = a_i A_i^{\frac{1-1/\sigma}{1-\alpha(1-1/\sigma)}} Y^{\frac{1}{\sigma(1-\alpha(1-1/\sigma))}} = a_i A_i^{\frac{1-1/\sigma}{1-\alpha(1-1/\sigma)}} \left( \sum_j \chi_j^{\frac{1}{\sigma}} A_j^{1-\frac{1}{\sigma}} n_j^{\alpha(1-1/\sigma)} \right)^{\frac{1}{(\sigma-1)(1-\alpha(1-1/\sigma))}}, \quad (8)$$

where  $a_i \equiv \left( \chi_i^{1/\sigma} (\lambda_i^{\alpha(1-1/\sigma)} - 1) / (\lambda_i - 1) \right)^{\frac{1}{1-\alpha(1-1/\sigma)}}$ .

We consider small shocks on productivities ( $A_i$ ) and the firms' response to the shocks. We start from an initial equilibrium input profile ( $n_i$ ). We define the gap

between the actual input and the lower threshold at the initial equilibrium as follows:

$$s_i \equiv \frac{\log n_i - \log n_i^*}{\log \lambda_i}. \quad (9)$$

Thus,  $s_i$  is the state variable associated with the initial equilibrium before the perturbation. Then, the optimal rule for a firm is to invest upon the perturbation if  $s_i$  is sufficiently close to 0:

$$s_i < \frac{d \log n_i^*}{\log \lambda_i} \approx \frac{1}{\log \lambda_i} \left( \frac{\beta(\sigma - 1)}{\alpha} d \log A_i + \beta \sum_{j=1}^N b_j \frac{\Delta \log n_j}{\log \lambda_j} \right), \quad (10)$$

where

$$\beta \equiv \frac{1}{\sigma(1/\alpha - 1) + 1}, \quad (11)$$

$$b_j \equiv \frac{\chi_j^{\frac{1}{\sigma}} A_j^{1-\frac{1}{\sigma}} n_{j,0}^{\alpha(1-1/\sigma)} \lambda_j^{\alpha(1-1/\sigma)} - 1}{\sum_l \chi_l^{\frac{1}{\sigma}} A_l^{1-\frac{1}{\sigma}} n_{l,0}^{\alpha(1-1/\sigma)} \alpha(1-1/\sigma)}. \quad (12)$$

The right-hand side of (10) is equal to the derivative on the left-hand side exactly at the limit  $N \rightarrow \infty$ , as shown in Appendix B.

The propagation effect stemming from  $n_j$  occurs both through the input-output relations among the firms and through the aggregate demand externality in which an investment increases the household's income, and thus, increases the consumption demand of other goods. An investment by firm  $j$  increases  $Y$ , which lowers the threshold and increases the likelihood of firm  $i$  investing. The magnitude of this impact ( $\Delta \log n_j$ ) on the gap of  $i$ , and thus, on the likelihood of  $i$ 's investment, through the complementarity is expressed by the following term in (10):

$$\frac{\beta b_j}{\log \lambda_i} \frac{\Delta \log n_j}{\log \lambda_j}. \quad (13)$$

Expression (13) is reduced to approximately  $\beta/N$  in the homogeneous case. We can see this by noting that  $\Delta \log n_j / \log \lambda_j = 1$  for an upward lumpy adjustment, that



the first fraction of  $b_j$  in (12) is  $1/N$  if the firms are homogeneous, and that the second fraction is approximately  $\log \lambda_j$  when the lumpiness is small ( $\lambda_j$  is close to 1). The coefficient  $\beta$  represents the impact of firm  $j$ 's lumpy investment on the threshold of firm  $i$ . Thus, we call  $\beta$  the degree of complementarity.

The degree of complementarity  $\beta$  also represents the asymptotic mean (as  $N \rightarrow \infty$ ) of the number of firms that are induced to adjust due to a firm's adjustment. Consider the case where  $s_i$  is a random variable uniformly distributed on the unit circumference of a circle as in a steady state of a one-sided (S,s) economy.<sup>5</sup> In this case, the probability that  $j$ 's adjustment induces  $i$  to adjust is equal to  $\beta b_j / \log \lambda_i$ . The probability is reduced to  $\beta/N$  for the homogeneous case with small lumpiness.<sup>6</sup> Thus, the mean number of firms that are induced to adjust due to a firm's adjustment is the probability  $\beta/N$  times population  $N - 1$ .

## 4 Estimation of the complementarity by firm-level data

In this section, we estimate the degree of complementarity  $\beta$ , which is the impact of a lumpy investment on the likelihood of lumpy investments by other firms within

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<sup>5</sup>Since a one-sided (S,s) rule has the adjustments only in one direction (e.g., an increase in capital or inaction), the invariant distribution of  $s_i$  becomes uniform in various setups. See Caplin and Spulber [7] and Caballero and Engel [5].

<sup>6</sup>We can also consider the case where  $s_i$  does not follow a uniform distribution, as in the case of transition to a steady state. We suppose, as a first-order approximation of such a general distribution of  $s_i$ , that the density around  $s_i = 0$  is constant  $q$ . Then, the probability that  $j$ 's adjustment induces  $i$  to adjust is equal to  $q\beta b_j / \log \lambda_i$ , and the remaining analysis continues to hold with a modified parameter.

the same region. In our model, the optimal decision rule for the firms' indivisible investment is  $d(i) = 1$  if  $s_i < \Delta \log n_i^* / \log \lambda$ . Thus, the optimal threshold rule for the firms' lumpy investments in our model, (7), provides us with a regression equation. Using (7), the firms choose  $d(i) = 1$  if

$$\frac{1}{1 - \alpha(1 - 1/\sigma)} \left( \log \left( \frac{\lambda_i^{\alpha(1-1/\sigma)} - 1}{\lambda_i - 1} A_i^{1-1/\sigma} \right) - s_i + \frac{1}{\sigma} \log Y_i - \log P_i \right) > 0. \quad (14)$$

Assuming that the stationary distribution of  $s_i$  is uniform over  $[0, 1)$  as in the standard one-sided (S,s) economy, the decision rule implies a linear probability model  $\Pr(d(i) = 1) = \Delta \log n_i^* / \log \lambda$ . However, we do not have a sufficient set of variables in our data set to fully specify  $\Delta \log n_i^*$ . Considering this, we employ the following probit model for our estimation:

$$\Pr(d(i, t) = 1) = \Phi(\gamma_D D_{i,t} + \gamma_Z Z_{i,t} + \beta^{OW} X_{i,t}^{OW} + \beta^{IW} X_{i,t}^{IW}). \quad (15)$$

The terms inside  $\Phi$  correspond to the left-hand side of (14).  $D_{i,t}$  and  $Z_{i,t}$  capture the changes in  $\log A_i$ .  $D$  denotes dummy variables for the industry and region-year pairs.  $Z_{i,t}$  includes the industry-wise growth in the productivity of capital, the cash flow in the previous period, and the growth in the real aggregate investment in the industry-region to which firm  $i$  belongs.<sup>7</sup>  $X_{i,t}^{OW}$  denotes the changes in  $\log Y_i$  in (14) caused by the lumpy investments by the firms in the same region as  $i$ , and  $X_{i,t}^{IW}$  captures a part of the changes in  $\log P_i$  caused by the lumpy investments. They are defined as

$$X_{i,t}^{OW} \equiv \sum_l \frac{\pi(l_i, l)}{\sum_h \pi(l_i, h)} \frac{\#G_{r_i, l}}{\#G_l} X(r_i, l, t), \quad (16)$$

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<sup>7</sup>The capital productivity is given by the residual  $\log Y - \alpha_K \log K - \alpha_L \log L$ , where  $Y$ ,  $K$ , and  $L$  are the average industry-wise output and inputs for each period, and  $\alpha_K$  and  $\alpha_L$  are the average expenditure shares during the sample periods in the industry to which  $i$  belongs. The cash flow is normalized by the firm's nominal size of capital.

$$X_{i,t}^{IW} \equiv \sum_l \frac{\pi(l, l_i)}{\sum_h \pi(h, l_i)} \frac{\#G_{r_i, l}}{\#G_l} X(r_i, l, t), \quad (17)$$

where  $\pi(l, h)$  denotes the  $(l, h)$  coordinate of the input-output matrix,  $G_l$  denotes the set of firms that belong to industry  $l$ ,  $G_{r, l}$  denotes the set of firms that belong to region  $r$  and industry  $l$ ,  $l_i$  denotes the industry that firm  $i$  belongs to, and  $r_i$  denotes the region that firm  $i$  belongs to.

We show that  $X_{i,t}^{OW}$  corresponds to the changes in  $\log Y_i$  caused by the lumpy investments in the structural model. When the vector  $(P_j)$  and consumption  $C$  are fixed, the effect of  $n_j$  on the threshold  $n_i^*$  can be derived as

$$\frac{\partial \log n_i^*}{\partial \log n_j} = \frac{\beta P_j^\sigma n_j \chi_{j,i}}{\alpha Y_i} = \beta \frac{z_{j,i} P_i^\sigma}{\alpha Y_i}. \quad (18)$$

To approximate this expression by data, we use the  $(l_j, l_i)$  coordinate of the input-output matrix normalized row-wise, divided by the number of firms in industry  $j$ . This approximation is exact if the firms in the same industry are homogeneous in size and if  $\sigma = 1$ .<sup>8</sup> Summing (18) across firms in region  $r_i$ , we obtain:

$$\left. \frac{\Delta \log n_{i,t}^*}{\log \lambda} \right|_{\bar{P}_{i,t}, \bar{A}_{i,t}, \bar{C}_t} = \beta \sum_{j \in G_{r_i}} \frac{\pi(l_i, l_j)}{\#G_l \sum_h \pi(l_i, h)} d(j, t) = \beta X_{i,t}^{OW}. \quad (19)$$

Following Bartelsman et al. [3], we call  $X_{i,t}^{OW}$  the output-weighted average of the fraction of investing firms in the industry-region groups. The output-weight represents the “demand-pull” or “customer” impact from a firm in industry  $l$  to firm  $i$ .

Equation (14) contains the input prices  $(P_i)$ , which are also affected by a change in  $n_j$ . This effect through the input prices is absent if the input weights  $\chi_{i,j}$  and  $\chi_{C,j}$  are homogeneous, since in the homogeneous case, the input prices are always equal to the

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<sup>8</sup>When we use the input-output matrix as the exogenous parameters, we implicitly assume that the share structure is stable over the sample periods. The input share is constant in the model only if  $\sigma = 1$ .

consumption goods price. The prices affect the decision rule for  $n_i^*$  directly through  $P_i$  or indirectly through  $Y_i$  in (7). The sum of these effects is complex, but a large impact comes directly through  $P_i$  from a change in  $p_j$ . This impact comes through the channel in which an investment by  $j$  decreases its own price  $p_j$  and thus decreases the input cost of  $i$  as per the amount of good  $j$   $i$  uses. This effect is linear in  $\partial \log P_i / \partial \log p_j = \chi_{i,j} p_j^{1-\sigma} / \sum_l \chi_{i,l} p_l^{1-\sigma} = z_{i,j} p_j / (n_i P_i)$ . The last expression is good  $j$ 's share of total derived demand of firm  $i$ . Thus, we can derive, as with  $X_{i,t}^{OW}$ , that the impacts on  $\log n_{i,t}^* / \log \lambda$  by the lumpy investments in region  $r_i$  through this channel is equal to  $X_{i,t}^{IW}$ .  $X_{i,t}^{IW}$  is called an input-weighted average of the fraction of investing firms in the industry-region groups. The input-weight  $(\pi(l, l_i) / \sum_h \pi(h, l_i)) (\#G_{r_i, l} / \#G_l)$  represents the “supply-push” or “supplier” impact from a firm in industry  $l$  to firm  $i$ .

When constructing  $X_{i,t}^{OW}$  and  $X_{i,t}^{IW}$ , we formed firm  $i$ 's share of the total demand by the transaction value in the input-output matrix divided by the number of firms (observed in our data set) in the same industry as  $i$ . By this procedure, we implicitly assume that Italy as a whole produces as much intermediate goods as it demands them. While our equilibrium model assumes that an economy (in this case, a regional economy) is closed, we do not make this closed-economy assumption on the regional economy for the micro-level estimation in this section. In the micro-level estimation, we do not fully specify where the demand  $Y_i$  in the decision rule (7) comes from, and thus,  $Y_i$  may include demand from other regions or abroad.

We estimate the degree of complementarity  $\beta$  in our model by the coefficient of  $X_{i,t}^{OW}$  in our probit estimation equation,  $\beta^{OW}$ . It is not possible to estimate the complementarity effect by simply regressing  $d(i, t)$  on  $\tilde{X}(r_i, t)$ , since such a regression may well pick up a common shock effect that affects all the firms in a region-year. Even though we include a region-year dummy to control for such a common shock, the simple

probit regression could still cause an endogeneity bias in the estimate of  $\beta$ , because  $X(r_i, t)$  is constructed by the left-hand-side variable  $d(i, t)$  in the same region. This is the reflection problem formulated by Manski [24]. Our strategy to avoid this issue is two-fold. First, we use the fact that increased derived demand by a firm has different effects across industries and regions, as proposed by Shea [27] and Bartelsman et al. [3]. The different effects can be seen in the output-weight in (16), in which the first fraction shows the heterogeneous input weight  $\pi(l_i, l)$  across industries, and the second fraction shows the heterogeneous industry composition across regions. Second, we employ a two-step estimation, in which  $X(r_i, l, t)$  is first regressed to exogenous variables, and then, the linear predictor  $\hat{X}(r_i, l, t)$  is used to form  $X_{i,t}^{OW}$  and  $X_{i,t}^{IW}$  in (16,17).

The auxiliary regression of the two-step estimation is formed as follows. First, as in Section 2, we construct  $\tilde{X}(l_i, t)$  by subtracting the year effects and industry (fixed) effects from the industry-year fraction  $X(l_i, t)$  (which is computed by excluding the firms in region  $r_i$ ). Thus,  $\tilde{X}(l_i, t)$  captures the industry-wide shocks (observed in the regions other than  $r_i$ ) on the lumpy investments, which we deem exogenous to the firm in the industries other than  $l_i$ . Next, we regress  $X(r_i, l_i, t)$  on  $\tilde{X}(l_i, t)$  and the aggregate investment growth rate for each industry in Italy and in the nearby nations<sup>9</sup> to obtain the linear predictors. The results of this auxiliary regression are reported in Table 3. These results show that the industry shocks  $\tilde{X}(l_i, t)$  have good explanatory power. The aggregate growth rates do not have a significant effect, which suggests that the lumpy investments are determined differently from the aggregate investments. Then, we substitute  $\hat{X}(r_i, l_i, t)$  for  $X(r_i, l_i, t)$  in (16) to obtain  $\hat{X}_{i,t}^{OW}$ , the exogenous portion

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<sup>9</sup>To construct the aggregate investment measure, we take the sum of the gross capital formation of France, the Netherlands, and Belgium. The reason for choosing these countries is that continuous data are available for them in the STAN database for the period 1983-1996.

	$\tilde{X}(l, t)$	$\Delta$ Inv. in Italy	$\Delta$ Inv. in FRA, NLD, BEL	Adjusted $R^2$	N. obs
$X(r, l, t)$	0.952*** (0.072)	-0.006 (0.015)	0.015 (0.014)	0.10	1686

Table 3: Result of auxiliary linear regression of the fraction of investing firms in industry-region  $X(r, l, t)$  on the fraction of investing firms in industry  $\tilde{X}(l, t)$  summed across regions excluding  $r$ .

of the variation in  $X_{i,t}^{OW}$ .<sup>10</sup> We obtain  $\hat{X}_{i,t}^{IW}$  similarly in (17). We then use  $\hat{X}_{i,t}^{OW}$  and  $\hat{X}_{i,t}^{IW}$  for the probit estimation (15). In the estimation, we include robust clustered standard errors with respect to firms.

The results of the probit estimation are shown in Table 4. We observe a positive and significant estimate of  $\beta^{OW}$ , whereas the estimate of  $\beta^{IW}$  is less significant and has a smaller value. This result is consistent with the estimate of Bartelsman et al. [3], which shows that in the short run, the inter-industry effects flow from the downstream to the upstream industries. We also find positive and significant coefficients for the lagged cash flow and the aggregate investment in the industry-region that the firm belongs to. The positive effect of the cash flow is consistent with the findings in the investment literature. The aggregate investment variable controls for the industry-region level common shocks, whereas the dummy variables for the region-year pairs control for the regional common shocks. The effect of productivity growth is not significantly different from zero.

<sup>10</sup>For the summation over  $l$  in (16), we do not include industry  $l_i$  when we construct  $\hat{X}_{i,t}^{OW}$ , in order to make sure that  $\hat{X}_{i,t}^{OW}$  does not contain  $d(i, t)$ , the left-hand side variable of the probit equation.

	$d\Phi/dx$	(s.e.)	$d\Phi/dx$	(s.e.)	$d\Phi/dx$	(s.e.)
$\Delta$ Productivity	-0.0166	(0.0113)	-0.0174	(0.0113)	-0.0168	(0.0113)
Lagged Cash Flow	0.0007*	(0.0004)	0.0007*	(0.0004)	0.0007*	(0.0004)
Inv. in industry-region	0.0339***	(0.0025)	0.0334***	(0.0025)	0.0338***	(0.0025)
$X^{OW}$	0.841***	(0.264)			0.824***	(0.265)
$X^{IW}$			0.359*	(0.206)	0.327	(0.206)
Log likelihood	-120428.4		-120435.6		-120426.5	

Table 4: Probit estimation of firms' binary investments. There are 183,896 observations for each estimation. \*\*\*, \*\*, and \* indicate significance at the 1%, 5%, and 10% levels, respectively.

## 5 Main results

In Section 2, we showed that the fraction of investing firms follows a double-exponential distribution. In this section, we show that the model in Section 3 calibrated with the parameter values estimated in Section 4 can explain this distribution. First, we show that the calibrated model can quantitatively explain the empirical double-exponential distribution observed in Section 2. Second, we show analytically that the model generates the exponential tail for the aggregate fluctuations. Third, we show that the model without the lumpy investment generates a normal distribution even if the investments are complements, and thus, argue that the lumpiness of the investments is the necessary ingredient for the model to generate the exponential distribution of the aggregate investments.

## 5.1 Simulated distributions of the fraction of investing firms

We first calibrate  $\beta$  to the point estimate  $\hat{\beta}^{OW}$  obtained in the third specification in Table 4 that yielded the highest log likelihood. We set the mark-up rate as 12.5% ( $\sigma = 9$ ). The total number of firms  $N$  is set to the median of the sample firms per region, 291. The distribution of firms across industries is set as in our data set. To calibrate  $\chi_{i,j}$  and  $A_i$ , we use the Italian input-output matrix (the 1985 version of the Italian total transaction table). First, we extract a 23-by-23 submatrix of the input-output matrix for the 23 industries we use. Then, we divide the transaction value for industry pair  $(i, j)$  by the number of firms in  $i$  and the number of firms in  $j$  observed in our data set. We regard the value computed above as the equilibrium input demand from a firm in  $j$  to a firm in  $i$ . Then, we collect these values in an  $N$ -by- $N$  matrix where  $N$  is the number of firms in the model. By normalizing the matrix to 1 column-wise and transposing it, we obtain the matrix for  $\chi_{i,j}$  that is equal to  $j$ 's share in  $i$ 's total inputs. Further, by taking the sum of the matrix column-wise, we obtain the distribution of the input demand. We then set  $A_i$  so that the obtained distribution matches the distribution of the equilibrium input demand  $Y_i$  across the firms implied by  $A_i$ . We calibrate the distribution of  $\lambda_i$  to an exponential distribution with lower bound 1.1 and mean 1.2, which mimics the histogram of the investment rate in our data shown in Figure 1.<sup>11</sup> Finally, we specify the perturbation shock to  $A_i$  as a normally distributed random variable. The standard deviation of the shock is set so that the mean fraction of firms that are induced to invest by the perturbation shocks is 3%. In other words, these firms decide to invest even if there is no propagation effect (i.e., if  $\beta = 0$  in (10)). This value is chosen to fit the empirical variance of the total fraction of

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<sup>11</sup>Our result of the double-exponential distribution of the fraction of investing firms does not depend on  $\lambda_i$  following an exponential distribution.



investing firms. However, it is not that we can fit any distribution by tuning the size of the shock arbitrarily. The shock is drawn from a normal distribution, and thus, it cannot generate an exponential tail by itself.

We compute the distribution of the number of investing firms using the Monte Carlo simulations of the model with the calibrated parameters. We first randomly draw the profiles of  $\lambda_i$  and  $A_i$  across the firms from the distribution functions calibrated in the previous paragraph. Next, we compute a smoothly-adjusting counterpart of our model, in which the firms are assumed to adjust smoothly without the indivisibility constraint. We then compute the equilibrium profile of our lumpy economy ( $n_i^0$ ). Next, we add a profile of small shocks to  $A_i$ , and compute a new smoothly-adjusting equilibrium and a new lumpy-adjusting equilibrium profile ( $n_i^1$ ). If multiple equilibria exist, we select an equilibria close to the smoothly-adjusting counterpart as described in Section 3.1. We repeat the procedure and compute the equilibria for 100,000 draws of a profile of  $A_i$  and shocks to  $A_i$ , while  $\lambda_i$  are fixed. The shocks to  $A_i$  correspond to  $d \log A_i$  in the decision rule (10), and the realization of  $(A_i)$  determines the distribution of gap variable  $s_i$ .

The left panel of Figure 3 compares the resulting simulated distribution of the fraction of investing firms with the empirical distribution of  $\tilde{X}(r, t)$ . Figure 3 shows the semi-log plots of the cumulative distributions for the positive and negative values of  $\tilde{X}$ . To produce the plot, we first divide  $\tilde{X}$  into two groups depending on the sign of  $\tilde{X}$ , and for each positive or negative group, we rank  $|\tilde{X}|$  in a descending order. Then, the log of the rank divided by the total number of observations is plotted against  $|\tilde{X}|$  for the positive and negative groups. Thus, the vertical line shows the percentiles of firms in a descending order. In the semi-log scale, an exponential distribution function would show as a straight line. Our plot demonstrates that both the positive and negative

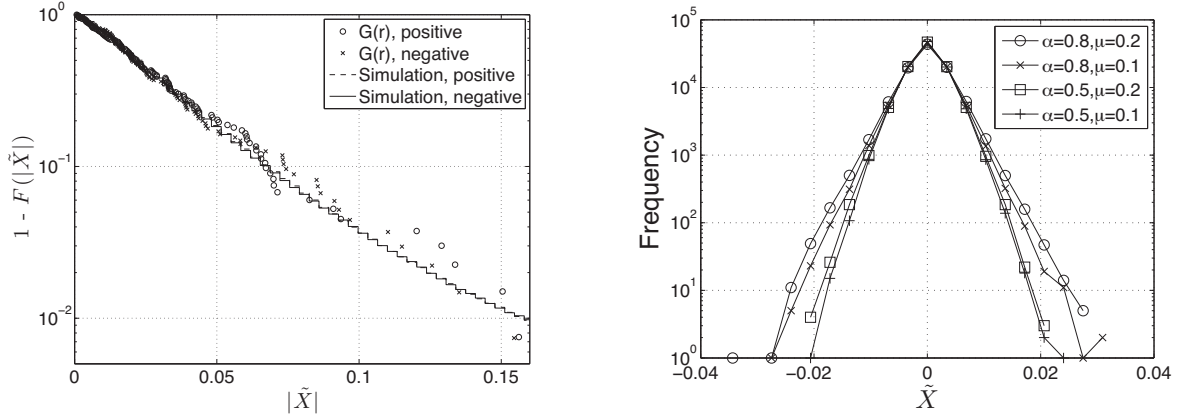


Figure 3: Left: the cumulative distribution of  $|\tilde{X}(r, t)|$  in our data and the simulated distribution. Right: simulated histogram of the fraction of investing firms for various values of returns to scale  $\alpha$  and mark-up rate  $1/(\sigma - 1)$ .

sides of the distribution of  $\tilde{X}(r, t)$  are fitted well by straight lines. As can be seen, the simulated distribution exhibits a good fit for the slope of the exponential decay of the empirical distribution.

The right panel of Figure 3 shows the simulated distributions for the various values of  $\alpha$  and  $\sigma$ . The returns to scale are set at  $\alpha = 0.5$  and  $0.8$ . The mark-up rate  $\mu$  is set as 10% and 20%, which corresponds to  $\sigma = 11$  and  $6$ , respectively. The variance of the perturbation to  $A_i$  is set so that the mean number of firms that invest due to the shock alone is 1. We observe that the simulated histogram conforms to the exponential pattern for all parameter sets. We also observe that the variance of the distribution is larger for the larger values of  $\alpha$  and  $\mu$ . This comparative statics is consistent with the analytical prediction of our model, as we will show in Section 5.2.

## 5.2 Analytical derivation of the exponential tail

We analytically show, under some approximation, that the model generates an exponential tail distribution for the fluctuations in the fraction of investing firms. To analytically characterize the equilibrium defined in the model, we consider a fictitious tatonnement process. Using the fictitious tatonnement process, the number of investing firms is computed as the sum of firms that choose to invest during the fictitious tatonnement from the initial equilibrium to the new equilibrium. When there exist multiple equilibria, we select an equilibrium close to the smoothly-adjusting equilibrium as described in Section 3.1. The selected equilibrium can be computed by the fictitious tatonnement process, if we set the smoothly-adjusting equilibrium to the initial state of the fictitious tatonnement, as shown in Vives [32] and Cooper [10]. The precise definition of the fictitious tatonnement is provided in Appendix C.

The fictitious tatonnement process depends on the realization of the initial productivity and the perturbation. Thus, before these random variables realize, the fictitious tatonnement process can be regarded as a stochastic process, and the difference in equilibrium output is expressed as the sum of this stochastic process.<sup>12</sup> The stochastic process is expressed using the probability that  $j$ 's adjustment induces  $i$  to adjust, given by (13). Consider the fictitious tatonnement process that starts from the initial state  $(s_i)$ . If there is only one firm  $j$  that adjusts capital in the first step of the tatonnement upon perturbation, the number of firms that are induced to adjust in the second step of the tatonnement, conditional on the adjustment of firm  $j$ , follows a convolution of Bernoulli trials with probability  $\beta b_i / \log \lambda_j$  across  $i \neq j$ . Then, the number of firms that adjust in step 2,  $m_2$ , conditional on  $m_1 = 1$  is an integer random variable that follows the above convolution and has mean  $\phi \equiv \beta E(\sum_{i=1}^N b_i) E(1 / \log \lambda_j)$ . If there are

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<sup>12</sup>The method to embed the fictitious tatonnement in a stochastic process is presented in Nirei [25].

$m_1 > 1$  firms that adjust in the first step, then the number of firms that are induced to adjust in the second step follows an  $m_1$ -convolution of the above integer random variable and has mean  $\phi m_1$ . The number of firms  $m_u$  that adjust in each step  $u$  of the tatonnement conditional on  $m_{u-1}$  follows a distribution that is represented as a  $m_{u-1}$ -times convolution of the integer random variable with mean  $\phi$ . The integer random variable is identically distributed across  $u$  as long as  $N$  is so large that  $\sum_{v=1}^u m_v/N$  is small, and thus, the density of  $s_i$  of the affected firms is constant.<sup>13</sup> We thus can use the following theorem:

**Theorem 1 (Otter)** *Consider a branching process  $m_u$ ,  $u = 1, 2, \dots$ , that starts from  $m_1 = 1$ . Let  $\phi$  denote the mean number of children in  $u + 1$  born to each parent in  $u$ . We define the total number as  $M = \sum_{u=1}^T m_u$ . Then,*

$$\Pr(M = m | m_1 = 1) = C_0 e^{-(\phi-1-\log \phi)m} m^{-1.5} \quad (20)$$

for a large integer  $m$ , where  $C_0$  is a constant.

This theorem can be further tuned in a homogeneous set up where the lumpiness  $\lambda_i$ , the productivity  $A_i$ , and the input weight  $\chi_{i,j}$  are common across  $i, j$ . In this case, the probability of the Bernoulli trial  $\beta b_i / \log \lambda_j$  becomes  $\beta/N$ . Thus, the number of firms  $m_u$  that are induced to adjust upward in  $u$  by  $m_{u-1}$  firms that adjusted in the previous step of the tatonnement  $u - 1$  asymptotically follows a Poisson distribution with mean  $\beta m_{u-1}$  as  $N \rightarrow \infty$ . Then, we obtain the following proposition.

**Proposition 1** *When  $\lambda_i$ ,  $A_i$ , and  $\chi_{i,j}$  are common across  $i, j$ ,  $M$  follows a symmetric probability distribution function:*

$$\Pr(|M| = m | m_1) = (m_1/m) e^{-\beta m} (\beta m)^{m-m_1} / (m - m_1)! \quad (21)$$

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<sup>13</sup>See Nirei [25] for further details and generalizations.

for  $m = m_1, m_1 + 1, \dots$ . The tail of the distribution function is approximated as

$$\Pr(|M| = m \mid m_1) \sim (m_1(e\beta)^{-m_1} / \sqrt{2\pi}) e^{-(\beta-1-\log\beta)m} m^{-1.5}. \quad (22)$$

Proof: See Appendix D.

Proposition 1 shows that  $M$  conditional on  $m_1$  follows a distribution that is a mixture of the power and exponential functions as seen in (22). Since the exponential function declines faster than the power function, the tail of the distribution is dominated by the exponential part. We argue that this corresponds to our empirical finding that the fraction of investing firms  $X$  follows an exponential distribution. The distribution obtained above is conditional on  $m_1$ , which is the net number of firms that adjust capital upward as a direct effect of the perturbation. If the perturbation is sufficiently small,  $m_1$  is much smaller than  $M$ .

Proposition 1 further shows that the slope of the exponential distribution is determined by  $\beta$ . Equation (22) implies that the exponential slope is equal to  $\beta - 1 - \log \beta$ ; the mean and standard deviation of the exponential part are determined by the inverse of the slope. Since the slope is decreasing in  $\beta$  in the region  $\beta < 1$ , a large  $\beta$  implies a large fluctuation of  $M$ . This is intuitive because  $\beta$  is the mean number of firms that are induced to invest when another firm conducts a lumpy investment in the fictitious tatonnement. The same implication that the degree of complementarity  $\beta$  determines the slope of the exponential decay holds for the case with heterogeneous firms as in (20), in which  $\beta$  is replaced by  $\phi$ .

Finally, the analysis shown here is also consistent with our simulations. From (11), small returns-to-scale (a small  $\alpha$ ) or a low mark-up (a large  $\sigma$ ) imply a small  $\beta$ , and thus, a steep slope by Proposition 1. This prediction is confirmed by our simulations,

as shown in the right panel of Figure 3. Thus, the exponential tail analytically derived by Proposition 1 under approximation seems to robustly characterize the distribution of the exact equilibrium.

### 5.3 Comparison between a smooth economy and a lumpy economy

In order to show that the lumpiness of the firms' capital adjustments is necessary to generate the exponential pattern of aggregate fluctuations in investment and output, we consider an economy in which capital is adjusted smoothly. We do so by eliminating the indivisibility constraint (2) from our model and keeping everything else unchanged. We refer to this as a smooth economy. The result is then compared with the case in which the capital adjustment is lumpy, which we refer to as a lumpy economy. In this comparison, we concentrate on the fluctuations of aggregate output  $Y = \sum_i p_i y_i$  rather than the fraction of investing firms, because all firms adjust in the smooth economy. The aggregate investment in the smooth economy is proportional to the aggregate output.

The left panel of Figure 4 shows the distribution of the log-deviation in  $Y$  in the lumpy economy. The plots show that the aggregate output inherits the exponential pattern in the distribution of the fraction of investing firms. The right panel of Figure 4 shows the log deviation of aggregate output  $Y$  in the smooth economy. When simulating the smooth economy, we use the same realizations of the random variables as in the simulation of the lumpy economy. In a semi-log plot, a normal distribution looks like a parabola, and we see that the plot clearly shows a parabola for the distribution of  $\Delta \log Y$  in the smooth economy. This presents a striking contrast to the exponential distribution in the lumpy economy shown in the left panel.

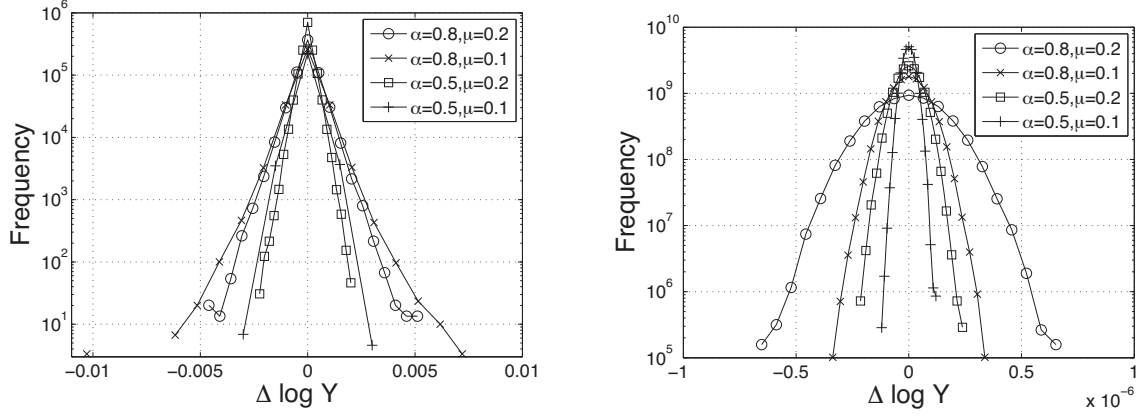


Figure 4: Simulated histograms of the log-deviations of the aggregate output when the adjustments are lumpy (left) and when the adjustments are smooth (right)

It can be analytically shown that the aggregate output distribution has an exponential tail if the firms are homogeneous. In the homogeneous case, the output is written as  $Y = A(\sum_i n_i^{\alpha(1-1/\sigma)})^{\sigma/(\sigma-1)} N^{1/(1-\sigma)}$ , and thus, the distribution of  $\log Y$  reflects the distribution of the sum of inputs.

We can also analytically derive the normal distribution for the smooth economy when  $\lambda_i$  and  $\chi_{i,j}$  are homogeneous across  $i$ . Considering the first-order condition with respect to  $n_i$  and combining it with the equilibrium conditions, we obtain the equilibrium aggregate output:

$$Y = \left( \left( \alpha \left( 1 - \frac{1}{\sigma} \right) \right)^{\left( 1 - \frac{1}{\sigma} \right) \frac{\alpha}{1 - \alpha(1-1/\sigma)}} \sum_i \left( A_i^{1-\frac{1}{\sigma}} \chi_i^{\frac{1}{\sigma}} \right)^{\frac{1}{1 - \alpha(1-1/\sigma)}} \right)^{\frac{\sigma(1-\alpha(1-1/\sigma))}{(\sigma-1)(1-\alpha(1-1/\sigma))-\alpha}}. \quad (23)$$

This shows that the equilibrium output  $\log Y$  is a weighted sum of the idiosyncratic growth shocks to  $A_i$ , and thus, follows a normal distribution due to the central limit theorem.

The smooth economy generates a normal distribution because the response of an

individual firm is locally linear in the aggregate investments. To see this, consider the simple case where decisions at the micro-level are positively dependent on the aggregate such as  $x_i = \beta \sum_{j=1}^N x_j/N + \epsilon_i$ . Then, the equilibrium aggregate is determined as  $\sum_{i=1}^N x_i/N = (1/(1 - \beta)) \sum_{i=1}^N \epsilon_i/N$ , and follows the normal distribution when the exogenous shocks  $\epsilon_i$  are independent. In contrast, in the lumpy model the response of an individual firm is non-linear (non-linearity exhibited as a threshold rule), and thus, the aggregation of individual behavior does not follow the central limit theorem.

Another salient difference between the distributions in Figure 4 is the magnitude of fluctuations. The standard deviation in the smooth economy is four orders of magnitude smaller than that in the lumpy economy, even though they both incur the same realization of random shocks. Thus, the micro-level lumpy adjustments have a strong effect in terms of propagating idiosyncratic shocks. The smooth economy is not completely without such a propagation mechanism. The simple model in the previous paragraph shows that an exogenous idiosyncratic shock  $\epsilon_i$  will be magnified by the multiplier  $1/(1 - \beta)$ . However, in the case of lumpy adjustments, the complementarity effect at the micro-level is non-linear because of the threshold rule. Due to non-linearity, the multiplier effect depends on capital and productivity profiles before shocks occur. In a dynamic setup, this will imply that the multiplier is history-dependent in an (S,s) economy. In our model where the finite profile of initial capital is drawn randomly from the invariant distribution of capital, the multiplier reflects the randomness in the capital profile, and thus, becomes stochastic. The exponential tail we have observed in the aggregates captures the nature of this stochastic multiplier effect. Returning to the domino analogy mentioned earlier, this is exactly the same as how small differences in the position of domino tiles can greatly affect the number of tiles that fall.<sup>14</sup>

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<sup>14</sup>It can be further argued that the domino effect constitutes a mechanism of endogenous deter-



In sum, this section showed that the simulated distribution in our calibrated model serves as a good approximation of the double-exponential distribution of the fraction of investing firms observed in the Italian data. This suggests that the empirical distribution favors the model of aggregate fluctuations that is driven by a propagation mechanism, rather than by exogenous shocks, which tend to generate the normal distribution due to the central limit theorem. Moreover, we showed that our model of lumpy investments generates the double-exponential distribution for a wide range of parameter values, while our model without lumpiness generates a normal distribution of the output fluctuations. This implies that the sectoral model with complementarity needs some non-linearity at the micro-level in order to generate the non-normal distribution with the considerable volatility observed empirically.

## 6 Conclusion

This paper argued that the distribution of the fraction of firms that engage in large investments provides a useful test for the existence of propagation effects among the firms' investment decisions. Testing for the endogenous propagation effects is often a challenge because the model of such propagation effects is observationally equivalent to the model with exogenous common shocks. We argued that in the class of binary choice models, the model with common shocks results in a normal distribution of the number of investing firms when the common shocks consist of many independent factors, whereas the model of propagation effects leads to a non-normal distribution

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ministic fluctuations. Consider a dynamic extension of the model as in Appendix A in which capital depreciates by  $\delta_i$  and the firms decide whether or not to increase capital by  $\lambda_i$  in each period. Then, the aggregate output follows a non-linear dynamic that can generate fluctuations even without the exogenous shocks to  $A_i$ . This possibility is explored in a separate paper (see Nirei [26]).

that is better characterized by a double-exponential distribution.

For our empirical analysis, we studied a panel of investment data for the Italian firms. We investigated the fraction of firms for which the investment-capital ratio exceeded a certain threshold. We showed that the empirical distribution of the fraction of investing firms across the region-year rejects the hypothesis that the distribution is normal, whereas the empirical distribution does not reject the hypothesis that the distribution is exponential.

In order to explain the exponential distribution of the fraction of investing firms, we presented a simple model of lumpy investments when the firms' investment decisions are interrelated through complementarity. The complementarity stems from the fact that the firms are linked by the input-output relations as well as the aggregate demand externality of consumption. In the model, when a firm increases production, it generates increased factor demand, and thus, provides an incentive for the upstream firms in the input-output relationship to produce more. We analytically showed that this complementarity generates an exponential distribution for the fluctuations of the fraction of investing firms.

We estimated the firms' decision rule derived in the model, by using the firm-level observations in our data set. In particular, we estimated the degree of complementarity, taking account of the issue of endogeneity bias by using a heterogeneous input-output matrix and a two-step estimation. We then used the degree of complementarity estimated at the firm level for the calibration of our model. The simulations of the calibrated model showed that with the estimated degree of complementarity, our model of lumpy investments explains the exponential pattern empirically observed. Finally, we showed that the model generates a normal distribution if the investments are not lumpy. Thus, we argued that the observed exponential pattern supports the view

that some aggregate fluctuations are generated by the interactions of firm-level lumpy investments.

## Appendix

### A Extension with capital adjustment costs

In this section, we incorporate a fixed adjustment cost of capital in our model in order to show a micro-foundation of the discreteness assumption (2). We consider a production function modified from (1):

$$y_i = A_i k_i^\gamma n_i^\alpha, \quad (24)$$

where  $k_i$  denotes capital. We assume diminishing returns to scale  $\gamma + \alpha < 1$ . We consider a two-period model. In period 0, the firms are endowed with initial capital  $k_{i,0}$  and decide the level of capital used in period 1. In period 1, production and transaction take place. We assume that the firms rent capital at fixed rental price  $r$ . Capital adjustments require fixed costs  $D_i$ . If no adjustment occurs, the capital in period 1 is determined by  $k_i = (1 - \delta_i)k_{i,0}$ , where  $\delta_i$  denotes the rate of capital depreciation.

The input  $n_i$  is produced as in (3). As in Section 3, the firms' cost minimization behavior leads to  $\sum_j p_j z_{i,j} = P_i n_i$ . Thus, the maximization problem of a firm as a monopolist is formulated as:

$$\max_{p_i, y_i, n_i, k_i} p_i y_i - P_i n_i - r k_i - D_i \mathbf{1}_{k_i \neq (1 - \delta_i) k_{0,i}} \quad (25)$$

subject to (24,5). Substituting in optimal choices for  $p_i, y_i, n_i$ , this problem can be

rewritten as  $\max_{k_i} \pi(k_i) - D \mathbf{1}_{k_i \neq (1-\delta_i)k_{i,0}}$ , where

$$\pi(k_i) \equiv \left( \left( \alpha \left( 1 - \frac{1}{\sigma} \right) \right)^{\alpha(1-1/\sigma)} \left( 1 - \alpha \left( 1 - \frac{1}{\sigma} \right) \right)^{1-\alpha(1-1/\sigma)} A_i^{1-1/\sigma} \frac{Y_i^{1/\sigma}}{P_i^{\alpha(1-1/\sigma)}} \right)^{\frac{1}{1-\alpha(1-1/\sigma)}} - r k_i. \quad (26)$$

Solving this, we obtain the optimal level of capital when the firm adjusts:

$$k_i^* = \left( \alpha^{\alpha(1-1/\sigma)} \gamma^{1-\alpha(1-1/\sigma)} \left( 1 - \frac{1}{\sigma} \right) A_i^{1-1/\sigma} \frac{Y_i^{1/\sigma}}{P_i^{\alpha(1-1/\sigma)} r^{1-\alpha(1-1/\sigma)}} \right)^{\frac{1}{1-(\alpha+\gamma)(1-1/\sigma)}}. \quad (27)$$

The optimal policy of firm  $i$  is to adjust capital to  $k_i^*$  if  $k_{i,0}$  is outside the region  $(\underline{k}_i, \bar{k}_i]$ , referred to as an inaction band, and not to adjust otherwise. The bounds of the inaction band  $\underline{k}_i$  and  $\bar{k}_i$  must satisfy  $\pi((1-\delta_i)\underline{k}_i) = \pi((1-\delta_i)\bar{k}_i) = \pi(k_i^*) - D_i$ . Such two bounds exist below and above  $k_i^*$ , since  $\pi$  is strictly concave. We can show that an increase in fixed cost  $D_i$  increases the width of the inaction band.

If we further assume that the capital adjustment is irreversible in the sense that  $k_i$  has to be greater than or equal to  $(1-\delta_i)k_{i,0}$ , the firms' optimal rule is to adjust to  $k_i^*$  if  $k_{i,0} < \underline{k}_i$  and not to adjust otherwise. In this case,  $k_i^*/\underline{k}_i$  corresponds to the lumpiness parameter  $\lambda_i$  in Section 3. This implies that the model of capital adjustment with fixed costs provides us with a micro-foundation of the discrete behavior that we assumed in the main model.

In a fully dynamic setup, a firm's objective function reflects not only the current profits  $\pi$  but also the value of capital  $k_i$  in the future. In order to address this issue, we extend the two-period model above to a recursive situation. We consider a firm's dynamic programming problem in continuous time in which firm  $i$  maximizes its discounted sum of profits:

$$V(k_{i,0}) = \max_{k_i, T_i} \int_0^{T_i} e^{-\rho t} \pi(k_{i,t}) dt + e^{-\rho T_i} (V(k_{i,T_i}) - D_i), \quad (28)$$

where  $\rho$  denotes the investors' required rate of return,  $T_i$  denotes the time taken by the firm to adjust capital, and  $k_{i,T_i}$  denotes the capital level chosen by the firm in  $T_i$ . Thus, for  $t < T_i$ , capital is naturally depreciated over time as  $k_{i,t} = e^{-\delta_i t} k_{i,0}$ . This type of dynamic programming with fixed adjustment costs was analyzed by Sheshinski and Weiss [29], and is widely used as in Stokey [30].

Here, we consider a stationary equilibrium with a continuum of firms in which  $Y_i$ ,  $P_i$ , and  $r$  are constant over time. The firms face the time-invariant dynamic optimization problem with state  $k_{i,t}$  in a stationary equilibrium, and thus, the optimal level of capital to which the firm adjusts is time-invariant. Let  $k_i^*$  denote the optimal level of capital to adjust to. For firm  $i$  who just adjusted capital, the capital levels at 0 and  $T_i$  are equal to  $k_i^*$ . Thus, at the adjustment point, the value function satisfies:

$$V(k_i^*) = \max_{k_i^*, T_i} \int_0^{T_i} e^{-\rho t} \pi(e^{-\delta_i t} k_i^*) dt + e^{-\rho T_i} (V(k_i^*) - D_i). \quad (29)$$

The first-order conditions for optimal  $T_i$  and  $k_i^*$  are

$$\pi(e^{-\delta_i T_i} k_i^*) - \rho(V(k_i^*) - D_i) = 0, \quad (30)$$

$$\int_0^{T_i} e^{-(\rho+\delta_i)t} \pi'(e^{-\delta_i t} k_i^*) dt + e^{-\rho T_i} V'(k_i^*) = 0. \quad (31)$$

For the interior of the inaction band, the value function satisfies the Bellman equation:

$$V(k) = \pi(k)dt + e^{-\rho dt} V(e^{-\delta_i dt} k) = \pi(k)dt + (1 - \rho dt)(V(k) - V'(k)\delta_i k dt) + o(dt). \quad (32)$$

Taking the limit  $dt \rightarrow 0$ , we obtain

$$0 = \pi(k) - \rho V(k) - \delta_i k V'(k). \quad (33)$$

We can solve this ordinary differential equation for the value function within the inaction band as a polynomial of the same degree as  $\pi(k)$ . Combining the value function

with the first-order conditions, we can determine  $T_i$  and  $k_i^*$ . The threshold point for adjustment (i.e., the capital level just before adjusting capital) is also determined as  $\underline{k}_i = e^{-\delta_i T_i} k_i^*$ . The lumpiness is determined by  $\lambda_i = k_i^*/\underline{k}_i = e^{-\delta_i T_i}$ , which is affected by the size of fixed costs  $D_i$ . The optimal capital dynamic for firm  $i$  is thus  $\dot{k}_{i,t}/k_{i,t} = -\delta_i$  for  $k_{i,t} \in (\underline{k}_i, k_i^*]$  and  $k_{i,t} = k_i^*$  when  $k_{i,t}$  reaches  $\underline{k}_i$ . This is the typical dynamics under a one-sided (S,s) rule. The gap variable  $s_{i,t} = (\log k_{i,t} - \log \underline{k}_i)/\log \lambda_i$  always stays within a unit interval as in our main model.

The model in this section provides the microfoundation for our main model in which  $\lambda$  is fixed. To be precise, our main model features a choice set that is an infinite sequence (2) rather than binary. However, the main model is reduced to a binary choice model, since we focus on the shifts in the equilibrium when the productivities are perturbed slightly enough not to induce a capital adjustment larger than  $\lambda$ .

## B Impact of a lumpy investment on total demand

We start from  $Y_0 = \left( \sum_j \chi_j^{\frac{1}{\sigma}} A_j^{1-\frac{1}{\sigma}} n_{j,0}^{\alpha(1-1/\sigma)} \right)^{\sigma/(\sigma-1)}$ . Suppose that firm  $i$  chooses to invest. Then,  $\log n_{i,1} - \log n_{i,0} = \log \lambda_i$ . We calculate the new aggregate output by the Taylor series expansion around  $Y_0$  as follows:

$$\begin{aligned} \log Y_1 - \log Y_0 &= \alpha \frac{\chi_i^{\frac{1}{\sigma}} A_i^{1-\frac{1}{\sigma}} n_{i,0}^{\alpha(1-1/\sigma)}}{\sum_j \chi_j^{\frac{1}{\sigma}} A_j^{1-\frac{1}{\sigma}} n_{j,0}^{\alpha(1-1/\sigma)}} \sum_{k=1}^{\infty} \frac{(\alpha(1-1/\sigma))^{k-1} (\log \lambda_i)^k}{k!} + O(N^{-2}) \quad (34) \\ &= \alpha \frac{\chi_i^{\frac{1}{\sigma}} A_i^{1-\frac{1}{\sigma}} n_{i,0}^{\alpha(1-1/\sigma)}}{\sum_j \chi_j^{\frac{1}{\sigma}} A_j^{1-\frac{1}{\sigma}} n_{j,0}^{\alpha(1-1/\sigma)}} \frac{\lambda_i^{\alpha(1-1/\sigma)} - 1}{\alpha(1-1/\sigma)} + O(N^{-2}). \quad (35) \end{aligned}$$

To obtain the second equation, we used the relation  $\sum_{k=0}^{\infty} a^k/k! = e^a$ . Note that the first term is of order  $N^{-1}$ . The term of order  $N^{-2}$  results from the derivative of the summation in the denominator of the first term with respect to  $n_i$ . From this we obtain

expression (13). By summing across all the firms that adjust  $n_i$ , we obtain (10).

## C Fictitious tatonnement

Consider a sequence of input profiles  $n_u = (n_{1,u}, n_{2,u}, \dots, n_{N,u})$  for discrete steps  $u = 1, 2, \dots$ . The sequence starts from the initial equilibrium that is defined for a realization of  $(A_{i,0}, \lambda_i)$ . The initial point is denoted by a profile  $n_0$ . There is a corresponding threshold rule at the initial equilibrium, determined by (7) and denoted by  $n_0^*$ . The initial gap variables are accordingly defined as  $(s_{i,0})$ .

Next, perturbation  $\epsilon_i$  is drawn independently across  $i$ . Productivity is perturbed such that  $\log A_{i,1} = \log A_{i,0} + \epsilon$ . We then solve for the optimal prices for firms  $(p_i)$  under  $(A_1, n_0)$ . Then, the threshold is altered according to (7), and the new threshold profile under  $(A_1, n_0)$  is denoted by  $n_1^*$ . The gap variable is also updated and denoted by  $s_1$ . We apply the new threshold rule, and some firms find it optimal to adjust their input. The best reply is denoted by  $n_1$ .

Then, the second step starts. Under  $(A_1, n_1)$ , we compute a new profile of threshold  $n_2^*$ , the gap variable  $s_2$ , and the best reply  $n_2$ . We iterate this process until  $n_u$  converges. The stopping step is denoted by  $T$ . Thus,  $n_{i,T} = n_{i,T-1}$  for all  $i$ . This sequence of input profiles,  $(n_0, n_1, \dots, n_T)$ , constitutes the fictitious tatonnement.

## D Proof of Proposition 1

The following proof largely draws on Nirei [25].  $m_u$  conditional on  $m_{u-1}$  for  $u \geq 2$  follows a Poisson distribution with mean  $\beta m_{u-1}$ . Since a Poisson distribution is infinitely divisible, the Poisson variable with mean  $\beta m_{u-1}$  is equivalent to an  $m_{u-1}$ -

times convolution of a Poisson variable with mean  $\beta$ . Thus,  $m_u$  for  $u \geq 2$  constitutes a branching process with a step random variable that is a Poisson variable with mean  $\beta$ . Since  $\beta \leq 1$ , the process  $m_u$  reaches 0 by a finite stopping time with probability one (see Feller [14]). Thus, this process is a valid algorithm of equilibrium selection in the sense that the convergence is achieved by a finite stopping time  $T$ . Then, as a consequence of the Poisson branching process (see Kingman [21]), the accumulated sum  $M$  conditional on the initial value  $m_1 > 0$  follows an infinitely divisible distribution called Borel-Tanner distribution, which is shown in (21). For a negative initial value  $m_1 < 0$ , the above analysis applies symmetrically to  $-M$ . Approximation (22) is obtained by applying Stirling's formula  $m! \sim \sqrt{2\pi}e^{-m}m^{m+0.5}$ .  $\square$

## References

- [1] Ruediger Bachmann, Ricardo J. Caballero, and Eduardo M.R.A. Engel. Aggregate implications of lumpy investment: New evidence and a DSGE model, 2010. mimeo.
- [2] Abhijit V. Banerjee. A simple model of herd behavior. *Quarterly Journal of Economics*, 107:797–817, 1992.
- [3] Eric J. Bartelsman, Ricardo J. Caballero, and Richard K. Lyons. Customer- and supplier-driven externalities. *American Economic Review*, 84:1075–1084, 1994.
- [4] William A. Brock and Steven N. Durlauf. Discrete choice with social interactions. *Review of Economic Studies*, 68:235–260, 2001.
- [5] Ricardo J. Caballero and Eduardo M. R. A. Engel. Dynamic (S,s) economies. *Econometrica*, 59:1659–1686, 1991.



- [6] Ricardo J. Caballero and Eduardo M. R. A. Engel. Explaining investment dynamics in U.S. manufacturing: A generalized (S,s) approach. *Econometrica*, 67:783–826, 1999.
- [7] Andrew S. Caplin and Daniel F. Spulber. Menu cost and the neutrality of money. *Quarterly Journal of Economics*, 102:703–726, 1987.
- [8] Vasco M. Carvalho. Aggregate fluctuations and the network structure of intersectoral trade, 2010. mimeo.
- [9] Federico Cingano and Fabiano Schivardi. Identifying the sources of local productivity growth. *Journal of the European Economic Association*, 2:720–742, 2004.
- [10] Russell Cooper. Equilibrium selection in imperfectly competitive economies with multiple equilibria. *Economic Journal*, 104:1106–1122, 1994.
- [11] Russell Cooper, John Haltiwanger, and Laura Power. Machine replacement and the business cycles: Lumps and bumps. *American Economic Review*, 89:921–946, 1999.
- [12] Mark Doms and Timothy Dunne. Capital adjustment patterns in manufacturing plants. *Review of Economic Dynamics*, 1:409–429, 1998.
- [13] Bill Dupor. Aggregation and irrelevance in multi-sector models. *Journal of Monetary Economics*, 43:391–409, 1999.
- [14] William Feller. *An Introduction to Probability Theory and Its Applications*, volume I. Wiley, NY, second edition, 1957.
- [15] Xavier Gabaix. The granular origins of aggregate fluctuations. *Econometrica*, 79:733–772, 2011.

- [16] Edward L. Glaeser, Bruce I. Sacerdote, and José A. Scheinkman. The social multiplier. *Journal of the European Economic Association*, 1:345–353, 2003.
- [17] François Gourio and Anil K Kashyap. Investment spikes: New facts and a general equilibrium exploration. *Journal of Monetary Economics*, 54:1–22, 2007.
- [18] Luigi Guiso and Fabiano Schivardi. Spillovers in industrial districts. *Economic Journal*, 117:68–93, 2007.
- [19] Michael Horvath. Sectoral shocks and aggregate fluctuations. *Journal of Monetary Economics*, 45:69–106, 2000.
- [20] Boyan Jovanovic. Micro shocks and aggregate risk. *Quarterly Journal of Economics*, 102:395–410, 1987.
- [21] J. F. C. Kingman. *Poisson Processes*. Oxford, NY, 1993.
- [22] John B. Long, Jr and Charles I. Plosser. Real business cycles. *Journal of Political Economy*, 91:39–69, 1983.
- [23] Robert E. Lucas, Jr. *Studies in business-cycle theory*. MIT Press, 1981.
- [24] Charles F. Manski. Identification of endogenous social effects: The reflection problem. *Review of Economic Studies*, 60:531–542, 1993.
- [25] Makoto Nirei. Threshold behavior and aggregate fluctuation. *Journal of Economic Theory*, 127:309–322, 2006.
- [26] Makoto Nirei. Aggregate fluctuations of discrete investments. *Hitotsubashi University IIR Working Paper*, 08-08, 2008.

- [27] John Shea. The input-output approach to instrument selection. *Journal of Business and Economic Statistics*, 11:145–155, 1993.
- [28] John Shea. Complementarities and comovements. *Journal of Money, Credit and Banking*, 34:412–433, 2002.
- [29] Eytan Sheshinski and Yoram Weiss. Inflation and costs of price adjustment. *Review of Economic Studies*, 44:287–303, 1977.
- [30] Nancy L. Stokey. *The economics of inaction*. Princeton, 2009.
- [31] Julia K. Thomas. Is lumpy investment relevant for the business cycle? *Journal of Political Economy*, 110(3):508–534, 2002.
- [32] Xavier Vives. Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19:305–321, 1990.
- [33] Quang H. Vuong. Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica*, 57:307–333, 1989.

