REVENUE COMPARISON IN ASYMMETRIC AUCTIONS
WITH DISCRETE VALUATIONS

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Abstract

We consider an asymmetric auction setting with two bidders such that the valuation of each bidder has a binary support. We prove that in this context the second price auction yields a higher expected revenue than the first price auction for a broad set of parameter values, although the opposite result is common in the literature on asymmetric auctions. For instance, the second price auction is superior both when a bidder’s valuation is more uncertain than the valuation of the other bidder, and in case of a not too large distribution shift or rescaling. In addition, we show that in some cases the revenue in the first price auction decreases when all the valuations increase [in doing so, we correct a claim in Maskin and Riley (1985)], and we derive the bidders’ preferences between the two auctions.

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Key words: Asymmetric auctions, First price auctions, Second price auctions.

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1 Introduction

This paper is about a seller’s preferences between a first price auction (FPA from now on) and a second price/Vickrey auction (SPA from now on) when the bidders’ valuations are independently but asymmetrically distributed. Precisely, we consider a setting with two bidders such that the valuation of each bidder has a binary support and prove that in this context the SPA yields a higher expected revenue than the FPA for a broad set of parameter values, although the opposite result is common in the literature on asymmetric auctions. In addition, we correct a claim in Maskin and Riley (1985), we show that in some cases the revenue in the FPA decreases when all the valuations increase, and we derive the bidders’ preferences among the FPA and the SPA.

As it is well known, with asymmetric distributions the revenue equivalence theorem does not apply, and only in very specific circumstances it is possible to derive the closed form of the equilibrium bidding functions for the FPA.¹ This complicates the comparison between the FPA and the SPA, but nevertheless some interesting results have been discovered. One of them is that the FPA is often more profitable than the SPA for the seller. On the basis of numeric analysis for some classes of continuous distributions, Li and Riley (2007) claim that “the ‘typical’ case leads to greater expected revenue in the sealed high-bid auction” [i.e., in the FPA]; a similar point of view is found in Klemperer (1999). Some general theoretical results are provided by Maskin and Riley (2000a),² which show that under suitable conditions on the distribution of valuations the FPA is superior to the SPA for a two-bidder setting in which a bidder’s distribution is obtained by shifting or stretching to the right the other bidder’s distribution; Kirkegaard (2011) generalizes these results.³ On the other hand, some papers identify settings in which the seller prefers the SPA, including Vickrey (1961), Maskin and Riley (2000a), Cheng (2010) and Gavious and Minchuk (2010). However, these results mostly refer to specific examples,⁴ whereas the result in Kirkegaard (2011) covers a relatively broad set of circumstances.

As we mentioned above, we study an environment with two bidders and binary distributions.⁵

²Lebrun (1996) and Cheng (2006) prove that the seller prefers the FPA for some classes of power distributions.
³Roughly speaking, Kierkegaard (2011) shows that the FPA is superior to the SPA if a bidder’s distribution is flatter and more disperse than the other bidder’s distribution. In Subsection 4.1.5 we describe with more details the main result in Kierkegaard (2011).
⁴Vickrey (1961) examines a setting in which a bidder’s valuation is common knowledge. Maskin and Riley (2000a) consider the case in which a bidder’s distribution is obtained from the other bidder’s distribution by shifting some probability mass to the lower end-point. Cheng (2010) analyzes environments such that the equilibrium bidding functions for the FPA are linear. Gavious and Minchuk (2010) study examples in which the bidders’ distributions are close to the uniform distribution.
⁵This is an extension of the set-up considered in Maskin and Riley (1983), which focus on the case in which the bidders’ low valuations coincide; we analyze different classes of asymmetries. Cheng (2011) employs the same discrete setting of Maskin and Riley (1983) in order to show that in some special cases the asymmetry increases the expected revenue in the FPA, unlike in the examples studied in Cantillon (2008).
In this environment we derive the unique equilibrium outcome and the expected revenue in the FPA for all parameter values, and then we compare the FPA with the SPA for some classes of asymmetries. We find that quite often the SPA is more convenient for the seller, despite the above citation from Li and Riley (2007). Precisely, this is the case when the bidders’ high valuations coincide. In alternative, when the probability of a high valuation is the same for the two bidders (but values may be different), the SPA dominates unless a bidder’s values are sufficiently large with respect to the other bidder’s values. More in detail, in this environment we prove that

- the SPA is more profitable if a bidder’s valuation is more variable than the other bidder’s valuation,\(^6\) and in the case of distribution shift (described above) – unlike in Maskin and Riley (2000a) – at least for not too large shifts;\(^7\)

- the revenue in the FPA may decrease when all the valuations increase, because increasing the high valuation of one bidder may induce his opponent to bid less aggressively. This makes the FPA inferior to the SPA, in contrast with a claim in Maskin and Riley (1985) for the particular case in which the only deviation from a symmetric setting is given by unequal high valuations [however, for this case Maskin and Riley (1983) agree with our ranking between the FPA and the SPA].

Finally, we show that the bidders’ preferences among the two auctions often go in the opposite direction with respect to the seller’s preferences.

The remainder of the paper is organized as follows. In Section 2 we describe the primitives of our model. In Section 3 we study equilibrium behavior in the SPA and in the FPA, and in Section 4 we present our results on the comparison between the FPA and the SPA.

### 2 The model

A (female) seller owns an indivisible object which is worthless to her and faces two (male) bidders. Let \(v_1\) (\(v_2\)) denote the monetary valuation for the object of bidder 1 (bidder 2), which he privately observes; \(v_1\) and \(v_2\) are independently distributed. The set \(\{v_{1L}, v_{1H}\}\) is the support for \(v_1\), with \(0 < v_{1L} < v_{1H}\) and \(\lambda_1 \equiv \Pr\{v_1 = v_{1L}\} \in (0, 1)\). Likewise, the support for \(v_2\) is \(\{v_{2L}, v_{2H}\}\) with \(0 < v_{2L} < v_{2H}\) and \(\lambda_2 \equiv \Pr\{v_2 = v_{2L}\} \in (0, 1)\). Without loss of generality we assume that \(v_{1L} \leq v_{2L}\). Both the seller and bidders are risk neutral, and a bidder’s utility if he wins is given by his valuation for the object minus the price paid to the seller; his utility if he loses is zero. We use \(i_j\) to denote bidder \(i\) when his valuation is \(v_{ij}\), thus for instance \(2_L\) is the type of bidder 2 with valuation \(v_{2L}\).

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\(^6\)After Vickrey (1961), this is the first ranking result in the theoretical literature which does not rely on first order stochastic dominance among the distributions of valuations.

\(^7\)Kirkegaard (2011) examines an example in which the distribution of the valuation of a bidder is obtained by rescaling the distribution of the other bidder’s valuation. In this case we show that the SPA dominates the FPA in our setting, unless the rescaling is large.
The main purpose of this paper is to evaluate the relative profitability of the FPA and the SPA for the seller. In either of these auctions each bidder submits simultaneously a nonnegative sealed bid, and the bidder who makes the highest bid wins the object (if the bidders tie, the winner is selected according to a specified tie-breaking rule: see next section). In the FPA the winning bidder pays the own bid; in the SPA he pays the loser’s bid (i.e., the second highest bid).

3 Equilibrium bidding

3.1 SPA

It is well known that when bidders have private values, in the SPA it is weakly dominant for each bidder to bid the own valuation. Thus the seller’s expected revenue $R^S$ is the expectation of $\min\{v_1, v_2\}$, which is straightforward to evaluate (recall that $v_1 L \leq v_2 L$):

$$R^S = \begin{cases} 
\lambda_1 v_1 L + (1 - \lambda_1) v_1 H & \text{if } v_1 H \leq v_2 L \\
\lambda_1 v_1 L + (1 - \lambda_1)(\lambda_2 v_2 L + (1 - \lambda_2) v_1 H) & \text{if } v_2 L < v_1 H \leq v_2 H \\
\lambda_1 v_1 L + (1 - \lambda_1)(\lambda_2 v_2 L + (1 - \lambda_2) v_2 H) & \text{if } v_2 H < v_1 H
\end{cases} \quad (1)$$

3.2 FPA

The analysis for the FPA is less immediate than for the SPA. In fact, finding the closed form for the equilibrium bidding strategies for an FPA with asymmetrically distributed valuations is often impossible when valuations are continuously distributed. However, this is not the case given our assumptions on the distributions of $v_1$ and $v_2$. We consider BNE in which no type of bidder bids above the own valuation.

In our setting no pure-strategy Bayes-Nash equilibrium exists [except in the case that condition (3) below is satisfied], and sometimes no mixed-strategy Bayes-Nash equilibrium (BNE in the following) exists either. Precisely, when $v_1 L = v_2 L$ we find that no BNE exists in the standard FPA in which each bidder wins with probability $\frac{1}{2}$ in case of tie.\(^8\) However, Proposition 2 in Maskin and Riley (2000b) establishes that a BNE exists under a suitable tie-breaking rule such that each bidder $i$ is required to submit both an “ordinary” bid $b_i \geq 0$ and a “tie-breaker” bid $c_i \geq 0$.\(^9\) If $b_1 \neq b_2$, then $c_1, c_2$ are irrelevant but if $b_1 = b_2$ then bidder $i$ wins if $c_i > c_j$ and pays $b_i + c_j$ (each bidder wins with probability $\frac{1}{2}$ if $b_1 = b_1$ and $c_1 = c_2$). Therefore $c_1, c_2$ are bids in a second price/Vickrey auction which takes place if and only if $b_1 = b_2$. In Proposition 1 we consider the first price auction with this “Vickrey tie-breaking rule”.

We want to stress that this particular tie-breaking rule is needed only when $v_1 L = v_2 L$, since existence is obtained for any tie-breaking rule if $v_1 L \neq v_2 L$. Precisely, when $v_1 L < v_2 L$ we find that multiple BNE exist regardless of the tie-breaking rule, but they are all outcome-equivalent. In particular, multiple BNE arise because type $1_L$ (and type $1_H$ in one case) never wins and needs to

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\(^8\)See the proof to Proposition 1 in the appendix (step 2 for the case of $v_1 L = v_2 L$).

\(^9\)A very similar idea appears in Lebrun (2002), in the auction he denotes with FPA.
bid weakly less than \( v_{1L} \) (weakly less than \( v_{1H} \)) with probability one, in such a way that no type of bidder 2 has incentive to bid below \( v_{1L} \) (below \( v_{1H} \)). Since there are many strategies of \( 1_L \) (of \( 1_H \)) which achieve this goal,\(^{10}\) multiple BNE exist. However, as we specified above, each BNE generates the same outcome in the sense that the allocation of the object, the payoff of each type of bidder and the expected revenue are the same;\(^{11}\) therefore multiplicity is not an issue.

Conversely, when \( v_{1L} = v_{2L} \) in each BNE both types \( 1_L \) and \( 2_L \) bid \( v_{1L} \), and (generically) also \( 1_H \) or \( 2_H \) bid \( v_{1L} \) with positive probability; suppose \( 2_H \) does so (to fix the ideas). Then \( 2_H \) ties with positive probability with \( 1_L \) by bidding \( v_{1L} \), and if \( 2_H \) does not win the tie-break with probability one, he has an incentive to bid slightly above \( v_{1L} \), which breaks the BNE. On the other hand, under the Vickrey tie-breaking rule, for a bidder \( i \) with valuation \( v_i \) submitting an ordinary bid \( b_i \), it is weakly dominant to choose \( c_i = v_i - b_i \), and in particular \( c_{1L} = 0, c_{2H} = v_{2H} - v_{1L} > 0 \) for the case we are considering; thus \( 2_H \) wins the tie-break paying \( v_{1L} \) in aggregate.\(^{12}\) Given this property on weak dominance for tie-breaking bids, when we describe a strategy of bidder \( i \) we implicitly assume that to each ordinary bid \( b_i \) is associated a tie-breaking bid \( c_i \) equal to \( v_i - b_i \). Therefore, whenever a tie occurs the bidder with the highest valuation wins and pays the valuation of the other bidder.

In Proposition 1(ii) below an important role is played by a specific bid \( \hat{b} \) which is the smaller solution to the following equation:

\[
\lambda_2 b^2 + ((1-\lambda_2)v_{1H} + (\lambda_1 - \lambda_2)v_{2L} - \lambda_1 v_{1L} - v_{2H})b + ((1-\lambda_1)v_{2H} - (1-\lambda_2)v_{1H})v_{2L} + \lambda_1 v_{1L} v_{2H} = 0 \tag{2}
\]

and assumption (4) in Proposition 1(ii) implies that \( \hat{b} \) satisfies \( v_{1L} \leq \hat{b} < \min \{v_{2L}, v_{1H} \} \).\(^{13}\)

**Proposition 1** Given \( v_{1L} \leq v_{2L} \), consider the FPA with the Vickrey tie-breaking rule. Although multiple BNE may exist, they are all outcome-equivalent to the following BNE.

Type \( 1_L \) always bids \( v_{1L} \) and the bids of the other types depend on the parameters as follows:

(i) If

\[
v_{1H} \leq \lambda_1 v_{1L} + (1 - \lambda_1)v_{2L} \tag{3}
\]

then types \( 2_L, 2_H \) bid \( v_{1H} \); type \( 1_H \) bids weakly less than \( v_{1H} \) with probability one and in such a way that no type of bidder 2 has incentive to bid below \( v_{1H} \).

(ii) If

\[
\lambda_1 v_{1L} + (1 - \lambda_1)v_{2L} < v_{1H} < \frac{(1 - \lambda_1)v_{2H} + (\lambda_1 - \lambda_2)v_{1L}}{1 - \lambda_2} \tag{4}
\]

then types \( 1_H, 2_L, 2_H \) play mixed strategies with support \([v_{1L}, \hat{b}] \) for \( 1_H \), \([v_{1L}, \hat{b}] \) for \( 2_L \), \([\hat{b}, \hat{b}] \) for \( 2_H \), in which \( \hat{b} \) is the smaller solution to (2) and \( \hat{b} = \lambda_2 \hat{b} + (1 - \lambda_2)v_{1H} \). The c.d.f. for the mixed

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\(^{10}\)One example is such that \( 1_L \) bids according to the uniform distribution on \([v_{1L}, v_{1L}] \) with \( \alpha < 1 \) and close to 1.

\(^{11}\)A slight modification of the proof of Proposition 1 (for the case of \( v_{1L} < v_{2L} \)) shows that all BNE are outcome-equivalent regardless of the tie-breaking rule. Details are available upon request.

\(^{12}\)In fact, whenever \( 1_L \) bids \( v_{1L} \) and ties with positive probability with type \( 2_j \) such that \( v_{2j} > v_{1L} \), in each BNE \( 1_L \) selects \( c_{1L} = 0 \), otherwise it is profitable for \( 2_j \) to bid slightly above \( v_{1L} \).

\(^{13}\)For more details see Proposition 1(ii) and its proof in the appendix.
strategies are

\[ G_{1H}(b) = \begin{cases} 
\frac{\lambda_1(b-v_{1L})}{(1-\lambda_1)(v_{2L}-b)} & \text{for } b \in [v_{1L}, \hat{b}] \\
\frac{1}{1-\lambda_1}(v_{2H}-b - \lambda_1) & \text{for } b \in (\hat{b}, \bar{b}) 
\end{cases} \]  

(5)

\[ G_{2L}(b) = \frac{v_{1H} - b}{\lambda_2(v_{1H} - b)} \]  

(6)

and from the definitions of \( \hat{b} \) and \( \bar{b} \) it follows that \( G_{1H} \) is continuous at \( b = \hat{b} \), \( G_{2L}(\hat{b}) = 1 \), \( G_{2H}(\hat{b}) = 0 \).\(^{14}\)

(iii) If

\[ \frac{(1-\lambda_1)v_{2H} + (\lambda_1 - \lambda_2)v_{1L}}{1 - \lambda_2} \leq v_{1H} \]  

(7)

then \( 2_L \) bids \( v_{1L} \) and \( 1_H, 2_H \) play mixed strategies with common support \([v_{1L}, \lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}]\) and the following c.d.f.

\[ G_{1H}(b) = \frac{\lambda_1 b - v_{1L}}{1 - \lambda_1 v_{2H} - b}, \quad G_{2H}(b) = \frac{1}{1 - \lambda_2} \left( \frac{v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H}}{v_{1H} - b} - \lambda_2 \right) \]  

(8)

When (3) holds, Proposition 1(i) establishes that each type of bidder 2 bids \( v_{1H} \) and wins for sure.\(^{15}\) This occurs because \( v_{2L} \) is sufficiently larger than \( v_{1H} \), which implies that each type of bidder 2 has so much to gain from winning that it is profitable for him to make a bid of \( v_{1H} \) in order to outbid each type of bidder 1. Precisely, (3) guarantees that type \( 2_L \) prefers winning for sure by bidding \( v_{1H} \) rather than bidding \( v_{1L} \) and winning only when facing type \( 1_L \), that is with probability \( \lambda_1 \).

Conversely, if \( v_{1H} \) is large then (3) is violated and \( 2_L \) is less aggressive since he prefers to bid \( v_{1L} \) and win only against \( 1_L \) rather than bidding \( v_{1H} \) and winning with certainty, as the latter alternative is too expensive. Indeed, \( 2_L \) bids in the interval \([v_{1L}, \hat{b}]\), with \( \hat{b} < v_{1H} \), and with an atom at \( b = v_{1L} \): \( G_{2L}(v_{1L}) = \frac{v_{1H} - \hat{b}}{\lambda_2(v_{1H} - v_{1L})} > 0 \). The less aggressive bidding of \( 2_L \) allows \( 1_H \) to win with positive probability by bidding in \([v_{1L}, \hat{b}]\), which makes his equilibrium payoff positive. This implies that the highest bid of \( 1_H \) is smaller than \( v_{1H} \), since each bid in the support of a bidder’s mixed strategy needs to maximize the expected payoff of the bidder given the strategies of the other types. Therefore also the highest bid of \( 2_H \) is smaller than \( v_{1H} \), as we see from Proposition 1(ii). As \( v_{1H} \) increases, \( 2_L \) becomes increasingly less aggressive: \( \hat{b} \) decreases and \( G_{2L}(b) \) increases for any \( b \in [v_{1L}, \hat{b}] \). This occurs because as \( v_{1H} \) increases, the equilibrium payoff of \( 1_H \) increases and this requires that \( G_{2L} \) puts more weight on \( v_{1L} \) and becomes flatter in \([v_{1L}, \hat{b}]\) to satisfy the indifference condition of \( 1_H \).\(^{16}\) For a large enough \( v_{1H} \) such that (7) is satisfied, this effect implies

\(^{14}\)In the case that \( \hat{b} = v_{1L} \) (which occurs if and only if \( v_{1L} = v_{2L} \)), \( 2_L \) bids \( v_{1L} \) and \( \hat{b} = \lambda_2 v_{1L} + (1 - \lambda_2)v_{1H} \), thus \( G_{1H}(b) = \frac{\lambda_1}{1-\lambda_1}(v_{2L} - \hat{b} - \lambda_1) \) and \( G_{2H}(b) = \frac{1}{1-\lambda_2}(v_{2H} - \hat{b} - \lambda_2) \) for each \( b \in [v_{1L}, \hat{b}] \).

\(^{15}\)In a setting with continuously distributed valuations, Maskin and Riley (2000a) identify an analogous BNE and provide the intuition we describe here and immediately after Proposition 2(i). In addition, Maskin and Riley (1983) identify the BNE we describe in Proposition 1 for the case of \( v_{1L} = v_{2L} = 0 \). Thus Proposition 1 is a new result for the case in which \( v_{1L} < v_{2L} \) and (3) is violated.

\(^{16}\)We describe a similar effect (with more details) in the intuition regarding condition (9) in Proposition 2(ii).
that $2_L$ bids $v_{1L}$ with certainty and also $2_H$ bids $v_{1L}$ with positive probability. In particular, when (7) holds the equilibrium strategies – and thus also the expected revenue – do not depend on $v_{2L}$.

A well known feature of the FPA when valuations are asymmetrically distributed is that an inefficient allocation of the object is implemented with positive probability. In our setting, suppose for instance that $v_{1L} < v_{2L}$, $v_{2L} \neq v_{1H}$ and (4) holds. Then $\hat{b} > v_{1L}$ and in the state of the world with types $1_H, 2_L$ each type has a positive probability to win and thus the highest valuation type may not win.

4 Comparison between the FPA and the SPA

In order to derive the seller’s preferences between the FPA and the SPA we need to evaluate the expected revenue $R^F$ in the FPA generated by the BNE described in Proposition 1. Although we can express $R^F$ in closed form (see Section 5.3 in the appendix), in many cases $R^F$ is a complicated function of the parameters [an exception occurs when (3) is satisfied]; this is largely due to the inefficiency of the FPA we mentioned above. In particular, it seems difficult to obtain insights from comparing $R^F$ with $R^S$ without any restriction on the parameters. We focus therefore on two particular cases which yield nevertheless quite interesting results. One is such that $\lambda_1 = \lambda_2$, and the other is such that $v_{1L} = v_{2L}$; the analysis of the case in which $v_{1L} = v_{2L}$ is performed in Maskin and Riley (1983).

4.1 The case in which $\lambda_1 = \lambda_2$

The following proposition describes our main results when $\lambda_1 = \lambda_2$ [in fact, Proposition 2(i) does not require $\lambda_1 = \lambda_2$]. The rest of this subsection is devoted to discussing these results and in providing intuitions.

**Proposition 2** (i) $R^F > R^S$ if (3) is satisfied;
(ii) $R^S > R^F$ if $\lambda_1 = \lambda_2 \equiv \lambda$ and at least one of the following conditions is satisfied:

\begin{align}
&v_{1L} = v_{2L} \quad \text{and} \quad v_{1H} \neq v_{2H} \quad \text{(9)} \\
&v_{1L} < v_{2L} \quad \text{and} \quad v_{2H} \leq v_{1H} \quad \text{(10)} \\
&v_{1L} < v_{2L} \leq v_{1H} < v_{2H} \quad \text{and} \quad v_{2L} - v_{1L} \simeq 0 \quad \text{or} \quad \lambda \geq \frac{1}{2} \quad \text{(11)}
\end{align}

Proposition 2(i) is very simple to interpret. Precisely, $R^F = v_{1H}$ when (3) is satisfied as both types of bidder 2 win the auction with a bid of $v_{1H}$; moreover, (3) implies $v_{1H} \leq v_{2L}$ and thus $R^S = \lambda_1 v_{1L} + (1 - \lambda_1) v_{1H}$. Then $R^F > R^S$ follows immediately. The intuition is that in both auctions bidder 2 always wins, thus $R^S$ is equal to the expected valuation of the loser, bidder 1, and $R^F$ is the high valuation of bidder 1.
Proposition 2(ii) describes a set of circumstances which imply $R^S > R^F$, and in order to facilitate its understanding it is useful to have in mind a benchmark symmetric environment which we now describe.

The benchmark symmetric setting Suppose that $v_{1L} = v_{2L} \equiv v_L$, $v_{1H} = v_{2H} \equiv v_H$ and $\lambda_1 = \lambda_2 \equiv \lambda$. We know from Maskin and Riley (1985) that in this case the unique BNE in the FPA is such that types 1,L,2,L both bid $v_L$ and types 1,H,2,H play the same mixed strategy with support $[v_L, E_v]$ – in which $E_v \equiv \lambda v_L + (1 - \lambda)v_H$ – and c.d.f. $G_H(b) = \frac{\lambda}{1 - \lambda} \frac{b-v_L}{v_H}$. Furthermore, $R^F = R^S = (2\lambda - \lambda^2)v_L + (1 - \lambda)^2v_H$.

4.1.1 Condition (9) Going back to Proposition 2(ii), we start by considering (9). This condition implies $R^S > R^F$, and in fact this result relies on the following property.

Proposition 3 Suppose that $v_{1L} = v_{2L} = v_L$, $v_{1H} \neq v_{2H} = v_H$ and $\lambda_1 = \lambda_2 = \lambda$. Then $R^F$ is increasing in $v_{1H}$ for $v_{1H} \in (v_L, v_H)$ and is decreasing in $v_{1H}$ for $v_{1H} \in (v_H, +\infty)$.

This proposition says that in a setting which is asymmetric only because $v_{1H} \neq v_H$, $R^F$ is maximized with respect to $v_{1H}$ at $v_{1H} = v_H$, and in particular increasing $v_{1H}$ above $v_H$ reduces $R^F$.\footnote{This fact may appear similar to the main message in Cantillon (2008), but in fact in our analysis the benchmark symmetric setting is fixed, whereas in Cantillon (2008) it is not.}

In order to obtain an intuition for Proposition 3 we start with the case of $v_{1H} > v_H$ and notice that given $\lambda_1 = \lambda_2$, (7) is satisfied when $v_{1H} > v_H$ and therefore Proposition 1(iii) applies. This reveals that the behavior of types 1,L,1,H,2,L is unchanged with respect to the benchmark symmetric setting, whereas now 2,H bids less aggressively. Precisely, $G_H$ and $G_{2H}$ have the same support $[v_L, E_v]$, but since $G_{2H}(b) = \frac{(1-\lambda)(v_{1H}-v_H)+\lambda(b-v_L)}{(1-\lambda)(v_{1H}-b)}$ it is simple to verify that $G_{2H}(b) > G_H(b)$ for any $b \in [v_L, E_v]$, and in particular $G_{2H}(v_L) > 0 = G_H(v_L)$. Since $2_H$ is less aggressive with respect to the symmetric setting, it follows that an increase in $v_{1H}$ has a negative effect on $R^F$. In fact, the larger is $v_{1H}$ the higher (lower) is the probability that $G_{2H}$ attaches to low (high) bids in $[v_L, E_v]$. As a consequence, $R^F$ is monotonically decreasing with respect to $v_{1H}$ for $v_{1H} > v_H$.

It is somewhat surprising that, starting from a symmetric setting, an increase of a single (high) valuation generates a decrease in $R^F$. In order to see what drives the result, suppose for a moment that 2,H still bids according to $G_H$ even though $v_{1H} > v_H$. Then the payoff of type 1,H from bidding $b \in [v_L, E_v]$ is $(v_{1H} - b)[\lambda + (1 - \lambda)G_H(b)]$. This is obviously higher than $(v_H - b)[\lambda + (1 - \lambda)G_H(b)]$, his payoff before the increase in $v_{1H}$, and – more importantly – is increasing in $b$. In order to make 1,H indifferent among the bids in an interval $(v_L, b^*)$ it is necessary that $G_{2H}$ is flatter than $G_H$.

\footnote{Obviously, an analogous result holds if $v_{1H}$ is kept fixed and $v_{2H}$ is allowed to vary.}
and indeed \( G_{2H}(b) = \frac{(1-\lambda)(v_H-v_B)+\lambda(b-v_L)}{(1-\lambda)(v_H-b)} \) has an atom at \( b = v_L \) and grows more slowly than \( G_H \) for \( b > v_L \). This is how a less aggressive behavior of \( 2_H \) results from an increase in \( v_{1H} \).

Given this result, it is straightforward to see that an increase in \( v_{1H} \) favors the SPA over the FPA since it does not affect the distribution of \( \min\{v_1, v_2\} \), and thus \( R^S \) does not change.

Maskin and Riley (1985) (in their Section III) consider the setting of Proposition 2(ii), except that they assume \( v_{1L} = v_{2L} = 0 \), and claim that an increase in \( v_{2H} \) favors the FPA over the SPA, in contrast with Proposition 2(ii). However, they do not provide a formal proof of their claim. On the other hand, Maskin and Riley (1983) conclude that \( R^S > R^F \), consistently with Proposition 2(ii): see their Figure 1 between pages 18 and 19.

For the case of \( v_{1H} < v_H \), Proposition 3 establishes the intuitive result that \( R^F \) is reduced with respect to when \( v_{1H} = v_H \). We can rely on Proposition 1(ii) (and in particular on footnote 14 since \( v_{1L} = v_{2L} \) and thus \( b = v_{1L} \)), but a simpler argument is also available. Given \( v_{1H} < v_H \), consider the symmetric setting with low valuations both equal to \( v_{1L} \) and high valuations both equal to \( v_{1H} \). Then \( R^F \) is smaller, by \( (1-\lambda)^2(v_H-v_{1H}) \), with respect to \( R^F \) for the benchmark symmetric setting. Now increase the valuation of type \( 2_H \) from \( v_{1H} \) to \( v_H \) to obtain the asymmetric setting we are considering. The same logic of Proposition 3 (see footnote 18) suggests that \( R^F \) further decreases. Therefore a decrease in the valuation of \( 1_H \) below \( v_H \) reduces \( R^F \) by more than \( (1-\lambda)^2(v_H-v_{1H}) \). On the other hand, from (1) we see that \( R^S \) decreases exactly by \( (1-\lambda)^2(v_H-v_{1H}) \). Hence, \( R^S > R^F \) holds both if \( v_{1H} > v_H \) and also if \( v_{1H} < v_H \).

**On the effect of increasing** \( v_{1L}, v_{2L}, v_{1H}, v_{2H} \) Proposition 3 suggests a simple observation. Suppose that we start from the benchmark symmetric setting and let \( R^{F*} \) denote the resulting expected revenue. Then suppose that the valuation of \( 1_H \) is increased; this reduces the revenue below \( R^{F*} \) by Proposition 3. Finally, increase slightly the valuations of \( v_{1L}, v_{2L}, v_{1H}, v_{2H} \) (but not each valuation by the same amount) we obtain a setting in which the revenue is reduced. For instance, suppose that \( v_{1L} = v_{2L} = 100, \ v_{1H} = v_{2H} = 200 \) and \( \lambda_1 = \lambda_2 = \frac{1}{2} \); then \( R^{F*} = 125 \). However, if \( v_{1L} = v_{2L} = 105, \ v_{1H} = 400 \) and \( v_{2H} = 205 \), then \( R^F \approx 123.12 \).

### 4.1.2 Condition (10)

The effects of (10) are almost straightforward. In case that \( v_{2H} = v_{1H} \), Proposition 1(iii) applies and as we mentioned in Subsection 3.2, \( R^F \) does not depend on \( v_{2H} \in [v_{1L}, v_{2H}] \); thus \( R^F \) is equal

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19Lebrun (1998) considers a setting with continuously distributed valuations and assumes that the valuation distribution of one bidder changes into a new distribution which dominates the previous one in the sense of reverse hazard rate domination (the support is unchanged). He show that, as a consequence, for each bidder the new bid distribution first order stochastically dominates the initial bid distribution, and thus the expected revenue increases.

20Since they assume \( v_{1L} = v_{2L} = 0 \), Maskin and Riley (1983) do not consider the various cases covered in Proposition 2(ii), and they do not have the results in Propositions 3 and 4.
to the revenue in the symmetric setting. On the other hand, (1) reveals that $R^S$ is increasing in $v_{2L}$ and therefore $R^S > R^F$.

In case that $v_{2H} < v_{1H}$, suppose first that $v_{1L} = v_{2L}$. We know from condition (9) that $v_{2H} < v_{1H}$ implies $R^S > R^F$, and the previous paragraph explains that an increase in $v_{2L}$ has no effect on $R^F$ (but increases $R^S$); hence the conclusion.

$v_1$ more uncertain than $v_2$ It is interesting to notice that (10) includes the case in which $v_{1L} < v_{2L} < v_{2H} < v_{1H}$, which means that $v_1$ has a wider range of variability than $v_2$; obviously, this includes the special case in which $v_1$ is a mean-preserving-spread of $v_2$. In each of these cases we obtain an unambiguous ranking between $R^S$ and $R^F$, that is the SPA is better than the FPA when the valuation of one bidder is more uncertain than the valuation of the other bidder.

Kirkegaard (2011) notices that only Vickrey (1961) provides a theoretical ranking result without the assumption of first order stochastic dominance between the bidders’ distributions of valuations.\textsuperscript{21} Precisely, Vickrey (1961) assumes that $v_1$ is uniformly distributed over $[0, 1]$ and $v_2$ is equal to a fixed value $a$, that is $v_2$ is common knowledge; he proves that the FPA is superior to the SPA for $a > 0.43$. Now consider in our framework the parameters $\lambda = \frac{1}{2}$ and $v_{1L} = 0$, $v_{1H} = 1$, $v_{2L} = a - x$, $v_{2H} = a + x$ with $x > 0$ and close to zero. This setting is in a sense similar to that in Vickrey (1961) since $v_1$ is uniformly distributed over $\{0, 1\}$, and $v_2$ is almost commonly known to be equal to $a$.\textsuperscript{22} However, Proposition 2(ii) establishes that $R^S > R^F$ for any $a \in (0, 1)$. This difference with respect to Vickrey (1961) arises because in our setting $R^F$ is considerably lower than in Vickrey (1961), due to the fact that type $2_L$ bids $v_{1L} = 0$ with certainty (and type $2_H$ bids 0 with positive probability), as bidding 0 suffices to win the auction if the opponent is type $1_L$, an event with probability $\frac{1}{2}$. Conversely, this does not occur when $v_1$ is uniformly distributed over $[0, 1]$ because if bidder 2 bids close to zero then he wins only against a small set of types of bidder 1. For instance, if $a = \frac{1}{2}$ then Vickrey (1961) proves that bidder 2’s equilibrium mixed strategy has support $[\frac{1}{2}, \frac{7}{10}]$, that is his minimum bid is $\frac{1}{4}$.

4.1.3 Condition (11)

Given the innocuous assumption that $v_{1L} \leq v_{2L}$, after (9) and (10) have been considered, the only class of asymmetry remaining is such that $v_{1L} < v_{2L}$ and $v_{1H} < v_{2H}$; the results for this case are quite simple to describe. First, from (9) and (10) it is intuitive that $R^S > R^F$ when $v_{2L} - v_{1L}$ is close to zero and $v_{2H} - v_{1H}$ is close to zero. On the other hand, Proposition 2(i) establishes that $R^F > R^S$ when $v_{2L}$ is sufficiently larger than $v_{1H}$ (which implies that also $v_{2H}$ is quite large). From (11) we see that as long as $\lambda \geq \frac{1}{2}$, $R^S > R^F$ holds provided that $v_{2L}$ is not larger than $v_{1H}$. We notice that this result is quite conservative because of the way it is obtained. Precisely, in our final

\textsuperscript{21}Gayle and Richard (2008), Li and Riley (1999) and Li and Riley (2007) apply numeric analysis to settings without first order stochastic dominance and obtain mixed results.

\textsuperscript{22}Proposition 1 still holds even though $v_{1L} = 0$ violates our assumption $v_{1L} > 0$. However, when $v_{1L} = 0$ the Vickrey tie-breaking rule is needed also if $v_{1L} \neq v_{2L}$.
remark in Subsection 3.2 we noticed that in the BNE described by Proposition 1(ii) the highest valuation bidder does not always win. Conversely, the efficient allocation is always achieved in the SPA. Therefore a sufficient condition for \( R^S > R^F \) is that the aggregate bidders’ rents in the FPA, \( U^F \), are (weakly) larger than the rents in the SPA, \( U^S \). Condition (11) guarantees indeed that \( U^F \geq U^S \). In words, when \( \lambda \geq \frac{1}{2} \) in order for \( R^F \geq R^S \) to hold it is not sufficient that the distribution of \( v_2 \) first order stochastically dominates the distribution of \( v_1 \), but it is actually necessary that \( v_1 \) \( H < v_2 \) \( L \); broadly speaking, we could say that there need to be no overlapping of supports.

In Figure 1 we fix \( \lambda = \frac{1}{4} \) and \( v_2L, v_2H \), and partition the space \((v_1L, v_1H)\) in two regions \( S \) and \( F \) such that \( R^S > R^F \) if \((v_1L, v_1H) \in S\), and \( R^F \geq R^S \) if \((v_1L, v_1H) \in F\). In particular, \( S(iii) \) is the set in which (10) is satisfied [in this case (7) holds and the BNE of Proposition 1(iii) applies]; \( F(i) \) is the set in which (3) holds [then the BNE of Proposition 1(i) applies]. The region between \( S(iii) \) and \( F(i) \) is such that (4) is satisfied – thus the BNE of Proposition 1(ii) applies – and the boundary between \( S \) and \( F \) is obtained numerically.\(^{23}\) For other values of \( \lambda < \frac{1}{2} \) a similar figure is obtained.\(^{24}\)

**Figure 1:** Comparison between the FPA and the SPA when \( \lambda_1 = \lambda_2 \).

In the dark region \( S = S(ii) \cup S(iii) \) the SPA dominates the FPA in terms of the seller’s revenue. Proposition 1(i) applies in the lower region \( F(i) \), 1(ii) in the region \( F(ii) \cup S(ii) \) in the middle, and 1(iii) in the upper region \( S(iii) \).

We notice that when \( \lambda_1 = \lambda_2 \), by Proposition 2(ii) the SPA is better than the FPA for any small deviation from the symmetric setting, that is when \( v_2L - v_1L \) and \( v_2H - v_1H \) are close to zero, but \( v_2L - v_1L > 0 \) and/or \( v_2H - v_1H \neq 0 \), as it is apparent from Figure 1.

\(^{23}\)The precise values of \( v_2L \) and \( v_2H \) do not affect the qualitative features of Figure 1 since (i) if all valuations are increased by a same amount \( \alpha \), then both \( R^F \) and \( R^S \) increase by \( \alpha \) and thus \( R^F - R^S \) is unaffected; (ii) if all valuations are multiplied by a same number \( \beta > 0 \), then both \( R^F \) and \( R^S \) are multiplied by \( \beta \) and the sign of \( R^F - R^S \) is unaffected. Using these two degrees of freedom, we can fix arbitrarily \( v_2L \) and \( v_2H \) without affecting the qualitative features of the sets \( S \) and \( F \). Conversely (of course), the value of \( \lambda \) affects substantially \( S \) and \( F \).

\(^{24}\)In fact, for \( \lambda < \frac{1}{2} \) but close to \( \frac{1}{2} \) each point on the boundary between \( S \) and \( F \) is such that \( v_1H < v_2L \).
Distribution shift and rescaling  A particular type of asymmetry considered in the literature is as follows. Given the c.d.f. $F_1$ for the valuation of bidder 1, the c.d.f. for $v_2$ is $F_2(v_2) = F_1(v_2 - a)$ with $a > 0$, that is $F_2$ is obtained by shifting $F_1$ to the right, which implies that bidder 2 is (ex ante) stronger than 1. In a setting with continuously distributed values, Maskin and Riley (2000a) prove that under suitable assumptions on $F_1$, the FPA generates a higher revenue than the SPA; Kirkegaard (2011) obtains the same result under weaker assumptions. In our context this sort of asymmetry is obtained by fixing $v_{1L}, v_{1H}$ and setting $v_{2L} = v_{1L} + a$, $v_{2H} = v_{1H} + a$, for some $a > 0$. From (11) we can find sufficient conditions for $R^S > R^F$, but in fact in the appendix we exploit this particular structure of asymmetry to modify the proof of Proposition 2(ii) and show that $R^S > R^F$ as long as $\frac{a}{v_{1H} - v_{1L}} \leq \frac{2\lambda}{2-3\lambda}$ (for $\lambda \leq \frac{2}{3}$) or $\frac{a}{v_{1H} - v_{1L}} \leq \frac{2(2+\lambda)}{3(2-\lambda)}$ (for $\lambda > \frac{2}{3}$). Actually, also this result is quite conservative, as numeric analysis shows that $R^S > R^F$ holds for the set of parameters in region $S$ in Figure 2.

Figure 2: Comparison between the FPA and the SPA in case of distribution shift.

The thin curve in the dark region is such that $U^F > U^S$ holds for the parameters below the curve. However, the SPA dominates the FPA in terms of the seller’s revenue in the whole dark region $S$.

Thus in our discrete setting a shift favors the FPA over the SPA only if the shift is sufficiently large; for instance, $R^F > R^S$ definitely holds if $a$ is such that (3) is satisfied, that is if $\frac{a}{v_{1H} - v_{1L}} \geq \frac{1}{1-\lambda}$. On the other hand, in their numeric analysis applied to continuous distributions, Li and Riley (2007) find that a shift “can result in economically very significant revenue differences [in favor of the FPA]” for examples with uniform or truncated normal distributions, and notice that “Analysis of other distributions also produces broadly similar results”.

Example 4 in Kirkegaard (2011) starts from $F_2$ such that $F_2(e^v)$ is convex and log-concave and obtains $F_1$ as $F_1(v) = F_2(\gamma v)$ for some $\gamma > 1$ and not too large; thus $v_1$ is a rescaling of $v_2$, and Kirkegaard (2011) proves that $R^F > R^S$. In our context this sort of asymmetry is obtained by fixing $v_{2L}, v_{2H}$ and setting $v_{1L} = \frac{1}{\gamma}v_{2L}$, $v_{1H} = \frac{1}{\gamma}v_{2H}$. The comparison between the SPA and the
FPA yields results similar to those obtained for a shift. Precisely, (11) reveals that $R_S > R_F$ if $\gamma$ is not much larger than 1, whereas a large $\gamma$ makes (3) satisfied and thus $R_F > R_S$.

4.1.4 The distribution of bids in the FPA and the bidders’ preferences

For $i = 1, 2$, let $G_i$ denote the ex ante c.d.f. of the equilibrium bids submitted by bidder $i$ in the FPA, that is $G_i(b) = \lambda G_{iL}(b) + (1 - \lambda)G_{iH}(b)$. Using Proposition 1 we can compare the equilibrium bid distributions of bidder 1 and 2 in the FPA, and we find that $G_2$ first order stochastically dominates $G_1$ when $v_{2H} > v_{1H}$; the opposite result obtains if $v_{1H} > v_{2H}$. Notice that when $v_{2H} > v_{1H}$, the distribution of $v_2$ first order stochastically dominates the distribution of $v_1$ and the result that $G_2$ first order stochastically dominates $G_1$ agrees with Corollary 1 in Kirkegaard (2009), for a setting with continuous distributions. On the other hand, when $v_{2H} < v_{1H}$ there is no first order stochastic dominance between the distribution of $v_1$ and $v_2$, but second order stochastic dominance applies if $v_{1H} \leq v_{2H} + \frac{1}{\lambda}(v_{2L} - v_{1L})$, that is if the expected value of $v_2$ is weakly larger than the expected value of $v_1$. Under second order stochastic dominance between the valuations distributions, Proposition 5 in Kirkegaard (2009) shows that the bid distributions must cross, whereas we find that $G_1$ first order stochastically dominates $G_2$.

Proposition 1 also allows us to compare the bidders’ payoffs in the FPA with their payoffs in the SPA: it turns out that bidder 1 weakly prefers the FPA, whereas bidder 2 weakly prefers the SPA. These results largely agree with the results in Propositions 3.3(ii) and 3.6 in Maskin and Riley (2000a).

4.1.5 Relationship with Kirkegaard (2011)

Proposition 2(ii) reveals that $R_S > R_F$ for a broad set of deviations from the benchmark symmetric setting, provided that $\lambda_1 = \lambda_2$. On the other hand, a frequent result in the literature on asymmetric auctions is that $R_F > R_S$. Since the most general theoretical results are obtained in Kirkegaard (2011), we explain why his analysis does not apply to our setting.

Kirkegaard (2011) considers a two-bidder environment with supports $[\beta_1, \alpha_1]$ for $v_1$ and $[\beta_2, \alpha_2]$ for $v_2$ such that $\beta_1 \leq \beta_2$ and $\alpha_1 < \alpha_2$. The c.d.f. $F_1, F_2$ have no atoms and have continuous and positive densities $f_1, f_2$ in the respective supports; moreover, 1 is ex ante weaker than 2 in the sense that $F_2$ first order stochastically dominates $F_1$. A crucial ingredient for the result is $r(v)$, which is defined as $F_2^{-1}[F_1(v)]$ for each $v \in [\beta_1, \alpha_1]$, that is $r(v)$ satisfies $\Pr\{v_2 \leq r(v)\} = \Pr\{v_1 \leq v\}$ and $r(v) \geq v$ as $F_2$ first order stochastically dominates $F_1$. The main result in Kirkegaard (2011),
Theorem 1, establishes that $R^F > R^S$ if
\[
\frac{f_2(v)}{\tilde{F}_2(v)} \geq \frac{f_1(v)}{\tilde{F}_1(v)} \quad \text{for any} \quad v \in [\beta_1, \alpha_1] \cap [\beta_2, \alpha_2] \tag{12}
\]
\[
f_1(v) \geq f_2(x) \quad \text{for any} \quad x \in [v, r(v)] \quad \text{and any} \quad v \in [\beta_1, \alpha_1] \tag{13}
\]

This theorem results from a clever application of the mechanism design techniques introduced by Myerson (1981), and precisely relies on the following argument – expressed only for the case of $\beta_1 = \beta_2$ for simplicity. In the SPA bidder 1 wins if and only if $v_2 < v_1$, but (12) implies that 1 wins more frequently in the FPA. Precisely, 1 wins as long as $v_2 < k^F(v_1)$ for a certain function $k^F$ such that $k^F(v_1) > v_1$, since bidder 1 (the weak bidder) bids more aggressively than 2 (the strong bidder) for a given valuation. Moreover, (12) also implies that the ex ante equilibrium bid distribution of 2 first order stochastically dominates the ex ante bid distribution of 1, which is equivalent to $k^F(v) \leq r(v)$. Since $\beta_1 = \beta_2$, the expected revenue is given by the expected virtual valuation of the winning bidder, and the FPA dominates the SPA if its inefficient allocation increases the expected virtual valuation of the winner. Condition (13) guarantees that this is the case, which establishes the result.26

The assumptions in Kirkegaard (2011) obviously rule out our discrete setting, but given the c.d.f.
\[
\tilde{F}_1(v_1) = \begin{cases} 
0 & \text{if } v_1 < v_{1L} \\
\lambda & \text{if } v_{1L} \leq v_1 < v_{1H} \\
1 & \text{if } v_{1H} \leq v_1
\end{cases} \quad \text{and} \quad \tilde{F}_2(v_2) = \begin{cases} 
0 & \text{if } v_2 < v_{2L} \\
\lambda & \text{if } v_{2L} \leq v_2 < v_{2H} \\
1 & \text{if } v_{2H} \leq v_2
\end{cases}
\]

for $v_1, v_2$ in our model, we can approximate $\tilde{F}_1, \tilde{F}_2$ using atomless c.d.f.27 Precisely, consider two sequences of atomless c.d.f. $\{F^n_1, F^n_2\}_{n=1}^{+\infty}$, with continuous and positive densities $f^n_1, f^n_2$ for each $n$, which converges weakly to $\tilde{F}_1, \tilde{F}_2$. We prove in the appendix that for any large $n$, (12) and/or (13) are violated by $F^n_1, F^n_2$.

4.2 The case in which $v_{1H} = v_{2H}$

In this subsection we remove the assumption $\lambda_1 = \lambda_2$ but we suppose that $v_{1H} = v_{2H}$. Then a very simple result holds, as stated by next proposition.

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25Condition (12) is a standard condition of dominance in terms of reverse hazard rates. On the other hand, (13) is innovative and Kirkegaard (2011) proves that it implies that $r(v) - v$ is increasing, which means that $F_2$ is more disperse than $F_1$ according to a specific order of dispersion between c.d.f. Moreover, Kirkegaard (2011) gives an economic interpretation to (13) linked to the relative steepness of the demand function of bidder 1 with respect to the demand function of bidder 2.

26Kirkegaard (2011) shows that Theorem 1 holds also when there are several bidders with c.d.f. $F_1$, and sometimes when there are several bidders with c.d.f. $F_2$.

27Lebrun (2002) establishes that the equilibrium correspondence is upper hemicontinuous with respect to the valuation distributions, for the weak topology. Given that all BNE are outcome-equivalent at each given information structure, it follows that the equilibrium correspondence is in fact continuous. Therefore also $R^F$ is continuous, as it is the expectation of a continuous function of bids (the maximum).
Proposition 4 Suppose that $v_{1H} = v_{2H}$. Then the inequality $R^S > R^F$ holds as long as $v_{1L} < v_{2L}$ and/or $\lambda_1 \neq \lambda_2$.

In a sense, Proposition 4 is quite intuitive since we know that $R^S > R^F$ when $v_{1H} = v_{2H}$ if (i) $v_{1L} < v_{2L}$ and $\lambda_1 = \lambda_2$ [from Proposition 2(ii)], or (ii) $v_{1L} = v_{2L}$ and $\lambda_1 \neq \lambda_2$ [from Maskin and Riley (1983)]. Proposition 4 essentially verifies that $R^S > R^F$ still holds if both inequalities $v_{1L} < v_{2L}$ and $\lambda_1 \neq \lambda_2$ hold. Precisely, when $v_{1H} = v_{2H}$ condition (3) is violated and (7) reduces to $\lambda_1 \geq \lambda_2$; therefore Proposition 1(iii) applies if $\lambda_1 \geq \lambda_2$, and Proposition 1(ii) applies if $\lambda_1 < \lambda_2$. In both cases the equality $v_{1H} = v_{2H}$ yields a simple expression for $R^F$ and it follows immediately that $R^S > R^F$.

The simplest way to see why $R^S > R^F$ when $v_{1H} = v_{2H}$ consists in arguing as in Subsection 4.1.3, and proving that the bidders’ rents are larger in the FPA than in the SPA. In fact, in the proof to Proposition 4 we show that bidder 1 (bidder 2) strictly (weakly) prefers the FPA to the SPA since (i) $1H$ earns zero in the SPA when facing $2H$, earns $v_{1H} - v_{2L}$ against $2L$; (ii) $1H$ can beat $2L$ in the FPA by bidding $v_{1L}$ or $\hat{b}$ (depending on whether $\lambda_1 \geq \lambda_2$ or $\lambda_1 < \lambda_2$), and both $v_{1L}$ and $\hat{b}$ are smaller than $v_{2L}$. Likewise, the payoff of bidder 2 in the SPA is zero against $1H$, is $v_2 - v_{1L}$ against $1_L$. The FPA is certainly not worse for 2 as he can beat $1_L$ by bidding $v_{1L}$.

We conclude with a remark on the equilibrium bid distributions of the two bidders, $G_1(b) = \lambda_1 G_{1L}(b) + (1 - \lambda_1) G_{1H}(b)$ and $G_2(b) = \lambda_2 G_{2L}(b) + (1 - \lambda_2) G_{2H}(b)$, which are obtained from Proposition 1. For the case in which $\lambda_1 \geq \lambda_2$, we find that $G_1(b) = G_2(b) = \lambda_1 \frac{v_H - v_L}{v_H - b}$ for each $b \in [v_{1L}, \lambda_1 v_{1L} + (1 - \lambda_1) v_{1H}]$, and therefore each bidder faces the same distribution of bids from his opponent even though the distribution of $v_2$ first order stochastically dominates the distribution of $v_1$. This occurs because $v_{1H} = v_{2H}$ implies that the payoffs of $1H$ and $2H$ are the same, hence either type needs to have the same probability of winning for any given bid.

5 Appendix

5.1 Proof of Proposition 1 for the case of $v_{1L} < v_{2L}$

For $i = 1, 2$ and $j = L, H$, let $G_{ij}$ denote the c.d.f. for the mixed strategy of type $j$ of bidder $i$, with $b_{ij} = \inf\{b : G_{ij}(b) > 0\}$ and $\bar{b}_{ij} = \sup\{b : G_{ij}(b) < 1\}$. Recall that in a mixed-strategy BNE any bid made by type $ij$ must generate the same expected payoff, that is the equilibrium payoff of type $ij$, which we denote by $u^e_{ij}$. We use $u_{ij}(b)$ and $p_{ij}(b)$ to denote the payoff of type $ij$ and his

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28In fact, if $\lambda_1 \geq \lambda_2$ then the FPA allocates the object efficiently since $v_{1H} = v_{2H} > v_{2L}$ and $1_H$ always wins against $2_L$. This further simplifies the expression of $R^F$.

29Here the bidders have the same preferences between the FPA and the SPA, whereas under the assumptions on Maskin and Riley (2000a) that is never the case.

30When instead $\lambda_1 < \lambda_2$, we obtain $G_1(b) = \lambda_1 \frac{v_H - v_L}{v_H - b} < G_2(b) = \lambda_2 \frac{v_H - \bar{b}}{v_H - b}$ for $b \in [v_{1L}, \bar{b}]$ and $G_1(b) = G_2(b) = \frac{v_H - v_L}{v_H - b}$ for $b \in [\bar{b}, \tilde{b}]$; hence $G_1$ first order stochastically dominates $G_2$. Since now $2_L$ randomizes over the interval $[v_{1L}, \bar{b}]$ (rather than bidding $v_{1L}$ with certainty), $G_1$ needs to make $2_L$ indifferent among bids in $[v_{1L}, \bar{b}]$ and thus $G_1(b) = G_2(b)$ does not hold for $b \in [v_{1L}, \bar{b}]$. 

15
Lemma 1 If a profile of strategies has the property that there is a bid \( b' \) such that with a positive probability type 1 \( j \) and type 2 \( k \) tie bidding \( b' \) and \( \min \{ v_{1j}, v_{2k} \} > b' \), then the profile of strategies is not a BNE.

Proof. By bidding \( b' \), at least one of these types loses the auction with positive probability; for instance type 1 \( j \). Since \( b' < v_{1j} \), type 1 \( j \) is better off bidding \( b' + \varepsilon \) rather than \( b' \) as in this way his probability of winning increases discretely, whereas his payment in case of victory increases only slightly. 

5.1.1 Step 1: When \( v_{1L} < v_{2L} \), any BNE is such that (i) \( b_{1L} \leq b_{1H} \), \( b_{2L} \leq b_{2H} \); (ii) either \( b_{1L} = b_{2L} = v_{1L} = b_{1L} \) or \( b_{1L} < b_{2L} \); (iii) \( u_{2L}^c = 0 \), \( u_{2L}^e > 0 \), \( v_{1L} \leq b_{2L} \); (iv) \( b_{1H} = \bar{b}_{2H} \)

(i) The monotonicity properties \( b_{1L} \leq b_{1H} \) and \( b_{2L} \leq b_{2H} \) follow from Proposition 1 in Maskin and Riley (2000b).

(ii) In order to prove that \( b_{1L} \leq b_{2L} \), suppose in view of a contradiction that \( b_{2L} < b_{1L} \). Since \( 2L \) bids in the interval \( [b_{2L}, b_{1L}] \) with positive probability, it follows that \( u_{2L}^c = 0 \). However, since \( b_{1L} < v_{1L} < v_{2L} \) we find that \( p_{2L}(b) > 0 \) and \( u_{2L}(b) > 0 \) if \( 2L \) bids \( b = b_{1L} + \varepsilon \): contradiction.

We now show that if \( b_{1L} = b_{2L} = \bar{b} \), then \( \bar{b} = v_{1L} \), and as a consequence we obtain \( b_{1L} = v_{1L} \).

Suppose \( \bar{b} < v_{1L} \). We distinguish several cases depending on whether \( G_{1L} \) and/or \( G_{2L} \) puts an atom on \( \bar{b} \): in each case we obtain a contradiction.

- \( G_{1L}(\bar{b}) = 0 \) \( [G_{2L}(\bar{b}) = 0 \) or \( G_{2L}(\bar{b}) > 0 \) does not matter]. In this case \( u_{2L}^c = 0 \) as \( p_{2L}(b) \) is about zero for \( b \) close to \( \bar{b} \) (as \( G_{1L} \) is right continuous). However, since \( \bar{b} < v_{1L} < v_{2L} \) we find that \( p_{2L}(b) > 0 \) and \( u_{2L}(b) > 0 \) if \( 2L \) bids \( b = \bar{b} + \varepsilon \).

- \( G_{1L}(\bar{b}) > 0 \) and \( G_{2L}(\bar{b}) > 0 \). This case is ruled out by Lemma 1.

- \( G_{1L}(\bar{b}) > 0 \) and \( G_{2L}(\bar{b}) = 0 \). In this case \( u_{1L}^c = 0 \) as \( p_{1L}(\bar{b}) = 0 \). However, since \( \bar{b} < v_{1L} \) we find that \( p_{1L}(b) > 0 \) and \( u_{1L}(b) > 0 \) if \( 1L \) bids \( b = \bar{b} + \varepsilon < v_{1L} \).

(iii) We notice that \( u_{1L}^c = 0 \) both if \( b_{1L} = b_{2L} = \bar{b}_{1L} = v_{1L} \) and if \( b_{1L} < b_{2L} \). Hence \( v_{1L} \leq b_{2L} \), since if \( b_{2L} < v_{1L} \) then any bid in \( (b_{2L}, v_{1L}) \) yields a positive payoff to \( 1L \). Finally, \( p_{2L}(b) \geq \lambda_1 \) for any \( b \geq v_{1L} + \varepsilon \), thus \( u_{2L}^c \geq \lambda_1(v_{2L} - v_{1L} - \varepsilon) > 0 \) for each small \( \varepsilon > 0 \).

(iv) If \( \bar{b}_{1H} > \bar{b}_{2H} \), then it is profitable for \( 1H \) to move some probability from \( (\bar{b}_{1H} - \varepsilon, \bar{b}_{1H}) \) to \( (\bar{b}_{2H}, \bar{b}_{2H} + \varepsilon) \), since the probability of winning remains 1 but his payment in case of victory is smaller. If \( \bar{b}_{1H} < \bar{b}_{2H} \), a symmetric argument applies to \( 2H \).
5.1.2 Step 2: When $v_{1L} < v_{2L}$, there exists a BNE such that $b_{1H} \leq b_{2L}$ if and only if (3) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(i)

We start by proving that $b_{1L} < b_{2L}$. Suppose in view of a contradiction that $b_{1L} = b_{2L}$. Then Step 1(i-ii) imply $b_{1L} = \bar{b}_{1L} = b_{1H} = b_{2L} = v_{1L}$. It is impossible that $G_{2L}(v_{1L}) > 0$, because in such a case $1_H$ and $2_L$ would tie with positive probability at $b = v_{1L}$, and then Lemma 1 would apply. As a consequence, $p_{1H}(v_{1L}) = 0$ and $u^*_H = 0$. However, if $1_H$ plays $b = v_{1L} + \varepsilon$ then $p_{1H}(b) > 0$ and $u_{1H}(b) > 0$ since $v_{1L} < v_{1H}$: contradiction.

From the inequality $\bar{b}_{1H} \leq b_{2L}$ it follows that $2_L$ wins with probability one; thus $u^*_H = 0$. Moreover, (i) $\bar{b}_{1H} = \bar{b}_{2H}$ by Step 1(iv) and thus $\bar{b}_{1H} = b_{2L} = \bar{b}_{2L} = \bar{b}_{2H}$; (ii) $v_{1H} \leq b_{2L}$ otherwise any bid in $(b_{2L}, v_{1H})$ yields a positive payoff to $1_H$. Hence, $u^*_L = v_{2L} - b_{2L}$ and $u^*_H = v_{2H} - b_{2L}$.

We need to examine the incentives of bidder 2 to bid below $b_{2L}$, and in particular we notice that bidding $b = b_{1L} + \varepsilon$ yields bidder 2 a probability of winning not smaller than $\lambda_1$. Thus the inequalities

$$\lambda_1(v_{2L} - b_{1L} - \varepsilon) \leq v_{2L} - b_{2L} \quad \text{and} \quad \lambda_1(v_{2H} - b_{1L} - \varepsilon) \leq v_{2H} - b_{2L}$$

need to hold for any $\varepsilon > 0$, and since $v_{2H} > v_{2L}$ it is simple to see that the first inequality is more restrictive than the second one. Given $b_{1L} \leq v_{1L}$ and $b_{2L} \geq v_{1H}$, the first inequality is most likely to be satisfied when $\bar{b}_{1L} = v_{1L}$ and $b_{2L} = v_{1H}$, and then it reduces to (3). This inequality is therefore a necessary condition for the existence of a BNE such that $\bar{b}_{1H} \leq b_{2L}$.

Bids above $v_{1H}$ are obviously suboptimal for bidder 2 because $u_{2L}(b) = v_{2L} - b < v_{2L} - v_{1H}$ if $b > v_{1H}$. On the other hand, for bids smaller than $v_{1H}$ the strategies of $1_L$ and $1_H$ need to be such that no $b < v_{1H}$ is a profitable deviation for type $2_L$.

For instance, we verify that this condition is satisfied if $G_{1H}$ is the uniform distribution over $[\alpha v_{1H}, v_{1H}]$, with $\alpha < 1$ and close to 1; recall that $1_L$ bids $v_{1L}$ with certainty. Then $p_{2L}(b) = 0$, $u_{2L}(b) = 0$ for $b < v_{1L}$, whereas $p_{2L}(v_{1L}) = \lambda_1$ (recall the Vickrey tie-breaking rule and $v_{2L} > v_{1L}$), $u_{2L}(v_{1L}) = \lambda_1(v_{2L} - v_{1L})$, but we know from (3) that this payoff is smaller than $v_{2L} - v_{1H}$, the payoff of $2_L$ if he bids $v_{1H}$.

For $b \in (v_{1L}, \alpha v_{1H})$ we find that $u_{2L}(b) = \lambda_1(v_{2L} - b)$ is decreasing. Finally, for $b \in [\alpha v_{1H}, v_{1H}]$, $u_{2L}(b) = (v_{2L} - b)[\lambda_1 + (1 - \lambda_1) \frac{b - \alpha v_{1H}}{v_{1H} - \alpha v_{1H}}]$ and is increasing for $\alpha > 1 - \frac{(1 - \lambda_1)(v_{2L} - v_{1H})}{v_{1H}}$, which implies that $b = v_{1H}$ is a best reply for $2_L$.

5.1.3 Step 3: When $v_{1L} < v_{2L}$, there exists no BNE such that $b_{2L} < \bar{b}_{1H} \leq b_{2L}$.

If $b_{2L} < b_{1H} \leq b_{2L}$, then $b_{2L} < \bar{b}_{1H} = b_{2L} = \bar{b}_{2H} = b^*$ by Step 1(iv). This implies $b^* \leq v_{1H}$, and thus $b_{2L} < b^*$ implies $u^*_H > 0$, and in turn $b^* < v_{1H}$. Since $2_H$ bids $b^*$ with certainty, it is

\[ \text{In particular, if } b_{1H} = b_{2L} \text{ and } 1_H \text{ and } 2_L \text{ tie with positive probability at } b_{2L}, \text{ then } 2_L \text{ needs to win the tie-break with probability } 1, \text{ otherwise it is profitable for him to bid } b_{2L} + \varepsilon \text{ rather than } b_{2L} (b_{2L} < v_{2L} \text{ since } u^*_L > 0). \]

\[ \text{If this property is satisfied, then no deviation is profitable for } 2_H \text{ since } (v_{2L} - b)p_{2L}(b) \leq v_{2L} - v_{1H} \text{ implies } (v_{2H} - b)p_{2H}(b) \leq v_{2H} - v_{1H}, \text{ as } p_{2L}(b) = p_{2H}(b) \text{ for any } b. \]
profitable for $1_H$ to bid $b^* + \varepsilon$ rather than $b^* - \varepsilon$, as in this way his probability of victory increases by at least $1 - \lambda_2 > 0$ and his payment in case of victory increases only slightly.

5.1.4 Step 4: When $v_{1L} < v_{2L}$, there exists a BNE such that $b_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$ if and only if (4) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(ii)

The inequality $b_{2L} < \bar{b}_{1H}$ implies $u_{1H}^c > 0$ because $\bar{b}_{1H} \leq v_{1L}$ and $p_{1H}(b) > 0$ for $b \in (b_{2L}, \bar{b}_{1H})$.

Next lemma provides a list of features of any BNE such that $b_{2L} < \bar{b}_{1H}$.

**Lemma 2** In any BNE such that $b_{2L} < \bar{b}_{1H}$ the following equalities hold: $\bar{b}_{1L} = \bar{b}_{1H} = b_{2L} = v_{1L}$, $b_{2L} = b_{2H}$; moreover, $G_{2L}(b_{2L}) > 0$.

**Proof.** The proof is split in two claims.

**Claim 1** $\bar{b}_{1L} = \bar{b}_{1H}$. In view of a contradiction, assume that $\bar{b}_{1L} < \bar{b}_{1H}$. If $G_{1H}(\bar{b}_{1H}) > 0$ and $G_{2L}(\bar{b}_{1H}) > 0$, then Lemma 1 applies since $u_{1H}^c > 0$ and $u_{2L}^c > 0$ imply $v_{1H} > \bar{b}_{1H}$ and $v_{2L} > \bar{b}_{1H}$. If $G_{1H}(\bar{b}_{1H}) > 0$ and $2$ puts no atom at $\bar{b}_{1H}$, then $2$ bids with zero probability in $(\bar{b}_{1L} + \varepsilon, \bar{b}_{1H}]$ and $1_H$ can increase his payoff by moving the atom from $\bar{b}_{1H}$ to any point in $(\bar{b}_{1L} + \varepsilon, \bar{b}_{1H}]$. If $G_{1H}(\bar{b}_{1H}) = 0$, then $2$ bids with zero probability in $(\bar{b}_{1L} + \varepsilon, \bar{b}_{1H}]$ (in particular, $2$ puts no atom in $\bar{b}_{1H}$) and then $1_H$ can increase his payoff by moving some probability from $[\bar{b}_{1H}, \bar{b}_{1H} + \varepsilon)$ to $(\bar{b}_{1L} + \varepsilon, \bar{b}_{1L} + 2\varepsilon)$.

**Claim 2** $\bar{b}_{1H} = b_{2L} = v_{1L}$, $G_{2L}(v_{1L}) > 0$ and $b_{2L} = b_{2H}$. If $\bar{b}_{1H} < b_{2L}$, then $1_H$ bids in $[\bar{b}_{1H}, b_{2L})$ with positive probability and thus $u_{1H}^c = 0$: contradiction. Thus $b_{2L} \leq \bar{b}_{1H}$ and since $\bar{b}_{1H} \leq v_{1L}$, $v_{1L} \leq b_{2L}$ [by Step 1(iii)] and $\bar{b}_{1L} = \bar{b}_{1H}$ (by Claim 1), we infer that $\bar{b}_{1L} = b_{2L} = b_{2H} = v_{1L}$. Moreover, given $\bar{b}_{1H} = b_{2L}$ if $G_{2L}(b_{2L}) = 0$ then $u_{1H}^c = 0$; thus $G_{2L}(b_{2L}) > 0$. The equality $b_{2L} = b_{2H}$ is proved along the same lines followed in Claim 1 to prove $\bar{b}_{1L} = \bar{b}_{1H}$. ■

**Lemma 3** In any BNE such that $b_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$, the mixed strategies of $1_H, 2_L, 2_H$ are given by (5)-(6), and they constitute a BNE if and only if (4) is satisfied.

**Proof.** In the following of this proof we use $\hat{b}$ and $\bar{b}$, respectively, instead of $\bar{b}_{2L}$ and of $b_{2H} = \bar{b}_{1H}$.

Given that $v_{1L} < \hat{b}$, types $1_H, 2_L, 2_H$ are all employing mixed strategies and we can argue like in the proof of Claim 1 in Lemma 2 to show that $G_{1H}, G_{2L}, G_{2H}$ are strictly increasing and continuous in the intervals $[v_{1L}, \hat{b}], [v_{1L}, \hat{b}], [\hat{b}, \bar{b}]$, respectively. This implies that the following conditions must be satisfied.

Indifference condition of type $1_H$:

\[
(v_{1H} - b)[\lambda_2 G_{2L}(b) + (1 - \lambda_2) G_{2H}(b)] = v_{1H} - \bar{b} \quad \text{for any} \quad b \in (v_{1L}, \bar{b}) \quad (14)
\]

\footnote{If we consider type $2_H$ instead of $2_L$, the same the argument applies.}
Indifference condition of type $2_L$:

\[
(v_{2L} - b)\left[\lambda_1 + (1 - \lambda_1)G_{1H}(b)\right] = \lambda_1(v_{2L} - v_{1L}) \quad \text{for any } b \in [v_{1L}, \hat{b}] 
\]  

(15)

Indifference condition of type $2_H$:

\[
(v_{2H} - b)\left[\lambda_1 + (1 - \lambda_1)G_{1H}(b)\right] = v_{2H} - \bar{b} \quad \text{for any } b \in [\bar{b}, \hat{b}] 
\]  

(16)

From (15) and (16) we obtain $G_{1H}$ in (5). For $b \in (v_{1L}, \hat{b})$, (14) reduces to $(v_{1H} - b)\lambda_2G_{2L}(b) = v_{1H} - \bar{b}$ and thus $G_{2L}$ satisfies (6). For $b \in [\bar{b}, \hat{b})$, (14) reduces to $(v_{1H} - b)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b}$ and then $G_{2H}$ satisfies (6).

Since $G_{2L}(\hat{b}) = 1$, we deduce that $\bar{b} = \lambda_2\hat{b} + (1 - \lambda_2)v_{1H}$, and since $G_{1H}$ needs to be continuous at $b = \hat{b}$ we infer that $\hat{b}$ solves (2); here we use $Z(b)$ to denote the left hand side of (2). The strategies in Proposition 1(ii) require that $\hat{b}$ satisfies $v_{1L} < \hat{b} < \min\{v_{2L}, v_{1H}\}$, and since $Z(v_{2L}) = -\lambda_1(v_{2L} - v_{1L})(v_{2H} - v_{2L}) < 0$ we infer that $\hat{b}$ is the smaller solution of (2); moreover, $Z(v_{1L}) = (1 - \lambda_2)(v_{2L} - v_{1L})\left(\frac{(\lambda_1 - \lambda_2)v_{1L} + (1 - \lambda_1)v_{2L}}{1 - \lambda_2} - v_{1H}\right)$ and thus $(\lambda_1 - \lambda_2)v_{1L} + (1 - \lambda_1)v_{2L} > v_{1H}$ needs to hold. The inequality $\hat{b} < v_{1H}$ is obviously satisfied if $v_{2L} \leq v_{1H}$, while if $v_{1H} < v_{2L}$ then it is equivalent to $Z(v_{1H}) < 0$. Since $Z(v_{1H}) = -[v_{1H} - \lambda_1v_{1L} - (1 - \lambda_1)v_{2L}](v_{2H} - v_{1H})$ and $v_{1H} < v_{2L} < v_{2H}$, we deduce that the converse of (3) needs to hold. Thus (4) is a necessary condition for the existence of a BNE such that $b_{2L} < b_{2L} < \hat{b}_{1H}$.

Now we verify that for each type of each bidder the strategy specified by Proposition 1(ii) is a best reply given the strategies of the two types of the other bidder. Notice that $p_{1H}(\hat{b}) = p_{2H}(\hat{b}) = 1$, thus we do not need to consider bids above $\hat{b}$. The same remark applies to the BNE described by Proposition 1(iii).

**Type 1L.** The strategies of types $2_L$ and $2_H$ are such that each type of bidder 2 bids at least $v_{1L}$ with probability one. Therefore the payoff of $1_L$ is zero if he bids $v_{1L}$ as specified by Proposition 1, and it is impossible for him to obtain a positive payoff.

**Type 1H.** We know from (14) that the payoff of $1_H$ is $v_{1H} - \hat{b} > 0$ for any $b \in (v_{1L}, \hat{b}]$. If $b < v_{1L}$, then $p_{1H}(b) = 0$ and $u_{1H}(b) = 0$. If $b = v_{1L}$, then $1_H$ loses against $2_H$ and loses also against $2_L$, unless $2_L$ bids $v_{1L}$, in which case $1_H$ ties with $2_L$ – an event with probability $G_{2L}(v_{1L})$. Consider the most favorable case for $1_H$, which means that he wins the tie-break against $2_L$ with probability one (this occurs if $v_{2L} < v_{1H}$): his expected payoff from bidding $v_{1L}$ is then $(v_{1H} - v_{1L})\lambda_2G_{2L}(v_{1L})$, which turns out to be equal to $v_{1H} - \hat{b}$.

**Type 2L.** We know from (15) that the payoff of $2_L$ is $\lambda_1(v_{2L} - v_{1L}) > 0$ for any $b \in [v_{1L}, \hat{b}]$. For bids smaller than $v_{1L}$, the payoff of $2_L$ is zero as $p_{2L}(b) = 0$ if $b < v_{1L}$. If $b \in [\hat{b}, \hat{b}]$, then $u_{2L}(b) = (v_{2L} - b)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = (v_{2L} - b)\frac{v_{2H} - \bar{b}}{v_{2L} - \bar{b}}$ which is decreasing in $b$, and therefore $u_{2L}(\hat{b}) > u_{2L}(b)$ for any $b \in [\hat{b}, \hat{b}]$.

**Type 2H.** We know from (16) that the payoff of $2_H$ is $v_{2H} - \bar{b} > 0$ for any $b \in [\bar{b}, \hat{b}]$. For bids smaller than $v_{1L}$, the payoff of $2_H$ is zero as $p_{2H}(b) = 0$ if $b < v_{1L}$. If $b \in [v_{1L}, \hat{b}]$, then $p_{2H}(b) = \lambda_1 + (1 - \lambda_1)G_{1H}(b) = \lambda_1\frac{v_{2L} - v_{1L}}{v_{2L} - \bar{b}}$ and $u_{2H}(b) = (v_{2H} - b)\lambda_1\frac{v_{2L} - v_{1L}}{v_{2L} - \bar{b}}$, which is increasing in $b$ and therefore $u_{2H}(b) < u_{2H}(\hat{b})$ for any $b \in [v_{1L}, \hat{b}]$. 

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5.1.5 Step 5: When $v_{1L} < v_{2L}$, there exists a BNE such that $b_{2L} = b_{2L} < b_{1H}$ if and only if (7) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(iii)

In this case Lemma 2 (in the proof of Step 4) applies, thus we infer that $b_{1L} = b_{2L} = b_{2L} = b_{1H} = v_{1L}$; this means that $2_L$ plays a pure strategy and bids $v_{1L}$. Conversely, types $1_H$ and $2_H$ employ mixed strategies and thus the following indifference conditions need to hold, in which we still use $b$ instead of $b_{2L} = b_{1H}$. For type $1_H$:

$$\left(v_{1H} - b\right)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b} \quad \text{for any} \quad b \in (v_{1L}, \bar{b}) \quad (17)$$

For type $2_H$:

$$\left(v_{2H} - b\right)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = v_{2H} - \bar{b} \quad \text{for any} \quad b \in (v_{1L}, \bar{b}) \quad (18)$$

Notice that $G_{1H}(v_{1L}) = 0$ since if $G_{1H}(v_{1L}) > 0$, then $1_H$ ties with $2_L$ with positive probability by bidding $v_{1L}$, and thus Lemma 1 applies. From $G_{1H}(v_{1L}) = 0$ and (18) we obtain $\bar{b} = \lambda_1 v_{1L} + (1 - \lambda_1) v_{2H}$, and then (17)-(18) yield $G_{1H}, G_{2H}$ in (8). The inequality (7) needs to hold since it is equivalent to $G_{2H}(v_{1L}) \geq 0$.

Now we verify that for each type of each bidder the strategy specified by Proposition 1(iii) is a best reply given the strategies of the two types of the other bidder.

**Type 1.L.** The same argument given in the proof of Lemma 3 in Step 4 applies.

**Type 1_H.** We know from (17) that the payoff of $1_H$ is $v_{1H} - \bar{b} > 0$ for any $b \in (v_{1L}, \bar{b}]$, and $b < v_{1L}$ implies $p_{1H}(b) = 0$, $u_{1H}(b) = 0$. If $b = v_{1L}$, then $1_H$ ties with type $2_L$ and loses against $2_H$, unless also $2_H$ bids $v_{1L}$ – an event with probability $G_{2H}(v_{1L})$. Consider the most favorable case for $1_H$, which means that he wins the tie-break against each type of bidder 2 with probability one (this occurs if $v_{2H} < v_{1H}$): his expected payoff from bidding $v_{1L}$ is then $(v_{1H} - v_{1L})[\lambda_2 + (1 - \lambda_2)G_{2H}(v_{1L})]$ which turns out to be equal to $v_{1H} - \bar{b}$.

**Type 2_L.** The payoff of $2_L$ is $\lambda_1(v_{2L} - v_{1L})$. For bids smaller than $v_{1L}$, we can argue exactly like in the proof of Lemma 3 in Step 4. If $b \in [v_{1L}, \bar{b}]$, then $p_{2L}(b) = \lambda_1 v_{2H} - v_{1L}$ and thus $u_{2L}(b) = (v_{2L} - b)\lambda_1 v_{2H} - v_{1L}$ is decreasing in $b$.

**Type 2_H.** The payoff of $2_H$ is $v_{2H} - \bar{b} > 0$ for any $b \in [v_{1L}, \bar{b}]$. For bids smaller than $v_{1L}$ we can argue exactly like in the proof of Lemma 3 in Step 4.

5.2 Proof of Proposition 1 for the case of $v_{1L} = v_{2L}$

5.2.1 Step 1: When $v_{1L} = v_{2L} = v_L$, any BNE is such that $b_{1L} = b_{2L} = b_{1H} = \bar{b}_{1H} = v_{1L}$

We start by proving that $b_{1L} = b_{2L} = v_{1L}$. In view of a contradiction, suppose that $b_{2L} < b_{1L}$. Since $2_L$ bids in $[2_{1L}, b_{1L})$ with positive probability, it follows that $u_{2L}^v = 0$. Then $v_L \leq b_{1L}$, since $b_{1L} < v_L$ implies that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ for any $b \in [b_{1L}, v_L)$. Moreover, $v_L \leq b_{2L}$ implies $u_{1L}^v = 0$.

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34Notice that $v_{1H} - \bar{b} > 0$ given (7).
but \( p_{1L}(b) > 0 \) and \( u_{1L}(b) > 0 \) for any \( b \in (b_{2L}, b_{1L}) \): contradiction. Therefore the inequality \( b_{2L} < b_{1L} \) cannot hold in equilibrium, and a similar argument applies to rule out \( b_{1L} < b_{2L} \).

Given that \( b_{1L} = b_{2L} \equiv b \), we prove that \( b = v_L \). In view of a contradiction, suppose that \( b < v_L \). In case that \( G_{1L}(b) > 0 \) and \( G_{2L}(b) > 0 \), Lemma 1 applies; thus \( G_{1L}(b) = 0 \) and/or \( G_{2L}(b) = 0 \). If \( G_{1L}(b) = 0 \), we find that \( u_{2L}^e \) is about 0 for \( b \) close to \( b \), but in fact \( 2_L \) can make a positive payoff by bidding in \( (b, v_L) \): contradiction. The same argument applies if \( G_{2L}(b) = 0 \). Thus \( b = v_L \), which implies \( \bar{b}_{1L} = \bar{b}_{2L} = v_L \): hence both \( 1_L \) and \( 2_L \) bid \( v_L \) with probability one.

5.2.2 Step 2: When \( v_{1L} = v_{2L} = v_L \), in the unique BNE \( 1_H, 2_H \) play the mixed strategies described by Proposition 1(iii) if (7) holds; if (7) is violated, then \( 1_H, 2_H \) play the mixed strategies described by (5) and (6) with \( b = v_L \).

As in the proof of Proposition 1(ii) (Lemma 2 in Step 4) we can prove that \( \bar{b}_{1L} = \bar{b}_{1H}(= v_L) \) and \( \bar{b}_{2L} = \bar{b}_{2H}(= v_L) \). Using again \( b \) instead of \( \bar{b}_{1H}, \bar{b}_{2H} \) we infer that \( G_{1H}, G_{2H} \) need to satisfy

\[
(v_{1H} - b)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b} \quad \text{for any} \quad b \in [v_L, \bar{b}]
\] (19)

and

\[
(v_{2H} - b)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = v_{2H} - \bar{b} \quad \text{for any} \quad b \in [v_L, \bar{b}]
\] (20)

From (19)-(20) we obtain \( G_{1H}(v_L) = \frac{1}{1 - \lambda_1} \left( \frac{v_{2H} - b}{v_{2H} - v_L} - \lambda_1 \right) \) and \( G_{2H}(v_L) = \frac{1}{1 - \lambda_2} \left( \frac{v_{1H} - b}{v_{1H} - v_L} - \lambda_2 \right) \). Lemma 1 implies that \( G_{1H}(v_L) > 0 \) and \( G_{2H}(v_L) > 0 \) cannot hold. Thus we consider the other cases.

If \( G_{1H}(v_L) > 0 = G_{2H}(v_L) \) we obtain \( b = \lambda_2 v_L + (1 - \lambda_2)v_{1H} \) and \( G_{1H}(v_L) > 0 \) is equivalent to the converse of (7); from (19)-(20) we obtain \( G_{1H}, G_{2H} \) as in footnote 14.\(^{35}\) Now we prove that no profitable deviation exists for any type. The payoff of \( 1_L \) (2L) is zero and he needs to bid above \( v_L \) in order to win. For \( 1_H \), we know from (19) that his payoff is \( v_{1H} - \bar{b} \) for any \( b \in [v_L, \bar{b}] \) and \( b < v_L \) yields \( u_{1H}(b) = 0 \). A similar argument applies to \( 2_H \).

In case that \( G_{2H}(v_L) \geq 0 = G_{1H}(v_L) \) we obtain \( b = \lambda_1 v_L + (1 - \lambda_1)v_{2H} \), and \( G_{2H}(v_L) \geq 0 \) is equivalent to (7); from (19)-(20) we obtain \( G_{1H}, G_{2H} \) as in (8). The proof that no profitable deviation exists for any type is exactly as when (7) is violated.

\(^{35}\)Step 1 and the proof of Step 2 up to this point apply for any tie-breaking rule. However, no BNE exists under the standard tie-breaking rule if (7) is violated since (i) \( G_{1H}(v_L) > 0 \) and \( 1_H \) and \( 2_L \) tie with positive probability at the bid \( v_L \); (ii) it is profitable for \( 1_H \) to bid \( v_L + \varepsilon \) rather than \( v_L \), which breaks the BNE [a similar argument applies if (7) holds with strict inequality]. On the other hand, with the Vickrey tie-breaking rule we have \( c_{1H} = v_{1H} - v_L > 0 \) and \( c_{2L} = 0 \); thus \( 1_H \) wins (paying \( v_L \) as aggregate price) in case of tie with \( 2_L \).
5.3 Derivation of $R^F$ given the BNE described by Proposition 1

5.3.1 The BNE of Proposition 1(ii) when \( v_{1L} < v_{2L} \)

We evaluate $R^F$ as the difference between the social surplus $S^F$ generated by the FPA minus the bidders’ rents $U^F$: $R^F = S^F - U^F$. Thus we need to derive $S^F$ and $U^F$:

\[
S^F = \lambda_1 \lambda_2 v_{2L} + \lambda_1 (1 - \lambda_2) v_{2H} + (1 - \lambda_1) \lambda_2 [v_{2L} + (v_{1H} - v_{2L}) \Pr\{1_H \text{ def } 2_L\}] \\
+ (1 - \lambda_1) (1 - \lambda_2) [v_{2H} + (v_{1H} - v_{2H}) \Pr\{1_H \text{ def } 2_H\}]
\]

and

\[
U^F = (1 - \lambda_1) (v_{1H} - \lambda_2 \hat{b} - (1 - \lambda_2) v_{1L}) + (1 - \lambda_2) (v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2) v_{1L}) + \lambda_2 \lambda_1 (v_{2L} - v_{1L})
\]

in which $\Pr\{1_H \text{ def } 2_j\}$, for $j = L, H$, is the probability that $1_H$ wins when he faces type $2_j$. Therefore

\[
R^F = \lambda_2 (2 - \lambda_1 - \lambda_2) \hat{b} + (1 + \lambda_2^2 + \lambda_1 \lambda_2 - 3 \lambda_2) v_{1H} + \lambda_2 (1 - \lambda_1) v_{2L} + \lambda_2 \lambda_1 v_{1L} \\
+ (1 - \lambda_1) \lambda_2 (v_{1H} - v_{2L}) \Pr\{1_H \text{ def } 2_L\} + (1 - \lambda_1) (1 - \lambda_2) (v_{1H} - v_{2H}) \Pr\{1_H \text{ def } 2_H\}
\]

**Derivation of $\Pr\{1_H \text{ def } 2_L\}$** For the case that $v_{1H} \neq v_{2L}$ we need to evaluate

\[
\Pr\{1_H \text{ def } 2_L\} = \int_{v_{1L}}^{b} G'_{1H}(b)G_{2L}(b)db + 1 - G_{1H}(\hat{b})
\]

and using $\hat{b} = \lambda_2 \hat{b} + (1 - \lambda_2) v_{1L}$ in $G_{2L}$ we find $G_{2L}(b) = \frac{v_{2H} - b}{v_{2H} - \hat{b}}$:

\[
\Pr\{1_H \text{ def } 2_L\} = \int_{v_{1L}}^{b} \frac{1}{1 - \lambda_1} \frac{v_{2L} - v_{1L} v_{1H} - \hat{b}}{(v_{2L} - \hat{b})^2 v_{1H} - \hat{b}} db + 1 - \frac{\lambda_1 (v_{1H} - v_{1L})}{(1 - \lambda_1)(v_{2L} - \hat{b})} \\
= \frac{\lambda_1 (v_{2L} - v_{1L})(v_{1H} - \hat{b})}{1 - \lambda_1} \int_{v_{1L}}^{b} \frac{1}{(v_{2L} - \hat{b})^2(v_{1H} - \hat{b})} db + 1 - \frac{\lambda_1 (v_{1H} - v_{1L})}{(1 - \lambda_1)(v_{2L} - \hat{b})}
\]

We exploit

\[
\int_{v_{1L}}^{b} \frac{1}{(v_{2L} - \hat{b})^2(v_{1H} - \hat{b})} db = \frac{1}{(v_{1H} - v_{2L})^2} \ln \frac{v_{2L} - b}{v_{1H} - b} + \frac{1}{(v_{1H} - v_{2L})(v_{2L} - b)}
\]

to obtain

\[
\int_{v_{1L}}^{b} \frac{1}{(v_{2L} - \hat{b})^2(v_{1H} - \hat{b})} db = \frac{1}{(v_{1H} - v_{2L})^2} \ln \frac{(v_{2L} - \hat{b})(v_{1H} - v_{1L})}{(v_{1H} - \hat{b})(v_{2L} - v_{1L})} + \frac{\hat{b} - v_{1L}}{(v_{1H} - v_{2L})(v_{2L} - b)(v_{2L} - v_{1L})}
\]

thus

\[
\Pr\{1_H \text{ def } 2_L\} = \frac{\lambda_1 (v_{1H} - \hat{b})(v_{2L} - v_{1L})}{(1 - \lambda_1)(v_{1H} - v_{2L})^2} \ln \frac{(v_{2L} - \hat{b})(v_{1H} - v_{1L})}{(v_{1H} - \hat{b})(v_{2L} - v_{1L})} + \frac{(1 - \lambda_1) (v_{1H} - v_{2L} + \lambda_1 \hat{b} - v_{1L})}{(1 - \lambda_1)(v_{1H} - v_{2L})}
\]

and

\[
(1 - \lambda_1) \lambda_2 (v_{1H} - v_{2L}) \Pr\{1_H \text{ def } 2_L\} = \frac{\lambda_1 \lambda_2 (v_{1H} - \hat{b})(v_{2L} - v_{1L})}{v_{1H} - v_{2L}} \ln \frac{(v_{2L} - \hat{b})(v_{1H} - v_{1L})}{(v_{1H} - \hat{b})(v_{2L} - v_{1L})} + \lambda_2 (1 - \lambda_1) (v_{1H} - v_{2L}) + \lambda_1 \lambda_2 (\hat{b} - v_{1L})
\]
**Derivation of** \( \text{Pr}\{1_H \ \text{def} \ 2_H\} \)  

For the case that \( v_{1H} \neq v_{2H} \) we need to evaluate

\[
\text{Pr}\{1_H \ \text{def} \ 2_H\} = \int_b^{\hat{b}} G_{1H}(b) G_{2H}(b) \, db
\]

and using \( \hat{b} = \lambda_2 \hat{b} + (1 - \lambda_2) v_{1H} \) in \( G_{2H} \) we find \( G_{2H}(b) = \frac{\lambda_2(b - \hat{b})}{(1 - \lambda_2)(v_{1H} - b)} \):

\[
\text{Pr}\{1_H \ \text{def} \ 2_H\} = \int_b^{\hat{b}} \frac{v_{2H} - \hat{b}}{(1 - \lambda_1)(v_{2H} - b)^2} \frac{\lambda_2(b - \hat{b})}{(1 - \lambda_2)(v_{1H} - b)} \, db
\]

\[
= \frac{\lambda_2(v_{2H} - \hat{b} - (1 - \lambda_2)v_{1H})}{(1 - \lambda_1)(1 - \lambda_2)} \int_b^{\hat{b}} \frac{b - \hat{b}}{(v_{1H} - b)(v_{2H} - b)^2} \, db
\]

We exploit

\[
\int \frac{b - \hat{b}}{(v_{1H} - b)(v_{2H} - b)^2} \, db = \frac{v_{1H} - \hat{b}}{(v_{1H} - v_{2H})^2} \ln \frac{v_{2H} - b}{v_{1H} - b} - \frac{v_{2H} - \hat{b}}{v_{1H} - v_{2H}}
\]

to obtain

\[
\int_b^{\hat{b}} \frac{b - \hat{b}}{(v_{1H} - b)(v_{2H} - b)^2} \, db = \frac{v_{1H} - \hat{b}}{(v_{1H} - v_{2H})^2} \ln \frac{v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H}}{\lambda_2(v_{2H} - b)} - \frac{v_{2H} - \hat{b}}{v_{1H} - v_{2H}} \frac{1 - \lambda_2}{v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H}}
\]

thus

\[
\text{Pr}\{1_H \ \text{def} \ 2_H\} = \frac{\lambda_2(v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H})}{(1 - \lambda_1)(1 - \lambda_2)} \frac{v_{1H} - \hat{b}}{(v_{1H} - v_{2H})^2} \ln \frac{v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H}}{\lambda_2(v_{2H} - b)} - \frac{\lambda_2(v_{2H} - \hat{b})}{(1 - \lambda_1)(v_{2H} - v_{1H})}
\]

and

\[
= \frac{(1 - \lambda_1)(1 - \lambda_2)(v_{1H} - v_{2H}) \text{Pr}\{1_H \ \text{def} \ 2_H\}}{v_{1H} - v_{2H}} \ln \frac{v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H}}{\lambda_2(v_{2H} - b)} + (1 - \lambda_2)\lambda_2(v_{1H} - \hat{b})
\]

**Evaluation of** \( R^F \)

\[
R^F = \lambda_2 \hat{b} + (1 - \lambda_2)v_{1H} + \frac{\lambda_1\lambda_2(v_{1H} - \hat{b})(v_{2L} - v_{1L})}{v_{1H} - v_{2L}} \ln \frac{(v_{2L} - \hat{b})(v_{1H} - v_{1L})}{(v_{1H} - b)(v_{2L} - v_{1L})}
\]

\[
+ \frac{\lambda_2(v_{1H} - \hat{b})(v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H})}{v_{1H} - v_{2H}} \ln \frac{v_{2H} - \lambda_2 \hat{b} - (1 - \lambda_2)v_{1H}}{\lambda_2(v_{2H} - b)}
\]

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5.3.2 The BNE of Proposition 1(ii) when $v_{1L} = v_{2L}$ (footnote 14)

\[ S^F = \lambda_1 \lambda_2 v_{1L} + \lambda_1 (1 - \lambda_2) v_{2L} + \lambda_2 (1 - \lambda_1) v_{1H} + (1 - \lambda_1)(1 - \lambda_2)(v_{1H} + (v_{2H} - v_{1H}) \Pr\{2_H \text{ def } 1_H\}) \]

\[ U^F = (1 - \lambda_1)(v_{1H} - \lambda_2 v_{2L} - (1 - \lambda_2)v_{1H}) + (1 - \lambda_2)(v_{2H} - \lambda_2 v_{2L} - (1 - \lambda_2)v_{1H}) \]

Therefore

\[ R^F = \lambda_2 (2 - \lambda_2) v_{1L} - (1 - \lambda_1) (1 - \lambda_2) v_{2L} + (2 - \lambda_1 - \lambda_2) (1 - \lambda_2) v_{1H} \]

\[ + (1 - \lambda_1)(1 - \lambda_2)(v_{2H} - v_{1H}) \Pr\{2_H \text{ def } 1_H\} \]

**Derivation of $\Pr\{2_H \text{ def } 1_H\}$** For the case that $v_{1H} \neq v_{2H}$ we need to evaluate

\[
\Pr\{2_H \text{ def } 1_H\} = \int_{v_{1L}}^{\lambda_2 v_{1L} + (1 - \lambda_2)v_{1H}} G_2^H(b) G_1^H(b) db \\
= \int_{v_{1L}}^{\lambda_2 v_{1L} + (1 - \lambda_2)v_{1H}} \frac{\lambda_2}{1 - \lambda_2} \frac{v_{1H} - v_{1L}}{(v_{1H} - b)^2} \frac{1}{1 - \lambda_1} \left( \frac{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}{v_{2H} - b} - \frac{\lambda_1}{(v_{1H} - b)^2} \right) db \\
= \frac{\lambda_2 (v_{1H} - v_{1L})}{(1 - \lambda_2)(1 - \lambda_1)} \int_{v_{1L}}^{\lambda_2 v_{1L} + (1 - \lambda_2)v_{1H}} \left( \frac{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}{(v_{2H} - b)(v_{1H} - b)^2} - \frac{\lambda_1}{(v_{1H} - b)^2} \right) db
\]

We exploit

\[
\int \frac{1}{(v_{2H} - b)(v_{1H} - b)^2} db = \frac{1}{(v_{2H} - v_{1H})^2} \ln \frac{v_{1H} - b}{v_{2H} - b} + \frac{1}{(v_{2H} - v_{1H})(v_{1H} - b)}
\]

to obtain

\[
\frac{\int_{v_{1L}}^{\lambda_2 v_{1L} + (1 - \lambda_2)v_{1H}} \frac{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}{(v_{2H} - b)(v_{1H} - b)^2} db}{\frac{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}{(v_{2H} - v_{1H})^2} \ln \frac{\lambda_2 (v_{2H} - v_{1L})}{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}} + \frac{(1 - \lambda_2)(v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H})}{\lambda_2 (v_{2H} - v_{1H})(v_{1H} - v_{1L})}}
\]

Moreover,

\[
\int_{v_{1L}}^{\lambda_2 v_{1L} + (1 - \lambda_2)v_{1H}} \frac{\lambda_1}{(v_{1H} - b)^2} db = \frac{\lambda_1 (1 - \lambda_2)}{\lambda_2 (v_{1H} - v_{1L})}
\]
thus
\[
\Pr\{2_H \text { def } 1_H\} = \frac{\lambda_2(v_{1H} - v_{1L})(v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H})}{(1 - \lambda_2)(1 - \lambda_1)(v_{2H} - v_{1H})^2} \ln \frac{\lambda_2 (v_{2H} - v_{1L})}{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}
\]
\[+ \frac{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}{(1 - \lambda_1)(v_{2H} - v_{1H})} \frac{\lambda_1}{1 - \lambda_1}
\]

and
\[
(1 - \lambda_1)(1 - \lambda_2)(v_{2H} - v_{1H}) \Pr\{2_H \text { def } 1_H\} = \frac{(v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H})\lambda_2(v_{1H} - v_{1L})}{v_{2H} - v_{1H}} \ln \frac{\lambda_2 (v_{2H} - v_{1L})}{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}
\]
\[+ (1 - \lambda_2)((\lambda_2 + \lambda_1 - 1)v_{1H} + (1 - \lambda_1)v_{2H} - \lambda_2 v_{1L})
\]

Evaluation of $R^F$
\[
R^F = \lambda_2 v_{1L} + (1 - \lambda_2)v_{1H} + \frac{(v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H})\lambda_2(v_{1H} - v_{1L})}{v_{2H} - v_{1H}} \ln \frac{\lambda_2 (v_{2H} - v_{1L})}{v_{2H} - \lambda_2 v_{1L} - (1 - \lambda_2)v_{1H}}
\]

5.3.3 The BNE in Proposition 1(iii)

$S^F = \lambda_1 \lambda_2 v_{2L} + \lambda_1 (1 - \lambda_2)v_{2H} + \lambda_2 (1 - \lambda_1)v_{1H} + (1 - \lambda_1)(1 - \lambda_2)(v_{2H} + (v_{1H} - v_{2H}) \Pr\{1_H \text { def } 2_H\})$

$U^F = (1 - \lambda_1)(v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H}) + (1 - \lambda_2)(v_{2H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H}) + \lambda_2 \lambda_1 (v_{2L} - v_{1L})$

Therefore
\[
R^F = \lambda_1(2 - \lambda_1)v_{1L} - (1 - \lambda_1)(1 - \lambda_2)v_{1H} + (1 - \lambda_1)(2 - \lambda_1 - \lambda_2)v_{2H}
\]
\[+ (1 - \lambda_1)(1 - \lambda_2)(v_{1H} - v_{2H}) \Pr\{1_H \text { def } 2_H\}
\]

Derivation of $\Pr\{1_H \text { def } 2_H\}$ For the case that $v_{1H} \neq v_{2H}$ we need to evaluate
\[
\Pr\{1_H \text { def } 2_H\} = \int_{v_{1L}}^{\lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}} G'_{1H}(b)G_{2H}(b)db
\]
\[
= \int_{v_{1L}}^{\lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}} \frac{\lambda_1}{1 - \lambda_1} \frac{v_{2H} - v_{1L}}{(v_{2H} - v_{1H})^2} \frac{1}{1 - \lambda_2} \frac{v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H}}{v_{1H} - b} \lambda_2 db
\]
\[
= \frac{\lambda_1(v_{2H} - v_{1L})}{(1 - \lambda_1)(1 - \lambda_2)} \int_{v_{1L}}^{\lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}} \frac{(v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H})(v_{1H} - b)(v_{2H} - b)^2}{(v_{2H} - b)^2} db - \frac{\lambda_2}{(v_{2H} - b)^2} db db
\]

We exploit
\[
\int \frac{1}{(v_{1H} - b)(v_{2H} - b)^2} db = \frac{1}{(v_{1H} - v_{2H})^2} \ln \frac{v_{2H} - b}{v_{1H} - b} + \frac{1}{(v_{1H} - v_{2H})(v_{2H} - b)}
\]

to obtain
\[
\int_{v_{1L}}^{\lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}} v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H} db
\]
\[
= \frac{(v_{1H} - v_{2H})\lambda_1(v_{1H} - v_{2H})}{(v_{1H} - v_{2H})^2} \ln \frac{v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H}}{v_{1H} - v_{1L} - (1 - \lambda_1)v_{2H}}
\]
\[+ (1 - \lambda_1)(v_{1H} - \lambda_1 v_{1L} - (1 - \lambda_1)v_{2H}) \frac{\lambda_1(v_{1H} - v_{2H})}{(v_{2H} - v_{1L})}
\]

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Moreover,

\[
\int_{v_1L}^{\lambda_1v_1L+(1-\lambda_1)v_2H} \frac{\lambda_2}{(v_2H-b)^2} db = \frac{\lambda_2(1-\lambda_1)}{\lambda_1(v_2H-v_1L)}
\]

thus

\[
\Pr\{1_H \text{ def } 2_H\} = \frac{\lambda_1(v_2H-v_1L)(v_1H-\lambda_1v_1L-(1-\lambda_1)v_2H)}{(1-\lambda_1)(1-\lambda_2)(v_1H-v_2H)^2} \ln \frac{\lambda_1(v_1H-v_1L)}{v_1H-\lambda_1v_1L-(1-\lambda_1)v_2H} + \frac{v_1H-\lambda_1v_1L-(1-\lambda_1)v_2H}{(1-\lambda_2)(v_1H-v_2H)} - \frac{\lambda_2}{1-\lambda_2}
\]

and

\[
= \frac{(1-\lambda_1)(1-\lambda_2)(v_1H-v_2H) \Pr\{1_H \text{ def } 2_H\}}{v_1H-v_2H} + \frac{(1-\lambda_1)(1-\lambda_2)v_1H-\lambda_1v_1L+(\lambda_1+\lambda_2-1)v_2H}{v_1H-\lambda_1v_1L-(1-\lambda_1)v_2H}
\]

**Evaluation of \( R^F \)**

\[
R^F = \lambda_1v_1L+(1-\lambda_1)v_2H + \frac{(v_1H-\lambda_1v_1L-(1-\lambda_1)v_2H)\lambda_1(v_2H-v_1L)}{v_1H-v_2H} \ln \frac{\lambda_1(v_1H-v_1L)}{v_1H-\lambda_1v_1L-(1-\lambda_1)v_2H}
\]

### 5.4 Proof of Proposition 2

(i) The proof is given in the text, immediately after the statement.

(ii) We consider the three conditions (9)-(11) separately.

#### 5.4.1 The proof when (9) is satisfied

Suppose that \( v_{1H} > v_{2H} \). Then Proposition 3 establishes that \( R^F \) is smaller than when \( v_{1H} \) satisfies \( v_{1H} = v_{2H} \); furthermore, from (1) it follows that an increase in \( v_{1H} \) above the level such that \( v_{1H} = v_{2H} \) has no effect on \( R^S \). Therefore \( R^S > R^F \) when \( v_{1H} > v_{2H} \).

Now suppose that \( v_{1H} < v_{2H} \). We know that \( R^S = R^F \) in the symmetric setting such that both high valuations are equal to \( v_{1H} \), and an increase in \( v_{2H} \) implies \( R^S > R^F \) by the argument in the previous paragraph (after reversing the bidders’ identities).

#### 5.4.2 The proof when (10) is satisfied

This proof is provided in the text.

#### 5.4.3 The proof when (11) is satisfied

When \( v_{1L} < v_{2L} \leq v_{1H} < v_{2H} \), Proposition 1(ii) applies and thus the aggregate bidders’ rents in FPA are \( U^F = (1-\lambda)(v_{1H}-\bar{b}) + (1-\lambda)(v_{2H}-\bar{b}) + \lambda^2(v_{2L}-v_{1L}) \) with \( \bar{b} = \lambda\bar{b} + (1-\lambda)v_{1H} \).

Since \( U^S = \lambda^2(v_{2L}-v_{1L}) + \lambda(1-\lambda)(v_{2H}-v_{1L}) + (1-\lambda)\lambda(v_{1H}-v_{2L}) + (1-\lambda)^2(v_{2H}-v_{1H}) \),
the difference $U^F - U^S$ is equal to $\lambda (1 - \lambda)(v_{2L} + v_{1L} - 2\hat{b})$. Given $\lambda_1 = \lambda_2 = \lambda$, (2) reduces to $\lambda b^2 + ((1 - \lambda)v_{1H} - \lambda v_{1L} - v_{2H})b + (1 - \lambda)(v_{2H} - v_{1H})v_{2L} + \lambda v_{1L}v_{2H} = 0$ and thus
\[
\hat{b} = \frac{1}{2\lambda}(\lambda v_{1L} + v_{2H} - (1 - \lambda)v_{1H} - Q)
\]
with $Q = \sqrt{((1 - \lambda)v_{1H} - \lambda v_{1L} - v_{2H})^2 - 4\lambda(1 - \lambda)(v_{2H} - v_{1H})v_{2L} - 4\lambda^2 v_{1L}v_{2H}}$. Therefore $U^F \geq U^S$ boils down to $Q \geq v_{2H} - (1 - \lambda)v_{1H} - \lambda v_{2L}$ and (after squaring – notice that $v_{2H} - (1 - \lambda)v_{1H} - \lambda v_{2L} > 0$) ultimately to
\[
\lambda(v_{2L} - v_{1L})[2(1 - \lambda)v_{1H} + 2(2\lambda - 1)v_{2L} - \lambda v_{1L} - \lambda v_{2L}] \geq 0
\] (21)

Setting $v_{2L} = v_{1L} + \varepsilon_L$ and $v_{2H} = v_{1H} + \varepsilon_H$, it is simple to see that (21) is satisfied for $\varepsilon_L > 0$, $\varepsilon_H > 0$ and close to zero. Furthermore, given $v_{2H} > v_{1H}$ and $\lambda \geq \frac{1}{2}$, we find that $2(1 - \lambda)v_{1H} + 2(2\lambda - 1)v_{2L} - \lambda v_{1L} - \lambda v_{2L} \geq \lambda(2v_{1H} - v_{1L} - v_{2L})$, which holds for any $v_{2L} \leq v_{1H}$.

**Proof for the case of distribution shift** In the case of shift, (21) reduces to $2\lambda(v_{1H} - v_{1L}) \geq (2 - 3\lambda)a$ for $a \leq v_{1H} - v_{1L}$. If instead $a > v_{1H} - v_{1L}$, then $v_{2L} > v_{1H}$ and $U^S = \lambda^2 a + \lambda(1 - \lambda)(a + v_{1H} - v_{1L}) + \lambda(1 - \lambda)(v_{1L} + a - v_{1H}) + (1 - \lambda)^2 a = a$; thus $U^F \geq U^S$ reduces to $2(2 + \lambda)(v_{1H} - v_{1L}) \geq 3(2 - \lambda)a$.

### 5.5 Proof of Proposition 3

Given $\lambda_1 = \lambda_2$ and $v_{1L} = v_{2L} = v_L$, when $v_{1H} < v_{2H} = v_H$ Proposition 1(ii) (footnote 14) applies and reveals that types $1_L, 2_L$ bid as in the benchmark symmetric setting, whereas $G_{1H}(b) = \frac{1}{\lambda} \left( \frac{b - v_{1H} - (1 - \lambda)v_{1L}}{\varepsilon_H + \hat{b}} \right)$ and $G_{2H}(b) = \frac{\lambda}{4\lambda} \left( \frac{b - v_{1L}}{v_{1H} - b} \right)$ with support $[v_L, \hat{b}]$, in which $\hat{b} = \lambda v_{1L} + (1 - \lambda)v_{1H}$. It is simple to see that both $G_{1H}(b)$ and $G_{2H}(b)$ are decreasing with respect to $v_{1H}$ for any $b \in (v_L, \hat{b})$, and this implies that $1_H$ and $2_H$ are both more aggressive, in the sense of first order stochastic dominance, the larger is $v_{1H}$ in $(v_L, v_H)$.

Given that
\[
R^F = \lambda^2 v_L + \lambda(1 - \lambda) \int_{v_L}^{\hat{b}} b dG_{2H}(b) + \lambda(1 - \lambda) \int_{v_L}^{\hat{b}} b dG_{1H}(b) + (1 - \lambda)^2 \int_{v_L}^{\hat{b}} b d(G_{1H}(b)G_{2H}(b))
\] (22)
we infer that $R^F$ is increasing in $v_{1H}$.

When $v_{1H} > v_H$, Proposition 1(iii) applies and reveals that types $1_L, 1_H, 2_L$ bid as in the benchmark symmetric setting, whereas $G_{2H}(b) = \frac{(1 - \lambda)(v_{1H} - v_H) + \lambda(b - v_L)}{(1 - \lambda)(v_{1H} - b)}$ for any $b \in [v_L, E_v]$. Since $G_{2H}(b)$ is strictly increasing in $v_{1H}$ for any $b \in [v_L, E_v]$, we infer that $2_H$ is less aggressive, in the sense of first order stochastic dominance, the larger is $v_{1H}$. Using again (22), after replacing $G_{1H}$ with $G_H$ and $\hat{b}$ with $E_v$, it follows that $R^F$ is strictly decreasing with respect to $v_{1H}$.

\[\text{Precisely, if } v_{1H} < v_{1H} < v_H, \text{ then } F_{1H} \text{ and } F_{2H} \text{ given } v_{1H} \text{ first order stochastically dominate, respectively, } F_{1H} \text{ and } F_{2H} \text{ given } v_{1H}.\]
5.6 Proof of the claims in Subsection 4.1.4

When (3) is satisfied, $G_2(b) \leq G_1(b)$ obviously holds for any $b$. Moreover, bidder 1 never wins in either auction when (3) holds. Conversely, 2 wins with probability one and in the FPA he pays $v_1H$; in the SPA his expected payment is the expected valuation of bidder 1, which is smaller than $v_1H$.

For $i = 1, 2$, let $U_i^F$ denote bidder $i$’s ex ante expected equilibrium payoff in the FPA; $U_i^S$ has a similar meaning with reference to the SPA. When (4) is satisfied we find $U_2^F = (1 - \lambda)(v_1H - \hat{b})$, $U_1^S = (1 - \lambda)\lambda \max\{v_1H - v_2L, 0\}$, and $U_1^F > U_2^F$ since $\hat{b} < \min\{v_2L, v_1H\}$. Moreover, $U_2^F = \lambda^2(v_2L - v_1L) + (1 - \lambda)[v_2H - \hat{b} - (1 - \lambda)v_1H]$, $U_2^S = \lambda\lambda(v_2L - v_1L) + (1 - \lambda)\max\{v_2L - v_1H, 0\} + (1 - \lambda)[v_2H - \lambda v_1L - (1 - \lambda)v_1H]$, and $U_2^S - U_2^F = (1 - \lambda)\lambda\max\{v_2L - v_1H, 0\} + \hat{b} - v_1L > 0$ since $\hat{b} > v_1L$.

For equilibrium bid distributions we find that $G_1(b) > G_2(b)$ for any $b \in [v_1L, \hat{b}]$ as $G_1(v_1L) = G_2(\hat{b}) = \lambda$. For $b \in (\hat{b}, \bar{b}]$, $G_1(b) = \frac{v_2H - \bar{b}}{v_2H - \hat{b}}$ and $G_2(b) = \frac{v_1H - \hat{b}}{v_1H - \bar{b}}$, hence $G_1(b) > G_2(b)$ for each $b \in (\hat{b}, \bar{b}]$.

When (7) holds we obtain $U_1^F = (1 - \lambda)(v_1H - \lambda v_1L - (1 - \lambda)v_2H), U_1^S = (1 - \lambda)(v_1H - \lambda v_2L - (1 - \lambda)v_2H)$, and $U_1^F \geq U_1^S$ since $v_1L \leq v_2L$. Moreover, $U_2^F = U_2^S = \lambda^2(v_2L - v_1L) + (1 - \lambda)\lambda(v_2H - v_1L)$. For equilibrium bid distributions we find that $G_1(b) = \lambda\lambda v_2H - \lambda v_1L$ and $G_2(b) = \frac{v_1H - \lambda v_1L}{v_1H - \hat{b}}$ with $\bar{b} = \lambda v_1L + (1 - \lambda)v_2H$ and $G_2(b) > G_1(b)$ for any $b \in [v_1L, \bar{b}]$.

5.7 Proof of the final claim in Subsection 4.1.5

We consider two sequences of atomless c.d.f. $F_{1n}^n, F_2^n, n=1^\infty$, with continuous and positive densities $f_1^n, f_2^n$ for each $n$, which converges weakly to $F_1, F_2$. We show that for any large $n$, (12) and/or (13) are violated by $F_{1n}^n, F_2^n$.

When $v_1L < v_2L$, select an arbitrary $\tilde{v} \in (v_1L, v_2L)$ and notice that given a small $\varepsilon > 0$, for a large $n$ the inequality $F_1^n(\tilde{v}) > \lambda - \varepsilon$ holds. Therefore $r^n(\tilde{v}) = (F_2^n)^{-1}[F_1^n(\tilde{v})] \geq v_2L - \varepsilon > \tilde{v}$ [because $\lim_{n \to +\infty} F_2^n(v) = 0$ for each $v < v_2L - \varepsilon$] and $\int_0^{r^n(\tilde{v})} f_2^n(x)dx = F_2^n[r^n(\tilde{v})] - F_2^n(\tilde{v}) > \lambda - 2\varepsilon$ for a large $n$. If $f_1^n(x) \geq f_2^n(x)$ for any $x \in [\tilde{v}, r^n(\tilde{v})]$, then $\lim_{n \to +\infty} F_1^n(\tilde{v}) = 0$ implies $\lim_{n \to +\infty} \int_0^{r^n(\tilde{v})} f_2^n(x)dx = 0$: contradiction. Hence (13) is violated if $F_1^n, F_2^n$ are close to $F_1, F_2$ and $v_1L < v_2L$.

Now assume that $v_1L = v_2L$ and $v_1H < v_2H$. Then given a small $\varepsilon > 0$ and a large $n$, the inequality $F_1^n(v_1H + \varepsilon) - F_1^n(v_1H - \varepsilon) = \int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_1^n(x)dx > 1 - \lambda - \varepsilon$ holds, and $F_2^n(v_1H + \varepsilon) - F_2^n(v_1H - \varepsilon) = \int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_2^n(x)dx$ tends to zero. Now notice that if there exists a number $t > 0$ such that $\frac{\int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_1^n(x)dx}{\int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_2^n(x)dx} \leq t$ for any $x \in (v_1H - \varepsilon, v_1H + \varepsilon)$ and any $n$, then $\int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_1^n(x)dx \leq t \int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_2^n(x)dx$ and $\lim_{n \to +\infty} \int_{v_1H - \varepsilon}^{v_1H + \varepsilon} f_1^n(x)dx = 0$. Thus for any $t > 0$, for any large $n$ there exists some $x_n \in (v_1H - \varepsilon, v_1H + \varepsilon)$ such that $\frac{f_1^n(x_n)}{f_2^n(x_n)} > t$, which implies that (12) cannot hold since $F_2^n(x_n) > \lambda - \varepsilon$. 28
5.8 Proof of Proposition 4

Suppose that $\lambda_1 < \lambda_2$. Then Proposition 1(ii) applies and the ex ante expected payoffs of bidders 1 and 2 in the FPA and in the SPA are

$$U^F_1 = (1 - \lambda_1) \lambda_2 (v_H - \hat{b}) \quad \text{and} \quad U^S_1 = (1 - \lambda_1) \lambda_2 (v_H - v_{1L})$$

$$U^F_2 = \lambda_2 \lambda_1 (v_{2L} - v_{1L}) + (1 - \lambda_2) \lambda_2 (v_H - \hat{b}) \quad \text{and} \quad U^S_2 = \lambda_2 \lambda_1 (v_{2L} - v_{1L}) + (1 - \lambda_2) \lambda_1 (v_H - v_{1L})$$

From (2) we obtain $\hat{b} = v_{2L} - \frac{\lambda_1}{\lambda_2} (v_{2L} - v_{1L})$, and this reveals that $U^F_1 > U^S_1$ and $U^F_2 > U^S_2$. In the opposite case such that $\lambda_1 \geq \lambda_2$, Proposition 1(iii) applies and

$$U^F_1 = (1 - \lambda_1) \lambda_1 (v_H - v_{1L}) > U^S_1 = (1 - \lambda_1) \lambda_2 (v_H - v_{2L})$$

$$U^F_2 = U^S_2 = \lambda_2 \lambda_1 (v_{2L} - v_{1L}) + (1 - \lambda_2) \lambda_1 (v_H - v_{1L})$$

In either case, $U^F = U^F_1 + U^F_2 > U^S = U^S_1 + U^S_2$ and thus $R^S > R^F$.

References


