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Revenue comparison in asymmetric auctions with discrete valuations

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Abstract

We consider an asymmetric auction setting with two bidders such that the valuation of each bidder has a binary support. We prove that in this context the second price auction yields a higher expected revenue than the first price auction for a broad set of parameter values, although the opposite result is common in the literature on asymmetric auctions. For instance, the second price auction is superior both when a bidder's valuation is more uncertain that the valuation of the other bidder, and in case of a not too large distribution shift or rescaling. In addition, we show that in some cases the revenue in the first price auction decreases when all the valuations increase [in doing so, we correct a claim in Maskin and Riley (1985)], and we derive the bidders' preferences between the two auctions.

JEL Classification: D44, D82.

Key words: Asymmetric auctions, First price auctions, Second price auctions.

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1 Introduction

This paper is about a seller's preferences between a first price auction (FPA from now on) and a second price/Vickrey auction (SPA from now on) when the bidders' valuations are independently but asymmetrically distributed. Precisely, we consider a setting with two bidders such that the valuation of each bidder has a binary support and prove that in this context the SPA yields a higher expected revenue than the FPA for a broad set of parameter values, although the opposite result is common in the literature on asymmetric auctions. In addition, we correct a claim in Maskin and Riley (1985), we show that in some cases the revenue in the FPA decreases when all the valuations increase, and we derive the bidders' preferences among the FPA and the SPA.

As it is well known, with asymmetric distributions the revenue equivalence theorem does not apply, and only in very specific circumstances it is possible to derive the closed form of the equilibrium bidding functions for the FPA.[1] This complicates the comparison between the FPA and the SPA, but nevertheless some interesting results have been discovered. One of them is that the FPA is often more profitable than the SPA for the seller. On the basis of numeric analysis for some classes of continuous distributions, Li and Riley (2007) claim that "the 'typical' case leads to greater expected revenue in the sealed high-bid auction" [i.e., in the FPA]; a similar point of view is found in Klemperer (1999). Some general theoretical results are provided by Maskin and Riley (2000a),[2] which show that under suitable conditions on the distribution of valuations the FPA is superior to the SPA for a two-bidder setting in which a bidder's distribution is obtained by shifting or stretching to the right the other bidder's distribution; Kirkegaard (2011) generalizes these results.[3] On the other hand, some papers identify settings in which the seller prefers the SPA, including Vickrey (1961), Maskin and Riley (2000a), Cheng (2010) and Gavious and Minchuk (2010). However, these results mostly refer to specific examples,[4] whereas the result in Kirkegaard (2011) covers a relatively broad set of circumstances.

As we mentioned above, we study an environment with two bidders and binary distributions.[5]

[1] See Cheng (2006), Cheng (2010), Kaplan and Zamir (2010), and Plum (1992) for examples. In order to circumvent this problem, some authors apply numerical methods: see for instance Fibich and Gavish (2011), Gayle and Richard (2008), Li and Riley (2007), and Marshall et al. (1994).

[2] Lebrun (1996) and Cheng (2006) prove that the seller prefers the FPA for some classes of power distributions.

[3] Roughly speaking, Kierkegaard (2011) shows that the FPA is superior to the SPA if a bidder's distribution is flatter and more disperse than the other bidder's distribution. In Subsection 4.1.5 we describe with more details the main result in Kierkegaard (2011).

[4] Vickrey (1961) examines a setting in which a bidder's valuation is common knowledge. Maskin and Riley (2000a) consider the case in which a bidder's distribution is obtained from the other bidder's distribution by shifting some probability mass to the lower end-point. Cheng (2010) analyzes environments such that the equilibrium bidding functions for the FPA are linear. Gavious and Minchuk (2010) study examples in which the bidders' distributions are close to the uniform distribution.

[5] This is an extension of the set-up considered in Maskin and Riley (1983), which focus on the case in which the bidders' low valuations coincide; we analyze different classes of asymmetries. Cheng (2011) employs the same discrete setting of Maskin and Riley (1983) in order to show that in some special cases the asymmetry increases the expected revenue in the FPA, unlike in the examples studied in Cantillon (2008).

In this environment we derive the unique equilibrium outcome and the expected revenue in the FPA for all parameter values, and then we compare the FPA with the SPA for some classes of asymmetries. We find that quite often the SPA is more convenient for the seller, despite the above citation from Li and Riley (2007). Precisely, this is the case when the bidders' high valuations coincide. In alternative, when the probability of a high valuation is the same for the two bidders (but values may be different), the SPA dominates unless a bidder's values are sufficiently large with respect to the other bidder's values. More in detail, in this environment we prove that

- the SPA is more profitable if a bidder's valuation is more variable than the other bidder's valuation,[6] and in the case of distribution shift (described above) – unlike in Maskin and Riley (2000a) – at least for not too large shifts;[7]
- the revenue in the FPA may decrease when all the valuations increase, because increasing the high valuation of one bidder may induce his opponent to bid less aggressively. This makes the FPA inferior to the SPA, in contrast with a claim in Maskin and Riley (1985) for the particular case in which the only deviation from a symmetric setting is given by unequal high valuations [however, for this case Maskin and Riley (1983) agree with our ranking between the FPA and the SPA].

Finally, we show that the bidders' preferences among the two auctions often go in the opposite direction with respect to the seller's preferences.

The remainder of the paper is organized as follows. In Section 2 we describe the primitives of our model. In Section 3 we study equilibrium behavior in the SPA and in the FPA, and in Section 4 we present our results on the comparison between the FPA and the SPA.

2 The model

A (female) seller owns an indivisible object which is worthless to her and faces two (male) bidders. Let v_1 (v_2) denote the monetary valuation for the object of bidder 1 (bidder 2), which he privately observes; v_1 and v_2 are independently distributed. The set $\{v_{1L}, v_{1H}\}$ is the support for v_1, with $0 < v_{1L} < v_{1H}$ and $\lambda_1 \equiv \Pr\{v_1 = v_{1L}\} \in (0, 1)$. Likewise, the support for v_2 is $\{v_{2L}, v_{2H}\}$ with $0 < v_{2L} < v_{2H}$ and $\lambda_2 \equiv \Pr\{v_2 = v_{2L}\} \in (0, 1)$. Without loss of generality we assume that $v_{1L} \leq v_{2L}$. Both the seller and bidders are risk neutral, and a bidder's utility if he wins is given by his valuation for the object minus the price paid to the seller; his utility if he loses is zero. We use i_j to denote bidder i when his valuation is v_{ij}, thus for instance 2_L is the type of bidder 2 with valuation v_{2L}.

[6] After Vickrey (1961), this is the first ranking result in the theoretical literature which does not rely on first order stochastic dominance among the distributions of valuations.

[7] Kirkegaard (2011) examines an example in which the distribution of the valuation of a bidder is obtained by rescaling the distribution of the other bidder's valuation. In this case we show that the SPA dominates the FPA in our setting, unless the rescaling is large.

The main purpose of this paper is to evaluate the relative profitability of the FPA and the SPA for the seller. In either of these auctions each bidder submits simultaneously a nonnegative sealed bid, and the bidder who makes the highest bid wins the object (if the bidders tie, the winner is selected according to a specified tie-breaking rule: see next section). In the FPA the winning bidder pays the own bid; in the SPA he pays the loser's bid (i.e., the second highest bid).

3 Equilibrium bidding

3.1 SPA

It is well known that when bidders have private values, in the SPA it is weakly dominant for each bidder to bid the own valuation. Thus the seller's expected revenue R^S is the expectation of $\min\{v_1, v_2\}$, which is straightforward to evaluate (recall that $v_{1L} \leq v_{2L}$):

$$R^S = \begin{cases} \lambda_1 v_{1L} + (1-\lambda_1)v_{1H} & \text{if } v_{1H} \leq v_{2L} \\ \lambda_1 v_{1L} + (1-\lambda_1)(\lambda_2 v_{2L} + (1-\lambda_2)v_{1H}) & \text{if } v_{2L} < v_{1H} \leq v_{2H} \\ \lambda_1 v_{1L} + (1-\lambda_1)(\lambda_2 v_{2L} + (1-\lambda_2)v_{2H}) & \text{if } v_{2H} < v_{1H} \end{cases} \quad (1)$$

3.2 FPA

The analysis for the FPA is less immediate than for the SPA. In fact, finding the closed form for the equilibrium bidding strategies for an FPA with asymmetrically distributed valuations is often impossible when valuations are continuously distributed. However, this is not the case given our assumptions on the distributions of v_1 and v_2. We consider BNE in which no type of bidder bids above the own valuation.

In our setting no pure-strategy Bayes-Nash equilibrium exists [except in the case that condition (3) below is satisfied], and sometimes no mixed-strategy Bayes-Nash equilibrium (BNE in the following) exists either. Precisely, when $v_{1L} = v_{2L}$ we find that no BNE exists in the standard FPA in which each bidder wins with probability $\frac{1}{2}$ in case of tie.[8] However, Proposition 2 in Maskin and Riley (2000b) establishes that a BNE exists under a suitable tie-breaking rule such that each bidder i is required to submit both an "ordinary" bid $b_i \geq 0$ and a "tie-breaker" bid $c_i \geq 0$.[9] If $b_1 \neq b_2$, then c_1, c_2 are irrelevant but if $b_1 = b_2$ then bidder i wins if $c_i > c_j$ and pays $b_i + c_j$ (each bidder wins with probability $\frac{1}{2}$ if $b_1 = b_1$ and $c_1 = c_2$). Therefore c_1, c_2 are bids in a second price/Vickrey auction which takes place if and only if $b_1 = b_2$. In Proposition 1 we consider the first price auction with this "Vickrey tie-breaking rule".

We want to stress that this particular tie-breaking rule is needed only when $v_{1L} = v_{2L}$, since existence is obtained for *any tie-breaking rule* if $v_{1L} \neq v_{2L}$. Precisely, when $v_{1L} < v_{2L}$ we find that multiple BNE exist regardless of the tie-breaking rule, but they are all outcome-equivalent. In particular, multiple BNE arise because type 1_L (and type 1_H in one case) never wins and needs to

[8] See the proof to Proposition 1 in the appendix (step 2 for the case of $v_{1L} = v_{2L}$).

[9] A very similar idea appears in Lebrun (2002), in the auction he denotes with $F\bar{P}A$.

bid weakly less than v_{1L} (weakly less than v_{1H}) with probability one, in such a way that no type of bidder 2 has incentive to bid below v_{1L} (below v_{1H}). Since there are many strategies of 1_L (of 1_H) which achieve this goal,[10] multiple BNE exist. However, as we specified above, each BNE generates the same outcome in the sense that the allocation of the object, the payoff of each type of bidder and the expected revenue are the same;[11] therefore multiplicity is not an issue.

Conversely, when $v_{1L} = v_{2L}$ in each BNE both types 1_L and 2_L bid v_{1L}, and (generically) also 1_H or 2_H bid v_{1L} with positive probability; suppose 2_H does so (to fix the ideas). Then 2_H ties with positive probability with 1_L by bidding v_{1L}, and if 2_H does not win the tie-break with probability one, he has an incentive to bid slightly above v_{1L}, which breaks the BNE. On the other hand, under the Vickrey tie-breaking rule, for a bidder i with valuation v_i submitting an ordinary bid b_i, it is weakly dominant to choose $c_i = v_i - b_i$, and in particular $c_{1L} = 0$, $c_{2H} = v_{2H} - v_{1L} > 0$ for the case we are considering; thus 2_H wins the tie-break paying v_{1L} in aggregate.[12] Given this property on weak dominance for tie-breaking bids, when we describe a strategy of bidder i we implicitly assume that to each ordinary bid b_i is associated a tie-breaking bid c_i equal to $v_i - b_i$. Therefore, whenever a tie occurs the bidder with the highest valuation wins and pays the valuation of the other bidder.

In Proposition 1(ii) below an important role is played by a specific bid $\hat{b}$ which is the smaller solution to the following equation:

$$\lambda_2 b^2 + ((1-\lambda_2)v_{1H} + (\lambda_1 - \lambda_2)\, v_{2L} - \lambda_1 v_{1L} - v_{2H})b + ((1-\lambda_1)v_{2H} - (1-\lambda_2)v_{1H})v_{2L} + \lambda_1 v_{1L} v_{2H} = 0 \quad (2)$$

and assumption (4) in Proposition 1(ii) implies that $\hat{b}$ satisfies $v_{1L} \leq \hat{b} < \min\{v_{2L}, v_{1H}\}$.[13]

Proposition 1 *Given $v_{1L} \leq v_{2L}$, consider the FPA with the Vickrey tie-breaking rule. Although multiple BNE may exist, they are all outcome-equivalent to the following BNE.*
Type 1_L always bids v_{1L} and the bids of the other types depend on the parameters as follows:

(i) If

$$v_{1H} \leq \lambda_1 v_{1L} + (1-\lambda_1)v_{2L} \quad (3)$$

then types 2_L,2_H bid v_{1H}; type 1_H bids weakly less than v_{1H} with probability one and in such a way that no type of bidder 2 has incentive to bid below v_{1H}.

(ii) If

$$\lambda_1 v_{1L} + (1-\lambda_1)v_{2L} < v_{1H} < \frac{(1-\lambda_1)v_{2H} + (\lambda_1 - \lambda_2)v_{1L}}{1-\lambda_2} \quad (4)$$

then types 1_H,2_L,2_H play mixed strategies with support $[v_{1L}, \bar{b}]$ for 1_H, $[v_{1L}, \hat{b}]$ for 2_L, $[\hat{b}, \bar{b}]$ for 2_H, in which $\hat{b}$ is the smaller solution to (2) and $\bar{b} \equiv \lambda_2 \hat{b} + (1-\lambda_2)v_{1H}$. The c.d.f. for the mixed

[10] One example is such that 1_L bids according to the uniform distribution on $[\alpha v_{1L}, v_{1L}]$ with $\alpha < 1$ and close to 1.

[11] A slight modification of the proof of Proposition 1 (for the case of $v_{1L} < v_{2L}$) shows that all BNE are outcome-equivalent regardless of the tie-breaking rule. Details are available upon request.

[12] In fact, whenever 1_L bids v_{1L} and ties with positive probability with type 2_j such that $v_{2j} > v_{1L}$, in each BNE 1_L selects $c_{1L} = 0$, otherwise it is profitable for 2_j to bid slightly above v_{1L}.

[13] For more details see Proposition 1(ii) and its proof in the appendix.

strategies are

$$G_{1H}(b) = \begin{cases} \frac{\lambda_1(b-v_{1L})}{(1-\lambda_1)(v_{2L}-b)} & \text{for } b \in [v_{1L}, \hat{b}] \\ \frac{1}{1-\lambda_1}(\frac{v_{2H}-\bar{b}}{v_{2H}-b} - \lambda_1) & \text{for } b \in (\hat{b}, \bar{b}] \end{cases} \tag{5}$$

$$G_{2L}(b) = \frac{v_{1H} - \bar{b}}{\lambda_2(v_{1H} - b)}, \qquad G_{2H}(b) = \frac{1}{1-\lambda_2}(\frac{v_{1H} - \bar{b}}{v_{1H} - b} - \lambda_2) \tag{6}$$

and from the definitions of $\hat{b}$ *and* $\bar{b}$ *it follows that* G_{1H} *is continuous at* $b = \hat{b}$, $G_{2L}(\hat{b}) = 1$, $G_{2H}(\hat{b}) = 0$.[14]

(iii) If

$$\frac{(1-\lambda_1)v_{2H} + (\lambda_1 - \lambda_2)v_{1L}}{1-\lambda_2} \leq v_{1H} \tag{7}$$

then 2_L *bids* v_{1L} *and* 1_H,2_H *play mixed strategies with common support* $[v_{1L}, \lambda_1 v_{1L} + (1-\lambda_1)v_{2H}]$ *and the following c.d.f.*

$$G_{1H}(b) = \frac{\lambda_1}{1-\lambda_1}\frac{b - v_{1L}}{v_{2H} - b}, \qquad G_{2H}(b) = \frac{1}{1-\lambda_2}(\frac{v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2H}}{v_{1H} - b} - \lambda_2) \tag{8}$$

When (3) holds, Proposition 1(i) establishes that each type of bidder 2 bids v_{1H} and wins for sure.[15] This occurs because v_{2L} is sufficiently larger than v_{1H}, which implies that each type of bidder 2 has so much to gain from winning that it is profitable for him to make a bid of v_{1H} in order to outbid each type of bidder 1. Precisely, (3) guarantees that type 2_L prefers winning for sure by bidding v_{1H} rather than bidding v_{1L} and winning only when facing type 1_L, that is with probability λ_1.

Conversely, if v_{1H} is large then (3) is violated and 2_L is less aggressive since he prefers to bid v_{1L} and win only against 1_L rather than bidding v_{1H} and winning with certainty, as the latter alternative is too expensive. Indeed, 2_L bids in the interval $[v_{1L}, \hat{b}]$, with $\hat{b} < v_{1H}$, and with an atom at $b = v_{1L}$: $G_{2L}(v_{1L}) = \frac{v_{1H}-\bar{b}}{\lambda_2(v_{1H}-v_{1L})} > 0$. The less aggressive bidding of 2_L allows 1_H to win with positive probability by bidding in $(v_{1L}, \hat{b}]$, which makes his equilibrium payoff positive. This implies that the highest bid of 1_H is smaller than v_{1H}, since each bid in the support of a bidder's mixed strategy needs to maximize the expected payoff of the bidder given the strategies of the other types. Therefore also the highest bid of 2_H is smaller than v_{1H}, as we see from Proposition 1(ii). As v_{1H} increases, 2_L becomes increasingly less aggressive: $\hat{b}$ decreases and $G_{2L}(b)$ increases for any $b \in [v_{1L}, \hat{b})$. This occurs because as v_{1H} increases, the equilibrium payoff of 1_H increases and this requires that G_{2L} puts more weight on v_{1L} and becomes flatter in $[v_{1L}, \hat{b}]$ to satisfy the indifference condition of 1_H.[16] For a large enough v_{1H} such that (7) is satisfied, this effect implies

[14] In the case that $\hat{b} = v_{1L}$ (which occurs if and only if $v_{1L} = v_{2L}$), 2_L bids v_{1L} and $\bar{b} = \lambda_2 v_{1L} + (1-\lambda_2)v_{1H}$, thus $G_{1H}(b) = \frac{1}{1-\lambda_1}(\frac{v_{2H}-\bar{b}}{v_{2H}-b} - \lambda_1)$ and $G_{2H}(b) = \frac{1}{1-\lambda_2}(\frac{v_{1H}-\bar{b}}{v_{1H}-b} - \lambda_2)$ for each $b \in [v_{1L}, \bar{b}]$.

[15] In a setting with continuously distributed valuations, Maskin and Riley (2000a) identify an analogous BNE and provide the intuition we describe here and immediately after Proposition 2(i). In addition, Maskin and Riley (1983) identify the BNE we describe in Proposition 1 for the case of $v_{1L} = v_{2L} = 0$. Thus Proposition 1 is a new result for the case in which $v_{1L} < v_{2L}$ and (3) is violated.

[16] We describe a similar effect (with more details) in the intuition regarding condition (9) in Proposition 2(ii).

that 2_L bids v_{1L} with certainty and also 2_H bids v_{1L} with positive probability. In particular, when (7) holds the equilibrium strategies – and thus also the expected revenue – do not depend on v_{2L}.

A well known feature of the FPA when valuations are asymmetrically distributed is that an inefficient allocation of the object is implemented with positive probability. In our setting, suppose for instance that $v_{1L} < v_{2L}$, $v_{2L} \neq v_{1H}$ and (4) holds. Then $\hat{b} > v_{1L}$ and in the state of the world with types $1_H, 2_L$ each type has a positive probability to win and thus the highest valuation type may not win.

4 Comparison between the FPA and the SPA

In order to derive the seller's preferences between the FPA and the SPA we need to evaluate the expected revenue R^F in the FPA generated by the BNE described in Proposition 1. Although we can express R^F in closed form (see Section 5.3 in the appendix), in many cases R^F is a complicated function of the parameters [an exception occurs when (3) is satisfied]; this is largely due to the inefficiency of the FPA we mentioned above. In particular, it seems difficult to obtain insights from comparing R^F with R^S without any restriction on the parameters. We focus therefore on two particular cases which yield nevertheless quite interesting results. One is such that $\lambda_1 = \lambda_2$, and the other is such that $v_{1H} = v_{2H}$; the analysis of the case in which $v_{1L} = v_{2L}$ is performed in Maskin and Riley (1983).

4.1 The case in which $\lambda_1 = \lambda_2$

The following proposition describes our main results when $\lambda_1 = \lambda_2$ [in fact, Proposition 2(i) does not require $\lambda_1 = \lambda_2$]. The rest of this subsection is devoted to discussing these results and in providing intuitions.

Proposition 2 *(i) $R^F > R^S$ if (3) is satisfied;*
(ii) $R^S > R^F$ if $\lambda_1 = \lambda_2 \equiv \lambda$ and at least one of the following conditions is satisfied:

$$v_{1L} = v_{2L} \quad \text{and} \quad v_{1H} \neq v_{2H} \tag{9}$$

$$v_{1L} < v_{2L} \quad \text{and} \quad v_{2H} \leq v_{1H} \tag{10}$$

$$v_{1L} < v_{2L} \leq v_{1H} < v_{2H} \quad \text{and} \quad \begin{array}{c} v_{2L} - v_{1L} \simeq 0 \\ v_{2H} - v_{1H} \simeq 0 \end{array}, \text{ or } \lambda \geq \frac{1}{2} \tag{11}$$

Proposition 2(i) is very simple to interpret. Precisely, $R^F = v_{1H}$ when (3) is satisfied as both types of bidder 2 win the auction with a bid of v_{1H}; moreover, (3) implies $v_{1H} \leq v_{2L}$ and thus $R^S = \lambda_1 v_{1L} + (1 - \lambda_1) v_{1H}$. Then $R^F > R^S$ follows immediately. The intuition is that in both auctions bidder 2 always wins, thus R^S is equal to the expected valuation of the loser, bidder 1, and R^F is the high valuation of bidder 1.

Proposition 2(ii) describes a set of circumstances which imply $R^S > R^F$, and in order to facilitate its understanding it is useful to have in mind a benchmark symmetric environment which we now describe.

The benchmark symmetric setting Suppose that $v_{1L} = v_{2L} \equiv v_L$, $v_{1H} = v_{2H} \equiv v_H$ and $\lambda_1 = \lambda_2 \equiv \lambda$. We know from Maskin and Riley (1985) that in this case the unique BNE in the FPA is such that types 1_L,2_L both bid v_L and types 1_H,2_H play the same mixed strategy with support $[v_L, E_v]$ – in which $E_v \equiv \lambda v_L + (1-\lambda)v_H$ – and c.d.f. $G_H(b) = \frac{\lambda}{1-\lambda}\frac{b-v_L}{v_H-b}$. Furthermore, $R^F = R^S = (2\lambda - \lambda^2)v_L + (1-\lambda)^2 v_H$.

4.1.1 Condition (9)

Going back to Proposition 2(ii), we start by considering (9). This condition implies $R^S > R^F$, and in fact this result relies on the following property.

Proposition 3 *Suppose that* $v_{1L} = v_{2L} = v_L$, $v_{1H} \neq v_{2H} = v_H$ *and* $\lambda_1 = \lambda_2 = \lambda$. *Then* R^F *is increasing in* v_{1H} *for* $v_{1H} \in (v_L, v_H]$ *and is decreasing in* v_{1H} *for* $v_{1H} \in [v_H, +\infty)$.

This proposition says that in a setting which is asymmetric only because $v_{1H} \neq v_H$, R^F is maximized with respect to v_{1H} at $v_{1H} = v_H$,[17] and in particular increasing v_{1H} above v_H reduces R^F.[18]

In order to obtain an intuition for Proposition 3 we start with the case of $v_{1H} > v_H$ and notice that given $\lambda_1 = \lambda_2$, (7) is satisfied when $v_{1H} > v_H$ and therefore Proposition 1(iii) applies. This reveals that the behavior of types $1_L, 1_H, 2_L$ is unchanged with respect to the benchmark symmetric setting, whereas now 2_H bids less aggressively. Precisely, G_H and G_{2H} have the same support $[v_L, E_v]$, but since $G_{2H}(b) = \frac{(1-\lambda)(v_{1H}-v_H)+\lambda(b-v_L)}{(1-\lambda)(v_{1H}-b)}$ it is simple to verify that $G_{2H}(b) > G_H(b)$ for any $b \in [v_L, E_v)$, and in particular $G_{2H}(v_L) > 0 = G_H(v_L)$. Since 2_H is less aggressive with respect to the symmetric setting, it follows that an increase in v_{1H} has a negative effect on R^F. In fact, the larger is v_{1H} the higher (lower) is the probability that G_{2H} attaches to low (high) bids in $[v_L, E_v]$. As a consequence, R^F is monotonically decreasing with respect to v_{1H} for $v_{1H} > v_H$.

It is somewhat surprising that, starting from a symmetric setting, an increase of a single (high) valuation generates a decrease in R^F. In order to see what drives the result, suppose for a moment that 2_H still bids according to G_H even though $v_{1H} > v_H$. Then the payoff of type 1_H from bidding $b \in [v_L, E_v]$ is $(v_{1H} - b)[\lambda + (1-\lambda)G_H(b)]$. This is obviously higher than $(v_H - b)[\lambda + (1-\lambda)G_H(b)]$, his payoff before the increase in v_{1H}, and – more importantly – is increasing in b. In order to make 1_H indifferent among the bids in an interval $(v_L, b^*]$ it is necessary that G_{2H} is flatter than G_H,

[17] This fact may appear similar to the main message in Cantillon (2008), but in fact in our analysis the benchmark symmetric setting is fixed, whereas in Cantillon (2008) it is not.

[18] Obviously, an analogous result holds if v_{1H} is kept fixed and v_{2H} is allowed to vary.

and indeed $G_{2H}(b) = \frac{(1-\lambda)(v_{1H}-v_H)+\lambda(b-v_L)}{(1-\lambda)(v_{1H}-b)}$ has an atom at $b = v_L$ and grows more slowly than G_H for $b > v_L$. This is how a less aggressive behavior of 2_H results from an increase in v_{1H}.[19]

Given this result, it is straightforward to see that an increase in v_{1H} favors the SPA over the FPA since it does not affect the distribution of $\min\{v_1, v_2\}$, and thus R^S does not change.

Maskin and Riley (1985) (in their Section III) consider the setting of Proposition 2(ii), except that they assume $v_{1L} = v_{2L} = 0$, and claim that an increase in v_{2H} favors the FPA over the SPA, in contrast with Proposition 2(ii). However, they do not provide a formal proof of their claim. On the other hand, Maskin and Riley (1983) conclude that $R^S > R^F$, consistently with Proposition 2(ii): see their Figure 1 between pages 18 and 19.[20]

For the case of $v_{1H} < v_H$, Proposition 3 establishes the intuitive result that R^F is reduced with respect to when $v_{1H} = v_H$. We can rely on Proposition 1(ii) (and in particular on footnote 14 since $v_{1L} = v_{2L}$ and thus $\hat{b} = v_{1L}$), but a simpler argument is also available. Given $v_{1H} < v_H$, consider the symmetric setting with low valuations both equal to v_L and high valuations both equal to v_{1H}. Then R^F is smaller, by $(1-\lambda)^2(v_H - v_{1H})$, with respect to R^F for the benchmark symmetric setting. Now increase the valuation of type 2_H from v_{1H} to v_H to obtain the asymmetric setting we are considering. The same logic of Proposition 3 (see footnote 18) suggests that R^F further decreases. Therefore a decrease in the valuation of 1_H below v_H reduces R^F by more than $(1-\lambda)^2(v_H - v_{1H})$. On the other hand, from (1) we see that R^S decreases exactly by $(1-\lambda)^2(v_H - v_{1H})$. Hence, $R^S > R^F$ holds both if $v_{1H} > v_H$ and also if $v_{1H} < v_H$.

On the effect of increasing $v_{1L}, v_{2L}, v_{1H}, v_{2H}$ Proposition 3 suggests a simple observation. Suppose that we start from the benchmark symmetric setting and let R^{F*} denote the resulting expected revenue. Then suppose that the valuation of 1_H is increased; this reduces the revenue below R^{F*} by Proposition 3. Finally, increase slightly the valuations of 1_L,2_L,2_H. Since R^F is a continuous function of the parameters, we infer that R^F remains smaller than R^{F*}. Therefore, starting from a symmetric setting and suitably increasing $v_{1L}, v_{2L}, v_{1H}, v_{2H}$ (but not each valuation by the same amount) we obtain a setting in which the revenue is reduced. For instance, suppose that $v_{1L} = v_{2L} = 100$, $v_{1H} = v_{2H} = 200$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$; then $R^{F*} = 125$. However, if $v_{1L} = v_{2L} = 105$, $v_{1H} = 400$ and $v_{2H} = 205$, then $R^F \simeq 123.12$.

4.1.2 Condition (10)

The effects of (10) are almost straightforward. In case that $v_{2H} = v_{1H}$, Proposition 1(iii) applies and as we mentioned in Subsection 3.2, R^F does not depend on $v_{2L} \in [v_{1L}, v_{2H})$; thus R^F is equal

[19] Lebrun (1998) considers a setting with continuously distributed valuations and assumes that the valuation distribution of one bidder changes into a new distribution which dominates the previous one in the sense of reverse hazard rate domination (the support is unchanged). He show that, as a consequence, for each bidder the new bid distribution first order stochastically dominates the initial bid distribution, and thus the expected revenue increases.

[20] Since they assume $v_{1L} = v_{2L} = 0$, Maskin and Riley (1983) do not consider the various cases covered in Proposition 2(ii), and they do not have the results in Propositions 3 and 4.

to the revenue in the symmetric setting. On the other hand, (1) reveals that R^S is increasing in v_{2L} and therefore $R^S > R^F$.

In case that $v_{2H} < v_{1H}$, suppose first that $v_{1L} = v_{2L}$. We know from condition (9) that $v_{2H} < v_{1H}$ implies $R^S > R^F$, and the previous paragraph explains that an increase in v_{2L} has no effect on R^F (but increases R^S); hence the conclusion.

$\mathbf{v}_1$ more uncertain than $\mathbf{v}_2$ It is interesting to notice that (10) includes the case in which $v_{1L} < v_{2L} < v_{2H} < v_{1H}$, which means that v_1 has a wider range of variability than v_2; obviously, this includes the special case in which v_1 is a mean-preserving-spread of v_2. In each of these cases we obtain an unambiguous ranking between R^S and R^F, that is the SPA is better than the FPA when the valuation of one bidder is more uncertain then the valuation of the other bidder.
Kirkegaard (2011) notices that only Vickrey (1961) provides a theoretical ranking result without the assumption of first order stochastic dominance between the bidders' distributions of valuations.[21] Precisely, Vickrey (1961) assumes that v_1 is uniformly distributed over $[0, 1]$ and v_2 is equal to a fixed value a, that is v_2 is common knowledge; he proves that the FPA is superior to the SPA for $a > 0.43$. Now consider in our framework the parameters $\lambda = \frac{1}{2}$ and $v_{1L} = 0$, $v_{1H} = 1$, $v_{2L} = a - x$, $v_{2H} = a + x$ with $x > 0$ and close to zero. This setting is in a sense similar to that in Vickrey (1961) since v_1 is uniformly distributed over $\{0, 1\}$, and v_2 is almost commonly known to be equal to a.[22] However, Proposition 2(ii) establishes that $R^S > R^F$ for any $a \in (0, 1)$. This difference with respect to Vickrey (1961) arises because in our setting R^F is considerably lower than in Vickrey (1961), due to the fact that type 2_L bids $v_{1L} = 0$ with certainty (and type 2_H bids 0 with positive probability), as bidding 0 suffices to win the auction if the opponent is type 1_L, an event with probability $\frac{1}{2}$. Conversely, this does not occur when v_1 is uniformly distributed over $[0, 1]$ because if bidder 2 bids close to zero then he wins only against a small set of types of bidder 1. For instance, if $a = \frac{1}{2}$ then Vickrey (1961) proves that bidder 2's equilibrium mixed strategy has support $[\frac{1}{4}, \frac{7}{16}]$, that is his minimum bid is $\frac{1}{4}$.

4.1.3 Condition (11)

Given the innocuous assumption that $v_{1L} \leq v_{2L}$, after (9) and (10) have been considered, the only class of asymmetry remaining is such that $v_{1L} < v_{2L}$ and $v_{1H} < v_{2H}$; the results for this case are quite simple to describe. First, from (9) and (10) it is intuitive that $R^S > R^F$ when $v_{2L} - v_{1L}$ is close to zero and $v_{2H} - v_{1H}$ is close to zero. On the other hand, Proposition 2(i) establishes that $R^F > R^S$ when v_{2L} is sufficiently larger than v_{1H} (which implies that also v_{2H} is quite large). From (11) we see that as long as $\lambda \geq \frac{1}{2}$, $R^S > R^F$ holds provided that v_{2L} is not larger than v_{1H}. We notice that this result is quite conservative because of the way it is obtained. Precisely, in our final

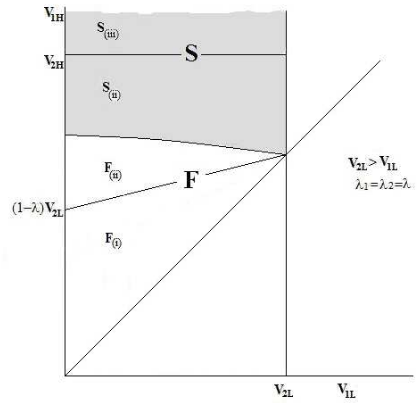
[21] Gayle and Richard (2008), Li and Riley (1999) and Li and Riley (2007) apply numeric analysis to settings without first order stochastic dominance and obtain mixed results.

[22] Proposition 1 still holds even though $v_{1L} = 0$ violates our assumption $v_{1L} > 0$. However, when $v_{1L} = 0$ the Vickrey tie-breaking rule is needed also if $v_{1L} \neq v_{2L}$.

remark in Subsection 3.2 we noticed that in the BNE described by Proposition 1(ii) the highest valuation bidder does not always win. Conversely, the efficient allocation is always achieved in the SPA. Therefore a sufficient condition for $R^S > R^F$ is that the aggregate bidders' rents in the FPA, U^F, are (weakly) larger than the rents in the SPA, U^S. Condition (11) guarantees indeed that $U^F \geq U^S$. In words, when $\lambda \geq \frac{1}{2}$ in order for $R^F \geq R^S$ to hold it is not sufficient that the distribution of v_2 first order stochastically dominates the distribution of v_1, but it is actually necessary that $v_{1H} < v_{2L}$; broadly speaking, we could say that there need to be no overlapping of supports.

In Figure 1 we fix $\lambda = \frac{1}{4}$ and v_{2L}, v_{2H}, and partition the space (v_{1L}, v_{1H}) in two regions S and F such that $R^S > R^F$ if $(v_{1L}, v_{1H}) \in S$, and $R^F \geq R^S$ if $(v_{1L}, v_{1H}) \in F$. In particular, $S_{(iii)}$ is the set in which (10) is satisfied [in this case (7) holds and the BNE of Proposition 1(iii) applies]; $F_{(i)}$ is the set in which (3) holds [then the BNE of Proposition 1(i) applies]. The region between $S_{(iii)}$ and $F_{(i)}$ is such that (4) is satisfied – thus the BNE of Proposition 1(ii) applies – and the boundary between S and F is obtained numerically.[23] For other values of $\lambda < \frac{1}{2}$ a similar figure is obtained.[24]

Figure1: Comparison between the FPA and the SPA when $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2$.



In the dark region $S = S_{(ii)} \cup S_{(iii)}$ the SPA dominates the FPA in terms of the seller's revenue. Proposition 1(i) applies in the lower region $F_{(i)}$, 1(ii) in the region $F_{(ii)} \cup S_{(ii)}$ in the middle, and 1(iii) in the upper region $S_{(iii)}$.

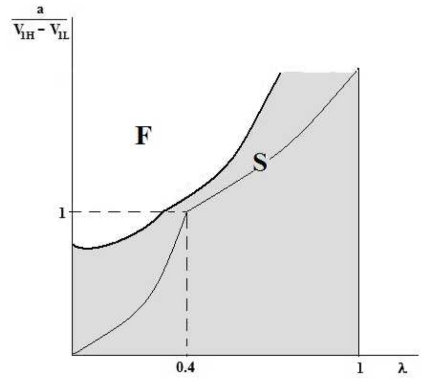
We notice that when $\lambda_1 = \lambda_2$, by Proposition 2(ii) the SPA is better than the FPA for any small deviation from the symmetric setting, that is when $v_{2L} - v_{1L}$ and $v_{2H} - v_{1H}$ are close to zero, but $v_{2L} - v_{1L} > 0$ and/or $v_{2H} - v_{1H} \neq 0$, as it is apparent from Figure 1.

[23] The precise values of v_{2L} and v_{2H} do not affect the qualitative features of Figure 1 since (i) if all valuations are increased by a same amount α, then both R^F and R^S increase by α and thus $R^F - R^S$ is unaffected; (ii) if all valuations are multiplied by a same number $\beta > 0$, then both R^F and R^S are multiplied by β and *the sign* of $R^F - R^S$ is unaffected. Using these two degrees of freedom, we can fix arbitrarily v_{2L} and v_{2H} without affecting the qualitative features of the sets S and F. Conversely (of course), the value of λ affects substantially S and F.

[24] In fact, for $\lambda < \frac{1}{2}$ but close to $\frac{1}{2}$ each point on the boundary between S and F is such that $v_{1H} < v_{2L}$.

Distribution shift and rescaling A particular type of asymmetry considered in the literature is as follows. Given the c.d.f. F_1 for the valuation of bidder 1, the c.d.f. for v_2 is $F_2(v_2) = F_1(v_2 - a)$ with $a > 0$, that is F_2 is obtained by shifting F_1 to the right, which implies that bidder 2 is (ex ante) stronger than 1. In a setting with continuously distributed values, Maskin and Riley (2000a) prove that under suitable assumptions on F_1, the FPA generates a higher revenue than the SPA; Kirkegaard (2011) obtains the same result under weaker assumptions. In our context this sort of asymmetry is obtained by fixing v_{1L}, v_{1H} and setting $v_{2L} = v_{1L} + a$, $v_{2H} = v_{1H} + a$, for some $a > 0$. From (11) we can find sufficient conditions for $R^S > R^F$, but in fact in the appendix we exploit this particular structure of asymmetry to modify the proof of Proposition 2(ii) and show that $R^S > R^F$ as long as $\frac{a}{v_{1H} - v_{1L}} \leq \frac{2\lambda}{2-3\lambda}$ (for $\lambda \leq \frac{2}{5}$) or $\frac{a}{v_{1H} - v_{1L}} \leq \frac{2(2+\lambda)}{3(2-\lambda)}$ (for $\lambda > \frac{2}{5}$). Actually, also this result is quite conservative, as numeric analysis shows that $R^S > R^F$ holds for the set of parameters in region S in Figure 2

Figure 2: Comparison between the FPA and the SPA in case of distribution shift.



The thin curve in the dark region is such that $U^F > U^S$ holds for the parameters below the curve. However, the SPA dominates the FPA in terms of the seller's revenue in the whole dark region S.

Thus in our discrete setting a shift favors the FPA over the SPA only if the shift is sufficiently large; for instance, $R^F > R^S$ definitely holds if a is such that (3) is satisfied, that is if $\frac{a}{v_{1H} - v_{1L}} \geq \frac{1}{1-\lambda}$. On the other hand, in their numeric analysis applied to continuous distributions, Li and Riley (2007) find that a shift "can result in economically very significant revenue differences [in favor of the FPA]" for examples with uniform or truncated normal distributions, and notice that "Analysis of other distributions also produces broadly similar results".

Example 4 in Kirkegaard (2011) starts from F_2 such that $F_2(e^v)$ is convex and log-concave and obtains F_1 as $F_1(v) = F_2(\gamma v)$ for some $\gamma > 1$ and not too large; thus v_1 is a rescaling of v_2, and Kirkegaard (2011) proves that $R^F > R^S$. In our context this sort of asymmetry is obtained by fixing v_{2L}, v_{2H} and setting $v_{1L} = \frac{1}{\gamma} v_{2L}$, $v_{1H} = \frac{1}{\gamma} v_{2H}$. The comparison between the SPA and the

FPA yields results similar to those obtained for a shift. Precisely, (11) reveals that $R^S > R^F$ if γ is not much larger than 1, whereas a large γ makes (3) satisfied and thus $R^F > R^S$.

4.1.4 The distribution of bids in the FPA and the bidders' preferences

For $i = 1, 2$, let G_i denote the ex ante c.d.f. of the equilibrium bids submitted by bidder i in the FPA, that is $G_i(b) = \lambda G_{iL}(b) + (1-\lambda)G_{iH}(b)$. Using Proposition 1 we can compare the equilibrium bid distributions of bidder 1 and 2 in the FPA, and we find that G_2 first order stochastically dominates G_1 when $v_{2H} > v_{1H}$; the opposite result obtains if $v_{1H} > v_{2H}$. Notice that when $v_{2H} > v_{1H}$, the distribution of v_2 first order stochastically dominates the distribution of v_1 and the result that G_2 first order stochastically dominates G_1 agrees with Corollary 1 in Kirkegaard (2009), for a setting with continuous distributions. On the other hand, when $v_{2H} < v_{1H}$ there is no first order stochastic dominance between the distribution of v_1 and v_2, but second order stochastic dominance applies if $v_{1H} \leq v_{2H} + \frac{\lambda}{1-\lambda}(v_{2L} - v_{1L})$, that is if the expected value of v_2 is weakly larger than the expected value of v_1. Under second order stochastic dominance between the valuations distributions, Proposition 5 in Kirkegaard (2009) shows that the bid distributions must cross, whereas we find that G_1 first order stochastically dominates G_2.

Proposition 1 also allows us to compare the bidders' payoffs in the FPA with their payoffs in the SPA: it turns out that bidder 1 weakly prefers the FPA, whereas bidder 2 weakly prefers the SPA. These results largely agree with the results in Propositions 3.3(ii) and 3.6 in Maskin and Riley (2000a).

4.1.5 Relationship with Kirkegaard (2011)

Proposition 2(ii) reveals that $R^S > R^F$ for a broad set of deviations from the benchmark symmetric setting, provided that $\lambda_1 = \lambda_2$. On the other hand, a frequent result in the literature on asymmetric auctions is that $R^F > R^S$. Since the most general theoretical results are obtained in Kirkegaard (2011), we explain why his analysis does not apply to our setting.

Kirkegaard (2011) considers a two-bidder environment with supports $[\beta_1, \alpha_1]$ for v_1 and $[\beta_2, \alpha_2]$ for v_2 such that $\beta_1 \leq \beta_2$ and $\alpha_1 < \alpha_2$. The c.d.f. F_1, F_2 have no atoms and have continuous and positive densities f_1, f_2 in the respective supports; moreover, 1 is ex ante weaker than 2 in the sense that F_2 first order stochastically dominates F_1. A crucial ingredient for the result is $r(v)$, which is defined as $F_2^{-1}[F_1(v)]$ for each $v \in [\beta_1, \alpha_1]$, that is $r(v)$ satisfies $\Pr\{v_2 \leq r(v)\} = \Pr\{v_1 \leq v\}$ and $r(v) \geq v$ as F_2 first order stochastically dominates F_1. The main result in Kirkegaard (2011), Theorem 1, establishes that $R^F > R^S$ if[25]

$$\frac{f_2(v)}{F_2(v)} \geq \frac{f_1(v)}{F_1(v)} \quad \text{for any} \quad v \in [\beta_1, \alpha_1] \cap [\beta_2, \alpha_2] \tag{12}$$

$$f_1(v) \geq f_2(x) \quad \text{for any} \quad x \in [v, r(v)] \quad \text{and} \quad \text{any} \quad v \in [\beta_1, \alpha_1] \tag{13}$$

This theorem results from a clever application of the mechanism design techniques introduced by Myerson (1981), and precisely relies on the following argument – expressed only for the case of $\beta_1 = \beta_2$ for simplicity. In the SPA bidder 1 wins if and only if $v_2 < v_1$, but (12) implies that 1 wins more frequently in the FPA. Precisely, 1 wins as long as $v_2 < k^F(v_1)$ for a certain function k^F such that $k^F(v_1) > v_1$, since bidder 1 (the weak bidder) bids more aggressively than 2 (the strong bidder) for a given valuation. However, (12) also implies that the ex ante equilibrium bid distribution of 2 first order stochastically dominates the ex ante bid distribution of 1, which is equivalent to $k^F(v) \leq r(v)$. Since $\beta_1 = \beta_2$, the expected revenue is given by the expected virtual valuation of the winning bidder, and the FPA dominates the SPA if its inefficient allocation increases the expected virtual valuation of the winner. Condition (13) guarantees that this is the case, which establishes the result.[26]

The assumptions in Kirkegaard (2011) obviously rule out our discrete setting, but given the c.d.f.

$$\tilde{F}_1(v_1) = \begin{cases} 0 & \text{if} \quad v_1 < v_{1L} \\ \lambda & \text{if} \quad v_{1L} \leq v_1 < v_{1H} \\ 1 & \text{if} \quad v_{1H} \leq v_1 \end{cases}, \qquad \tilde{F}_2(v_2) = \begin{cases} 0 & \text{if} \quad v_2 < v_{2L} \\ \lambda & \text{if} \quad v_{2L} \leq v_2 < v_{2H} \\ 1 & \text{if} \quad v_{2H} \leq v_2 \end{cases}$$

for v_1, v_2 in our model, we can approximate $\tilde{F}_1, \tilde{F}_2$ using atomless c.d.f.[27] Precisely, consider two sequences of atomless c.d.f. $\{F_1^n, F_2^n\}_{n=1}^{+\infty}$, with continuous and positive densities f_1^n, f_2^n for each n, which converges weakly to $\tilde{F}_1, \tilde{F}_2$. We prove in the appendix that for any large n, (12) and/or (13) are violated by F_1^n, F_2^n.

4.2 The case in which $v_{1H} = v_{2H}$

In this subsection we remove the assumption $\lambda_1 = \lambda_2$ but we suppose that $v_{1H} = v_{2H}$. Then a very simple result holds, as stated by next proposition.

[25] Condition (12) is a standard condition of dominance in terms of reverse hazard rates. On the other hand, (13) is innovative and Kirkegaard (2011) proves that it implies that $r(v) - v$ is increasing, which means that F_2 is more disperse than F_1 according to a specific order of dispersion between c.d.f. Moreover, Kirkegaard (2011) gives an economic interpretation to (13) linked to the relative steepness of the demand function of bidder 1 with respect to the demand function of bidder 2.

[26] Kirkegaard (2011) shows that Theorem 1 holds also when there are several bidders with c.d.f. F_1, and sometimes when there are several bidders with c.d.f. F_2.

[27] Lebrun (2002) establishes that the equilibrium correspondence is upper hemicontinuous with respect to the valuation distributions, for the weak topology. Given that all BNE are outcome-equivalent at each given information structure, it follows that the equilibrium correspondence is in fact continuous. Therefore also R^F is continuous, as it is the expectation of a continuous function of bids (the maximum).

Proposition 4 *Suppose that $v_{1H} = v_{2H}$. Then the inequality $R^S > R^F$ holds as long as $v_{1L} < v_{2L}$ and/or $\lambda_1 \neq \lambda_2$.*

In a sense, Proposition 4 is quite intuitive since we know that $R^S > R^F$ when $v_{1H} = v_{2H}$ if (i) $v_{1L} < v_{2L}$ and $\lambda_1 = \lambda_2$ [from Proposition 2(ii)], or (ii) $v_{1L} = v_{2L}$ and $\lambda_1 \neq \lambda_2$ [from Maskin and Riley (1983)]. Proposition 4 essentially verifies that $R^S > R^F$ still holds if both inequalities $v_{1L} < v_{2L}$ and $\lambda_1 \neq \lambda_2$ hold. Precisely, when $v_{1H} = v_{2H}$ condition (3) is violated and (7) reduces to $\lambda_1 \geq \lambda_2$; therefore Proposition 1(iii) applies if $\lambda_1 \geq \lambda_2$, and Proposition 1(ii) applies if $\lambda_1 < \lambda_2$. In both cases the equality $v_{1H} = v_{2H}$ yields a simple expression for R^F and it follows immediately that $R^S > R^F$.[28]

The simplest way to see why $R^S > R^F$ when $v_{1H} = v_{2H}$ consists in arguing as in Subsection 4.1.3, and proving that the bidders' rents are larger in the FPA than in the SPA. In fact, in the proof to Proposition 4 we show that bidder 1 (bidder 2) strictly (weakly) prefers the FPA to the SPA since (i) 1_H earns zero in the SPA when facing 2_H, earns $v_{1H} - v_{2L}$ against 2_L; (ii) 1_H can beat 2_L in the FPA by bidding v_{1L} or $\hat{b}$ (depending on whether $\lambda_1 \geq \lambda_2$ or $\lambda_1 < \lambda_2$), and both v_{1L} and $\hat{b}$ are smaller than v_{2L}. Likewise, the payoff of bidder 2 in the SPA is zero against 1_H, is $v_2 - v_{1L}$ against 1_L. The FPA is certainly not worse for 2 as he can beat 1_L by bidding v_{1L}.[29]

We conclude with a remark on the equilibrium bid distributions of the two bidders, $G_1(b) = \lambda_1 G_{1L}(b) + (1-\lambda_1)G_{1H}(b)$ and $G_2(b) = \lambda_2 G_{2L}(b) + (1-\lambda_2)G_{2H}(b)$, which are obtained from Proposition 1. For the case in which $\lambda_1 \geq \lambda_2$, we find that $G_1(b) = G_2(b) = \lambda_1 \frac{v_H - v_{1L}}{v_H - b}$ for each $b \in [v_{1L}, \lambda_1 v_{1L} + (1-\lambda_1)v_H]$, and therefore each bidder faces the same distribution of bids from his opponent even though the distribution of v_2 first order stochastically dominates the distribution of v_1. This occurs because $v_{1H} = v_{2H}$ implies that the payoffs of 1_H and 2_H are the same, hence either type needs to have the same probability of winning for any given bid.[30]

5 Appendix

5.1 Proof of Proposition 1 for the case of $v_{1L} < v_{2L}$

For $i = 1, 2$ and $j = L, H$, let G_{ij} denote the c.d.f. for the mixed strategy of type j of bidder i, with $\underline{b}_{ij} = \inf\{b : G_{ij}(b) > 0\}$ and $\bar{b}_{ij} = \sup\{b : G_{ij}(b) < 1\}$. Recall that in a mixed-strategy BNE any bid made by type i_j must generate the same expected payoff, that is the equilibrium payoff of type i_j, which we denote by u^e_{ij}. We use $u_{ij}(b)$ and $p_{ij}(b)$ to denote the payoff of type i_j and his

[28] In fact, if $\lambda_1 \geq \lambda_2$ then the FPA allocates the object efficiently since $v_{1H} = v_{2H} > v_{2L}$ and 1_H always wins against 2_L. This further simplifies the expression of R^F.

[29] Here the bidders have the same preferences between the FPA and the SPA, whereas under the assumptions on Maskin and Riley (2000a) that is never the case.

[30] When instead $\lambda_1 < \lambda_2$, we obtain $G_1(b) = \lambda_1 \frac{v_{2L} - v_{1L}}{v_{2L} - b} < G_2(b) = \lambda_2 \frac{v_H - \hat{b}}{v_H - b}$ for $b \in [v_{1L}, \hat{b})$ and $G_1(b) = G_2(b) = \frac{v_H - \bar{b}}{v_H - b}$ for $b \in [\hat{b}, \bar{b}]$; hence G_1 first order stochastically dominates G_2. Since now 2_L randomizes over the interval $[v_{1L}, \hat{b}]$ (rather than bidding v_{1L} with certainty), G_1 needs to make 2_L indifferent among bids in $[v_{1L}, \hat{b}]$ and thus $G_1(b) = G_2(b)$ does not hold for $b \in [v_{1L}, \hat{b})$.

probability to win – respectively – as a function of his bid b, given the strategies of the two types of the other bidder.

This proof is organized in several steps, and throughout the proof ε denotes a number which is positive and close to zero. We start by recording a feature of any BNE.

Lemma 1 *If a profile of strategies has the property that there is a bid b' such that with a positive probability type 1_j and type 2_k tie bidding b' and $\min\{v_{1j}, v_{2k}\} > b'$, then the profile of strategies is not a BNE.*

Proof. By bidding b', at least one of these types loses the auction with positive probability; for instance type 1_j. Since $b' < v_{1j}$, type 1_j is better off bidding $b' + \varepsilon$ rather than b' as in this way his probability of winning increases discretely, whereas his payment in case of victory increases only slightly. ■

5.1.1 Step 1: When $v_{1L} < v_{2L}$, any BNE is such that (i) $\bar{b}_{1L} \leq \underline{b}_{1H}$, $\bar{b}_{2L} \leq \underline{b}_{2H}$; (ii) either $\underline{b}_{1L} = \underline{b}_{2L} = v_{1L} = \bar{b}_{1L}$ or $\underline{b}_{1L} < \underline{b}_{2L}$; (iii) $u^e_{1L} = 0$, $u^e_{2L} > 0$, $v_{1L} \leq \underline{b}_{2L}$; (iv) $\bar{b}_{1H} = \bar{b}_{2H}$

(i) The monotonicity properties $\bar{b}_{1L} \leq \underline{b}_{1H}$ and $\bar{b}_{2L} \leq \underline{b}_{2H}$ follow from Proposition 1 in Maskin and Riley (2000b).

(ii) In order to prove that $\underline{b}_{1L} \leq \underline{b}_{2L}$, suppose in view of a contradiction that $\underline{b}_{2L} < \underline{b}_{1L}$. Since 2_L bids in the interval $[\underline{b}_{2L}, \underline{b}_{1L})$ with positive probability, it follows that $u^e_{2L} = 0$. However, since $\underline{b}_{1L} \leq v_{1L} < v_{2L}$ we find that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ if 2_L bids $b = \underline{b}_{1L} + \varepsilon$: contradiction.

We now show that if $\underline{b}_{1L} = \underline{b}_{2L} \equiv \underline{b}$, then $\underline{b} = v_{1L}$, and as a consequence we obtain $\bar{b}_{1L} = v_{1L}$. Suppose $\underline{b} < v_{1L}$. We distinguish several cases depending on whether G_{1L} and/or G_{2L} puts an atom on $\underline{b}$; in each case we obtain a contradiction.

- $G_{1L}(\underline{b}) = 0$ [$G_{2L}(\underline{b}) = 0$ or $G_{2L}(\underline{b}) > 0$ does not matter]. In this case $u^e_{2L} = 0$ as $p_{2L}(b)$ is about zero for b close to $\underline{b}$ (as G_{1L} is right continuous). However, since $\underline{b} < v_{1L} < v_{2L}$ we find that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ if 2_L bids $b = \underline{b} + \varepsilon$.

- $G_{1L}(\underline{b}) > 0$ and $G_{2L}(\underline{b}) > 0$. This case is ruled out by Lemma 1.

- $G_{1L}(\underline{b}) > 0$ and $G_{2L}(\underline{b}) = 0$. In this case $u^e_{1L} = 0$ as $p_{1L}(\underline{b}) = 0$. However, since $\underline{b} < v_{1L}$ we find that $p_{1L}(b) > 0$ and $u_{1L}(b) > 0$ if 1_L bids $b = \underline{b} + \varepsilon < v_{1L}$.

(iii) We notice that $u^e_{1L} = 0$ both if $\underline{b}_{1L} = \underline{b}_{2L} = \bar{b}_{1L} = v_{1L}$ and if $\underline{b}_{1L} < \underline{b}_{2L}$. Hence $v_{1L} \leq \underline{b}_{2L}$, since if $\underline{b}_{2L} < v_{1L}$ then any bid in $(\underline{b}_{2L}, v_{1L})$ yields a positive payoff to 1_L. Finally, $p_{2L}(b) \geq \lambda_1$ for any $b \geq v_{1L} + \varepsilon$, thus $u^e_{2L} \geq \lambda_1(v_{2L} - v_{1L} - \varepsilon) > 0$ for each small $\varepsilon > 0$.

(iv) If $\bar{b}_{1H} > \bar{b}_{2H}$, then it is profitable for 1_H to move some probability from $(\bar{b}_{1H} - \varepsilon, \bar{b}_{1H}]$ to $(\bar{b}_{2H}, \bar{b}_{2H} + \varepsilon)$, since the probability of winning remains 1 but his payment in case of victory is smaller. If $\bar{b}_{1H} < \bar{b}_{2H}$, a symmetric argument applies to 2_H.

5.1.2 Step 2: When $v_{1L} < v_{2L}$, there exists a BNE such that $\bar{b}_{1H} \leq \underline{b}_{2L}$ if and only if (3) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(i)

We start by proving that $\underline{b}_{1L} < \underline{b}_{2L}$. Suppose in view of a contradiction that $\underline{b}_{1L} = \underline{b}_{2L}$. Then Step 1(i-ii) imply $\underline{b}_{1L} = \bar{b}_{1L} = \underline{b}_{1H} = \bar{b}_{1H} = \underline{b}_{2L} = v_{1L}$. It is impossible that $G_{2L}(v_{1L}) > 0$, because in such a case 1_H and 2_L would tie with positive probability at $b = v_{1L}$, and then Lemma 1 would apply. As a consequence, $p_{1H}(v_{1L}) = 0$ and $u^e_{1H} = 0$. However, if 1_H plays $b = v_{1L} + \varepsilon$ then $p_{1H}(b) > 0$ and $u_{1H}(b) > 0$ since $v_{1L} < v_{1H}$: contradiction.

From the inequality $\bar{b}_{1H} \leq \underline{b}_{2L}$ it follows that 2_L wins with probability one;[31] thus $u^e_{1H} = 0$. Moreover, (i) $\bar{b}_{1H} = \bar{b}_{2H}$ by Step 1(iv) and thus $\bar{b}_{1H} = \underline{b}_{2L} = \bar{b}_{2L} = \underline{b}_{2H} = \bar{b}_{2H}$; (ii) $v_{1H} \leq \underline{b}_{2L}$ otherwise any bid in $(\underline{b}_{2L}, v_{1H})$ yields a positive payoff to 1_H. Hence, $u^e_{2L} = v_{2L} - \underline{b}_{2L}$ and $u^e_{2H} = v_{2H} - \underline{b}_{2L}$.

We need to examine the incentives of bidder 2 to bid below $\underline{b}_{2L}$, and in particular we notice that bidding $b = \bar{b}_{1L} + \varepsilon$ yields bidder 2 a probability of winning not smaller than λ_1. Thus the inequalities

$$\lambda_1(v_{2L} - \bar{b}_{1L} - \varepsilon) \leq v_{2L} - \underline{b}_{2L} \qquad \text{and} \qquad \lambda_1(v_{2H} - \bar{b}_{1L} - \varepsilon) \leq v_{2H} - \underline{b}_{2L}$$

need to hold for any $\varepsilon > 0$, and since $v_{2H} > v_{2L}$ it is simple to see that the first inequality is more restrictive than the second one. Given $\bar{b}_{1L} \leq v_{1L}$ and $\underline{b}_{2L} \geq v_{1H}$, the first inequality is most likely to be satisfied when $\bar{b}_{1L} = v_{1L}$ and $\underline{b}_{2L} = v_{1H}$, and then it reduces to (3). This inequality is therefore a necessary condition for the existence of a BNE such that $\bar{b}_{1H} \leq \underline{b}_{2L}$.

Bids above v_{1H} are obviously suboptimal for bidder 2 because $u_{2L}(b) = v_{2L} - b < v_{2L} - v_{1H}$ if $b > v_{1H}$. On the other hand, for bids smaller than v_{1H} the strategies of 1_L and 1_H need to be such that no $b < v_{1H}$ is a profitable deviation for type 2_L.[32] For instance, we verify that this condition is satisfied if G_{1H} is the uniform distribution over $[\alpha v_{1H}, v_{1H}]$, with $\alpha < 1$ and close to 1; recall that 1_L bids v_{1L} with certainty. Then $p_{2L}(b) = 0$, $u_{2L}(b) = 0$ for $b < v_{1L}$, whereas $p_{2L}(v_{1L}) = \lambda_1$ (recall the Vickrey tie-breaking rule and $v_{2L} > v_{1L}$), $u_{2L}(v_{1L}) = \lambda_1(v_{2L} - v_{1L})$, but we know from (3) that this payoff is smaller than $v_{2L} - v_{1H}$, the payoff of 2_L if he bids v_{1H}. For $b \in (v_{1L}, \alpha v_{1H})$ we find that $u_{2L}(b) = \lambda_1(v_{2L} - b)$ is decreasing. Finally, for $b \in [\alpha v_{1H}, v_{1H}]$, $u_{2L}(b) = (v_{2L} - b)[\lambda_1 + (1-\lambda_1)\frac{b - \alpha v_{1H}}{v_{1H} - \alpha v_{1H}}]$ and is increasing for $\alpha > 1 - \frac{(1-\lambda)(v_{2L} - v_{1H})}{v_{1H}}$, which implies that $b = v_{1H}$ is a best reply for 2_L.

5.1.3 Step 3: When $v_{1L} < v_{2L}$, there exists no BNE such that $\underline{b}_{2L} < \bar{b}_{1H} \leq \bar{b}_{2L}$

If $\underline{b}_{2L} < \bar{b}_{1H} \leq \bar{b}_{2L}$, then $\underline{b}_{2L} < \bar{b}_{1H} = \bar{b}_{2L} = \underline{b}_{2H} = \bar{b}_{2H} \equiv b^*$ by Step 1(iv). This implies $b^* \leq v_{1H}$, and thus $\underline{b}_{2L} < b^*$ implies $u^e_{1H} > 0$, and in turn $b^* < v_{1H}$. Since 2_H bids b^* with certainty, it is

[31] In particular, if $\bar{b}_{1H} = \underline{b}_{2L}$ and 1_H and 2_L tie with positive probability at $\underline{b}_{2L}$, then 2_L needs to win the tie-break with probability 1, otherwise it is profitable for him to bid $\underline{b}_{2L} + \varepsilon$ rather than $\underline{b}_{2L}$ ($\underline{b}_{2L} < v_{2L}$ since $u^e_{2L} > 0$).

[32] If this property is satisfied, then no deviation is profitable for 2_H since $(v_{2L} - b)p_{2L}(b) \leq v_{2L} - v_{1H}$ implies $(v_{2H} - b)p_{2H}(b) \leq v_{2H} - v_{1H}$, as $p_{2L}(b) = p_{2H}(b)$ for any b

profitable for 1_H to bid $b^* + \varepsilon$ rather than $b^* - \varepsilon$, as in this way his probability of victory increases by at least $1 - \lambda_2 > 0$ and his payment in case of victory increases only slightly.

5.1.4 Step 4: When $v_{1L} < v_{2L}$, there exists a BNE such that $\underline{b}_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$ if and only if (4) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(ii)

The inequality $\underline{b}_{2L} < \bar{b}_{1H}$ implies $u^e_{1H} > 0$ because $\bar{b}_{1H} \leq v_{1H}$ and $p_{1H}(b) > 0$ for $b \in (\underline{b}_{2L}, \bar{b}_{1H})$. Next lemma provides a list of features of any BNE such that $\underline{b}_{2L} < \bar{b}_{1H}$.

Lemma 2 *In any BNE such that $\underline{b}_{2L} < \bar{b}_{1H}$ the following equalities hold: $\bar{b}_{1L} = \underline{b}_{1H} = \underline{b}_{2L} = v_{1L}$, $\bar{b}_{2L} = \underline{b}_{2H}$; moreover, $G_{2L}(\underline{b}_{2L}) > 0$.*

Proof. The proof is split in two claims.

Claim 1 $\bar{b}_{1L} = \underline{b}_{1H}$.
In view of a contradiction, assume that $\bar{b}_{1L} < \underline{b}_{1H}$. If $G_{1H}(\underline{b}_{1H}) > 0$ and $G_{2L}(\underline{b}_{1H}) > 0$,[33] then Lemma 1 applies since $u^e_{1H} > 0$ and $u^e_{2L} > 0$ imply $v_{1H} > \underline{b}_{1H}$ and $v_{2L} > \underline{b}_{1H}$. If $G_{1H}(\underline{b}_{1H}) > 0$ and 2 puts no atom at $\underline{b}_{1H}$, then 2 bids with zero probability in $(\bar{b}_{1L} + \varepsilon, \underline{b}_{1H}]$ and 1_H can increase his payoff by moving the atom from $\underline{b}_{1H}$ to any point in $(\bar{b}_{1L} + \varepsilon, \underline{b}_{1H}]$. If $G_{1H}(\underline{b}_{1H}) = 0$, then 2 bids with zero probability in $(\bar{b}_{1L} + \varepsilon, \underline{b}_{1H}]$ (in particular, 2 puts no atom in $\underline{b}_{1H}$) and then 1_H can increase his payoff by moving some probability from $[\underline{b}_{1H}, \underline{b}_{1H} + \varepsilon)$ to $(\bar{b}_{1L} + \varepsilon, \bar{b}_{1L} + 2\varepsilon)$.

Claim 2 $\underline{b}_{1H} = \underline{b}_{2L} = v_{1L}$, $G_{2L}(v_{1L}) > 0$ and $\bar{b}_{2L} = \underline{b}_{2H}$.
If $\underline{b}_{1H} < \underline{b}_{2L}$, then 1_H bids in $[\underline{b}_{1H}, \underline{b}_{2L})$ with positive probability and thus $u^e_{1H} = 0$: contradiction. Thus $\underline{b}_{2L} \leq \underline{b}_{1H}$ and since $\bar{b}_{1L} \leq v_{1L}$, $v_{1L} \leq \underline{b}_{2L}$ [by Step 1(iii)] and $\bar{b}_{1L} = \underline{b}_{1H}$ (by Claim 1), we infer that $\bar{b}_{1L} = \underline{b}_{2L} = \underline{b}_{1H} = v_{1L}$. Moreover, given $\underline{b}_{1H} = \underline{b}_{2L}$, if $G_{2L}(\underline{b}_{2L}) = 0$ then $u^e_{1H} = 0$; thus $G_{2L}(\underline{b}_{2L}) > 0$. The equality $\bar{b}_{2L} = \underline{b}_{2H}$ is proved along the same lines followed in Claim 1 to prove $\bar{b}_{1L} = \underline{b}_{1H}$. ■

Lemma 3 *In any BNE such that $\underline{b}_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$, the mixed strategies of 1_H,2_L,2_H are given by (5)-(6), and they constitute a BNE if and only if (4) is satisfied.*

Proof. In the following of this proof we use $\hat{b}$ and $\bar{b}$, respectively, instead of $\bar{b}_{2L}$ and of $\bar{b}_{2H} = \bar{b}_{1H}$. Given that $v_{1L} < \hat{b}$, types $1_H, 2_L, 2_H$ are all employing mixed strategies and we can argue like in the proof of Claim 1 in Lemma 2 to show that G_{1H}, G_{2L}, G_{2H} are strictly increasing and continuous in the intervals $[v_{1L}, \bar{b}]$, $[v_{1L}, \hat{b}]$, $[\hat{b}, \bar{b}]$, respectively. This implies that the following conditions must be satisfied.
Indifference condition of type 1_H:

$$(v_{1H} - b)[\lambda_2 G_{2L}(b) + (1 - \lambda_2) G_{2H}(b)] = v_{1H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in (v_{1L}, \bar{b}] \tag{14}$$

[33] If we consider type 2_H instead of 2_L, the same the argument applies.

Indifference condition of type 2_L:

$$(v_{2L}-b)[\lambda_1+(1-\lambda_1)G_{1H}(b)]=\lambda_1(v_{2L}-v_{1L}) \qquad \text{for} \quad \text{any} \quad b\in[v_{1L},\hat{b}] \tag{15}$$

Indifference condition of type 2_H:

$$(v_{2H}-b)[\lambda_1+(1-\lambda_1)G_{1H}(b)]=v_{2H}-\bar{b} \qquad \text{for} \quad \text{any} \quad b\in[\hat{b},\bar{b}] \tag{16}$$

From (15) and (16) we obtain G_{1H} in (5). For $b\in(v_{1L},\hat{b}]$, (14) reduces to $(v_{1H}-b)\lambda_2 G_{2L}(b)=v_{1H}-\bar{b}$ and thus G_{2L} satisfies (6). For $b\in[\hat{b},\bar{b}]$, (14) reduces to $(v_{1H}-b)[\lambda_2+(1-\lambda_2)G_{2H}(b)]=v_{1H}-\bar{b}$ and then G_{2H} satisfies (6).

Since $G_{2L}(\hat{b})=1$, we deduce that $\bar{b}=\lambda_2\hat{b}+(1-\lambda_2)v_{1H}$, and since G_{1H} needs to be continuous at $b=\hat{b}$ we infer that $\hat{b}$ solves (2); here we use $Z(b)$ to denote the left hand side of (2). The strategies in Proposition 1(ii) require that $\hat{b}$ satisfies $v_{1L}<\hat{b}<\min\{v_{2L},v_{1H}\}$, and since $Z(v_{2L})=-\lambda_1(v_{2L}-v_{1L})(v_{2H}-v_{2L})<0$ we infer that $\hat{b}$ is the smaller solution of (2); moreover, $Z(v_{1L})=(1-\lambda_2)(v_{2L}-v_{1L})\left(\frac{(\lambda_1-\lambda_2)v_{1L}+(1-\lambda_1)v_{2H}}{1-\lambda_2}-v_{1H}\right)$ and thus $\frac{(\lambda_1-\lambda_2)v_{1L}+(1-\lambda_1)v_{2H}}{1-\lambda_2}>v_{1H}$ needs to hold. The inequality $\hat{b}<v_{1H}$ is obviously satisfied if $v_{2L}\leq v_{1H}$, while if $v_{1H}<v_{2L}$ then it is equivalent to $Z(v_{1H})<0$. Since $Z(v_{1H})=-[v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2L}](v_{2H}-v_{1H})$ and $v_{1H}<v_{2L}<v_{2H}$, we deduce that the converse of (3) needs to hold. Thus (4) is a necessary condition for the existence of a BNE such that $\underline{b}_{2L}<\bar{b}_{2L}<\bar{b}_{1H}$.

Now we verify that for each type of each bidder the strategy specified by Proposition 1(ii) is a best reply given the strategies of the two types of the other bidder. Notice that $p_{1H}(\bar{b})=p_{2H}(\bar{b})=1$, thus we do not need to consider bids above $\bar{b}$. The same remark applies to the BNE described by Proposition 1(iii).

Type 1_L. The strategies of types 2_L and 2_H are such that each type of bidder 2 bids at least v_{1L} with probability one. Therefore the payoff of 1_L is zero if he bids v_{1L} as specified by Proposition 1, and it is impossible for him to obtain a positive payoff.

Type 1_H. We know from (14) that the payoff of 1_H is $v_{1H}-\bar{b}>0$ for any $b\in(v_{1L},\bar{b}]$. If $b<v_{1L}$, then $p_{1H}(b)=0$ and $u_{1H}(b)=0$. If $b=v_{1L}$, then 1_H loses against 2_H and loses also against 2_L unless 2_L bids v_{1L}, in which case 1_H ties with 2_L – an event with probability $G_{2L}(v_{1L})$. Consider the most favorable case for 1_H, which means that he wins the tie-break against 2_L with probability one (this occurs if $v_{2L}<v_{1H}$): his expected payoff from bidding v_{1L} is then $(v_{1H}-v_{1L})\lambda_2 G_{2L}(v_{1L})$, which turns out to be equal to $v_{1H}-\bar{b}$.

Type 2_L. We know from (15) that the payoff of 2_L is $\lambda_1(v_{2L}-v_{1L})>0$ for any $b\in[v_{1L},\hat{b}]$. For bids smaller than v_{1L}, the payoff of 2_L is zero as $p_{2L}(b)=0$ if $b<v_{1L}$. If $b\in[\hat{b},\bar{b}]$, then $u_{2L}(b)=(v_{2L}-b)[\lambda_1+(1-\lambda_1)G_{1H}(b)]=(v_{2L}-b)\frac{v_{2H}-\bar{b}}{v_{2H}-b}$ which is decreasing in b, and therefore $u_{2L}(\hat{b})>u_{2L}(b)$ for any $b\in(\hat{b},\bar{b}]$.

Type 2_H. We know from (16) that the payoff of 2_H is $v_{2H}-\bar{b}>0$ for any $b\in[\hat{b},\bar{b}]$. For bids smaller than v_{1L}, the payoff of 2_H is zero as $p_{2H}(b)=0$ if $b<v_{1L}$. If $b\in[v_{1L},\hat{b}]$, then $p_{2H}(b)=\lambda_1+(1-\lambda_1)G_{1H}(b)=\lambda_1\frac{v_{2L}-v_{1L}}{v_{2L}-b}$ and $u_{2H}(b)=(v_{2H}-b)\lambda_1\frac{v_{2L}-v_{1L}}{v_{2L}-b}$, which is increasing in b and therefore $u_{2H}(b)<u_{2H}(\hat{b})$ for any $b\in[v_{1L},\hat{b})$. ■

5.1.5 Step 5: When $v_{1L} < v_{2L}$, there exists a BNE such that $\underline{b}_{2L} = \bar{b}_{2L} < \bar{b}_{1H}$ if and only if (7) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(iii)

In this case Lemma 2 (in the proof of Step 4) applies, thus we infer that $\bar{b}_{1L} = \underline{b}_{1H} = \underline{b}_{2L} = \bar{b}_{2L} = \underline{b}_{2H} = v_{1L}$; this means that 2_L plays a pure strategy and bids v_{1L}. Conversely, types 1_H and 2_H employ mixed strategies and thus the following indifference conditions need to hold, in which we still use $\bar{b}$ instead of $\bar{b}_{2H} = \bar{b}_{1H}$. For type 1_H:

$$(v_{1H} - b)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in (v_{1L}, \bar{b}] \tag{17}$$

For type 2_H:

$$(v_{2H} - b)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = v_{2H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in (v_{1L}, \bar{b}] \tag{18}$$

Notice that $G_{1H}(v_{1L}) = 0$ since if $G_{1H}(v_{1L}) > 0$, then 1_H ties with 2_L with positive probability by bidding v_{1L}, and thus Lemma 1 applies. From $G_{1H}(v_{1L}) = 0$ and (18) we obtain $\bar{b} = \lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}$, and then (17)-(18) yield G_{1H}, G_{2H} in (8). The inequality (7) needs to hold since it is equivalent to $G_{2H}(v_{1L}) \geq 0$.

Now we verify that for each type of each bidder the strategy specified by Proposition 1(iii) is a best reply given the strategies of the two types of the other bidder.

Type 1_L. The same argument given in the proof of Lemma 3 in Step 4 applies.

Type 1_H. We know from (17) that the payoff of 1_H is $v_{1H} - \bar{b} > 0$ for any $b \in (v_{1L}, \bar{b}]$,[34] and $b < v_{1L}$ implies $p_{1H}(b) = 0$, $u_{1H}(b) = 0$. If $b = v_{1L}$, then 1_H ties with type 2_L and loses against 2_H, unless also 2_H bids v_{1L} – an event with probability $G_{2H}(v_{1L})$. Consider the most favorable case for 1_H, which means that he wins the tie-break against each type of bidder 2 with probability one (this occurs if $v_{2H} < v_{1H}$): his expected payoff from bidding v_{1L} is then $(v_{1H} - v_{1L})[\lambda_2 + (1 - \lambda_2)G_{2H}(v_{1L})]$ which turns out to be equal to $v_{1H} - \bar{b}$.

Type 2_L. The payoff of 2_L is $\lambda_1(v_{2L} - v_{1L})$. For bids smaller than v_{1L} we can argue exactly like in the proof of Lemma 3 in Step 4. If $b \in [v_{1L}, \bar{b}]$, then $p_{2L}(b) = \lambda_1 \frac{v_{2H} - v_{1L}}{v_{2H} - b}$ and thus $u_{2L}(b) = (v_{2L} - b)\lambda_1 \frac{v_{2H} - v_{1L}}{v_{2H} - b}$ is decreasing in b.

Type 2_H. The payoff of 2_H is $v_{2H} - \bar{b} > 0$ for any $b \in [v_{1L}, \bar{b}]$. For bids smaller than v_{1L} we can argue exactly like in the proof of Lemma 3 in Step 4.

5.2 Proof of Proposition 1 for the case of $v_{1L} = v_{2L}$

5.2.1 Step 1: When $v_{1L} = v_{2L} = v_L$, any BNE is such that $\underline{b}_{1L} = \underline{b}_{2L} = \bar{b}_{1L} = \bar{b}_{2L} = v_L$

We start by proving that $\underline{b}_{1L} = \underline{b}_{2L}$. In view of a contradiction, suppose that $\underline{b}_{2L} < \underline{b}_{1L}$. Since 2_L bids in $[\underline{b}_{2L}, \underline{b}_{1L})$ with positive probability, it follows that $u_{2L}^e = 0$. Then $v_L \leq \underline{b}_{1L}$, since $\underline{b}_{1L} < v_L$ implies that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ for any $b \in (\underline{b}_{1L}, v_L)$. Moreover, $v_L \leq \underline{b}_{1L}$ implies $u_{1L}^e = 0$,

[34] Notice that $v_{1H} - \bar{b} > 0$ given (7).

but $p_{1L}(b) > 0$ and $u_{1L}(b) > 0$ for any $b \in (\underline{b}_{2L}, \underline{b}_{1L})$: contradiction. Therefore the inequality $\underline{b}_{2L} < \underline{b}_{1L}$ cannot hold in equilibrium, and a similar argument applies to rule out $\underline{b}_{1L} < \underline{b}_{2L}$.

Given that $\underline{b}_{1L} = \underline{b}_{2L} \equiv \underline{b}$, we prove that $\underline{b} = v_L$. In view of a contradiction, suppose that $\underline{b} < v_L$. In case that $G_{1L}(\underline{b}) > 0$ and $G_{2L}(\underline{b}) > 0$, Lemma 1 applies; thus $G_{1L}(\underline{b}) = 0$ and/or $G_{2L}(\underline{b}) = 0$. If $G_{1L}(\underline{b}) = 0$, we find that $u^e_{2L} = 0$ since $p_{2L}(b)$ is about 0 for b close to $\underline{b}$, but in fact 2_L can make a positive payoff by bidding in $(\underline{b}, v_L)$: contradiction. The same argument applies if $G_{2L}(\underline{b}) = 0$. Thus $\underline{b} = v_L$, which implies $\bar{b}_{1L} = \bar{b}_{2L} = v_L$: hence both 1_L and 2_L bid v_L with probability one.

5.2.2 Step 2: When $v_{1L} = v_{2L} = v_L$, in the unique BNE 1_H,2_H play the mixed strategies described by Proposition 1(iii) if (7) holds; if (7) is violated, then 1_H,2_H play the mixed strategies described by (5) and (6) with $\hat{b} = v_L$

As in the proof of Proposition 1(ii) (Lemma 2 in Step 4) we can prove that $\bar{b}_{1L} = \underline{b}_{1H} (= v_L)$ and $\bar{b}_{2L} = \underline{b}_{2H} (= v_L)$. Using again $\bar{b}$ instead of $\bar{b}_{1H}, \bar{b}_{2H}$ we infer that G_{1H}, G_{2H} need to satisfy

$$(v_{1H} - b)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in [v_L, \bar{b}] \tag{19}$$

and

$$(v_{2H} - b)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = v_{2H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in [v_L, \bar{b}] \tag{20}$$

From (19)-(20) we obtain $G_{1H}(v_L) = \frac{1}{1-\lambda_1}(\frac{v_{2H}-\bar{b}}{v_{2H}-v_L} - \lambda_1)$ and $G_{2H}(v_L) = \frac{1}{1-\lambda_2}(\frac{v_{1H}-\bar{b}}{v_{1H}-v_L} - \lambda_2)$. Lemma 1 implies that $G_{1H}(v_L) > 0$ and $G_{2H}(v_L) > 0$ cannot hold. Thus we consider the other cases.

If $G_{1H}(v_L) > 0 = G_{2H}(v_L)$ we obtain $\bar{b} = \lambda_2 v_L + (1 - \lambda_2)v_{1H}$ and $G_{1H}(v_L) > 0$ is equivalent to the converse of (7); from (19)-(20) we obtain G_{1H}, G_{2H} as in footnote 14.[35] Now we prove that no profitable deviation exists for any type. The payoff of 1_L (2_L) is zero and he needs to bid above v_L in order to win. For 1_H, we know from (19) that his payoff is $v_{1H} - \bar{b}$ for any $b \in [v_L, \bar{b}]$ and $b < v_L$ yields $u_{1H}(b) = 0$. A similar argument applies to 2_H.

In case that $G_{2H}(v_L) \geq 0 = G_{1H}(v_L)$ we obtain $\bar{b} = \lambda_1 v_L + (1 - \lambda_1)v_{2H}$, and $G_{2H}(v_{1L}) \geq 0$ is equivalent to (7); from (19)-(20) we obtain G_{1H}, G_{2H} as in (8). The proof that no profitable deviation exists for any type is exactly as when (7) is violated.

[35] Step 1 and the proof of Step 2 up to this point apply for any tie-breaking rule. However, no BNE exists under the standard tie-breaking rule if (7) is violated since (i) $G_{1H}(v_L) > 0$ and 1_H and 2_L tie with positive probability at the bid v_L; (ii) it is profitable for 1_H to bid $v_L + \varepsilon$ rather than v_L, which breaks the BNE [a similar argument applies if (7) holds with strict inequality]. On the other hand, with the Vickrey tie-breaking rule we have $c_{1H} = v_{1H} - v_L > 0$ and $c_{2L} = 0$; thus 1_H wins (paying v_L as aggregate price) in case of tie with 2_L.

5.3 Derivation of R^F given the BNE described by Proposition 1

5.3.1 The BNE of Proposition 1(ii) when $v_{1L} < v_{2L}$

We evaluate R^F as the difference between the social surplus S^F generated by the FPA minus the bidders' rents U^F: $R^F = S^F - U^F$. Thus we need to derive S^F and U^F:

$$\begin{aligned} S^F &= \lambda_1\lambda_2 v_{2L} + \lambda_1(1-\lambda_2)v_{2H} + (1-\lambda_1)\lambda_2[v_{2L} + (v_{1H} - v_{2L})\Pr\{1_H \text{ def } 2_L\}] \\ &\quad +(1-\lambda_1)(1-\lambda_2)[v_{2H} + (v_{1H} - v_{2H})\Pr\{1_H \text{ def } 2_H\}] \end{aligned}$$

and

$$U^F = (1-\lambda_1)(v_{1H} - \lambda_2\hat{b} - (1-\lambda_2)v_{1H}) + (1-\lambda_2)(v_{2H} - \lambda_2\hat{b} - (1-\lambda_2)v_{1H}) + \lambda_2\lambda_1(v_{2L} - v_{1L})$$

in which $\Pr\{1_H \text{ def } 2_j\}$, for $j = L, H$, is the probability that 1_H wins when he faces type 2_j. Therefore

$$\begin{aligned} R^F &= \lambda_2(2-\lambda_1-\lambda_2)\hat{b} + (1+\lambda_2^2+\lambda_1\lambda_2 - 3\lambda_2)v_{1H} + \lambda_2(1-\lambda_1)v_{2L} + \lambda_2\lambda_1 v_{1L} \\ &\quad +(1-\lambda_1)\lambda_2(v_{1H} - v_{2L})\Pr\{1_H \text{ def } 2_L\} + (1-\lambda_1)(1-\lambda_2)(v_{1H} - v_{2H})\Pr\{1_H \text{ def } 2_H\} \end{aligned}$$

Derivation of $\Pr\{1_H$ def $2_L\}$ For the case that $v_{1H} \neq v_{2L}$ we need to evaluate

$$\Pr\{1_H \text{ def } 2_L\} = \int_{v_{1L}}^{\hat{b}} G'_{1H}(b)G_{2L}(b)db + 1 - G_{1H}(\hat{b})$$

and using $\bar{b} = \lambda_2\hat{b} + (1-\lambda_2)v_{1H}$ in G_{2L} we find $G_{2L}(b) = \frac{v_{1H}-\hat{b}}{v_{1H}-b}$:

$$\begin{aligned} \Pr\{1_H \text{ def } 2_L\} &= \int_{v_{1L}}^{\hat{b}} \frac{\lambda_1}{1-\lambda_1}\frac{v_{2L}-v_{1L}}{(v_{2L}-b)^2}\frac{v_{1H}-\hat{b}}{v_{1H}-b}db + 1 - \frac{\lambda_1(\hat{b}-v_{1L})}{(1-\lambda_1)(v_{2L}-\hat{b})} \\ &= \frac{\lambda_1(v_{2L}-v_{1L})(v_{1H}-\hat{b})}{1-\lambda_1}\int_{v_{1L}}^{\hat{b}} \frac{1}{(v_{2L}-b)^2(v_{1H}-b)}db + 1 - \frac{\lambda_1(\hat{b}-v_{1L})}{(1-\lambda_1)(v_{2L}-\hat{b})} \end{aligned}$$

We exploit

$$\int \frac{1}{(v_{2L}-b)^2(v_{1H}-b)}db = \frac{1}{(v_{1H}-v_{2L})^2}\ln\frac{v_{2L}-b}{v_{1H}-b} + \frac{1}{(v_{1H}-v_{2L})(v_{2L}-b)}$$

to obtain

$$\int_{v_{1L}}^{\hat{b}} \frac{1}{(v_{2L}-b)^2(v_{1H}-b)}db = \frac{1}{(v_{1H}-v_{2L})^2}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} + \frac{\hat{b}-v_{1L}}{(v_{1H}-v_{2L})(v_{2L}-\hat{b})(v_{2L}-v_{1L})}$$

thus

$$\Pr\{1_H \text{ def } 2_L\} = \frac{\lambda_1(v_{1H}-\hat{b})(v_{2L}-v_{1L})}{(1-\lambda_1)(v_{1H}-v_{2L})^2}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} + \frac{(1-\lambda_1)(v_{1H}-v_{2L}) + \lambda_1(\hat{b}-v_{1L})}{(1-\lambda_1)(v_{1H}-v_{2L})}$$

and

$$\begin{aligned} (1-\lambda_1)\lambda_2(v_{1H}-v_{2L})\Pr\{1_H \text{ def } 2_L\} &= \frac{\lambda_1\lambda_2(v_{1H}-\hat{b})(v_{2L}-v_{1L})}{v_{1H}-v_{2L}}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} \\ &\quad +\lambda_2(1-\lambda_1)(v_{1H}-v_{2L}) + \lambda_1\lambda_2(\hat{b}-v_{1L}) \end{aligned}$$

Derivation of $\Pr\{1_H$ **def** $2_H\}$ For the case that $v_{1H} \neq v_{2H}$ we need to evaluate

$$\Pr\{1_H \text{ def } 2_H\} = \int_{\hat{b}}^{\bar{b}} G'_{1H}(b) G_{2H}(b) db$$

and using $\bar{b} = \lambda_2 \hat{b} + (1-\lambda_2) v_{1H}$ in G_{2H} we find $G_{2H}(b) = \frac{\lambda_2(b-\hat{b})}{(1-\lambda_2)(v_{1H}-b)}$:

$$\begin{aligned}
\Pr\{1_H \text{ def } 2_H\} &= \int_{\hat{b}}^{\bar{b}} \frac{v_{2H}-\bar{b}}{(1-\lambda_1)(v_{2H}-b)^2} \frac{\lambda_2(b-\hat{b})}{(1-\lambda_2)(v_{1H}-b)} db \\
&= \frac{\lambda_2(v_{2H}-\bar{b})}{(1-\lambda_1)(1-\lambda_2)} \int_{\hat{b}}^{\bar{b}} \frac{b-\hat{b}}{(v_{1H}-b)(v_{2H}-b)^2} db
\end{aligned}$$

We exploit

$$\int \frac{b-\hat{b}}{(v_{1H}-b)(v_{2H}-b)^2} db = \frac{v_{1H}-\hat{b}}{(v_{1H}-v_{2H})^2} \ln \frac{v_{2H}-b}{v_{1H}-b} - \frac{v_{2H}-\hat{b}}{(v_{2H}-v_{1H})(v_{2H}-b)}$$

to obtain

$$\begin{aligned}
\int_{\hat{b}}^{\bar{b}} \frac{b-\hat{b}}{(v_{1H}-b)(v_{2H}-b)^2} db &= \frac{v_{1H}-\hat{b}}{(v_{1H}-v_{2H})^2} \ln \frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})} \\
&\quad - \frac{(1-\lambda_2)(v_{1H}-\hat{b})}{(v_{2H}-v_{1H})\left(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}\right)}
\end{aligned}$$

thus

$$\begin{aligned}
\Pr\{1_H \text{ def } 2_H\} &= \frac{\lambda_2(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H})}{(1-\lambda_1)(1-\lambda_2)} \frac{v_{1H}-\hat{b}}{(v_{1H}-v_{2H})^2} \ln \frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})} \\
&\quad - \frac{\lambda_2(v_{1H}-\hat{b})}{(1-\lambda_1)(v_{2H}-v_{1H})}
\end{aligned}$$

and

$$\begin{aligned}
&(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H}) \Pr\{1_H \text{ def } 2_H\} \\
= &\frac{\lambda_2(v_{1H}-\hat{b})(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H})}{v_{1H}-v_{2H}} \ln \frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})} \\
&+(1-\lambda_2)\lambda_2(v_{1H}-\hat{b})
\end{aligned}$$

Evaluation of R^F

$$\begin{aligned}
R^F &= \lambda_2\hat{b} + (1-\lambda_2)v_{1H} + \frac{\lambda_1\lambda_2(v_{1H}-\hat{b})(v_{2L}-v_{1L})}{v_{1H}-v_{2L}} \ln \frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} \\
&\quad + \frac{\lambda_2(v_{1H}-\hat{b})(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H})}{v_{1H}-v_{2H}} \ln \frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})}
\end{aligned}$$

An expression for $\hat{b}$ is found by solving (2):

$$\hat{b} = \frac{1}{2\lambda_2}\left(v_{2H} + \lambda_1 v_{1L} - (1-\lambda_2)v_{1H} + (\lambda_2 - \lambda_1)v_{2L} - Q\right)$$

with

$$Q = \sqrt{((1-\lambda_2)v_{1H} + (\lambda_1 - \lambda_2)\, v_{2L} - \lambda_1 v_{1L} - v_{2H})^2 - 4\lambda_2(((1-\lambda_1)v_{2H} - (1-\lambda_2)v_{1H})v_{2L} + \lambda_1 v_{1L} v_{2H})}$$

5.3.2 The BNE of Proposition 1(ii) when $v_{1L} = v_{2L}$ (footnote 14)

$$\begin{aligned} S^F &= \lambda_1\lambda_2 v_{1L} + \lambda_1(1-\lambda_2)v_{2H} + \lambda_2(1-\lambda_1)v_{1H} + (1-\lambda_1)(1-\lambda_2)(v_{1H} + (v_{2H} - v_{1H})\Pr\{2_H \text{ def } 1_H\}) \\ U^F &= (1-\lambda_1)(v_{1H} - \lambda_2 v_L - (1-\lambda_2)v_{1H}) + (1-\lambda_2)(v_{2H} - \lambda_2 v_L - (1-\lambda_2)v_{1H}) \end{aligned}$$

Therefore

$$\begin{aligned} R^F &= \lambda_2\,(2-\lambda_2)\,v_{1L} - (1-\lambda_1)\,(1-\lambda_2)\,v_{2H} + (2-\lambda_1-\lambda_2)\,(1-\lambda_2)\,v_{1H} \\ &\quad + (1-\lambda_1)(1-\lambda_2)(v_{2H} - v_{1H})\Pr\{2_H \text{ def } 1_H\} \end{aligned}$$

Derivation of $\Pr\{2_H$ def $1_H\}$ For the case that $v_{1H} \neq v_{2H}$ we need to evaluate

$$\begin{aligned} \Pr\{2_H \text{ def } 1_H\} &= \int_{v_{1L}}^{\lambda_2 v_{1L} + (1-\lambda_2)v_{1H}} G'_{2H}(b) G_{1H}(b)\, db \\ &= \int_{v_{1L}}^{\lambda_2 v_{1L} + (1-\lambda_2)v_{1H}} \frac{\lambda_2}{1-\lambda_2}\frac{v_{1H} - v_{1L}}{(v_{1H} - b)^2}\frac{1}{1-\lambda_1}\left(\frac{v_{2H} - \lambda_2 v_{1L} - (1-\lambda_2)v_{1H}}{v_{2H} - b} - \lambda_1\right) db \\ &= \frac{\lambda_2(v_{1H} - v_{1L})}{(1-\lambda_2)(1-\lambda_1)} \int_{v_{1L}}^{\lambda_2 v_{1L} + (1-\lambda_2)v_{1H}} \left(\frac{v_{2H} - \lambda_2 v_{1L} - (1-\lambda_2)v_{1H}}{(v_{2H} - b)(v_{1H} - b)^2} - \frac{\lambda_1}{(v_{1H} - b)^2}\right) db \end{aligned}$$

We exploit

$$\int \frac{1}{(v_{2H} - b)(v_{1H} - b)^2} db = \frac{1}{(v_{2H} - v_{1H})^2} \ln\frac{v_{1H} - b}{v_{2H} - b} + \frac{1}{(v_{2H} - v_{1H})(v_{1H} - b)}$$

to obtain

$$\begin{aligned} &\int_{v_{1L}}^{\lambda_2 v_{1L} + (1-\lambda_2)v_{1H}} \frac{v_{2H} - \lambda_2 v_{1L} - (1-\lambda_2)v_{1H}}{(v_{2H} - b)(v_{1H} - b)^2} db \\ &= \frac{v_{2H} - \lambda_2 v_{1L} - (1-\lambda_2)v_{1H}}{(v_{2H} - v_{1H})^2} \ln \frac{\lambda_2\,(v_{2H} - v_{1L})}{v_{2H} - \lambda_2 v_{1L} - (1-\lambda_2)v_{1H}} \\ &\quad + \frac{(1-\lambda_2)(v_{2H} - \lambda_2 v_{1L} - (1-\lambda_2)v_{1H})}{\lambda_2\,(v_{2H} - v_{1H})\,(v_{1H} - v_{1L})} \end{aligned}$$

Moreover,

$$\int_{v_{1L}}^{\lambda_2 v_{1L} + (1-\lambda_2)v_{1H}} \frac{\lambda_1}{(v_{1H} - b)^2} db = \frac{\lambda_1(1-\lambda_2)}{\lambda_2\,(v_{1H} - v_{1L})}$$

thus

$$\begin{aligned}\Pr\{2_H \text{ def } 1_H\} &= \frac{\lambda_2(v_{1H}-v_{1L})(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})}{(1-\lambda_2)(1-\lambda_1)\left(v_{2H}-v_{1H}\right)^2}\ln\frac{\lambda_2\left(v_{2H}-v_{1L}\right)}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}} \\ &\quad +\frac{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}{(1-\lambda_1)\left(v_{2H}-v_{1H}\right)}-\frac{\lambda_1}{1-\lambda_1}\end{aligned}$$

and

$$\begin{aligned}&(1-\lambda_1)(1-\lambda_2)(v_{2H}-v_{1H})\Pr\{2_H \text{ def } 1_H\} \\ =\ &\frac{(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})\lambda_2(v_{1H}-v_{1L})}{v_{2H}-v_{1H}}\ln\frac{\lambda_2\left(v_{2H}-v_{1L}\right)}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}} \\ &+(1-\lambda_2)\left((\lambda_2+\lambda_1-1)\,v_{1H}+(1-\lambda_1)v_{2H}-\lambda_2 v_{1L}\right)\end{aligned}$$

Evaluation of R^F

$$R^F=\lambda_2 v_{1L}+(1-\lambda_2)v_{1H}+\frac{(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})\lambda_2(v_{1H}-v_{1L})}{v_{2H}-v_{1H}}\ln\frac{\lambda_2\left(v_{2H}-v_{1L}\right)}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}$$

5.3.3 The BNE in Proposition 1(iii)

$$\begin{aligned}S^F &= \lambda_1\lambda_2 v_{2L}+\lambda_1(1-\lambda_2)v_{2H}+\lambda_2(1-\lambda_1)v_{1H}+(1-\lambda_1)(1-\lambda_2)(v_{2H}+(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\}) \\ U^F &= (1-\lambda_1)(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})+(1-\lambda_2)(v_{2H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})+\lambda_2\lambda_1(v_{2L}-v_{1L})\end{aligned}$$

Therefore

$$\begin{aligned}R^F &= \lambda_1(2-\lambda_1)v_{1L}-(1-\lambda_1)(1-\lambda_2)v_{1H}+(1-\lambda_1)(2-\lambda_1-\lambda_2)v_{2H} \\ &\quad +(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\}\end{aligned}$$

Derivation of $\Pr\{1_H$ def $2_H\}$ For the case that $v_{1H}\neq v_{2H}$ we need to evaluate

$$\begin{aligned}\Pr\{1_H \text{ def } 2_H\} &= \int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} G'_{1H}(b)G_{2H}(b)db \\ &= \int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} \frac{\lambda_1}{1-\lambda_1}\frac{v_{2H}-v_{1L}}{(v_{2H}-b)^2}\frac{1}{1-\lambda_2}\left(\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{v_{1H}-b}-\lambda_2\right)db \\ &= \frac{\lambda_1(v_{2H}-v_{1L})}{(1-\lambda_1)(1-\lambda_2)}\int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}}\left(\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(v_{1H}-b)(v_{2H}-b)^2}-\frac{\lambda_2}{(v_{2H}-b)^2}\right)db\end{aligned}$$

We exploit

$$\int\frac{1}{(v_{1H}-b)(v_{2H}-b)^2}db=\frac{1}{(v_{1H}-v_{2H})^2}\ln\frac{v_{2H}-b}{v_{1H}-b}+\frac{1}{(v_{1H}-v_{2H})(v_{2H}-b)}$$

to obtain

$$\begin{aligned}&\int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}}\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(v_{2H}-b)^2(v_{1H}-b)}db \\ =\ &\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(v_{1H}-v_{2H})^2}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}} \\ &+\frac{(1-\lambda_1)(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})}{\lambda_1(v_{1H}-v_{2H})(v_{2H}-v_{1L})}\end{aligned}$$

Moreover,

$$\int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} \frac{\lambda_2}{(v_{2H}-b)^2}db = \frac{\lambda_2(1-\lambda_1)}{\lambda_1(v_{2H}-v_{1L})}$$

thus

$$\begin{aligned}\Pr\{1_H \text{ def } 2_H\} &= \frac{\lambda_1(v_{2H}-v_{1L})(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})}{(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})^2}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}} \\ &+\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(1-\lambda_2)(v_{1H}-v_{2H})}-\frac{\lambda_2}{1-\lambda_2}\end{aligned}$$

and

$$\begin{aligned}&(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\} \\ =\ &\frac{\lambda_1(v_{2H}-v_{1L})(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})}{v_{1H}-v_{2H}}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}} \\ &+(1-\lambda_1)((1-\lambda_2)v_{1H}-\lambda_1 v_{1L}+(\lambda_1+\lambda_2-1)v_{2H})\end{aligned}$$

Evaluation of R^F

$$R^F=\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}+\frac{(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})\lambda_1(v_{2H}-v_{1L})}{v_{1H}-v_{2H}}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}$$

5.4 Proof of Proposition 2

(i) The proof is given in the text, immediately after the statement.

(ii) We consider the three conditions (9)-(11) separately.

5.4.1 The proof when (9) is satisfied

Suppose that $v_{1H} > v_{2H}$. Then Proposition 3 establishes that R^F is smaller than when v_{1H} satisfies $v_{1H} = v_{2H}$; furthermore, from (1) it follows that an increase in v_{1H} above the level such that $v_{1H} = v_{2H}$ has no effect on R^S. Therefore $R^S > R^F$ when $v_{1H} > v_{2H}$.

Now suppose that $v_{1H} < v_{2H}$. We know that $R^S = R^F$ in the symmetric setting such that both high valuations are equal to v_{1H}, and an increase in v_{2H} implies $R^S > R^F$ by the argument in the previous paragraph (after reversing the bidders' identities).

5.4.2 The proof when (10) is satisfied

This proof is provided in the text.

5.4.3 The proof when (11) is satisfied

When $v_{1L} < v_{2L} \le v_{1H} < v_{2H}$, Proposition 1(ii) applies and thus the aggregate bidders' rents in FPA are $U^F = (1-\lambda)(v_{1H}-\bar{b}) + (1-\lambda)(v_{2H}-\bar{b}) + \lambda^2(v_{2L}-v_{1L})$ with $\bar{b} = \lambda\hat{b} + (1-\lambda)v_{1H}$. Since $U^S = \lambda^2(v_{2L}-v_{1L}) + \lambda(1-\lambda)(v_{2H}-v_{1L}) + (1-\lambda)\lambda(v_{1H}-v_{2L}) + (1-\lambda)^2(v_{2H}-v_{1H})$, the difference $U^F - U^S$ is equal to $\lambda(1-\lambda)(v_{2L} + v_{1L} - 2\hat{b})$. Given $\lambda_1 = \lambda_2 = \lambda$, (2) reduces to $\lambda b^2 + ((1-\lambda)v_{1H} - \lambda v_{1L} - v_{2H})b + (1-\lambda)(v_{2H} - v_{1H})v_{2L} + \lambda v_{1L}v_{2H} = 0$ and thus

$$\hat{b} = \frac{1}{2\lambda}\left(\lambda v_{1L} + v_{2H} - (1-\lambda)v_{1H} - Q\right)$$

with $Q = \sqrt{((1-\lambda)v_{1H} - \lambda v_{1L} - v_{2H})^2 - 4\lambda(1-\lambda)(v_{2H} - v_{1H})v_{2L} - 4\lambda^2 v_{1L}v_{2H}}$. Therefore $U^F \geq U^S$ boils down to $Q \geq v_{2H} - (1-\lambda)v_{1H} - \lambda v_{2L}$ and (after squaring – notice that $v_{2H} - (1-\lambda)v_{1H} - \lambda v_{2L} > 0$) ultimately to

$$\lambda(v_{2L} - v_{1L})[2(1-\lambda)v_{1H} + 2(2\lambda - 1)v_{2H} - \lambda v_{1L} - \lambda v_{2L}] \geq 0 \tag{21}$$

Setting $v_{2L} = v_{1L} + \varepsilon_L$ and $v_{2H} = v_{1H} + \varepsilon_H$, it is simple to see that (21) is satisfied for $\varepsilon_L > 0$, $\varepsilon_H > 0$ and close to zero. Furthermore, given $v_{2H} > v_{1H}$ and $\lambda \geq \frac{1}{2}$, we find that $2(1-\lambda)v_{1H} + 2(2\lambda-1)v_{2H} - \lambda v_{1L} - \lambda v_{2L} \geq \lambda(2v_{1H} - v_{1L} - v_{2L})$, which holds for any $v_{2L} \leq v_{1H}$.

Proof for the case of distribution shift In the case of shift, (21) reduces to $2\lambda(v_{1H} - v_{1L}) \geq (2-3\lambda)a$ for $a \leq v_{1H} - v_{1L}$. If instead $a > v_{1H} - v_{1L}$, then $v_{2L} > v_{1H}$ and $U^S = \lambda^2 a + \lambda(1-\lambda)(a + v_{1H} - v_{1L}) + \lambda(1-\lambda)(v_{1L} + a - v_{1H}) + (1-\lambda)^2 a = a$; thus $U^F \geq U^S$ reduces to $2(2+\lambda)(v_{1H} - v_{1L}) \geq 3(2-\lambda)a$.

5.5 Proof of Proposition 3

Given $\lambda_1 = \lambda_2$ and $v_{1L} = v_{2L} = v_L$, when $v_{1H} < v_{2H} = v_H$ Proposition 1(ii) (footnote 14) applies and reveals that types 1_L,2_L bid as in the benchmark symmetric setting, whereas $G_{1H}(b) = \frac{1}{1-\lambda}(\frac{v_H - \lambda v_L - (1-\lambda)v_{1H}}{v_H - b} - \lambda)$ and $G_{2H}(b) = \frac{\lambda}{1-\lambda}\frac{b - v_L}{v_{1H} - b}$ with support $[v_L, \bar{b}]$, in which $\bar{b} = \lambda v_L + (1-\lambda)v_{1H}$. It is simple to see that both $G_{1H}(b)$ and $G_{2H}(b)$ are decreasing with respect to v_{1H} for any $b \in (v_L, \bar{b})$, and this implies that 1_H and 2_H are both more aggressive, in the sense of first order stochastic dominance, the larger is v_{1H} in $(v_L, v_H]$.[36] Given that

$$R^F = \lambda^2 v_L + \lambda(1-\lambda)\int_{v_L}^{\bar{b}} bdG_{2H}(b) + \lambda(1-\lambda)\int_{v_L}^{\bar{b}} bdG_{1H}(b) + (1-\lambda)^2\int_{v_L}^{\bar{b}} bd(G_{1H}(b)G_{2H}(b)) \tag{22}$$

we infer that R^F is increasing in v_{1H}.

When $v_{1H} > v_H$, Proposition 1(iii) applies and reveals that types $1_L, 1_H, 2_L$ bid as in the benchmark symmetric setting, whereas $G_{2H}(b) = \frac{(1-\lambda)(v_{1H} - v_H) + \lambda(b - v_L)}{(1-\lambda)(v_{1H} - b)}$ for any $b \in [v_L, E_v]$. Since $G_{2H}(b)$ is strictly increasing in v_{1H} for any $b \in [v_L, E_v)$, we infer that 2_H is less aggressive, in the sense of first order stochastic dominance, the larger is v_{1H}. Using again (22), after replacing G_{1H} with G_H and $\bar{b}$ with E_v, it follows that R^F is strictly decreasing with respect to v_{1H}.

[36] Precisely, if $v_{1H} < v'_{1H} < v_H$, then F_{1H} and F_{2H} given v'_{1H} first order stocastically dominate, respectively, F_{1H} and F_{2H} given v_{1H}.

5.6 Proof of the claims in Subsection 4.1.4

When (3) is satisfied, $G_2(b) \leq G_1(b)$ obviously holds for any b. Moreover, bidder 1 never wins in either auction when (3) holds. Conversely, 2 wins with probability one and in the FPA he pays v_{1H}; in the SPA his expected payment is the expected valuation of bidder 1, which is smaller than v_{1H}.

For $i = 1, 2$, let U_i^F denote bidder i's ex ante expected equilibrium payoff in the FPA; U_i^S has a similar meaning with reference to the SPA. When (4) is satisfied we find $U_1^F = (1-\lambda)\lambda(v_{1H} - \hat{b})$, $U_1^S = (1-\lambda)\lambda \max\{v_{1H} - v_{2L}, 0\}$, and $U_1^F > U_1^S$ since $\hat{b} < \min\{v_{2L}, v_{1H}\}$. Moreover, $U_2^F = \lambda^2(v_{2L} - v_{1L}) + (1-\lambda)[v_{2H} - \lambda\hat{b} - (1-\lambda)v_{1H}]$, $U_2^S = \lambda[\lambda(v_{2L} - v_{1L}) + (1-\lambda)\max\{v_{2L} - v_{1H}, 0\}] + (1-\lambda)[v_{2H} - \lambda v_{1L} - (1-\lambda)v_{1H}]$, and $U_2^S - U_2^F = (1-\lambda)\lambda[\max\{v_{2L} - v_{1H}, 0\} + \hat{b} - v_{1L}] > 0$ since $\hat{b} > v_{1L}$.
For equilibrium bid distributions we find that $G_1(b) > G_2(b)$ for any $b \in [v_{1L}, \hat{b}]$ as $G_1(v_{1L}) = G_2(\hat{b}) = \lambda$. For $b \in (\hat{b}, \bar{b}]$, $G_1(b) = \frac{v_{2H}-\bar{b}}{v_{2H}-b}$ and $G_2(b) = \frac{v_{1H}-\bar{b}}{v_{1H}-b}$, hence $G_1(b) > G_2(b)$ for each $b \in (\hat{b}, \bar{b})$.

When (7) holds we obtain $U_1^F = (1-\lambda)(v_{1H} - \lambda v_{1L} - (1-\lambda)v_{2H})$, $U_1^S = (1-\lambda)(v_{1H} - \lambda v_{2L} - (1-\lambda)v_{2H})$, and $U_1^F \geq U_1^S$ since $v_{1L} \leq v_{2L}$. Moreover, $U_2^F = U_2^S = \lambda^2(v_{2L} - v_{1L}) + (1-\lambda)\lambda(v_{2H} - v_{1L})$.
For equilibrium bid distributions we find that $G_1(b) = \lambda\frac{v_{2H}-v_{1L}}{v_{2H}-b}$ and $G_2(b) = \frac{v_{1H}-\bar{b}}{v_{1H}-b}$ with $\bar{b} = \lambda v_{1L} + (1-\lambda)v_{2H}$ and $G_2(b) > G_1(b)$ for any $b \in [v_{1L}, \bar{b})$.

5.7 Proof of the final claim in Subsection 4.1.5

We consider two sequences of atomless c.d.f. $\{F_1^n, F_2^n\}_{n=1}^{+\infty}$, with continuous and positive densities f_1^n, f_2^n for each n, which converges weakly to $\tilde{F}_1, \tilde{F}_2$. We show that for any large n, (12) and/or (13) are violated by F_1^n, F_2^n.

When $v_{1L} < v_{2L}$, select an arbitrary $\hat{v} \in (v_{1L}, v_{2L})$ and notice that given a small $\varepsilon > 0$, for a large n the inequality $F_1^n(\hat{v}) > \lambda - \varepsilon$ holds. Therefore $r^n(\hat{v}) = (F_2^n)^{-1}[F_1^n(\hat{v})] \geq v_{2L} - \varepsilon > \hat{v}$ [because $\lim_{n\to+\infty} F_2^n(v) = 0$ for each $v < v_{2L} - \varepsilon$] and $\int_{\hat{v}}^{r^n(\hat{v})} f_2^n(x)dx = F_2^n[r^n(\hat{v})] - F_2^n(\hat{v}) > \lambda - 2\varepsilon$ for a large n. If $f_1^n(\hat{v}) \geq f_2^n(x)$ for any $x \in [\hat{v}, r^n(\hat{v})]$, then $\lim_{n\to+\infty} f_1^n(\hat{v}) = 0$ implies $\lim_{n\to+\infty} \int_{\hat{v}}^{r^n(\hat{v})} f_2^n(x)dx = 0$: contradiction. Hence (13) is violated if F_1^n, F_2^n are close to $\tilde{F}_1, \tilde{F}_2$ and $v_{1L} < v_{2L}$.

Now assume that $v_{1L} = v_{2L}$ and $v_{1H} < v_{2H}$. Then given a small $\varepsilon > 0$ and a large n, the inequality $F_1^n(v_{1H} + \varepsilon) - F_1^n(v_{1H} - \varepsilon) = \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_1^n(x)dx > 1 - \lambda - \varepsilon$ holds, and $F_2^n(v_{1H} + \varepsilon) - F_2^n(v_{1H} - \varepsilon) = \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_2^n(x)dx$ tends to zero. Now notice that if there exists a number $t > 0$ such that $\frac{f_1^n(x)}{f_2^n(x)} \leq t$ for any $x \in (v_{1H} - \varepsilon, v_{1H} + \varepsilon)$ and any n, then $\int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_1^n(x)dx \leq t \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_2^n(x)dx$ and $\lim_{n\to+\infty} \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_1^n(x)dx = 0$. Thus for any $t > 0$, for any large n there exists some $x_n \in (v_{1H} - \varepsilon, v_{1H} + \varepsilon)$ such that $\frac{f_1^n(x_n)}{f_2^n(x_n)} > t$, which implies that (12) cannot hold since $F_2^n(x_n) > \lambda - \varepsilon$.

5.8 Proof of Proposition 4

Suppose that $\lambda_1 < \lambda_2$. Then Proposition 1(ii) applies and the ex ante expected payoffs of bidders 1 and 2 in the FPA and in the SPA are

$$\begin{aligned} U_1^F &= (1-\lambda_1)\lambda_2(v_H - \hat{b}) \quad \text{and} \quad U_1^S = (1-\lambda_1)\lambda_2(v_H - v_{2L}) \\ U_2^F &= \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)\lambda_2(v_H - \hat{b}) \quad \text{and} \quad U_2^S = \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)\lambda_1(v_H - v_{1L}) \end{aligned}$$

From (2) we obtain $\hat{b} = v_{2L} - \frac{\lambda_1}{\lambda_2}(v_{2L} - v_{1L})$, and this reveals that $U_1^F > U_1^S$ and $U_2^F > U_2^S$.
In the opposite case such that $\lambda_1 \geq \lambda_2$, Proposition 1(iii) applies and

$$\begin{aligned} U_1^F &= (1-\lambda_1)\lambda_1(v_H - v_{1L}) > U_1^S = (1-\lambda_1)\lambda_2(v_H - v_{2L}) \\ U_2^F &= U_2^S = \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)\lambda_1(v_H - v_{1L}) \end{aligned}$$

In either case, $U^F = U_1^F + U_2^F > U^S = U_1^S + U_2^S$ and thus $R^S > R^F$.

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