The Short-Run Dynamics

of Optimal Growth Models with Delays

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December 9, 2003

Abstract

Differential equations with advanced and delayed time arguments may arise in the optimality conditions of simple growth models with delays. Models with investment gestation lags (time-to-build), consumption gestation lags (habit formation) or learning by using lie in this category. In this paper, we propose a shooting method to deal with leads and lags in the Euler system associated to dynamic general equilibrium models in continuous time. We introduce the discussion describing the dynamics that emerge under various assumptions on learning by using and gestation lags. Then, we implement the numerical method we propose to solve for the short run dynamics of a neoclassical growth model with a simple time–to–build lag.

JEL codes: O40, E32, C63

Key words: Time–to–build, Shooting method, DDEs

1 Introduction

Modern growth theory extensively uses differential equations and dynamic optimization. The long-run and short-run dynamics of the variables of interest are described by the optimality conditions associated to fully specified general equilibrium optimal growth models. Those optimality conditions yield, in general, a system of ordinary differential equations (ODEs) that can be studied with standard analytical and numerical tools.

On the other hand, many actual systems have the property of aftereffect, i.e. the future states depend not only on the present but also on the past history. The existing literature refers to models with such a property as models with delay. There is a great variety of problems exhibiting this property and a corresponding variety of models describing them. Models with investment gestation lags (time-to-build), consumption gestation lags (habit formation) or learning by using lie in this category. When those features of reality are embedded in an optimal growth framework the optimality conditions associated to the growth problem may yield advanced and delayed time arguments. As far as those features are studied in the context of discrete time models the state space augments to include states at different time-horizons and the application of standard analytical and numerical tools to solve for the dynamics is just constrained by the course of dimensionality. However, in a more standard growth framework in continuous time the possibility of modelling real phenomena with delays by using functional differential equations (FDEs) constitutes a powerful device.[1]

In particular, delayed differential equations (DDEs) have proven to be useful in understanding the internal dynamics of capital accumulation [cf. Benhabib and Rustichini (1991), Boucekkine, Germain and Licandro (1997) and Asea and Zhak (1999), among others]. But unless specific assumptions are placed on individual objective functions, the quantitative evaluation of

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[1] As is well known, an ODE is an equation connecting the values of an unknown function and some of its derivatives for one and the same argument value, e.g. $F(t, x(t), \dot{x}(t), \ddot{x}(t)) = 0$. A functional equation (FE) involves an unknown function for different arguments. The differences between the argument values and t in a FE are called argument deviations. If all argument deviations are discrete and constant the FE is called a difference equation. By increasing the number of summands and decreasing the differences between neighboring argument values one arrives at FEs with continuous (and mixed) argument deviations. Combining the notions of differential and functional equations we obtain the notion of FDE: an equation connecting the unknown function and some of its derivatives for, in general, different argument values. Correspondingly, one arrives to the notion of differential-difference equation of delayed (DDE) or advanced type (ADE).

these models largely remains unexplored as most of the available methods to solve DDEs restrict to backward looking dynamics.

This paper is an attempt to fill in this gap. We introduce the discussion by examining a variety of growth models for which the dynamics are characterized by the appearance of advanced and delayed time arguments. Then, we propose a numerical method to solve dynamic general equilibrium models involving forward looking behavior with delays. The numerical procedure combines a Runge-Kutta type of algorithm adapted to solve DDEs by direct application of the method of steps — commonly used to solve this functional problems —, with a shooting method that iterates on a guess on forward-looking behavior. We implement this shooting method for the resolution of the short-run dynamics of a neoclassical growth model with a simple time–to–build lag. Then, we evaluate the performance of the algorithm and the quality of the approximation over this particular specification.

It should be stressed that there exists no numerical method available to solve for the type of Euler-equations system we are describing in this paper, to the best of our knowledge. In particular, the implications of different time lags for the short-run dynamics of the neoclassical growth model in continuous time are rigorously examined. Also, an attractive feature of the method is that it is relatively easy to handle and therefore should be of interest for a good number of related applications.

The paper is organized as follows. Next section is devoted to sketching up the general framework of optimal growth models with delays together with a description of a variety of models belonging to this framework. In Section 3 we present the neoclassical growth model with a time–to–build lag lag as well as the characterization of optimal solutions. Section 4 describes the algorithm and discusses some implementation issues, whereas in Section 5 we examine our numerical results and the short-run dynamics of the time–to–build model. A last section concludes.

2   Optimal growth with delays

An optimal growth model with delays can be written as the following social planner problem:

$$\max \int_0^\infty U\left(y(t), x(t)\right) \, \mathrm{e}^{-\rho t} \, \mathrm{d}t \text{ with } \rho > 0,$$

subject to

$$\dot{x}(t) = F\left(x(t), x(t-a), y(t), y(t-b)\right) \tag{1}$$

with given initial conditions $x(t) = x_0(t)$ for $t \in [-a, 0]$ and $y(t) = y_0(t)$ for $t \in [-b, 0[$, where $a > 0$ and $b > 0$ are delays in the dynamic system. Function $U\left(y(t), x(t)\right)$ represents instantaneous utility and the delayed differential equation system (1) is the feasibility constraint. The vector $y(t)$ of controls typically includes consumption and the state vector $x(t)$ may include physical and human capital as well as an indicator of habit formation.

This family of models can be solved using optimal control theory with delays.[2] The resulting necessary optimality conditions are a system of mixed delayed differential equations taking the following general form

$$\dot{y}(t) = G\left(x(t), x(t+a), y(t), y(t+b)\right) \tag{2}$$

$$\dot{x}(t) = F\left(x(t), x(t-a), y(t), y(t-b)\right). \tag{1}$$

Equation (2) is the Euler-type condition associated to optimal growth models. Given that time t decisions on the control (resp. the state) affect the state at time $t + b$ (resp. $t + a$), through (1), an advanced term is associated to the optimal condition. The examples below describe some well-known deterministic representative agent economies in continuous time belonging to this family of growth models with delay:

Example 1. The neoclassical growth model.

In this example we consider the particular case where $U\left(.\right) = u\left(c(t)\right)$ and $F\left(.\right) = f\left(k(t)\right) - \delta k(t) - c(t)$, under some assumptions on functions $u\left(c\right)$ and $f\left(k\right)$. Delays do not appear in this case.

Example 2. Delivery lags or time–to–build.

In the framework of the neoclassical growth model with standard preferences $U\left(.\right) = u\left(c(t)\right)$ we consider a simple time-to-build lag. In this economy capital equipment produced at time t is assumed to become productive at time $t + d$, $d > 0$ [cf. Asea and Zak (1999)]. Under this assumption, the feasibility condition becomes $F\left(.\right) = f\left(k(t-d)\right) - \delta k(t-d) - c(t)$.

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[2]See Kolmanovskii and Myshkis (1998) for finite time problems. An extension to infinite time models with one-hoss shay depreciation and AK technology is in Boucekkine et al (2003).

Example 3. Habit formation.

In this example we consider a delay in the control. Habit formation may be introduced by assuming that $U(.) = u(c(t), h(t))$, $u_2 < 0$, where $h(t)$ represents the stock of habits. The habit stock is assumed to depend on a simple average of past consumption over some relevant interval, say $[t-b, t]$, $b > 0$ [cf. Carroll, Overland and Weil (2000)]. Under this assumption, the stock of habits evolves according to $\dot{h}(t) = \frac{1}{b}(c(t) - c(t-b))$. Technology is assumed to be neoclassical, so that $\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$.

Example 4. Non-exponential depreciation.

Vintage capital models may be characterized by non-exponential rates of depreciation and technical change and can incorporate gestation lags as well as learning by using [cf. Benhabib and Rustichini (1991)]. The technology is again given by a neoclassical production function of the capital stock but

$$f\left(\int_{-\infty}^{t} i(z) \, \mathrm{d}\mu(z-t)\right)$$

where it is assumed $\mathrm{d}\mu(z) = m(z) \, \mathrm{d}(z)$, with $m(0) \neq 0$, under alternative depreciation schedules, $m(z)$. In particular, $m(z) = e^{\delta z}$ corresponds to the standard model of exponential depreciation discussed in Example 1 above. Alternatively, we can consider for instance the case in which capital equipment does not depreciate but has a lifetime $d > 0$, i.e. $m(z) = \chi_{[-d,0]}(s)$ where $\chi_A(t) \equiv \{1 \text{ if } t \in A, 0 \text{ if } t \notin A\}$, the one-hoss shay depreciation assumption. Under this assumption the feasibility condition takes the form: $\dot{k}(t) = f(k(t) - k(t-d)) - c(t)$. The case of one-hoss shay is a limit situation for non-exponential decay that might take place at a more regular pace.

In the following sections, we analyze precisely the particular case of a simple time–to–build lag. Essentially, in such a model there is a time lag after which capital equipment is available for production. The time–to–build technology embedded in an optimal growth framework is shown to yield a system of functional differential equations of the mixed type. Firstly, optimal control theory is applied and the corresponding Euler system of mixed delayed differential equations is derived. Secondly, the stability properties of the system are analyzed. Finally, a shooting algorithm is proposed to solve for the short-run dynamics of a neoclassical growth model with a time–to–build lag.

3 The case of time–to–build

Let us consider an economy populated by infinitely–lived households with unit aggregate measure. In this economy, a social planner chooses at each moment in time the amounts of consumption and investment so as to maximize the infinite stream of discounted instantaneous utilities derived from consumption, subject to the resource constraint, that is

$$\max \int_0^\infty u\left(c(t)\right) \, \mathrm{e}^{-\rho t} \, \mathrm{d}t \text{ with } \rho > 0,$$

subject to

$$\dot{k}(t) = k(t-d)^\alpha - \delta k(t-d) - c(t) \tag{3}$$

with initial conditions $k(t) = k_0(t)$ for all $t \in [-d, 0]$. $k_0(t)$ is the initial capital function, which is taken as given by the social planner. $d > 0$ is a parameter that determines a simple time–to–build lag — *i.e.* machines produced at time t are available for production at time $t + d$, such that the production function is given by $k(t-d)^\alpha$, $\alpha \in]0, 1[$. Hereafter, the utility function is assumed to take the following form

$$u\left(c\right) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \text{ for } \sigma > 0.$$

The necessary conditions associated to this problem are

$$c(t)^{-\sigma} \, \mathrm{e}^{-\rho t} = \phi(t)$$

$$-\phi(t+d)\left(\alpha \, k(t)^{\alpha-1} - \delta\right) = \dot{\phi}(t),$$

and the transversality conditions,

$$\lim_{t\to\infty} \phi(t) \geq 0 \qquad \text{and} \qquad \lim_{t\to\infty} \phi(t)k(t) = 0$$

where $\phi(t)$ is the co-state variable representing the marginal value of capital produced at time t but available at time $t + d$. Consequently, consumption is found to satisfy the following Euler-type equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left(\left(\alpha \, k(t)^{\alpha-1} - \delta\right)\left(\frac{c(t)}{c(t+d)}\right)^\sigma \mathrm{e}^{-\rho d} - \rho\right), \tag{4}$$

where the marginal productivity of capital available at time $t + d$ is weighted by the ratio of the marginal utility of consumption at $t + d$ to the marginal utility of consumption at t. Through investment, the social planner substitutes consumption at t for consumption at $t + d$. Notice that the optimal conditions converge to the solution of the standard neoclassical growth model when $d \to 0$.

Let us define a steady state as an optimal allocation for which $\dot{c}(t) = \dot{k}(t) = 0$. In this case, the capital stock k_s and consumption c_s are given by

$$k_s = \left(\frac{\alpha}{\rho\, \mathrm{e}^{\rho d} + \delta} \right)^{\frac{1}{1-\alpha}}$$

$$c_s = k_s^{\alpha} - \delta k_s.$$

where the only difference with respect to the standard neoclassical growth model relies on the ratio of marginal utilities, which is represented by the term $\mathrm{e}^{\rho d}$. Consequently, the steady state is unique given the Inada conditions.

Linearizing equations (3) and (4) about steady state (c_s, k_s) we compute the associated characteristic function

$$h(\lambda) = \lambda^2 - A\, \mathrm{e}^{-\rho d} \lambda + \left(B - A^2\, \mathrm{e}^{-\rho d} \right) - A\lambda\, \mathrm{e}^{-d\lambda} + A^2\, \mathrm{e}^{-\rho d}\, \mathrm{e}^{-d\lambda} + A\, \mathrm{e}^{-\rho d} \lambda\, \mathrm{e}^{d\lambda},$$

where $A = \alpha k_s^{\alpha-1} - \delta$ and $B = \frac{\alpha(\alpha-1)}{\sigma} k_s^{\alpha-2} c_s\, \mathrm{e}^{-\rho d}$. Since the characteristic equation $h(\lambda) = 0$ has an infinite number of roots, the steady state is generally a saddle. Convergence will be governed by the smallest negative real eigenvalue.

4   Solving for the short-run dynamics

System (3)–(4) is a mixed delayed differential equation (MDDE) system, with initial condition $k(t) = k_0(t)$ for all $t \in [-d, 0]$. There exist methods to solve DDEs, most of them being based on the so–called method of steps (See Paul (1997) for instance). However, these methods cannot be applied directly to solve the MDDE system (3)–(4), which involves expectations on future consumption and is therefore forward looking. To our knowledge, there exists no numerical method to solve this type of system. We therefore propose a simple method that combines *(i)* a standard method of steps and *(ii)* a shooting algorithm.

In order to provide with a better understanding of the difficulties we face in solving such a system, let us decompose the problem into several parts. Let us first focus on equation (3) that determines the accumulation of capital and let us assume that $c(t) = \widetilde{c}(t)$ is given exogenously

$$\dot{k}(t) = Ak(t-d)^\alpha - \widetilde{c}(t) - \delta k(t-d) \tag{5}$$

The main difficulty that rises in solving such a differential equation lies in the presence of a constant delay between the time at investment is decided and the time capital becomes operative. When $d = 0$, equation (5) would be a simple ordinary differential equation (ODE) that could be solved by using a standard Runge–Kutta type of algorithm. The existence of a delay complicates things. However, a standard Runge–Kutta method can still be used once adapted to this case. Indeed, since the function $k(t)$, $t \in (-d, 0]$ is known in t, we can substitute the initial function $k_0(t - d)$ for $k(t - d)$, $t \in (0, d]$ in (5). The DDE is then turned into a standard ODE that can be solved using a standard Runge–Kutta method. Then, $k(t)$ is known for the time span $(0, d]$. The same method can then be applied for $d < t \leqslant 2d$. The whole dynamics can be solved recursively for all subsequent time span $((i-1)d, id)$, $i = 1, \ldots$ This is a simple application of the well-known method of steps for solving DDEs (see Boucekkine, Licandro and Paul (1997) for instance).

The method of steps can be simply implemented in a backward looking dynamical system. However, most optimal control systems we deal with in economics give rise to fundamentally forward looking behavior for which the direct implementation of the method of steps does not work. In order to understand this phenomenon, let us now relax the simplifying assumption of an exogenously given consumption path, but let us impose that the delay, d, is equal to 0. In such a case, the model reduces to a standard optimal growth model and we therefore have to solve

$$
\begin{aligned}
\dot{k}(t) &= Ak(t)^\alpha - c(t) - \delta k(t) \\
\dot{c}(t) &= \frac{c(t)}{\sigma} \left(\alpha A k(t)^{\alpha-1} - \delta - \rho \right)
\end{aligned}
$$

It is well known that this system possesses the saddle path property. This type of ODEs system then calls for specific methods that can deal with its forward looking component. Methods such as the linearization of the system, a time–elimination method (reverse shooting) or a projection method can be used to recover the path of both capital and consumption. However, as soon

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as the delay is strictly positive these methods cannot be applied as they rely on the fact that the consumption policy rule is a function of the capital stock in period t. In our case, this is not the case anymore as the history of capital between $t - d$ and t determines future states. Therefore, consumption is a function of the functional of capital between $t - d$ and t. We therefore have to rely on another method. We chose to simply use a shooting method. This implies that the initial consumption function between $t - d$ and t is selected such that it guarantees convergence of the economy to the steady state along the saddle path.

The presence of a delay introduces an additional complication in the system, as consumption in period $t + d$ enters the Euler-type equation. This triggers the joint determination of period t and $t + d$ consumption levels, but period $t + d$ consumption requires the knowledge of period t capital stock. In order to circumvent this problem, we iterate on a guess for expected consumption. We investigate two strategies. The first one amounts to iterate on a pointwise approximation of expectations and using cubic spline interpolation when necessary. The second one iterates on the coefficients of a polynomial approximation to the level of expected consumption. In this case, we formulate the approximation of expectations in iteration l as

$$\widetilde{c}_\ell(t + d) = \exp\left(\sum_{i=0}^{n} \theta_{\ell,i} T_i(\varphi(t + d))\right)$$

where T_i is a polynomial of order i and $\varphi(\cdot)$ is a function that maps the time span into an appropriate interval: $[-1; 1]$. For a time span $[0, T + d]$, the mapping is given by $\varphi(t) = 2t/(T + d) - 1$.

4.1 The algorithm

The objective of the algorithm is then to solve the system(3)-(4) for $c(t)$ and $k(t)$ paths using the method of steps for $t \in [0, S \times d]$, given $k(t) = k_0(t)$ for all $t \in [-d, 0]$. S is the number of steps. The algorithm works as follows.

Initialization: As far as the expectation function is concerned, the initial guess for $\widetilde{c}_0(t + d)$ depends on the approximation. In the pointwise approximation, we set the initial guess equal to the steady state consump-

tion for all $t \geq 0$.[3] In the case of a polynomial approximation, choosing an approximation involves choosing *i)* a basis for polynomials, *ii)* an order of approximation, and *iii)* an initial vector of parameters, θ_0.

The computation of the saddle path solution rests on a truncated–horizon shooting algorithm that involves setting a time span $[0; T]$ and an initial condition for the consumption path. Setting a time span actually amounts to set a number of steps, such that $T = S \times d$, where S is the number of steps and d is the delay.

We follow Judd [1998] and use a bracketing algorithm to determine the initial consumption function. This triggers setting an upper bound $c_H > 0$ an a lower bound $c_L \geqslant 0$ for the consumption level. c_H is selected such that the solution diverges from below, while $c_L \geq 0$ such that the solution diverges from above.

Set the iteration counter ℓ to 0. Finally stopping criteria $\epsilon^e > 0$ (for the expectation guess) and $\varepsilon^s > 0$ (for the shooting part) are chosen.

Step 1 (Solving the system conditional on an expectation function):
Given an expectation guess, $\widetilde{c}_\ell(t)$, perform a shooting algorithm.

1. Set $c_0 = (c_H + c_L)/2$

2. Given c_0 and a guess for the expectation function, $\widetilde{c}_\ell(t)$, solve the dynamic system by the method of steps:

   (a) Set $k(t) = k_0(t)$ for $t \in (-d, 0]$. Set $i = 0$

   (b) Given the functions $k_i(t)$, $t \in ((i-1) \times d, i \times d]$, $\widetilde{c}_\ell(t)$, and initial c_0 solve

   $$\dot{k}(t) = Ak_i(t - d)^\alpha - c(t) - \delta k_i(t - d)$$
   $$\dot{c}(t) = \frac{c(t)}{\sigma} \left(e^{-\rho d} \left(\alpha A k(t)^{\alpha - 1} - \delta \right) \left(\frac{c(t)}{\widetilde{c}_\ell(t + d)} \right)^\sigma - \rho \right)$$

   and set $k_i(t) = k(t)$ for $t \in (i \times d, (i+1) \times d]$.

   (c) Set $i = i + 1$ and go back to (b) until $i = S$.

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[3]We investigated other formulations of the initial guess, such as $c^\star + \exp(-\lambda t)(c_0 - c^\star)$ where λ is the real stable eigenvalue of the system, but no significant differences with the simplest initial guess were found.

3. If $|c(t) - c^\star| < \varepsilon^s$ for $t \in ((S-1) \times d, S \times d)$ stop; else

- If $c(t) > c^\star$ for $t \in ((S-1) \times d, S \times d)$, set $c_H = c_0$ and go back to (a).

- If $c(t) < c^\star$ for $t \in ((S-1) \times d, S \times d)$, set $c_L = c_0$ and go back to (a).

Hence, given any $c(0)$, apart from the stable one, the solution of the DDE system must diverge. Call c_H at any $c(0)$ for which the solution converges to infinity. Otherwise, call it c_L ($c_L < c_H$). In order to find an approximation of the optimal $c(0)$, we take a simple average of the last c_H and c_L. This defines a very simple and natural way to converge toward the stable c_0 value.

Step 2 (Revision of expectations): If $\|c(t) - \widetilde{c}_\ell(t)\| < \varepsilon^e$, stop, else revise the expectation function and go back to 1.

The revision of the expectation function depends on the approximation procedure.

- In the case of the pointwise approximation, we set

$$\widetilde{c}_{\ell+1}(t) = \lambda \widetilde{c}_\ell(t) + (1-\lambda)c_\ell(t), \text{ with } 0 \leqslant \lambda \leqslant 1$$

- In the case of the polynomial approximation, a new set of parameters is obtained as

$$\theta_{\ell+1} = \lambda\theta_\ell + (1-\lambda)\widehat{\theta}_\ell, \text{ with } 0 \leqslant \lambda \leqslant 1$$

where

$$\widehat{\theta}_\ell \in \underset{\theta}{\text{Argmin}} \left\| \log(c(t)) - \sum_{j=0}^{n} \theta_i T_i \varphi(t) \right\|^2$$

Implementing the algorithm involves making several important choices in terms of initial conditions, tolerance criteria and approximating functional forms. From a heuristic point of view, none of these choices were found to fundamentally question the overall convergence of the algorithm, but each can accelerate it substantially. These decisions are discussed in the next section.

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4.2 Practical Implementation

The practical implementation of the algorithm we described in the previous section requires to take several decisions concerning the model parameters, the technical parameters of the algorithm and its initialization. This section discusses these decisions.

Parameterization: The first step in setting up the practical implementation of the method is to determine a set of structural parameters. We take the year as our unit of time. The elasticity of output with respect to the capital stock, α, is set at the value 0.3. The depreciation rate of capital is set such that 10% of the capital stock depreciates within a year, $\delta = 0.1$. Households are assumed to discount the future at a psychological rate of 5% per year, $\rho = 0.05$. The elasticity of intertemporal substitution, $1/\sigma$, is set at 2/3, which corresponds to a value of $\sigma = 1.5$. These parameter values which are standard in the growth literature fully determine the steady state of the model, which is reported in Table 1 for several values of the delay, d. We consider three cases. The first one, $d = 0$, corresponds to the standard optimal growth model á la Ramsey. The second and third one introduce time–to–build in the model. We consider two alternative values, $d = 2$ which corresponds to a time–to–build of 2 years and $d = 20$.

Table 1: Steady state

	Ramsey (d=0)	T–to–B (d=2)	T–to–B (d=20)
$k^\star$	2.6918	2.5625	1.4096
$c^\star$	1.0767	1.0699	0.9675

Algorithm Parameters: We need to set a value for the tolerance parameter of the shooting algorithm (ε^s), and that associated to the revision of expectations (ε^e). We set both values to `1e-6`. We also need to set values for the tolerance parameters in the method of steps for the ODE solver. We used Matlab's `ODE23` function and adopted its default tolerance parameters.

Concerning the choice of parameter S, a large value of it gives a more precise choice of $c(0)$, but at a higher computational cost. In the optimal

growth model, the speed of convergence is relatively high. Consequently, if the delay is large, we should expect the optimal solution being close to the steady state at time d. For this reason, when d is large, the number of step is set to a low value. For instance, when $d = 20$, we set the number of steps to 6, which was sufficient to achieve convergence. Conversely, for $d = 2$, the number of steps was set to 25.

Guess on expectations: We investigate two alternative forms of guesses for expectation. In the first one, the expectation function guess is a pointwise guess, implying that consumption expectations out of the grid are approximated by a spline interpolation scheme. An alternative guess we use is based on a polynomial approximation, described above

$$\widetilde{c}_\ell(t + d) = \exp\left(\sum_{i=0}^{n} \theta_{\ell,i} T_i(\varphi(t + d))\right)$$

In this case, we chose a basis of Chebychev polynomials that has the advantage of being an orthogonal basis. Since Chebychev polynomials are defined over the interval $[-1; 1]$ only, we need to map the time span into the same interval. We therefore apply the $\varphi(\cdot)$ transformation that maps $[0; S \times d]$ into $[-1; 1]$. $\varphi(t)$ takes the form

$$\varphi(t) = 2\frac{t}{S \times d} - 1$$

The order of approximation is set to 20.

Initial Conditions: Initial conditions are particularly important for the dynamics of the model, as the initial capital functional determines the rest of the dynamics. We set $k(t) = \Delta k^\star$, for $t \in (-d, 0]$ as an initial function for the capital stock. Δ is a constant that set the initial percentage deviation from the steady state. This was set to 0.95 indicating that the economy starts 5% below its steady state capital stock.

Another important initial condition is the initial guess for the expectation function. When a pointwise approximation is used, we set the approximation function equal to $c^\star$ for any t. We investigated over initial guesses, taking advantage of a linear approximation of the dynamic system, but found n o major improvement in doing so. When a polynomial approximation was used, we took advantage of the fact that the Chebychev polynomial of order

0 is $T_0(x) = 1$, and therefore set the parameter α_0 equal to $\log(c^\star)$, while setting $\alpha_i = 0$ for $i = 1 \ldots, n$. The initial guess is therefore the same as the pointwise approximation.

A last important initial condition is the pair (c_L, c_H) that is used to bracket the initial consumption level c_0 in the shooting algorithm. We could always set $c_H = k_0(-d)^\alpha$ and $c_L = 0$ at the initialization stage. However, it turns out to be useful to c_H and c_L as deviations from the steady state. More precisely, we set c_H close to the steady state $(0.99 \times c^\star)$ and c_L a little bit further away $(0.85 \times c^\star)$

5   Results

This section discusses our numerical results and the short–run dynamics of the time–to–build model when the economy experiments an unexpected 5 negative shock on its steady state capital level.

First of all, the algorithm does not require a lot of iterations on the expectations to converge. Indeed, convergence is attained after 6 iterations when $d = 2$, and 8 iterations when $d = 20$ in the case of a pointwise approximation. In the case of a polynomial approximation everything depends on the size of the polynomial. In Table 2, we report the number of iterations on the expectation function, as well as the coefficients of the approximation for different order of approximation, when $d = 20$. It appears that the number of required iterations to achieve convergence at the accuracy level is a bit lower (7 iterations) when a polynomial approximation is used. Furthermore, the coefficients quickly vanish as the order of the polynomial increase. For instance, they are all lower than `1e-4` as the order of approximation, n, is greater than 10.

As an accuracy check, we computed the difference between the pointwise approximation, which can be taken to be exact (at the `1e-6`) level on the nodes of approximation and the polynomial approximation to the expectation function. This difference is reported in Figure 1 for several orders of the approximating polynomials ($n =$2, 5, 10, 20). As can be seen from the figure and as expected, approximation errors can be large when too small orders of polynomials are used. When the order of approximation is large enough (n=20) the error becomes small. Interestingly, the largest error is obtained when time is close to a multiple of the delay ($i \times d$), and it is the largest at d. This indicates that the most difficult part of the approximation lies

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Table 2: Coefficients of the Expectation Approximation ($d = 20$)

	n=2	n=5	n=20	n=20
α_0	-0.037509	-0.037574	-0.037627	-0.037614
α_1	0.006290	0.007474	0.007527	0.007551
α_2	-0.004045	-0.004202	-0.004309	-0.004280
α_3	–	0.001748	0.001810	0.001830
α_4	–	-0.000526	-0.000657	-0.000627
α_5	–	0.000091	0.000190	0.000211
α_6	–	–	-0.000197	-0.000169
α_7	–	–	0.000191	0.000215
α_8	–	–	-0.000266	-0.000241
α_9	–	–	0.000217	0.000245
$\alpha_1 0$	–	–	-0.000178	-0.000149
$\alpha_1 1$	–	–	–	0.000032
$\alpha_1 2$	–	–	–	0.000076
$\alpha_1 3$	–	–	–	-0.000119
$\alpha_1 4$	–	–	–	0.000058
$\alpha_1 5$	–	–	–	0.000046
$\alpha_1 6$	–	–	–	-0.000069
$\alpha_1 7$	–	–	–	0.000030
$\alpha_1 8$	–	–	–	0.000018
$\alpha_1 9$	–	–	–	-0.000017
$\alpha_2 0$	–	–	–	0.000006
Iterations	7	7	7	7

Figure 1: Approximation error (Expectation function)
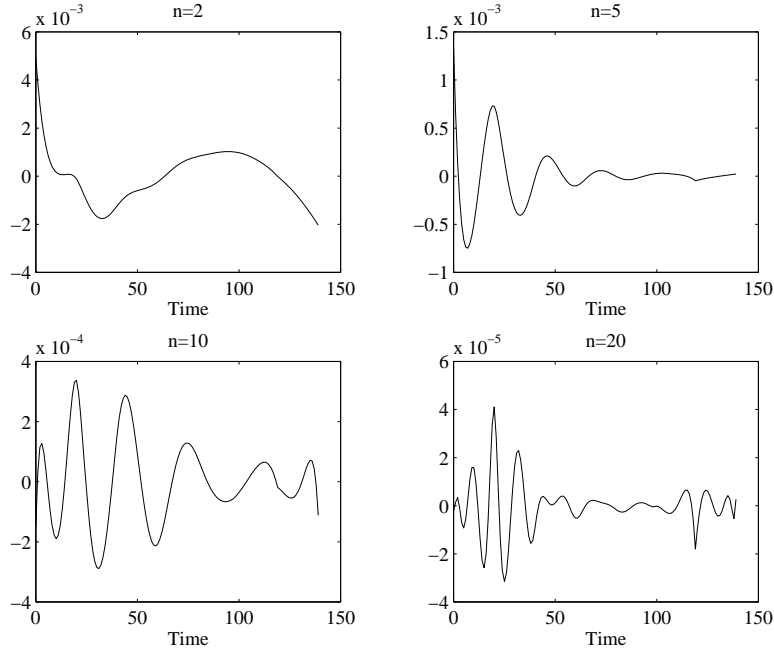
in capturing potential reversals in the behavior of the economy when capital becomes operative for the first time.

Figure 2 reports the dynamics of the economy for the several cases we are investigating. All reported dynamics are expressed in percentage deviation from the steady state and are obtained using a polynomial approximation of consumption expectation function with $n = 20$.[4] The grey line corresponds to the standard optimal growth model ($d = 0$) which will the used as a benchmark, the dark dashed line corresponds to the short time–to–build model ($d = 2$) and finally the dark plain line refers to the long time–to–build situation ($d = 20$). As is now well–known the standard optimal growth model displays monotonic convergence to the steady state. Since the capital stock is lower, output drops while the real interest rate rises. This triggers a instantaneous decrease in consumption both by wealth and intertemporal substitution motives. Conversely, the increase in the real interest rate creates an incentive to accumulate and investment rises, therefore increasing the pace of accumulation. Increase in the capital stock puts downward pressure on the interest rate and enables an increase in output. Both effects make

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[4]They do not significantly differ from the results obtained using a pointwise approximation.

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Figure 2: Dynamics

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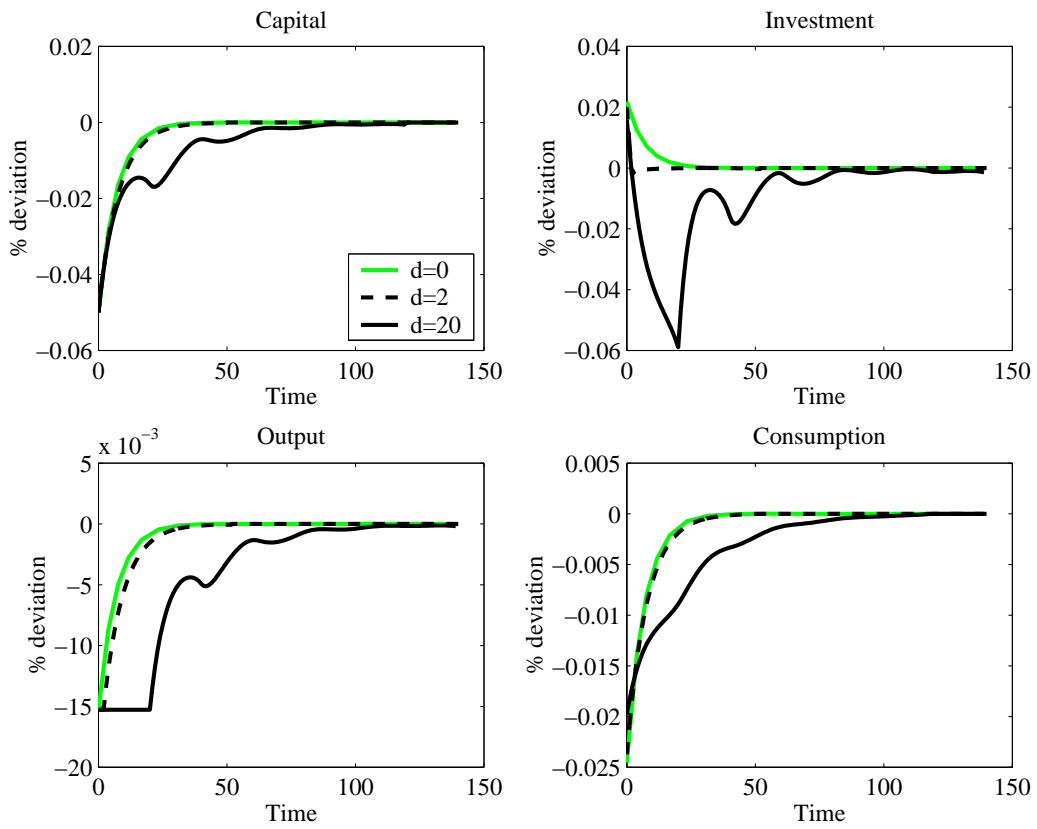
the household consume more while the decrease in the marginal efficiency of capital triggers a slowdown in investment. This makes the economy to converge back to the steady state monotonically.

The model with a short horizon time–to–build looks almost identical to the standard optimal growth model, in that it also converges monotonically to the steady state. However, there are significant differences related to the existence of time–to–build. Indeed, with a time–to–build of length 2, the economy is stuck with an output level of k_0 for the time span $(0, d]$, and any extra investment will only become productive in period d. In other words, the household cannot benefit from accumulation within the time interval $(0, d]$ as her income remains fixed. This therefore constrains investment which then responds less. For instance, as can be seen from table 3, investment increases

Table 3: Impact effect (% deviation from steady state)

	Ramsey (d=0)	T–to–B (d=2)	T–to–B (d=20)
k	-0.0500	-0.0500	-0.0500
c	-0.0245	-0.0237	-0.0198
i	0.0218	0.0200	0.0158
y	-0.0153	-0.0153	-0.0153

by 2% in the time–to–build model, whereas it increases by only 2.18% in the optimal growth model. Likewise, consumption is less responsive as the foregone consumption needed to support investment is lower. As soon as time reaches period d, new capital becomes productive, and output suddenly rises while decreasing returns bring the real interest rate down. As can be seen from the Euler equation defining the household's consumption/saving behavior, this makes it possible to smooth consumption. Indeed, the lower expected marginal efficiency in period d puts strong downward pressure on the investment effort which makes it possible to increase consumption. From this period on, the dynamics of the economy is close to that of the optimal growth model albeit smoother because of the time–to–build hypothesis.

Increasing the size of the delay considerably alters the internal dynamics of the model. Setting it to 20, the convergence path is not monotonic anymore but rather displays oscillations that accounts for echoes effects related to the time–to–build assumption. Since capital is lower than its steady state value, it is optimal for the household to increase its investment effort so as to fasten

the pace of accumulation. But, because of time–to–build, output is stuck at a low level for a longer period of time. This makes it possible for the household to smooth her investment effort for a larger period of time. Therefore, investment responds to a lesser extent (1.58% for $d = 20$ to be compared to the earlier 2% in the case $d = 2$). Output being given, consumption drops by a lower amount (nearly 2% to be compared to 2.4% when $d = 2$). As capital accumulation takes place, household expectations concerning future interest rate are downward sloping, such that as time approaches period d, investment becomes less and less attractive — even becoming lower than its steady state value. Consequently, the household can consume more, which brings back consumption closer to its steady state level. When accumulated capital in period 0 becomes operative (period d), investment is at its lower level — 6% below its steady state. This slowdown in the pace of accumulation makes it beneficial for the household to raise her investment effort. Investment then starts increasing again, triggering a slowdown in increase in consumption. But once again, the faster pace of accumulation exerts negative pressures on the interest rate that weaken household's desire to invest. Investment then starts declining until period $2d$ (period 40) allowing for greater increase in consumption. This oscillating process takes place until convergence of the economy to its steady state value.

6   Concluding remarks

This paper proposes a simple shooting method to deal with advanced and delayed time arguments in the Euler system associated to dynamic general equilibrium models. We implement it successfully to solve for the short-run dynamics of the optimal growth model in continuous time augmented to incorporate a simple time–to–build lag. The numerical properties of the algorithm and the dynamics of the model under different time lags are discussed. An attractive feature of the numerical method is that it is relatively easy to handle and therefore should be of interest for a good number related applications.

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