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A Comparison of Alternative Methods

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The detection of hidden periodicities : A comparison of alternative methods *

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Abstract

"Fixed frequency effect models" represent a powerful tool for analyzing time series exhibiting strong periodicities. However, in spite of their appeal to the practitioner, their use has been constrained by ignorance about their statistical properties. This paper attempts to offer a comparison among alternative methods via extensive simulation studies. The methods are compared across several performance characteristics most notably bias, variance power and RMSE (root mean square error). By way of illustration, two empirical examples are also included.

Keywords: Fixed frequency effect models, mixed spectrum, maximum periodogram ordinates, amplified harmonics, simulations, power comparisons.

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1 Introduction

Many series occurring in nature exhibit strong periodicities. The search for "hidden periodicities" is a problem with a long history, recurring time and again in several subjects such as seismology, astronomy, oceanography, acoustics and medicine. Early formal statements of the problem may be found in Schuster (1898), Slutsky (1937) and Yule (1927), for example; though the modern treatment of this problem may be said to begin with the seminal contributions of, among others,Whittle (1952), Bartlett (1954), Moran (1953) and Grenander and Rosenblatt (1957).

The problem has been termed in the literature the "harmonic regression problem" or "fixed frequency effects models". In the general statement of this problem, we have T observations $X_1, X_2, ..., X_T$ on a discrete parameter series generated by:

$$X_T = \mu + \sum_{i=1}^{k} (A_i \cos \omega_i t + B_i \sin \omega_i t) + u(t) \qquad (t = 1, ..., T) \qquad (1)$$

where $u(t)$ is a stationary process (capable of ARMA representation) and the A_i, B_i and ω_i are unknown parameters to be estimated. The number of "harmonics" k may either be assumed known or unknown (though most of the methods that we discuss allow for it to be unknown). The major methods suggested in the literature to deal with the problem (1), may be grouped under six headings:

1. Whittle's approximate least squares (WALS) method (Whittle (1952), Hannan (1973), Campbell and Walker (1977)).

2. Pisarenko's Harmonic Decomposition method (Pisarenko (1973), Kay and Marple (1981)).

3. Extended Prony method (Marple (1987), Candy (1988)).

4. Mixed Spectrum methods (Priestley (1964), Bhansali (1979)).

5. Methods based on the maximum of the periodogram ordinate (Fisher (1929), Bartlett (1957) and Chiu (1989)).

6. Method of Amplified Harmonics (Truong Van (1990)).

Thanks to the works of Hannan (1971), Hannan (1973) and Walker (1973), the asymptotic properties of WALS are well understood. Briefly, this method yields fairly accurate estimates of the frequencies ω_i in (1) (with variance of order T^{-3}), if the $u(t)$ are Gaussian. Its main drawback (as noted in Bloomfield (1976)) is that the criterion function is likely to have several local minima, so that the final estimates are likely to have large biases, unless the iterations happen to commence at values close to the true frequencies. The Pisarenko and Prony methods have proved extremely popular in signal processing, though their statistical properties have not been fully investigated. These techniques additionally presume that $u(t)$ in (1) is white noise (see Candy (1988) and Tiao and Tsay (1983)). Because methods (1)-(3) make rather restrictive assumptions on the disturbance term, their use in applications can be problematic. Our paper therefore, focusses on methods (4) to (6) listed above. The next three sections are devoted to each of these methods in succession. Simulation comparisons are presented in Section 5, as well as two applications. Conclusions are gathered in Section 6. Before introducing our methods formally, it is best to recapitulate a few basic but key definitions. We denote the process of interest as X_t, and assume it to be real and discrete.

Definition 1 *1. X_t is said to be* covariance-stationary *if*

(a) $\mathbf{E}(X_t) = \mu$

(b) $\mathbf{Var}(X_t) = \sigma_x^2$

(c) $\mathbf{Cov}(X_t, X_s) = g(\mid t - s \mid), s \neq t$

2. If X_t is covariance-stationary, then its autocovariance *at lag k is defined as*

$$R_k = E[(X_t - \mu), (X_{t+k} + \mu)]$$

and the autocorrelation *at lag k is simply*

$$\rho_k = \frac{R_k}{R_0}$$

3. The spectrum $F(\omega)$ *of a covariance-stationary process X_t is defined as*

$$F(\omega) = (\frac{1}{2\pi}) \sum_{r=-\infty}^{r=+\infty} \rho_k e^{-i\omega r}, \ \omega \in [-\pi, \pi]$$

2 Mixed spectrum methods

2.1 Mixed Spectrum

It is well known that the spectrum of a stationary ARMA process is continuous, whereas that of a sinusoid consists of a sharp peak at the frequency of the sinusoid. The spectrum of a sum of k sinusoids would thus be a step function containing k jumps. This leads Priestley (1964) and Priestley (1981), to the notion of a "mixed" spectrum. For a process such as X_t described in (1), the spectrum $F(\omega)$ could be decomposed as follows,

$$F(\omega) = F_1(\omega) + F_2(\omega) \tag{2}$$

Where $F_1(\omega)$ is a discrete spectrum (corresponding to the trigonometric sum) and $F_2(\omega)$ is the continuous spectrum corresponding to the ARMA process $u(t)$. $F(\omega)$ is then called a mixed spectrum.

2.2 Priestley's $P(\lambda)$ test

To motivate the test we reformulate (1) as

$$X_t = \mu + \sum_{i=1}^{k} D_i \cos(\lambda_i t + \phi_i) + u(t) \tag{3}$$

where now μ, D_i, λ_i, and k are unknown parameters, the ϕ_i, are independent and rectangularly distributed on $(-\pi,\ \pi)$ and $u(t)$ is a stationary linear process with a continuous spectrum. It is important to note that the only source of non-stationarity in X_t are the sinusoids. In particular, we assume that X_t has no unit roots and further that μ is independent of time. The introduction of these complications may modify the results stated below in ways about which we are currently ignorant. The first step is to test the null hypothesis,

$$H_0:\ D_i = 0,\ i = 1, 2, ..., k \tag{4}$$

i.e. that harmonic terms are absent from (3). Non-rejection of the null implies that X_t is a stationary ARMA process with a purely continuous spectrum. The $P(\lambda)$ test rests on a simple intuition, *viz.* that under H_0, the correlogram of X_t will eventually decay to zero. On the other hand, if one or more D_i are non-zero, then the correlogram will exhibit sinusoidal behaviour in its tail. The great advantage of this test is that in the event of the rejection of H_0 , it also suggests estimates of D_i, λ_i, etc. The analytics of the method is described in (Priestley (1981), chapter 8). Let f_m and f_n denote two "window" estimates of the spectrum of X_t, obtained using truncation points m and n respectively where $n > 2m$ (the window used for forming these estimates may be the same or different).

Remark 2 *Let X_1, X_2,..., X_T be a real discrete covariance stationary process with spectrum $F(\omega)$.*

By a "window" estimate of the spectrum we mean the estimate

$$\hat{F}(\omega) = \left(\frac{1}{2\pi}\right) \sum_{s=-(T-1)}^{s=T-1} \lambda(s) r(s) \cos(s\omega) \tag{5}$$

where $r(s)$ is the sample autocorrelation at lag s and $\lambda(s)$ is the so-called lag window generator. Two commonly used windows are the Bartlett and Parzen windows, with the following lag window generators.

Barlett

$$\lambda = \begin{cases} 1 - \frac{|s|}{M} & \text{if } |\ s\ | \leq M, \\ 0 & \text{if } |\ s\ | > M. \end{cases}$$

where M is referred to as the truncation parameter and $M < T$.

Parzen

$$\lambda = \begin{cases} 1 - 6\left(\frac{s}{M}\right)^2 + 6\left(\frac{|s|}{M}\right)^3 & \text{if } |\ s\ | \leq \frac{M}{2}, \\ 2\left(1 - \frac{|s|}{M}\right)^3 & \text{if } \frac{M}{2} <|\ s\ |< M. \\ 0 & \text{if } |\ s\ | \geq M. \end{cases}$$

with M being once again the truncation parameter.

We next put

$$P(\lambda) = \hat{f}_n(\lambda) - \hat{f}_m(\lambda) \tag{6}$$

at the Fourier frequencies $\lambda = (2\pi j/T),\ j = 0, 1, ..., [T/2]$.

If the D_i's are not all zero, $P(\lambda)$ will have several well-defined peaks say $\omega_1 < \omega_2 < ... < \omega_k$. These peaks are tested for significance (in the order of their occurrence), until a significant peak is found. If none of the peaks is significant, we conclude in favour of H_0. The procedure may be illustrated as follows.

Suppose we are testing the first peak at $\omega_1 = (2\pi p/T),\ \ p \neq 0,\ (T/2)$. We define the following quantities:

$r(s)$ - sample autocorrelation at lag s

$C(s)$ - sample autocovariance at lag s

$$P^*(\lambda) = \frac{P(\lambda)}{C(0)} \tag{7}$$

and

$$\hat{g}(\pi) = (4\pi)^{-1}[2 \sum_{u=-(m-1)}^{u=m-1} r^2(u) - \sum_{u=-(2m-1)}^{u=2m-1} r^2(u)] \tag{8}$$

Further we choose δ such that,

$$\omega_1 = \frac{2\pi p}{T} = (\frac{2\pi s}{m}) + \delta \tag{9}$$

for some integer s and

$$\Lambda_{(n,m)} = (\frac{2n}{3}) - (\frac{4m}{3}) + \frac{2m^3}{3n}. \tag{10}$$

The above expression for $\Lambda_{(n,m)}$ applies when $\hat{f}_n(\lambda)$ and $\hat{f}_m(\lambda)$ are both based on the Bartlett window. For other windows, slightly different expressions apply.

We next define the statistic

$$J_q = [\left(\frac{T}{m}\right) \Lambda_{(n,m)}^{-1}]^{(\frac{1}{2})} \sum_{s=0}^{q} P^*[\frac{2\pi s}{m} + \delta][\frac{g(\pi)}{2\pi}]^{(-\frac{1}{2})} \quad q = 0, ..., [\frac{T}{2}].$$

Let α be the chosen level of significance:

- if $\max_q(J_q) \leq \alpha_0$, ($\alpha_0 = 100$, $\alpha\%$ ordinate of $N(0,1)$) then the first peak at ω_1 is deemed insignificant and we pass on to the second peak of $P(\lambda)$ at ω_2 and so on.
- if $\max_q(J_q) > \alpha_0$, ($\alpha_0 = 100$, $\alpha\%$ ordinate of $N(0,1)$) then the peak at ω_1 is deemed significant and the amplitude of the corresponding harmonic term is estimated by:

$$\widehat{D}_1^2 = \frac{8\pi P(\omega_1)}{(n-m)} \tag{11}$$

We now remove the effect of the harmonic by defining,

$$c^{(1)}(s) = c(s) - 0.5\hat{D}_1^2 cos(s\omega_1), \tag{12}$$

and

$$r^{(1)}(s) = [\frac{c^{(1)}(s)}{c^{(1)}(0)}]. \tag{13}$$

Let ϖ_1 denote the first peak selected by the above procedure. The amplitude of the corresponding harmonic term is now estimated by (11). The process is repeated using $c^{(1)}(s)$ and $r^{(1)}(s)$. The successive iterations of $P(\lambda)$ may be denoted by $P^{(1)}(\lambda)$, $P^{(2)}(\lambda)$. At the k_{th} stage the chosen level of significance for testing the peaks in $P^{(k)}(\lambda)$ has however to be adjusted to $(\frac{\alpha}{k+1})$ in view of degrees of freedom corrections. If none of the peaks in $P^{(m)}(\lambda)$ is significant, then the procedure is terminated at this stage with m harmonics being identified at the frequencies ϖ_j $(j = 1, ..., m)$.

2.3 Bhansali's correction

Bhansali (1979) has noted that the correction formulae (12) and (13) due to Priestley are not sufficiently accurate as $c^{(1)}(s)$ and $r^{(1)}(s)$ are unstable for large values of s. He proposes the following correction instead

$$c^*(s) = \left(\frac{1}{T}\right) \sum_{i=1}^{T-|s|} X_t^* X_{t+|s|}^* \tag{14}$$

where

$$X_t^* = X_t - \hat{A}_1 \cos(t\varpi_1) - \hat{B}_1 \sin(t\varpi_1) \tag{15}$$

$$\hat{A}_1 = \left(\frac{2}{T}\right) \sum_{t=1}^{T} X_t \cos(t\varpi_1) \tag{16}$$

$$\hat{B}_1 = \left(\frac{2}{T}\right) \sum_{t=1}^{T} X_t \sin(t\varpi_1) \tag{17}$$

2.4 Continuous spectrum

Suppose by following the above procedure, we identify m harmonics at the frequencies ω_j $(j = 1, ..., m)$. We then estimate the following model by OLS

$$X_t = \mu + \sum_{1=1}^{m} (\hat{A}_i \cos\omega_i t + \hat{B}_i \sin\omega_i t) + u(t), \quad (t = 1, ..., T) \tag{18}$$

The fact that $u(t)$ may be correlated is not much of a cause for concern, since Durbin (1960) has shown that for harmonic regressions of the type (18), OLS estimates of A_i and B_i are asymptotically efficient.

We have already seen that the residual term $u(t)$ will have a continuous spectrum, and Bhansali (1979) shows that an autoregressive model may be fitted to $u(t)$ by a suitable lag selection criterion. His preference is for the FPE_α criterion developed in Bhansali and Downhan (1977), though the use of other criteria such as AIC or BIC is also, of course, possible. The selected order p of $u(t)$ is the one which minimises $FPE_\alpha(p)$, for $p = 0, 1, ..., L$ and L being the maximum lag considered.

3 Maximum periodogram methods

This group of tests has its origins in the classic test proposed by Fisher (1929), based on the maximum of the periodogram ordinate. Extensions were proposed by Whittle (1952), Bartlett (1957) and Hannan (1961).This strand of literature has been brought up to date and synthesized by Chiu (1989), who also proposes two new variants. We discuss Chiu's tests here, since they are both an improvement on, and a generalisation of, the earlier tests. As compared to the tests in the previous section, however, they suffer from two major limitations:

1. firstly, they are applicable only in the special case where $u(t)$ in (1) is a Gaussian white noise process and

2. the methods are exclusively focused on estimating k, the number of harmonics, although Chiu (1989) also suggests a method for estimating the significant frequencies based on the trimmed mean. This method has several similarities with the method suggested by Quinn and Thomson (1991). Little attention is devoted to the estimation of the other parameters in (1) viz. μ, A_i, and B_i $(i = 1, \ldots, k)$. It is thus not very clear whether the estimates ω_i should be directly substituted in (1) and the remaining parameters estimated by OLS or whether some kind of pretesting procedure should be applied.

These limitations should, however, be counterbalanced by the fact that the tests have good power properties, whereas the power properties of the $P(\lambda)$ test are somewhat inferior (see Table 1). The procedure may be described as follows:

Let $I_x(\omega_j)$ denote the periodogram of $X_t,\ t = 1, \ldots, T$ at the Fourier frequencies

$$\omega_j = \left(\frac{2\pi j}{T}\right), \quad j = 0, 1, \ldots [\frac{T}{2}] \tag{19}$$

We arrange the $n (= \frac{T}{2})$ periodogram ordinates in ascending order as $I_1 < I_2 < \ldots < I_n$. Then the original Fisher g-statistic may be defined as:

$$F = \frac{I_n}{\sum_{j=1}^{n} I_x(\omega_j)}$$

This statistic, however has several well-known limitations (see Priestley (1981) and Chiu (1989)) and hence Chiu (1989) suggests two new statistics to test the null of

$$H_0\text{: zero harmonics in } X_t$$

against the alternative

$$H_1\text{: } r \text{ harmonics in } X_t$$

The statistics are defined as follows:

$$U(r) = \frac{I_{n-r+1}}{\sum_{i=1}^{n} I_i}, \tag{20}$$

$$V(r) = \frac{I_{n-r+1}}{\sum_{i=1}^{n-r} I_i}. \tag{21}$$

Chiu (1989) derives the asymptotic distribution of $U(.)$ and $V(.)$ for testing H_0 as follows:

Define

$$Z_1(r) = nU(r) - \ln(n - r + 1) \tag{22}$$

$$Z_2(r) = c(n - r)V(r) - \ln(n - r + 1) \tag{23}$$

where,

$$c = 1 + \frac{r[\ln(\frac{r}{n})]}{n - r} \tag{24}$$

Also let:

$$P_i(r) = \exp\{-\exp[-Z_i(r)]\} \sum_{j=0}^{r-1} \exp\{-jZ_i(r)/j!\} \qquad i = 1, 2 \tag{25}$$

If the selected level of significance is α, we reject H_0 in favour of H_1 if

$$P_i(r) > (1 - \alpha), i = 1, 2$$

Suppose r peaks are indicated by the statistics $U(r)$ or $V(r)$, then the estimates of the frequencies are taken to be the frequencies associated with the r periodogram ordinates $I_{n-r+1}, \ldots, I_n$. The other parameters in (1) are then estimated along the lines discussed above in Section 2.4.

4 Method of amplified harmonics

This method was introduced into the literature by Truong Van (1990). The underlying model is taken to be (3) with $u(t)$ having a stationary ARMA representation. The number of data points is taken to be T as before. The logic of the method relies on the notion of amplification of harmonics at various frequencies. Truong-Van (op. cit.) defines for each frequency ω_j ,a process $\xi_t(\omega_j^*)$ by the the following recursion where w_j^* is near to w_j.

$$\xi_t(\omega_j^*) = 2\cos(\omega_j^*)\xi_{t-1}(\omega_j^*) + \xi_{t-2}(\omega_j^*) + X_t \quad (t = 1, \ldots, T), \tag{26}$$

with $\xi_0 = \xi_{-1} = 0$. It is then shown that the process $\xi_t(\omega_j^*)$ amplifies the harmonic of frequency ω_j, selectively relative to the others. Truong-Van (1990, Theorem 3) is then led to demonstrate that among the harmonic amplifiers $\xi_t(\hat{\omega}_j^*)$ of ω_j, there exists an amplifier s.t.,

$$\sum_{t=2}^{T} \xi_{t-1}(\hat{\omega}_j)X_t = 0, \tag{27}$$

i.e. $\xi_{t-1}(\hat{\omega}_j)$ is orthogonal to X_t. A suggested estimate of ω_j is then $\hat{\omega}_j$. These estimates are strongly consistent and asymptotically normal (see Hannan (1973) and Truong Van (1990), Theorem 4). From the computational point of view, the following result is important (Truong-Van, op. cit. Equation (10)).

Proposition 1 *Consider the problem of estimating α in the following regression by OLS,*

$$\xi_t(\omega_j^*) + \xi_{t-2}(\omega_j^*) = 2\alpha\xi_{t-1}(\omega_j^*) + \epsilon_t \tag{28}$$

(where ω_j^* is near to ω_j) Let $e_t(\omega_j^*)$ denote the OLS residuals of (26). Then $\widehat{\omega}_j$ is the solution to the following minimisation problem:

$$\min_{\omega_j^* \in V(\omega_j)} \sum_{t=1}^{T} (e_t(\omega_j^*) - X_t)^2. \tag{29}$$

Remark 3 *The neighbourhood $V(\omega_j)$ of ω_j is defined as follows. Let $u(t)$ in (3) have the ARMA representation,*

$$\Phi(B)u(t) = \Theta(B)a(t), \tag{30}$$

where $a(t)$ are i.i.d. with mean 0 and and variance σ^2. Let $(\frac{2\pi}{\sigma^2})f(\omega)$ be the spectrum of $u(t)$, then $V(\omega_j)$ is defined as

$$V(\omega_j) = \frac{24\omega_j\sigma^2}{D_j^2}, \tag{31}$$

where D_j^2 are as defined in (3). The solution of this somewhat intricate problem can proceed along either of the two lines suggested by Truong Van (1990)

- *signal orthogonal amplifiers by double amplification of harmonics (SODAH)*
- *recursive least squares on amplified harmonics (RLSOAH).*

The SODAH algorithm requires starting values of ω_j^* fairly close to the true values ω_j, while RLSOAH is more robust to the choice of initial values. We therefore resort to the RLSOAH algorithm (described in the Appendix). This algorithm will lead to estimates $\widehat{\omega}_j$ $(j = 1, \ldots m)$, of the harmonics- the algorithm *also identifying m, the number of harmonics. The estimation of model (3) is now straightforward. We estimate the following equation by OLS:*

$$X_t = \mu + \sum_{i=1}^{m} (A_i \cos \hat{\omega}_i t + B_i \sin \hat{\omega}_i t) + u(t). \qquad (32)$$

The residuals $u(t)$ from this model can then be used to identify an ARMA(p,q) *model for $u(t)$ in the standard fashion (one should possibly test $u(t)$ for stationarity to establish the reliability of the estimates $(\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_m)$*

5 Simulation and empirical results

5.1 Simulation experiments

In analysing "fixed frequency effects" models such as (1) or (3), there are at least 3 considerations involved viz.

1. diagnostic tests for the presence of harmonics
2. estimation of the number of harmonics and the corresponding frequencies and
3. estimation of the amplitudes and phases i.e. A_i , B_i in (1) or D_i and ϕ in (3).

So far as the last aspect (3) is concerned, all methods are unanimous in recommending the application of OLS to equations such as (14), once the number of harmonics k and the corresponding frequencies $\hat{\omega}_i$, $(i = 1, \ldots, k)$ have been estimated. Both Priestley's $P(\lambda)$ test, as well as the $U(r)$ and $V(r)$ tests of Chiu, pay attention to problems (1) and (2), though the former is far more systematic in its approach to problem (2) than the latter. The Truong-Van method presents no formal test for the presence of harmonics or for estimating the number of harmonics -this being done by a visual inspection of the graph of $Z(r)$ (see step 3 of the Appendix)-though it does focus a great deal on obtaining refined estimates of the frequencies. In judging the empirical performance of the tests, we may thus focus on the following three aspects:

1. the power of those tests which are concerned with detecting the presence of harmonics, viz. $P(\lambda)$, $U(r)$ and $V(r)$ tests.
2. examining whether the number of harmonics is correctly predicted and whether there is a tendency for a test to under- or over-estimate this number. Once again these statistics are meaningful for the $P(\lambda)$, $U(r)$ and $V(r)$ tests only. However, we also present results in this regard for the Truong-Van procedure, where the number of harmonics is determined in every simulation via the following admittedly mechanical rule: We looked at a window of the $2[T/10]$ Fourier frequencies centered around r_i. $Z(r_i)$ was defined to be a maximum whenever:

- $Z(r_i) > Z(r_j)$ for all $j \neq i$ in the window
- $Z(r_i) > T^{-3/2}$

3. estimating the empirical bias, variance and RMSE (root mean square error) of the frequencies, amplitudes and phases of the harmonics generated by the different methods.

5.1.1 Power considerations

Table 1 presents comparisons of the power properties of the $P(\lambda)$, $U(r)$ and $V(r)$ tests. Three alternative models are used for this purpose: a white noise, an $AR(1)$ model and an $AR(2)$ model. None of these models has a fixed frequency component.

Model 1 X_t is i.i.d. $N(0, 1)$.

Model 2 $X_t = 0.7X_{t-1} + u(t)$, where $u(t)$ is i.i.d. $N(0, 1)$.

Model 3 $X_t = 0.37X_{t-1} + 0.34X_{t-2} + u(t)$, with $u(t)$ being i.i.d. $N(0, 1)$.

We also consider three alternative sample sizes viz. $T = 150$, 300 and 500 and generate $10,000$ replications in each case. In the $P(\lambda)$ test, the null is of no peaks (see (4)) with the implicit alternative of at least one peak. The $U(r)$ and $V(r)$ tests also suppose a null of zero peaks but the alternative is of exactly r peaks. In Table 1, we fix the alternative at $r = 1$. The message of Table 1 is fairly clear-cut. The Priestley-Bhansali $P(\lambda)$ test seems to have fairly good power properties when the underlying process is white-noise and the power increases with the sample size. However for $AR(1)$ or $AR(2)$ processes, the power performance of the test deteriorates appreciably. The converse seems to be true of the $U(r)$ and $V(r)$ tests. Both are quite powerful for low order AR processes (even for relatively small samples) but their performance for white noise data generating processes is noticeably inferior to the Priestley-Bhansali test.

5.1.2 Bias, variance and RMSE

We now turn our attention to to investigating the bias, variance and RMSE of all the tests we have considered:

Test I: Priestley's $P(\lambda)$ test, incorporating Bhansali's correction

Test IIA: Chiu's $U(r)$ test

Test IIB: Chiu's $V(r)$ test

Test III: Truong-Van procedure

For this purpose, we set up three alternative benchmark models viz. a single harmonic, a double harmonic and a triple harmonic.

Model A: Single Harmonic

$$X_t = A_i \cos(\omega t) + B_i \sin(\omega t) + u(t) \tag{33}$$

where

1. $u(t)$ is i.i.d. $N(0,1)$
2. $A = 0.50,\ B = -0.35,\ \omega = (20\pi/T)$

Model B: Double Harmonic

$$X_t = \sum_{i=1}^{2} A_i \cos(\omega_i t) + B_i \sin(\omega_i t) + u(t) \tag{34}$$

where:

1. $u(t)$ is i.i.d. $N(0,1)$.
2. $\omega_1 = (20\pi/T)$ and $\omega_2 = (50\pi/T)$.
3. $A_1 = 0.50,\ B_1 = -0.35,\ A_2 = -0.20,\ B_2 = 0.45$.

Model C: Triple Harmonic

$$X_t = \sum_{i=1}^{3} A_i \cos(\omega_i t) + B_i \sin(\omega_i t) + u(t) \tag{35}$$

where:

1. $u(t)$ is i.i.d. $N(0,1)$.
2. $\omega_1 = (20\pi/T)$, $\omega_2 = (50\pi/T)$ and $\omega_3 = (60\pi/T)$.
3. $A_1 = 0.50,\ B_1 = -0.35; A_2 = -0.20,\ B_2 = 0.45;\ A_3 = 1.25,\ B_3 = 1.10$.

Once again three alternative sample sizes are considered viz. $T = 150$, 300 and 500 and the aggregate number of replications fixed, as before at $10,000$. We also note for each test, the proportion of replications in which the number of harmonics is correctly identified, under-identified and over-identified in each of the above models. The bias, variance and RMSE are well-defined only for those replications where the number of harmonics is correctly identified. The results for models A, B and C are presented in Tables 2, 3 and 6 respectively. Tables 2 to 6 present several conclusions of interest:

1. Focussing first on the issue of correct identification of the number of harmonics, (i.e. the first 3 rows of each table) it is evident that the $U(r)$ and $V(r)$ tests clearly emerge as superior to the other two tests, with $U(r)$ having a slight edge over $V(r)$ especially for large samples. The only exception to this general tendency is when the true data contain 3 harmonics (model 35) where the $U(r)$ test fares badly for the smallest

sample size considered ($T = 150$ in Table 6). Even here, the performance of the $V(r)$ test remains fairly good. The Priestley-Bhansali test has a pronounced tendency to overestimate the number of harmonics (exception being for the model 35, $T = 500$); by contrast, the Truong-Van test veers towards under-identification. In discussing the forecasting performance, it becomes necessary to distinguish the underlying models carefully, for the performance patterns seem to be dependent on this aspect.

2. For the single harmonic case (model 33), the RMSE for the $U(r)$ and $V(r)$ methods is much lower than the other 2 methods, irrespective of the sample size. Once again the $U(r)$ test seems to hold a small edge over the $V(r)$ test.

3. For the double harmonic case (model 34), forecasting ability (as judged by the RMSE) seems to be significantly better for Method I (Priestley-Bhansali) than for all the other methods, both for the low frequency (ω_1) and the higher frequency (ω_2). Additionally, Method III (Truong-Van) comfortably out-performs Methods II A and B (Chiu) in the case of ω_1 with the reverse holding for ω_2. Interestingly, Methods I and III seem to be better at predicting the lower frequency ω_1 than the higher frequency ω_2 (comparison of last two rows of Table 3) while the opposite seems to hold for Methods IIA and B.

4. For the triple harmonic case (model 35), the Priestley-Bhansali method emerges as the best at the lower frequencies (ω_1 and ω_2) for all sample sizes, and also for the high frequency ω_3 for $T = 500$. So far as the other 2 methods are concerned, the Truong-Van is noticeably superior to the Chiu methods for the low frequencies ω_1 and ω_2 but loses out for the higher frequency ω_3.

5.2 Empirical examples

Two empirical illustrations are now presented, using the above techniques. The first pertains to the standard sunspot data set, Wolfer's annual sunspot series (1700-1955) (see Waldheimer (1961)), while the second is concerned with total daily production of electricity (measured in mega watts per hour) in Spain for the year 1998 (01-01-1998 to 31-12 1998) obtained from OMEL (the Spanish market operator). A brief discussion of each case follows.

5.2.1 Sunspot data

In this case, the number of harmonics identified was as follows:

Priestley-Bhansali Method: =5
Chiu Method: =6
Truong-Van Method : =4

The actual frequencies were estimated as suggested in Bhansali (1979), Chiu (1989) and Truong Van (1990) respectively. The relevant results are summarised in Table 5.

The Priestley-Bhansali method suggests 5 cycles in the data, with periods ranging between 10.05 years to 91.39 years; the range for the Chiu method is 2.69 years to 85.37 years, while Truong-Van's procedure yields cycles in the range 2.02 years to 52.58 years. Using these frequency estimates, we obtained OLS estimates of (14) in each of the 3 cases (as discussed in Section 2 above). The original data and the fitted models are displayed in Figures 1 to 3. A visual inspection of the figures shows that three methods are quite successful in picking up the peaks and troughs of the original data. The Chiu procedure, however, mistakenly identifies 2 troughs around the year 1800 as peaks, while the Truong-Van procedure besides failing in the same fashion, also additionally misses the trough around 1880. The Priestley-Bhansali method is more accurate in the early period, though its performance slips in the period 1900-1925. All these methods, however, seriously underestimate the amplitudes in the original data. This is reflected in the relatively modest correlations (see Table 5) between the fitted model and the original data.

5.2.2 Electricity data

The number of harmonics identified and the associated frequencies in this case are presented in Table 6. The results of the three methods are very similar, with an especially close correspondence between the Priestley-Bhansali and Truong Van methods. Both these methods identify 6 harmonics with periodicities ranging from 2.34 to 80.69 days (Priestley-Bhansali method) and from 2.34 to 73.0 days (Truong Van method). The Chiu method, however, identifies 2 harmonics less, and the range of the periods is now from 3.50 to 75.93 days. Thus, all three methods unanimously support the existence of 2 cycles (both of which seem intuitively quite reasonable) - a near-quarterly cycle of about 73-80 days duration (corresponding to the frequencies in the neighbourhood of 0.08) and a weekly cycle (frequencies clustered around 0.89). All three methods also indicate intra-weekly cycles, which admit of two plausible explanations - electricity usage varying as between weekdays and weekends, and intra-weekly cycles as "aliases" of extremely short intra-day cycles.

The associated harmonic models are estimated as in the earlier example, and the graphs displayed in Figures 4 to 6. The data are highly volatile, with frequent peaks and troughs, which are often quite sharp. The Chiu method is undoubtedly superior to the other two for this data set, as is evident both, from the correlations between "fitted" and actual values (reported in the last row of Table 6), as well as visually from the graphs in Figures 4 to 6. It does an excellent job of picking up the "turning points", which is especially creditable considering the fact that they are so closely clustered. Of the remaining two methods, the Truong Van method captures the turning points well but seriously underestimates the "amplitudes" throughout. The Priestley-Bhansali method (as in the sunspots example above) performs well in (roughly) the first quarter, slips badly in the next two, and picks up again in the final quarter. One possible reason for the superior overall performance of the Chiu method could be that it identifies an additional half-quarterly cycle of 44.5 days (corresponding to the frequency 0.13682), which the other two methods miss out.

6 Conclusions

The paper has been concerned with a discussion of some of the methods suggested in the literature to deal with series exhibiting strong periodicities. However, the power properties of these tests do not seem to have been systematically documented. We address this problem in extensive simulations, in which -apart from the power properties- predictability characteristics such as bias, variance and RMSE are also investigated. The simulations are replicated for alternative data generating processes (DGPs) covering low-order harmonics of order up to three. In general the power properties of the Priestley-Bhansali test are moderate except when the DGP is white noise. By contrast, the Chiu tests yield exceptionally good power for low-order AR models but fare poorly for white-noise DGPs. So far as forecasting ability is concerned, the Priestley-Bhansali and Truong-Van methods are noticeably superior to the Chiu method for low-order harmonics (2 or 3). In the case of a single harmonic, the Chiu method seems to yield better results. Another feature to emerge strongly is that the Priestley-Bhansali and Truong-Van methods are better at predicting low frequency harmonics, higher frequency harmonics being better predicted by the Chiu method.

It is of course tempting to conclude from this analysis that the Chiu method should be preferred when there is reason to believe that the underlying data indicate a single harmonic of high frequency, with preference given to the other two methods when the 'a priori' presumption is in favour of more low-frequency harmonics. But such an inference would be somewhat adventurous, considering that our results are derived on the basis of simulations rather than theoretical considerations. The two empirical examples seem to indicate that the methods yield reasonable results in real-world situations.

Appendix

RLSOAH Algorithm (Truong-Van (1990))

Let X_t , $t = 1, 2, \ldots, T$ denote the observations on the given series. The major steps in this algorithm may be defined as follows.

Step 1: Define a new series Y_t as a function of a variable r.

$$\begin{aligned} Y_1(r) &= X_1 \\ Y_2(r) &= X_2 + 2X_1 \cos(r) \end{aligned}$$

and for $t = 3, \ldots, T$.

$$Y_t(r) = X_t + 2Y_{t-1}(r)\cos(r) - Y_{t-2}(r).$$

Step 2: Define

$$Z(r) = \sum_{t=1}^{T} Y_t^2(r),$$

for $r = (2\pi)/T, (4\pi)/T, \ldots, \pi$

Step 3: Let the plot of $Z(r)$ have k significant peaks say,

$$\omega_1^0 < \omega_2^0 < \ldots < \omega_k^0,$$

Further let

$$\alpha_j^0 = \cos(\omega_j^0) \qquad j = 1, 2, \ldots, k.$$

Step 4: We now iterate on α_j and ω_j , $(j = 1, \ldots, k)$ denoting their values at the p^{th} iteration by $\alpha_j(p)$ and $\omega_j(p)$ respectively. The iteration rule is as follows:

$$\alpha_j^{(p+1)} = \alpha_j^p + \frac{\sum_{t=2}^T X_t Y_{t-1}(\omega_j^{(p)})}{\sum_{t=2}^T Y_{t-1}^2(\omega_j^{(p)})},$$

for $p = 1, 2, \ldots$where,

$$\omega_j^{(p)} = \cos^{-1}\alpha_j^{(p)}.$$

The iterations are stopped at stage m if

$$|\alpha_j^{(m+1)} - \alpha_j^{(m)}| < \varepsilon_0,$$

where ε_0 is a pre-determined constant.

Note that the above iterations are done for each $j = 1, 2, \ldots, k$ Let the final values be denoted by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_k$ and $\widehat{\omega}_1, \ldots, \widehat{\omega}_k$.

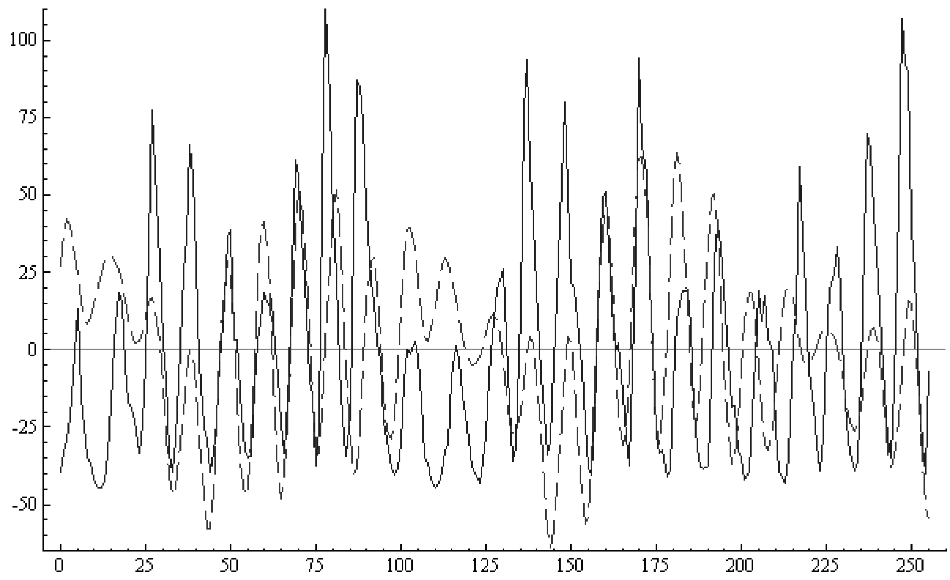

Figure 1: Sunspot data analyzed by Priestley-Bhansali procedure. The dotted line represents the original data and the solid line the fitted data

Step 5: Estimate the following equation by OLS

$$X_t = \mu + \sum_{j=1}^{k} A_j \cos(t\widehat{\omega}_j) + B_j \sin(t\widehat{\omega}_j) + u_t,$$

and let $\widehat{u_t}$ denote residuals of the above regression.

Step 6: We now repeat Step 5 with $\widehat{u_t}$ replacing X_t.

Step 7: The estimates obtained from Step 6 may be denoted by $\alpha_1^\sharp, \ldots, \alpha_l^\sharp$ and are essentially refinements of the estimates $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_k$, obtained in Step 4 (note that l and k may be different). They may be treated as the final estimates or we may have one more repetition of the cycle.

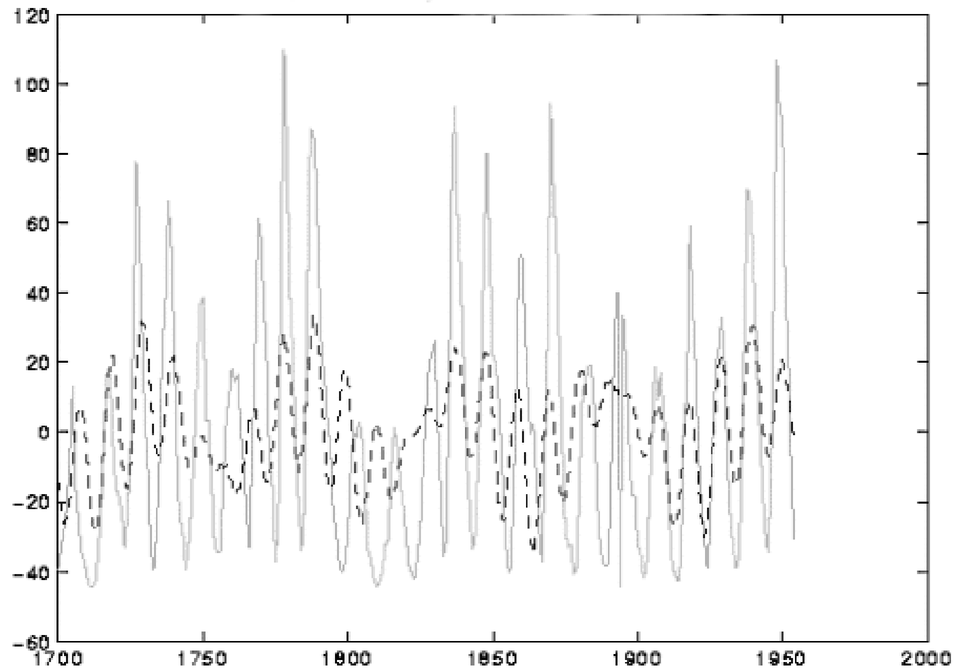

Figure 2: Sunspot data analyzed by Truong-Van procedure. The dotted line represents the original data and the solid line the fitted data

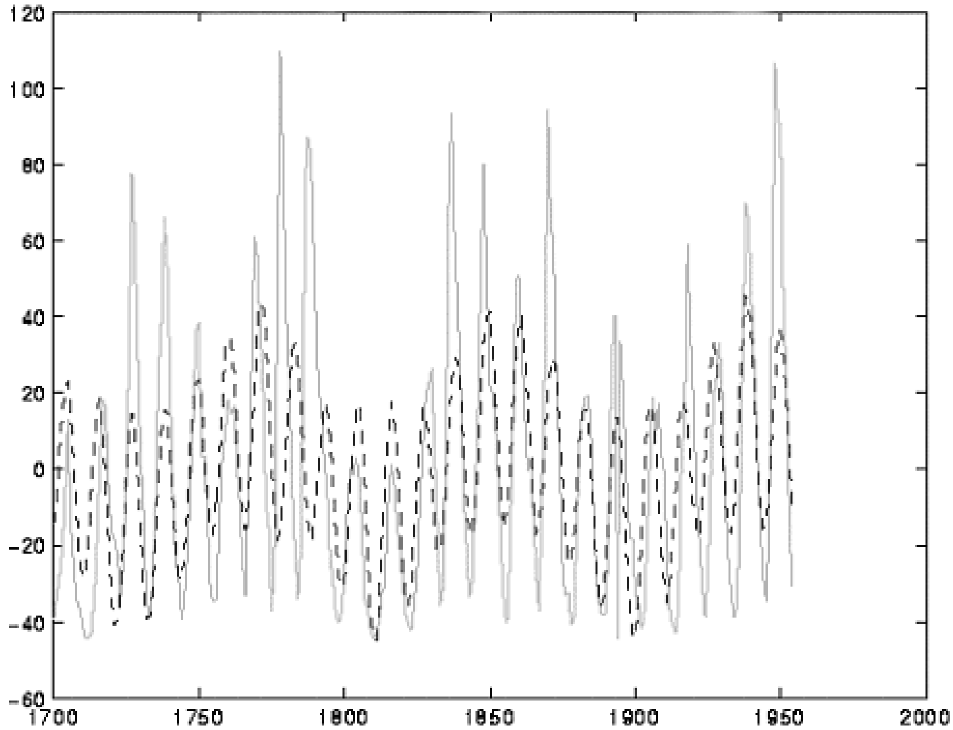

Figure 3: Sunspot data analyzed by Chiu procedure. The dotted line represents the original data and the solid line the fitted data

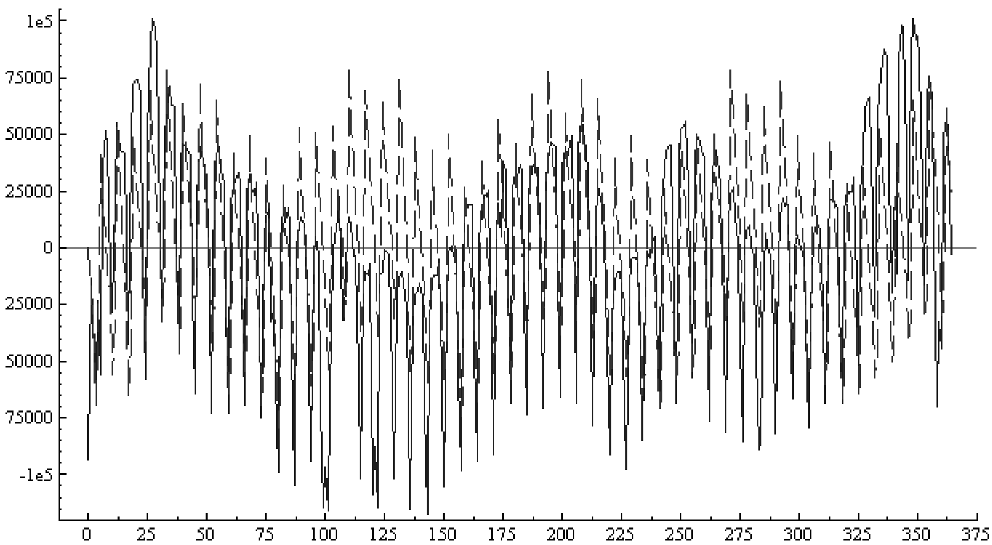

Figure 4: Electricity demand analyzed by Priestley-Bhansali. The dotted line represents the original data and the solid line the fitted data

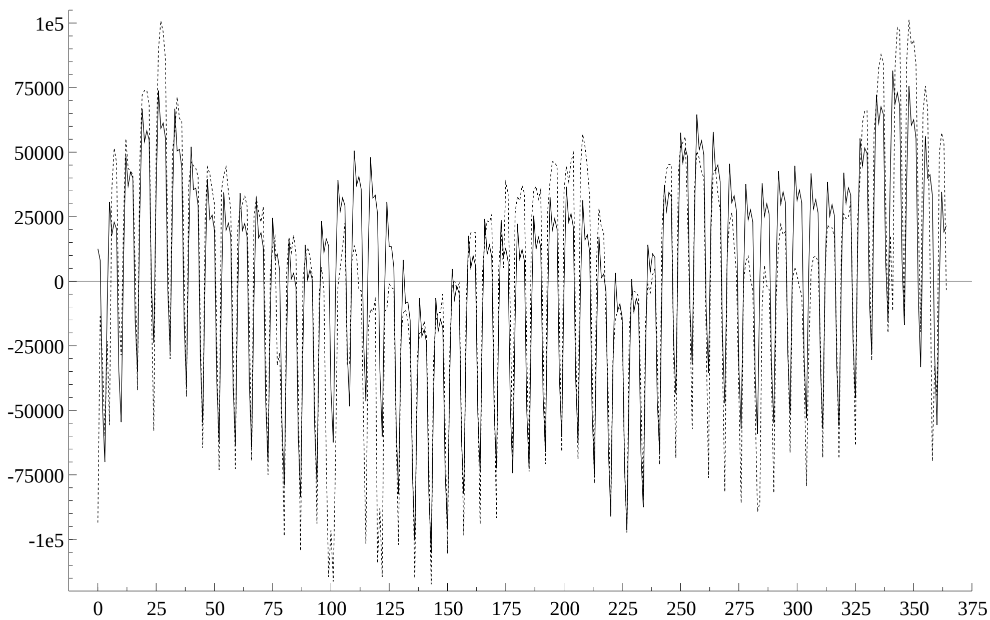

Figure 5: Electricity demand analyzed by Chiu procedure. The dotted line represents the original data and the solid line the fitted data

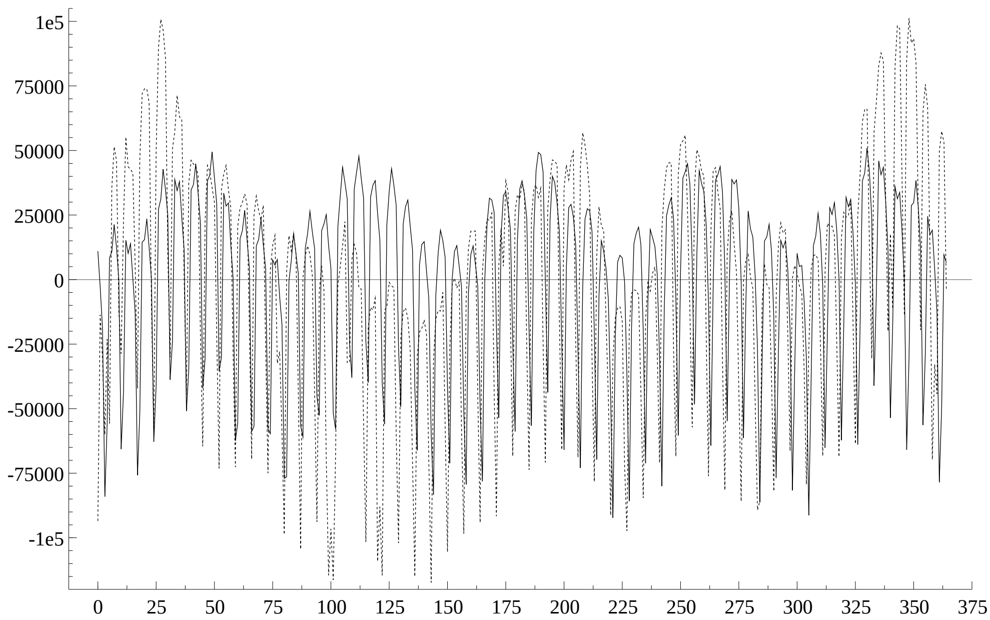

Figure 6: Electricity demand analyzed by Troung Van procedure. The dotted line represents the original data and the solid line the fitted data

Table 1: Power comparisons
Entries in the cell are the proportion of rejection
frequencies of the null (i.e. the power of the test)

	Model 1			Model 2			Model 3		
Test	T=150	T=300	T=500	T=150	T=300	T=500	T=150	T=300	T=500
Priestley − Bhansali	63.09	78.48	81.33	28.35	37.46	59.48	21.72	30.12	39.86
$U(r)$	18.72	20.04	19.57	71.57	79.96	92.13	68.62	74.44	88.82
$V(r)$	19.36	20.44	19.79	70.79	83.46	89.6	66.37	73.52	84.03

Table 2: Tests' performance: Model A

HCI is the % of harmonics correctly identified, HOI is the % of harmonics over identified, HUI is the % of harmonics under identified, $B(\omega)$ is the bias of the test for harmonic with frequency ω, $V(\omega)$ is the variance of the test for harmonic with frequency ω and $RMSE(\omega)$ is the root mean square error of the test for harmonic with frequency ω

	Test I			Test IIA			Test IIB			Test III		
	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500
HCI	15.92	16.12	14.03	90.92	93.65	94.89	82.61	82.78	82.98	19.58	19.74	21.58
HOI	81.83	83.65	85.96	6.08	6.35	5.11	14.53	17.22	17.02	26.54	34.58	38.44
HUI	2.25	0.23	0.01	3	0	0	2.86	0	0	53.88	45.68	39.98
$B(\omega)$	0.02	-0.004	-0.003	0.0055	0.0038	0.0023	0.0056	0.0035	0.0028	-0..226	-0.125	-0.077
$V(\omega)$	0.051	0.024	0.0091	0.011	0.0082	0.0067	0.011	0.0096	0.0074	0.0199	0.0011	0.0047
$RMSE(\omega)$	0.227	0.155	0.095	0.1051	0.091	0.0819	0.1051	0.098	0.086	0.2665	0.129	0.1031

Table 3: Tests' performance: Model B

HCI is the % of harmonics correctly identified, HOI is the % of harmonics over identified, HUI is the % of harmonics under identified, $B(\omega)$ is the bias of the test for harmonic with frequency ω, $V(\omega)$ is the variance of the test for harmonic with frequency ω and $RMSE(\omega)$ is the root mean square error of the test for harmonic with frequency ω

	Test I			Test IIA			Test IIB			Test III		
	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500
HCI	15.65	8.31	14.51	66.41	95.68	97.8	69.23	84.7	84.98	12.82	15.33	16.67
HOI	73.04	90.65	83.93	1.58	2.75	2.17	9.49	14.61	15.02	14.97	20.09	22.42
HUI	11.31	1.04	1.56	32.01	1.57	0.03	21.28	0.69	0	72.21	64.58	60.91
$Bias(\omega_1)$	-0.001	-0.005	-0.003	0.4595	0.2648	0.1702	0.48	0.2649	0.1696	-0.243	0.1304	-0.081
$Bias(\omega_2)$	0.074	0.014	-0.002	0.446	0.264	-0.17	0.4608	-0.264	-0.17	-0.683	0.3431	-0.227
$Var(\omega_1)$	0.009	0.0015	-0.0011	0.1055	0.0147	0.0031	0.1146	0.0146	0.0032	0.005	0.0009	0.0002
$Var(\omega_2)$	0.158	0.029	0.002	0.0843	0.0132	0.0031	0.0793	0.0132	0.0032	0.1729	0.1428	0.0457
$RMSE(\omega_1)$	0.096	0.005	0.003	0.5627	0.2912	0.1791	0.5873	0.2913	0.1788	0.258	0.1338	0.0823
$RMSE(\omega_2)$	0.404	0.171	0.049	0.5321	0.288	0.1791	0.54	0.288	0.1788	0.7992	0.5104	0.3119

Table 4: Tests performance: Model C

HCI is the % of harmonics correctly identified, HOI is the % of harmonics over identified, HUI is the % of harmonics under identified, $B(\omega)$ is the bias of the test for harmonic with frequency ω, $V(\omega)$ is the variance of the test for harmonic with frequency ω and $RMSE(\omega)$ is the root mean square error of the test for harmonic with frequency ω

	Test I			Test IIA			Test IIB			Test III		
	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500	T= 150	T= 300	T= 500
HCI	4.72	1.89	64.4	3.46	56.29	95.95	68.17	86.54	86.33	8.29	10.82	13.31
HOI	92.07	95.85	8	0	0	0	7.33	12.71	13.67	12.05	9.68	8.92
HUI	3.21	2.26	27.6	96.54	43.71	4.05	24.5	0.75	0	79.66	79.5	77.77
$Bias(\omega_1)$	0.084	0.024	-0.003	0.4013	0.2639	0.1674	0.4461	0.2571	0.1675	-0.245	-0.113	-0.073
$Bias(\omega_2)$	0.113	0.095	-0.001	-0.4011	-0.2641	-0.1676	-0.423	-0.257	-0.168	-0.702	-0.332	-0.202
$Bias(\omega_3)$	0.038	0.014	0.0005	0.0003	0.0008	0.0001	0.0001	0.0005	0.0003	-0.066	0.1534	0.2404
$Var(\omega_1)$	0.054	0.008	0.0005	0.0914	0.0183	0.0035	0.1176	0.0159	0.0035	0.0058	0.0035	0.0011
$Var(\omega_2)$	0.0136	0.0004	0.0007	0.0914	0.0183	0.0035	0.0896	0.0148	0.0035	0.0595	0.0279	0.0342
$Var(\omega_3)$	0.0275	0.0001	0.0002	0.0002	0.0004	0.0006	0.0003	0.0002	0.0005	0.9724	0.9147	0.87
$RMSE(\omega_1)$	0.247	0.093	0.022	0.5024	0.2728	0.1776	0.5627	0.2865	0.1777	0.2567	0.1271	0.0806
$RMSE(\omega_2)$	0.163	0.097	0.026	0.5024	0.2729	0.1776	0.5231	0.2839	0.1777	0.7436	0.3721	0.2739
$RMSE(\omega_3)$	0.17	0.013	0.014	0.014	0.02	0.024	0.017	0.014	0.022	0.9883	0.9686	0.9632

Table 5: Harmonic frequencies for sunspot data

Correlation coefficient pertains to that between the actual and fitted values

	Priestley - Bhansali	Chiu	Truong-Van
	0.068747	0.0736	0.1195
	0.1164	0.3682	0.5375
	0.22585	0.4172	0.628
	0.5653	0.5645	3.1082
	0.62519	2.2089	—-
	—-	2.3317	—-
Correlation Coefficient	0.2749	0.5979	0.5314

Table 6: Harmonic frequencies for electricity data

Correlation coefficient pertains to that between the actual and fitted values

	Priestley - Bhansali	Chiu	Truong-Van
	0.077904	0.082780	0.08607
	0.30842	0.13682	0.30986
	0.58078	0.89736	0.58528
	1.7952	0.89745	0.89514
	1.7948		1.7903
	2.6915		2.6854
Correlation Coefficient	0.5598	0.83880	0.35531

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