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Risk-Sharing and Retrading in Incomplete Markets

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Abstract

At a competitive equilibrium of an incomplete-markets economy agents' marginal valuations for the tradable assets are equalized ex-ante. We characterize the finest partition of the state space conditional on which this equality holds for any economy. This leads naturally to a necessary and sufficient condition on information that would lead to retrade, if such information were to become publicly available after the initial round of trade.

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1 Introduction

Consider a two-period single-good economy with incomplete asset markets. It is well-understood that competitive equilibria in this setting are constrained efficient in the sense that a Pareto improvement cannot be achieved by reallocating the existing assets (Diamond (1967)), while being generically Pareto inefficient (see, for example, Magill and Quinzii (1996)). In other words, at a competitive equilibrium, agents' marginal valuations for the tradable assets are equal when evaluated ex-ante, but are typically not equal conditional on the true state of the world, for every realization of the uncertainty.

In this paper, we provide a characterization of the events conditional on which marginal valuations for assets are equalized in equilibrium. We define an *insurable event* to be a subset of states $\hat{S}$ with the property that, at a competitive equilibrium, agents' marginal valuations for assets are equal conditional on $\hat{S}$ for any economy, but generically not equal conditional on a strict subset of $\hat{S}$. Thus the set of insurable events is the finest partition of the state space conditional on which agents' asset valuations are equal in equilibrium for a generic economy. An insurable event is a generalization of the notion of an insurable state, i.e. a state for which the corresponding Arrow security can be replicated with the existing assets. We describe later the precise sense in which a claim that pays off only in an insurable event can be replicated.

The notion of an insurable event is closely tied to the kinds of information that would induce agents to rebalance their portfolios, if such information were to become publicly available after the initial round of trade. Retrading occurs after the arrival of information if and only if it generates disagreement among agents regarding the marginal value of assets. We show that information that affects only the relative probabilities of insurable events does not lead to retrade, while retrade does generically occur if the information alters the relative probabilities of states within an insurable event. For a generic economy, therefore, the latter condition is both necessary and sufficient for the information to lead to retrade.

While there is a substantial literature on trading in financial markets in response to news, little has been said on the characteristics of news that induces agents to retrade. A class of no-trade results can be traced back to Milgrom and Stokey (1982) who show that the arrival of information does not lead to retrade if the initial allocation is Pareto efficient. This leaves open the question of retrading in a competitive economy with incomplete markets. In this setting, Blume et al. (2006) provide sufficient conditions on a public signal such that retrade occurs for a generic economy. We generalize their result (in Theorem 4.3) by providing a weaker sufficient condition, for a broader class of public signals (including signals that induce a partition of the state space), and for an arbitrary asset structure.

The paper is organized as follows. We describe the economy in the next section. In Section 3, we introduce the notion of an insurable event and analyze its properties. Then, in Section 4, we consider a public signal observed by agents after the initial round of trade, and characterize the set of signals that lead to retrade.

2 The Economy

There are two periods, 0 and 1, and a single physical consumption good. The economy is populated by $H \geq 2$ agents, with typical agent $h \in H$ (here, and elsewhere, we use the same symbol for a set and its cardinality). Uncertainty, which is resolved at date 1, is described by S states of the world. The probability of state s is $\overline{\pi}_s$ ($\overline{\pi}_s > 0$ for all s, and $\sum_s \overline{\pi}_s = 1$).

Agent $h \in H$ has endowments $\omega_0^h > 0$ in period 0 and $\omega^h \in \mathbb{R}^S_{++}$ in period 1, and time-separable expected utility preferences with von Neumann-Morgenstern utility functions $u_0^h : \mathbb{R}_{++} \to \mathbb{R}$ for period 0 consumption and $u^h : \mathbb{R}_{++} \to \mathbb{R}$ for period 1 consumption. We assume that u^h is twice continuously differentiable, $u^{h\prime} > 0, u^{h\prime\prime} < 0$, and $\lim_{c\to 0} u^{h\prime}(c) = \infty$; the same assumptions apply to u_0^h.

Asset markets, in which $J \geq 2$ securities are traded, open at date 0. At date 1 assets pay off. Asset payoffs are given by the $S \times J$ matrix R, whose (s, j)'th element is r_s^j, the payoff of asset j in state s. We denote the j'th column of R by r^j and the s'th row of R by $r_s^\top$ (by default all vectors are column vectors, unless transposed). Thus r^j is the vector of payoffs of asset j, and r_s is the vector of asset payoffs in state s. We assume that $r_s \neq 0$ for all $s \in S$, and R has full column rank J. Markets are complete if $J = S$, and incomplete if $J < S$.

We parametrize economies by agents' date 1 endowments $\omega := \{\omega^h\}_{h\in H} \in \Omega := \mathbb{R}^{SH}_{++}$. Let $p \in \mathbb{R}^J$ be the vector of asset prices (date 0 consumption serves as the numeraire), and $y^h \in \mathbb{R}^J$ the portfolio of agent h. The consumption of agent h is then given by $c_0^h := \omega_0^h - p \cdot y^h$ at date 0, and $c_s^h := \omega_s^h + r_s \cdot y^h$ in state s at date 1. Let $y := \{y^h\}_{h\in H}$. A competitive equilibrium is defined as follows:

Definition 2.1 *Given an economy $\omega \in \Omega$, a competitive equilibrium consists of a portfolio allocation y, and prices p, satisfying the following two conditions:*

(a) *Agent optimization: $\forall h \in H$, y^h solves*

$$\max_{y^h \in \mathbb{R}^J} \Big(u_0^h\left[\omega_0^h - p \cdot y^h\right] + \sum_{s\in S} \overline{\pi}_s \, u^h\left[\omega_s^h + r_s \cdot y^h\right] \Big). \tag{1}$$

(b) *Market clearing:*

$$\sum_{h\in H} y^h = 0. \tag{2}$$

Notation. In our analysis we use the following shorthand notation for matrices. Given an index set $\mathcal{N}$ with typical element n, and a collection $\{z_n\}_{n\in\mathcal{N}}$ of vectors or matrices, we denote by $\text{diag}_{n\in\mathcal{N}}[z_n]$ the (block) diagonal matrix with typical entry z_n, where n varies across all elements of $\mathcal{N}$. In similar fashion, we write $[\ldots z_n \ldots_{n\in\mathcal{N}}]$ to denote the row block with typical element z_n, and analogously for column blocks. We drop reference to the index set if it is obvious from the context: for example $\mathrm{diag}_{h\in H}$ is shortened to diag_h, and $[\ldots z_s \ldots_{s\in S}]$ to $[\ldots z_s \ldots_s]$. We use the same symbol 0 for the zero scalar and the zero matrix; in the latter case we occasionally indicate the dimension in order to clarify the argument. A "$*$" stands for any term whose value is immaterial to the analysis.

3 Insurable Events and Risk-Sharing

We formalize the notion of an insurable event as follows. Consider a partition of S given by $\{S_1, \ldots, S_K\}$. For each $k \in K := \{1, \ldots, K\}$, let L_k be the subspace of $\mathbb{R}^J$ spanned by the vectors $\{r_s\}_{s\in S_k}$. We say that the subspaces $L_1, \ldots, L_K$ are linearly independent if $\sum_{k\in K} \ell_k = 0$, $\ell_k \in L_k$, implies $\ell_k = 0$ for all k. Henceforth, we choose the partition for which $L_1, \ldots, L_K$ are linearly independent, and K is maximal[1] (it is easy to check that there is a unique such partition). We denote this partition by $\mathcal{S}(R)$ and call $S_k \in \mathcal{S}(R)$ an *insurable event*.

We will show below (Theorems 3.1 and 3.2) that an insurable event as defined above is a subset of S satisfying the two properties stated in the Introduction, namely that (a) conditional on this event, agents' asset valuations are equal in equilibrium, and (b) conditional on a strict subset of this event, agents' asset valuations are not equal at any equilibrium, for a generic (i.e. open and dense)[2] subset of endowments. Thus, for a generic subset of endowments, $\mathcal{S}(R)$ is the finest partition of S conditional on which asset valuations are equalized across agents in equilibrium.

We denote the dimension of L_k by J_k. Thus we have $\sum_{k\in K} J_k = J$. Without loss of generality we can order the states in S so that the first S_1 states correspond to the event S_1, the following S_2 states correspond to the event S_2, and so on. The following lemma shows that the partition $\mathcal{S}(R)$ is invariant to changes in asset payoffs that do not affect the column span of R. Moreover, R is column-equivalent to a block-diagonal matrix, with each block corresponding to an insurable event S_k:

Lemma 3.1 *Suppose the asset payoff matrices R and R' are column-equivalent. Then $\mathcal{S}(R) = \mathcal{S}(R')$. Furthermore, R is column-equivalent to* $\mathrm{diag}_{k\in K}[R_k]$, *where R_k is an $S_k \times J_k$ matrix with* $\mathrm{rank}(R_k) = J_k$.

The proof is in the Appendix. We say that an insurable event S_k is *trivial* if it is a singleton, and *nontrivial* otherwise. A trivial insurable event consists of a single insurable state, while a nontrivial insurable event consists of two or more states, none of which is insurable. Note that a nontrivial insurable event exists if and only

[1]The same partition is employed by Geanakoplos and Mas-Colell (1989) (Section III) in order to characterize the degree of indeterminacy of equilibria with nominal assets.

[2]More precisely, given a subset E of Euclidean space, endowed with the relative Euclidean topology, we say that $E' \subset E$ is a generic subset of E if it is open and dense in E.

if markets are incomplete. Moreover, an insurable event S_k is nontrivial if and only if $S_k > J_k$.[3]

Lemma 3.1 implies in particular that, for each insurable event, a portfolio can be found that pays off only in that event. Unless an insurable event is trivial, however, not every payoff in that event can be replicated with the existing assets.

Example 1. Suppose there are four states of the world: $S = \{s_1, s_2, s_3, s_4\}$. Consider a nontraded cashflow that pays

$$d = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

There are two traded assets, a debt claim on d with face value 2, and a residual equity claim. Thus the asset payoff matrix is

$$R = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 2 & 1 \\ 2 & 2 \end{pmatrix}.$$

It is easy to verify there is only one insurable event, i.e. $\mathcal{S}(R) = \{S\}$. ||

Example 2. Consider the asset structure in Example 1. Suppose that, in addition to risky debt and levered equity, a riskfree asset is also available. Thus the asset payoff matrix is

$$R = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix},$$

which is column-equivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Therefore the set of insurable events is $\mathcal{S}(R) = \{S_1, S_2\}$, where S_1 is a trivial insurable event consisting of the single insurable state s_1, and $S_2 = \{s_2, s_3, s_4\}$ is a nontrivial insurable event. ||

An assumption commonly employed in the incomplete-markets literature is that the asset payoff matrix R is in general position, meaning that every $J \times J$ submatrix

[3] If $S_k = J_k$, the fact that $\text{rank}(R_k) = J_k$ implies that R_k is column-equivalent to the identity matrix, so that S_k is not an insurable event unless it is trivial.

of R is nonsingular. If R is in general position, and markets are incomplete, there is only one insurable event.[4] The argument is as follows. Suppose S_k is a nontrivial insurable event. Since, by the general position of R, any collection of J' rows of R, with $J' \leq J$, is linearly independent, we must have $S_k > J$. But this implies that $\dim(L_k) = J$. Hence there is no insurable event other than S_k. The converse is not true, however: the asset payoff matrix in Example 1 is not in general position; yet there is only one insurable event.

Henceforth, we assume that R takes the block-diagonal form $\text{diag}_k[R_k]$. Due to Lemma 3.1, this assumption is without loss of generality.

We now characterize risk-sharing at a competitive equilibrium in term of insurable events. The first-order conditions for the utility-maximization program (1),

$$\sum_{s\in S} \overline{\pi}_s\, u^{h\prime}\big[\omega_s^h + r_s \cdot y^h\big]\, r_s - u_0^{h\prime}\big[\omega_0^h - p \cdot y^h\big]\, p = 0, \qquad \forall h \in H, \tag{3}$$

imply that

$$\frac{\sum_{s\in S} \overline{\pi}_s\, u^{h\prime}\big[\omega_s^h + r_s \cdot y^h\big]\, r_s}{u_0^{h\prime}\big[\omega_0^h - p \cdot y^h\big]} = \frac{\sum_{s\in S} \overline{\pi}_s\, u^{\hat{h}\prime}\big[\omega_s^{\hat{h}} + r_s \cdot y^{\hat{h}}\big]\, r_s}{u_0^{\hat{h}\prime}\big[\omega_0^{\hat{h}} - p \cdot y^{\hat{h}}\big]}, \quad \forall h, \hat{h} \in H. \tag{4}$$

Thus asset valuations (by which we mean the marginal rates of substitution between assets and period 0 consumption) are equalized across agents when evaluated ex-ante, i.e. at the time of trading. This is just the standard result that competitive equilibria are constrained Pareto efficient. In order to economize on notation, we let

$$\mu^{h\hat{h}}(y,p) := \frac{u_0^{h\prime}\big[\omega_0^h - p \cdot y^h\big]}{u_0^{\hat{h}\prime}\big[\omega_0^{\hat{h}} - p \cdot y^{\hat{h}}\big]},$$

and use the shorthand $u_s^{h\prime} := u^{h\prime}\big[\omega_s^h + r_s \cdot y^h\big]$ and $u_s^{h\prime\prime} := u^{h\prime\prime}\big[\omega_s^h + r_s \cdot y^h\big]$. Then (4) can be written as

$$\sum_{s\in S} \overline{\pi}_s \left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime}\right) r_s = 0, \qquad \forall h, \hat{h} \in H. \tag{5}$$

Since the subspaces $L_1, \ldots, L_K$ are linearly independent, the following result is immediate:

Theorem 3.1 *At any equilibrium (y,p) of $\omega \in \Omega$,*

$$\sum_{s\in S_k} \overline{\pi}_s \left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime}\right) r_s = 0, \qquad \forall h, \hat{h} \in H;\ S_k \in \mathcal{S}(R).$$

[4] It is also worth noting that if R is in general position, so is any R' that is column-equivalent to R.

In other words, at a competitive equilibrium, asset valuations are equalized across agents not only unconditionally, but also conditional on any insurable event (and hence also conditional on a union of insurable events). Specializing to an insurable state, we have the standard result:

Corollary 3.1 *If s is an insurable state, then at any equilibrium (y,p) of $\omega \in \Omega$, $u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} = 0$, for all $h, \hat{h} \in H$.*

For a generic subset of endowments, the converse of Theorem 3.1 is true as well, so that we can strengthen the result as follows:

Theorem 3.2 *There is a generic subset $\hat{\Omega}$ of Ω, such that at any equilibrium (y,p) of $\omega \in \hat{\Omega}$,*

$$\sum_{s \in \hat{S}} \overline{\pi}_s \left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \right) r_s = 0, \qquad \forall h, \hat{h} \in H, \tag{6}$$

if and only if $\hat{S}$ is a union of insurable events.

Thus, for a generic subset of endowments, the set of insurable events $\mathcal{S}(R)$ is the finest partition of S conditional on which agents' asset valuations are equal in equilibrium.

The proof of Theorem 3.2 uses the transversality theorem. Since we also exploit transversality in the proofs of Theorems 4.1 and 4.2 in the next section, it is useful to summarize the argument here. Consider a function $\Psi : \mathbb{R}^n \times \mathcal{E} \to \mathbb{R}^m$, where $\mathcal{E}$ is an open subset of Euclidean space and $m > n$. For $e \in \mathcal{E}$, let Ψ_e be the function $\Psi(\cdot, e)$. The argument involves identifying such a function Ψ, such that the desired result can be formulated as $\Psi_e^{-1}(0) = \varnothing$, for every e in a generic subset of $\mathcal{E}$. We show that the Jacobian $D_{x,e}\Psi$ has full row rank at all zeros (x,e) of Ψ, i.e. Ψ is transverse to zero. By the transversality theorem, there is then a dense subset $\hat{\mathcal{E}}$ of $\mathcal{E}$ such that, for each $e \in \hat{\mathcal{E}}$, $\Psi_e : \mathbb{R}^n \to \mathbb{R}^m$ is transverse to zero. It follows that $\Psi_e^{-1}(0) = \varnothing$. In other words, the equation system $\Psi_e(x) = 0$ has no solution since the number of (locally) independent equations m exceeds the number of unknowns n. A standard argument (see, for example, Citanna et al. (1998)) establishes that the set $\hat{\mathcal{E}}$ is open, and hence a generic subset of $\mathcal{E}$.

Proof of Theorem 3.2:

We begin by characterizing a competitive equilibrium as a solution to a system of (locally) independent equations. Let $g(y) = 0$ and $f(y,p,\omega) = 0$ denote the equation systems (2) and (3) respectively. The tuple (y,p) is a competitive equilibrium for economy ω if and only if it satisfies

$$F(y,p,\omega) := \begin{pmatrix} f(y,p,\omega) \\ g(y) \end{pmatrix} = 0, \tag{7}$$

which consists of $JH + J$ equations, equal to the number of unknowns (y,p). The Jacobian of F can be written as follows:

$$D_{y,p,\omega}F = \begin{pmatrix} D_{y,p}f & D_\omega f \\ D_{y,p}g & 0 \end{pmatrix},$$

with

$$D_\omega f = \operatorname{diag}_h\big[\ldots \overline{\pi}_s\, u_s^{h\prime\prime} r_s \ldots_s\big]$$

and

$$D_{y,p}g = (\ldots I_J \ldots_h \quad 0),$$

where I_J is the $J \times J$ identity matrix. The matrix $D_\omega f$ has full row rank since R has full column rank. Clearly, $D_{y,p}g$ has full row rank as well.

We now proceed with the proof of the theorem. If $\hat{S}$ is a union of insurable events, the result follows from Theorem 3.1. To prove the converse, suppose $\hat{S}$ is not a union of insurable events. Then there is a nontrivial insurable event, which we can take to be S_1 without loss of generality, such that $\hat{S}$ contains some but not all elements of S_1. Hence we can write S_1 as the union of two nonempty and disjoint sets, $\hat{S}_1 := S_1 \cap \hat{S}$ and $\check{S}_1 := S_1 \backslash \hat{S}_1$. We reorder the set S_1 so that the states in $\hat{S}_1$ appear before the states in $\check{S}_1$.

Recall that R_1 is the first diagonal block of R corresponding to the insurable event S_1. Let

$$\hat{R}_1^* := \left(\begin{array}{c|c} \hat{R}_1 & \hat{R}_1 \\ \hline 0 & \check{R}_1 \end{array}\right),$$

where $\hat{R}_1$ consists of the rows of R_1 corresponding to the states in $\hat{S}_1$, while $\check{R}_1$ consists of the remaining rows of R_1, i.e. those corresponding to the states in $\check{S}_1$. Deleting the rows of $\hat{R}_1^*$ corresponding to the redundant rows of $\hat{R}_1$ and of $\check{R}_1$, we are left with a matrix whose diagonal blocks have full row rank. It follows that $\operatorname{rank}(\hat{R}_1^*) \geq \operatorname{rank}(\hat{R}_1) + \operatorname{rank}(\check{R}_1)$. Since S_1 is an insurable event, the row spaces of $\hat{R}_1$ and $\check{R}_1$ have a nontrivial intersection, implying that $\operatorname{rank}(\hat{R}_1) + \operatorname{rank}(\check{R}_1) > \operatorname{rank}(R_1) = J_1$. It follows that $\operatorname{rank}(\hat{R}_1^*) > J_1$.

Let $\hat{r}^j := [\ldots r_s^j \ldots_{s\in\hat{S}_1} \quad 0_{1\times\check{S}_1}]^\top$, a vector in $\mathbb{R}^{S_1}$. Since $\operatorname{rank}(\hat{R}_1^*) > J_1$, we can pick $j \in J_1$ such that $\hat{r}^j$ lies outside the column span of R_1. We fix such a value of j for the remainder of the proof. Due to the block-diagonal structure of R, the vector $[\hat{r}^j \quad 0_{1\times(S-S_1)}]^\top = [\ldots r_s^j \ldots_{s\in\hat{S}_1} \quad 0_{1\times(S-\hat{S}_1)}]^\top$ lies outside the column span of R. In other words, the matrix

$$A := \left(\begin{array}{cc} \ldots r_s^j \ldots_{s\in\hat{S}_1} & 0_{1\times(S-\hat{S}_1)} \\ \hline \multicolumn{2}{c}{\ldots r_s \ldots_{s\in S}} \end{array}\right) \tag{8}$$

has full row rank $J+1$.

It suffices to establish the theorem for the first two agents, h_1 and h_2. We will show that, for ω in a generic subset of Ω, there is no solution to the equation system

$$\Psi_1(y,p,\omega) := \begin{pmatrix} F(y,p,\omega) \\ \sum_{s\in\hat{S}} \overline{\pi}_s\big(u_s^{h_1\prime} - \mu^{h_1h_2}\, u_s^{h_2\prime}\big) r_s^j \end{pmatrix} = 0.$$

Since $j \in J_1$, $r_s^j = 0$ for all $s \notin S_1$, so that the sum over $\hat{S}$ in this equation system can be restricted to $\hat{S}_1$. Hence, the Jacobian $D_{y,p,\omega}\Psi_1$, evaluated at a zero (y, p, ω) of Ψ_1, is

$$\left(\begin{array}{c|cc} * & \multicolumn{2}{c}{\mathrm{diag}_h\left[\ldots \overline{\pi}_s\, u_s^{h\prime\prime} r_s \ldots_s\right]} \\ \hline D_{y,p}g & \multicolumn{2}{c}{0} \\ \hline * & \left[\ldots \overline{\pi}_s\, u_s^{h_1\prime\prime} r_s^j \ldots_{s\in\hat{S}_1} \quad 0_{1\times(S-\hat{S}_1)}\right] & * \end{array}\right).$$

The Jacobian is row-equivalent to

$$\left(\begin{array}{c|c|c} * & \ldots \overline{\pi}_s\, u_s^{h_1\prime\prime} r_s^j \ldots_{s\in\hat{S}_1} \quad 0_{1\times(S-\hat{S}_1)} & * \\ \hline * & \ldots \overline{\pi}_s\, u_s^{h_1\prime\prime} r_s \ldots_s & 0 \\ \hline * & 0 & \mathrm{diag}_{h\neq h_1}\left[\ldots \overline{\pi}_s\, u_s^{h\prime\prime} r_s \ldots_s\right] \\ \hline D_{y,p}g & 0 & 0 \end{array}\right),$$

which in turn is column-equivalent to

$$\left(\begin{array}{c|c|c} * & * & A \\ \hline * & \mathrm{diag}_{h\neq h_1}\left[\ldots \overline{\pi}_s\, u_s^{h\prime\prime} r_s \ldots_s\right] & 0 \\ \hline D_{y,p}g & 0 & 0 \end{array}\right),$$

where A is defined by (8). This matrix has full row rank since each of the diagonal blocks has that property. Therefore, so does the Jacobian $D_{y,p,\omega}\Psi_1$, at every zero of Ψ_1. Thus Ψ_1 is transverse to zero, and $\Psi_{1\omega}^{-1}(0) = \varnothing$ for all ω in a generic subset of Ω. $\square$

The following result is an immediate consequence of Theorem 3.2:

Corollary 3.2 *At any equilibrium* (y, p) *of* $\omega \in \hat{\Omega}$, $u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} = 0$, *for all* $h, \hat{h} \in H$, *if and only if* s *is an insurable state.*

This result can be established directly for a generic subset of endowments using standard arguments.

4 Information and Retrading

We wish to describe the kinds of (unanticipated) information that will induce agents to retrade. We assume that the information takes the form of a public signal correlated with the state of the world s that agents observe after trading at date 0, but before consumption takes place,[5] and before the uncertainty regarding endowments

[5]The assumption that information arrives before date 0 consumption is essentially just an analytical convenience. In our setup retrade occurs when the marginal rates of substitution between assets and date 0 consumption are not equal for a pair of agents. If information arrives after date 0 consumption, we can replace this by the equivalent condition that the marginal rate of substitution between a pair of assets is not equal for a pair of agents.

and asset payoffs is resolved. We consider the class of public signals that take finitely many values. Accordingly, we fix a finite set of possible "signal realizations" Σ, $\#\Sigma \geq 2$, with a typical element of Σ denoted by σ. A public signal can then be described by a probability measure on $S \times \Sigma$, i.e. by the probabilities $\pi := \{\pi_{s\sigma}\}_{s \in S, \sigma \in \Sigma} \in \mathbb{R}_+^{S\Sigma}$, where $\pi_{s\sigma}$ denotes $\text{Prob}(s, \sigma)$. Let $\pi_s := \text{Prob}(s) = \sum_\sigma \pi_{s\sigma}$, $\pi_\sigma := \text{Prob}(\sigma) = \sum_s \pi_{s\sigma}$, and $\pi_{s|\sigma} := \text{Prob}(s|\sigma) = \pi_{s\sigma}/\pi_\sigma$.

Since a public signal is completely described by the associated vector π, we refer to π itself as a public signal. Formally, a public signal lies in the set

$$\Pi := \left\{ \pi \in \mathbb{R}_+^{S\Sigma} \,\middle|\, \pi_s = \overline{\pi}_s, \forall s \in S;\ \pi_\sigma > 0, \forall \sigma \in \Sigma \right\}.$$

In other words, any public signal in Π is consistent with the uncertainty over fundamentals given by $\{\overline{\pi}_s\}_{s \in S}$. The condition on the marginal distribution over Σ is without loss of generality. This specification admits a range of possible signals. It includes those that have full support, with $\{s \in S \,|\, \pi_{s|\sigma} > 0\} = S$, for all σ. It also includes signals for which the support $\{s \in S \,|\, \pi_{s|\sigma} > 0\}$ is a strict subset of S for some signal realizations. A special case of the latter is one where the signal induces a partition of S.[6]

In the remainder of this section, we characterize the set of public signals that lead to retrade. Given an equilibrium (y, p), there is no retrade at π if and only if the equality of asset valuations which holds in equilibrium (condition (5)) also holds at π, i.e.

$$\sum_{s \in S} \pi_{s|\sigma} \left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \right) r_s = 0, \qquad \forall h, \hat{h} \in H;\ \sigma \in \Sigma.$$

As in Theorem 3.1, we can exploit the linear independence of the subspaces $L_1, \ldots, L_K$ to refine this no-retrade condition:

Lemma 4.1 *Given an equilibrium (y, p), there is no retrade at π if and only if*

$$\sum_{s \in S_k} \pi_{s|\sigma} \left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \right) r_s = 0, \qquad \forall h, \hat{h} \in H;\ S_k \in \mathcal{S}(R);\ \sigma \in \Sigma. \tag{9}$$

It is clear from (9) that a public signal that affects only the relative likelihood of insurable events does not generate retrade. Agents' asset valuations remain equal after the arrival of such a signal. For example, a public signal that induces a partition of S that contains $\mathcal{S}(R)$, or is equal to $\mathcal{S}(R)$, does not generate any retrade since it leaves the conditional distribution over S_k unchanged for every k. More generally, if π leads to retrade, it must belong to the following set:

$$\hat{\Pi} := \left\{ \pi \in \Pi \,\middle|\, \exists\, \sigma \in \Sigma,\ S_k \in \mathcal{S}(R)\ s.t.\ \{\pi_{s|\sigma}\}_{s \in S_k} \neq \alpha \{\overline{\pi}_s\}_{s \in S_k}, \forall \alpha \geq 0 \right\}.$$

This is the set of public signals π for which $\{\pi_{s|\sigma}\}_{s \in S_k}$ is not proportional to $\{\overline{\pi}_s\}_{s \in S_k}$ for some σ and some insurable event S_k. Of course, S_k must be nontrivial for this to

[6]We provide an example of such a signal in Example 3 at the end of this section.

be the case. Thus $\hat{\Pi}$ is empty if markets are complete. On the other hand, if markets are incomplete, $\hat{\Pi}$ is a generic subset of Π.[7]

Moreover, if markets are incomplete, for the generic subset of endowments $\hat{\Omega}$ for which Theorem 3.2 (and hence Corollary 3.2) holds, $u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \neq 0$ for every state s in a nontrivial insurable event S_k. While the no-retrade condition (9) is not necessarily violated for every $\pi \in \hat{\Pi}$ and $\omega \in \hat{\Omega}$, we show that an appropriate perturbation of either π or ω ensures that it is violated. More precisely, Theorem 4.1 establishes that, for every $\pi \in \hat{\Pi}$, retrade occurs for ω in a generic subset of $\hat{\Omega}$ (and hence of Ω).[8] Analogously, Theorem 4.2 shows that, for every $\omega \in \hat{\Omega}$, retrade occurs for π in a generic subset of $\hat{\Pi}$ (and hence of Π). Finally, Theorem 4.3 strengthens Theorem 4.2 by showing that retrade occurs for every public signal that is "sufficiently rich," in a sense that we shall make precise.

We say that an economy ω *admits a π-retrade* if at every equilibrium of this economy the public signal π leads to retrade for at least one value of σ.

Theorem 4.1 *Suppose markets are incomplete. Then, for any $\pi \in \hat{\Pi}$, there is a generic subset $\check{\Omega}(\pi)$ of $\hat{\Omega}$ such that every economy $\omega \in \check{\Omega}(\pi)$ admits a π-retrade.*

Proof:

Consider a π in $\hat{\Pi}$, and fix a σ and a nontrivial insurable event, which we can take to be S_1 without loss of generality, such that $\{\pi_{s|\sigma}\}_{s\in S_1}$ is not proportional to $\{\overline{\pi}_s\}_{s\in S_1}$. Let

$$R_1^* := \begin{pmatrix} \mathrm{diag}_{s\in S_1}[\pi_{s|\sigma}]R_1 & \mathrm{diag}_{s\in S_1}[\overline{\pi}_s]R_1 \end{pmatrix}.$$

We claim that $\mathrm{rank}(R_1^*) > J_1$. If $\pi_{s|\sigma} > 0$ for all $s \in S_1$, this is immediate from the following result, which can be deduced from Lemma 5 of Geanakoplos and Mas-Colell (1989):

FACT 1 *Let R_k be the diagonal block of R corresponding to a nontrivial insurable event $S_k \in \mathcal{S}(R)$. Consider nonzero scalars θ_s, θ'_s, $s \in S_k$, such that $\{\theta_s\}_{s\in S_k}$ is not proportional to $\{\theta'_s\}_{s\in S_k}$. Then, $\mathrm{diag}_{s\in S_k}[\theta_s]R_k$ and $\mathrm{diag}_{s\in S_k}[\theta'_s]R_k$ do not have the same column span.*

Suppose, on the other hand, that $\pi_{s|\sigma} = 0$ for some $s \in S_1$. Let $\mathring{S}_1$ be the set of states in S_1 for which $\pi_{s|\sigma} = 0$, and let $\mathring{R}_1$ be the $\mathring{S}_1 \times J_1$ submatrix of R_1 corresponding to these states. Similarly, let $\breve{S}_1$ be the remaining states in S_1, and $\breve{R}_1$ the submatrix of R_1 corresponding to these states. Then R_1^* is row-equivalent to

$$\left(\begin{array}{c|c} \mathrm{diag}_{s\in\breve{S}_1}[\pi_{s|\sigma}]\,\breve{R}_1 & \mathrm{diag}_{s\in\breve{S}_1}[\overline{\pi}_s]\,\breve{R}_1 \\ \hline 0 & \mathrm{diag}_{s\in\mathring{S}_1}[\overline{\pi}_s]\,\mathring{R}_1 \end{array}\right).$$

[7] Notice that since Π is not an open subset of $\mathbb{R}^{S\Sigma}$, a generic subset of Π is open in Π but not necessarily open in $\mathbb{R}^{S\Sigma}$ (an open subset of Π is the intersection of Π with an open subset of $\mathbb{R}^{S\Sigma}$; see footnote 2). In particular, a generic subset of Π may include public signals that induce a partition of S and hence lie on the boundary of Π.

[8] A special case of this result, when R is in general position (so that there is only one insurable event), can be found in Gottardi and Rahi (2011).

By the argument used in the proof of Theorem 3.2 to establish the rank condition for $\hat{R}_1^*$, we see that $\operatorname{rank}(R_1^*) > J_1$. It follows that the rank of

$$R^* := \Big(\operatorname{diag}_{s\in S}[\pi_{s|\sigma}]R \quad \operatorname{diag}_{s\in S}[\overline{\pi}_s]R\Big)$$

is strictly greater than J. Therefore, we can pick j such that $\operatorname{diag}_{s\in S}[\pi_{s|\sigma}]\, r^j$ lies outside the column span of $\operatorname{diag}_{s\in S}[\overline{\pi}_s]R$, and hence the matrix

$$B := \begin{pmatrix} \ldots \pi_{s|\sigma}\, r_s^j \ldots_{s\in S} \\ \ldots \overline{\pi}_s\, r_s \ldots_{s\in S} \end{pmatrix} \tag{10}$$

has full row rank $J+1$. We fix such a value of j for the remainder of the proof.

Recall that the equations describing an equilibrium are given by $F(y,p,\omega) = 0$ (equation (7)). We will show that, for a generic subset of Ω, there is no solution to the equation system

$$\Psi_2(y,p,\omega;\pi) := \begin{pmatrix} F(y,p,\omega) \\ \sum_{s\in S} \pi_{s|\sigma}\big(u_s^{h_1\prime} - \mu^{h_1 h_2} u_s^{h_2\prime}\big) r_s^j \end{pmatrix} = 0,$$

i.e. the no-retrade condition (9) is violated for the first two agents, h_1 and h_2. The Jacobian of Ψ_2 is

$$D_{y,p,\omega}\Psi_2 = \left(\begin{array}{c|cc} * & \operatorname{diag}_h\big[\ldots \overline{\pi}_s\, u_s^{h\prime\prime} r_s \ldots_s\big] & \\ \hline D_{y,p}g & 0 & \\ \hline * & \big[\ldots \pi_{s|\sigma}\, u_s^{h_1\prime\prime} r_s^j \ldots_s\big] & * \end{array}\right).$$

The Jacobian is row-equivalent to

$$\left(\begin{array}{c|c|c} * & \ldots \pi_{s|\sigma}\, u_s^{h_1\prime\prime} r_s^j \ldots_s & * \\ \hline * & \ldots \overline{\pi}_s\, u_s^{h_1\prime\prime} r_s \ldots_s & 0 \\ \hline * & 0 & \operatorname{diag}_{h\neq h_1}\big[\ldots \overline{\pi}_s\, u_s^{h\prime\prime} r_s \ldots_s\big] \\ \hline D_{y,p}g & 0 & 0 \end{array}\right),$$

which in turn is column-equivalent to

$$\left(\begin{array}{c|c|c} * & * & B \\ \hline * & \operatorname{diag}_{h\neq h_1}\big[\ldots \pi_{s|\sigma}\, u_s^{h\prime\prime} r_s \ldots_s\big] & 0 \\ \hline D_{y,p}g & 0 & 0 \end{array}\right),$$

where B is defined by (10). This matrix has full row rank since each of the diagonal blocks has that property. Therefore, so does the Jacobian $D_{y,p,\omega}\Psi_2$, at every zero of Ψ_2. Thus Ψ_2 is transverse to zero, and $\Psi_{2\omega}^{-1}(0) = \varnothing$, for every ω in a generic subset of Ω. This generic subset depends on π, which is a parameter of Ψ_2. Moreover, by taking the intersection of this set with $\hat{\Omega}$, we obtain the generic subset $\check{\Omega}(\pi)$.[9] $\square$

Theorem 4.1 shows that for π to lead to retrade it is not only necessary that it belong to $\hat{\Pi}$ but, for a generic subset of endowments, sufficient as well. Next we present our second retrading result which involves perturbing probabilities.

Theorem 4.2 *Suppose markets are incomplete. Then, every economy $\omega \in \hat{\Omega}$ admits a π-retrade for every $\pi \in \hat{\Pi}_1(\omega)$, a generic subset of $\hat{\Pi}$.*

Proof:

Fix a pair of agents h and $\hat{h}$, a nontrivial insurable event S_k, and a signal realization σ. It suffices to show that the no-retrade condition (9) is violated for these given values. Let $\hat{s}$ be a state in S_k, and let $j \in J$ be an asset which has a nonzero payoff in $\hat{s}$, i.e. $r^j_{\hat{s}} \neq 0$ (our assumption that $r_s \neq 0$ for all s ensures that there is such an asset).

Consider an economy $\omega \in \hat{\Omega}$. We will show that, for $\pi_{\hat{s}\sigma}$ in a generic subset of the interval $(0, \overline{\pi}_{\hat{s}})$, at every equilibrium (y, p), there is no solution to the equation system

$$\Psi_3(y, p, \pi_{\hat{s}\sigma}; \omega) := \begin{pmatrix} F(y, p; \omega) \\ \sum_{s \in S_k} \pi_{s\sigma} \big(u^{h'}_s - \mu^{h\hat{h}}\, u^{\hat{h}'}_s\big) r^j_s \end{pmatrix} = 0,$$

and thus the no-retrade condition (9) is violated. For any choice of $\pi_{\hat{s}\sigma} \in (0, \overline{\pi}_{\hat{s}})$, we can always choose $\{\pi_{\hat{s}\sigma'}\}_{\sigma' \neq \sigma}$, so that $\sum_\sigma \pi_{\hat{s}\sigma} = \overline{\pi}_{\hat{s}}$. Moreover, if $\pi_{\hat{s}\sigma}$ is in a generic subset of $(0, \overline{\pi}_{\hat{s}})$, a corresponding π is in a generic subset of Π. Clearly π must also lie in $\hat{\Pi}$, and hence belongs to a generic subset of $\hat{\Pi}$.

The Jacobian of Ψ_3, evaluated at a zero $(y, p, \pi_{\hat{s}\sigma})$ of Ψ_3, is

$$D_{y,p,\pi_{\hat{s}\sigma}} \Psi_3 = \left(\begin{array}{c|c} D_{y,p}F & 0 \\ \hline * & \big(u^{h'}_{\hat{s}} - \mu^{h\hat{h}}\, u^{\hat{h}'}_{\hat{s}}\big) r^j_{\hat{s}} \end{array}\right). \tag{11}$$

Since $\hat{s}$ is not an an insurable state, $u^{h'}_{\hat{s}} - \mu^{h\hat{h}}\, u^{\hat{h}'}_{\hat{s}} \neq 0$, by Corollary 3.2. Also, we have chosen asset j for which $r^j_{\hat{s}}$ is nonzero. Hence the lower right block of (11) is a nonzero scalar. Moreover, for $\omega \in \hat{\Omega}$, we see from the proof of Theorem 3.2 that $D_{y,p}\Psi_1$ has full rank, and therefore so does $D_{y,p}F$, at all zeros of F.

Therefore, the Jacobian $D_{y,p,\pi_{\hat{s}\sigma}}\Psi_3$ has full row rank, at every zero of Ψ_3. Thus Ψ_3 is transverse to zero, and $\Psi^{-1}_{3\pi_{\hat{s}\sigma}}(0) = \varnothing$, for every $\pi_{\hat{s}\sigma}$ in a generic subset of $(0, \overline{\pi}_{\hat{s}})$. This generic subset depends on ω, which is a parameter of Ψ_3. $\square$

In Theorem 4.1, the no-retrade condition (9) is violated by fixing a π in $\hat{\Pi}$ and perturbing endowments. The generic set $\check{\Omega}(\pi)$ therefore depends on π. In Theorem

[9] We choose to state Theorem 4.1 for a generic set of endowments that is a subset of $\hat{\Omega}$, even though this is not required by our argument, in order to facilitate comparison with our other results.

4.2, on the other hand, a violation of the no-retrade condition is achieved by fixing an ω in $\hat{\Omega}$ and perturbing π. The generic set $\hat{\Pi}_1(\omega)$ therefore depends upon ω. In our final result we consider economies in $\hat{\Omega}$, as in Theorem 4.2, and identify a subset of $\hat{\Pi}$ of "sufficiently rich" public signals, which does not depend on the economy under consideration, such that retrade occurs for every π in this set. A signal is "sufficiently rich" if it changes the relative probabilities of states in some nontrivial insurable event S_k not just for one value of σ (as is the case for $\pi \in \hat{\Pi}$), but independently for a number of values of σ that exceeds the degree of market incompleteness, $S_k - J_k$, in the event S_k. More precisely, we establish the result for the following set of public signals:

$$\hat{\Pi}_2 := \left\{ \pi \in \Pi \,\middle|\, \exists\, S_k \in \mathcal{S}(R) \; s.t. \; \operatorname{rank}(\Lambda_{\pi,S_k}) > S_k - J_k > 0 \right\},$$

where

$$\Lambda_{\pi,S_k} := \left(\begin{array}{c} \vdots \\ \ldots \pi_{s|\sigma} \ldots_{s \in S_k} \\ \vdots_{\sigma} \end{array} \right). \tag{12}$$

Clearly $\hat{\Pi}_2$ is a generic subset of $\hat{\Pi}$.[10]

Theorem 4.3 *Suppose markets are incomplete. Then, every economy $\omega \in \hat{\Omega}$ admits a π-retrade, for every $\pi \in \hat{\Pi}_2$.*

Proof:

Consider an economy $\omega \in \hat{\Omega}$, an equilibrium (y, p), a nontrivial insurable event S_k, and a $\pi \in \hat{\Pi}$ satisfying $\operatorname{rank}(\Lambda_{\pi,S_k}) > S_k - J_k$. Suppose there is no retrade at π. Then, the no-retrade condition (9) holds for an arbitrary pair of agents h and $\hat{h}$:

$$\sum_{s \in S_k} \pi_{s|\sigma} \left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \right) r_s = 0, \qquad \forall \sigma \in \Sigma,$$

which can be rewritten as follows:

$$\Lambda_{\pi,S_k} \operatorname{diag}_{s \in S_k} \left[\left(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \right) \right] R_k = 0. \tag{13}$$

By Corollary 3.2, $u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime} \neq 0$, for all $s \in S_k$. Therefore, the rank of $D_k := \operatorname{diag}_{s \in S_k}\big[\big(u_s^{h\prime} - \mu^{h\hat{h}}\, u_s^{\hat{h}\prime}\big)\big] R_k$ is J_k. Let $\mathcal{D}_k$ be the column space of D_k. Equation (13) implies that the rows of Λ_{π,S_k} lie in $\mathcal{D}_k^{\perp}$, the orthogonal complement of $\mathcal{D}_k$ in $\mathbb{R}^{S_k}$. It follows that $\operatorname{rank}(\Lambda_{\pi,S_k}) \leq \dim(\mathcal{D}_k^{\perp}) = S_k - J_k$, a contradiction. $\square$

The theorem generalizes Theorem 5 of Blume et al. (2006). They impose a stronger full rank condition on the public signal; in our notation, their assumption

[10] The rank condition in the definition of $\hat{\Pi}_2$ allows for the possibility that $\{\pi_{s|\sigma}\}_{s \in S_k}$ is proportional to $\{\overline{\pi}_s\}_{s \in S_k}$ for some values of σ.

is that $\text{rank}(\Lambda_{\pi,S_1} \dots \Lambda_{\pi,S_K}) = S$, or that the matrix (12) defined over S rather than S_k has full column rank. Moreover, they only consider public signals that have full support, and hence do not include those that induce a partition of S. They also assume that there is an insurable state, and that for each state there is at least one asset whose payoff is positive in that state.

While in Theorems 4.1 and 4.2 it sufficed to consider a public signal for only two values of σ, for example an appropriate choice of $\{\pi_{s|\sigma_1}\}_{s\in S_k}$ for which there is retrade conditional on σ_1, and a corresponding choice of $\{\pi_{s|\sigma_2}\}_{s\in S_k}$, π_{σ_1} and π_{σ_2} in order to ensure that $\pi_{s\sigma_1} + \pi_{s\sigma_2} = \overline{\pi}_s$ for all $s \in S_k$, Theorem 4.3 requires an independent change in information across at least $S_k - J_k$ values of σ.

The sets $\hat{\Pi}_1(\omega)$ and $\hat{\Pi}_2$, i.e. the generic subsets of $\hat{\Pi}$ identified by Theorems 4.2 and 4.3 for which retrade occurs, are not nested in general.

Example 3. Suppose there are three equally likely states, i.e. $S = \{s_1, s_2, s_3\}$ and $\overline{\pi}_s = 1/3$ for all s, and the asset payoff matrix is given by

$$R = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the set of insurable events is $\mathcal{S}(R) = \{S_1, S_2\}$, where $S_1 = \{s_1, s_2\}$ is a non-trivial insurable event and S_2 consists of the single insurable state s_3. Suppose $\Sigma = \{\sigma_1, \sigma_2\}$. Consider a public signal that induces the partition $\{S_1, S_2\}$, i.e. the conditional probabilities over the three states are given by $(1/2, 1/2, 0)$ for one value of σ and $(0, 0, 1)$ for the other. This signal is not in $\hat{\Pi}$ and therefore does not generate any retrade. On the other hand, consider a signal that induces the partition $\{\{s_1\}, \{s_2, s_3\}\}$, e.g. with the conditional probabilities given by $(1, 0, 0)$ for σ_1 and $(0, 1/2, 1/2)$ for σ_2. This signal does lie in $\hat{\Pi}$. Moreover,

$$\Lambda_{\pi,S_1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

which has rank equal to 2, greater than $S_1 - J_1 = 1$. By Theorem 4.3, there is retrade for every economy in $\hat{\Omega}$. Of course, it is not necessary that the public signal induce a partition of S in order to generate retrade. $\|$

A Appendix

Proof of Lemma 3.1:

The matrices R and R' are column-equivalent if and only if $R' = RX$, for some $J \times J$ nonsingular matrix X. Let $\mathcal{S}(R) = \{S_1, \ldots, S_K\}$ be the set of insurable events for R, and let $\bar{R}_k$ be the $S_k \times J$ submatrix of R consisting of the rows of R corresponding to the states in S_k. Similarly, let $\bar{R}'_k$ be the $S_k \times J$ submatrix of R' corresponding to S_k. Consider a vector $a \in \mathbb{R}^S$, and let $a_k \in \mathbb{R}^{S_k}$ be the elements of a corresponding to S_k. We have $a^\top R' = a^\top RX$ and $a_k^\top \bar{R}'_k = a_k^\top \bar{R}_k X$.

Now suppose $a^\top R' = 0$. Then $a^\top R = \sum_{k \in K} a_k^\top \bar{R}_k = 0$. Since the subspaces $\{L_k\}$ are linearly independent, we must have $a_k^\top \bar{R}_k = 0$, for all k. It follows that $a_k^\top \bar{R}'_k = 0$, for all k, and hence the subspaces $\{L'_k\}$ are linearly independent. Moreover, since $\{L_k\}$ is a maximal set of linearly independent subspaces, so is $\{L'_k\}$. This establishes that $\mathcal{S}(R) = \mathcal{S}(R')$.

We now show that there exists a $J \times J$ nonsingular matrix X such that RX has the block-diagonal structure in the statement of the theorem. Let M_k be the J_k-dimensional subspace of $\mathbb{R}^J$ that is the orthogonal complement of the subspace generated by $\{L_{\hat{k}}\}_{\hat{k} \neq k}$. We claim that the subspaces $\{M_k\}$ are linearly independent. Indeed, consider $m_k \in M_k$ such at $\sum_k m_k = 0$. Then, $\ell_k \cdot \sum_k m_k = 0$, for all $\ell_k \in L_k$. But $\ell_k \cdot m_{\hat{k}} = 0$, for all $\hat{k} \neq k$. Therefore, $\ell_k \cdot \sum_k m_k = \ell_k \cdot m_k = 0$, for all $\ell_k \in L_k$, i.e. m_k is orthogonal to L_k. By the definition of M_k, m_k is orthogonal to $L_{\hat{k}}$, for all $\hat{k} \neq k$. Consequently, m_k is orthogonal to $\mathbb{R}^J$, implying that it is zero. The same argument applies for all values of k.

Let X_k be a $J \times J_k$ matrix whose columns are a basis of M_k. Thus every column of X_k is orthogonal to every row of R that does not correspond to the states in S_k. Therefore, $\bar{R}_{\hat{k}} X_k = 0$, for all $\hat{k} \neq k$. Let $X := (X_1 \ldots X_K)$. Then $RX = \text{diag}_k[R_k]$, where $R_k := \bar{R}_k X_k$, an $S_k \times J_k$ matrix. Since the subspaces $\{M_k\}$ are linearly independent, X is nonsingular. This proves that R is column-equivalent to $\text{diag}_k[R_k]$. Moreover, $\text{rank}(R_k) = \text{rank}(\bar{R}_k) = J_k$. $\square$

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