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Loyola de Palacio Programme on Energy Policy

A REAL OPTIONS MODEL  
FOR ELECTRICITY CAPACITY EXPANSION

Joachim Gahungu and Yves Smeers



**EUROPEAN UNIVERSITY INSTITUTE, FLORENCE**  
**ROBERT SCHUMAN CENTRE FOR ADVANCED STUDIES**  
**LOYOLA DE PALACIO PROGRAMME ON ENERGY POLICY**

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## **Abstract\***

This paper proposes a real option capacity expansion model for power generation with several technologies that differ in operation and investment costs. The economy is assumed perfectly competitive and the instantaneous payoff accruing from the generation system is the instantaneous welfare defined as the usual sum of consumer and producer surplus. The computation of this welfare requires the solution of a multi- technology optimization problem and the obtained optimal function value is not additively separable in generation capacities, contrary to what is generally assumed in multi asset real option models to prove the optimality of a myopic behavior. Using the geometric Brownian motion as uncertainty driver we propose two regression models to approximate the instantaneous welfare. A first, additively separable approximation implies the optimality of myopia. The second approximation is non separable and hence forces to take myopic behavior as an assumption. Using myopia as an assumption, we propose a semianalytic method which combines Monte Carlo simulations (used to compute the value of the marginal capacity) and analytical treatment (to solve an optimal stopping problem on a regression scheme).

## **Keywords**

Real options, capacity expansion, power investment, optimal dispatch.

**JEL Classification:** L11, L94, C61

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# 1 Introduction

Increasing demand and price risks have pervaded the power system since many decades. Their development motivated the recourse to real options models and reasoning in the eighties. The flexibility of these models and their intuitive financial interpretation compared advantageously with the heavy and often opaque capacity expansion planning models of the time and motivated their development in practice. Risks are much higher today, due to global economic issues but also to new regulatory risks caused by the introduction of competition an environmental and renewable policies. This justifies revisiting former models of real options to explore their potential in the new context.

Applied real options models generally treat the problem of a single asset investment opportunity, using closed form solutions, binomial lattices or backward Monte Carlo.<sup>1</sup> In all cases they measure the value of an hypothetical additional plant and the optimal investment policy for that plant. These models rely on two critical simplifications:

1. The electricity price (or plant value) process is exogenous: it thus implicitly excludes strategic<sup>2</sup> interaction among firms and the impact of new capacities on the price process.
2. Assets are independent, in the sense that cash flows accruing to one asset are independent of the capacities of other assets.

These assumptions do not fit well with the realities of the power market. While interactions between investment decisions may be irrelevant for infinitesimal energy generators in a perfectly liquid market, a realistic price process has at least to take into account the impact of total generation capacity. Power markets are affected by price spikes and more generally by price distributions that critically depend on the existing capacity mix and the load and can thus be profoundly affected by the addition of a single new asset. Adding a new capacity creates a feedback effect on the price that is neglected by many real options models.

Referring to the second assumption, it is also clear that the value of a generation unit is affected by the presence of others. If cash flows independence can be an acceptable approximation in some cases, it is less defensible in multi-technology systems where synergies between technologies (generally neglected in real options models) can be important. For instance, it is well known that coupling wind farm and hydro with pump storage increases the value of the wind farm as potential wind overproduction can be stored in hydro installations.

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<sup>1</sup>Closed form solutions are used e.g. by Siddiqui and Fleten (2008) and Fleten and Näsäkkälä (2005, 2010) via calculus on spark spread options. Binomial lattices are used in Abadie and Chamorro (2008a, 2008b). Tseng and Barz (2002) and Gardner and Zhuang (2000) showed programming skills in this direction by implementing technical constraints. Monte Carlo using Longstaff and Schwartz (2001) procedure includes Abadie and Chamorro (2009).

<sup>2</sup>The use of “strategic” here is the one of the real options literature and differs from the common economic interpretation which generally implies an exercise of market power.

A second class of real options models goes beyond the single project evaluation and considers a schedule of capacity expansions. These models<sup>3</sup> do not assume individual values of plants irrespective of other assets capacities: the “value of the project” is determined simultaneously with the optimal expansion plan via the resolution of a stochastic control problem in which the price process is endogenous. These models offer a remedy to critics 1 and 2 in theory but they rely on restrictive simplifying assumptions (homogeneous agents and symmetric technologies) to be made mathematically tractable. For this reason, they can be categorized as *stylized real options* literature.<sup>4</sup>

Real options models, for all their attractive properties, may thus make some important simplifications that, combined with other effects such as construction lags and indivisibilities may distort the investment process away from optimality, possibly inducing cycles of capacity shortage or excesses.<sup>5</sup> But the obvious question is whether there is a clearly dominating alternative: capacity expansion models of the optimization type clearly offer an alternative to real options models and are not subject to critics 1 and 2. Their drawback is to quickly become intractable when extended to a stochastic setting.

This paper starts from the dichotomy between intuitive and tractable real options models that make considerable assumptions and more realistic stochastic capacity expansion optimization or equilibrium models that are however often opaque and quickly become intractable. The paper attempts to adapt stylized capacity expansion real options models to investments in generation assets. The analysis begins by exploring the reasons that make real options capacity expansion models so attractive and finds that they rely on a myopic behavior that is proved optimal under certain conditions but may not be justified in general.<sup>6</sup> We investigate the assumption of myopia<sup>7</sup> and find that it can be adapted rel-

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<sup>3</sup>See e.g. Pindyck (1988), Bertola (1989), Leahy (1993), Grenadier (2002), Aguerrevere (2003).

<sup>4</sup>Let us note that sequential investment problems (investments that needs several stages to complete or entry/exit problems) use exogenous price processes. Thus they are classified in one-time investment models. Their sequential nature make that they represent an interesting alternative to capacity expansion models. For readers interested by sequential investment problems under uncertainty, we mention a few works dealing with power capacity generation. Bar-Ilan and Strange (1997) propose a sequential model of investment with construction lags. Dangl (1999) and Bar-Ilan and Strange (1999) consider an investment model where the first stage decides the timing of investment and the second stage its size. Décamps et al. (2006) focus on the choice between two mutually exclusive projects. In all these models, the price process is exogenous and the investment opportunity is a staged one-shot project (to be distinguished with entry/exit investment projects—i.e., hysteresis sequence—like Dixit, 1989, 1992; and Bar-Ilan and Strange, 1996).

<sup>5</sup>System dynamic literature on energy investment pointed out the importance of forward-looking. See Ford (1997), Bunn and Larsen (1992, 1997) and the literature therein.

<sup>6</sup>Needless to say one can also expect that what can be proved to hold true under certain (particular) conditions remains valid even if one cannot prove it true under different (possibly more general) settings.

<sup>7</sup>The optimality of myopia was stated by Leahy (1993) for perfect competition under decreasing return to scale technologies. See Baldursson and Karatzas (1997) for a rigorous mathematical derivation of this result in perfect competition, Grenadier (2002) for an extension in symmetric Cournot games, and Back and Paulsen (2009) for a critique of the result in Cournot games.

atively easily to situations where its optimality cannot be proved but remains intuitively reasonable in the current highly risky economic environment. These models are intuitive adaptations or substitutes of rational expectations models. Needless to say real options models based on this myopic principle may distort investments compared to rational expectation models, possibly resulting in cycles of under or over investments. But capacity expansion models that deal with uncertainties also require simplifications techniques to be solved and their outcome in these conditions is not guaranteed. All this should at some point be checked empirically; but this is beyond the scope of this paper.

The power sector departs in at least three ways from assumptions commonly found in real options capacity expansion models (e.g. Leahy, 1993; Baldursson and Karatzas, 1997 and Aguerrevere, 2003). We contribute to the literature by implementing these particularities.

1. *Profits accruing from new capacities are not given in closed form but need to be computed numerically by an optimization problem.* Plant revenues are determined by an auction that takes the form of an optimization problem. Optimization is not new in the real options literature as small optimization problems were used to determine cash flows at least since Brennan and Schwartz (1985), Pindyck (1988) and He and Pindyck (1992). These optimization models were always small enough for the revenue to be written in closed form, therefore facilitating or even enabling the construction of the solution. We believe that we offer the first treatment of a problem that does not lead to a closed form but requires the computation of a cash flow that is not additively separable.
2. *Electricity is a differentiated product.* Electricity is not storable and its demand differs through time. At minimal product differentiation distinguishes between peak and off-peak load. In this model we differentiate electricity in different time segments.
3. *Technologies differ by both operation and investment costs.* Product differentiation requires an efficient generation mix involving technologies that differ in operation and investment costs. In contrast, real options capacity expansion models usually assume one technology or at best several technologies that all have identical investment cost in order to ensure mathematical tractability.

As we will see in Section 2.2, product differentiation and different technologies combine to give a non additively separable profit flow that considerably complicates the underlying mathematics of real options capacity expansion models.

These ideas structure the paper. Section 2 introduces the power capacity expansion model and discusses the optimality of myopia. Section 3 presents an analytic method to tackle an approximate problem; it establishes investment triggers and discusses implications in terms of optimality of myopia. Section 4

proposes semi-analytic method, based on myopia and using Monte Carlo simulations. Section 5 discusses the benefits of solving a capacity expansion problem using an assumption of myopia. Section 6 treats a four technologies example and Section 7 concludes.

## 2 The basic model

### 2.1 A capacity expansion model for competitive power market

Consider a competitive power market characterized at time  $t$  by a set of technologies  $\mathbb{K}$ , a capacity vector  $K_t \in \mathfrak{R}_+^{d(\mathbb{K})}$  ( $K(k)$  is the capacity in technology  $k \in \mathbb{K}$ ;  $d(\mathbb{K})$  is the dimension of the set  $\mathbb{K}$ ), an activity index  $Y_t$  (e.g. the GDP) and a instantaneous welfare  $\Psi(Y_t, K_t)$ .<sup>8</sup>

We assume that the evolution of the activity index  $Y$  through time is subject to random disturbances. The mathematical model of  $Y$  is (as often in real options model) the Ito integral (which is time-continuous); we further assume that  $Y$  is a diffusion. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space, and  $\{\mathcal{F}_t \subset \mathcal{F}, t \in [0, \infty[ \}$  a family of  $\sigma$ -algebras contained by  $\mathcal{F}$ , increasing in  $t$ , right-continuous and completed by sets of  $\Omega$  having  $\mathbb{P}$ -measure zero.  $\mathcal{F}_t$  represents the information available to agents at time  $t$ . We note  $Y_s^y(\omega) : \Omega \times \mathfrak{R}^+ \rightarrow \mathfrak{R}$  an Ito-diffusion  $Y$  starting at  $Y_0 = y$ . We will often consider  $Y_s^Y(\omega)$  i.e. the Ito-diffusion  $Y$  starting at  $Y_0 = Y$ . This slight abuse of notation ( $Y$  simultaneously designates a stochastic process and its initial value) allows one to formulate problems as in other real options papers and should not lead to confusion.

It is well known since Lucas and Prescott (1971) that real options models of capacity expansion under uncertainty in perfectly competitive markets can be described by an optimal control problem where a benevolent social planner develops the capital stock  $K$  in order to maximize the expected time integrated discounted instantaneous welfare (that is consumer willingness to pay) net of expansion costs. We adopt this formulation that considerably facilitates the discussion (see Dixit and Pindyck, 1994, Chapter 9 for a detailed argument on the equivalence perfect competition/maximization of the welfare in the real options context).

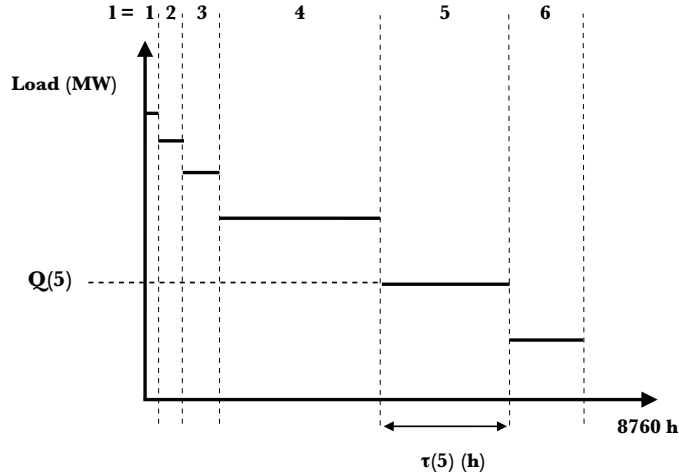
#### 2.1.1 The instantaneous welfare problem

We model the instantaneous welfare created by the generation system as the sum of consumer and producer surplus of satisfying a time-dependent demand represented by a load duration curve. The load duration curve gives the load (the demand of energy per unit time in MW) during a reference period such as a year, a month or a day. The load duration curve refers to a year in this paper. We use a standard piecewise constant approximation of the load duration curve of the power market as depicted in Figure 1: demand data are ordered in

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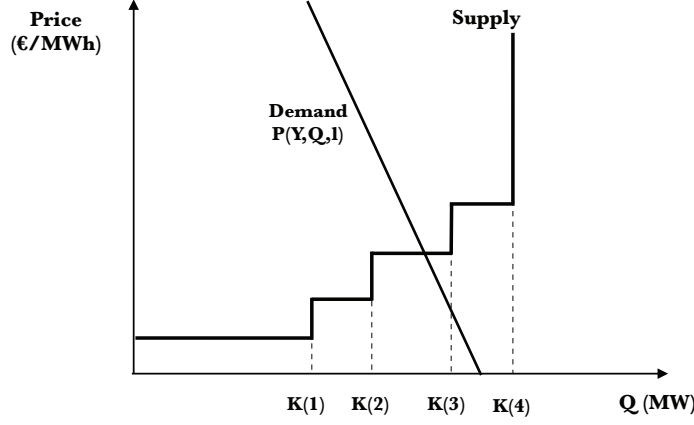
<sup>8</sup>The welfare is the sum of producer and consumer surplus.

descending order of magnitude. Let  $\mathbb{L}$  be the set of sub-periods for which the load level is supposed constant and  $\tau(l)$ ,  $l \in \mathbb{L}$  the duration of these sub-periods in hour per year ( $\sum_{l \in \mathbb{L}} \tau(l) = 8760h/y$ ). Let  $P(Y, q, l)$  be the inverse demand curve in segment  $l$ .



**Figure 1 :** Load duration curve for  $\text{card}(\mathbb{L}) = 6$ .  $\mathbb{L}$  is the set of sub-periods for which the load level is supposed constant.  $\tau(l)$ ,  $l \in \mathbb{L}$  is the duration of these sub-periods in hour per year ( $h/y$ ) s.t.  $\sum_{l \in \mathbb{L}} \tau(l) = 8760$ .

Generators have operating flexibility: a generator using technology  $k$  during time segment  $l$  may produce any quantity  $q(k, l)$  up to capacity  $K(k)$ . The optimal plant dispatch maximizes over a year the consumer and producer surplus net of operation costs and operation and maintenance cost, respectively  $\sum_{k,l} c(k)q(k, l)$  and  $\sum_k OMC(k)K(k)$ . Figure 2 offers a representation of demand and supply curves for a power industry with 4 technologies and linear demand curve. The instantaneous welfare has the following mathematical expression.



**Figure 2** : Supply and demand curves in an industry with 4 technologies and linear demand curves  $P(Y, Q, l)$ ,  $l \in \mathbb{L}$ .

**Problem 1** (The social planner's instantaneous welfare). *The social planner instantaneous welfare  $\Psi(Y, K)$  is the value function of the following program*

$$\Psi(Y, K) \triangleq \max_q \sum_{l \in \mathbb{L}} \tau(l) \left\{ \int_0^{Q(l)} P(Y, x, l) dx - \sum_{k \in \mathbb{K}} c(k) q(k, l) \right\} - \sum_{k \in \mathbb{K}} OMC(k) K(k) \quad (1)$$

s.t.

$$q(k, l) \geq 0 \quad k \in \mathbb{K}, l \in \mathbb{L} \quad (2)$$

$$q(k, l) \leq K(k) \quad k \in \mathbb{K}, l \in \mathbb{L} \quad (3)$$

$$\sum_{k \in \mathbb{K}} q(k, l) = Q(l) \quad l \in \mathbb{L}. \quad (4)$$

where  $q \in \mathfrak{R}_+^{d(\mathbb{K}) \times d(\mathbb{L})}$  is the annual dispatch.

The welfare  $\Psi(Y, K)$  is expressed in  $\text{€}/y^9$  and does not explicitly depend on time. Note that a change of  $Y$  modifies the dispatch  $q$  in a complex way that depends on how  $Y$  affects the different demand segments via the functions  $P(Y, x, l)$ . Moreover,  $K$  also impacts the objective function through capacity constraints. It is easy to see that  $\Psi(Y, K)$  is not additively separable in  $K$ .

Since the demand function is downward sloping,  $\int_0^Q P(Y, x, l) dx$  is concave in  $Q$  for all  $Y$  and all  $l$  and the Problem 1 is convex. Thus,  $\Psi(Y, K)$  can easily

<sup>9</sup>The time sub periods  $\tau(l)$  are in  $h/y$ ;  $P(Y, x, l)$  and  $c(k)$  are in  $\text{€}/\text{MWh}$ ;  $OMC$  is in  $\text{€}/\text{MWy}$ ;  $q$ ,  $dx$  and  $K$  are in  $\text{MW}$  thus  $\Psi(Y, K)$  is a instantaneous welfare in  $\text{€}/y$ .

be solved numerically, for given  $Y$  and  $K$ . Moreover, the value function  $\Psi(Y, K)$  is concave in  $K$  (see Appendix A for a proof).

### 2.1.2 The capacity expansion problem

Working in continuous time and with an infinite horizon, we assume that investment is incremental and irreversible and that future cash flows are discounted at an annual rate  $\rho$ . Assume that the shock process and installed capacities are initially (at time zero) respectively  $Y$  and  $K$ . While observing the evolution of  $Y_s^Y(\omega)$ , the social planner develops (controls) its capital stock  $K_s^K : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+^{d(\mathbb{K})}$  at investment cost  $I \in \mathfrak{R}_+^{d(\mathbb{K})}$  so that the economic value of the industry expansion is maximized, under the constraint that investment is irreversible.  $K_s^K(\omega)$  (the controlled process) should thus be a non-decreasing left-continuous  $\mathcal{F}_s$ -measurable process. Call  $F(Y, K) : \mathfrak{R} \times \mathfrak{R}_+ \rightarrow \mathfrak{R}$  the value function of this expansion plan. The “future” of the initial configuration  $(Y, K)$  is driven by the two stochastic processes  $Y_s^Y(\omega)$  and  $K_s^K(\omega)$  and we call  $\mathbb{E}^{Y, K}$  the expectation w.r.t. the probability law  $\mathbb{P}^{Y, K}$  generated by these two processes (i.e.  $\mathbb{E}^{Y, K} [h(Y_t, K_t)] = \mathbb{E} [h(Y_t, K_t) | Y_0 = Y, K_0 = K]$ ). The optimal control problem to be solved by the social planner is the following.

**Problem 2** (The social planner’s capacity expansion). *Find the value function  $F(Y, K)$  and a non-decreasing, left-continuous and  $\mathcal{F}_s$ -measurable process  $K_s$ ,  $s \geq 0$  such that*

$$F(Y, K) = \sup_{\{K_s\}} \mathbb{E}^{Y, K} \left[ \int_0^{+\infty} \Psi(Y_s, K_s) e^{-\rho s} ds - \sum_{k \in \mathbb{K}} \int_0^{+\infty} I(k) e^{-\rho s} dK_s(k) \right].$$

## 2.2 Myopia as a link between (some) stochastic control problems and optimal stopping

The social planner’s capacity expansion problem is a stochastic control problem. But real options theory often states this problem as an optimal stopping problem on a marginal investment. Indeed, in the real options framework the relation between the two problems is not always made clear and the two problems are often implicitly assumed equivalent. The following clarifies this question that will become important later in the discussion.

Note that the control process  $K_s$  is constrained to be non decreasing, which makes the stochastic control Problem 2 non trivial. This type of control problem is called *singular* because the optimal control is usually “*bang bang*”.<sup>10</sup> A direct determination of the solution is difficult, but the bang bang nature of the control suggests proceeding through optimal stopping: solve first a related optimal stopping problem, and then verify that its value function can be integrated into the value function of the control problem. This connection (when it holds) is of

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<sup>10</sup>In our setting, the cost of control is linear with the control amplitude and there are no fix cost of controlling. We thus expect instantaneous control and not impulse control. See Harrison and Taksar (1983) and Harrison et al. (1983).

considerable interest: the optimal control is treated as a succession of optimal stopping problems which (by definition) take the control variables as constants. Needless to say this connection does not hold in general for stochastic control problems (nor for dynamic optimization or equilibrium). It is thus not surprising to find that this connection is not trivial to analyze and has only been rigorously established for particular cases.<sup>14</sup>

Economists developed an intuition for this equivalence that we refer to as the optimality of myopia.<sup>11</sup> This property—when it holds—is central to this paper: it assumes that an investment in a new incremental capacity is evaluated assuming that it is the last one in the horizon. This implies that the assessment does not consider other investments after the investment takes place. In other words the usual dynamic programming backward induction is limited to the (unknown) time when the investment takes place.

The optimality of myopia only holds under certain assumptions.

- I. Investments should be defined in an incremental way (optimality refers to a marginal investment).
- II. The economy is convex (return to scale are non increasing).
- III. Agents are homogeneous. This is trivially the case for a monopoly (e.g. Bertola, 1989, Pindyck 1988, 1993). This is also the case for perfect competition when all agents invest in the same technology.<sup>12</sup>
- IV. If the model incorporates several technologies, the profit should be additively separable (see He and Pindyck, 1992 and Benth and Reikvam, 1999) or the technologies should have same investment cost (Baldursson and Karatzas, 1997 and Dixit and Pindyck, 1994, Chapter 9, Section 1.B).<sup>13</sup>

### 2.2.1 An illustration of this equivalence in a single technology model

We now discuss why the multiplicity of technologies (assumption IV) can affect optimality of myopia without going too far in analytical details. Consider Problem 2 assuming a single technology: Let  $K \in \mathfrak{R}_+$  and  $\Psi(Y, K) \in \mathcal{C}^{2,1}(Y, K)$  (i.e. twice continuously differentiable in  $Y$  and once continuously differentiable in  $K$ ) is concave in  $K$  (i.e. the technology is subject to constant or decreasing return to scale). These assumptions are those of Pindyck (1988) and Bertola (1989)

<sup>11</sup>The term myopia in this context seems to appear for the first time in Leahy (1993). For a formal definition of myopia and a strict proof of optimality for perfect competition, see Baldursson and Karatzas (1997).

<sup>12</sup>In symmetric Cournot competition, one can also *construct* a planner problem. For this artificial planner, myopia is optimal (Grenadier, 2002). However, each agent has an incentive to preempt this myopic trigger (see Back and Paulsen, 2009).

<sup>13</sup>According to the work of Baldursson and Karatzas (1997), a symmetry in investment costs appears necessary to solve the problem when the profit flow is not additively separable. Under these conditions, Dixit and Pindyck (1994, Chapter 9, Section 1.B) show that investment occur in the various technologies so as to equalize marginal operation cost across firms. Their model is actually the only multi-technology capacity expansion model with non additively separable profit that provides an explicit calculation of the investment threshold.



models where myopia is proved optimal (for a formal proof, see Baldursson and Karatzas, 1997). In fact, myopia expresses an equivalence between the optimal control Problem 2 and a sequence of optimal stopping problems “invest myopically in marginal values of the expansion plan  $\frac{\partial F}{\partial K}(Y, K_i)$ ” for a non-decreasing sequence of capacities  $\{K_i\}_{i=1,2,\dots,\infty}$ . We now explain a bit more precisely this equivalence and characterize the timing and the scale of this sequence of investments.

Let  $\mathcal{S}$  be the set of random times that are stopping times w.r.t. the filtration generated by  $Y_s^Y$ . Recall that  $\mathbb{E}^Y$  is the expectation w.r.t. the probability law  $\mathbb{P}^Y$  generated by  $Y_s^Y$  (i.e.  $\mathbb{E}^Y[h(Y_t)] = \mathbb{E}[h(Y_t)|Y_0 = Y]$ ). For a stopping time  $\tau$ , define the function

$$C(Y, K_1, \tau) \triangleq \mathbb{E}^Y \left[ \int_0^\tau \frac{\partial \Psi}{\partial K}(s, K_1) e^{-\rho s} ds + e^{-\rho \tau} I \cdot 1_{\tau < \infty} \right]. \quad (5)$$

$C(Y, K_1, \tau)$  is the cost of *not investing in one unit up to a certain time  $\tau$  and then invest in this unit* when the initial capacity and random shock are  $K_1$  and  $Y$ , respectively. Indeed by maintaining the capacity level to  $K_1$ , one gives up the change of welfare  $(\partial \Psi / \partial K)(s, K_1)$  which becomes the opportunity cost of not investing. This function is myopic in the sense that it does not consider what happens after  $\tau$ . Now consider the optimal stopping problem

$$\tau^*(Y, K_1, \omega) \triangleq \arg \inf_{\tau \in \mathcal{S}} C(Y, K_1, \tau) \quad (6)$$

that is the problem of finding the optimal stopping time (i.e. policy) of not investing. Following the stochastic control literature, we call *risk function* and denote by  $r$  the value function of this optimal stopping problem, i.e.

$$r(Y, K_1) \triangleq C(Y, K_1, \tau^*(Y, K_1)). \quad (7)$$

Observe that if  $\tau^*(Y, K_1) = 0$  a.s.  $P^Y$  then  $r(Y, K_1) = I$  i.e. the minimal cost of not investing equals the investment cost. In this situation it might be relevant to invest. If  $\tau^*(Y, K_1) > 0$  a.s.  $P^Y$  then  $r(Y, K_1) < I$ . Thus  $r(Y, K_1) \leq I$ . Intuition suggests that the risk function  $r$  equals the *marginal value function* i.e.

$$r(Y, K) \triangleq \frac{\partial F}{\partial K}(Y, K). \quad (8)$$

A formal connection between  $r(Y, K)$  and  $\frac{\partial F}{\partial K}(Y, K)$  has been established in the singular stochastic control literature<sup>14</sup> for particular problems (e.g. the monotone, finite fuel or bounded velocity follower). Such a connection is proved

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<sup>14</sup>The seminal paper on singular stochastic control is attributed to Bather and Chernoff (1967) who study the optimal control of a spaceship with a restriction on the total exerted control (finite fuel follower). They solve an optimal stopping problem (determined on a heuristic ground) whose solution is after related with the optimal policy of the initial optimal control problem, by analytical arguments. The same approach is used in several consecutive papers; e.g. Harrison and Taylor (1978), Chernoff and Petkau (1978), Benes et al. (1980), Karatzas (1981), Harrison and Taksar (1983), Harrison et al. (1983), Karatzas (1983). In contrast, Karatzas and Shreve (1984, 1985, 1986) use probabilistic arguments to demonstrate

formally for our problem in Baldursson and Karatzas (1997) which establishes  $r(Y, K) \triangleq \frac{\partial F}{\partial K}(Y, K)$ . This formal connection is usually not established nor proved properly in real options capacity expansion models (Baldursson and Karatzas, 1997 and Back and Paulsen, 2009 being exceptions). Instead, real options capacity expansion models usually furnish heuristic arguments aimed at establishing a so called optimality of myopic behavior. We show now that this optimality of myopia, un-rigorously stated in most real options capacity expansion models is equivalent to the connection (8), rigorously proved in stochastic control literature.

The optimal stopping problem (6) leads to the free boundary problem ( $\mathcal{A}_Y$  is the generator of the diffusion  $Y$ )

$$(\mathcal{A}_Y - \rho) r(Y, K_1) + \frac{\partial \Psi}{\partial K}(Y, K_1) = 0 \quad Y \leq Y^*(K_1) \quad (9)$$

$$r(Y^*(K_1), K_1) = I \quad (10)$$

$$\frac{\partial r}{\partial Y}(Y^*(K_1), K_1) = 0 \quad (11)$$

that we refer to as the singular stochastic control formulation of the myopic investment. Suppose we have been able to prove that (8) holds. Then to solve the stochastic control Problem 2 requires to find the function  $F$  and the level  $Y^*(K)$  such that (8), (9), (10) and (11) hold. Two alternative (but equivalent) formulations of this problem (8)-(9)-(10)-(11) are found in the real options literature. They were derived heuristically: the connexion (8) were never formally proved.

A first formulation is found in Bertola (1989). It is trivially obtained by substituting  $r$  by  $\frac{\partial F}{\partial K}(Y, K)$  in (9)-(10)-(11) to give

$$(\mathcal{A}_Y - \rho) \frac{\partial F}{\partial K}(Y, K_1) + \frac{\partial \Psi}{\partial K}(Y, K_1) = 0 \quad Y \leq Y^*(K_1)$$

$$\frac{\partial F}{\partial K}(Y^*(K_1), K_1) = I$$

$$\frac{\partial^2 F}{\partial Y \partial K}(Y^*(K_1), K_1) = 0.$$

A second formulation is due to Pindyck (1988). Note that a solution of (9) is

$$r(Y, K_1) = m(Y, K_1) - f(Y, K_1) \quad (12)$$

where  $f$  and  $m$  are respectively the homogeneous solution and the particular

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the equivalence in more general settings (see also El Karoui and Karatzas 1988, 1989, 1991). For generalization of monotone followers type of control problems to evolved stochastic processes, see e.g. Alvarez (2000), Boetius and Kohlmann (1998) and Dufour and Miller (2004). Advanced singular stochastic control problems applied to investment were proposed by e.g. Kobila (1993), Baldursson and Karatzas (1997), Duckworth and Zervos (2000), Øksendal (2000), Guo and Pham (2005), Chiarolla and Haussmann (2005, 2007) and Merhi and Zervos (2007).

integral of (9) i.e.

$$(\mathcal{A}_Y - \rho) f(Y, K_1) = 0 \quad (13)$$

$$(\mathcal{A}_Y - \rho) m(Y, K_1) + \frac{\partial \Psi}{\partial K}(Y, K_1) = 0 \quad (14)$$

(we justify the minus sign in front of  $f(Y, K_1)$  in (12) at the end of the section). It is well known in stochastic analysis that (14) in fact *implies*<sup>15</sup> that

$$m(Y, K_1) = \mathbb{E}^Y \left[ \int_0^\infty e^{-\rho s} \frac{\partial \Psi}{\partial K}(Y_s, K_1) ds \right] \quad (15)$$

which is the *myopic value of the marginal unit* i.e. the value of a unit for  $K$  fixed at  $K_1$  over the entire time horizon. Using (12), (13) and (15), the free boundary problem (9)-(10)-(11) can be reformulated as

$$(\mathcal{A}_Y - \rho) f(Y, K_1) = 0 \quad Y \leq Y^*(K_1) \quad (16)$$

$$m(Y, K_1) = \mathbb{E}^Y \left[ \int_0^\infty e^{-\rho s} \frac{\partial \Psi}{\partial K}(Y_s, K_1) ds \right]$$

$$f(Y^*(K_1), K_1) = m(Y^*(K_1), K_1) - I \quad (17)$$

$$\frac{\partial f}{\partial Y}(Y^*(K_1), K_1) = \frac{\partial m}{\partial Y}(Y^*(K_1), K_1). \quad (18)$$

Equations (17) and (18) imply that  $f$  should be positive which justifies the minus sign in (12).

The first formulation is often (wrongly) referred to as a stochastic control resolution; in opposition to the second one which—through (15)—is seen as a myopic behavior formulation. But both formulations are a consequence (8), thus they are both myopic. The question is whether (8) can be extended to a more general problem, for instance multi-technology problems. This question is treated in the next subsection (Subsection 2.2.2).

Before we come to that and for sake of completeness, now knowing the timing of capacity additions, it remains to describe their amplitude. Decreasing return to scale (i.e. concavity of  $\Psi$ ) ensures that  $Y^*(K)$  is strictly increasing in  $K$  (and, in fact, that investment will be incremental and not lumpy; see Dixit, 1995 or Dixit and Pindyck, 1994, Chapter 11, Section 2). At the random time  $\tau$  when  $Y(\tau, \omega) > Y^*(K_1)$ ,  $K_1$  is increased to  $K_2 > K_1$  i.e. the (left-continuous) control process  $K_s$  makes a jump at  $\tau$ . The precise amplitude  $dK_\tau = K_{\tau+} - K_\tau = K_2 - K_1$  of this jump is such that  $Y(\tau, \omega) = Y(\tau_+, \omega) \leq Y^*(K_{\tau+})$  i.e. such that the stochastic process  $Y$  is brought back under the investment threshold of the current capacity  $K_{\tau+} = K_2$ . The procedure repeats itself on the new capital stock  $K_2$ .

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<sup>15</sup> $m$  is the *resolvent* operator of killing rate  $\rho$  applied to  $\frac{\partial \Psi}{\partial K}(K, Y)$ . See e.g. Øksendal (2007), Theorem 8.1.5 b).

### 2.2.2 Integrability in multi-technology models

We showed that optimality of myopia is a consequence of the connection (8). Suppose that the free boundary problem (9)-(10)-(11) has been solved to give the risk function  $r(Y, K)$ . The question is then whether (8) holds.

- ◊ If it does (i.e. myopia is optimal) then  $r(Y, K)$  can be integrated with respect to the state variable  $K$  to find the value function  $F(Y, K)$ .
- ◊ If it does not (i.e. myopia is not optimal),
  - either the function  $r$  (which is defined independently of optimality of myopia) is not integrable with respect to  $K$ ;
  - or it is integrable and its integral is a function  $G$  different from the value function  $F$ .

We thus see that integrability of the function  $r$  is a necessary but not a sufficient condition for optimality of myopia. In multi-technology models, for each technology  $k \in \mathbb{K}$ , one has to solve a free boundary problem (9)-(10)-(11) in order to determine a “candidate” risk function  $r_k(Y, K_k)$  (note that the PDE (9) for a particular technology  $k$  will depend on the partial derivative  $\partial\Psi(Y, K)/\partial K_k$ ). The integrability necessary condition requires to find a function  $G$  such that  $\partial G/\partial K_k = r_k$  for all  $k \in \mathbb{K}$  i.e.

$$\nabla_K G(Y, K) = r(Y, K) \tag{19}$$

componentwise where  $\nabla_K \triangleq (\partial/\partial K_1, \dots, \partial/\partial K_{d(\mathbb{K})})$  is the gradient operator with respect to the vector  $K$ . It turns out that if technologies have different investment costs and  $\Psi(Y, K)$  is not additively separable in  $K$ , one cannot find a  $G$  such that (19) holds (see He and Pindyck, 1992, page 586 and Benth and Reikvam, 1999). This explains assumption IV.

Assumption IV is quite restrictive regarding power generation since the profit flow of a power system is not separable in individual capacities and generation technologies differ in operation and investment costs as well as in other characteristics. The direct application of results drawn from existing real options models is thus generally impossible. This last remark hold true in particular for our model where  $\Psi(Y, K)$  is not additively separable and the technologies differ in investment cost.

## 3 An analytic resolution

Real options capacity expansion models are appealing because they propose closed-form optimal investment rule whose attractive economic interpretation and tractability make intuitive in small models and formally implementable in more complex macro types of models (e.g. system dynamics). Thus it makes sense to work out an analytical solution of an approximated version of our capacity expansion model.

The proposed resolution procedure is natural.  $\Psi(Y, K)$  is concave in  $K$ , but cannot be written in closed form. Thus an analytical resolution implies to approximate  $\Psi(Y, K)$  by a smooth analytical function  $\bar{\Psi}(Y, K) \in \mathcal{C}^{2,1}(Y, K)$ , using linear combination of basis functions. Of course, one can not just use any basis functions in the  $Y \times K$  space but one should find a sufficiently large span of basis functions enabling the analytic resolution of the stochastic control Problem 2.

Since  $\Psi$  is concave, it is relevant to require from  $\bar{\Psi}$  that it is also concave. The concavity of the immediate profit is an important property guaranteeing unicity of the optimal control, if an optimal control exists. We consider the two following situations.

1. Supposing a regression model that is concave and additively separable in the technologies. Then myopia is (de facto) optimal. This case is treated in Section 3.1.
2. Supposing a regression model concave but not additively separable, there is no theory to treat the problem. One can still use optimality of myopia as a heuristic. This case is treated in Section 3.2.

Note that the distinction of assumptions between the two approaches is somewhat blurred: starting from an additively separable fit  $\bar{\Psi}$  of  $\Psi$  does not require any myopia optimality assumption because separability implies myopia optimality. In contrast a non separable fit  $\bar{\Psi}$  of  $\Psi$  does not imply this optimality of myopia that needs thus be introduced explicitly. In both cases myopia optimality plays a central role, so we need basis functions allowing the analytical resolution of an optimal stopping problem.

We discuss these two options in this section, assuming that the Ito-diffusion  $Y$  driving the demand is a geometric Brownian motion

$$dY_t(\omega) = \mu Y_t dt + \sigma Y_t dB_t(\omega), \quad Y_0 \geq 0$$

whose infinitesimal generator is

$$\mathcal{A}_Y \triangleq \mu Y \frac{\partial}{\partial Y} + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2}{\partial Y^2}. \quad (20)$$

Recall that a perpetual American call option on  $Y$  has a value of the form  $aY^{\beta_1}$  where

$$\beta_1 = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2\rho}{\sigma^2}} \quad (21)$$

and  $a$  is a positive coefficient depending on the exercise price. If  $0.5\sigma^2 < \mu < \rho$ ,  $\beta_1$  is greater than 1 and tends to 1 as  $\sigma$  increases (see McDonald and Siegel, 1986).

The modeling of uncertainty by a GBM allows for a simple characterization of investment triggers. However, one can show (see ?, Appendix D) that the introduced solution procedure is general and can be used without loss of generality on other regular Ito diffusions; e.g. the standard Brownian motion, the Schwartz process or the geometric mean reverting process.

### 3.1 A separable instantaneous welfare

Assume that the regression  $\bar{\Psi}$  of the welfare function  $\Psi$  is concave and additively separable. Then the singular stochastic control/optimal stopping equivalence (i.e. myopia) holds using  $\bar{\Psi}$  instead of  $\Psi$  and the value of the project  $\bar{F}(Y, K)$  obtained using this approximation is given by

**Problem 3** (The social planner's optimal stopping problem). *The value of the project  $\bar{F}(Y, K)$  and the trigger vector  $Y^*(K) \in \mathfrak{R}^{d(\mathbb{K})}$  are solution of the free boundary problem*

$$(\mathcal{A}_Y - \rho) F(Y, K) + \bar{\Psi}(Y, K) = 0 \quad Y \leq Y^*(K), \quad (22)$$

$$\nabla_K F(Y^*(K), K) = I \quad (23)$$

$$\nabla_{K,Y}^2 F(Y^*(K), K) = 0. \quad (24)$$

where  $\mathcal{A}_Y$  is given by (20).

*Proof.* See He and Pindyck (1992) page 584, equation (14) with the assumptions that there is a single uncertainty and that the profit flow is additively separable.  $\square$

As discussed in Section 2.2, Problem 3 is Bertola's formulation of myopia. Another equivalent formulation (provided by Pindyck) is the following.

**Problem 4** (Myopic behavior of each agent). *Find the values  $f(Y) \in \mathfrak{R}^{d(\mathbb{K})}$  of perpetual American call options on the next marginal units  $m(Y, K) \in \mathfrak{R}^{d(\mathbb{K})}$  and the triggers  $Y^*(K) \in \mathfrak{R}^{d(\mathbb{K})}$  such that*

$$(\mathcal{A}_Y - \rho) f(Y, K) = 0 \quad Y \leq Y^*(K),$$

$$m(Y, K) = \mathbb{E}^Y \left[ \int_0^{+\infty} (\nabla_K \bar{\Psi})(Y_s, K) e^{-\rho s} ds \right]. \quad (25)$$

$$f(Y^*(K), K) = m(Y^*(K), K) - I \quad (26)$$

$$\nabla_Y f(Y^*(K), K) = \nabla_Y m(Y^*(K), K). \quad (27)$$

Here, myopia is explicitly expressed by the very definition of the function  $m(Y, K)$  in (25) where  $K$  is assumed forever constant in the computation of the right hand side integral. The two Problems 3 and 4 return the same  $Y^*(K)$ , and one sees that  $\nabla_K F(Y, K) = r(Y, K) = m(Y, K) - f(Y, K) \in \mathfrak{R}^{d(\mathbb{K})}$ .

Note however than Problem 4 (the myopic problem) can be defined even if myopia is not optimal i.e. if the value function  $\bar{F}(Y, K)$  is not given by Problem 3. In that case, either there is no function  $G$  such that  $\nabla_K G(Y, K) = m(Y, K) - f(Y, K)$ , or, if such  $G$  exists, it is different from  $\bar{F}$ . In order to use this separable formulation, we propose the following regression model.

**Regression Model 1** (additively separable). *Consider an interpolation of the profit flow  $\Psi$  by the additively separable function  $\bar{\Psi}$  given by*

$$\bar{\Psi}(Y, K) = \sum_{k=1}^{d(\mathbb{K})} \sum_{i,j=1}^{d(\gamma), d(\alpha)} b_{k,ij} Y^{\gamma_i} K^{\alpha_j}(k) - \sum_{k=1}^{d(\mathbb{K})} OMC(k) K(k) \quad (28)$$

where  $\gamma$  and  $\alpha$  are respectively positive base vectors of dimension  $d(\gamma)$  and  $d(\alpha)$  such that  $\forall i, 0 < \gamma_i < \beta_1; \forall j, 0 < \alpha_j \leq 1; b \geq 0$  and  $\beta_1$  given by (21).

The motivation of this approximating formula is that one can find the particular integral of each term of this interpolation. The homogeneous solution of the differential equation (22) is well known:  $F_h(Y, K) = A(K)Y^{\beta_1}$  with  $\beta_1$  given by (21). It is also well known in the literature that if  $\Psi(Y, K) = b_{t,ij}Y^{\gamma_i}K_t^{\alpha_j}$  (resp.  $c_{ut,ijk}Y^{\gamma_i}K_t^{\alpha_j}K_u^{\alpha_k}$ ) with  $0 \leq \gamma_i < \beta_1$ , the particular integral of (22) is  $F_p(Y, K) = \bar{b}_{t,ij}(\gamma_i)Y^{\gamma_i}K_t^{\alpha_j}$  (resp.  $\bar{c}_{ut,ijk}(\gamma_i)Y^{\gamma_i}K_t^{\alpha_j}K_u^{\alpha_k}$ ) with

$$\bar{b}_{t,ij}(\gamma_i) = \frac{b_{t,ij}}{\rho - \mu\gamma_i - \frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1)} \quad (29)$$

$$\left( \text{resp. } \bar{c}_{ut,ijk}(\gamma_i) = \frac{c_{ut,ijk}}{\rho - \mu\gamma_i - \frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1)} \right). \quad (30)$$

Note that it is compulsory that every power of  $Y$  is strictly lower than  $\beta_1$ .

Taking stock of these developments, the solution of the Bellman partial differential equation of the generation plan can be written as:

$$F(Y, K) = A(K)Y^{\beta_1} + \sum_{k=1}^{d(\mathbb{K})} \sum_{i,j=1}^{d(\gamma), d(\alpha)} \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j} - \sum_{k=1}^{d(\mathbb{K})} \frac{OMC(k)K(k)}{\rho}$$

and because the instantaneous welfare is now additively separable, one can write

$$F(Y, K) = \sum_{k=1}^{d(\mathbb{K})} F_k(Y, K_k)$$

with

$$F_k(Y, K_k) = A_k(K_k)Y^{\beta_1} + \sum_{i,j=1}^{d(\gamma), d(\alpha)} \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j} - \frac{OMC(k)K(k)}{\rho} \quad (31)$$

where  $F_k(Y, K_k)$  is interpreted as the value of the project to invest in technology  $k$ . The first term  $A_k(K_k)Y^{\beta_1}$  in this expression is the option value of capacity expansion in technology  $k$  and the second term  $\sum_{i,j=1}^{d(\gamma), d(\alpha)} \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j} - OMC(k)K(k)/\rho$  is the value of the installed capital for the same technology. One can solve the problem independently for each technology (for each  $k \in \mathbb{K}$ ) as treated in Bertola (1989) and Pindyck (1988).

**Proposition 1** (Investment trigger for the separable case). *The investment trigger  $Y_k^*(K_k)$  for technology  $k$  when capacity in place is  $K$  is given by the equation*

$$\sum_{i,j=1}^{d(\gamma), d(\alpha)} \left\{ \alpha_j \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j-1} \left( \frac{\beta_1 - \gamma_i}{\beta_1} \right) \right\} = I_k + \frac{OMC(k)}{\rho}. \quad (32)$$

*Proof.* See Appendix B. □

Appendix B shows that  $Y_k^*(K_k)$  defined by (32) is uniquely defined as an increasing function of  $K_k$ . This is sufficient to our purpose: one is not able to find a closed form solution for  $Y_k^*(K_k)$ , but in real time one can observe  $Y(t)$  and check if (32) holds or not. The same appendix provides the sufficient condition

$$\frac{\beta_1 - \hat{\gamma}}{\beta_1} \geq \hat{\alpha}, \quad \hat{\gamma} \triangleq \max \gamma, \quad \hat{\alpha} \triangleq \max \alpha \quad (33)$$

under which the coefficient  $A_k(K_k) = - \int_{K_k}^{\infty} A'_k(x) dx$  converges.

### 3.2 A non separable instantaneous welfare

The mathematical programming formulation of the instantaneous welfare  $\Psi$  shows that this welfare is not a separable function: the economic value accruing from the portfolio of generation plants is not the sum of economic values accruing from each technology. Investing in a given technology actually depreciates the option value of investing in others. A precise fit of the instantaneous welfare will therefore likely require a non additively separable regression model.

This introduces a theoretical complication because there is no theory to rely on if the instantaneous welfare of the singular stochastic control problem faced by the social planner is not additively separable. Baldursson and Karatzas (1997) work suggests that a singular stochastic control/optimal stopping equivalence does not hold in this case. And since myopia relies on this equivalence, it is unlikely to find a myopia result in this context.

There are however two good reasons to heuristically rely on myopia in order to derive an investment rule. First, practitioners are likely to use myopia as a proxy for the true optimal behavior under uncertainty, because the myopic problem can be solved in any situation while the true problem cannot. Second, myopia is unquestionably optimal for the two benchmarks—symmetric—cases that are the monopoly (Pindyck, 1988) and the perfect competition with identical agents (Leahy, 1993 and Baldursson and Karatzas, 1997). It is moreover close<sup>16</sup> to being optimal in symmetric oligopolies (see Grenadier, 2002). One can thus argue that myopia can be used as an assumption to simplify the problem and derive a useful heuristic.

Having in mind the use of myopia—this time as an assumption—, we propose the following regression model for  $\Psi(Y, K)$ .

**Regression Model 2 (Non Additively Separable).** *Consider an interpola-*

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<sup>16</sup>See Back and Paulsen (2009) and the distinction between open-loop and closed-loop equilibria under uncertainty.



tion of the profit flow  $\Psi$  by the non additively separable function  $\bar{\Psi}$

$$\begin{aligned} \bar{\Psi}(Y, K) &= \sum_{t=1}^{d(\mathbb{K})} \sum_{i,j=1}^{d(\gamma), d(\alpha)} b_{t,ij} Y^{\gamma_i} K^{\alpha_j}(t) \\ &+ \sum_{\substack{t,u=1 \\ u \neq t}}^{d(\mathbb{K})} \sum_{i,j,k=1}^{d(\gamma), d(\lambda), d(\lambda)} c_{ut,ijk} Y^{\gamma_i} K^{\lambda_j}(t) K^{\lambda_k}(u) - \sum_{t=1}^{d(\mathbb{K})} OMC(t) K(t) \end{aligned} \quad (34)$$

with  $\gamma$ ,  $\alpha$  and  $\lambda$  respectively positive base vectors of dimension  $d(\gamma)$ ,  $d(\alpha)$  and  $d(\lambda)$  such that  $\forall i, 0 < \gamma_i < \beta_1$ ;  $\forall j, 0 < \alpha_j \leq 1$ ;  $\forall k, 0 < \lambda_k < 1$ ; with the restriction that any interaction terms should pick two powers  $\lambda_i$  and  $\lambda_j$  such that  $i \neq j$  and  $\lambda_i + \lambda_j \leq 1$ ;  $b \geq 0$ ; and  $\beta_1$  given by (21).

This regression scheme is the natural non separable extension of the Regression Model 1. We now assume that competitive agents are myopic i.e. that they assume no future entry after their own investment. They therefore solve the free boundary Problem 4 on the only  $Y$  variable, assuming the current industry capital stock  $K$  fixed forever at its current level. Proposition 2 gives optimal investment triggers under this myopia assumption.

**Proposition 2.** *The myopic investment trigger  $Y_k^*(K)$  for technology  $k \in \mathbb{K}$  when capacities in places are  $K$  is given by the equation*

$$\begin{aligned} \sum_{i=1}^{d(\gamma)} Y^{\gamma_i} \left( \frac{\beta_1 - \gamma_i}{\beta_1} \right) \left\{ \sum_{j=1}^{d(\alpha)} \alpha_j \bar{b}_{k,ij}(\gamma_i) K_k^{\alpha_j - 1} \right. \\ \left. + \sum_{\substack{t=1 \\ t \neq k}}^{d(\mathbb{K})} \sum_{j,l=1}^{d(\lambda), d(\lambda)} \lambda_k \bar{b}_{kt,ijl}(\gamma_i) K_t^{\lambda_j} K_k^{\lambda_l - 1} \right\} = I_k + \frac{OMC(k)}{\rho}. \end{aligned}$$

*Proof.* See Appendix C. □

### 3.3 Regression schemes on uncertainty $\times$ technology space in practice

We construct two regression schemes allowing for the determination of investment triggers for a power capacity expansion. Let us recall that we use myopia differently in the two cases. By assuming additive separability, myopia is effectively optimal. By assuming non additive separability one can obtain a better regression of the instantaneous welfare but optimality of myopia is not guaranteed and is thus an assumption. The question is which regression scheme is the most useful in practice. This comes down to compare the exact solution of an approached problem with the approached solution of an exact problem which is an empirical and practical issue. In any case, the difference of performance between the two regression schemes is a measure of the validity of the

assumption of optimality of myopia. If the regression error of the non additively separable regression scheme is significantly lower, this suggests that the assumption of myopia—which includes exogenous price models as a particular case—is mistaken.

Dimensionality is a major practical issue in these regression schemes. Dimensionality weight on computations before the regression step i.e. during data production as one needs to compute  $\Psi(Y, K)$  on a grid of the  $Y \times K$  space (a hyperspace of dimension equal to the number of uncertainties times the number of technologies). In the non separable regression scheme, the number of basis functions required increases exponentially with the number of technologies. Thus it becomes difficult to solve the constrained mean square error minimization problem.

To sum up, the appealing simplicity of real options is more apparent than real in the capacity expansion problem. First, the application of the principle requires an assumption of myopia, which even though holding in the single asset, single uncertainty problem, does not necessarily carry through to an asymmetric multidimensional problem. Second, even when making this assumption, the application of the method requires computations that are still of an exponential nature. The following tries to simplify this latter point.

## 4 A semi-analytic algorithm

It is possible to adapt the procedure developed in the preceding Section into a less dimension-sensitive heuristic, and still find a tractable investment rule of the real options type. We shall see that myopia allows us to conduct the regression work in the sole uncertainty space, thereby bypassing the wide uncertainty  $\times$  technology space.

### 4.1 Myopia and optimization as motivations

Suppose that the current installed capacity is  $K_0$ . By inspecting the two previous resolution procedures, one finds that  $\bar{\Psi}(Y, K)$  is only used through its  $K$ -gradient which in turn serves to compute the vector  $m(Y, K_0) = \mathbb{E}^Y \left[ \int_0^\infty \nabla_K \bar{\Psi}(Y_s, K_0) e^{-\rho s} ds \right]$  of (myopic) marginal values. In other words myopia only requires the marginal profit of each unit i.e. the vector  $\nabla_K \bar{\Psi}(Y_s, K_0)$  for each possible future realization  $Y_s$  of the shock. The general idea of the heuristic is to use a numerical proxy of  $\nabla_K \bar{\Psi}(Y, K_0)$  in a Monte Carlo computation of  $m(Y, K_0)$  as follows.

To find a proxy of  $\nabla_K \bar{\Psi}(Y, K)$ , we use Lagrange multipliers of capacity constraints as follows. Let  $\lambda_{k,l}(Y, K)$ ,  $k \in \mathbb{K}$ ,  $l \in \mathbb{L}$  be Lagrange multipliers of capacity constraints (3) on  $K(k)$  at the optimum for  $(Y, K)$ . Define the vector  $\lambda(Y, K) \in \mathbb{R}^{d(\mathbb{K})}$  componentwise by

$$\lambda_k(Y, K) \triangleq \sum_{l \in \mathbb{L}} \lambda_{k,l}(Y, K).$$

A natural proxy of the  $k$ -th component of the marginal welfare  $\nabla_K \Psi(Y, K)$  is  $\lambda_k(Y, K) - OMC(k)$ . We thus consider the (vectorial) approximation

$$\nabla_K \Psi(Y, K) \approx \lambda(Y, K) - OMC \quad (35)$$

(see e.g. Secomandi, 2010, for a similar use of Lagrange multipliers in a one-shot investment problem). Note that  $\lambda$  and  $OMC$  are in  $\text{€}/\text{MWy}$ . The optimization software used to find  $\Psi(Y, K)$  can provide  $\lambda(Y, K)$  as a collateral output upon request.  $\lambda$  is thus really easy to obtain.

We solve the myopic problem using the proxy (35) and Monte Carlo simulations as follows. Since myopia assumes that the current capacity  $K_0$  remains constant over the entire horizon, one can use Monte Carlo to compute  $m(Y, K_0)$  on a grid of  $Y$  by  $m(Y, K_0) = \mathbb{E}^Y [\int_0^\infty \nabla_K \Psi(Y_s, K_0) e^{-\rho s} ds] \approx \mathbb{E}^Y [\int_0^\infty (\lambda(Y_s, K_0) - OMC) e^{-\rho s} ds]$ . We can then regress on  $Y$  and solve analytically an optimal stopping problem on  $Y$  to find the trigger  $Y^*(K_0)$ . When this trigger is reached, new capacities are added i.e.  $K_0$  is updated to  $K_1 > K_0$  and we apply the same procedure on the updated capacity level  $K_1$ . Note that the Monte Carlo evaluation of  $m(Y, K_0)$  in the second step is enabled by myopia. The next subsection gives a precise description of the algorithm.

## 4.2 The procedure

The semi-analytic algorithm described in the preceding discussion proceeds as follows.

1. Assume that the current capacity and shock are  $K$  and  $Y_0$ , respectively. On a time grid

$$T^{\text{grid}} = (0, \Delta t, 2\Delta t, \dots, T) = (s_1, \dots, s_{d(T^{\text{grid}})})$$

of horizon  $T$ , we generate a sample  $\hat{\Omega} \triangleq \{\omega_1, \dots, \omega_N\}$  of  $N$  trajectories  $\{Y_{s_j}(\omega_i)\}_{s \in T^{\text{grid}}}$  starting from  $Y_0$ .

For each trajectory  $\omega_i$ , one computes:

- (a) The parametrized vector of Lagrange multipliers  $\{\lambda(Y_{s_j}(\omega_i), K)\}_{s \in T^{\text{grid}}} \in \mathfrak{R}^{d(\mathbb{K}) \times d(T^{\text{grid}})}$ ;
- (b) the vector of marginal values  $M(Y_0, K, \omega_i) \in \mathfrak{R}^{d(\mathbb{K})}$  by

$$M(Y_0, K, \omega_i) \triangleq \sum_{j=1}^{d(T^{\text{grid}})} \lambda(Y_{s_j}(\omega_i), K) e^{-\rho s_j} - \frac{OMC}{\rho}. \quad (36)$$

By averaging on the sample  $\hat{\Omega}$  (the statistic expectation is noted  $\hat{\mathbb{E}}$ ), one determines the value of the marginal units at the current point  $Y_0$

$$\begin{aligned} \hat{m}(Y_0, K) &= \hat{\mathbb{E}}[M(Y_0, K, \omega)] \\ &= \hat{V}(Y_0, K) - \frac{OMC}{\rho} \end{aligned}$$

with  $\hat{V}(Y_0, K)$  the vector of expected cumulated income

$$\hat{V}(Y_0, K) \triangleq \hat{\mathbb{E}} \left[ \sum_{j=1}^{d(T^{\text{grid}})} \lambda(Y_{s_j}(\omega), K) e^{-\rho s_j} \right].$$

2. The preceding procedure is applied at each point of a grid of initial values of  $Y$

$$Y_0^{\text{grid}} = (Y_0^1, \dots, Y_0^g) \in \mathfrak{R}_+^g.$$

One finds  $\hat{m}(Y_0^i, K)$ ,  $i = 1, \dots, g$ .

3. For each technology  $k \in \mathbb{K}$ , one regresses  $\{\hat{V}_k(Y_0^i, K)\}_{i=1, \dots, g}$  by power functions of  $Y$

$$\hat{V}_k(Y, K) \approx \sum_{i=1}^{d(\gamma)} c_i(k, K) Y^{\gamma_i}$$

with  $\gamma \in \mathfrak{R}_+^{d(\gamma)}$  such that  $\forall i, 0 < \gamma_i < \beta_1$  and  $c(k, K) \in \mathfrak{R}_+^{d(\gamma)}$ . Define

$$R(w, Y) \triangleq \sum_{i=1}^{d(\gamma)} w_i Y^{\gamma_i} \quad (\text{Regression}) \quad (37)$$

$$e_k(w, Y, K) = \hat{V}_k(Y, K) - R(w, Y) \quad (\text{Error}) \quad (38)$$

$R(w, Y)$  defined by (37) is the regression function using a weight vector  $w$  and a base of powers  $\gamma \in \mathfrak{R}_+^{d(\gamma)}$ . The term  $e_k(w, Y, K)$  defined by (38) is the regression error of using  $R(w, Y)$  as approximation of the myopic value  $\hat{V}_k(Y, K)$  of a unit of technology  $k$ . If we choose to minimize the square of the regression error,  $c(k, K)$  is given by

$$c(k, K) = \arg \min_{w \in \mathfrak{R}_+^{d(\gamma)}} \sum_{j=1}^g \left\| e_k(w, Y_0^j, K) \right\|^2.$$

4. For each technology  $k \in \mathbb{K}$ , one solves analytically the optimal stopping problem

$$J(k) \triangleq I(k) + \frac{OMC(k)}{\rho}$$

$$\tau^*(Y, K, k, \omega) = \arg \sup_{\tau \in \mathcal{S}} \mathbb{E}^Y \left[ e^{-\rho \tau} \left( R(c(k, K), Y_\tau) - J(k) \right) \right] \quad (39)$$

for any  $Y \in [Y_0^1, Y_0^g]$ . The investment trigger for technology  $k$  is thus the unique root  $Y^*(k, K)$  of the equation

$$\sum_{i=1}^{d(\gamma)} c_i(k, K) Y^{\gamma_i} \left( \frac{\beta_1 - \gamma_i}{\beta_1} \right) = I(k) + \frac{OMC(k)}{\rho} \quad (40)$$

where (40) is determined using Appendix C from equation (66) to the end, using  $m_k(Y, K) = \sum_{i=1}^{d(\gamma)} c_i Y^{\gamma_i}$ .

## 5 Discussion

The marginal instantaneous welfare is a wealth per unit time and capacity (in our example, in €/MWh) i.e. the profit flow of a unit capacity. Since we systematically used myopia, a natural question is to delineate the benefits (if any) of using myopia on a capacity expansion model instead of an exogenous price process (i.e. an exogenous profit) model. The gain is twofold.

### 5.1 $\nabla_K \Psi$ is the spread between electricity and fuel prices

A system in perfect competition cannot normally be represented by closed form expressions of equilibrium asset values as function of the market input data. Taking our model as example one can numerically compute  $q^*(Y, K)$  for particular instances of  $Y$  and  $K$ , but one cannot derive a corresponding closed form function. In the context of the power market, the marginal welfare (i.e. the Lagrange multipliers of the capacity constraints) is the generic expression of equilibrium plants profit; this marginal welfare generalizes the concept of spark spread (for gas plant) and dark spread (for coal plant) as we shall now see.

Taking our model as an example, it is directly visible from the KKT conditions of Problem 1 that

$$\lambda_{k,l} = \tau(l) \left[ P \left( Y, \sum_{k \in \mathbb{K}} q^*(k, l), l \right) - c(k) \right] \quad \text{if } q(k, l) > 0. \quad (41)$$

Thus

$$\begin{aligned} (\nabla_K \Psi)_k(Y, K) &= \sum_l \lambda_{k,l} - OMC(k) \\ &= \sum_{l \in \mathbb{L}} \tau(l) \left[ P \left( Y, \sum_{k \in \mathbb{K}} q^*(k, l), l \right) - c(k) \right] - OMC(k) \end{aligned}$$

which is a spread.  $\nabla_K \Psi(Y, K)$  is however not closed form determinable since the matrix

$$q^*(Y, K) = (q_{k,l}^*(Y, K))_{\substack{k \in \mathbb{K} \\ l \in \mathbb{L}}}$$

is not.

The spread  $\nabla_K \Psi(Y, K)$  sometimes takes particular values. For Problem 1, it is easy to see that technologies will always be used in the order of their increasing marginal cost. Thus, for given  $k$  and  $l$ , if

1.  $K(k) > q^*(k, l, Y) > 0$ , then
  - (a)  $k$  is the most expensive unit running in segment  $l$ . This implies that capacities are tight for technologies having lower marginal cost.
  - (b)  $K(k) > q^*(k, l, Y)$  also implies  $\lambda_{k,l} = 0$  which in turn implies (by Eq. 41) that the power price in segment  $l$  equals the marginal cost of this

type of unit. In other words, the price equals the marginal cost of the most expensive running unit if capacity for this type of unit is not tight.

2. If  $q^*(k, l, Y) = K(k)$  and  $k$  is the most expensive unit running in segment  $l$ , then  $\lambda_{k,l}$  can take arbitrary values in  $[0, c(k+1) - c(k)[$ . This is in fact the typical situation, since  $Y$  varies continuously.

Note that the advantage of this generic modeling of the profit flow goes far beyond our rather naive model. Many variants of this optimization problem are indeed relevant to model more sophisticated competitive power markets. A spatial extension where power plants and demand are both geographically allocated is the first one that comes to mind. The problem is then to maximize the welfare under production and transmission lines capacity constraints. Moreover, transmission is subject to physical laws that can also easily be embedded in an optimization problem, therefore leading to rather straightforward generalizations of the above discussion.

An economically different optimization model can also serve to model profit flow. Suppose that consumers do not react to price i.e. that demand is inelastic. It is well known that, for this case, a competitive equilibrium is obtained by minimizing the total operation cost (note that the consumer surplus—thereby the welfare—is not defined, since there is no demand function). Assume capacity constraints and an obligation to satisfy load for each demand segment  $D(l)$ ,  $l \in \mathbb{L}$  and (just for exposition simplicity) no operation and maintenance cost. The competitive equilibrium is described by the optimization problem

$$\Psi(Y, K) = \max_q \left\{ - \sum_{\substack{k \in \mathbb{K} \\ l \in \mathbb{L}}} \tau(l)c(k)q(k, l) \right\} \quad (42)$$

s.t.

$$q(k, l) \geq 0 \quad \forall k, l \in \mathbb{K}, \mathbb{L} \quad (p(k, l)) \quad (43)$$

$$q(k, l) \leq K(k) \quad \forall k, l \in \mathbb{K}, \mathbb{L} \quad (\lambda(k, l)) \quad (44)$$

$$\sum_{k \in \mathbb{K}} q(k, l) \geq D(l) \quad \forall l \in \mathbb{L} \quad (\Pi(l)\tau(l)) \quad (45)$$

where  $\lambda(k, l)$  and  $\Pi(l)\tau(l)$  are respectively Lagrange multipliers of constraints (44) and (45) in €/MWh.  $\Pi(l)$  is interpreted as the power price in demand segment  $l$  in €/MWh (recall that  $\tau(l)$  is in h/y). Here again, it is easy to show that if the most expensive technology operating in demand segment  $l$  is not at full capacity, its marginal cost during the segment  $l$ , namely  $\tau(l)c(k)$ , is the profit  $\tau(l)\Pi(l)$ . Except in this particular configuration,  $\Pi(l)$  is not closed form determinable and one has

$$\lambda(k, l) = \tau(l) \max(\Pi(l) - c(k), 0) \quad (46)$$

which is (again) the spread with option to suspend operation.

## 5.2 Dynamic consistency and micro founded models

The marginal welfare describes the market as a function of its inputs which, in our model, are the demand functions  $P(Y, \cdot, l)$ , the industry activity indicator  $Y$ , the costs  $c$  and  $OMC$  and the capacity  $K$ . The market outputs are the operation variables  $q^*$  and the realized price  $P(Y, \sum_{k,l} q_{k,l}^*, l)$ .

Working with a capacity expansion model and assuming myopia logically requires to calibrate the market input so that the output of the model reproduces market data. Taking the example of our model, the calibration procedure should use as input past time series of costs, activity indicator (the GDP or an industry specific uncertainty) and capacities to calibrate the demand functions such that the model replicates past time series of power price. This calibration technique is robust to capacity addition (i.e. one does not need recalibrate the model when capacities are added). It is used notably in system dynamics models which pay close attention to feedback effects of investment on the price process.

In contrast, by deciding to use an exogenous price process for each considered investment, one is required to recalibrate the price process each time capacity addition takes place. This leads to difficulties as the time series relevant for each of these calibrations are of small size. The calibration process is thus less robust.

To sum up, capacity expansion models under myopia allow to model plant profits in greater generality. Moreover, because of their advanced microeconomic foundation, they allow a coherent and robust calibration of the price process.

## 6 An example: 4 technologies

To fix ideas, we consider a capacity expansion problem with 4 technologies. We refer to  $K \in \mathbb{R}_+^4$  as the capacity of the system and  $K(1)$ ,  $K(2)$ ,  $K(3)$  and  $K(4)$  respectively as the capacity in nuclear, coal, combined cycle gas and open cycle gas plants. Costs for these technologies are given in Table 3.

Demand is split in 6 segments ( $d(\mathbb{L}) = 6$ ). For each load segment  $l \in \mathbb{L}$ , we assume a linear inverse demand curve

$$P_l(Y, Q_l) = A_l Y_t(\omega) - b_l Q_l \quad (47)$$

where  $Y_t(\omega)$  is a geometric Brownian motion (with  $Y_0 = 1$ ) affecting all the demand segments. We note  $E(l)$  the elasticity of the demand function  $P_l(Y, Q_l)$ .  $|E(l)|$  is increasing in  $l$  i.e. peak load consumers are less sensitives to price variations. The  $A_l$  and  $b_l$  in (47) are given in Table 2; the calibration procedure is explained in Appendix E. The parameters of  $Y$  and the discount rate  $\rho$  are given in Table 1.

shock process	
$Y_0$	1
$\mu$	0.02
$\sigma$	0.03
discount rate	
$\rho$	0.1
$\beta_1$	4.62

**Table 1 :** The industry wide activity indicator  $Y$  follows a GBM  $dY(t, \omega) = \mu Y(t)dt + \sigma Y(t)d\mathbb{B}_t(\omega)$ .  $\beta_1$  is given by (21).

Demand Segment $l$	1	2	3	4	5	6
$\tau(l)$ (kh)	0.01	0.04	0.31	4.4	3	1
$\bar{Q}_l$ (MW)	86000	83000	80000	60000	40000	20000
$P_l(\bar{Q}_l)$ (€/MWh)	300	60	55	40	30	20
$E_l$	-0.1	-0.3	-0.4	-0.5	-0.6	-0.7
$A_l$ for $Y = 1$	900	180	165	120	90	60
$b_l$	0.0070	0.0014	0.0014	0.0013	0.0015	0.0020

**Table 2 :** Demand calibration data.  $\bar{Q}_l$  is the average load in demand segment  $l$ .

Technology	Nuclear	Coal	CCGT	OCGT
Indice	1	2	3	4
Fuel Cost (€/MWh)	5	25.714	37.241	55.385
Emission Cost (€/MWh)	0	18.729	7.994	14.012
Marginal Cost (€/MWh)	5	44.44	45.23	69.397
O. & M. Cost (€/MWy)	87600	70080	43800	43800
Overnight Cost (€/MW)	2 755 000	1 300 000	1 000 000	100 000

**Table 3 :** Power generation technologies. These prices are based on input costs of respectively 3 and 6 €/GJ for the coal and the gas; 1.85 €/MWh for the nuclear input; and finally 23 €/t of CO<sub>2</sub> (see Risto and Aija, 2008). The real discount rate is 5 %.

## 6.1 Investment trigger

### 6.1.1 Monte Carlo on a point $Y_0$

For a given initial capacity vector  $K_0 \in \mathfrak{R}_+^4$  and initial demand shock  $Y_0$ , we compute the (statistical) value of a marginal unit  $\hat{m}(Y_0, K_0) \in \mathfrak{R}_+^4$  by averaging the economic value (the sum of discounted cash flows) of each unit on  $N$  scenarios of the shock process  $Y_s(\omega)$ ,  $s \geq T$ . For  $Y_0 = 1$  and  $K_0 = (10000, 10000, 0, 0) \in \mathfrak{R}_+^4$ , the marginal plant values are  $\hat{m}(Y_0, K_0) = (9.060, 0.5478, 0.5412, 0.3610) \times 10^6 \in \mathfrak{R}_+^4$  with relevant statistics (variance, standard deviation, confidence interval, value at risk and conditional value at risk) given in Table 4.



Technology	Nuclear	Coal	CCGT	OCGT
Indice	1	2	3	4
$\hat{m}(Y_0, K_0) \triangleq \hat{\mathbb{E}}[m(Y_0, K_0)]$ (€)	$9.06 \cdot 10^6$	$5.48 \cdot 10^6$	$5.41 \cdot 10^6$	$3.61 \cdot 10^6$
vâr $[m(Y_0, K_0)]$ (€)	$7.29 \cdot 10^{11}$	$7.06 \cdot 10^{11}$	$7.05 \cdot 10^{11}$	$5.99 \cdot 10^{11}$
std dev. $[m(Y_0, K_0)]$ (€)	$8.54 \cdot 10^5$	$8.40 \cdot 10^5$	$8.4 \cdot 10^5$	$7.74 \cdot 10^5$
C.I. (99.5 %)	$6.95 \cdot 10^4$	$6.84 \cdot 10^4$	$6.84 \cdot 10^4$	$6.30 \cdot 10^4$
VaR 95 % (€)	$7.79 \cdot 10^6$	$4.23 \cdot 10^6$	$4.17 \cdot 10^6$	$2.50 \cdot 10^6$
CVaR 95 % (€)	$7.52 \cdot 10^6$	$3.98 \cdot 10^6$	$3.91 \cdot 10^6$	$2.29 \cdot 10^6$

**Table 4 :** Marginal value for  $Y_0 = 1$  and  $K_0 = (10000, 10000, 0, 0)$ .  
 $\hat{\mathbb{E}}$ , vâr and std dev. are statistic expectancy, variance and standard deviation on a sample of size  $N = 1000$ .  $\hat{m}$  is estimated using (25) over an horizon  $T = 50$  years.

### 6.1.2 Monte Carlo on a grid

The computation of the marginal values  $m(Y, K_0)$  is applied to each point of a grid  $Y = (Y_{\min}, Y_{\min} + \Delta Y, Y_{\min} + 2\Delta Y, \dots, Y_{\max})$ . Let  $d(Y)$  be the dimension of the grid of  $Y$ ; we thus obtain a matrix  $\hat{m}(Y, K_0) \in \mathbb{R}^4 \times \mathbb{R}^{d(Y)}$  or, for  $Y = (0.25, 0.5, 0.75, \dots, 2)$ ,

$$\hat{m}(Y, K_0) = 10^7 \times \begin{bmatrix} 0.1235 & 0.3661 & 0.6218 & 0.9060 & 1.1952 & 1.4893 & 1.7835 & 2.0719 \\ 0.0048 & 0.0661 & 0.2749 & 0.5478 & 0.8323 & 1.1265 & 1.4206 & 1.7090 \\ 0.0047 & 0.0643 & 0.2699 & 0.5412 & 0.8256 & 1.1193 & 1.4134 & 1.7018 \\ 0.0032 & 0.0322 & 0.1395 & 0.3610 & 0.6239 & 0.9112 & 1.1962 & 1.4797 \end{bmatrix}$$

where  $\hat{m}(Y, K_0)$  is in €/MW. Relevant statistics are matrices of the same size.

### 6.1.3 Regression

For each row  $k \in \mathbb{K}$  of the matrix  $\hat{m}$  (i.e. for each technology), we perform a regression on  $Y$  for the base of powers

$$\gamma = (0.25, 0.5, \dots, 4.5)$$

(note that we chose  $\gamma$  such that  $\gamma_{\max} \leq \beta_1$ ) and compute the corresponding investment threshold. We obtain a vector of relative regression errors

$$e(Y_{\text{grid}}, K_0, \gamma) = (0.0154, 0.0547, 0.0548, 0.0581),$$

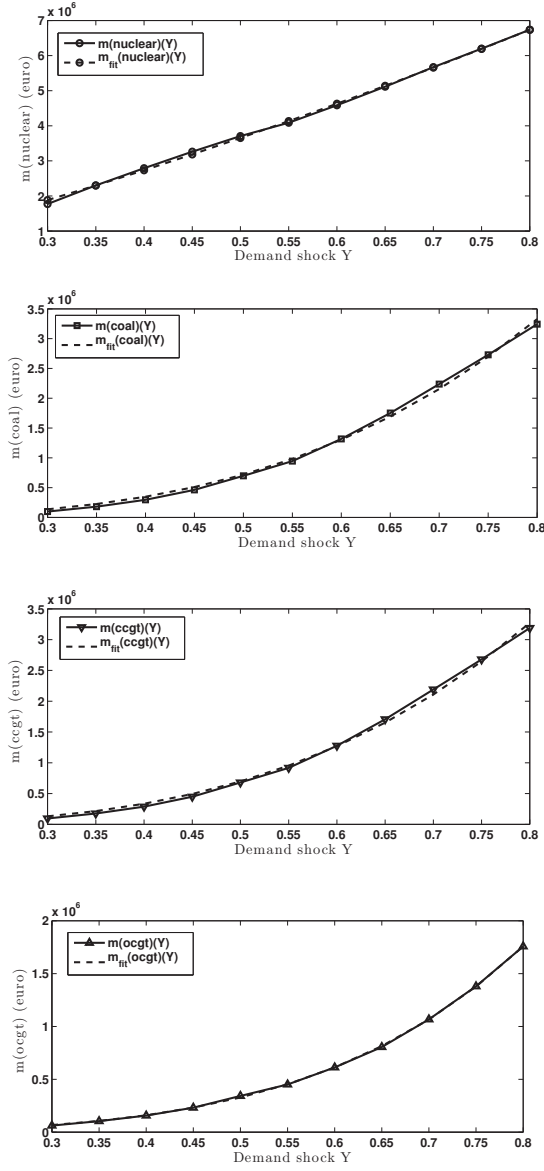
a regression R-square

$$R^2(Y_{\text{grid}}, K_0, \gamma) = (0.9888, 0.9752, 0.9771, 0.9989)$$

(see Figure 3) and a vector of investment thresholds for the reference scenario

$$Y^*(K_0, \gamma) = (0.4939, 0.5782, 0.5027, 0.2220).$$

Note that the vector of investment thresholds depends on the entire capital in place. In particular, here, the system needs peakers (the trigger of OCGT is the lowest).



**Figure 3 :** Regression of  $\hat{m}(Y, K_0)$  for  $Y = (0.25, 0.5, 0.75, \dots, 2)$  and for nuclear, coal, CCGT and OCGT technologies. For these four technologies, the regression R-square is 0.9888, 0.9752, 0.9771 and 0.9989, respectively.

## 6.2 Comparative statics

### 6.2.1 Variation of the trigger with the average growth of $Y$

Departing from the base case scenario (with  $\mu = 0.02$ ) we computed investment triggers vectors for  $\mu = 0.01, 0.03, 0.04, 0.05$ . Figure 4 shows that, for each technology, the investment trigger decreases with the growth rate  $\mu$  of  $Y$ .

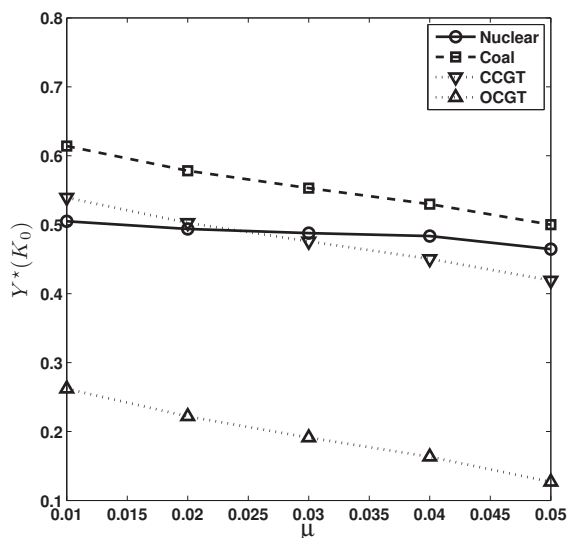


Figure 4 : Variation of the trigger vector  $Y^*(K_0)$  with  $\mu$ .

This trend is in line with simpler real options models (see Dixit and Pindyck (1994), Page 194, Figure 6.6). The effect of the growth rate on the investment trigger (40) is subtle: an increase of  $\mu$  raises the uncertainty multiplier  $\beta_1/(\beta_1 - \gamma_i)$  but, at the same time, increases the expected value of capital i.e. the coefficients  $\{c_i(k, K)\}_{i=1, \dots, d(\gamma)}$ . The latter effect generally dominates the former so that the investment trigger decreases with  $\mu$ . Relative regression errors have the same magnitude than in the base case scenario.

### 6.2.2 Variation of the trigger with the volatility of $Y$

Departing from the base case scenario ( $\sigma = 0.3$ ) we compute investment triggers vectors for  $\sigma = 0.1, 0.2, 0.4, 0.5$ . As Figure 5 shows, the investment trigger increases with the volatility rate  $\sigma$  of  $Y$  for each technology.

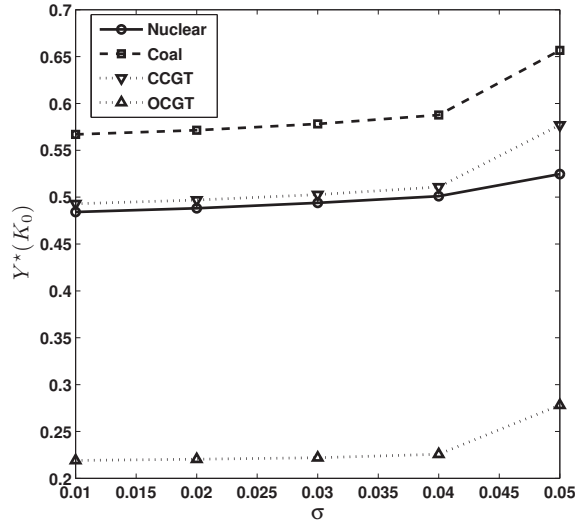


Figure 5 : Variation of the trigger vector  $Y^*(K_0)$  with  $\sigma$ .

This trend is in line with the view of investment as an option exercise: because the value of an option increases with uncertainty, an optimal investment that requires to pay the “full cost” (investment cost plus abandonment of the option) demands a higher net present value.

### 6.3 Variation of the trigger with the capacity vector

Departing from the base case scenario (which was considering  $K_0 = (10000; 10000; 0; 0)$ ) we computed investment triggers vectors for the 30 points of the grid:  $K_1 \otimes K_2$  with  $K_1 = (10000; 20000; 30000; 40000; 50000)$  and  $K_2 = (5000; 10000; 15000; 20000; 25000; 30000)$ . We choose to vary only two components of  $K$  for exposition convenience. Figures 6, 7, 8 and 9 show that the investment trigger is increasing with respect to  $K_1$  and  $K_2$  for each technology.

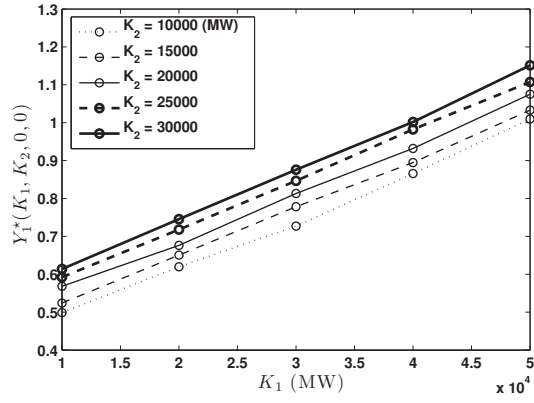


Figure 6 : Investment triggers for nuclear.

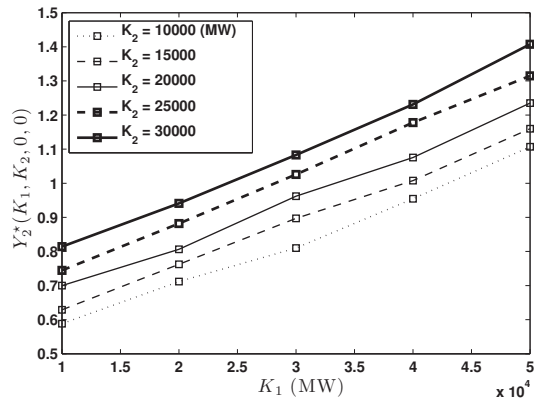


Figure 7 : Investment triggers for coal.

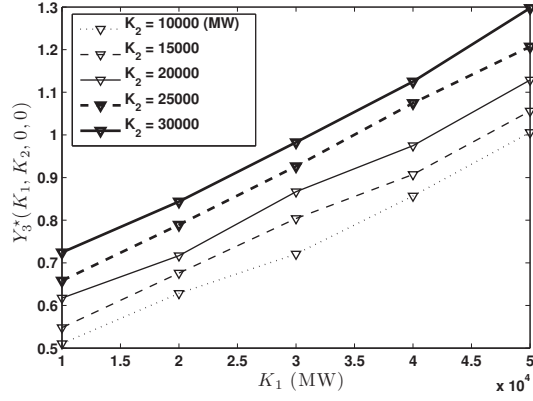


Figure 8 : Investment triggers for *CCGT*.

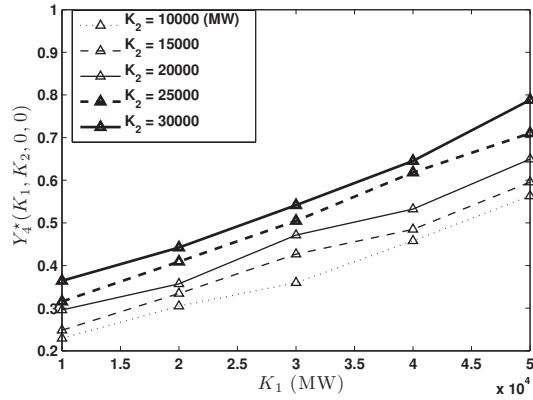


Figure 9 : Investment triggers for *OCGT*.

One observes that the investment trigger of a technology is increasing with its own capital stock, as well as with the capital stock of other technologies; this is true for all technologies. Note that this simple and intuitive property does not appear with model constructed on the basis of an additively separable regression, or with exogenous price processes.

## 6.4 Alternative configurations

### 6.4.1 Variation of the trigger with the elasticity in a given segment

Departing from the base case scenario (with  $E(1) = -0.1$ ), we compute the vector of investment triggers for  $E(1) = -0.3$  that is, we increase the demand response in peak load. One finds

$$Y^*(K_0) = (0.5, 0.59, 0.52, 1.07).$$

One sees that the investment trigger increased dramatically for the peak load technology ( $Y_4^*(K_0)$  went from 0.22 to 1.07) but only slightly increased for other technologies. It is easy to explain this very localized variation of the trigger vector. Note that our system is initially already tight ( $Y_1^*(K_0) = 0.5$ ) i.e., that investments are required. The first load segment has the shortest duration. If we plan to use a technology only during small portion of its lifetime, it is preferable that this technology has a lower fixed cost. Thus the first load segment is the most important user of peak load technologies. When increasing the elasticity in this load segment, consumers are less willing to support high operation costs. Thus we decrease the usefulness of peakers in their principal demand segment.

### 6.4.2 Extension to risk aversion

The computation of the marginal value  $m(X, K)$  in (25) implicitly assumes risk neutrality. Since the practical computation  $\hat{m}(Y, K)$  of method 2 relies on Monte Carlo, it is as easy to compute statistically risk indicators (e.g. value at risk (VaR) and conditional value at risk (CVaR)) of the marginal value  $M(X, K, \omega)$  defined by (36).

Consider for instance the simple extension to risk aversion

$$\begin{aligned} \hat{m}_2(X, K) &= \hat{m}(X, K) - \phi \text{CVaR}_{0.05}(m) \\ &= \hat{\mathbb{E}}[M(X, K, \omega)] - \phi \text{CVaR}_{0.05}[M(X, K, \omega)] \end{aligned}$$

where  $\phi$  represents a market price of risk. This extension to risk aversion does not require additional Monte Carlo simulations since CVaR is a statistic which only depends on the distribution of  $M(X, K, \omega)$ . One just needs to compute  $\hat{m}_2(Y, K)$  for each point of the previously used grid  $Y_{\text{grid}} = (Y_1, \dots, Y_l)$  and apply the regression step on  $\{\hat{m}_2(Y_i, K)\}_{i=1, \dots, l}$ .

For  $\phi = 0.5$ , one obtains

$$Y^*(K_0, \gamma) = (0.7595, 0.7308, 0.6289, 0.2425).$$

One sees that, for each technology, the investment trigger increases with the risk aversion.

## 7 Conclusion

This paper proposes a real options capacity expansion model for power generation in a competitive market with several technologies. Main assumptions of

this model are incremental irreversible investments and non-increasing return to scale. Thus, our work is in the stream of previous models by Pindyck (1988), Bertola (1989), Leahy (1993) and Baldursson and Karatzas (1997). For these types of models, the optimal investment behavior is usually determined using a mathematical equivalence between the stochastic control describing the capacity expansion problem and an optimal stopping problem.

Our model differs in at least three ways from the aforementioned works. First one uses optimal dispatch to compute the instantaneous welfare of the social planner. Second power is treated as a differentiated product. Third our model uses several technologies that may also differ by their investment cost. These features are incorporated so that our model fits better power generation real characteristics. The counterpart is that they significantly complicate the underlying mathematics of the model in at least two ways. First a welfare defined by an optimal dispatch is not additively separable in its different technologies. This combines with the asymmetry of investment costs to destroy the stochastic control/optimal stopping equivalence and hence the optimality of myopia. Second, even if we take myopia as an assumption, the welfare function is defined by the numerically obtained solution of an optimization problem and is not given in closed form. This requires to resort to a regression of the computed costs by convenient analytical functions in view of applying the standard real options machinery (i.e. the formulation of the free boundary problem of the myopic agent).

An explanation of real options theory popularity is its ability to provide closed-form investment rules that are economically intuitive. Thus, in a first part of the paper, we try to work out analytical investment criteria. In the process of fitting the instantaneous welfare by convenient analytical functions, we in fact have two possibilities. Either we assume a regression model which is additively separable in the various technologies; this implies a less accurate regression but guarantees the optimality of the myopic behavior. Alternatively, we use a non separable regression model, in which case we are more precise at the regression step but myopia, which can no longer be proved optimal, is used as an assumption. Using the geometric Brownian motion as uncertainty driver in both cases we determine analytic investment conditions. This first part leads to the two following observations: (i) all real options roads lead to myopia; (ii) the regression in the space of uncertainties and technologies is subject to curse of dimensionality in practice.

The second part of the paper addresses the problem of implementation. The theoretical work developed in the first part suggests a semi-analytic method which is not sensitive to the dimension of the technology space. Firstly, we use the vector of Lagrange multipliers of capacity constraints to evaluate the profit flow of each type of plant. Secondly, we use systematically myopia as an assumption to compute the value of marginal units by Monte Carlo simulations. Thirdly, the value of the marginal units are regressed by polynomial functions. Finally, one solves analytically an optimal stopping problem on these regressions of the marginal units. Note that contrarily to the three first steps which involve numerical methods, the last step is purely analytical. We illustrate the appli-



cation of this semi-analytic technique on a four-technologies power generation capacity expansion model.

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## Appendices

### A $\Psi(Y, K)$ is concave in $K$

Indeed, rewriting Problem 1 as a minimization, one can write

$$\Psi(Y, K) + \sum_{k \in \mathbb{K}} OMC(k)K(k) =$$

$$- \min_q \sum_{l \in L} \tau(l) \left\{ \sum_{k \in \mathbb{K}} c(k)q(k, l) - \int_0^{Q(l)} P(Y, q, l) dq \right\} \quad (48)$$

$$\text{subject to } 0 \leq q(k, l) \leq K(k) \quad \forall k, \forall l \quad (49)$$

$$\sum_{k \in \mathbb{K}} q(k, l) = Q(l) \quad \forall l. \quad (50)$$

We note that (48) is the minimization of a convex function subject to two convex constraints (49) and (50). Standard optimization theory guarantees that the objective cost function at the optimum is convex with respect to the constraint vector  $K$  i.e. that  $\Psi(Y, K)$  is concave in  $K$ .

### B Proof of Proposition 1

One will solve Problem 3. Recall that one could alternatively solve Problem 4 and then integrate the risk function  $r(Y, K) = m(Y, K) - f(Y, K)$  to find the function  $F$  (see Section 2.2). One has:

$$F_k(Y, K_k) = A_k(K_k)Y^{\beta_1} + \sum_{i,j} \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j} - \frac{OMC(k)K_k}{\rho} \quad (51)$$

$$\frac{\partial F_k}{\partial K_k}(Y_k^*(K_k), K_k) = I_k \quad (52)$$

$$\frac{\partial^2 F_k}{\partial K_k Y}(Y_k^*(K_k), K_k) = 0. \quad (53)$$

The following decomposition is insightful.

$$F_k(Y, K_k) = \underbrace{A_k(K_k)Y^{\beta_1}}_{O_k(Y, K_k)} + \underbrace{\sum_{i,j} \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j}}_{V_k(Y, K_k)} - \frac{OMC(k)K_k}{\rho}$$

The term  $O_k(Y, K_k)$  is the expansion option in technology  $k$  for a given  $Y$ . It is decreasing in  $K_k$ .  $V_k(Y, K_k)$  is the myopic value of installed capital  $K_k$  for drift  $Y$ . A necessary (but not sufficient) condition to increase capacity is  $\frac{\partial V_k}{\partial K_k}(Y, K_k) > 0$ .

### B.1 Preliminary study: the function $V_k(Y, K_k)$

$$V_k(Y, K_k) \triangleq \sum_{i,j} \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j} - \frac{OMC(k)K_k}{\rho};$$

$$V_k(Y, 0) = 0;$$

$$V_k(0, K_k) = -\frac{OMC(k)K_k}{\rho};$$

$$\lim_{K_k \rightarrow +\infty} V_k(Y, K_k) = -\infty.$$

We compute the partial derivatives of the function  $V_k$ .

$$\frac{\partial V_k}{\partial K_k}(Y, K_k) = \sum_{i,j} \alpha_j \bar{b}_{k,ij}(\gamma_i)Y^{\gamma_i}K_k^{\alpha_j-1} - \frac{OMC(k)}{\rho} \quad (54)$$

$$\frac{\partial^2 V_k}{\partial K_k^2}(Y, K_k) = \sum_{i,j} \underbrace{\alpha_j}_{>0} \underbrace{(\alpha_j - 1)}_{\leq 0} \underbrace{\bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{Y^{\gamma_i}K_k^{\alpha_j-2}}_{>0} \leq 0$$

→  $V_k(Y, K_k)$  is a concave function of  $K_k$ .

$$\frac{\partial^2 V_k}{\partial K_k \partial Y}(Y, K_k) = \sum_{i,j} \underbrace{\alpha_j \gamma_i}_{>0} \underbrace{\bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{Y^{\gamma_i-1}}_{>0} \underbrace{K_k^{\alpha_j-1}}_{>0, \searrow K_k} > 0 \quad (55)$$

→  $\frac{\partial^2 V_k}{\partial K_k \partial Y}$  is positive and decreasing in  $K_k$ .

One also computes

$$\frac{\partial F_k}{\partial K_k}(Y, K_k) = \frac{\partial A_k}{\partial K_k}(K_k)Y^{\beta_1} + \frac{\partial V_k}{\partial K_k}(Y, K_k) \quad (56)$$

$$\frac{\partial^2 F_k}{\partial K_k \partial Y}(Y, K_k) = \beta_1 \frac{\partial A_k}{\partial K_k}(K_k)Y^{\beta_1-1} + \frac{\partial^2 V_k}{\partial K_k \partial Y}(Y, K_k) \quad (57)$$

### B.2 Resolution of the free boundary problem and trigger

The smooth pasting (53) gives (using (55) and (57))

$$\frac{\partial A_k}{\partial K_k}(K_k) = \frac{-\sum_{i,j} \alpha_j \gamma_i \bar{b}_{k,ij}(\gamma_i)Y^{*\gamma_i-1}K_k^{\alpha_j-1}}{\beta_1 Y^{*(\beta_1-1)}} < 0 \quad (58)$$

As said earlier, the option value decreases along with  $K_k$ .

The value matching (52) gives (introducing (58) and (54) in (56))

$$\frac{-\sum_{i,j} \alpha_j \gamma_i \bar{b}_{k,ij}(\gamma_i)Y^{*\gamma_i-1}K_k^{\alpha_j-1}}{\beta_1 Y^{*(\beta_1-1)}} Y^{*\beta_1} + \sum_{i,j} \alpha_j \bar{b}_{k,ij}(\gamma_i)Y^{*\gamma_i}K_k^{\alpha_j-1} - \frac{OMC(k)}{\rho} = I_k.$$

We finally obtain the trigger

$$\sum_{i,j} \alpha_j \bar{b}_{k,ij}(\gamma_i)Y^{*\gamma_i}K_k^{\alpha_j-1} \left( \frac{\beta_1 - \gamma_i}{\beta_1} \right) = I_k + \frac{OMC(k)}{\rho}.$$

### B.3 The trigger is uniquely defined for each $K_k$ and is increasing in $K_k$

Choose a particular technology  $k$ , and define

$$T(Y, K_k) \triangleq \sum_{i,j} \underbrace{\alpha_j \bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{Y^{*\gamma_i}}_{>0} \underbrace{K_k^{\alpha_j-1}}_{>0, \searrow K_k} \underbrace{\left(\frac{\beta_1 - \gamma_i}{\beta_1}\right)}_{>0}$$

so that the optimal investment trigger  $Y^*(K_k)$  is given by the zero of the function

$$G(Y, K_k) \triangleq I_k + \frac{OMC(k)}{\rho} - T(Y, K_k).$$

We note that

$$\begin{aligned} G(0, K_k) &= I_k + \frac{OMC(k)}{\rho} > 0; \\ G(+\infty, K_k) &\triangleq \lim_{Y \rightarrow +\infty} G(Y, K_k) = -\infty. \end{aligned}$$

Moreover  $G$  is strictly continue and decreasing in  $Y$  for all  $K_k$ . Bolzano's theorem therefore ensures that  $G(\cdot, K_k)$  as only one zero for all  $K_k$ , noted  $Y_k^*(K_k)$ . Furthermore, as the function  $G(Y, K_k)$  is strictly increasing in  $K_k$ , so is  $Y_k^*(K_k)$ .

### B.4 The coefficient $A_k(K_k)$ converges.

From (58), we had

$$\begin{aligned} \frac{\partial A_k}{\partial K_k}(K_k) &= \frac{-\sum_{i,j} \alpha_j \gamma_i \bar{b}_{k,ij}(\gamma_i) Y_k^{*(\gamma_i-1)} K_k^{\alpha_j-1}}{\beta_1 Y_k^{*(\beta_1-1)}} < 0 \\ &= -\sum_{i,j} \underbrace{\frac{\alpha_j \gamma_i}{\beta_1} \bar{b}_{k,ij}(\gamma_i)}_{\geq 0} \underbrace{\frac{Y_k^{*\gamma_i-\beta_1}(K_k)}{\gamma_i < \beta_1, \searrow Y_k^*}}_{\searrow K_k} \underbrace{\frac{K_k^{\alpha_j-1}}{\alpha_j < 1, \searrow K_k}}_{\geq 0} \\ &\quad \underbrace{\hspace{10em}}_{\geq 0, \searrow K_k} \end{aligned}$$

Thus  $\frac{\partial A_k}{\partial K_k}(K_k) \leq 0, \nearrow K_k$  and  $\lim_{K_k \rightarrow \infty} \frac{\partial A_k}{\partial K_k}(K_k) = 0$ . Define  $\Delta_{ijk} \triangleq \frac{\alpha_j \gamma_i}{\beta_1} \bar{b}_{k,ij}(\gamma_i)$ . Observe that

$$\begin{aligned} A_k(\infty) &= 0 = A(K_k) + \int_{K_k}^{\infty} \frac{\partial A_k}{\partial K_k}(x) dx; \\ A_k(K_k) &= -\int_{K_k}^{\infty} -\sum_{i,j} \Delta_{ijk} Y_k^{*\gamma_i-\beta_1}(x) x^{\alpha_j-1} dx \\ &= \sum_{i,j} \Delta_{ijk} \int_{K_k}^{\infty} Y_k^{*\gamma_i-\beta_1}(x) x^{\alpha_j-1} dx. \end{aligned}$$

$\int_{K_k}^{\infty} Y_k^{*\gamma_i-\beta_1}(x) x^{\alpha_j-1} dx$  converges if  $Y_k^{*\gamma_i-\beta_1}(x) x^{\alpha_j-1} \approx x^{-(1+\epsilon)}$  for  $\epsilon \geq 0$ .

We first try to determine the rate of growth of the trigger  $Y_k^*(K_k)$  w.r.t.  $K_k$ . We know that  $G(Y^*(K_k), K_k) \triangleq 0$  for all  $K_k$ . So

$$\frac{dG(Y^*(K_k), K_k)}{dK_k} = \frac{\partial G(v, w)}{\partial v} \Big|_{\substack{v=Y_k^*(K_k) \\ w=K_k}} \times \frac{\partial Y_k^*(K_k)}{\partial K_k} + \frac{\partial G(v, w)}{\partial w} \Big|_{\substack{v=Y_k^*(K_k) \\ w=K_k}} = 0$$

where the first equality comes from the envelop theorem. Noting  $\Lambda_{ijk} \triangleq \alpha_j \bar{b}_{k,ij}(\gamma_i) \left( \frac{\beta_1 - \gamma_i}{\beta_1} \right)$  we obtain

$$\begin{aligned} \frac{\partial Y_k^*(K_k)}{\partial K_k} &= \frac{-\frac{\partial G(v,w)}{\partial w} \Big|_{\substack{v=Y_k^*(K_k) \\ w=K_k}}}{\frac{\partial G(v,w)}{\partial v} \Big|_{\substack{v=Y_k^*(K_k) \\ w=K_k}}} \\ &= \frac{\frac{\partial T}{\partial K_k}(Y^*(K_k), K_k)}{-\frac{\partial T}{\partial Y}(Y^*(K_k), K_k)} = \frac{\sum_{i,j} \Lambda_{ijk} (\alpha_j - 1) Y^{*\gamma_i}(K_k) K_k^{\alpha_j - 2}}{-\sum_{i,j} \Lambda_{ijk} Y^{*(\gamma_i - 1)}(K_k) K_k^{\alpha_j - 1} \gamma_i}. \end{aligned} \quad (59)$$

But  $Y^*(K_k)$  is increasing in  $K_k$ , so, from (59), we find that

$$\lim_{K_k \rightarrow \infty} \frac{\partial Y_k^*(K_k)}{\partial K_k} = \frac{(1 - \hat{\alpha}) Y_k^*(K_k)}{\hat{\gamma} K_k} \quad (60)$$

with  $\hat{\gamma} \triangleq \max_i \gamma_i$  and  $\hat{\alpha} \triangleq \max_j \alpha_j$  which means that, for arbitrarily high  $K_k$ , the trigger  $Y^*(K_k) \approx c K_k^{(1 - \hat{\alpha})/\hat{\gamma}}$  for a positive  $c$ . Therefore, for high  $x$ ,

$$Y_k^{*\gamma_i - \beta_1}(x) x^{\alpha_j - 1} \approx x^{-(1 + \{(1 - \hat{\alpha})(\beta_1 - \gamma_i)/\hat{\gamma} - \alpha_j\})} = x^{-(1 + \epsilon)}$$

for  $\epsilon = (1 - \hat{\alpha})(\beta_1 - \gamma_i)/\hat{\gamma} - \alpha_j$ . Thus a sufficient convergence condition for  $A_k(K_k)$  is

$$\frac{1 - \hat{\alpha}}{\hat{\gamma}} > \frac{\alpha_j}{\beta_1 - \gamma_i} \quad \forall i, j$$

which turns out to be, more simply

$$\frac{\beta_1 - \hat{\gamma}}{\beta_1} \geq \hat{\alpha}.$$

As  $\hat{\gamma} < \beta_1$ , it is always possible to find a base  $\alpha$  such that this condition holds.

## C Proof of Proposition 2

One has to solve

$$\mu Y \frac{\partial f_k}{\partial Y}(Y) + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2 f_k}{\partial Y^2} - \rho f_k(Y) = 0 \quad Y \leq Y_k^*, \quad (61)$$

with

$$f(Y_k^*) = m_k(Y_k^*) - I_k \quad (62)$$

$$\frac{\partial f}{\partial Y}(Y_k^*) = \frac{\partial m_k}{\partial Y}(Y_k^*) \quad (63)$$

$$m_k(Y) = \mathbb{E}^Y \left[ \int_0^{+\infty} \frac{\partial \bar{\Psi}}{\partial K_k}(Y_s, K) e^{-\rho s} ds \right]. \quad (64)$$

Our first step will be to compute  $m_k(Y)$ .<sup>17</sup> Using  $\bar{\Psi}(Y, K)$  given by equation (34), one computes

$$\begin{aligned} \frac{\partial \bar{\Psi}}{\partial K_k}(Y, K) &= \sum_{i,j=1}^{d(\gamma), d(\alpha)} \alpha_j b_{k,ij} Y^{\gamma_i} K_k^{\alpha_j - 1} \\ &\quad + \sum_{\substack{t=1 \\ t \neq k}}^{d(\mathbb{K})} \sum_{i,j,l=1}^{d(\gamma), d(\lambda), d(\lambda)} \lambda_l c_{kt,ijl} Y^{\gamma_i} K_t^{\lambda_j} K_k^{\lambda_l - 1} - OMC(k) \end{aligned}$$

<sup>17</sup>In this appendix we compute this expectation explicitly using the distribution of the geometric Brownian motion. One can alternatively compute  $m_k(Y)$  by solving the PDE (14). See equations 14, 15 and footnote 15 page 11.

but

$$\mathbb{E}^Y \left[ \int_0^{+\infty} \frac{\partial \bar{\Psi}}{\partial K_k}(Y_s, K) e^{-\rho s} ds \right] = \int_0^{+\infty} \mathbb{E}^Y \left[ \frac{\partial \bar{\Psi}}{\partial K_k}(Y_s, K) \right] e^{-\rho s} ds$$

so we just need to evaluate

$$\begin{aligned} \mathbb{E}^Y [Y_s^{\gamma_i}] &= \mathbb{E} \left[ Y^{\gamma_i} e^{\gamma_i(\mu - \frac{1}{2}\sigma^2)s + \gamma_i \sigma B_s(\omega)} \right] \\ &= Y^{\gamma_i} \mathbb{E} \left[ e^{(\mu\gamma_i - \frac{1}{2}\gamma_i^2\sigma^2)s + (\frac{1}{2}\gamma_i^2\sigma^2 - \frac{1}{2}\gamma_i\sigma^2)s + \gamma_i\sigma B_s(\omega)} \right] \\ &= Y^{\gamma_i} e^{\frac{1}{2}\gamma_i\sigma^2(\gamma_i-1)s} e^{\gamma_i\mu s}. \end{aligned}$$

We have

$$\int_0^{+\infty} \mathbb{E}^Y [Y_s^{\gamma_i}] e^{-\rho s} ds = Y^{\gamma_i} \int_0^{+\infty} e^{-[\rho - \gamma_i\mu - \frac{1}{2}\gamma_i\sigma^2(\gamma_i-1)]s} ds$$

and this integral converges if  $\rho - \gamma_i\mu - \frac{1}{2}\gamma_i\sigma^2(\gamma_i - 1) > 0$ , taking value  $Y^{\gamma_i}/(\rho - \gamma_i\mu - \frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1))$ . Remembering of notations (29) and (30), one can note:

$$\begin{aligned} m_k(Y) &= \sum_{i,j=1}^{d(\gamma),d(\alpha)} \alpha_j \bar{b}_{k,i,j}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j-1} \\ &+ \sum_{\substack{t=1 \\ t \neq k}}^{d(\mathbb{K})} \sum_{i,j,l=1}^{d(\gamma),d(\lambda),d(\lambda)} \lambda_l \bar{c}_{kt,i,jl}(\gamma_i) Y^{\gamma_i} K_t^{\lambda_j} K_k^{\lambda_l-1} - \frac{OMC(k)}{\rho} \end{aligned} \quad (65)$$

The general solution of (61) is  $f_k(Y) = AY^{\beta_1}$ . The value matching and smooth pasting conditions take the form:

$$f_k(Y_k^*) = AY_k^{*\beta_1} = m_k(Y_k^*) - I_k \quad (66)$$

$$\frac{\partial f_k}{\partial Y}(Y_k^*) = \beta_1 AY_k^{*\beta_1-1} = \frac{\partial m_k}{\partial Y}(Y_k^*). \quad (67)$$

As usual, we start by exploiting the smooth pasting conditions; using (67), one obtains

$$A = \frac{\frac{\partial m_k}{\partial Y}(Y_k^*(K_k))}{\beta_1 Y_k^{*\beta_1-1}(K_k)} \quad (68)$$

with

$$\begin{aligned} \frac{\partial m_k}{\partial Y}(Y) &= \sum_{i,j=1}^{d(\gamma),d(\alpha)} \gamma_i \alpha_j \bar{b}_{k,i,j}(\gamma_i) Y^{\gamma_i-1} K_k^{\alpha_j-1} \\ &+ \sum_{\substack{t=1 \\ t \neq k}}^{d(\mathbb{K})} \sum_{i,j,l=1}^{d(\gamma),d(\lambda),d(\lambda)} \gamma_i \lambda_l \bar{c}_{kt,i,jl}(\gamma_i) Y^{\gamma_i-1} K_t^{\lambda_j} K_k^{\lambda_l-1}. \end{aligned}$$

Using (68) into the value matching (66), one gets

$$\frac{\frac{\partial m_k}{\partial Y}(Y_k^*(K_k))}{\beta_1} Y_k^*(K_k) = m_k(Y_k^*(K_k)) - I_k.$$

Introducing the expressions of  $m_k(Y)$  and  $\frac{\partial m_k}{\partial Y}(Y)$  leads directly to the expression:

$$\begin{aligned} \sum_{i=1}^{d(\gamma)} \underbrace{Y^{\gamma_i}}_{>0, \nearrow Y} \underbrace{\left( \frac{\beta_1 - \gamma_i}{\beta_1} \right)}_{>0} \left\{ \sum_{j=1}^{d(\alpha)} \underbrace{\alpha_j \bar{b}_{k,i,j}(\gamma_i)}_{>0} \underbrace{K_k^{\alpha_j-1}}_{>0, \searrow K_k} \right. \\ \left. + \sum_{\substack{t=1 \\ t \neq k}}^{d(\mathbb{K})} \sum_{j,l=1}^{d(\lambda),d(\lambda)} \underbrace{\lambda_k \bar{b}_{kt,i,jl}(\gamma_i)}_{>0} \underbrace{K_t^{\lambda_j}}_{>0} \underbrace{K_k^{\lambda_l-1}}_{>0, \searrow K_k} \right\} = I_k + \frac{OMC(k)}{\rho}. \end{aligned}$$



The left hand expression is positive, increasing in  $Y$  and decreasing in  $K_k$  for any values  $K_t$ ,  $t \neq k$ . The Bolzano's theorem ensures that this equation has a unique zero  $Y_k^*(K_k)$  for any given  $K_k$ . And because this left hand expression is decreasing in  $K_k$ , the trigger is increasing in  $K_k$ .

## D The verification theorem

All optimal stopping problems treated in this paper<sup>18</sup> are of the general form

$$\tau^*(Y, \omega) = \arg \sup_{\tau \in \mathcal{S}} \mathbb{E}^Y \left[ e^{-\rho\tau} \left( \sum_{i=1}^n c_i Y_\tau^{\alpha_i} - I \right) \right] \quad (69)$$

where  $\mathcal{S}$  is the set of random time that are stopping times<sup>19</sup> w.r.t. the filtration  $\mathcal{F}_t$  generated by the geometric Brownian motion  $Y_t^Y(\omega)$ .

This optimal stopping problem leads to the free boundary problem (variational inequality formulation): find  $f(Y)$  and  $Y^*(I)$  such that

$$\begin{aligned} \left( \mu Y \frac{\partial}{\partial Y} + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2}{\partial Y^2} - \rho \right) f(Y) &= 0 & Y \leq Y^* \\ f(Y) &= \sum_{i=1}^n c_i Y^{\alpha_i} - I & Y \geq Y^* \\ \frac{\partial}{\partial Y} f(Y) &= \frac{\partial}{\partial Y} \left( \sum_{i=1}^n c_i Y^{\alpha_i} - I \right) & Y = Y^* \end{aligned} \quad (70)$$

with  $\alpha_i < \beta_1$ .

We found

$$\begin{aligned} f(Y) &= v(Y) \triangleq a Y^{\beta_1} & Y \leq Y^* \\ f(Y) &= g(Y) \triangleq \sum_{i=1}^n c_i Y^{\alpha_i} - I & Y \geq Y^* \\ Y^* &= \arg_Y \left\{ I - \sum_{i=1}^n c_i Y^{\alpha_i} \left( \frac{\beta_1 - \alpha_i}{\beta_1} \right) = 0 \right\} \\ a &= \frac{1}{\beta_1} \sum_{i=1}^n \alpha_i c_i Y^{*\alpha_i - \beta_1}. \end{aligned}$$

We solved the problem assuming that smooth pasting condition (70) held. However, there exists simple<sup>20</sup> optimal stopping problems where the smooth pasting condition fails to hold.<sup>21</sup> A useful result of Brekke and Øksendal (1991) ensures that an stopping time found via a smooth pasting condition is the optimal stopping time. For sake of completeness, let us show that this verification theorem is easy to apply for (69). The candidate optimal stopping time determined using the smooth pasting principle satisfies the following properties

- (i)  $f \in \mathcal{C}^1(\mathfrak{R}_+)$ .
- (ii)  $f \geq g$  on  $\mathfrak{R}_+$ .

<sup>18</sup>The optimal stopping problems described by Problem 4 with  $\bar{\Psi}$  given by (1), by Problem 4 with  $\bar{\Psi}$  given by (2) and the optimal stopping problem (39).

<sup>19</sup>A random time  $\tau$  is a stopping time w.r.t. a certain filtration  $\mathcal{F}_t$  if  $\tau(\omega) \leq t$  is  $\mathcal{F}_t$ -measurable.

<sup>20</sup>Involving regular stochastic processes and differentiable reward functions

<sup>21</sup>See for instance Peskir, 2007.

*Proof.* It suffices to prove that  $f \geq g$  on  $Y \leq Y^*$ . We compute

$$\begin{aligned} g'(Y) &= \sum_{i=1}^n \alpha_i c_i Y^{\alpha_i - 1} \\ v'(Y) &= a \beta_i Y^{\beta_1 - 1} \\ &= \left( \sum_{i=1}^n \alpha_i c_i Y^{*\alpha_i - \beta_1} \right) Y^{\beta_1 - 1} \\ &\leq \left( \sum_{i=1}^n \alpha_i c_i Y^{\alpha_i - \beta_1} \right) Y^{\beta_1 - 1} \quad \text{for } Y \leq Y^* \\ &\leq g'(Y) \quad \text{for } Y \leq Y^*. \end{aligned}$$

Thus

$$\begin{aligned} v(Y) &= v(Y^*) - \int_Y^{Y^*} v'(x) dx \\ &= g(Y^*) - \int_Y^{Y^*} v'(x) dx \\ &\geq g(Y^*) - \int_Y^{Y^*} g'(x) dx = g(Y) \quad \text{for } Y \leq Y^*. \end{aligned}$$

□

- (iii) The geometric Brownian motion  $Y(t, \omega)$  spends no time on  $Y^*$ .
- (iv)  $Y^*$  is (trivially) a Lipschitz surface.
- (v)  $f \in \mathcal{C}^2(\mathfrak{R}_+ \setminus Y^*)$
- (vi) It is clear that, as long as  $\alpha_i \in [\beta_2, \beta_1]$  for all  $i$ , we have  $\left( \mu Y \frac{\partial}{\partial Y} + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2}{\partial Y^2} - \rho \right) f(Y) \leq 0$  for  $Y \geq Y^*$ .
- (vii) By hypothesis,  $\left( \mu Y \frac{\partial}{\partial Y} + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2}{\partial Y^2} - \rho \right) f(Y) = 0$  for  $Y \leq Y^*$
- (viii)  $\inf\{t : Y(t, \omega) \geq Y^*(I)\} < \infty$  a.s.  $P^Y, \forall Y = Y_0 \in \mathfrak{R}_+$  (the geometric Brownian motion is a regular process).
- (ix)  $\{v(Y_\tau); \tau \leq \tau^*; \tau \in \mathcal{S}\}$  is uniformly integrable<sup>22</sup> w.r.t.  $Q^Y, \forall Y = Y_0 \in \mathfrak{R}_+$  since  $Y_{\tau^*} = Y^*$  is bounded.

By the verification theorem (see Øksendal (2007), chapter. X, theorem 10.18 in the particular case when the value function  $f$  takes value in  $\mathfrak{R}_+$ ) for optimal stopping,  $\tau^* = \inf\{t : Y(t, \omega) \geq Y^*\}$  with  $Y^*$  defined by (71) is the optimal stopping time of (69).

## E Calibration of the demand function

For each  $l$ , the initial calibration of the demand function requires one observed point  $(\bar{P}(l), \bar{Q}(l))$  and elasticity  $\bar{E}(l)$ . One finds  $A(l)$  and  $b(l)$  by solving the simultaneous equations:

$$\left\{ \bar{P}(l) = A(l) - b(l)\bar{Q}(l), \quad \bar{E}(l) = \frac{\bar{P}(l)}{\bar{P}(l) - A(l)} \right\}.$$

<sup>22</sup>A family  $\{f_j\}_{j \in J}$  of real, measurable functions  $f_j$  on  $\Omega$  is called uniformly integrable if

$$\lim_{M \rightarrow \infty} \left( \sup_{j \in J} \left\{ \int_{|f_j| > M} |f_j| d\mathbb{P} \right\} \right) = 0.$$

*A real options model for electricity capacity expansion*

After time 0, there are two observed processes:  $\bar{P}_t(l)$  and  $\bar{Q}_t(l)$  respectively annual average values of prices and loads for a given demand segment  $l$ . The shift parameter  $\bar{Y}_t(l)$  for demand segment  $l$  is estimated by

$$\bar{Y}_t(l) = \frac{\bar{P}_t(l) + b(l)\bar{Q}_t(l)}{A(l)} \quad (71)$$

then averaged over  $l$  using weights  $\tau(l)$ .

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