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A NON-LINEAR MODEL OF FLUCTUATIONS
IN OUTPUT IN A MIXED ECONOMY

by

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"So far, our efforts to scale the non-linear
barrier have consisted of chiselling a few
footholds which are low enough so that we
can always keep one foot on linear ground....
There is no general theory for non-linear
problems, so that it is necessary to develop
solutions by means of special techniques for
each type.....The non-linear barrier appears
to be one of nature's least vulnerable strong-
holds"

"Life Can Be So Non Linear"
by Ladis D.Kovach in
the American Scientist

BADIA FIESOLANA, SAN DOMENICO (FI)

1. Introduction

In an interesting outline of 'Alternative approaches to macroeconomic theory' that underlie three current approaches to theoretical macroeconomics, Solow indicated his own preference (subtitling his article as 'a partial view' - i.e., not impartial!, cf. Solow (1979)) for the fixed (or sticky) price models as being the most promising avenue for research in elucidating the policy dilemma facing political economists. The other two he called the 'post-Keynesian approach' and 'equilibrium theory'. The choice-theoretic framework - or, optimizing economic agents - seems to be the (only) common factor in the first and third approaches. However, in the latter approach, an additional strong hypothesis is made regarding equilibrium: fluctuations in aggregate variables is predicated upon an assumption of market clearance. In the post-Keynesian approach, on the other hand, a sense of nihilism seems to prevail in that optimization, equilibrium and even dynamics - at least in the conventional sense of formalization in terms of differential equations - are all rejected in the abstractions implied in the modelling exercises approved by that school. Though it is difficult to be totally unsympathetic to the post-Keynesian criticisms, particularly in the forms in which Joan Robinson has developed them, on the unsuitability and even irrelevance of conventional concepts and tools, the difficulty remains, as Solow (op.cit., p. 344) astutely observes, that this school has provided no systematic description or example of what it conceives to be the right way to do macroeconomic theory!

The policy dilemma itself is the problem of stagflation: its cures and origins. Disequilibrium theorists, using fixed-price models claim to have contributed, at least, to an elucidation of the origins of, if not the cure to, one half of the stagflation problem; i.e., the distinction, important for policy purposes, between unemployment which results from a lack of effective demand to that which results from inappropriate levels of the real wage. Between the Scylla of lack of effective demand (not least due to low real wages and unemployment!) and the Charybdis of high real wages (and a third, generally neglected classification) the economy must steer a careful course which, in general, implies in fact unemployment.

Equilibrium theorists, on the other hand, emphasizing monetary disturbances, attribute fluctuations in levels of output and employment to imperfect and non-uniform information (at least subjectively so). In this approach, however, the problem of dynamics - persistent fluctuations in aggregate variables - is tackled from the very outset. This is, of course, natural in view of the methodology adopted by this school, but also because the problem of stagflation is, by definition, a dynamic problem.

This has, unfortunately, not been the case in the disequilibrium approach to macroeconomics (cf. Fitoussi (1983)). The problem of stagflation, in fixed-price models, has been tackled, at least until very recently, in two almost totally separate

compartments - corresponding, in fact, to the twin horns that
characterize stagflation: inflation and unemployment. In the
first instance fixed-price models analysing the various permu-
tations that are possible when more than one market is allowed
to exhibit disequilibria, have given almost deceptively clear
characterizations of unemployment in the price-wage space.
Only then, in the next stage have wage-price dynamics been con-
sidered, but within each regime. This dichotomy leads, quite
naturally, to considerations of switching problems. The traj-
ectory of a representative point, in the wage-price space (which
is, usually, partitioned into the famous 3 regimes of Classical
Unemployment, Keynesian Unemployment and Repressed Inflation,
in the first, static, stage of analysis) will, in general,
'hit' one of the boundaries characterizing the different re-
gimes. At any such boundary the well known conditions for
existence and uniqueness of solutions fails to hold because the
initial conditions will be compatible with dynamics in more
than one regime - unless, naturally, restrictive conditions are
imposed. In recent attempts to combine disequilibrium statics
and macrodynamics the technical problem of switching from one
regime (say 'Classical Unemployment') to another (say 'Keynesian
Unemployment') has been resolved by one of two ways. In the
one case, where disequilibrium statics of the macromodel is
combined with the assumptions of neoclassical growth theory,
(cf. for e.g. Ito (1980)) some variant of the original Marchaud
(1934) or Zaremba (1936) technique of the theory of contingent

equations (cf. Filippov (1962)), Roxin (1965) or the survey
by Bridgeland (1967 (a)(b) for recent revivals of this tradition -
or, for an alternative exposition, the elegant discussion in
Hajek (1968) esp. pp. 31-34). In the other case, inspired by
Zeeman's work on the heartbeat and nerve impulses (cf. Zeeman
(1973)), as in Blad (1981), regime switching has been formulated
as a generalized relaxation oscillation problem by assuming the
existence of different speeds of adjustment with respect to
quantities and prices (cf. Guckenheimer (1973) esp. pp. 888-
889).

The attempt, therefore, to combine disequilibrium statics
and macrodynamics revolves around switching from one regime to
another. The analytical solution presupposes, in this case,
that the boundaries are invariant with respect to trajectories
that approach and attain values defining coordinates on the
boundary. In the 'old fashioned' terminology of 'trade cycle'
theory, as the full employment ceiling is approached and then
attained, growth is blunted and cyclical elements are activated.
The point, however, is that the ceiling itself is a fuzzy entity
though it is, indisputably, a constraint (cf. below), just as
the boundaries of the regions in the price-wage space in fix-
price literature. It is inconceivable that the vector-field
of macrodynamic variables in the direction of the boundaries
('ceilings') will leave invariant the partition defining the
boundaries in the price-wage space. Indeed, in his pioneering
essay, Malinvaud explicitly considered some of the conditions

under which 'the partition of the price-wage space moves'
(Malinvaud) (1977) esp. ch. III). These conditions turn out
to be as can be expected, those that are typical of macro-
dynamic models (technical progress, variations in the distribu-
tion of income induced by inflation, etc.). If, therefore,
macrodynamics is taken seriously, the conflation with disequi-
librium statics to discuss stagflation, particularly from a
policy point of view, cannot proceed satisfactorily by formal-
izing regime switches. The formalizing should consider the
'movement of the partition of the w-p space' (Malinvaud (op.
cit., p. 88) as a function of the macrodynamic coordinates
characterizing the point of the boundary to which the vector
field collapses - if at all.

Admirable as the above efforts are, and have been in
clarifying several crucial problems in technique and concepts,
it appears as if the 'disequilibrium static' starting point
was inappropriate - particularly in view of the fact that the
whole apparatus of disequilibrium macroeconomics seems to have
been (at least in recent years, even if not in inception) a
tool for the analysis of an inherently dynamic problem: i.e.,
stagflation. The uneasy feeling persists that 'dynamics' has
been grafted onto an inherently static system, almost as an
afterthought, in several ad-hoc ways.

This, of course, is not the case in the framework adopted
by the 'equilibrium school'. However, quite apart from the
sweeping assumptions about the economy always being in equi-
librium, the existence of persistent fluctuations in aggregate

variables is, itself, explained largely, if not solely, by
(subjective) expectational errors. It is, however, possible
to show, by means of a simple variant of the model we present
below, that irregular fluctuations may persist in a non-random
model (so-called "chaotic" dynamics or 'strange attractors' of
dynamical systems) even if we grant all the assumptions under-
lying the models generating 'equilibrium business cycles':
i.e., granting all assumptions with the exception of so-called
(subjective) expectational errors. This result would seem to
leave the explanation of the problem of persistent fluctuations
in unemployment hanging by its own bootstrap in 'equilibrium
business cycle' models.

'Stylised facts' in economics, like truth, according to
Dr. Stockman in Ibsen's 'Folkefiende' (An Enemy of the People),
seem to have a life of about twenty years. The one exception,
almost confirming, by being the exception, Dr. Stockman's
profound observation, at least in capitalist economies of the
mixed type, seems to be the indisputable observational fact of
growth cycles in employment and output, against a backdrop of
rising government activity. To be more precise, the interaction
between real wages, profits, investment and capacity utilization
seems to be such that, with increasing government activity
(measured, for example, by the ratio of government expenditure
in total output), growth cycles in output and employment have
become endemic in so-called mixed economies of the advanced
capitalist type. The dynamical relationsthip between real
wages, profits, investment and capacity utilization are such

that, whatever the policies adopted, i.e., whether Keynesian, classical or Monetarist, fluctuations in functional income distribution seem to be the result. It is, of course, possible that misplaced policies, eg., Keynesian policies to a classical situation, are the source of these fluctuations of an economy incessantly in search of "An Appropriate Income Distribution" (Malinvaud (1980)). If this were the case, the contributions of the fix-price models, even with ad-hoc dynamics, should be considered fundamental. On the other hand, if it was the case that no natural constraints exist in an economy (say, no "floors" and/or "ceilings" in the classic Hicksian sense) then the fluctuations may well have been caused by the rigidities induced by discretionary economic policy (whether monetary or fiscal) rather than sticking to 'rules'. The fact that neither of these properties are true makes it necessary to formulate models in such a way that dynamics in the presence of various constraints are considered from the very outset. The existence of constraints, for eg., the indisputable one of "full employment", implies that "rules" will <u>have</u> to be changed which, in turn, means that "rules" without "discretion" will be meaningless. On the other hand, the dynamical implications of sustained policies, as so clearly and controversially brought to the forefront of macroeconomics by the rational expectations school, is that behavioural functions change as learning and adaptation proceeds (thus modifying the partitions of the w-p space and in-

ducing shifts in the boundaries).[1]

The model we present here is based on a return to the tradition of old-fashioned cycle theory but with growth - i.e., a theory of growth cycles.

In this paper, at least methodologically, we try to combine from the very outset, in a simple way, the two central themes of the 'equilibrium' and 'fixed-price' macroeconomists: dynamics and disequilibria. It has always seemed to us that much is to be gained by emphasizing more one part of the funda- mental message of the 'correspondence principle': 'one interested only in fruitful statics must study dynamics' (Samuelson (1947) p. 5) or 'in order for the comparative-static analysis to yield

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[1] Either way, the result will be fluctuations in growth rates of output and employment now nudging the share of wages one way and again nudging it another way; now to restore profit- ability and then, again, to re-establish levels of demand compatible with productive potential; all the while imply- ing, and implied by, the existence of the state for social, political and economic reasons. Socially, the necessity and desire to achieve fairer distributions of income and wealth implies, in the simplest of models, variations in the propen- sity to consume. Politically, the need to retain power may induce governments to activate fiscal and monetary policies such that the elements conducive to the formation of 'Political Business Cycles' will be compounded; and, economically, the need to utilize efficiently, and create optimally, appropriate capacity, may call forth policies not compatible with the social and political aims. In this complex maze of social conflicts, political ideologies and economic efficiency to concentrate exclusively on one or the other of the three spheres, as existing models seem to do, is quite clearly myopic.

Encouraging signs that both fixed-price theorists and equi- librium methodologists are moving in directions that seem to recognize the existence of such an interaction are emerging (cf. in particular Malinvaud (op.cit.), especially the dis-

fruitful results, we must first develop a theory of dynamics'
(Samuelson, op.cit., pp. 262-263, italics added).[2]

Working backwards from the inherent dynamics of defini-
tional equations stated in terms of ratios so as to circum-
vent some technical difficulties arising out of working in
terms of levels we develop a simple model of disequilibrium
and persistent fluctuations in the labour and product markets
resolved by variations in the functional distribution of
income. Our starting point, though it may not be evident
in the formalization, is the class of models in the tradition
of Goodwin (1967), Solow-Stiglitz (1968) and Akerlöf-Stiglitz
(1969 - cf. esp. p. 273, footnote 1).

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1) cont.nd
   cussion on "An Appropriate Income Distribution" and Lucas
   (1981), especially pp. 289-291). But this, after all, was
   the message of old-fashioned trade cycle theory! Indeed
   some of the modern theorists do acknowledge this - Malinvaud
   ·in particular; others pass on, dismissing that paradigm on
   the grounds that behavioural relations (in old-fashioned
   trade cycle theory) have not been derived from a choice-
   theoretic framework. (They do not, of course, explain the
   transfer from individual choice-theoretic frameworks to
   aggregate relations except by assuming "representative
   agents" - without realizing that these assumptions are
   equivalent to assumptions about "homogeneous labour" and
   "equal organic composition of capital" in that, other, much
   maligned tradition of Marxian economics.)

2) Yet another point to be emphasized in developing a theory
   of dynamics for the macroeconomy - also elegantly described
   in Samuelson's Nobel Prize Lecture - is that only very
   special dynamical systems are implied by useful underlying
   maximizing principles (cf. Samuelson (1970), p. 12 ff.
   in particular).

In these early and pioneering approaches some assumptions were made in the interests of retaining the possibility of planar dynamics so that graphical (phase-plane) techniques could be utilized. This was explicit, for example, in the case of Solow-Stiglitz (op.cit., p. 320) and implicit in the case of Goodwin (disequilibrium only in the labour market).

In the model we develop these restrictive assumptions are relaxed whilst retaining the focus on the central problem of stagflation. In addition an attempt is made to avoid some of the weak points we have outlined with respect to the fixed-price and equilibrium models.

Thus, in §2, the framework and notations of the model are made precise. In §3, some general results for the complete model are presented. In §4, some simple special cases of the model will be analyzed to illustrate not only the techniques but also to elucidate the role of the assumptions in the dynamic workings of the model. Finally, in a concluding §5, some directions for further developments, within the class of models and techniques we have presented, are discussed.

§2.   The Model

2.1 :   The Framework

Instead of the usual procedure of beginning from the
behavioural, technical, and definitional equations of an ag-
gregative economy, we commence, from the very outset, with
identities, expressed as ratios, from which we derive the in-
herent dynamics of a closed economy of a 'more-or-less' capital-
istic variety. The canonical framework will not include, ex-
plicitly, government activities; we will also abstract from the
complications of an open economy. However, we will indicate,
at the end of this section, the nature of the modifications that
must be considered if our basic model must be extended to in-
clude these extra complications.

The essential forces characterizing stagflationary econo-
mies are determined (in the simplest case of a one-good, closed
economy, without an explicit role for the state) by the disequi-
libria in the labour market, product market and the dynamic
relationship between productivity and real wages. These, in
turn, depend crucially on capitalists' decisions to create
new capacity in conjunction with their decisions on the de-
gree to which utilization of existing capacity is compatible
with considerations of profitability. The demand side of the
disequilibria depends, on the other hand, on the forces deter-

mining the choice of technique in the economy. Clearly, this too is a decision variable of the capitalists in our simplified economy. Three dimensions, therefore, make up the decision space of the capitalists (if we abstract from price formation), viz: level of investment (or creation of capacity), choice of technique (or capital intensity) and rate of utilization of existing capacity. Of these three, we will ignore explicit considerations of the problem of capacity utilization; it will, however, become clear that the extent to which capacity is sub-optimally utilized is reflected in the disequilibrium in the product market and the decisions, by capitalists, to create new capacity.

The crucial ratios are, therefore, the following three:

$$u = \frac{wL}{pY_s} \qquad \qquad \dots\dots \quad (1)$$

$$v = \frac{L}{N} \qquad \qquad \dots\dots \quad (2)$$

$$\text{and} \quad y = \frac{Y_d}{Y_s} \qquad \qquad \dots\dots \quad (3)$$

where:

    u : share of wages

    v : (un-)employment ratio

    y : ratio of demand to supply in the product market

    w : money wage rate

    p : price level

    L : Labour demand

N : Labour supply

Y_s : Supply of Output or Productive Potential (real)

Y_d : Demand for Output (real)

Taking logarithmic derivatives of the identities (1) ~ (3)
that have been expressed as ratios, we get:

$$\frac{\dot{u}}{u} = \left(\frac{\dot{w}}{w} - \frac{\dot{p}}{p}\right) - \left(\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L}\right) \qquad \ldots \ldots \quad (4)$$

$$\frac{\dot{v}}{v} = \frac{\dot{L}}{L} - \frac{\dot{N}}{N} \qquad \ldots \ldots \quad (5)$$

and,

$$\frac{\dot{y}}{y} = \frac{\dot{Y}_d}{Y_d} - \frac{\dot{Y}_s}{Y_s} \qquad \ldots \ldots \quad (6)$$

It is quite clear from these dynamic relations that the
aggregate dynamics of a stagflationary closed economy (with
no explicit role for the government) will depend on the deter-
minants of the rates of growth of: real wages $\left(\frac{\dot{w}}{w} - \frac{\dot{p}}{p}\right)$, product-
ivity, $\left(\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L}\right)$, labour supply $\left(\frac{\dot{N}}{N}\right)$, output $\left(\frac{\dot{Y}_s}{Y_s}\right)$ and demand
$\left(\frac{\dot{Y}_d}{Y_d}\right)$.

In the next subsection we discuss the specifications of
the relations determining the dynamics of income distribution
and labour- and product-market disequilibria.

2.2 : Model Specifications

2.2.a : Productivity or Productive Potential:

In the usual neo-classical models of an aggregative
economy productive potential is formalized in terms of the
well-known production function. However, as Solow (op.cit.)
has persuasively argued, it is no longer sufficient to have only
a "technical" interepretation of the productive potential in
an economy.[3] Growth in real wages, according to Solow, en-
capsulating "morale, productivity and quality effects" must
be an argument in any relation depicting the productive potential
of an economy. In looking, therefore, for a specification of
the development of productive potential of an economy we have,
in addition to the above requirements, to take into account
also consistency with useful and feasible microfoundations
both from an engineering and an economic point of view. From
an economic point of view, once the level of capacity is deter-
mined, the decision on choice of technique, we assume, is based
on considerations of the (expected) benefits of cost reduction.

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3) "It has always struck me as a more substantial mystery that
   employers do not more aggressively push for wage reductions
   in a buyers' market.
       An extension of the argument (i.e., a rationale for in-
   cluding the relative wages as an argument in the preference
   functions (which amounts to including income distribution as
   an argument in the preference function)), can close the
   gap.... One can .... formalize this .... by the unconven-
   tional devise of including the wage as an argument in the
   firm's production function to represent the morale, product-
   ivity, and quality effects in a summary way"
                   (Solow, op.cit., p. 347 ;cf. also Solow (1979a).

   Clearly, if the wage enters the production function in an
   essential way, it is not possible to resort to the usual
   marginal productivity relationship to determine wages.

On the other hand, from an engineering point of view, taking a hint from Johansen's lucid discussion of the Kurz-Manne study (cf. Johansen (1972) pp. 190-195 and Kurz and Manne (1963)), it seems reasonable to attempt to relate output per unit of labour to capital per unit of labour. This relation, at the firm and engineering level, can be interpreted as Johansen's "technique relation" (cf. Johansen (op.cit.), p. 21, equation 2-17) from which a Johansen-type short-run macro production function can be derived.

Combining a relationship between output per unit of labour and capital per unit of labour with the influence of factor costs that have implications both for factor proportions and productivity (i.e., choice of technique on the basis of economic considerations) we obtain a specification which is a hybrid of the Kaldor (1957), Kennedy (1964), Solow (op.cit.) vision of the development of productivity and productive potential in an economy:[4]

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \Omega \quad \left(\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right), \left(\frac{\dot{w}}{w} - \frac{\dot{p}}{P}\right) \right) \qquad \ldots \ldots \quad (7)$$

where

K : Capital[5] and $\Omega_1 > 0$, $\Omega_2 > 0$

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4) All functions, unless otherwise stated, will be assumed to be continuous and smooth - i.e., c' functions. It is easy to show that 'Verdoon'-type considerations can also be encapsulated in our formalization without undue extra complications.

5) Note: Ω is a functional relation, i.e., (7) is a general, non-linear function. It must also be noted that it is a relation about the supply potentials of the economy and, hence, the relevant output variable is the supply of output, Y_s.

we can rewrite (7) as follows:

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \Omega \left((\frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{\dot{Y}_s}{Y_s} + \frac{\dot{Y}_s}{Y_s}), (\frac{\dot{w}}{w} - \frac{\dot{p}}{p}) \right) \quad \ldots \ldots \quad (8)$$

and, we get, after rearranging:

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \Omega \left(((\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L}) - (\frac{\dot{Y}_s}{Y_s} - \frac{\dot{K}}{K})), (\frac{\dot{w}}{w} - \frac{\dot{p}}{p}) \right) \ldots \quad (9)$$

2.2.b : <u>Investment Behaviour</u>

Paralleling the dominance of the concept of the production function to describe productive potentials and productivity, the theory of investment behaviour has, almost without exception, been based on the neo-classical theory originated by Jorgensen (1963). It is, however, also well known that the rate of investment, whether optimal or not, is indeterminate in the class of models descending from the approach formalized along the lines suggested by Jorgenson. This remark applies also to the models incorporating classical versions of the flexible accelerator. Several ad-hoc methods to circumvent the problem of indeterminateness have been suggested, notably by Lucas (1967) and Uzawa (1969). These ad-hoc variants rely on giving more structure to neo-classical models in terms of adjustment or installation costs that constrain the achievement of a desired capital stock infinitely fast. It has recently been shown by Hayashi (1982) and Yoshikawa (1980) that the neo-classical theory of investment behaviour with adjustment costs is, under

some conditions, equivalent to the so-called "q" theory of
investment. Thus, in recent macrodynamic models, the "q" theory
of investment behaviour has been directly incorporated (cf.
Buiter (1979), p. 125, equation 12, for example). Following
the type of analysis indicated in Hayashi, it can be shown
that, at the micro level, if economic behaviour is modelled as
maximization of the (expected) present value of the net profits
stream subject to an Uzawa-type installation function (cf.
Uzawa (1969), pp. 639-641 and figure 4, p. 640), then invest-
ment is simply a function of "q" or the "valuation ratio" (cf.
Kaldor (1966), appendix pp. 316-19, and Kahn (1972), chapter
10).[6]

In the Cambridge version, the relationship between in-
vestment behaviour and the valuation ratio[7] is derived on the
basis of equilibrium in the securities markets, whereas in the
hands of Uzawa, Hayashi and Yoshikawa the derivation is based
on some form of inter-temporal optimization. If, within the
context of our model - especially taking into account equation
(9) - we follow the latter method, it will not only be necessary
to increase the dimensions of the dynamical system ((4) - (6)),

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6) It is interesting to note that not only is the neo-classical
   theory of investment modified by including adjustment or
   installation costs equivalent to Tobin's "q" theory of in-
   vestment, but also the Cambridge theory of investment, at
   least in the hands of Kaldor (op. cit.), Kahn (op.cit.) and
   Wood (1972), when based on microlevel considerations in a
   rigorous fashion, reduces to the same formulation.

7) There is some confusion, in the early literature on linking
   investment behaviour to a valuation ratio, between marginal
   and average concepts (cf. however, Hayashi (op.cit.), for a
   lucid classification of concepts).

but it will also lead to technical complexities without cor-
responding analytical benefits for the main problems of this
paper. If we follow the former method, we not only incorporate,
in an essential way, conditions that determine clearance of
the securities market but also are able to consider the sav-
ings-investment identity without violating our basic assumption
about disequilibrium in the product market. We assume, in any
case:

$$\frac{I}{K} = \frac{\dot{K}}{K} = I(q) \quad , \quad I \geq 0, I' > 0 \qquad \ldots \ldots \quad (10)$$

and I : real gross investment. (cf. also Oulton (1981),
in particular §2.3, pp. 183-185 and equation (30), p. 185.)

We have, however, claimed that our exercise is a return
to the tradition of old-fashioned trade cycle theory. In that
sense, the fact that a q-theory of investment can be derived
from a flexible-accelerator type of model (or desired stock
of capital model - cf. Oulton (op.cit.)) substantiates, to
some extent, our claim. It must be noted, however, that implied
in the "Cambridge" derivation is some concept of an "equilibrium"
or "natural" rate of profit - i.e., a rate of profit compatible
with equilibrium in the securities market and a savings-invest-
ment identity parametrized by that rate of profit. Clearly,
then, a realized rate of profit given, in our model, by $(1-u)\frac{Y}{K}$,
not compatible with such an "equilibrium" rate must imply price
dynamics on the nominal side or output dynamics on the real side -

or both.[8]   We turn, therefore, now to price - and wage -

dynamics.

2.2.c :   Price-Wage Dynamics

We make no effort to go beyond conventional wisdom in

postulating (behavioural) relationships for price- and money-

wage dynamics - except, of course, that since money is purely

accommodating we do not consider one of the 'conventional' al-

ternative: i.e., that of money supply and money demand deter-

mining the price level. It is not, however, difficult to re-

formulate the model (after including some features of an open

economy and an essential role for government) so as to proceed

to the determination of the price level and its dynamics along

these neo-monetarist lines. This we leave for a different

exercise. On the other hand, we are sufficiently unconventional

to reconsider the price dynamics proposed by Wicksell (1898).

Clearly, the valuation ratio (which is, in any version,

a relation between the market value of securities and the ac-

counting value of assets) encapsulates, when changing, the

central idea of Wicksell: the discrepancy between the market

rate and the 'natural' rate of interest. (Though Wicksell was

disturbed, all through his life in economics, about the concept

of the natural rate of interest - derived from his deep and

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8) This way of posing the problem meets some of the important
    objections raised by Leijonhufvud in his outstanding essay
    on "Wicksellian Themes" (cf. Leijonhufvud (1981), chapter 7)
    regarding the loanable funds vs. liquidity preference theories.

dedicated study of Böhm-Bawerk's theory of capital - and continued to shift ground regarding the theoretical necessity and soundness of such a concept[9], he never deviated from his original interpretation of the level and rate of change of money prices.[10]) Investment, we have seen, is an increasing function of the valuation ratio. It is then only a natural step in consistency as well to consider specifications of price dynamics in terms of the valuation ratio, where, now, the latter could be used as a proxy for the

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9) Indeed in our opinion the definite statement with respect to the 'natural rate of interest' was made by Sraffa in his celebrated discussions with Hayek in the pages of the Economic Journal in the early 30's (cf. Sraffa (1932a, 1932b), and Hayek (1932)). In fact the important chapter 17 of the General Theory also derives from these Wicksellian themes, as is clear from Keynes' footnote in that chapter referring to Sraffa.

10) "The exchange of commodities in itself, and the conditions of production and consumption on which it depends, affect only exchange values or RELATIVE prices: they can exert NO DIRECT INFLUENCE WHATEVER ON THE ABSOLUTE LEVEL OF MONEY PRICES.

   For a single commodity or group of commodities, the establishment on the market of an incorrect relative price results in an inequality between supply and demand, between production and consumption, and this sooner or later effects the necessary correction. But if, ..., the prices of all commodities, or the average price level, is for any reason forced up or depressed, there is nothing in the conditions of the COMMODITY market that is calculated to bring about a reaction......

   .....If there is any reaction whatever away from a GENERAL level of prices that is too high or too low, it must originate somehow or other from OUTSIDE the commodity market proper......

   .....one thing is certain: money prices, as opposed to relative prices, can never be governed by the conditions of the commodity market itself (or of the production of goods); it is rather in the relations of this market to the MONEY MARKET, in the widest sense of the term, that it is necessary to search for the causes that regulate money prices."
                                    (Wicksell (1936), pp. 23-24)

discrepancy between the so-called market rate of interest and an accounting rate of profit. It is, however, also true that specifications of price dynamics, in aggregative models have, almost always, been based on one of two considerations: either on excess demand ruling in the product market <u>or</u> on some form of mark-up principle due, in turn, to imperfect competition in the product market (or a combination of the two). Retaining agnosticism for theoretical purposes we postulate, therefore:

$$\frac{\dot{p}}{p} = \int (u,y;q) \qquad\qquad \ldots\ldots \quad (11)$$

where $\int_1 > 0$ and $\int_2 > 0$.

The problem of wage formation and wage dynamics must, of course, be subjected to the same strictures as those hoisted upon specifications of price dynamics. In particular, when no meaningful marginal productivity determinants of the wage rate can be extracted, it seems natural to resort to bargaining theoretic rationale. This must be particularly compelling in highly

---

10) cont.nd

Though there seems to be a minor contradiction between the third and fourth quoted paragraphs, the message is unambiguously clear. And, then, in discussing propositions trying to base the specification of the absolute level of money prices on the basis of cost of production reasonings, he was equally categorical in the denial:

"In any case, the proposition that prices of commodities depend on their costs of production and rise and fall with them has a meaning only in connection with RELATIVE prices. To apply this proposition to the general level of money prices involves a generalization which is not only fallacious but of which it is in fact impossible to give any clear account."
(Wicksell (op.cit.), p. 99)

industrialized, heavily unionized mixed-capitalist economies
of the Western variety - the type of economy we have in mind
in our modelling and specification considerations. We make no
effort, therefore, in this case as well to go beyond conventional
wisdom in postulating a behavioural function for money-wage
dynamics (indeed, not even as far as that, though no particular
analytical difficulties are involved if the full paraphernalia of
"expectation economics" must also be included). An attempt is
made, however, to be as general as possible so that not only
Phillips-curve wisdom but extensions in terms of various forms
of expectations can, if necessary, be extracted. Thus, we
assume:

$$\frac{\dot{w}}{w} = g\left(u,v,y,\frac{\dot{p}}{p}, (\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L})\right) \qquad \ldots\ldots \quad (12)$$

where $g_i > 0 \quad \nabla i, \quad i = 2\ldots.4$

and $g_1 \lessgtr 0$

Wage dynamics, therefore, is assumed to depend - on the basis
of a bargaining rationale - on disequilibria in the labour and
product markets, inflation and productivity; in addition, re-
taining the bargaining theoretic rationale, the income distrib-
ution variable is introduced to encapsulate the type of trade-
off envisaged in Fitoussi-Nuti (1983) based on wage-fund argu-
ments.

2.2.d : <u>Labour Supply</u>

Labour <u>demand</u> in a mixed capitalistic economy should be
an endogenous consequence of government expenditure decisions,
capitalistic decisions on creation and utilization of capacity
and, finally, also based on capitalists' criteria for choice
of technique. On the other hand, we have argued for the in-
clusion, based on wage-fund reasonings and bargaining rationale,
of functional income distribution as a proxy for the wage bill
in the relation for money wage dynamics. It will be stretching
the domains of economics too far to claim that labour <u>supply</u>
should also be endogenous. Demographic, social and political
factors are, surely, far more important as determinants of
labour supply than purely economic factors. It has, however,
been a tradition - even up to the time of Wicksell (cf. the
first edition of Wicksell's lectures (Wicksell (1901)) - to
include population dynamics in macrodynamic analysis. Until
the recent revival of neo-classical economists' interest in
problems of work-leisure choice, labour supply was considered
as purely exogenously determined or alternatively some variant
of the classical assumption of dependence upon growth in real
wages was used. If we follow the neo-classicals all the way,
the choice-theoretic basis would be something like: maximize
utility subject to an upper limit on feasible supply of labour
(and the other, usual, budget constraints) with the contributions
of work interpreted as disutility. This latter approach seems,
to us, at least as simplistic as the classical versions -

especially for the type of economies for which these relations
are being considered: advanced, mixed capitalist economies of
the Western type. Ideally, our macrodynamic model should be
coupled to a demographic-aggregative no doubt - model deter-
mining population dynamics. Since this task, though not im-
possible, will detract from the main purpose of this paper
we opt for an assumption of exogeneity for labour supply.

It must, however, be pointed out that classical assumptions
on labour supply can easily be included; indeed, even a neo-
classical, choice-theoretic assumption would be consistent with
the model being developed.[11] The latter we have eschewed be-
cause it is too simplistic and violates too many of the true
dimensions of population dynamics. The former we have ignored
because it seems unrealistic for modern economies (cf. Tirelli
(1983) for a lucid discussion on the development of thought
regarding the problem of labour supply). Thus, we assume:

$$N = N_o e^{\gamma t} \qquad \dots\dots \quad (13)$$

where $\gamma \geq o$

2.2.e : Model Reduction

In 2.2.a $\sim$ 2.2.d we have considered all the necessary
building blocks for our model - no more is necessary. Some

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11) It is easy to see that classical assumptions such as
$$\frac{\dot{N}}{N} = j_1(\frac{\dot{w}}{w} - \frac{\dot{p}}{p}) \qquad \dots\dots \quad (a)$$
or $\frac{\dot{N}}{N} = j_2\left(\left[(\frac{\dot{w}}{w} - \frac{\dot{p}}{p})\right], u\right)$ $\qquad \dots\dots \quad (b)$

fit into the framework that we are considering. Neo-
classical alternatives are also easy to include in our
framework. However, since we have nowhere specified the
functions of a utility maximizing individual, some re-
specifications would be required.

elementary mathematical manipulations is all that is now required to derive a reduced form dynamical system in u, v and y. This we proceed to do now. Substituting (11) and (12) in (9), we get:

$$\frac{\dot{Y}_s}{Y_s} = \frac{\dot{L}}{L} = \Omega \left\{ \left[(\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L}) - (\frac{\dot{Y}_s}{Y_s} - \frac{\dot{K}}{K}) \right], \left(g\left[u,v,y,f(.;q), (\frac{Y_s}{Y_s} - \frac{\dot{L}}{L}) \right] - f(.;q) \right) \right\} \quad \ldots\ldots \quad (14)$$

We can rewrite (13) as:

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \Omega \left\{ \left[(\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L}) - (\frac{\dot{Y}_s}{Y_s} - \frac{\dot{K}}{K}) \right], (\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L}), u,v,y,q \right\} \quad \ldots\ldots \quad (15)$$

Assuming an appropriate form of the implicit function theorem with respect to labour productivity, we get:

$$\frac{\dot{Y}_s}{Y_s} - \frac{\dot{L}}{L} = \theta \left[(\frac{\dot{Y}_s}{Y_s} - \frac{\dot{K}}{K}), u,v,y,q \right] \quad \ldots\ldots \quad (16)$$

Denote labour and capital productivity as:

$$\beta = \frac{Y_s}{K} \quad \ldots\ldots \quad (17)$$

$$\alpha = \frac{Y_s}{L} \quad \ldots\ldots \quad (18)$$

Using (17) and (18) we can rewrite (16) as:

$$\frac{\dot{\alpha}}{\alpha} = \theta \left(\frac{\dot{\beta}}{\beta}, u,v,y,q \right) \quad \ldots\ldots \quad (19)$$

To be consistent with our two-stage decomposition of the capitalists decision process where, at one stage (and apparant-ly exogenously), the level of capacity (creation) or investment

is determined and at another stage the problem of choice of technique is resolved, we now consider the latter problem.[12]

The basic capitalist program in choice of technique or capital intensity is assumed to be a behavioural rule which attempts to <u>maximize the reduction in unit costs</u>[13]. In words, the rule amounts to maximizing the weighted reduction in unit labour and materials costs where, in our one good closed economy model, the latter reduce to capital costs and the weights are the respective productivities. Thus, we have, as the capitalists' program on capital intensity and its determination, the following:

---

12) It is of course conceivable that this is an improper procedure - i.e., the two-part decomposition of the capitalists' decision process. However, inspired by Kuczynski's (1983) "Letter from No" we have endeavoured to try to capture the essence of those features that also reflect the separation of control from ownership in modern firms. <u>That</u>, is our main justification for this separation. Analytically, no great difficulties will be involved in a simultaneous solution. There is no implication of strategic considerations with respect to the sequential nature of the decision process.

13) The problem is to account for the impact of changing relative factor prices on the choice of techniques. The expected benefit from a cost reducing choice is, clearly, the sum of reductions in input requirements weighted by the price of each factor. This, in turn, amounts to, the sum of the reductions in labour requirements, weighted by the unit costs of labour, plus the reduction in capital requirements, weighted by capital costs. We have, in the text, made an inversion in the role for weights for obvious reasons. A similar approach to the problem of capital intensity, though in a somewhat broader context of <u>research and innovation</u>, is taken by Hans P. Binswanger in a series of interesting contributions (cf. for example, Binswanger (1974)).

Maximize the reduction in:

$$\frac{\dot{\alpha}}{\alpha} u + \frac{\dot{\beta}}{\beta} (1-u) \qquad \ldots\ldots (20)$$

subject to the technical progress function (given by (19)):

$$\frac{\dot{\alpha}}{\alpha} = \theta (\frac{\dot{\beta}}{\beta}, u, v, y; q) \qquad \ldots\ldots (19)$$

This program results in an optimal factor intensity given, implicitly, by:

$$\theta'_1 = \frac{(1-u)}{u} \qquad \ldots\ldots (20)$$

Assuming, once again, an appropriate form of the implicit function theorem, we can write the above as:

$$\frac{\dot{\beta}}{\beta} = \zeta (u, v, y; q) \qquad \ldots\ldots (21)$$

Given, now, (21) we get, by straightforward substitution, from (19):

$$\frac{\dot{\alpha}}{\alpha} = \tau (u, v, y; q) \qquad \ldots\ldots (22)$$

There is a long tradition, both in the theoretical and empirical literature, of discussion regarding the effects on factor intensity of so-called factor-price distortions. The way we have formulated our model makes it clear that there is no unicausal relation with respect to direction between the problem of choice of techniques and factor prices. In the older and more neo-classical literature it was possible to discuss this relationship in a meaningful way because the

choice-theoretic problem was subject to the constraints of a standard production function. In our case, on the wage side, distortion - even if meaningful - can only be considered if the criterion which is the basis of wage bargaining is known. On the price side, as would be clear from our discussion in §2.2.c, it is almost a meaningless question. Thus, on the wage side, if the bargaining on the side of labour is based on a criterion function weighting employment against real wages, then distortion could be measured, from their (i.e., labour's) point of view, in terms of those weights. Three, at least, additional complications would then have to be considered:

a) The determination of the weights of the criterion function(s) for wage bargaining;

b) A more thorough consideration of wage-fund problems;

c) The conflict not only between labour and capital but that between employed and unemployed labour (or, almost equivalently, between unionized and non-unionized labour).

In related, but not yet integrated, work we are attempting to come to grips with those three aspects. Thus, in Rustem and Velupillai (1983) we have considered (a). In Fitoussi-Nuti (1983) the problem of wage-fund theory is considered and, implicitly, in Velupillai (1982) the conflict situation described in (c) is considered. The task, however, is to integrate all these issues in one macrodynamic model.

Now, from (11), (12) and (22) and substitutions in (4), we can determine the reduced form dynamics of functional income distribution:

$$\frac{\dot{u}}{u} = F(u,v,y;q) \qquad \ldots\ldots \quad (23)$$

By an accounting identity, on the basis of some simplifications that do not lead to loss of generality, we have:

$$pY_d = \{1-s_w(v,u)\} \, wL + \{1-s_c(u)\} \, \{pY_s-wL\} \ldots \quad (24)$$

where:

$s_w(v,u)$: savings 'coefficient' out of (contractual) wage income

$s_c(u)$: savings 'coefficient' out of profits income.

Dividing (24) by pY_s, we get:

$$y = 1 + u \, \{s_c(u) - s_w(v,u)\} - s_c(u) \qquad \ldots\ldots \quad (25)$$

Taking the time derivative of (25) we get:

$$\frac{dy}{dt} = \frac{du}{dt} \, \{s_c(u) - s_w(v,u)\} + u\left\{\frac{\delta s_c(u)}{\delta u} \cdot \frac{dy}{dt} - \left(\frac{\delta s_w(v,u)}{\delta v} \cdot \frac{dv}{dt}\right.\right.$$

$$\left.\left. + \frac{\delta s_w(v,u)}{\delta u} \cdot \frac{dy}{dt}\right)\right\}$$

$$- \frac{\delta s_c(u)}{\delta u} \cdot \frac{du}{dt} \qquad \ldots \quad (26)$$

Rearranging and simplifying we get:

$$\frac{dy}{dt} = \frac{du}{dt} \left\{\left[s_c(u) - s_w(v,u)\right] + u\frac{\delta s_c}{\delta u} - \frac{\delta s_w(v,u)}{\delta u} - \frac{\delta s_c}{\delta u}\right\}$$

$$- \frac{\delta s_w(v,u)}{\delta v} \cdot \frac{dv}{dt} \qquad \ldots \quad (27)$$

However, substituting, in turn, (10) in (21) and then the
result in (22) and using (13), the reduced form dynamics of
disequilibria in the labour market can be determined. Thus,
we have:

$$\frac{\dot{v}}{v} = G(u,v,y;q) \qquad \ldots\ldots \text{(28)}$$

From (23), (26) and (28) we can determine, finally, the
disequilibrium dynamics of the product market:

$$\frac{\dot{y}}{y} = H(u,v,y;q) \qquad \ldots\ldots \text{(29)}$$

The dynamical system represented by (23), (28) and (29) is
the complete reduced form version for our analysis of stag-
flation. Before we make some remarks concerning extensions
to an open economy with an essential role for governmental
activity, it may be appropriate to discuss, in very broad
terms, the mathematical structure of the dynamical system
represented by (23), (28) and (29).[14]

First of all, the way we have formulated the problem and
derived the dynamical system, it can be seen that it is a
system of three (non-linear) ordinary differential equations,
parametrized by q, the valuation ratio (or Tobin's "q"). But,
"richer" parametrizations are possible, for example, if the price equation
is explicitly parametrized by the adjustment coefficient and the
mark-up factor associated with a mark-up rule, then the dynamical
system is parametrized by q, λ and Π. It is, of course, well

14) All further references to a dynamical system will be to
the simultaneous consideration of (23), (28) and (29).

known that even deceptively simple three-dimensional systems
of ordinary differentail equations give rise to <u>strange attractors</u>
(the Lorenz equation for example: cf. Lorenz (1963) and Sparrow
(1983)).

Secondly, in view of the fact that it is a system of three
non-linear ordinary differential equations, we cannot proceed
in the usual way to investigate the dynamics of the model:
viz., appeal to the Poincaré-Bendixson theorem and the associated
concepts of planar dynamics.

Thirdly, the natural mathematical approach to the study
of the dynamics of systems that may undergo abrupt changes in
the characteristics of equilibrium states as certain parameters
are varied (endogenously or exogenously) seems to be <u>bifurcation</u>
<u>theory</u> (and its sub-branches: Catastrophe Theory, Synergetics,etc.)
In the dynamical system we have derived, the parametrization with
respect to q seems natural. Both policy - by influencing the
prices and, therefore, yields on financial assets - and spon-
taneous exogenous factors influence the value of q and, in the
aggregate, there is no need for the restriction $q \geq 1$. Thus
investment and price dynamics depend, continuously, on q in
some bounded interval - though hyper-inflationary possibilities
need not be ruled out. Thus, a possible mathematical approach
would be bifurcation analysis of the dynamical system with
respect to q. In this way we can circumvent the unnatural and
difficult analysis based on switching regimes.

Fourthly, perhaps we should take seriously Lorenz's pioneering methodology and investigate, by numerical integration, the possible dynamics of the system we have derived for various permutations of plausible values of the parameter(s).

Finally, there are intriguing possibilities if the dynamics of u, v and y can be associated with negative phase-space flow divergence (i.e., when $\delta F/\delta u + \delta G/\delta v + \delta H/\delta y$ is a negative constant). It may then be necessary and useful to consider the exotics of Mandelbrot's fractal dimensions (cf. Mandelbrot (1983)).

Before we proceed to a few general results for the complete model some notes on possible extensions to an open economy with an essential role for government seems almost imperative if the model is not ot be considered an empty box from a policy point of view.

Define:

$$h \equiv \frac{G}{Y_s} \qquad \qquad \ldots\ldots \quad (30)$$

where G : government expenditure (real)

and, similarly, define:

$$\Delta \equiv \frac{x}{Y_s} \qquad \qquad \ldots\ldots \quad (31)$$

$$\Lambda \equiv \frac{M}{Y_s} \qquad \qquad \ldots\ldots \quad (32)$$

where x : value of exports (nominal)

and M : value of imports (nominal)

and h, Δ and Λ are the respective ratios of government expend-
iture, value of exports and value of imports to the level of
output. To introduce government expenditure into our system,
for example, we can rewrite (24) as follows:

$$pY_d = \{1-s_w(v,u)\}wL + \{1-s_c(u)\}\{pY_s-wL\} + pG \quad \ldots\ldots \quad (33)$$

Proceeding as before and dividing by pY_s we get:

$$y = 1-u\{s_c(u) - s_w(v,u)\}- s_c(u) + \frac{G}{Y_s} \quad \ldots\ldots \quad (34)$$

and thus:

$$y = 1-u\{s_c(u) - s_w(v,u)\}- s_c(u) + h \quad \ldots\ldots \quad (35)$$

From (30) we have:

$$\frac{\dot{h}}{h} = \frac{\dot{G}}{G} - \frac{\dot{Y}_s}{Y_s} \quad \ldots\ldots \quad (36)$$

If it is assumed that government expenditure is given exogenously
then the dynamics of (23), (28) and (29) augmented by (36) can
be analysed in the 3-dim. space determined by the disequilibria
in the product market, labour market and the functional distrib-
ution of income. If government expenditure is circumscribed by
the possibilities due to feasible rates of direct taxation G
becomes, at least in part, endogenous and a 4-dim. dynamical
system must be analysed. If, finally, government revenue can
be enhanced by monetary means further complications in the form

of an explicit interest rate, at the least, must be considered.
Clearly the valuation ratio comes into its own in the latter
case and the model will have to be complicated even more - though
not in increasing the dimension. In a different exercise (cf.
Fitoussi-Velupillai (1984) we explore some questions of the
structure of public finance, so-called 'crowding-out' and ex-
change-rate dynamics. The hint to proceed along these lines is
to note that investment is, in our model, a function of the
valuation ratio which, in turn, depends almost directly on some
concept of the market-rate of interest.

We do not go further into the problem of 'open-economy
dynamics' except to note that from (31) and (32) we have:

$$\frac{\dot{\Delta}}{\Delta} = \frac{\dot{x}}{x} - \frac{\dot{Y}_s}{Y_s} \qquad \ldots\ldots (37)$$

and
$$\frac{\dot{\Lambda}}{\Lambda} = \frac{\dot{M}}{M} - \frac{\dot{Y}_s}{Y_s} \qquad \ldots\ldots (38)$$

Depending on the basis adopted for including the trade flows
in the national accounts the modification to (33) is purely
formal. The rest proceeds as in the previous closed-economy
case or as in the system augmented by government expenditure.
It is also possible to proceed along IS-LM type of methodology
by assuming, as a constraint, continuous trade balance so that
$\frac{x}{M}$, consistently valued, is identically equal to unity. Such
an assumption helps to reduce the dimensions of the system.
However, in the short-run, the implications of the above as-
sumption would be too unrealistic ($\frac{\dot{x}}{x} - \frac{\dot{M}}{M} = 0$). If, however,

an hierarchy of adjustment speeds can be assumed, in the sense
that accounting conventions (or natural causes) constrain certain
markets to be in 'balance' then the dimension of the system can
be reduced. Thus, if we can assume that the exchange-rate is
flexible to some degree and interest-rate variations are sufficient-
ly elastic, then, as in traditional macrodynamics (and indeed
classical osciallation theory - viz. relaxation oscillations),
we can assume the various money-markets 'clear' first and se-
quentially (cf. Leijonhufcud (1968), ch. II, in particular Ap-
pendix to ch. II, pp. 60-66). These notes we explore in Fitoussi-
Velupillai (op. cit.).

However, a sketch of the possible way of considering these
factors can easily be provided. Assume, for example, that de-
ficit financing (of the government budget) by means of simple
money creation or bond financing (or a combination of both).
Complications of the means, are, of course, possible. Then,
the government budget constraint and its dynamics can be con-
sidered as follows:

$$p \left\{ G + \frac{B}{p} - t(Y) \right\} = \dot{M} + \frac{\dot{B}}{R} \qquad \ldots \ldots \text{(38-a)}$$

where: t : tax rate

R : rate of interest

B : government bonds (units of and each of annual
yield "$ 1")

put: $\dfrac{dM}{dt} = j \left[p \left\{ G + \dfrac{B}{p} - t(Y) \right\} \right]$ $\qquad \ldots \ldots \text{(38-b)}$

and : $\dfrac{dB}{dt} \cdot \dfrac{1}{R} = (1-j)\left[p\left\{G + \dfrac{B}{p} - t(Y)\right\}\right]$ ...... (38-c)

Thus: $j = 0 \Rightarrow$ pure bond financing of the government account

$\quad\quad j = 1 \Rightarrow$ pure money creation for the government account.

Again, introducing, say, traditional LM-dynamics, we have, for example:

$\quad \dot{R} = L(R,Y) - M$ ...... (38-d)

where : L : liquidity preference function

If, therefore, as reasoned above, we assume the existence of a hierarchy of adjustment speeds in the system, reflected in the 'instantaneous' clearance of the money market, we can solve for R to get:

$\quad R = J(Y,M)$ ...... (38-e)

Given, α, G, t and R it is clear from (38-b) and (38-c) that M and B are functions of the endogenous variables of the dynamical system we developed earlier. In particular M and B are explicit functions of p and Y. Since we have not, explicitly, considered utility analysis for the 'consumer' the problem of asset choice does not surface in any dramatic way. Under these conditions, it is not clear that so-called 'crowding out' discussions have any sense at all. So long as no pathological factors enter into the demand for government securities and the q ratio behaves reasonably well there does not seem to be any justification for dramatic and drastic strictures against government activity at any level.

Similar extensions to exchange-rate dynamics are also possible.

§3.   The Workings of the Complete Model

3.1 :   Further Assumptions

We will proceed to the analysis of the dynamics of the complete model, in this section, in three stages. First, in §3.1 further assumptions about the nature and magnitude of local variations in the functions characterizing the model will be discussed. Next, in §3.2 we consider local stability analysis along well-known Routh-Hurwitz lines. Finally, in §3.3 we study the oscillatory properties of the system, parametrized by q, using Hopf's theorem. In an appendix to this section some unpleasant technicalities, necessary for the main text, can be found.

Variations in the distribution of income, the (un)employment ratio and the disequilibrium in the product market have effects on the respective dynamics that can be computed - at least for signs and (relative) magnitudes fairly directly, on the basis of assumptions in §2, from (A-3-2) $\sim$ (A-3-10); with, however, one exception: itis clear from (A-3-8) $\sim$ (A-3-10) that computation is not straightforward in the case of the effects on the dynamics of the product market. For our immediate purposes we indulge in some heuristics - the rigorous basis of which, though not explicit, can be provided.[15] Though the effects of variations in the distribution of income on the disequilibrium dynamics of the product market is ambiguous ($\frac{\delta H}{\delta u} \gtrless 0$), the other two

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15)   These computations will be provided on request.

effects are unambiguous: viz. $\frac{\delta H}{\delta v} < 0$ and $\frac{\delta H}{\delta y} > 0$. In words,

as the disequilibrium in the labour market is reduced, product

market disequilibria also diminishes, at least in proportional

terms, in the former case; in the latter case heuristics are

less transparent. It amounts to the paradoxical statement that

attempts to reduce the discrepancy between Y_s and Y_d increase

the difference in proportional growth between supply and demand.

This is really an aggregative and dynamic version of the so-called

paradox of thrift and enters our model because of the way in

which savings (and, hence, consumption) have been specified in

the accounting relation defining 'aggregate' demand.[16] We

now resolve the ambiguity in $\frac{\delta H}{\delta u}$ by assuming this to be <u>negative</u>

but almost negligible in magnitude[17] to neutralize some of the

paradoxes inherent in the immediately preceding assumption.

All the other partials have signs determined on the basis

of the assumptions in §2. However, we still have to consider

relative magnitudes before any proposition regarding the

stability of equilibrium can be made. This we consider in the next

sub-section.

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16) It must be remembered that 'aggregate demand' is defined
     without including investment in (24). Total demand, there-
     fore, must be distinguished from the definition of Y_d.

17) Mainly because it is a compounding of long-term factors on
     a very short-term variable.

3.2 : <u>(Local) Stability of an Equilibrium Solution</u>

Define A_i, $i = 1,2,3$ as follows:

$$A_1 \equiv \frac{u\delta F}{\delta u} + \frac{v\delta G}{\delta v} + \frac{y\delta F}{\delta y} \qquad \ldots \ldots \text{(39)}$$

$$A_2 \equiv \begin{vmatrix} \dfrac{u\delta F}{\delta u} & \dfrac{u\delta F}{\delta v} \\[2ex] \dfrac{v\delta G}{\delta u} & \dfrac{v\delta G}{\delta v} \end{vmatrix} + \begin{vmatrix} \dfrac{v\delta G}{\delta v} & \dfrac{v\delta G}{\delta y} \\[2ex] \dfrac{y\delta H}{\delta v} & \dfrac{y\delta H}{\delta y} \end{vmatrix} + \begin{vmatrix} \dfrac{u\delta F}{\delta u} & \dfrac{u\delta F}{\delta y} \\[2ex] \dfrac{y\delta H}{\delta u} & \dfrac{y\delta H}{\delta y} \end{vmatrix} \quad \text{..} \text{(38)}$$

$$A_3 \equiv \begin{vmatrix} \dfrac{u\delta F}{\delta u} & \dfrac{u\delta F}{\delta v} & \dfrac{u\delta F}{\delta y} \\[2ex] \dfrac{v\delta G}{\delta u} & \dfrac{v\delta G}{\delta v} & \dfrac{v\delta G}{\delta y} \\[2ex] \dfrac{y\delta H}{\delta u} & \dfrac{y\delta H}{\delta v} & \dfrac{y\delta H}{\delta y} \end{vmatrix} \qquad \ldots \ldots \text{(39)}$$

On the basis of the assumptions in §2 and the discussion in the appendix to this section supplemented by footnote 17 (and the main text referring to this footnote) it is clear that $A_3 > 0$. (We will not state the results referring to equilibrium and (local) stability as formal propositions - reserving that status to a statement on the oscillatory properties of the model.) We can, therefore, by the inverse function theorem proceed to a discussion of the local stability of a singular point. By the well-known conditions of Routh-Hurwitz local stability requires:

$$A_1 > 0, \ A_2 > 0 \ \text{ and } \ A_1 A_2 - A_3 > 0 \qquad \ldots \ldots \text{(40)}$$

Once again, from the assumptions in §2 and appendix to §3, it is clear that the conditions for local stability are satisfied.

A minor digression may not be out of place at this point. Rewrite, in a compact way, the dynamical system defining the model (i.e., (23), (28) and (29)) as:

$$\frac{dx}{dt} = xM(x;\epsilon) \qquad \qquad \ldots\ldots \quad (41)$$

Our aim now is to study <u>the singular point(s)</u> of the dynamical system defining movements in the functional distribution of income in response to disequilibria in the labour and product markets (and in terms of the reverse causality). This means we want to investigate the structure of zeroes for:

$$M(x;\epsilon) = 0 \qquad \qquad \ldots\ldots \quad (42)$$

where :

> $x \in X$
>
> and X is the cartesian product of the feasible region defining u, v and y.

and :

$\epsilon \in \xi$ where ξ is the parameter space (defined by q and, say, the adjustment coefficient and the mark-up factor if the price equation was specified - as will be done in the next section, §4, appropriately).

In general we call a SOLUTION of (42) a point $(x;\epsilon) \in X \times \xi$ such that relation (42) will be satisfied. If we define

$$Z = \{x \in X; \ (x;\epsilon) \in Z\} \qquad \qquad \ldots\ldots \quad (43)$$

where Z denotes the set of solutions of (42) given by $Z \in X \times \xi$ and

relation (43) holds for any $\varepsilon \in \xi$. In this form it is possible
to recognize a respectable tradition in economic analysis -
beginning with the 'Correspondence Principle' and progressing
through any number of variants of so-called 'Comparative Statics'
studies of models and their equilibria. Clearly, if Z is <u>any</u>
<u>closed set</u> in $X \times \xi$ it would be impossible to systematically study
Z_ε as ε is varied. On the other hand, if we confine our attention
to the characterization of the solution set Z in a neighbourhood
of <u>a</u> solution $(x_o; \varepsilon_o)$ then, invoking some form of the implicit
function theorem (as we have done above in appealing to the in-
verse function theorem), it would be possible to study Z_ε as ε is
varied in a neighbourhood of ε_o for x_o. The procedure we have
just described is the method of (static) bifurcation theory.
It is <u>static</u> because we compare the structure of the solution
set for nearby values of the parameter. In slightly more tech-
nical language this means that an equivalence relation is
defined <u>on the solution set</u> for investigations in terms of
variations in the parameter. On the other hand, if the equi-
valence relation is defined on the vector field defining the
system and if a study of its variations w.r.t. some parameter
set is made, then the procedure is usually called dynamic bi-
furcation theory. Questions referring to structural stability
belong to the domain of dynamic bifurcation theory. In all
cases the real problem is to choose useful, intuitive and meaning-
ful equivalence relations. It is customary to consider topo-
logical equivalence relations. Elsewhere we explore these latter

questions (cf. Velupillai (1983), (1984)). It may now be
more clear that bifurcation theory circumvents difficult analysis
where switching between regimes must be explicitly considered.
The price, of course, is to be paid in terms of the ingenuity
with which equivalence relations can be usefully chosen. We
shall proceed with the method of static bifurcation theory in
the rest of our discussion in this section (§3) and for part
of the discussion in §4.

Though we have not specified, explicitly, conditions guaran-
teeing uniqueness of the singular point (i.e., the solution set
Z is to be a singleton) it may, for expository purposes, be use-
ful to imagine that this is so.[18] Thus, from (39) and (40)
we know that an equilibrium is locally stable implying, for
this third order system in u, v and y, eigenvalues with negative
real parts. To probe further into the local oscillatory char-
acteristics of the singular point further investigations into
the structure of the eigenvalues will be necessary. To this
we now turn.

_____

18)   The reason for not attempting a characterization of the
      uniqueness of the singular point will be evident from even
      a cursory inspection of the simple Lorenz system (op.cit.).
      But this does not mean that we cannot isolate the study
      of any one singular point - neglecting others - locally.

3.3 : <u>Oscillations in the 3-dim. system in u, v and y</u>

In 2-dimensional systems the powerful Poincare-Bendixson theorem simplifies many difficulties in the investigation of oscillatory properties. The Poincare-Bendixson theorem (a version of which we use, implicitly, in §4 to prove a 'Classical' equilibrium) has no analogue in higher dimensions. As discussed in the previous sub-section it is, perhaps, more useful to approach the analysis of oscillatory properties as static or dynamic bifurcation problems. This we attempt now using one of the most famous theorems in this field: viz. the Hopf bifurcation theorem (cf. Marsden-McCracken (1976) for historical and analytical notes; also Hassard et.al. (1981) and Arnold (1983)).

<u>Proposition 3.1:</u>

Under the assumptions of §2, Appendix to §3 and 3.1, 3.2, there exist values of $q = q^*$ such that the Jacobian of the dynamical system defined by (23), (28) and (29) has a pair of pure imaginary eigenvalues.

<u>Proof</u>

From 3.1, 3.2 and Appendix to §3 it follows that the eigenvalues have at most negative real parts. On the other hand, the discriminant of a third order equation:

$$f(x) \equiv ax^3 + 3bx^2 + 3cx + d \qquad \ldots\ldots (44)$$

$$\text{for } f(x) = 0 \qquad \ldots\ldots (45)$$

is given by $\dfrac{B^2 + 4E^3}{a^2}$ $\ldots\ldots (46)$

In this case:

$$B = A_3 - \frac{A_1 A_2}{3} + \frac{2}{9} A_1^3 \qquad \dots\dots \text{(47)}$$

and $\quad E = \frac{A_2}{3} - \frac{A_1^3}{9} \qquad \dots\dots \text{(48)}$

The criterion for pure imaginary roots is:

$$B^2 + 4E^3 > 0 \qquad \dots\dots \text{(49)}$$

Clearly, given the order of magnitude of the elements of A_1^3 and A_2, and the fact that investment depends continuously on q (with no restriction for the feasible range of q in the aggregate), relation (49) is easily satisfied[19] for some $q = q^*$. Q.E.D.

It is now a matter of brute computation only to show that the conditions of Hopf theorem are satisfied for the dynamical system parametrized by q.

Proposition 3.2

Under the assumptions of proposition 3.1 the dynamical system (23), (28), (29) parametrized by q exhibits a Hopf bifurcation (for the singular point under proposition 3.1).

Proof

For a precise statement of the Hopf theorem cf. Marsden-McCracken (op.cit.) or Hassard et.al. (op.cit.). It is only necessary to verify that the hypothesis of the Hopf theorem is satisfied and

---

19) The tedious computation will be supplied on request.

this requires only one extra complication. (The example in the next section is more explicit and intuition does not falter!). Due to discussions in §2 we know that all constituent functions are sufficiently smooth and hence the dynamical system (23), (28) and (29) are continuous and differentiable. By 3.1, 3.2 the singular point under discussion is stable (locally, of course). From proposition 3.1, for $q = q^*$ the Jacobian of the dynamical system at the singular point has a pair of pure imaginary eigenvalues. It is now necessary only to show that the real part of the imaginary roots - complex roots off the imaginary axis - crosses the imaginary axis with non-zero speed: i.e., we have to show for

$$\lambda_i = \alpha_i + i\beta_i \qquad i = 1,2 \qquad \ldots\ldots \quad (50)$$

$$\lambda_3 = -\gamma \qquad \ldots\ldots \quad (51)$$

(where λ_i, i = 1,2,3 are the three eigenvalues of the Jacobian at a singular point)

that:

$$\frac{d.re\ \alpha(q^*)}{dq} \neq 0 \qquad \ldots\ldots \quad (52)$$

Once again, we omit computations but it is easy to show, using any one of the possible relations between the roots of an equation and the coefficients characterizing it that (52) holds. Thus all the conditions of the Hopf theorem are verified and the dynamical system defined by (23), (28) and (29) is characterized by a one-parameter family of closed orbits in a neighbourhood of the equilibrium for $q = q^*$.

Q.E.D.

Some heuristic remarks are in order concerning the economic
background to such an oscillatory behaviour branching off from
a stable singular point. As the valuation ratio varies the
system loses the characteristics of stable behaviour and oscil-
lates. For some range of values of q, any deviation away from
the singular point sets up 'self-correcting' forces so that the
system seeks automatically a return to those values of u, v and y
characterizing a dynamic equilibrium (which, of course, does
not necessarily mean, in the short run, equilibrium of supply
and demand). As q varies beyond a certain fixed radius of
values, the system loses its simple stability characteristics
and oscillations are induced. Further variation of q can lead
to exotic dynamics like a 'jump' to quite another equilibrium.
In this sense this type of analysis substantiates Leijonhufvud's
important 'corridor' hypothesis (Leijonhufvud (1981), ch. 6,
p. 109): i.e., for some 'corridor' of values of q 'optimal'
policy may well be 'laissez-faire'. But the vagaries of q do
not warrant 'rules' at any cost: discretion will be necessary,
but only after the rough magnitudes that determine the 'size
of the corridor' defining q can be ascertained. In the usual
literature on Hopf's theorem it is not easy to find discussions
on the possibilities of computing explicitly a range for q
such that our 'corridor' can be determined in some meaningful
sense (but cf.: Swinnerton-Dyer (1977)). Secondly, it is
clearly not necessary to formulate 'regime changes' as difficult
'switching problems'. As in classical relaxation oscillation

methods, we have an elegant and simple way to view switches
in regimes as well as oscillations: a switch from (asymptotically)
stable behaviour to persistent oscillations as q varies due to
exogenous factors.

How does the above analysis relate to fix-price analysis and
New Classical Macroeconomics? We have discussed some of the
relations above and will consider other affinities in §4. How-
ever, at this point, some remarks are in order.

The main characteristics of New Classical Macroeconomics
depend on <u>assuming</u> prices and wages such that the product and
labour markets clear. It is not always clear in this literature
what meaning can be attached to the concept of equilibrium in
the product (or, indeed, in the labour) market.

Assuming, however, that these definitional problems
can be resolved, we can consider the above requirement in terms
of setting G and H simultaneously to zero. Then the burden of
adjustment, as always, falls upon that much misused concept:
functional income distribution. This, in turn, is defined in
terms of productivity and real wages. Now, adding the further
requirement of rational expectations - in a deterministic con-
text, naturally, implying, perfect foresight - can we also infer
<u>invariance</u> propositions? Clearly, not. The <u>theoretical technology</u>
(in colourful Lucasian terminology) of non-linear dynamics allows

many more exotica than those allowed in planar dynamics and saddle-point characterizations. It is _a priori_ impossible, even under conditions of perfect foresight for q and the parameters characterizing market clearing values for p and w, to choose dynamics analogous to saddle-point paths. To substantiate, analytically, these observations would take us beyond the limited scope of this exercise, but these are explored in Fitoussi-Velupillai (op.cit.).

A reverse procedure is adopted in the fix-price tradition. Clearly, even as the generic name of this tradition implies, setting $\int$ ((eq. (11)) and g (eq.(12)) simultaneously to zero, we obtain the main characteristics of the analysis. If, in addition, productivity is assumed to be constant - as is usually done - then for F = 0 we have to consider the planar dynamics of G and H. In this case almost complete characterization of the dynamics is possible. A slightly modified approach, as an example of modelling in this tradition, is given in §4. But even without going into details it can be seen that the search for an 'appropriate income distribution' simply means find the locus of p and w (for given productivity) such that G and H have desired dynamics. It is, of course, not necessary to assume given productivity.

Since efficiency requirements dictate that fix-price analysts 'search' for the income distribution that will characterize a Walrasian equilibrium, and the New Classicals predicate their

analysis upon this requirement (of market clearing values), there does not seem to be much dividing the two schools. On the other hand, without assumptions guaranteeing uniqueness, the former approach falters on the impossibility or at least infelicity of comparing alternative Pareto-optima; the latter, Equilibrium Dynamics, analysis becomes meaningless when, in three dimensions (and higher) strange attractors generate exotic, but non-random and aperiodic fluctuations.

At least one more feature of New Classical Macroeconomics remains to be explored: the problem of monetary shocks generating observed fluctuations. Within the framework we have developed there is no need to have to appeal to ad-hoc stochastics though we are, of course, guilty of other ad-hockeries. An investigation, similar to that developed above, of a dynamical system depending on a parameter would suffice; in this case, all we need to do is to study the dynamics w.r.t. variations in j to encapsulate the simplest considerations of monetary shocks. It is possible to go beyond. As for example, when the valuation ratio, for its market determinants, depends upon the extent to which government activity 'distorts' financial markets. These considerations lead, more than has been indicated earlier, to direct and conventional approaches to macroeconomic policy analysis. We explore these directions in greater detail in Fitoussi-Velupillai (1984).

Appendix to §3

We have:

$$\frac{\dot{u}}{u} = F(u,v,y;q) \qquad \qquad \dots\dots \quad (23)$$

$$\frac{\dot{v}}{v} = G(u,v,y;q) \qquad \qquad \dots\dots \quad (28)$$

$$\frac{\dot{y}}{y} = H(u,v,y;q) \qquad \qquad \dots\dots \quad (29)$$

To investigate, locally, stability characteristics of the model and to utilize the Hopf(-Friedrichs) theorem, we have to investigate appropriate partial derivatives. These we derive in this appendix:

For (23), (28) and (29) the Jacobian evaluated at a singular point will be:

$$\begin{pmatrix} \overset{+}{u\dfrac{\delta F}{\delta u}} & \overset{+}{u\dfrac{\delta F}{\delta v}} & \overset{+}{u\dfrac{\delta F}{\delta y}} \\[2em] \overset{-}{v\dfrac{\delta G}{\delta u}} & \overset{+}{v\dfrac{\delta G}{\delta v}} & \overset{+}{v\dfrac{\delta G}{\delta y}} \\[2em] \overset{-}{y\dfrac{\delta H}{\delta u}} & \overset{-}{y\dfrac{\delta H}{\delta v}} & \overset{+}{y\dfrac{\delta H}{\delta y}} \end{pmatrix} \qquad \dots\dots \quad (A\text{-}3\text{-}1)$$

where:

$$\overset{+}{\frac{\delta F}{\delta u}} = \overset{\cdot 0}{\frac{\delta q}{\delta u}} + \overset{+}{\frac{\delta q}{\delta \int}} \cdot \overset{+}{\frac{\delta \int}{\delta u}} + \overset{+}{\frac{\delta q}{\delta \tau}} \cdot \overset{\pm}{\frac{\delta \tau}{\delta u}} - \overset{+}{\frac{\delta \int}{\delta u}} - \overset{\pm}{\frac{\delta \tau}{\delta u}} \qquad \dots\dots \quad (A\text{-}3\text{-}2)$$

$$\overset{+}{\frac{\delta F}{\delta v}} = \overset{+}{\frac{\delta q}{\delta v}} + \overset{+}{\frac{\delta q}{\delta \int}} \cdot \overset{\cdot 0}{\frac{\delta \int}{\delta v}} + \overset{+}{\frac{\delta q}{\delta \tau}} \cdot \overset{\pm}{\frac{\delta \tau}{\delta v}} - \overset{\cdot 0}{\frac{\delta \int}{\delta v}} - \overset{\pm}{\frac{\delta \tau}{\delta v}} \qquad \dots\dots \quad (A\text{-}3\text{-}3)$$

$$\frac{\delta \overset{+}{F}}{\delta y} = \frac{\delta \overset{-}{q}}{\delta y} + \frac{\delta \overset{+}{q}}{\delta \int} \cdot \frac{\delta \overset{+}{\int}}{\delta y} + \frac{\delta \overset{+}{q}}{\delta \tau} \cdot \frac{\delta \overset{+}{\tau}}{\delta y} - \frac{\delta \overset{\neq 0}{\int}}{\delta y} - \frac{\delta \overset{+}{\tau}}{\delta y} \qquad \ldots\ldots \text{(A-3-4)}$$

$$\frac{\delta \overset{-}{G}}{\delta u} = \frac{\delta \overset{-}{\xi}}{\delta u} + \frac{\delta \overset{+}{\tau}}{\delta u} \qquad \ldots\ldots \text{(A-3-5)}$$

$$\frac{\delta \overset{+}{G}}{\delta v} = \frac{\delta \overset{+}{\xi}}{\delta v} + \frac{\delta \overset{\pm}{\tau}}{\delta v} \qquad \ldots\ldots \text{(A-3-6)}$$

$$\frac{\delta \overset{+}{G}}{\delta y} = \frac{\delta \overset{+}{\xi}}{\delta y} + \frac{\delta \overset{\pm}{\tau}}{\delta y} \qquad \ldots\ldots \text{(A-3-7)}$$

$$\frac{\delta \overset{-}{H}}{\delta u} = \frac{\delta}{\delta u}\left(\frac{dy}{dt}\right)\{\cdot\} + \frac{du}{dt}\frac{\delta}{\delta u}\{\cdot\} - \left[\frac{\delta}{\delta u}\left(\frac{\delta s_w}{\delta v}\right)\cdot\frac{dv}{dt} + \frac{\delta s_w}{\delta v}\cdot\frac{\delta}{\delta u}\left(\frac{dv}{dt}\right)\right]$$
$$\ldots\ldots \text{(A-3-8)}$$

$$\frac{\delta \overset{-}{H}}{\delta v} = \frac{\delta}{\delta v}\left(\frac{du}{dt}\right)\{\cdot\} + \frac{du}{dt}\frac{\delta}{\delta v}\{\cdot\} - \left[\frac{\delta}{\delta v}(.)\cdot\frac{dv}{dt} + \frac{\delta s_w}{\delta v}\cdot\frac{\delta}{\delta v}\left(\frac{dv}{dt}\right)\right]$$
$$\ldots\ldots \text{(A-3-9)}$$

$$\frac{\delta \overset{+}{H}}{\delta y} = \frac{\delta}{\delta y}\left(\frac{du}{dt}\right)\{\cdot\} + \frac{du}{dt}\cdot\frac{\delta}{\delta y}\{\cdot\} - \left[\frac{\delta}{\delta y}(.)\cdot\frac{dv}{dt} + \frac{\delta s_w}{\delta v}\cdot\frac{\delta}{\delta y}\left(\frac{dv}{dt}\right)\right]$$
$$\ldots\ldots \text{(A-3-10)}$$

§4   Some Simple Special Cases

We propose to illustrate, with two simple examples, the versatility of the methodology and the workings of a model of the type we have developed. Anyone familiar with Goodwin's famous model of 'A Growth Cycle' (Goodwin (1967)) or the Akerlöf-Stiglitz (op.cit.) model would recognize that the model we have developed in §2 is nothing more than a generalization of these pioneering works.

In Goodwin's model, for example, for initial conditions not compatible with the long-run equilibrium values, the economy generates fluctuations in functional income distribution and the (un-)employment ratio to resolve the conflict between capital and labour. Since Goodwin's model is a particular realization of our model in §2, we will use it as the first example to illustrate the use of the Hopf bifurcation theorem in an explicit way.[20]

Example 1

Assume the following particular case of "Model §2":

   i) Equilibrium in the product market

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[20] This example was presented by the second author at a seminar in honour of Goodwin held in Siena in April 1981. In the proceedings of that conference a similar result is given in the excellent paper by Cugno and Montrucchio. The proceedings were published in late 1982 and became available to the present authors in 1983 when the first version of this paper had already taken shape. It would be possible to choose another of Goodwin's models to illustrate the use of the Hopf theorem (cf. Velupillai (1982) - same proceedings as above) - but the relation with the model in §2 would be less clear.

ii) Add <u>all other assumptions</u> in the original Goodwin model
(Goodwin (op.cit.)) <u>except</u>:

iii) Money-wage dynamics given by

$$\frac{\dot{w}}{w} = \int(v) + \lambda\frac{\dot{p}}{p} \qquad \ldots \ldots \text{(53)}$$

$$\int' > 0$$

iv) Price dynamics given by:

$$\frac{\dot{p}}{p} = g\{\left[\begin{array}{cc} \pi & \frac{wL}{Y} - p\end{array}\right]/p\} \qquad \ldots \ldots \text{(54)}$$

$$\text{i.e. } \frac{\dot{p}}{p} = g\{\begin{array}{cc} \pi & u - 1\end{array}\} \qquad \ldots \ldots \text{(55)}$$

$$g' > 0, \; g(0) = 0.$$

Clearly, for $\lambda = 0$ and linearization of $\int(v)$ the model reduces
to Goodwin's original 'Growth Cycle'. (We can parametrize
also in terms of savings propensies for a gneralization to
illustrate the use of the Hopf-theorem.) Using, then, the
notations for the particular assumptions as in Goodwin's original
presentation, the dynamics of the share of wages and the (un-)
employment ratio will be given by:

$$\frac{\dot{u}}{u} = \int(v) - \alpha - (1-\lambda)g\{\begin{array}{cc} \pi & u - 1\end{array}\} \qquad \ldots \ldots \text{(56)}$$

and

$$\frac{\dot{v}}{v} = \frac{(1-u)}{\sigma} - (\alpha+\beta) \qquad \ldots \ldots \text{(57)}$$

Let us, somewhat misleadingly, denote λ as a "money-illusion"
parameter. As pointed out above, when $\lambda=0$ and $\int(v)$ is linearized
(56) and (57) collapse to the original system of 'Lotka-Volterra'
equation.

From the characteristic equation for (56) and (57) we get:

$$\rho^2 - \text{trace } J(u,v).\rho + \det J(u,v) = 0 \qquad \ldots\ldots \quad (58)$$

where for $J(u,v)$ we have:

$$J(u,v) = \begin{pmatrix} -(1-\lambda)\dfrac{\delta g}{\delta u} & \dfrac{\delta f}{\delta v} \\[2ex] -\dfrac{1}{\sigma} & 0 \end{pmatrix} \qquad \ldots\ldots \quad (59)$$

and, therefore, the roots are:

$$\rho_{1,2} = \frac{-(1-\lambda)\,\delta g/\delta u}{2} \pm \sqrt{\frac{(1-\lambda)^2\,(\delta g/\delta u)^2}{4} - \frac{\delta f}{\delta v}\left(\frac{-1}{\sigma}\right)} \quad \ldots \quad (60)$$

By the assumptions of the original Goodwin model we have:

$$\frac{\delta g}{\delta u} > 0 \qquad \ldots\ldots \quad (61)$$

$$\frac{\delta f}{\delta v} > 0 \qquad \ldots\ldots \quad (62)$$

Clearly, $\nabla\,\lambda\,(\varepsilon\,\mathbb{R})$ such that

$$|1-\lambda| < 2\sqrt{\frac{\delta f}{\delta v}\cdot\left(\frac{-1}{\sigma}\right)} \qquad \ldots\ldots \quad (63)$$

the roots $\rho_{1,2}$ are (conjugate) complex.

Also, when:

$$\lambda = 1,\quad r\,e.\,\rho_{1,2} = 0 \qquad \ldots\ldots \quad (64)$$

and $\dfrac{d}{d\lambda}$ re. $\rho(\lambda)$ at $\lambda = 1$

$$\text{gives } \frac{1}{2}\frac{\delta g}{\delta u} > 0 \qquad \ldots\ldots \quad (65)$$

i.e., the dynamical system is such that the real part of the complex roots cross the imaginary axis with non-zero speed.

All the conditions of the Hopf bifurcation theorem hold, which, therefore, implies that there exists a one-parameter family of closed orbits in a neighbourhood of the equilibrium for $\lambda = 1$. A conjecture: it is well known that the Lotka-Volterra system, and hence the system in the original Goodwin model for u and v is structurally unstable. It has generally been assumed that this is an undesirable property for systems depicting some aspects of real life situations. The above example, however, casts a surprisingly positive aspect of the property of structural (in-)stability: by studying the directions in which perturbation of such a system leads to qualitatively different phase paths, it will be possible to obtain information on useful parametrizations. There may, therefore, be a case for beginning with structurally unstable systems in the first stage of an analytical study. Classical Mechanics may be considered to be a canonical example for a case study of such a strategy.

Example 2:

In the previous simple example it is easy to verify that for $\lambda < 1$ we get a convergent, albeit fluctuating path, to the long-run equilibrium configuration in the share of wages and the (un-)employment ratio. Even though the economics of the motion of the dynamical system has very many classical elements in it, a crucial component of the thought of that school is not incorporated in the simplified example 1: i.e., endogenization of labour supply as a function of real wages (cf. above pp. 23-24).

This example, incorporating such a feature, is mainly for the purpose of illustrating the so-called 'Classical Unemployment' - i.e., the generation of an unemployment equilibrium mainly due to real wages being high (but in conjunction with explicit dynamics for productivity, pricing, labour supply and investment so that the Keynesian case due to too low effective demand can, if necessary, be discussed with minor modifications).

Assume the following particular case of "Model §2": (in addition to assumption i) and ii) of example 1 above).

i) Investment relation is given by a simple conjunction of an accelerator principle and a profits terms. For example as in Kaldor (1957):

$$\frac{\dot{K}}{Y} = \frac{K}{Y} \cdot \dot{Y} + f\{(1-u)\frac{Y_s}{K}\} \qquad \ldots\ldots (66)$$

(where, of course, due to assumption i) of example 1 Y_s is replaced by Y).

Under these assumptions, the dynamics of the share of wages and the unemployment ratio reduce to:

$$\frac{\dot{u}}{u} = G(u,v) \qquad \ldots\ldots (67)$$

$$\frac{\dot{v}}{v} = H(u,v) \qquad \ldots\ldots (68)$$

such that the following assumptions hold:

$$\frac{\delta H}{\delta u} < 0 \qquad \ldots\ldots (69)$$

Assuming also that real wages and the share of wages move with money wages, but less rapidly so, (i.e., in terms of example 1,

the 'money illusion' parameter $\lambda < 1$), we get:

$$\frac{\delta G}{\delta v} < 0 \qquad \qquad \ldots \ldots \quad (70)$$

We also assume:

$$\lim_{u \to 0} H(u,v) > 0 \text{ for } v < \bar{v} \text{ and } H(u,v) < 0 \text{ for } v > \bar{v}$$

$$\qquad \qquad \ldots \ldots \quad (71)$$

$$\lim_{v \to 0} G(u,v) > 0 \text{ for } u < \bar{u} \text{ and } G(u,v) < 0 \text{ for } u > \bar{u}$$

$$\qquad \qquad \ldots \ldots \quad (72)$$

Roughly speaking these assumptions (valid, for example, in the original Goodwin 'Growth Cycle') mean the following: (in true 'classical' - in the Keynesian sense - vein):

a) Decreasing share of wages, leading to a rise in profitability, increase more than proportionately the employment ratio, in some finite region of the (u,v) space.

b) Similarly with the roles of u,v reversed.

For mathematical convenience, which does not violate any of the economic assumptions of the analysis, we assume that the H and G curves of the equilibrium state are such that their inverse images have distinct tangent lines. We can then state the following 'theorem', for the assumptions made in this section:

Proposition 4.1

The equilibrium state of the dynamical system represented by (67) and (68) are ω-limit sets that do not contain closed orbits, and all the ω-limit points are equilibria.

(Intuitively, this means, for any initial point in the (u,v)-plane, where $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 1$, both u and v tend to some equilibrium state, i.e., an existence proposition for feasible initial conditions. Thus, the intersection points of $G^{-1}(0)$ and $H^{-1}(0)$ determine the distribution of income and degree of unemployment compatible with any level of real wages. The interesting point, however, is that the same configuration of an equilibrium in the distribution of income and degree of unemployment can be realized with a model with more emphasis on Keynesian elements - i.e., lack of effective demand. The point of the morale of the story is to cast some doubt on the classification implied by methods of disequilibrium statics.)

Proof:

Consider the non-intersecting points of $G^{-1}(0)$ and $H^{-1}(0)$. If v is taken along the horizontal axis and u along the vertical, any point on $G^{-1}(0)$ which is not an intersection point, moves vertically; conversely, any non-intersection point on $H^{-1}(0)$ moves horizontally. Moreover, in the subspaces (in fact they are open sets) determined by the intersection points of $G^{-1}(0)$ and $H^{-1}(0)$, the non-intersection points of $G^{-1}(0)$ and $H^{-1}(0)$ point in the same direction. This means that they are monotone, hence ruling out closed orbits.

Clearly, the dynamical system defined by (67) and (68) is defined for all $t \geqslant t_o = 0$. Furthermore, $\nabla t \geqslant t_o$, the system (67) and (68) lie in a closed and bounded plane region, because

of the assumptions represented by (71) and (72). Thus, the
set defined by $0 \leqslant v \leqslant 1$ and $0 \leqslant u \leqslant 1$ is an INVARIANT SET; i.e.
since we have all along assumed that the various functions are
continuous and smooth, we have a continuous mapping from a com-
pact set into itself. Then, by the Poincare-Bendixson theorem,
since closed orbits are ruled out, there exists some equilibrium.
In fact, because the domain of definition is an invariant set, all
the ω. limit points are equilibria and the proposition is proved.

<div align="center">Q.E.D.</div>

We have, with some design, altered the investment function: partly
to encapsulate so-called 'classical', 'Keynesian', etc. type of
classification popularized by the 'fix-price' school; partly
to show an alternative approach, using the Poincare-Bendixson
theorem, to prove the existence of a singular point.

§5   Concluding Notes

In the lengthy introduction some critical remarks were
made regarding the methodology adopted by the fix-price macro-
economists. In that approach unnecessarily complicated mathema-
tical formalizations have been adopted to analyse so-called
regime switches. We have tried to show that bifurcation theory
provides, possibly, a conceptually much simpler and mathematically
somewhat richer framework for analysing disequilibrium dynamics.
In addition, as illustrated in the second example of §4, even the
economic classification of the methods of fix-price macroecono-
mists may not be unambiguous.

On the other hand, the excessive claims of New Classical
Macroeconomists on 'Methods' to analyse aggregate fluctuations
seem highly misplaced. The 'theoretical technology' of
Lucasian macroeconomics does not seem to have caught up with
the possibilities of non-linear analysis in general and bifurca-
tion theory in particular.

Clearly the most immediate and glaring short-comings of
the model that was developed in §2 relates to the abstractions
from the complications of introducing in an essential way govern-
ment activity and behaviour in addition to open economy consider-
ations. However, as discussed at the end of §2, such essential
features can be incorporated in an extended version of the model
of that section. The main reason for not going all the way and

including the complications of government activity and open
economy macrodynamics was technical. Beyond systems of third
order, to proceed with the sort of computations we have already
considered in §3, become difficult to interpret in an economical-
ly meaningful way. Other, more formal, techniques - within a
Hopf-type approach - will have to be used. We indicated some
possibilities in the general discussion of §2: Centre Manifold
Theory, Generalized Relaxation Oscillations (or the existence
of non-identical time constants in the system: Synergetics),
Liapunov-Schmidt reduction, etc.

The political economy of austerity and its analysis requires
the full consideration of an open economy with government activity
so that policies that are competitively depressive can be .ex-
plicitly considered in dynamic and disequilibrium models from
the outset. We consider the model developed in §2 and the
analysis in §3 as a preface to this more important task.

Even though, scattered, there are some hints about the
way in which the model we have developed can be extended to
consider elements of the 'political economy of austerity'
this is neither made explicit nor is it sufficient. The more
glaring of the deficiencies are of course clear: money and
monetary policy are no more than permissive; wealth effects
are totally ignored; capacity utilization and its analysis
remains implicit; in addition, the state and relations charac-
terizing an open economy were mentioned only in passing,

albeit with explicit formal hints on the way in which our model can be extended to encapsulate such elements.

However, our main objective was to develop a model integrating, from the very outset, disequilibrium _and_ dynamic elements. In this, at least to some extent, we believe that we have succeeded. The extent to which we had to compromise in neglecting monetary considerations, wealth effects and trade cum exchange rate dynamics remains to be seen when an extended model is developed. It is, however, somewhat heartening to note that this model can be considered canonical in the sense that standard fix-price and conventional New Classical elements drop out as special cases.

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