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TURNOUT AND POLICY: THE ROLE OF CANDIDATES

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the Role of Candidates*

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Abstract

Turnout is an important determinant of which candidate wins an election. Since candidates know this, it follows that they will consider turnout when choosing their policy platforms. In this paper I formally examine the effect voter turnout has on candidates' policy positions by characterizing the equilibrium policy choices of candidates given the subsequent equilibrium turnout. I show that alienation among extreme voters, which occurs only when citizens' utility over policy is convex or linear, is a necessary condition for divergent, positive-turnout equilibria. My model suggests that candidate polarization is correlated with the importance of the election outcome and, counterintuitively, predicts that the level of turnout can increase with the cost of voting.

Keywords

Voter turnout; policy separation.

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1 Introduction

The possibility that parties will be kept from converging ideologically in a two-party system depends upon the refusal of extremist voters to support either party if both become alike – not identical, but merely similar. (Downs (1957), p. 118)

Underlying Downs’ theory is a rather simple and direct concept: turnout is sensitive to candidate characteristics – therefore, candidates will consider turnout when choosing political platforms. This logic implies that turnout determines which candidate wins *and* shapes the policy options from which voters select. While the importance of turnout to election outcomes is widely recognized, the formal literature on turnout almost exclusively focuses on its effect on who wins the election, taking candidate behavior as exogenous. This paper studies the effect of turnout considerations on candidates’ policy positions, and establishes the theoretical structure necessary to address questions such as whether abstention amongst citizens with extreme policy preferences can keep candidates from converging ideologically, and how lower voting costs will impact the candidates’ policy positions.

To formally explore the effect of turnout on the policy positions of the candidates, I use a two-stage model of elections. The first stage is a standard model of candidate policy choice: two office-motivated candidates choose policy on a one-dimensional space. The second stage is a standard model of turnout: given the candidates’ policy positions from stage one, citizens choose to vote according to the pivotal voter model.¹ Explicitly modeling both candidate policy choice and citizen turnout allows me to detail how candidate behavior is influenced by the citizens’ underlying preferences over the policy space and the costs of turnout.

The focus of this paper is purely positive; I characterize the equilibrium policy choices of the candidates in the first stage given the resulting equilibrium turnout in the second stage. Downs (1957) predicted that abstention among citizens with extreme ideal points provides a centrifugal incentive that counteracts the centripetal incentive of the Hotelling model of elections (the standard median voter result). Downs’ logic is formalized here; I find that the incentive to converge to the center can be counterbalanced, and even outweighed, by considerations of voter turnout. This result, however, is not general: the equilibria of the model are sensitive to the shape of citizens’ utility functions and, in the case of convex utility, the distribution citizens’ ideal points.

Given the sensitivity of the equilibria to assumptions regarding the shape of utility, a

¹Other models of turnout produce similar results as long as turnout depends on the policy positions of the candidates; in a related paper (Valasek (2011)) I examine the effect of subsidizing voting on election outcomes using a similar two-stage model of elections, but with a group model of turnout in the second stage.

brief discussion of utility over policy is warranted. Concave utility, particularly the quadratic loss function, is standard in the spatial voting literature. It is not clear, however, that this assumption accurately describes citizen preferences over the policy spectrum. Uncertainty regarding the shape of utility is expressed by Osborne (1995):

The assumption of concavity is often adopted, first because it is associated with ‘risk aversion’ and second because it makes it easier to show that an equilibrium exists. However, I am uncomfortable with the implication of concavity that extremists are highly sensitive to differences between moderate candidates...Further, it is not clear that evidence that people are risk averse in economic decision making has any relevance here. I conclude that in the absence of any convincing empirical evidence, it is not clear which of the assumptions is more appropriate. (p. 22)

Reflecting this uncertainty, I do not make a specific assumption regarding the utility function; instead, I characterize the equilibria with concave, convex, and linear utility.²

With concave utility, candidate policy will converge and turnout will be zero. The intuition behind convergence in the concave case lies in the general result (formalized in Lemma 3) that candidates will always converge to the median voter *unless* citizens in the extremes of the preference distribution abstain due to alienation. With concave utility, the utility difference between the two candidates’ policy positions is greatest for citizens at the extreme ends of the distribution. Therefore, citizens with extreme ideal points have the highest incentive to vote, which precludes alienation in the extremes.

With convex utility, however, the utility difference between the candidates is greatest for citizens with ideal points that coincide with a candidate’s policy platform. Therefore, convex utility admits alienation among the extreme voters, which in turn allows for divergent, positive-turnout equilibria. Equilibria in the convex case are sensitive to the distribution of citizens’ ideal points. With a uniform distribution, as long as policy is sufficiently close to the median citizen’s ideal point to induce alienation among extreme voters, then candidates have no incentive to shift policy towards the center or towards the extremes. Therefore, any two policy points located in an interval centered about the median citizen is an equilibrium.

With a single-peaked distribution, candidates will converge and turnout will be zero. Citizens in the extremes of the distribution do abstain if candidates are close enough, but

²John Aldrich, among others, has suggested that sigmoid utility, an S-shaped utility function that is at first concave and then convex, best captures citizen preferences. While I do not present the sigmoid case formally, as long as the utility function turns convex ‘soon enough’, then the results in the sigmoid case mirror the convex case.

candidates still have an incentive to converge at the median since the density of citizen ideal points is highest at the center.

With a bimodal distribution, a Nash Equilibrium in pure strategies does not always exist. A unique symmetric Local Equilibrium with positive turnout does exist, and this equilibrium is also Nash as long as voting costs are high enough. Interestingly, in direct contrast to the Hotelling-Black median voter result, it will not be an equilibrium for candidates to set policy at the median. At the median, candidates will have a best response to set policy closer to one of the modes of the distribution to increase the density of their support.

With linear utility, the range of equilibria are similar to the convex-uniform case: any policy pair in an interval centered about the median citizen is an equilibrium. In contrast to the convex-uniform case, however, turnout is only positive when candidates set policy at the endpoints of this interval. This positive turnout equilibrium is very robust to the distribution of citizens as it exists for any continuous distribution, or any finite distribution of citizens drawn from a continuous distribution.

Related Literature:

An earlier paper that considers a model of rational turnout and office motivated candidates is Ledyard's (1984) seminal paper. Ledyard shows that if citizens have strictly concave utility and candidates are office-motivated, then the unique equilibrium of the model is for candidates to converge and for turnout to equal zero.³ In this paper I use a similar setup to Ledyard, but consider utility functions other than concave: specifically linear and convex. I find that Ledyard's convergence result is not general to other functional forms of utility. In fact, concave is the only class of utility in which a polarized, positive-turnout equilibrium cannot be found.

Most formal models of turnout have focused on the citizens' decision to vote, given exogenous candidate policy positions (for example Palfrey and Rosenthal (1985), Uhlaner (1989), Feddersen and Sandroni (2006); see Aldrich (1993), Blais (2000), and Feddersen (2004) for a review of the turnout literature). While this literature has established the effect of turnout on who wins an election, it has not addressed the effect of turnout on who runs in an election. This is the question I address here.

Most formal models of candidate policy choice assume full turnout, or directly assume turnout as a function of the candidates' policy positions rather than explicitly modeling the decision to vote. Callander (2005) uses a model of expressive voting, where citizens decide

³Morton (1987) considers office motivated candidates in a group model of turnout and shows an analogous result to Ledyard (1984).

to vote based on the proximity of their preferred candidate to their ideal point. Expressive voting, however, could also take a negative form: citizens might choose to vote if the opposing candidate is too far from their ideal point (extreme leftist voters might abstain if Obama were to run against Romney, but vote for Obama over Gingrich or Palin). A model of expressive voting would replicate the results of the pivotal voter model used in this paper, as long as citizens turn out based on the proximity of their preferred candidate *relative* to the opposing candidate.

Callander predicts that candidate polarization will increase with voting costs. In contrast, when the policy positions of both candidates are considered the comparative static depends on shape of the utility function. In the convex-uniform and linear cases, the range of equilibria grows with the cost of voting, which suggests that polarization is ‘weakly’ increasing with voting costs. In the convex-bimodal case, increasing the cost of voting could either increase or decrease polarization, depending on relative steepness of the slope of the distribution of citizen ideal points. In the concave and convex-unimodal cases, there is no connection between the cost of voting and polarization since the candidates will always converge in equilibrium.

McKelvey (1975) explores how turnout could lead to candidate polarization by formally defining how voters must behave for office-motivated candidates to set divergent policy positions in equilibrium. The explicit nature of these equilibria, and the microfoundations that would lead voters to turn out in this manner, however, have remained largely unexplored until now.

This paper shows that turnout considerations can cause candidates to diverge from the median voter’s ideal position; other models of elections have demonstrated that candidate policy polarization occur even with full turnout, if candidates have motivations other than winning office or if voters care about candidate characteristics other than policy. Candidate policy separation has been achieved in models with policy-motivated candidates and an uncertain median (Wittman (1983), and Calvert (1985)), where candidates cannot commit to policy (Alesina (1988), Osborne and Slivinski (1996), and Besley and Coate (1997)), and with uncertainty regarding candidate characteristics (Kartik and McAfee (2007), and Callander and Wilkie (2007)). Calvert (1985) demonstrates that without significant uncertainty and differences in ideal policy, candidate differentiation will be marginal. Alesina (1988) shows how the repeated nature of elections could cause candidates to approximate commitment through reputational mechanisms. Osborne and Slivinski (1996) and Besley and Coate (1997) develop a model of citizen candidates who institute their ideal policy if elected and make the choice of whether to run for office (at a cost).

The paper proceeds as follows: Section 2 introduces the model, Section 3 examines

equilibria under different assumptions on utility, and Section 4 concludes.

2 The Model

There are 2 candidates, $j \in \{A, B\}$, who are able to commit to policy, $g_j \in [0, 1]$, prior to the election. Candidates receive a utility of 1 if elected and 0 otherwise, making their expected utility equal to their probability of winning the election. I assume (without loss of generality) that $g_A \leq g_B$. Take $g \equiv (g_A, g_B)$, and g_m to be the average candidate policy; $g_m = \frac{g_B + g_A}{2}$.

There is a continuum of citizens of measure one whose ideal policy points, α_i , are distributed over $[0, 1]$ according to the function f . f is symmetric about $\frac{1}{2}$, differentiable, strictly positive over $[0, 1]$, and equal to 0 elsewhere. Take α_m to be the ideal point of the median citizen, equal to $\frac{1}{2}$ for all symmetric distributions. Take “interior” to refer to the set of citizens with ideal points between g_A and g_B , and “exterior” to refer to the set of citizens not in the interior. All agents have complete information.

Citizens have a common cost of voting, c , and have preferences over policy that are a strictly decreasing function of the distance of policy from their ideal point; their (von Neuman-Morgenstern) utility functions are of the form:

$$U_i(g^*, \alpha_i) = u(|g^*, \alpha_i|) - c,$$

where g^* is the realized policy. $u(\cdot)$ is continuous and differentiable, and $u'(\cdot) < 0$.

Take $\beta(g, \alpha_i)$ to be the net utility that citizen i receives if their preferred candidate wins; $\beta(g, \alpha_i) = |u(|g_A, \alpha_i|) - u(|g_B, \alpha_i|)|$. Note that $\beta(g, \alpha_i)$ is twice the benefit of voting when pivotal.

Take $V_A(g)$ to be the set of citizens who vote for candidate A ; $V_B(g)$ is defined analogously.

The support set for candidate A , $S_A(g)$, is the set of citizens who prefer candidate A and for whom voting is not a strictly dominated action; $S_A(g) = \{\alpha_i; u(|g_A, \alpha_i|) - u(|g_B, \alpha_i|) \geq 2c\}$. $S_B(g)$ is defined analogously. The support sets are significant since citizens in the support set will vote as a best response when pivotal, while citizens outside the support set will always abstain. Take $|S|$ to be the Lebesgue measure of set S , and $n_f[S]$ to be the measure of citizens with ideal points in S given f . I refer to $n_f[S_A]$ as the size of candidate A 's support set.

Since I use a continuous distribution of citizens as an approximation of a large N election, I assume citizens are pivotal whenever $n_f[V_A] = n_f[V_B]$.⁴ In the appendix I show that the

⁴Individual pivotalness can formally be restored in the model with a continuum of citizens with the

linear model can be extended to a distribution of a finite number of voters, where the problem of zero-mass voters is alleviated.⁵

Election Rules

- (1) *If $n_f[V_A] > n_f[V_B]$ then candidate A wins the election; If $n_f[V_A] < n_f[V_B]$ then candidate B wins the election.*
- (2) *If $n_f[V_A] = n_f[V_B]$ then each candidate wins with equal probability.*

Stages of the Game

- (1) *Candidates set g_j simultaneously.*
- (2) *Citizens choose to vote or abstain. The winning candidate is determined by the election rules outlined above.*

I simplify by considering only the case where the candidate who has the support of the largest number of citizens wins an expected plurality: $n_f[S_A] > n_f[S_B] \rightarrow n_f[V_A] > n_f[V_B]$.⁶ This eliminates situations where candidates tie regardless of position or where candidates have an incentive to decrease their relative support. Since candidates can always equalize their relative support by setting policy equal to the opposing candidates policy, unequal support is never equilibrium play (I formalize this in Lemma 1 below). This simplification, however, requires that I use Nash Equilibrium as my equilibrium concept, rather than Subgame Perfect Nash Equilibrium.

3 Equilibrium Analysis

In this section I will first detail some general results. Following subsections examine the equilibria of the election model under different assumptions of the shape of utility. All proofs are relegated to Appendix A.

following assumption: Take \hat{V}_A to be the closure of all subsets of V_A that are not separated by closed neighborhoods. Candidates tie if $n_f[V_A] = n_f[V_B]$ and all citizens with ideal points in \hat{V}_A and \hat{V}_B vote; if all citizens in \hat{V}_A vote, but not all citizens in \hat{V}_B vote, then candidate A wins an expected plurality. This reintroduces the notion of each voter being pivotal, since every citizen with an ideal point in \hat{V}_A and \hat{V}_B must vote for the candidates to tie.

⁵Additionally, the results of this paper hold without modification with an expressive model of voting as long as citizens turn out based on the proximity of their preferred candidate relative to the opposing candidate; i.e., if citizens receive a utility of *voting* equal to $|u(|g_A, \alpha_i|) - u(|g_B, \alpha_i|)| - c$.

⁶With a finite number of voters this is equilibrium behavior, but it does not always hold asymptotically (see Taylor and Yildirim (2010)). Since I am using a continuous distribution only as an approximation of a large N election, I assume that the candidate who has the support of the largest number of citizens wins an expected plurality to approximate equilibrium behavior in finite N elections.

3.1 General Results

In this section, I establish three general lemmas that will be helpful for characterizing the equilibria under the different assumptions on citizens' utility over policy.

Lemma 1. *In equilibrium, $n_f[V_A(g)] = n_f[V_B(g)]$. Moreover, if $n_f[S_A(g)] = n_f[S_B(g)]$ then it is an equilibrium for the citizens in the support set to vote ($S_k(g) = V_k(g)$) and for all other citizens to abstain.*

The first result follows from candidates' ability to always guarantee a payoff of $\frac{1}{2}$ by choosing the same policy as the opposing candidate. Citizens are all pivotal when $n_f[S_A(g)] = n_f[S_B(g)]$ and if all citizens in the support sets vote, then voting is an equilibrium strategy, since abstaining will cause their preferred candidate to lose the election. Lemma 1 allows easy identification of equilibria: an equilibrium is a policy pair where $n_f[S_A(g)] = n_f[S_B(g)]$ and neither candidate can secure a relatively larger support set by choosing a different policy.

Lemma 2 provides some geometric results that will be useful for determining the set of equilibria for the different cases.

Lemma 2. *(i) If neither support set includes an endpoint of the distribution, then $|S_A(g)| = |S_B(g)|$.
(ii) If $\beta(g, \alpha = 0 < 2c)$ and $\beta(g, \alpha = 1 > 2c)$, then $|S_A(g)| > |S_B(g)|$.
(iii) If both endpoints are in the support sets and $g_m < (>, =)\alpha_m$, then $n_f[S_A(g)] < (>, =)n_f[S_B(g)]$.*

The intuition behind the proof is as follows:

(i) If neither support set includes an endpoint of the distribution, then both support sets are intervals interior to $[0, 1]$ (see Appendix A for a proof that the support sets are intervals). $S_A(g)$ and $S_B(g)$ are symmetric about g_m and must therefore have the same length.

(ii) If $S_A(g)$ is interior and a subset of $S_A(g)$ has a symmetric (about g_m) subset that falls outside of $[0, 1]$, then $|S_B(g)|$ will be smaller than $|S_A(g)|$. This will be the case when $\alpha = 1$ is strictly greater than $2c$, due to the continuity of citizens' utility in α_i .

(iii) Since the support sets are intervals on $[0, 1]$, they can be represented as $S_A(g) = [0, \alpha_A^+]$ and $S_B(g) = [\alpha_B^+, 1]$. α_A^+ and α_B^+ are symmetric about g_m ; therefore, if g_m is smaller than α_m , then α_A^+ is farther from α_m than α_B^+ . Since f is symmetric and $S_B(g)$ extends farther towards α_m than $S_A(g)$ it follows that $n_f[S_A(g)] < n_f[S_B(g)]$. The other results follow from the same logic.

Lemma 3 shows that for a policy to be an equilibrium, citizens with ideal points at the extremes of the distribution must have voting as a weakly dominated strategy.

Lemma 3. *If citizens with ideal points at 0 and 1 strictly prefer to vote when pivotal ($\beta(g, \alpha) > 2c$ for $\alpha = 0, 1$), then (g_A, g_B) is not an equilibrium.*

Suppose $\beta(g, \alpha) > 2c$ for $\alpha = 0, 1$. Since the distribution of voters is symmetric and the support sets are intervals that include the endpoints of the policy spectrum, g_A and g_B must be symmetric about α_m otherwise the size of the support sets will not be equal. Since $\alpha = 0, 1$ have $\beta(g, \alpha)$ strictly greater than $2c$, A can move g_A marginally towards α_m and $\alpha = 0, 1$ will still be in the support sets. Following this deviation, however, g_A is slightly closer to the median voter ($g_m > \alpha_m$) and, by Lemma 2 (iii), the size of candidate A 's support set is relatively larger. This shows that if $\beta(g, \alpha) > 2c$ for $\alpha = 0, 1$, then at least one candidate always has a strictly profitable deviation.

Before discussing the significance of Lemma 3, I distinguish between abstention due to alienation and abstention due to indifference. Intuitively, alienation occurs if both candidates' policy choices are too far from a citizen's ideal point (ideal points at the extreme), while indifference occurs when a citizen's ideal point lies close to the candidate (ideal points near the center). The distinction between alienation and indifference is largely semantic: both result from the citizen's net utility between the candidates being too low to vote. Since the set of citizens who abstain due to alienation are affected differently by moves in a candidate's policy than the set of citizens who abstain from indifference, it will be useful to distinguish between the two.

I formalize the distinction between alienation and indifference with the following definitions:

Definition 1. $A_A(g)$ is the set of α_i such that:

$$u(|g_B, \alpha_i|) \leq u(|g_A, \alpha_i|), \beta(g, \alpha_i) < 2c, \text{ and } \partial\beta(g, \alpha_i)/\partial\alpha_i > 0.$$

I refer to $A_A(g)$ as the alienation set for candidate A ; $A_B(g)$ defined analogously.

$I_A(g)$ is the set of α_i such that:

$$u(|g_B, \alpha_i|) \leq u(|g_A, \alpha_i|), \beta(g, \alpha_i) < 2c, \text{ and } \partial\beta(g, \alpha_i)/\partial\alpha_i \leq 0.$$

I refer to $I_A(g)$ as the indifference set for candidate A ; $I_B(g)$ defined analogously.

If citizens at the endpoints of the distribution abstain due to indifference, then all citizens abstain due to indifference, since the set of indifferent citizens is convex and always contains citizens with $\alpha_i = g_m$. Therefore, Lemma 3 shows that, without alienation among the extremes, office-motivated candidates will converge to the point where no citizens will bother to vote. This result allows us to characterize the general shape of any positive-turnout equilibrium: two candidate support sets, with $n_f[S_A(g)] = n_f[S_B(g)]$, separated by non-empty indifference sets, and bounded away from the extremes by sets of alienation (illustrated in Figure 1).

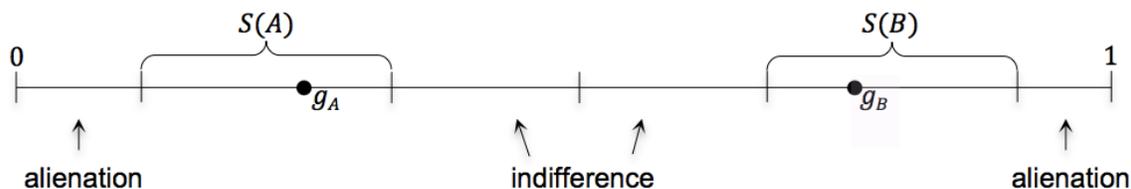


Figure 1: Positive turnout equilibrium

3.2 Concave Utility

Proposition 1 provides an analogous result to Ledyard’s proof of no turnout in equilibrium with strictly concave preferences.

Proposition 1. *If $u(\cdot)$ is strictly concave, then no equilibrium with positive turnout exists; i.e. for any equilibrium value of g , voting is a strictly dominated strategy for all citizens.*

Lemma 3 specifies that alienation must occur for a positive turnout equilibrium to exist. Concave utility, however, precludes alienation since $\beta(g, \alpha_i)$ is the highest for citizens with ideal points at the extremes. Therefore, it follows that positive turnout equilibria cannot exist with concave utility.

The concave model predicts that candidates will set policy close enough to the ideal point of the median voter that turnout will equal zero (all citizens are indifferent). While this is not enough to dismiss concave utility over policy, as shown below, the model does produce more realistic predictions with alternative forms of utility.

3.3 Convex Utility

The equilibria with convex utility are sensitive to the distribution of citizen ideal points. I therefore examine three different distributions separately: uniform, single peaked, and bimodal. With a uniform distribution, any pair of policy points within a certain distance of the median citizen are equilibria. With a single-peaked distribution, the equilibrium replicates the zero-turnout result from the concave model. With a bimodal distribution, a unique Nash equilibrium with positive turnout, alienation and indifference, and policy separation exists in some cases. Generally, however, there exists a Local Equilibrium (defined formally in the Bimodal section) with positive turnout.

As I will catalogue throughout this subsection, the equilibria described here were intuited by Downs (1957). While Downs did not formally model turnout, he reasoned that abstention

of extremists would counteract the centripetal incentive of the Hotelling model of elections. Even without the benefit of a formal model, the equilibria predicted by Downs given the different distributions of citizen ideal points are strikingly similar to the equilibria found in the convex-utility case.

With a formal model, however, I am able to give a more complete description of the equilibria and also look at the comparative statics of the model. The main comparative static is the relationship between the cost of voting and turnout, and between the cost of voting and candidate polarization. When interpreting this comparative static, it is important to consider the implicit normalization of utility over policy. While voting costs are likely to remain relatively constant between elections, the benefit of voting will likely change depending upon the office the election concerns. Since the benefit of winning the election is normalized in my model, the cost of voting, c , should actually be interpreted as the cost divided by the benefit of winning the election. This allows us to restate the comparative static as the relationship between the relative importance of an election and candidate polarization.

3.3.1 Uniform Distribution

With strictly convex utility and a uniform distribution, all (g_A, g_B) within a certain distance of α_m are equilibria. All equilibria feature alienation (or marginal alienation) for citizens with ideal points at the extremes, and as long as candidates locate far enough apart that voting is not a dominated strategy for all voters, then turnout is positive.

Before proving the existence of equilibria in the convex-uniform model, it is useful to characterize the maximal equilibrium distance from α_m , δ .

Definition δ : Take $\delta = \min[\frac{1}{2}, \min\{d \geq 0 : \beta(\alpha_m - d, \alpha_m + d, \alpha = 0) = 2c\}]$

In words, δ is the maximum distance that candidates can be from α_m before citizens at the endpoints have a strict preference for voting (given (g_A, g_B) symmetric about α_m).

When $\delta = \frac{1}{2}$, then voting is a dominated strategy for all positive measures of citizens, regardless of candidate policy. To see why this is the case, note that with convex utility $\beta(g, \alpha_i)$ is highest for citizens with ideal points equal to candidate policy; also, $\beta(g, \alpha_i)$ is increasing for citizens with ideal points at candidate policy as the distance between candidate positions increase. Therefore, since the distance between candidate positions is maximized at $(g_A, g_B) = (0, 1)$, if $\beta(g, \alpha_i) \leq 2c$ for citizens with ideal points at the endpoint of the distribution, then voting is strictly dominated for all other citizens ($\beta(g, \alpha_i) < 2c \forall \alpha_i \in (0, 1)$), and turnout will be zero regardless of candidate positions.

Proposition 2. *If $u(\cdot)$ is strictly convex and f is uniform, then a necessary and sufficient*

condition for an equilibrium is $(g_A, g_B) \in [\alpha_m - \delta, \alpha_m + \delta]^2$. Equilibria with positive turnout exist iff $\delta < \frac{1}{2}$.

If one candidate sets policy outside of $[\alpha_m - \delta, \alpha_m + \delta]$, then the opposing candidate can deviate to either $\alpha_m - \delta$ or $\alpha_m + \delta$, whichever maximizes the distance between candidates. At this new point, citizens at the extremes will be in the support sets; the deviating candidate, however, will be closer to α_m and, by Lemma 2 (iii), will receive an expected plurality. This means that the original policy pair cannot be an equilibrium.

For any g_A and g_B in $[\alpha_m - \delta, \alpha_m + \delta]$, citizens with α equal to 0 and 1 will be alienated. By Lemma 2 (i) the length of the support sets will therefore be equal, and, since length equals size in the uniform case, the candidates will tie. No deviation can leave a candidate better off.

Turnout is positive for a range of equilibria in this model. Specifically, turnout is positive as long as candidates set policy so that $\beta(g, \alpha_i = g_A) > 2c$. In other words, as long as the candidate policy is distinct enough that at least one voter would pay c to break a tie between the candidates, then turnout is positive.

Note that $\beta(g, \alpha_i = 0)$ decreases as the candidates move closer together, which implies that $\delta(c)$ will be increasing in c . This gives the following comparative static: as the relative importance of an election to the citizens increases (c decreases), candidate positions will move (weakly) closer together. Also, turnout will increase as the relative importance of an election to the citizens increases.

The uniform-convex model formalizes Downs's (1957) intuition that the convergence of politicians to the median voter in the (uniform) Hotelling model of elections would be checked by abstention at the political extremes. Downs goes on to say:

At exactly what point this leakage checks the convergence of A and B depends upon how many extremists each loses by moving towards the center compared with how many moderates it gains thereby. (p. 117)

As explicitly modeled above, candidates' incentive to converge disappears as soon as they are close enough to the median voter that alienation occurs at the ends of the political spectrum.

Example: $u(|g_j, \alpha_i|) = -(|g_j, \alpha_i|)^{1/2}$

The definition of δ gives the following equation:

$$(|\alpha_m + \delta, 0|)^{1/2} - (|\alpha_m - \delta, 0|)^{1/2} = 2c$$

Solving for δ with respect to c gives:

$$\delta = c(2 - 4c^2)^{\frac{1}{2}}$$

With a voting cost of 0.1, for example, δ is equal to 0.14 and any policy pair with g_A and g_B in $[0.36, 0.64]$ is an equilibrium.

Continuing with the example of $c = 0.1$, take (g_A, g_B) equal to $(0.37, 0.63)$. With this policy pair, the support set for A consists of all citizens with ideal policy points in $[0.068, 0.431]$. The citizens in $[0, 0.068)$ abstain due to alienation, and those in $(0.431, \alpha_m]$ abstain due to indifference.

The size of the support set is increasing as the candidates move farther apart; for (g_A, g_B) equal to $(0.32, 0.68)$, approximately 83.5% of citizens vote. It is also possible to find a closed form solution for the minimum distance between candidates at which turnout is positive: $d = 2c^2$. For $c = 0.1$, turnout is positive for all g_A and g_B that are farther apart than 0.02.

Candidates do not need to be placed symmetrically about α_m to be in equilibrium. In the above example, $g_A = 0.40$ and $g_B = 0.65$ is an equilibrium with positive turnout.

3.3.2 Single-Peaked Distribution

If utility over policy is strictly convex and f is single-peaked, then, equivalent to the concave case, no equilibrium with positive turnout exists. The intuition behind the candidates' incentive to move towards the middle, however, is different: in the concave case, candidates moved inward to press the opponent's support set towards the endpoint of the distribution; in the convex-uniform case, a move inward will leave the Lebesgue measure of the support sets equalized, but will increase the relative size of the deviating candidate's support set.

Proposition 3. *If $u(\cdot)$ is strictly convex and f is single-peaked, then no equilibrium with positive turnout exists.*

Since the number of citizens over an interval of a given length is higher the closer it is to the median citizen, candidates will always have an incentive to deviate closer to α_m to increase the relative size of their support set. Therefore, the only equilibria are for candidate support sets to be empty and turnout equal to zero.

Proposition 3 formalizes Downs's statement that with a single-peaked distribution:

The possible loss of extremists will not deter their movement toward each other, because there are so few voters to be lost at the margins compared with the number to be gained in the middle. (p. 118)

3.3.3 Bimodal Distribution

With a bimodal distribution and convex utility, a Nash Equilibrium in pure strategies does not always exist. If candidates are constrained to incremental changes, however, equilibria

do exist. Below, I detail the conditions under which a unique symmetric Local Equilibrium with positive turnout exists. Since Nash Equilibria are also Local Equilibria, the Local Equilibrium is the only possible location of a Nash Equilibrium with positive turnout. Below I will discuss under which conditions the unique symmetric Local Equilibrium is the unique Nash Equilibrium with positive turnout.

Local Equilibrium: A policy pair (g_A^*, g_B^*) is a local equilibrium if there exists an open neighborhood of (g_A^*, g_B^*) in which (g_A^*, g_B^*) is a best response.

I focus on bimodal distributions with interior modes.⁷ Take α_A^- equal to the minimum of $S_A(g)$ and α_A^+ equal to the maximum of $S_A(g)$.

Proposition 4. *If $u(\cdot)$ strictly convex and F is bimodal, then take (g'_A, g'_B) such that $\alpha_i = 0, 1$ both have $\beta(g'_A, g'_B, \alpha_i) = 2c$:*

Case 1: If $f(0) \leq f(\alpha_A^+)$ at (g'_A, g'_B) , then a sufficient and necessary condition for a symmetric local equilibrium with positive turnout (g_A^, g_B^*) is $f(\alpha_A^-) = f(\alpha_A^+)$.*

Case 2: If $f(0) > f(\alpha_A^+)$ at (g'_A, g'_B) , then (g'_A, g'_B) is the unique symmetric local equilibrium.

Moreover, a symmetric local equilibrium with positive turnout exists iff $\beta(p_A, p_B, \alpha = p_A) > 2c$, where p_A is the left mode of f and p_B is the right mode of f ; if it exists, then the symmetric equilibrium is unique.

The logic behind Proposition 4 is that if the candidates are at a symmetric policy pair and $\alpha_A^- < \alpha_A^+$, then candidate A will have a centripetal incentive, since the region gained has a higher probability measure than the region lost. If $\alpha_A^- > \alpha_A^+$, then, similarly, candidate A will have a centrifugal incentive as long as $\alpha_A^- \neq 0$. If $\alpha_A^- = 0$ and $\beta(g, \alpha_i = 0) = 2c$ then, by Lemma 2 (iii), candidate A will not have an incentive to move outward or inward (Case 1 equilibrium). Otherwise, the only symmetric equilibrium with positive turnout will be where $\alpha_A^- = \alpha_A^+$ (Case 2).

Case 1 gives a local equilibrium with marginal alienation at the extremes (citizens with ideal points at 0 and 1 get equal utility from voting and abstaining). Case 2 gives equilibria with a set of alienated voters in each extreme, as illustrated in Figure 2:

While Proposition 4 only gives the existence of a local equilibrium, generally, a unique Nash Equilibrium with positive turnout will exist if the cost of voting is high enough.⁸ The intuition is perhaps best explained by detailing why the local equilibrium can fail to

⁷A bimodal distribution with modes at 0 and 1 gives a unique Nash Equilibrium with positive turnout (the proofs closely follow the proof of nonexistence of positive turnout equilibria in the single-peaked model). I do not cover this model, however, since it implies that extremists are the largest electoral group.

⁸As an example, if $c = 0.25$, $u(x) = x^{\frac{1}{2}}$ and $f = (\frac{1}{2}\text{beta}(5, 2.2) + \frac{1}{2}\text{beta}(2.2, 5))$, then $g = (0.27, 0.73)$ is the unique Nash Equilibrium and turnout is positive.

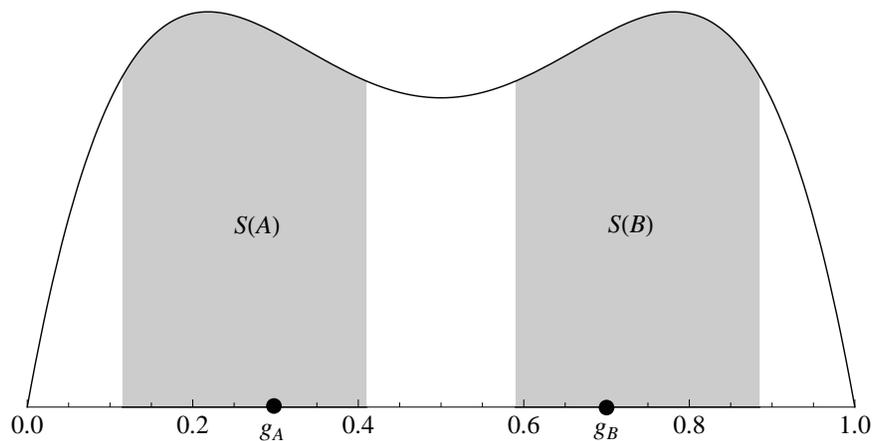


Figure 2: Case 2 Equilibrium

be Nash. The local equilibrium will rarely coincide with the modes of the distribution. Therefore candidate A can deviate to the right-hand-side mode of the distribution. If the cost of voting is low enough, then $S_A(g)$ and $S_B(g)$ will be greater than zero (turnout will be positive) and $n[S_A(g)]$ will be greater than $n[S_B(g)]$. If the cost of voting is high enough, however, then turnout will be zero after the deviation, and candidate A will not be better off.

It is also interesting to note that in many cases it is not an equilibrium for both candidates to set policy at the median voter's ideal point. Since f is low at the median, a candidate who deviates to a point closer to one of the modes of f can guarantee a relatively larger support. The only cases for which this will not be true is if the cost of voting is very large, or if the modes of the distribution are very close to the median, so that any deviation which results in non-empty support sets gives the deviator a support set which lies on the outside of the mode of f , which could result in smaller support.

Again, this style of equilibrium was intuited by Downs (1957). Downs stated that with a symmetric bimodal distribution:

...the two parties will not move away from their initial positions at 25 and 75 at all; if they did, they would lose far more voters at the extremes than they could possibly gain in the center.

Downs's logic shows that the bimodal distribution and abstention in the extremes leads to a situation where candidates do not have an incentive to deviate inward. As shown above, however, we must also consider the incentive to deviate outward; only at one symmetric policy

pair are the centripetal incentives perfectly balanced by the centrifugal incentives (although the equilibrium is unlikely to be exactly at the modes of the distribution as predicted by both Downs (1957) and Osborne (1995)).

While the convex-bimodal model gives a unique symmetric local equilibrium with alienation and indifference, the comparative static of candidate positions and costs depends on relative steepness of the slope of bimodal distribution at the equilibrium values of α_A^- and α_A^+ . If $f'(\alpha_A^-) > -f'(\alpha_A^+)$, then a marginal drop in c will cause candidates to move closer together (since $f(\alpha_A^-) < f(\alpha_A^+)$ at the old equilibrium). If $f'(\alpha_A^-) < -f'(\alpha_A^+)$, however, then candidates move farther apart with a marginal drop in c . This also leads to an ambiguous relationship between the cost of voting and voter turnout.

3.4 Linear Utility

Linear utility over policy is certainly a knife-edge assumption, but, as I show in this section, the results and comparative statics of the linear model are quite similar to the convex and sigmoid model with uniform distributions of citizen ideal points. The linear model, however, benefits from analytical ease: the equilibrium is easy to calculate and is the same for all symmetric distributions. The linear model also extends easily to the full-information model with finite voters. It might therefore be useful to use as an approximation of the more complex convex and sigmoid cases.

Proposition 5. *A necessary and sufficient condition for an equilibrium is $g_A, g_B \in [\alpha_m - c, \alpha_m + c]^2$. At $(g_A = \alpha_m - c, g_B = \alpha_m + c)$ turnout is positive; all other equilibria have zero turnout.*

The logic of the proof is similar to that for Proposition 3 (convex-uniform case). Note, however, that Proposition 5 holds for any symmetric distribution of voters.

If either g_A or g_B is interior to $[\alpha_m - c, \alpha_m + c]$, then turnout is zero, which is not very appealing from an empirical viewpoint. If candidates have a secondary concern of maximizing turnout, or even just a secondary preference for non-zero turnout, then $g_A = \alpha_m - c, g_B = \alpha_m + c$ becomes the unique equilibrium of the model. To see how a preference for positive turnout arises, consider the following modification to the setup: if no citizens vote then the election is rerun and candidates will pay an additional election cost in the second election. If this is the case (and if cheap talk is allowed), then $g_A = \alpha_m - c, g_B = \alpha_m + c$ becomes the unique equilibrium of the model.⁹

⁹With a continuous distribution of citizens, note that citizens in the exterior only vote as a weak best response. With a finite number citizens drawn from a continuous distribution, however, there will almost surely exist an equilibrium where exterior citizens vote as a strict best response. Proof available on request.

With $g_A = \alpha_m - c$, $g_B = \alpha_m + c$ as the unique equilibrium, the distance between candidates is strictly decreasing in the benefit of the election (increasing in c), and turnout will be strictly decreasing in the cost of voting.

While Proposition 5 holds only for symmetric distributions of citizen ideal points, an analogous result holds for any continuous distribution over $[0, 1]$. Even with an asymmetric distribution, (g_A^*, g_B^*) such that $|g_A^*, g_B^*| = 2c$ and $n_f[S_A(g)] = n_f[S_B(g)]$ will be an equilibrium where the exterior citizens vote and the interior voters abstain (proof analogous to Proposition 5). Note that such a point exists for all continuous distributions, but need not be centered about the median citizen.

As discussed in the previous section, I use an infinite number of voters only as an approximation of a large election. In the case of linear preferences, however, the equilibria found in the infinite population case also easily generalize to any N greater than one. In particular, Proposition 6 shows that with linear utility, a positive turnout equilibrium where the exterior citizens vote and the interior citizens abstain exists almost surely for any finite population drawn from any continuous distribution (symmetry is not needed).

Proposition 6. *A sufficient condition for the existence of an equilibrium with positive turnout given a finite distribution of citizens is that there is no overlap in citizens' policy preferences; i.e. $\alpha_i \neq \alpha_j \forall i \neq j$.*

With no overlap in citizens' policy preferences, a policy pair g_A^* and g_B^* can be found such that the distance between g_A^* and g_B^* is equal to $2c$ and the number of citizens in each candidate's support set is equal, which gives a set of equilibria akin to those given in Proposition 5. The formal proof of Proposition 6 requires the introduction of a different set of notation and is therefore left to Appendix B.

4 Conclusion

This paper formally studies how turnout considerations affect the candidates' policy positions and lays the theoretical foundation for further work in this area. I find that the effect of turnout is dependent on the shape of citizens' utility over the distance between their ideal policy and realized policy. If utility is concave, then turnout introduces a centripetal incentive; since only citizens in the interior abstain, when a candidate moves closer to the median position they increase the number of citizens who prefer their position *and* increase their relative turnout rate. If utility is convex, then turnout introduces a centrifugal incentive; since citizens in the extremes abstain, when a candidate moves closer to the median position

they still increase the number of citizens who prefer their position, but their relative turnout rate decreases.

I find that positive turnout equilibria exist when utility is linear, or when utility is convex and the distribution of citizens' ideal points is uniform or bimodal. Positive turnout equilibria generally have the following properties: candidate policy is divergent and lies close to the median voter, citizens with policy preferences close to a candidate's policy position will vote, citizens with preferences close to the center will abstain due to indifference, and citizens at the extremes of the distribution will abstain due to alienation.

An important implication of this paper is that the standard assumption of concave utility is not without loss of generality. As this paper and other recent contributions (Bade (2004) and Kamada and Kojima (2009)) have shown, the equilibrium outcomes of some of the most standard spacial models of politics are sensitive to assumptions on the shape of utility (and distribution of citizens' ideal points over the policy space). This begs the following empirical question: what is the shape of citizens' utility over policy, and what is the distribution their preferences? The answers to these questions could have broad implications for both the theoretical and empirical literature of spacial politics. Aldrich and McKelvey (1977), using data from the 1968 and 1972 US presidential elections, conclude that citizens' preferences follow a unimodal distribution; Palfrey and Poole (1987), however, find that heteroscedasticity can introduce bias that "...causes the scaled distribution to be very centrally tended (unimodal), even in strongly bimodal populations."

Testing the shape of utility directly using electoral data might not be possible since the distinction between concave and convex utility (given fixed candidate positions) is empirically moot in most elections: voters generally choose between two viable candidates, but three points are needed to identify the shape of preferences. An experimental approach could be used to test candidate choice over three or more options. Such an approach would likely require two steps: first, generating a metric on the policy space (perhaps a monetary metric); and second, testing choice over three policy points (with probabilistic outcomes). Another option would be to use theoretical identification to test the shape of utility indirectly: per my model, a positive correlation between election costs and polarization would suggest that utility is (weakly) convex.

This paper suggests possible avenues for further research concerning the interaction between policy and turnout. To allow for greater complexity, the basic framework of this paper could be extended to a partisan model of elections with policy motivated candidates and non-homogenous costs and benefits. In such a model, turnout would still induce centrifugal or centripetal incentives: even candidates that are purely policy motivated consider the probability of winning when choosing policy platforms. A partisan model of elections with

endogenous candidates and voter turnout would further illuminate the interaction between citizen characteristics and candidates' policy positions.

Take, for example, an election where leftists are highly motivated (represented by a decrease in their normalized cost of voting, which is equivalent to a proportional increase in their utility over election outcomes). Counterintuitively, fervent support amongst leftists could induce the leftist candidate to take a policy position closer to the center: with high turnout guaranteed in the extremes, even a policy-motivated leftist candidate could have an incentive to shift to the center to garner support, and motivate turnout, among centrist citizens. Similarly, it is possible that decreasing voting costs for a particular segment of the population could result in a candidate taking a policy position that is farther away from that segment's preferred policy point.

Explicitly modeling the primary elections could be an important extension of the general-election model presented here. In certain cases, the model allows for a wide range of equilibrium policy positions. In this case, the selection of candidates in the primary elections is of great importance in determining final policy outcomes. Also, the predictions of the model suggest an important tool for manipulating polarization, which, according to certain policy makers, has risen above optimal levels in the US. In a working paper (Valasek (2011)) I use a related model to examine the effect of measures to increase turnout, such as mandatory voting, on candidate polarization and election outcomes.

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5 Appendix A: Proofs

The following result will be needed for the proofs of Lemmas 1-3:

Result 1. $A_A(g)$, $I_A(g)$, and $S_A(g)$ are all convex sets; i.e. they are all intervals on $[0, 1]$.

Proof: I focus my attention on $S_A(g)$ without loss of generality and therefore restrict my attention to $[0, g_m]$. I show that the alienation set is convex and, if nonempty, includes $\alpha = 0$ and that the indifference set is also convex and always includes g_m . Therefore, $S_A(g)$, which is just the complement of $A_A(g) \cup I_A(g)$ on $[0, g_m]$, must also be convex. Before proving the result, it will be useful examine the curvature of $\beta(g, \alpha_i)$.

Properties of $\partial\beta(g, \alpha_i)/\partial\alpha_i$:

Note that:

$$\partial\beta(g, \alpha_i)/\partial\alpha_i = \partial u(|g_A, \alpha_i|)/\partial\alpha_i - \partial u(|g_B, \alpha_i|)/\partial\alpha_i \quad (1)$$

A marginal change in α_i is equivalent to a marginal change in distance. In the interior (between g_A and g_m), a marginal increase in α_i moves α_i farther away from g_A and closer to g_B . This implies $u(|g_A, \alpha_i|)$ decreases, and $u(|g_B, \alpha_i|)$ increases, with α_i . By Equation 1

$$\partial\beta(g, \alpha_i)/\partial\alpha_i < 0 \text{ when } \alpha_i \in (g_A, g_m].$$

In the exterior (between 0 and g_A), a marginal increase in α_i moves α_i closer to both g_A and g_B . Therefore, both $u(|g_A, \alpha_i|)$ and $u(|g_B, \alpha_i|)$ are increasing with α_i . The sign of

$\partial\beta(g, \alpha_i)/\partial\alpha_i$ will depend on relative magnitude $\partial u(|g_A, \alpha_i|)/\partial\alpha_i$ and $\partial u(|g_B, \alpha_i|)/\partial\alpha_i$, and hence the curvature of $u(\cdot)$.

If $u(\cdot)$ is concave, then $\partial u(|g_A, \alpha_i|)/\partial\alpha_i < \partial u(|g_B, \alpha_i|)/\partial\alpha_i$, and by Equation 1:

$$\partial\beta(g, \alpha_i)/\partial\alpha_i < 0$$

Similarly if $u(\cdot)$ is convex, then:

$$\partial\beta(g, \alpha_i)/\partial\alpha_i > 0$$

If $u(\cdot)$ is linear, then:

$$\partial\beta(g, \alpha_i)/\partial\alpha_i = 0$$

$A_A(g)$ *Convex*:

This part of the proof must be done for each class of utility functions separately:

Concave: For $u(\cdot)$ concave, $\partial\beta(g, \alpha_i)/\partial\alpha_i < 0$ in both the interior and the exterior. By definition, this implies $A_A(g)$ will be empty.

Linear: For $u(\cdot)$ linear, $\partial\beta(g, \alpha_i)/\partial\alpha_i \leq 0$ in both the interior and the exterior. By definition, this implies $A_A(g)$ will be empty.

Convex: For $u(\cdot)$ convex, alienation can occur, but only in the exterior, since $\partial\beta(g, \alpha_i)/\partial\alpha_i > 0$ in the exterior. Given $A_A(g)$ non-empty, take α_A^- to be the supremum of $A_A(g)$. $\alpha_i < \alpha_A^-$ must be in the exterior, since $A_A(g)$ is in the exterior. Therefore, since $\partial\beta(g, \alpha_i)/\partial\alpha_i > 0$ for $\alpha_i < \alpha_A^-$ and $\beta(g, \alpha_A^-) \leq 2c$, then $\beta(g, \alpha_i) < 2c$ for all $\alpha_i < \alpha_A^-$. This shows that $A_A(g) = [0, \alpha_A^-)$ or \emptyset .

Sigmoid: For $u(\cdot)$ sigmoid, take any g_A, g_B . Because $u(\cdot)$ is concave initially and then convex, the exterior can be broken down into an interval, $[0, \alpha^*)$, where $\partial\beta(g, \alpha_i)/\partial\alpha_i > 0$ and an interval, $(\alpha^*, g_A]$, where $\partial\beta(g, \alpha_i)/\partial\alpha_i < 0$. Since alienation can only occur in $[0, \alpha^*)$ the proof that $A_A(g) = [0, \alpha_A^-]$ or \emptyset follows from the convex case.

$I_A(g)$ *Convex*:

Indifference can only occur in the subset of the policy space where $\partial\beta(g, \alpha_i)/\partial\alpha_i \leq 0$. Call this subset X ; for all utility functions considered, X is an interval on the policy space (a convex subset of $[0, g_m]$). For $u(\cdot)$ concave or linear, $X = [0, g_m]$; for $u(\cdot)$ convex, X is equal to the interior only; and for $u(\cdot)$ sigmoid, $X = [\alpha^*, g_m]$, where α^* denotes the lower bound of the concave portion of the exterior.

Also note that g_m is always in $I_A(g)$ since g_m is in the interior and $\beta(g, \alpha_i = g_m) = 0$. Take α_A^+ to be the infimum of $I_A(g)$. Since $\beta(g, \alpha_i) < \beta(g, \alpha_A^+) \leq 2c \forall \alpha_i \in (\alpha_A^+, g_m]$ and $(\alpha_A^+, g_m] \subset X$ we can use the same logic as used above to show that $I_A(g) = (\alpha_A^+, g_m]$

$S_A(g)$ Convex:

$A_A(g)$, $S_A(g)$, and $I_A(g)$ are a partition of the policy spectrum from 0 to g_m ; i.e. they are disjoint but their union covers $[0, g_m]$. Therefore, if $S_A(g)$ is non-convex then for some $x, y \in S_A(g)$ ($x < y$) there exists z in either $A_A(g)$ or $I_A(g)$ such that $\lambda x + (1 - \lambda)y = z$. Since $A_A(g)$ and $I_A(g)$ are convex and $x < z < y$, the definition of convexity implies that either x or y must be in $A_A(g)$ or $I_A(g)$, clearly a contradiction.

◇

Proof of Lemma 1:

If $n_f[V_A(g)] \neq n_f[V_B(g)]$ then either $n_f[V_A(g)] > n_f[V_B(g)]$ or $n_f[V_A(g)] < n_f[V_B(g)]$. Supposing (without loss of generality) that $n_f[V_A(g)] > n_f[V_B(g)]$, then candidate B will receive an expected utility of less than $\frac{1}{2}$, and will have an incentive to deviate to $g_A = g_B$, where $n_f[V_A(g)] = n_f[V_B(g)] = \emptyset$.

If a citizen is pivotal, then their benefit from voting is equal to $\frac{\beta(g, \alpha_i)}{2}$; if a citizen is not pivotal, it is equal to zero. For a citizen outside of a support set, the benefit from voting is less than c by definition, and abstaining is therefore a dominant strategy. For a citizen in a support set, the benefit of voting is greater or equal to c if they are pivotal. Therefore, if $n_f[S_A(g)] = n_f[S_B(g)]$ and all other citizens in the support sets are voting, it is a best response for i to vote, since their vote will move the candidates into a tie.

◇

Proof of Lemma 2:

The following fact will be helpful for the proof of Lemma 2:

Fact 1: *If α_1, α_2 are equidistant to g_m , then $\beta(g, \alpha_1) = \beta(g, \alpha_2)$.*

This fact follows directly from $u(\cdot)$ being a function of distance only, and since citizens with ideal points symmetric about g_m have the same distance between their ideal point, the candidate policy they prefer, and the candidate policy they oppose.

(i) *If neither support set includes an endpoint of the distribution, then $|S_A(g)| = |S_B(g)|$:*

If neither support set includes an endpoint of the distribution, then $S_A(g)$ and $S_B(g)$ are interior to $[0, 1]$, by convexity. By Fact 1, any point α_1 in $S_A(g)$ has a corresponding symmetric point α_2 in $S_B(g)$, since $\beta(g, \alpha_2) = \beta(g, \alpha_1) \geq 2c$. Since $S_A(g)$ and $S_B(g)$ are interior, they must be intervals symmetric about g_m and therefore have the same lebesgue measure.

(ii) *If one of the support sets includes an endpoint, and the other is interior, then the length of the interior support set is weakly greater, and strictly greater if $\beta(g, \alpha = 0 \text{ or } 1) > 2c$:*

Assume, without loss of generality, that $0 \in S_A(g)$ and $S_B(g)$ is interior. Take $S_B(g)'$ to be the interval symmetric, about g_m , to $S_B(g)$; note that $|S_B(g)'| = |S_B(g)|$. By Fact 1, $S_B(g)'$ must cover $S_A(g)$, which gives $|S_B(g)'| = |S_B(g)| \geq |S_A(g)|$. If $\beta(g, \alpha = 0) > 2c$, then $S_B(g)'$ must cover $S_A(g) \cup [\epsilon, 0]$, where $\epsilon < 0$. This implies that the lebesgue measure of $S_B(g)'$ is greater than that of $S_A(g)$ ($|S_B(g)'| = |S_B(g)| \geq |S_A(g)|$).

(iii) If both endpoints are in the support sets and $g_m < (>, =)\alpha_m$ then $n_f[S_A(g)] < (>, =)n_f[S_B(g)]:$

Take $S_B(g)'$ to be the interval symmetric, about α_m (not g_m), to $S_B(g)$. By the symmetry of f , $n_f[S_B(g)'] = n_f[S_B(g)]$. By Fact 1 and since $g_m < \alpha_m$, $S_B(g)'$ must cover $S_A(g) \cup [g_A^+, \epsilon]$, where g_A^+ is the max of $S_A(g)$ and $\epsilon > 0$. This implies that $n_f[S_A(g)] < n_f[S_B(g)'] = n_f[S_B(g)]$, since f is strictly positive. The proofs of $(>, =)$ are analogous.

◇

Proof of Lemma 3: If $\beta(g, \alpha) > 2c$ for $\alpha = 0, 1$ then (g_A, g_B) is not an equilibrium:

Suppose an equilibrium, (g_A^*, g_B^*) , exists with $\beta(g_A^*, g_B^*, \alpha) > 2c$ for $\alpha = 0, 1$. Since the support sets are convex and include the endpoints, $S_A(g) = [0, \alpha_A^+]$, where $\alpha_A^+ > 0$ since $\beta(g, \alpha = 0) > 2c$ and $\beta(g, \alpha)$ is continuous in α . Symmetrically, $S_B(g) = [\alpha_B^+, 1]$ with $\alpha_B^+ < 1$.

By Lemma 1 $n_f[S_A(g)] = n_f[S_B(g)]$ which implies, by Lemma 2 (iii), that $g_m = \alpha_m$. Suppose candidate A deviates to $g'_A = g_A^* + \epsilon$ where $\epsilon > 0$ but is small enough that $\beta(g'_A, g_B^*, \alpha) > 2c$ for $\alpha = 0, 1$. Note that such an ϵ exists because $\beta(g, \alpha)$ is continuous (and decreasing) in g_A . Now $g'_m > \alpha_m$ and $\alpha = 0, 1$ are still in the support sets. By Lemma 2 (iii), therefore, $n_f[S_A(g)] > n_f[S_B(g)]$. This contradicts the assumption that (g_A^*, g_B^*) is an equilibrium, since candidate A receives an expected plurality if she deviates to g'_A .

◇

Proof of Proposition 1: Suppose an equilibrium, (g_A^, g_B^*) , exists where $|S_A(g)| > 0$ for at least one support set (assume without loss of generality $S_A(g)$). By $|S_A(g)| > 0$, there exist some α' in $S_A(g)$ where $\alpha' \in (0, g_m]$. Since $u(\cdot)$ concave, $\beta(g, \alpha = 0) > \beta(g, \alpha_i) \forall \alpha_i \in (0, g_m]$ so $\beta(g, \alpha = 0) > \beta(g, \alpha') \geq 2c$. By Lemma 1, $|S_A(g)| = |S_B(g)|$, and following the same argument as above $\beta(g, \alpha = 1) > 2c$. Then by Lemma 3, (g_A^*, g_B^*) cannot be an equilibrium.*

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Proof of Proposition 2: Note that the proof is trivial for $\delta = \frac{1}{2}$ since voting is a strictly dominated strategy for all sets of voters with positive mass for all (g_A, g_B) . Therefore, for the remainder of the proof, assume $\delta < \frac{1}{2}$.

Necessity: Assume an equilibrium, (g_A^*, g_B^*) , exists where $g_B^* > \alpha_m + \delta$. Suppose candidate A sets policy to $g'_A = \alpha_m - \delta$. Since $\beta(\alpha_m - \delta, \alpha_m + \delta, \alpha = 1) = 2c$ and $\beta(g, \alpha = 1)$ is increasing in g_B , $g_B^* > \alpha_m + \delta$ implies that $\beta(g'_A, g_B^*, \alpha = 1) > 2c$ which in turn means that $S_B(g)$ is non-empty. Note, however, that $g'_m > \alpha_m$. By Lemma 2 (ii), $|S_A(g)'| > |S_B(g)'|$ since $\alpha = 1$ is interior to $S_B(g)$. Since $|S| = n_f[S]$ with a uniform distribution, this gives g'_A as a strictly profitable deviation.

Sufficiency: If $(g_A^*, g_B^*) = (\alpha_m - \delta, \alpha_m + \delta)$ then $g_m = \alpha_m$ and by Lemma 2 (ii), $|S_A(g)| = |S_B(g)|$. For any $g_A^*, g_B^* \in [\alpha_m - \delta, \alpha_m + \delta]$ with at least one policy point interior, $\beta(g_A^*, g_B^*, \alpha = 0 \text{ and } 1) < 2c$, and by Lemma 2 (i) $|S_A(g)| = |S_B(g)|$.

To see that neither candidate has an incentive to deviate from any $g_A^*, g_B^* \in [\alpha_m - \delta, \alpha_m + \delta]^2$, note that any deviation that such that $|S_A(g)| \neq |S_B(g)|$ will leave the deviator's policy farther from α_m than the other candidate, and by Lemma 2 (iii) the deviator will receive a utility of less than $\frac{1}{2}$.

Positive turnout for $\delta < \frac{1}{2}$: If $\delta < \frac{1}{2}$, then $(g_A^*, g_B^*) = (\alpha_m - \delta, \alpha_m + \delta)$ is an equilibrium, and since $\beta(g, \alpha)$ is increasing in the exterior, $\beta(g, \alpha_m - \delta) > \beta(g, \alpha = 0) = 2c$. Therefore, $[0, \alpha_m - \delta] \subset S_A(g)$.

◇

Proof of Proposition 3: Note that since f is single peaked and symmetric, $f(\alpha_m) > f(\alpha) \forall \alpha \neq \alpha_m$. Also, $f(\alpha) > f(\alpha')$ iff $|\alpha, \alpha_m| < |\alpha', \alpha_m|$. Therefore, for any two intervals S, S' , if $|S| \geq |S'|$ and S closer to α_m than S' , then $n_f[S] > n_f[S']$.

First, I show that no equilibrium exists where $\beta(g, \alpha) > 2c$ for α equal to either 0 or 1. By Lemma 3, $\beta(g, \alpha) > 2c$ for $\alpha = 0$ and 1. Assume an equilibrium exists where $\beta(g, \alpha) > 2c$ for only one of the endpoints, without loss of generality $\alpha = 0$, and one of the support sets has a positive lebesgue measure. By Lemma 2 (ii), $|S_A(g)| < |S_B(g)|$. It is also the case that $S_B(g)$ is closer to α_m than $S_A(g)$ and, as showed above, this implies $n_f[S_A(g)] < n_f[S_B(g)]$, which contradicts Lemma 1.

Secondly, I show that $\beta(g_A^*, g_B^*, \alpha) \leq 2c$ for $\alpha = 0$ and 1 and cannot be an equilibrium if the lebesgue measure of the support sets is non-zero. Suppose an equilibrium exists, if $\beta(g_A^*, g_B^*, \alpha) \leq 2c$ for $\alpha = 0$ and 1, then $|S_A(g)| = |S_B(g)|$ by Lemma 2 (i). By single-peakedness of f and since $n_f[S_A(g)] = n_f[S_B(g)]$ in equilibrium, $S_A(g)$ and $S_B(g)$ must be symmetric about α_m . By the continuity of $\beta(\cdot)$, there exists an $\epsilon > 0$ small enough that a deviation to $g'_A = g_A^* + \epsilon$ will leave the lebesgue measure of the support sets greater than zero. The deviation will leave $\beta(g'_A, g_B^*, \alpha) \leq 2c$ for $\alpha = 0$ and 1 and hence $|S_A(g)'| = |S_B(g)'|$. Since, $|g'_A, \alpha_m| < |g_B^*, \alpha_m|$, however, $S_A(g)'$ will be closer to α_m than $S_B(g)'$ which implies $n_f[S_A(g)'] < n_f[S_B(g)']$ and precludes g_A^* as a best response.

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Proof of Proposition 4:

Sufficiency:

Since the best response functions are continuous in g , (g_A^*, g_B^*) is a local equilibrium as long as payoffs are strictly decreasing for any marginal deviation from (g_A^*, g_B^*) .

Case 1: I will consider a deviation by candidate A, without loss of generality. Since $S_A(g)$ is an interval, $|S_A(g)| = \alpha_A^+ - \alpha_A^-$, which gives $\partial|S_A(g)|/\partial g_A = \partial\alpha_A^+/\partial g_A - \partial\alpha_A^-/\partial g_A$. And since $S_A(g)$ and $S_B(g)$ are interior, $|S_A(g)'| = |S_B(g)'|$ after a marginal change in g_A , which gives $\partial|S_A(g)|/\partial g_A = \partial|S_B(g)|/\partial g_A$, or, equivalently:

$$\partial\alpha_A^+/\partial g_A + \partial\alpha_B^+/\partial g_A = \partial\alpha_B^-/\partial g_A + \partial\alpha_A^-/\partial g_A \quad (2)$$

$n_f[S_A(g)] = n_f[S_B(g)]$ at (g_A^*, g_B^*) by Lemma 2 (iii). Therefore, for (g_A^*, g_B^*) to be a local equilibrium, $\partial n_f[S_A(g)]/\partial g_A = \partial n_f[S_B(g)]/\partial g_A$. Since $S_A(g)$ is an interval, $n_f[S_A(g)] = F(\alpha_A^+) - F(\alpha_A^-)$ and:

$$\partial n_f[S_A(g)]/\partial g_A = \partial F(\alpha_A^+)/\partial g_A - \partial F(\alpha_A^-)/\partial g_A = f(\alpha_A^+)\partial\alpha_A^+/\partial g_A - f(\alpha_A^-)\partial\alpha_A^-/\partial g_A \quad (3)$$

For any (g_A, g_B) symmetric about α_m , $f(\alpha_A^+) = f(\alpha_B^+) = f(\alpha^+)$ and $f(\alpha_A^-) = f(\alpha_B^-) = f(\alpha^-)$. Plugging Equation 3 into $\partial n_f[S_A(g)]/\partial g_A = \partial n_f[S_B(g)]/\partial g_A$ and rearranging gives:

$$f(\alpha^+)[\partial\alpha_A^+/\partial g_A + \partial\alpha_B^+/\partial g_A] = f(\alpha^-)[\partial\alpha_A^-/\partial g_A + \partial\alpha_B^-/\partial g_A] \quad (4)$$

As the terms within the brackets are equal by Equation 2, Equation 3 is true iff $f(\alpha^+) = f(\alpha^-)$, which gives (g_A^*, g_B^*) as a local equilibrium.

Case 2: At (g'_A, g'_B) , neither candidate has an incentive to deviate outward, since both endpoints will be in the support sets, and by Lemma 2 (iii) the deviator will receive a utility of less than $\frac{1}{2}$.

A deviation inward from (g'_A, g'_B) will leave $|S_A(g)| = |S_B(g)|$, since both support sets will be interior. Therefore, Equation 2 will hold, and since $f(\alpha^-) > f(\alpha^+)$ at (g'_A, g'_B) , the LHS of equation 3 will be greater than the RHS, which implies $\partial n_f[S_A(g)]/\partial g_A < \partial n_f[S_B(g)]/\partial g_A$. This shows that candidate A will also be strictly worse off with a marginal inward deviation (candidate B has analogous payoffs).

Existence and Uniqueness:

Take g_A and g_B symmetric about α_m and $|g_A, \alpha_m| = d$. The existence and uniqueness of a symmetric local equilibrium (done simultaneously for Case 1 and 2) follows from Equation

3 and that α_A^- and α_A^+ are strictly decreasing in d , the distance between the policy positions and the median ideal point.

First, I show that no positive turnout equilibrium exists if $\beta(p_A, p_B, \alpha = p_A) \leq 2c$ (where p_A, p_B are the location of the left and right modes of f , respectively). $\beta(p_A, p_B, \alpha = p_A) \leq 2c$ implies that $S_A(g)$ and $S_B(g)$ are empty, or have no mass, for $d = p$, where p is the distance between the mode of f and α_m . Turnout can only be positive for $d > p$, but in this case, $S_A(g)$ is located on the increasing portion of f , so $f(\alpha^-) > f(\alpha^+)$. By Equation 3, $\partial n_f[S_A(g)]/\partial g_A > \partial n_f[S_B(g)]/\partial g_A$, and for any d with positive turnout, candidate A will have an incentive to make a marginal inward deviation.

If $\beta(p_A, p_B, \alpha = p_A) > 2c$, then $S_A(g)$ and $S_B(g)$ have positive mass for $d = p$. Since α_A^- and α_A^+ are on opposite sides of p_A , and α_A^- and α_A^+ are strictly decreasing in d , $\partial f(\alpha_A^-)/\partial d > 0$ and $\partial f(\alpha_A^+)/\partial d < 0$. Therefore, if $f(\alpha_A^-) < f(\alpha_A^+)$ at $d = p$, then by the continuity of f , there exists $d^* < p$ such that $f(\alpha_A^-) = f(\alpha_A^+)$.

Similarly, if $f(\alpha_A^-) > f(\alpha_A^+)$ at $d = p$, then there exists either a $d^* > p$ such that $f(\alpha_A^-) = f(\alpha_A^+)$ (Case 1), or $f(\alpha_A^- = 0) > f(\alpha_A^+)$ (Case 2). Also, if $f(\alpha_A^-) = f(\alpha_A^+)$ for $d = p$ then existence is trivial.

Uniqueness for both cases follows from the proof of existence. Case 1: If $f(\alpha_A^-) = f(\alpha_A^+)$ at d^* , it follows from above that, $f(\alpha_A^-) < f(\alpha_A^+)$ for all $d > d^*$ (excluding a Case 2-type equilibrium), and $f(\alpha_A^-) > f(\alpha_A^+)$ for all $d < d^*$. Case 2: If $f(\alpha_A^- = 0) > f(\alpha_A^+)$ at d^* , $f(\alpha_A^-) > f(\alpha_A^+)$ for all $d < d^*$ (excluding a Case 1-type equilibrium); we showed above that $d > d^*$ cannot be an equilibrium.

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Proof of Proposition 5:

This proof follows the proof of Proposition 2.

Necessity: Assume an equilibrium, (g_A^*, g_B^*) , exists with $g_B^* > \alpha_m + c$. If candidate A deviates to $g'_A = g_B^* - 2c$, then all exterior voters will be in the support sets, including $\beta(g'_A, g_B^*, \alpha = 0, 1) = 2c$ and $g_m > \alpha_m$, which gives $n_f[S_A(g)] > n_f[S_B(g)]$ by Lemma 2 (iii). Other results are analogous.

Sufficiency: If g_A^* or g_B^* are interior to $[\alpha_m - c, \alpha_m + c]$, then $|g_A^*, g_B^*| < 2c$ and $n_f[S_A(g)] = n_f[S_B(g)] =$ and turnout is zero. If $(g_A^*, g_B^*) = (\alpha_m - c, \alpha_m + c)$, the support sets will consist of exterior voters only, and since $g_m = \alpha_m$, $n_f[S_A(g)] = n_f[S_B(g)]$ by Lemma 2 (iii).

As in Proposition 2, any deviation that leaves $n_f[S_A(g)'] \neq n_f[S_B(g)']$ will leave the deviator worse off.

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6 Appendix B: Finite Number of Citizens in the Linear Model

As discussed in the introduction, I use an infinite number of voters only as an approximation of a large N election. In this section, I show that with linear utility over policy, an equilibrium with positive turnout exists for any finite distribution of citizens.

First, some notation and setup:

There are N citizens ($N \geq 2$); the citizens and candidates are identical to those in the previous model. For simplicity, I normalize the policy space so that $\alpha_1 = 0$ and $\alpha_N = 1$. I only consider the case in which $2c < 1$ (after normalization).

Definitions

- (1) $n[(a, b)]$ is now the number of citizens with ideal policy points in $[a, b]$, rather than the probability measure of $[a, b]$. Let $n(g_A) \equiv n(0, g_A)$ and $n(g_B) \equiv n(0, g_B)$.
- (2) Let α_{g_A} equal the maximum ideal point in $[0, g_A]$ (i.e. $\max\{\alpha_i \in [0, g_A]\}$), and α_{g_B} equal the minimum ideal point in $[g_B, 1]$ (i.e. $\min\{\alpha_i \in [g_B, 1]\}$).

Lemma 4 *A sufficient condition for an equilibrium with positive turnout in pure NE strategies is the existence of an interval on $[0, 1]$, S^* , such that (i) $n(0, \inf(S^*)) = n(\sup(S^*), 1)$ and (ii) $|\inf(S^*), \sup(S^*)| = 2c$.*

Proof: Take $\inf(S^*) \equiv g_A^*$ and $\sup(S^*) \equiv g_B^*$. Again, interior citizens will abstain due to indifference and exterior citizens will vote. By the same logic of Proposition 2, candidates cannot gain additional votes by moving farther away from the median (in fact they can only lose votes by doing this). If they move closer to the median, then all citizens will abstain and the candidates will remain in a tie.

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Lemma 4 gives a sufficient condition for an equilibrium with positive turnout (S^*), but does not show when an S^* exists. The following proposition shows that under fairly general conditions (no perfect overlap of policy preferences) there exists an S^* that satisfies Lemma 4.

Proposition 6 *A sufficient condition for the existence of an equilibrium with positive turnout given a finite distribution of citizens is that there is no overlap in citizen's policy preferences; i.e. $\alpha_i \neq \alpha_j \forall i \neq j$.*

Proof: The proof proceeds as follows: take any $S \subset [0, 1]$ with $\inf(S) \equiv g_A$ and $\sup(S) \equiv g_B$ and $|g_A, g_B| = 2c$. I will show that, given no overlap, S can always be “shifted” (I use shift

to indicate a move to new interval S' , also with $|g'_A, g'_B| = 2c$) to increase (or decrease) $|n(g_A^*) - n(g_B^*)|$ by one. Therefore, by an induction-type argument, we can always find a set S^* s.t. $|n(g_A) - n(g_B)| = 0$.

To show that $|n(g_A) - n(g_B)|$ can always be increased by one, I consider two cases separately:

Case 1: $g_B = \alpha_B$. Shift S rightward by less than $\min\{|g_A, \alpha_{A+1}|, |g_B, \alpha_{B+1}|\}$. $|n(g_A) - n(g_B)|$ will increase by one since $n(g_A)$ stays constant and $n(g_B)$ decreases by one (α_{g_B} is now in the interior).

Case 2: $g_B \neq \alpha_B$. Shift S rightward by $\min\{|g_A, \alpha_{A+1}|, |g_B, \alpha_{B+1}| + \epsilon\}$ where ϵ is small enough.¹⁰ If the first term is smaller, then $n(g_A)$ increases by one and $n(g_B)$ stays constant. If the second term is smaller then $n(g_A)$ stays constant and $n(g_B)$ will decrease by one.

Together, Cases 1 and 2 show that $|n(g_A) - n(g_B)|$ can always increase by one. The proof for decreasing $|n(g_A) - n(g_B)|$ by one is symmetric.

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Proposition 6 shows a sufficient condition on the distribution of citizen's preferences such that an equilibrium with positive turnout exists. I argue that the condition of no overlap is actually quite general, since it will be satisfied almost surely for any finite set of citizens whose preferences are drawn from a continuous distribution.

While S^* need not be unique, note that $n(g_A)$ and $n(g_B)$ move in opposite directions as S is shifted. This, in turn, implies that $\alpha_{g_A}^*$ and $\alpha_{g_B}^*$ are unique; i.e. the set of citizens who vote will be the same for all S^* .

¹⁰Where $\epsilon < \min\{||g_A + ||g_B, \alpha_{B+1}||, \alpha_{A+1}||, ||\alpha_{B+1}, \alpha_{B+2}||\}$ to ensure that $n(g_A)$ stays constant and $n(g_B)$ decreases by no more than one. Also, note that $||g_A + ||g_B, \alpha_{B+1}||, \alpha_{A+1}|| > 0$ when $|g_A, \alpha_{A+1}| > |g_B, \alpha_{B+1}| + \epsilon$.

