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DEPARTMENT OF ECONOMICS

Three Essays in Macroeconomics

Miguel Sánchez-Martínez

Thesis submitted for assessment with a view to obtaining the degree
of Doctor of Economics of the European University Institute

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Department of Economics

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A mi madre.

Abstract

This thesis comprises three thorough investigations into some of the economic issues that rank high in the current policy agenda. The common nexus between them is the long-horizon nature of the subjects covered, in which economic growth plays a central role. The time and effort devoted into these pieces of research has aimed at casting light on yet open questions, such as: to what extent financial integration among countries is favorable for economic development, what implications aggregate population growth has for the natural environment via its interaction with the economic system and how fiscal policy should be designed to correct for market failures when the latest insights on individuals' consumption behavior are considered. The study of the specific topics addressed throughout each chapter of this thesis delivers a number of important results that are summarized subsequently.

The first chapter provides an analysis of the relationship between aggregate economic uncertainty, growth and welfare. The conditions under which it is beneficial for a country to liberalize its financial markets are examined, contrary to the previous theoretical literature, in the framework of a simple growth model featuring an endogenous R&D sector. The model generates two main predictions. First, consumption growth is, *ceteris paribus*, higher on average in a country that participates in the international financial markets compared to one that is financially autarkic, contradicting the main result of the reference model in the literature. Second, the sign of the net effect of financial integration on overall social welfare is ambiguous. Furthermore, the result that higher uncertainty has a neutral impact on optimal savings when preferences are logarithmic, established in preceding contributions, no longer holds in the context of the model presented here.

In the second chapter, an inquiry is made into the implications of population pressure for an optimal economic management problem with pollution externalities. The presence

of a potential irreversibility in the dynamics for the stock of pollution is responsible for an outcome overlooked by the previous literature, namely, the possibility that the economy lands an economic-environment trap for high population values. In line with the available empirical evidence, the model implies that a growing population leads to an increasingly deteriorated state of environmental quality. In addition, we examine, following a demographic shock, the shape of the transition paths and their interaction with threshold effects .

Finally, the third chapter explores an optimal taxation problem in the presence of both congestion and habituation effects in consumption in an economy featuring endogenous growth. The quantity of the final good, which can be thought of as a public facility, is subject to a certain degree of rivalry, losing thus its purely public nature. This introduces a negative consumption externality that is internalized via fiscal policy. In addition, individuals exhibit a process of habituation in their consumption standard, whereby the utility they derive from current consumption is negatively affected by past consumption. This, in turn, implies a strong preference for smoothing out the evolution of consumption across time, irrespective of the functional form of the utility function. The results show that the second-best solution entails a constant consumption tax rate in the long-run and a counter-cyclical consumption tax growth rate in the short-run whose level hinges on the importance of both congestion and habit effects in preferences.

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Contents

Chapter 1

Risk diversification and growth with endogenous technological change

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Keywords: Autarky, Integration, Endogenous Growth, R&D, Aggregate Risk, Precautionary Savings, Welfare.

1

1.1   Introduction

The question of the three-way interdependence between aggregate risk, economic growth and social welfare has been approached in the literature from angles different from the one proposed here. The results of these studies differ significantly from one another due to the use of disparate frameworks and a distinct set of modelling assumptions. Next, a brief review of the main literature, both theoretical and empirical, on the topic of risk and growth, broadly defined, is provided.

Devereux and Smith (1994) analyze the impact on the consumption growth rate of risk diversification among countries in the context of an endogenous growth model of human capital accumulation. Employing a simple structure of uncertainty and focusing on completely antagonistic environments, they conclude that risk sharing is detrimental to growth because it leads agents to cut back on precautionary savings, the accumulation of which is the engine of growth in the economy they present. The effect of the transition towards financial market integration on overall welfare is, however, ambiguous. The model presented below borrows the structure of risk used by these authors and applies it to a model of growth featuring endogenous technical change. The mechanisms at work in this model differ in important ways from the ones present in Devereux and Smith (1994). In particular, the result concerning the effect of integration on consumption growth is reversed and, while we find that the effect on welfare is also indeterminate in our case, this is driven by other economic factors than the ones at play in their analysis.

Obstfeld (1994) presents a model of growth in which expected consumption growth hinges on the share of stochastic wealth that agents invest in high-yielding risky activities. Portfolios with a higher weight in these risky investments translate into higher average growth. The source of risk is in the level of wealth of the representative agent. Two possible equilibria are isolated: one in which a positive amount of riskless assets is held in equilibrium and one in which only risky assets are held. In the first type of equilibrium, a decrease in risk due

to economic integration with other countries unambiguously leads to an increase in average consumption growth, while in the second possible equilibrium the effect of a reduction in risk on growth is ambiguous and depends on the intertemporal elasticity of substitution. Nevertheless, economic integration increases welfare in either case. Although it can be accredited as the first model to point out the possibility that integration can lead to an increase in equilibrium consumption growth, the mechanisms at work and the conclusions on welfare differ substantially when growth in the economy is driven by the accumulation of productive innovations developed in an R&D sector.

Following the two theoretical models above, Ramey and Ramey (1995) conduct an empirical study consisting of a cross-country analysis where mean growth is regressed against different measures of volatility, finding a strong and negative relationship between them. Their main findings are: first, there is a strong negative growth-risk correlation in all samples, second, the relationship between volatility and growth does not change after including investment[1], and, third, unpredictable policies hurt growth in a statistically significant way. A number of robustness checks are carried out, all of them corroborating the negative link.

A two-sector model with human capital as a factor of production is presented in Feeney (1998). In this study, the existence of complete international financial markets speeds up convergence to the steady-state growth rate because it enables agents to fully specialize in the sector in which the country has a comparative advantage more quickly than when they cannot diversify risk. Agents supply a strictly positive amount of labor to the two sectors in the autarkic equilibrium as a means to diversify away income risk. When this diversification is provided by international markets upon liberalization, the latter motive for providing labor services to the two sectors does not exist and there remains only one incentive for individuals: namely, work only in the sector where they are most productive. The main

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[1]meaning that just a small part of the effect of risk on growth can be attributed to physical capital investment alone. This is in stark contrast with the precautionary savings argument put forward by Devereux and Smith (1994).

drawback is that the learning process is a pure externality and, thus, the economic intuition behind the growth process in this model is left unexplained.

Another contribution featuring two sectors (consumption and education) is Femminis (2001). A key assumption in this model is that the production of goods uses physical capital more intensively than the production of education. Output in the education sector is human capital. Human capital accumulation is the engine of growth. Risk is detrimental to growth because it leads to higher precautionary savings, which increase the physical relative to human capital stock and renders labor more productive in the goods sector. Thus, risk amelioration has a positive impact on growth by allowing more labor resources to be employed in the production of human capital. This model is an extension of the baseline model in Devereux and Smith (1994); it preserves the same main mechanism but augments it by making the dynamics of human capital accumulation endogenous.

Krebs (2003) introduces the government into a very similar model of human-capital-accumulation-driven growth. A government-induced reduction in the risk of return to human capital risk induces agents to invest a higher share of their portfolio in human capital at the expense of physical capital, thereby increasing the expected return of this portfolio. This generates higher expected income growth Thus, the key difference from the rest of the investigations discussed so far is that risk is reduced by government action instead of by participating in insurance markets. The fact that the main result of the paper holds only for a reduced set of preferences is an important shortcoming.

A departure from the assumption of independently and identically distributed (i.i.d.) shocks is done in Wilson (2004), where returns on risky assets are, instead, equally distributed and bear some degree of correlation. Proceeding this way, the author argues that countries which have inherently less diversification opportunities than others may actually have higher growth rates, all else equal. Financial integration, which is synonym with lower correlation of asset returns, can, by means explained in their model, lead to lower expected growth

compared to a situation with undiversifiable income risk. The key for this to hold is that the stochastic processes that drive the returns on the assets exhibit some degree of positive correlation.

The model presented below proposes a new mechanism through which aggregate risk may affect consumption growth and social welfare by combining the structure of risk present in Devereux and Smith (1994) with a discrete-time standard growth model of endogenous technological change. This attempt is, to my knowledge, not yet part of the previously mentioned literature. While most of the models feature endogenous growth, the majority of them are based on learning-by-doing spillovers or human capital accumulation; an examination of the potential consequences of uncertainty has not been conducted, as of yet, in a model-economy where growth is driven by endogenous investment in R&D. Such a study sheds light on new channels through which risk sharing can influence growth and welfare. First, the analysis below shows that perfect risk sharing drives the variance of the returns to R&D to zero. This makes it more attractive to allocate labor to this sector instead of the consumption good sector, thus fostering economic growth. As it will become apparent in the next section, although the R&D technology is not intrinsically stochastic, the future returns to this activity are subject to uncertainty due to unknown future demand arising from final good producers. This uncertainty vanishes completely once agents can insure themselves perfectly against it. Second, the presence of different market failures, as discussed below, will yield welfare outcomes that depend on the relative strength of these failures and, hence, on the fundamental parameters of the model. Lastly, as an ancillary result that helps explain the first one, it is shown that risk in the return to investment does have an impact on the optimal consumption behaviour of agents even in the case of logarithmic preferences. In particular, greater uncertainty leads to lower R&D investment in equilibrium and higher current consumption, thus hampering growth in consumption per capita. This finding modifies the previously held view that optimal saving decisions are independent of the level of uncertainty for this type of utility function (Levhari and Srinivasan (1969)).

This chapter is organized as follows. In Section 1.2, the model is described and the equilibria of the two hypothetical scenarios considered are derived. In Section 1.3, the social optimum is obtained by solving the Social Planner's problem. In Section 1.4, a comparative analysis of equilibrium growth, socially optimal growth and welfare in the two case-scenarios is made. Finally, Section 1.5 contains concluding remarks.

1.2   Model

A stylized general equilibrium model belonging to the class of variety-expanding endogenous growth models *a la* Romer (1990) is presented in this section. The key new ingredient is the presence of aggregate risk in the form of a final good endowment shock that is proportional to the economy's state in every period. For simplicity, both time and the number of varieties are discrete. Results are not sensitive to this choice of modelling strategy. The economy is populated by a continuum of homogeneous households of measure one whose size is assumed constant (zero population growth). In any given period, the individual labor endowment is normalized to unity. Finally, the households labor supply is assumed to be infinitely inelastic and hence equal to one in every period.

We will distinguish two polar opposite market environments: one in which the countries' financial markets are shut to the outside world (Financial Market Autarky or FMA) and another one in which assets can be costlessly exchanged among countries (Financial Market Integration or FMI). We begin by studying the former scenario and then move on to the analysis of the latter so as to examine how equilibrium outcomes differ between them in the next section.

Financial Market Autarky

In this scenario, all idiosyncratic risk borne by countries is uninsurable. An exogenous endowment shock takes place every period which determines the final total quantity available

of the consumption good on top of what has been produced during the same period. This total quantity can either be equal, lower or greater than the output produced by final good firms. In turn, this means that, in equilibrium, the growth rate of consumption is stochastic and revolves around a deterministic mean. Let us first describe the structure of the economy and solve the optimization problems faced by all agents.

Preferences   The period felicity function is assumed to be logarithmic, with parameter $\beta \in (0,1)$ representing the subjective time discount factor. The problem solved by the representative household is:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \ln (c_t)$$

$$s.t. \ a_{t+1} = (1 + r_t) a_t - c_t + w_t + \varepsilon_t (S_t)$$

where a_t are assets owned by individuals at time t and r_t is the return on these assets in terms of units of the final good. These assets are in the form of patents because it is households that own the firms engaged in R&D. Upon invention of a new variety, they are entitled to a patent of infinite length that protects them against competitors in the production of the newly invented intermediate good. We also abstract from the possibility of leapfrogging. Each individual is endowed with one unit of labor which is supplied inelastically in every period. Hence, w_t stands for labor income.

Let $\varepsilon_t (S_t) = \gamma_t (S_t) n_t$ denote the endowment shock, which can effectively be thought of as *manna from heaven*. It is equally distributed among households and can either be negative (i.e., *stolen* units of the final good), positive or zero. We assume γ to be a (discrete) random variable whose value depends on the realized state of nature, S_t, at time t. There is a finite number N of possible, equally likely states for the realization of the endowment shocks, which follow an i.i.d. mean-zero process over time. In other words, each γ_t for each country corresponds to only one possible realization of S_t, which is drawn period after period from

a uniform distribution, and, on average, γ_t takes on the value zero. This amounts to saying that $\gamma_t(S_t) \sim U[a,b]$ with $E_t[\gamma_t(S_t)] = 0$ for all $t = 0, ..., \infty$, where a and b are chosen so as to be consistent with interior solutions. As regards the dynamics of $n_{t,}$, the number of varieties available at time t, they will become clear below. The reason why the shock is made proportional to the state of the economy, $n_{t,}$ is that it would otherwise become negligible as the economy grows large.

The first order conditions associated with this maximization problem read:

$[c_t]$

$$\lambda_t = \frac{1}{c_t}$$

$[a_{t+1}]$

$$\lambda_t = \beta E_0 [\lambda_{t+1}(1 + r_{t+1})] \implies \frac{1}{c_t} = \beta E_0 \left[\frac{1 + r_{t+1}}{c_{t+1}}\right]$$

where the latter equation is the usual Euler equation for consumption.

Technology

Final good sector: The production technology used to transform intermediate inputs into units of the final good is given by:

$$y_t = n_t^{\eta + 1 - \frac{1}{\alpha}} \left(\sum_{j=0}^{n_t} x_t(j)^\alpha\right)^{\frac{1}{\alpha}}$$

where $n_t \in \mathbb{N}$ is the number of varieties of intermediate goods available at date t and $x_t(j)$ are the units of intermediate good j employed at time t. This specification implies that the productivity of each type of input depends positively on the total number of different inputs developed up until date t. This is to say that there is a positive externality going from the R&D sector to the final good sector. Parameter η captures the size of this externality and will be assumed, for simplicity, equal to one from now onwards. The presence of such

positive externality is required to generate output growth over time. Let also the elasticity of technical substitution between any two varieties, $(\frac{1}{1-\alpha})$, be larger than one. This is the same as imposing $\alpha \in (0,1)$. The way in which intermediate goods are symmetrically aggregated inside the term in brackets is standard.

Competition in this sector is perfect and the representative firm, letting the consumption good act as the numeraire (so that its price is normalized to one), faces the following problem:

$$\max_{x_t(j)} \quad n_t^{2-\frac{1}{\alpha}} \left(\sum_{j=0}^{n_t} x_t(j)^\alpha \right)^{\frac{1}{\alpha}} - \sum_{j=0}^{n_t} p_t(j) x_t(j)$$

where we have already made use of the assumption that $\eta = 1$ and where $p_t(j)$ stands for the equilibrium price set by the monopolistically competitive firm in industry j. There is a number n_t of such industries at any time t (i.e., $j \in [0, n_t]$). The solution to this problem is:

$$n_t^{\frac{2\alpha-1}{1-\alpha}} y_t p_t(j)^{\frac{1}{\alpha-1}} = x_t(j)$$

Imposing symmetry of input demands (a result that will be shown later) gives $y_t = n_t^2 x_t$.

Intermediate goods sector: The technology here is simply assumed to be linear on labor: $x_t(j) = l_t(j)$. This assumption is useful to keep matters simple. As it was previously mentioned, firms in the input-producing sector are not price-takers but, rather, have some monopoly power over their prices which depends on the elasticity of substitution between varieties, α. The monopolistically competitive firms solve:

$$\max_{p_t} \quad \pi_t = p_t x_t - w_t l_t$$

$$s.t. \quad x_t = B_t p_t^{\frac{1}{\alpha-1}}$$

$$l_t = x_t$$

where firm indices have been omitted given the symmetry of the problem across firms and where $B_t = n_t^{\frac{2\alpha-1}{1-\alpha}} y_t$. By substituting the constraints directly into the objective function, the solution to this simple static problem yields $p_t = \frac{w_t}{\alpha}$, which is independent of j, hence proving the symmetry of prices charged and quantities produced by all intermediate good firms. Period profits stay: $\pi_t = (1-\alpha) p_t x_t$.

R&D sector: Innovation, in the form of successive increments in the variety index n_t, happens according to the following technology of production:

$$n_{t+1} = bn_t \left(1 - L_t + \tfrac{1}{b}\right)$$

where $b > 0$ is the marginal productivity of labor devoted to R&D and $L_t = \sum_{j=0}^{n_t} l_t(j)$ denotes the share of labor allocated to the production of the final good. Note the positive externality of the *standing on the shoulder of giants* type; all else equal, a higher number of varieties today automatically translates into a higher number of varieties tomorrow (except in the corner solution $L_t = 1$). That is, even if the number of researchers and their productivity remained fixed, the innovation process would never stop because n_t would still grow, in this case, at a constant rate.

There is also perfect competition in this sector, meaning that, in equilibrium, the following arbitrage or zero-profit condition must be satisfied: $v_t = \frac{w_t}{bn_t} = \frac{\alpha p_t}{bn_t}$. The interpretation of this equation is that the entry-exit flow of firms into R&D will differ from zero (so that equilibrium is not yet reached) until the benefit or value of a new patent (the left-hand side) exactly equalizes the total cost of creating a new invention (the right-hand side).

Given that patents are assumed to be infinitely-lived and we abstract from the possibility of leapfrogging, the (symmetric) present discounted market value of a patent for R&D firms, v_t, is:

$$v_t = \sum_{s=t}^{\infty} \prod_{x=t}^{s} \left(\tfrac{1}{1+r_x}\right) \pi_s$$

where time is discounted exponentially . Using this definition to compute the difference $v_{t+1} - v_t$ yields the following expression for the interest rate:

$$r_t = \frac{v_{t+1} - v_t + \pi_t}{v_t} = \frac{v_{t+1}}{v_t} - 1 + \frac{\pi_t}{v_t}$$

Making use of the following relationships derived from the optimality condition for final good firms, the optimality condition for intermediate good firms, and the zero-profit condition for R&D firms,

$$\begin{cases} \frac{\pi_t}{v_t} = \left(\frac{1-\alpha}{\alpha}\right) bL_t \\ \frac{v_{t+1}}{v_t} = 1 \end{cases}$$

the equilibrium interest rate finally reads: $r_t = r(L_t) = \left(\frac{1-\alpha}{\alpha}\right) bL_t$

Equilibrium   Risk will influence the economic outcomes at the aggregate level through the resource constraint, which reads:

$$c_t \leq y_t + \varepsilon_t(S_t)$$

An important equilibrium result is that the value of assets owned by households at any date t must be equal to the market value of all patents developed up to time t: $a_t = n_t v_t$. In effective terms, the representative household is thus the owner of all patent-holding firms (i.e., all intermediate good producing firms). Nonetheless, since v_t is constant across all time periods and inputs, the only one state variable in this economy is the index of input types, n_t, with law of motion as previously specified.

Taking into account the symmetric behavior of the manufacturers of inputs regarding their output, x, and labor, l, decisions, the labor market clearing condition can be written as $x_t n_t = L_t$. This, in turn, simplifies the expression of the final output production function to $y_t = n_t L_t$. At this point, we assume that the same wage is paid to workers in the final good

and R&D sectors. After some simple algebraic manipulation of the objects derived so far, we obtain the following set of equilibrium equations:

$$\begin{cases} \frac{1}{c_t} = \beta E_0 \left[\frac{1+r_{t+1}}{c_{t+1}} \right] \\ n_{t+1} = bn_t \left(1 - L_t + \frac{1}{b} \right) \\ c_t = n_t \left(L_t + \gamma_t(S_t) \right) \\ r_t = \left(\frac{1-\alpha}{\alpha} \right) bL_t \end{cases}$$

In order to study the equilibrium dynamics of the variables of interest, let us define the stationary variable $z_t = \frac{n_t}{c_t}$ and reduce the system of two difference equations on (c_t, n_t) to a single difference equation on the newly defined variable, which, after substituting away the equilibrium interest rate and the resource constraint, reads:

$$bz_t \left(1 - \frac{1}{z_t} + \gamma_t + \frac{1}{b} \right) = \beta E_0 \left[\left(1 + \left(\frac{1-\alpha}{\alpha} \right) b \left(\frac{1}{z_{t+1}} - \gamma_{t+1} \right) \right) z_{t+1} \right]$$

where we have slightly abused notation by implicitly using the identity $\gamma_t \equiv \gamma_t(S_t)$ (which applies henceforth). Given the non-linear nature of the latter equation on z_t, we can proceed by using a first-order Taylor approximation of the term inside the expectations operator around the deterministic steady state of the economy[2]. The result of this linearization yields:

$$z_t = \frac{1}{1 + \gamma_t + \frac{1}{b}} + \frac{\beta}{b \left(1 + \gamma_t + \frac{1}{b} \right)} E_0 \left[z_{t+1} \right] + \frac{\beta \left(1 - \alpha \right)}{\alpha \left(1 + \gamma_t + \frac{1}{b} \right)}$$

Forward iteration on this equation gives:

$$z_t = \frac{b(\alpha + \beta(1-\alpha))}{\alpha(1+b-\beta)} - \left(\frac{b^2(\alpha + \beta(1-\alpha))}{\alpha(1+b)(1+b-\beta)} \right) \gamma_t + \left(1 + b \left(1 - \gamma_t \right) \right) \lim_{T \to \infty} \frac{\beta^T}{(1+b)^T} E_0 \left[z_T \right]$$

From the transversality condition associated with the consumer's problem, it follows that the last term of the right hand side is zero and thus we have obtained x_t as a function of γ_t only. Recalling that $z_t = \frac{n_t}{c_t}$, an analytical expression for the policy function of consumption ensues immediately from the previous equation:

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[2]see Appendix A for derivations

$$c_t = f\left(\gamma_t, n_t\right) = \frac{n_t\alpha(1+b)(1+b-\beta)}{b(\alpha+\beta(1-\alpha))(1+b(1-\gamma_t))}$$

In turn, the derivations of the expressions for gross consumption growth and the policy function for labor are straightforward:

$$\frac{c_{t+1}}{c_t} = f\left(\gamma_t, \gamma_{t+1}\right) = \frac{(1+b(2+b(1-\gamma_t)(1+\gamma_t)))(\alpha+\beta(1-\alpha))-\alpha(1+b)(1+b-\beta)}{\left(1+b\left(1-\gamma_{t+1}\right)\right)(\alpha+\beta(1-\alpha))}$$

$$L_t = f\left(\gamma_t\right) = \frac{\alpha(1+b)(1+b-\beta)}{b(\alpha+\beta(1-\alpha))(1+b(1-\gamma_t))} - \gamma_t$$

Financial Market Integration

This scenario corresponds to full integration of international asset markets. This entails the absence of barriers in the exchange of assets across countries, which effectively amounts to the idea that worldwide financial markets work indeed as one. The aggregate world endowment shock in equilibrium is assumed to be equal to zero. This means that country-specific shocks must be perfectly negatively correlated since, for the former assumption to hold, it must be the case that, when summing up in every period the purely random units of the final good available across countries, they exactly cancel out with one another. Simplicity and our main goal of comparing completely antagonistic scenarios are the reasons why the baseline model is set up such that $\sum_{i=1}^{T} \varepsilon_{it}\left(S^j\right) = 0$ for all $j = 1, ..., N$, $t = 1, ..., \infty$. The other important assumption we make is the existence of a complete set of state-contingent securities equal in number to the states of nature, N.

<u>Preferences</u> Define $q_t\left(S_{t+1}, S_t\right)$ as the price of a bond that pays one unit of the final good if state S_{t+1} occurs tomorrow and zero otherwise, given state S_t today. Let $b_{it}(S_{t+1}, S_t)$ be the number of such bonds held by the representative household in country $i = 1, ..., T$. The budget constraint can thus be written as:

$$a_{it+1} = (1 + r_t) a_{it} - c_{it} + w_{it} - \sum_{j=1}^{N} q_t \left(S^j, S_t\right) b_{it}(S^j, S_t) + b_{it-1}(S_t, S_{t-1})$$

The representative household solves the following programme:

$$\max_{\{c_{it}, a_{it+1}, b_{it}(S^j, S_t)\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \ln \left(c_{it}\right)$$

$$\text{s.t.} \quad a_{it+1} = (1 + r_t) a_{it} - c_{it} + w_{it} - \sum_{j=1}^{N} q_t \left(S^j, S_t\right) b_{it}(S^j, S_t) + b_{it-1}(S_t, S_{t-1}) + \varepsilon_{it} \left(S_t\right)$$

The first-order conditions of this problem are:

$[c_{it}]$

$$\lambda_t = \frac{1}{c_{it}}$$

$[a_{it+1}]$

$$\lambda_t = \beta E_0 \left[\lambda_{t+1} \left(1 + r_{it+1}\right)\right] \implies \frac{1}{c_{it}} = \beta E_0 \left[\frac{1 + r_{it+1}}{c_{it+1}}\right]$$

$[b_{it}(S^j, S_t)]$

$$-\lambda_t q_t \left(S^j, S_t\right) + \beta \frac{1}{N} E_0 \left[\lambda_{t+1}\right] = 0$$

where the third equation follows from the fact that agents expect to get $\frac{1}{N} b_{it}(S^j, S_t)$ units of the final good tomorrow from their investment in bonds. Combining these optimality conditions yields:

$$\frac{1}{c_{it}} = \beta \frac{1}{N} E_0 \left[\frac{1}{q_t(S^j, S_t) c_{it+1}}\right]$$

This last expression is the arbitrage equation for the two types of assets available in the open economy, namely bonds and patents.

Technology   All countries share identical technologies in all sectors and these technologies are exactly the same as the ones in autarky, so all that was derived before carries over to the new open-economy scenario. We can then proceed to the analysis of the decentralized markets equilibrium.

Equilibrium Assuming no source of worldwide aggregate uncertainty, as previously mentioned (i.e., $\sum_{i=1}^{T} \varepsilon_{it}(S^j) = 0$ for all $j = 1, ..., N$) and taking into account that $\sum_{i=1}^{T} b_{it}(S^j, S_t) = 0$ for all (S^j, S_t) has to hold in a world market equilibrium, the world's resource constraint stays:

$$\sum_{i=1}^{T} C_{it}(S_t) \le \sum_{i=1}^{T} y_{it}(S_t) + \sum_{i=1}^{T} \varepsilon_{it}(S_t) + \sum_{i=1}^{T} b_{it}(S^j, S_t) \iff \sum_{i=1}^{T} C_{it}(S_t) \le \sum_{i=1}^{T} y_{it}(S_t)$$

The fact that all states of nature are equally likely and the fact that we assume away a common source of risk among countries imply that the price function is independent of the realized shock, and, therefore the same for all types of bonds. This, in turn, implies that consumption paths are symmetric and state-invariant. If we further assume a symmetric equilibrium with perfect pooling, then, from the resource constraint, we have $NC_{it} = Ny_{it}$ and $c_{it} = n_{it}L_t = y_{it}{}^3$. This means that the equilibrium structures of all the economies boil down to the one of the riskless, closed-economy baseline model.

After noticing the symmetry of countries (i.e., dropping country indices), the equilibrium set of equations reads:

$$\begin{cases} \frac{1}{c_t} = \beta \left[\frac{1+r_{t+1}}{c_{t+1}} \right] \\ n_{t+1} = bn_t \left(1 - L_t + \frac{1}{b} \right) \\ r_t = \left(\frac{1-\alpha}{\alpha} \right) bL_t \\ c_t = n_t L_t \end{cases}$$

To analyze this equilibrium, we proceed in the same way as we did before for FMA. Let $z_t = \frac{n_t}{c_t}$. This system can then be summarized in the following equation:

$$z_{t+1} = \underbrace{\left(\frac{1+b}{\beta} \right)}_{>1} z_t - \frac{b}{\beta} - \left(\frac{1-\alpha}{\alpha} \right) b$$

---

[3]satisfied with strict equality for, otherwise, idle resources would remain, which would violate optimality

The coefficient attached to x_t being greater than one implies, first, that there exists a unique solution to the difference equation (i.e., the system is non-explosive) as shown in Blanchard and Kahn (1980) and, second, that the steady state is globally asymptotically stable, as shown in Acemoglu (2009). These properties imply that there are no transitional dynamics and, hence, that the economy is always at is stationary steady state.

Forward iteration on the latter equation yields:

$$z_t = \frac{b(\beta(1-\alpha)+\alpha)}{\alpha(1+b-\beta)} + \lim_{T\to\infty} \left(\frac{\beta}{1+b}\right)^T z_T$$

Again, it follows from the transversality condition of the consumer problem that the last term in the right-hand side is zero.

The policy functions for consumption and labor as well as the gross growth rate of consumption are then given by:

$$c_t = \frac{n_t\,(1 + b - \beta)\,\alpha}{b\,(\alpha + \beta(1-\alpha))}$$

$$L_t = \frac{\alpha\,(1 + b - \beta)}{b\,(\alpha + \beta(1-\alpha))} = L_{ss}$$

$$\frac{c_{t+1}}{c_t} = \frac{n_{t+1}}{n_t} = \frac{\beta\,(1 + b\,(1-\alpha))}{\alpha + \beta(1-\alpha)}$$

A parameter restriction, $\beta \geq \frac{\alpha}{b(1-\alpha)+\alpha}$, is needed for $0 \leq L_{ss} \leq 1$ and for consumption tomorrow to be greater than consumption today.

In brief, we conclude that the economy is always at its steady state and, therefore, growth is constant.

1.3   Social Optimum

Next, we solve the social planner's problem for the non-stochastic economy in order to analyze whether equilibrium consumption growth under FMI is *too* high or *too* low, depending on where it stands relative to its Pareto optimal counterpart.

The benevolent social planner maximizes social welfare subject to both the resource constraint and the R&D technology constraint. The problem that she solves can be written in recursive form as:

$$V\left(n\right) = \max_{\left\{c,n'\right\}} \left\{\log\left(c\right) + \beta V\left(n'\right)\right\}$$

$$\begin{aligned} s.t. \quad n' &= bn\left(1 - L + \frac{1}{b}\right) \\ c &= nL \end{aligned}$$

We can simplify this problem by combining the two constraints into one, which yields $n' = (n-c)b + n$. The original problem can then be rewritten as an unconstrained one:

$$V\left(n\right) = \max_{n'} \left\{\log\left(\frac{n-n'}{b} + n\right) + \beta V\left(n'\right)\right\}$$

The first order condition is:

$$\frac{-1/b}{c} + \beta\frac{\partial V\left(n'\right)}{\partial n'} = 0$$

The corresponding envelope condition reads:

$$\frac{\partial V\left(n\right)}{\partial n} = \frac{1/b + 1}{c} \implies \frac{\partial V\left(n'\right)}{\partial n'} = \frac{1/b + 1}{c'}$$

Putting these two conditions together gives the Pareto optimal growth rate:

$$\frac{c'}{c} = \beta(1 + b)$$

The sign of the difference between this growth rate and the competitive equilibrium one is ambiguous; the decentralized equilibrium growth rate may be inefficiently high, inefficiently low or efficient depending on a number of parametric conditions. The reason why these three possible cases emerge can be traced to the technology and market properties of the model. First, there is a positive external effect: the higher the output in the R&D sector, the higher the productivity of the intermediate inputs (or, equivalently, the higher the productivity of labor) in final good production. The size of this externality is captured by parameter η, which was set equal to 1. Since there are yet two more market imperfections, it is not possible to know whether $\eta = 1$ corresponds to too much or too little growth. A discussion of the role played by this externality will be provided later when commenting on the other fundamental parameters of the model. The second distortion is the monopolistically competitive nature of the intermediate-good sector. The size of this inefficiency is reflected in the mark-up charged by intermediate firms to final good producing firms and depends negatively on the degree of substitution between inputs in the final good production function. Finally, there is the *standing on the shoulder of giants* externality, which is present in the research sector and whose size is captured by the value of parameter b.

The factors that determine whether the competitive equilibrium growth rate is equal, higher or lower than the socially optimal one can be investigated by computing for which values of the elasticity of substitution between varieties (α), the subjective time discount factor (β) and the productivity of labor in the R&D sector (b), these three cases emerge, taking into

account the initial fundamental restrictions placed on them:

- Pareto optimal growth higher than competitive equilibrium growth: $\beta(1+b) > \frac{\beta(b(1-\alpha)+1)}{\beta(1-\alpha)+\alpha}$

  occurs when:

$$\begin{cases} 0 < \alpha \leq \frac{1}{2} \quad \cup \quad \frac{2\alpha-1}{\alpha-1} < \beta < 1 \quad \cup \quad b > \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)} \\ \qquad\qquad\qquad \frac{1}{2} < \alpha < 1 \end{cases}$$

- Optimal growth lower than competitive equilibrium growth: $\beta(1+b) < \frac{\beta(b(1-\alpha)+1)}{\beta(1-\alpha)+\alpha}$

  occurs when:

$$0 < \alpha < \frac{1}{2} \begin{cases} \qquad\qquad 0 < \beta \leq \frac{2\alpha-1}{\alpha-1} \\ \frac{2\alpha-1}{\alpha-1} < \beta < 1 \quad \cup \quad b < \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)} \end{cases}$$

- Optimal growth equal to competitive equilibrium growth: $\beta(1+b) = \frac{\beta(b(1-\alpha)+1)}{\beta(1-\alpha)+\alpha}$ only

  occurs when:

$$0 < \alpha < \frac{1}{2} \quad \cup \quad \frac{2\alpha-1}{\alpha-1} < \beta < 1 \quad \cup \quad b = \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)}$$

These conditions have an intuitive economic interpretation. First, optimal growth can never be lower than or equal to decentralized market equilibrium growth for high degrees of substitutability across varieties (i.e., for $\alpha \geq \frac{1}{2}$), since these are associated with low equilibrium mark-ups set on the input variety prices and, hence, a relatively unimportant presence of inefficiencies due to imperfect competition. In this case, the market produces *too* little growth, because the equilibrium growth rate lies on a region where the marginal social benefit of an additional unit of consumption is strictly higher than the marginal cost it generates in terms of the subsequent loss in consumer surplus. Therefore, this corresponds to the case where the positive welfare impact of the externality of output in the R&D sector, whose size is given by the marginal product of labor in this sector, b, outweighs the negative welfare impact of monopoly pricing. Second, the competitive equilibrium growth rate will be

higher than the Pareto optimal one in the presence of high monopoly power $(0 < \alpha < \frac{1}{2})$ either when the productivity level in R&D is not sufficiently high to compensate for the negative monopolistic distortion when the time discount rate is high or when this discount factor is sufficiently low no matter what the productivity level in the R&D sector is. Thus, the market delivers *too* much growth because either the welfare burden that monopolistic competition imposes on consumers is either too acute compared to the positive effect due to the productivity of researchers or the representative agent is too impatient for her to prefer higher levels of consumption in the future when there is such an important loss of consumer surplus that accrues to intermediate good producing firms. Third, the market equilibrium growth rate will fall short of the optimal one whenever the productivity of researchers and the patience of households are sufficiently high, no matter the level of monopoly power (α). This is what one would expect to happen, inasmuch as a high productivity in the R&D sector translates into further increments in both the productivity of the inputs used in the final good sector and in the productivity of future researchers[4].

1.4   Comparing FMA and FMI

In this section, an analysis of the differences in the growth and welfare outcomes between the two scenarios is provided.

Let us first begin with the comparison of the average growth rates in the two cases. Up to a second-order Taylor approximation of $E_0 \left[\frac{c_{t+1}}{c_t} \right]$ around the deterministic, saddle-path-stable steady state, consumption growth is higher in FMI than in FMA for all possible values of the structural parameters of the economy:

$$\left[\frac{c_{t+1}}{c_t} \right]_{FMI} - E_0 \left[\frac{c_{t+1}}{c_t} \right]_{FMA} = b^2 var\left(\gamma \right) \frac{\alpha(1+b-\beta)}{(1+b)^2(\alpha+\beta(1-\alpha))} > 0$$

---

[4]This effect is not fully internalized by the market

This in sharp contrast to the main result in Devereux and Smith (1994), where savings are depressed when a country opens up its financial borders. Instead, what is at work in our model is that the risk reduction that capital market integration permits causes people to invest more in R&D, thereby creating new patents at a faster rate and ultimately bolstering growth in per capita consumption. Even though the arrival rate of new inventions is in itself not subject to uncertainty, the return to R&D effort, reflected in the interest rate one period ahead, is. In particular, as apparent from the equation for the equilibrium interest rate, this return depends in a non-linear way on the one-period-ahead's endowment shock. It can be shown that, up to a second-order approximation, the interest rate is on average higher in FMA than in FMI. However, an increase in the volatility of the shock leads to both a higher mean (due to an increase in the risk premium, which is equal to the difference between the expected interest rate in FMA and FMI) and a higher variance of the return. Yet, the consumption growth gap between the two scenarios widens after such increase. This can only be true because second-order effects (i.e., increase in the asset's rate-of-return risk) dominate first-order effects (i.e., increase in the expected return), implying that, compared to the riskless case, the representative agent will allocate more labor resources to the final good sector at the expense of the R&D sector and will thus consume more units of the final good today .

These effects and the mechanisms behind them can be analyzed in the framework proposed in Sandmo (1970). In this context, an increase in risk, captured by an increase in the volatility of the return to R&D, has two effects on current consumption, the relative sizes of which reflect the importance of the precautionary motive for savings. First, there is a substitution effect, by which

> "...an increase in the degree of risk makes the consumer less inclined to expose his resources to the possibility of loss; hence the positive substitution effect on consumption..."

Second, there is an income effect, by which

> "...higher riskiness makes it necessary to save more in order to protect oneself
> against very low levels of future consumption; hence the negative income effect
> on consumption..."[1]

As proved in Appendix B, the substitution effect outweighs the income effect and, thus, an increase in current period consumption always follows after a mean-preserving spread in the distribution of the endowment shock. This outcome is however contrary to what both Sandmo (1970) and Levhari and Srinivasan (1969) find, namely, that the substitution and income effects exactly cancel out when the instantenous utility function is logarithmic. The reason behind these opposite predictions is related to the fundamental differences in the income sources of the representative household. As opposed to the structure of the budget constraint in Sandmo (1970), in our model consumption and savings can also be financed via an additional, safe income source accruing from labor services sold to both the final good and R&D sectors. Labor income in $t + 1$ is known in period t, since $w_{t+1} = \alpha n_{t+1}$ and n_{t+1} is a function of γ_t. The latter is realized before optimal decisions are made in t and, hence, it is known. This labor income, on top of the risky *capital* income, is not present in any of the two previous studies. The presence of a deterministic wage helps to compensate for the stochastic part of income and, thus, makes the risk-averse representative agent even less inclined to save for a precautionary motive given that the only one savings technology available is risky.

It can further be shown that, ceteris paribus:

$$\begin{cases} \dfrac{\partial \left[\frac{c_{t+1}}{c_t}\right]_{FMI} - E_0\left[\frac{c_{t+1}}{c_t}\right]_{FMA}}{\partial \beta} < 0 \\ \dfrac{\partial^2 \left[\frac{c_{t+1}}{c_t}\right]_{FMI} - E_0\left[\frac{c_{t+1}}{c_t}\right]_{FMA}}{\partial \beta^2} < 0 \end{cases}$$

This means that the more patient the representative agent, the smaller the positive difference between growth in FMI and FMA and the higher the rate at which this difference shrinks.

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[1] Sandmo (1970), pages 357-358

In the limit, as $\beta \to 1$, the difference remains still positive but reaches a minimum. The explanation for this is that, in FMA, the risk and R&D investment trade-off shifts to a more favourable attitude towards investment as the representative agent places a greater value in future payoffs for high values of β. This is the same as saying that the risk premium demanded decreases with the time discount factor, which can be shown to be the case. Hence, she will tend to find it optimal to engage in more research projects when she does not discount the future highly, since these investments will yield high returns in the periods to come and they weigh almost as much as today's returns in utility terms.

The previous discussion notwithstanding, it does not follow that welfare in FMI is necessarily higher than in FMA precisely because of the factors at play highlighted before on the analysis of the Pareto optimality of the competitive equilibrium growth rate: higher consumption growth does not automatically make individuals better off, for this growth may be excessive. In actuality, after second-order approximating the total welfare function in FMA, the difference in welfare levels across the two scenarios can be written as[5]:

$$ WFMA - WFMI = b^2 var\left(\gamma\right) \left[\frac{(1+b(1-\alpha))(2\beta-1)+2\alpha(1+b-\beta)}{2(\beta-1)(1-\beta)(1+b)^2(1+b(1-\alpha))} \right] $$

where $WFMA$ ($WFMI$) stands for total welfare in FMA (FMI). The sign of this expression is indeterminate and, in turn, closely related to the difference between the socially optimal and competitive equilibrium growth rates in FMI. To illustrate this point, let us proceed now by plotting two graphs in the (β, α, b) parameter space. The first one contains the set of values of these parameters for which welfare is higher in the integrated economy than in the autarkic one while the second one shows the set of values for which optimal growth is higher than or equal to decentralized equilibrium growth. Then, I will argue[6] that the second region is a subset of the first one. This means that it can never be the case that the representative agent is made worse off by moving from FMA to FMI when optimal growth is higher than or equal to decentralized equilibrium growth. This is due to the fact that there

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[5]see Appendix C for derivations

[6]an analytical proof is provided in Appendix D

is a net gain from integrating in this case: on one hand, financial integration is synonym with aggregate risk amelioration (let me refer to this effect as the "*risk channel*" henceforth) and, thus, it enhances the well-being of risk-averse individuals and, on the other hand, it leads to an equilibrium consumption growth rate which is closer to the Pareto optimal one compared to the equilibrium growth rate in FMA ("*growth channel*"). The latter effect also improves upon the welfare of the representative agent. Therefore, in this case, we have that both of these channels work in the same pro-integration direction.

An alternative interpretation of the fact that the second set is *smaller* than the first is that we may find combinations of the fundamental parameters of the economy for which the FMI market equilibrium delivers *too* much consumption growth from a social point of view but, still, there is a welfare gain to be made from the liberalization of capital markets. This is because there must exist parameter vectors for which WFMI>WFMA holds and, at the same time, optimal growth is strictly lower than FMI competitive equilibrium growth.

The regions in (β, α, b) space for which $WFMA < WFMI$ holds and for which optimal growth is higher than or equal to FMI equilibrium growth, respectively, are plotted in the following two graphs:
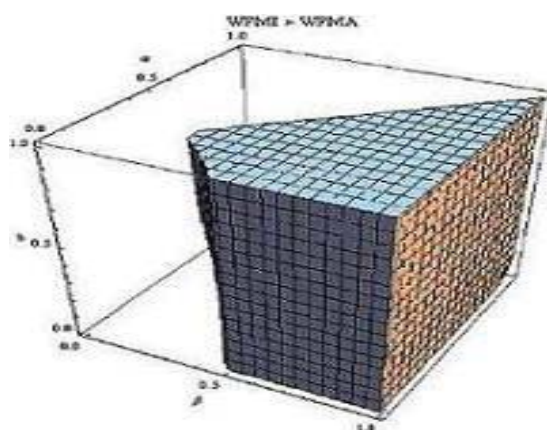

Figure 1: Parametric region in (β, b, α)-space in which welfare in FMI is higher than welfare in FMA.
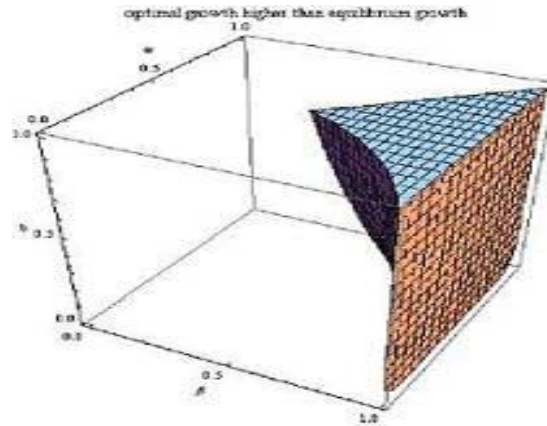
Figure 2: Parametric region in (β, b, α)-space in which optimal growth is higher than or equal to (FMI) equilibrium growth.

Visual inspection of these two graphs already gives a hint that the second set is indeed contained in the first one, meaning that our results are consistent with the previously discussed risk and growth effects. The ranges of parameter values shown in the figures have been chosen without loss of generality.

The following chart summarizes the arguments put forward thus far:

Possible cases	Pro-integration sign of channels		Welfare ordering
	risk channel ,	growth channel	
FMA growth<FMI growth≤ optimal growth	>0 ,	>0	WFMI>WFMA always
optimal growth≤FMA growth<FMI growth	>0 ,	?	WFMI>WFMA if |risk channel| > |growth channel| or if growth channel>0
FMA growth< optimal growth≤FMI growth	>0 ,	?	WFMI>WFMA if |risk channel| > |growth channel| or if growth channel>0

It follows from the previous analysis that we can find vectors of the fundamental parameters of the economy such that the FMI decentralized market equilibrium growth rate is exactly at the level where total welfare in the two scenarios coincides; at this rate, the risk and growth effects exactly cancel out and, therefore, the representative agent must be equally well-off in both scenarios. The expression for this growth rate, which lies above the socially optimal one, is:

$$\left[\frac{c_{t+1}}{c_t}\right]_{FMI, WFMA=WFMI} = \frac{\beta(1-3\alpha+2\beta(\alpha-1)+(1-\alpha)(2\beta+2\alpha(1-\beta)-1))}{(\beta(1-\alpha)+\alpha)(1-3\alpha+2\beta(\alpha-1))}$$

The level of productivity in the R&D sector implied by this growth rate is:

$$b_{WFMA=WFMI} = \frac{2\alpha(1-\beta)+2\beta-1}{1-3\alpha+2\beta(\alpha-1)}$$

It can be shown that, holding the other two parameters constant and departing from this productivity level of research, increases in it drive a positive and widening wedge between welfare under FMI and welfare under FMA, which means that $b > b_{WFMA=WFMI}$ ($b < b_{WFMA=WFMI}$) are associated with $WFMI > WFMA$ ($WFMA > WFMI$). This is also true for the remaining two parameters: for both α and β, low levels are associated, ceteris paribus, with higher welfare in FMA compared to FMI. In the case of α, both effects reinforce each other because an increase in this parameter starting from the level that equalizes total welfare in the two scenarios is followed by a reduction of both FMA and FMI growth, while having no effect on the optimal growth rate, which makes it clear that, above that critical value, it must always pay off to integrate on efficiency grounds. However, as regards the subjective time discount factor, β, we are unable to distinguish the interplay in the sign of the growth and risk effects because, whereas the FMA, FMI and optimal growth rates increase as a result of an increase in the critical β, the relative movement in the FMI and optimal growth rates is ambiguous, leaving the *pro* or *anti* integration sign of the growth effect indeterminate.

1.5   Concluding remarks

The model and the comparison between the two case-scenarios presented in this paper generate a number of interesting results, some of which confirm what has been previously addressed and some of which go against the predictions of a number of the most prominent examples in the literature. I will comment briefly on the two most important findings and point to possible extensions for future research.

First, we find that, no matter what the primitives of the economy are, average gross consumption growth is always higher in a financially integrated economy than in an autarkic one thanks to the elimination of risk that integration permits. This reduction in risk buttresses investment in R&D activities, the ultimate engine of growth in this model. However, when the time discount factor used by agents approaches one, the incentive to conduct research for future dividends is so high that the amount of resources allocated to research in FMA converges to the level in FMI, meaning that the growth difference between the two polar opposite environments can get arbitrarily small, albeit still positive. This is in line with the results of previous theoretical papers summarized in the first section.

Second, a higher growth rate of consumption does not necessarily imply higher overall welfare. In this model there are three market failures at work: two positive externalities stemming from the production of research, monopolistic competition in the intermediate good sector and imperfect risk sharing. The combination of the three makes it possible that the competitive equilibrium growth rate be equal, higher or lower than the socially optimal one depending on how important each of the effects at play is. The relevance of each of these effects is, in turn, reflected in the values of the different fundamental parameters. This is just as true for the possible welfare orderings that may arise; whenever integrating financial markets leads to an excessive growth rate according to the Pareto optimum criterion, a benevolent social planner has to weigh, in welfare units, the advantage of integration (namely, the elimination of the imperfect risk sharing inefficiency) against this *too-much-growth* disadvantage, conditional on the relative weight of the aforementioned market failures. To analyze this, a *growth* and a *risk* channel were identified in the previous section. From that analysis, it followed that it may well be the case that integration is called for when there are strong risk amelioration arguments for it even when it leads to a growth rate above the optimum.

Hence, from a policy perspective, in light of this stylized model, a country's efforts to become

participant in the international capital markets may or may not be justified from a social welfare point of view. As discussed above, the answer will depend mainly on the inner technology and market characteristics of the economy in question. For instance, whenever the profits of monopolistically competitive firms are high, the overall productivity in the R&D sector is low and there are relatively small income fluctuations in the economy, then this model predicts that financial integration may not give rise to a welfare improvement; another economy with similar other features, but with higher intrinsic risk may, in contrast, find it beneficial to escape financial autarky. Also, when there are reasons to believe that consumption growth is far behind its potential (optimal) level and individuals are facing a high overall income uncertainty, then market integration will most probably raise the welfare of the representative agent, since in this case the *growth* and *risk* channels point in the same pro-integration direction.

As potential avenues for future research, the introduction of a more sophisticated risk structure into the economy that does not necessarily yield perfect risk sharing could be interesting to explore. Another extension of the model that could arguably shed further light on the topics of this paper is to deviate from the assumption of perfectly symmetric countries by assuming structural differences *a la* North-South. The incoporation of this feature would be especially interesting for analyzing the case for financial integration when there are heterogenous regions involved.

Appendix

A

Here we describe how to obtain the linearized version of the Euler equation on x_t. For this purpose, a first-order Taylor expansion around the deterministic steady state is used:

$$f\left(\gamma_{t+1}, x_{t+1}(n_{t+1}, \gamma_{t+1})\right) \simeq$$
$$\left[f\left(\gamma_{t+1}, x_{t+1}(n_{t+1}, \gamma_{t+1})\right)\right]_{ss}$$
$$+ \left[f_{\gamma_{t+1}}\left(\gamma_{t+1}, x_{t+1}(n_{t+1}, \gamma_{t+1})\right) + f_{x_{t+1}}\left(\gamma_{t+1}, x_{t+1}(n_{t+1}, \gamma_{t+1})\right)\frac{\partial x_{t+1}}{\partial \gamma_{t+1}}\right]_{ss} \left(\gamma_{t+1} - \gamma_{ss}\right)$$
$$+ \left[f_{x_{t+1}}\left(\gamma_{t+1}, x_{t+1}(n_{t+1}, \gamma_{t+1})\right)\right]_{ss} \left(x_{t+1} - x_{ss}\right)$$

where $f(.)$, $f_{\gamma_{t+1}}(.)$, $f_{x_{t+1}}(.)$, $[.]_{ss}$ stand for, respectively, the generic expressions of the two sides of the non-linear equation on x_t (which are functions of the variables x_{t+1} and γ_{t+1}), the first-derivatives of these expressions with respect to γ_{t+1} and x_{t+1} and the evaluation of the corresponding expression at the steady-state values.

Taking into account that $x_{ss} = \frac{b(\beta(1-\alpha)+\alpha)}{\alpha(1+b-\beta)}$ (as shown in the analysis of FMI in the text) and $\gamma_{ss} = 0$, we proceed with the linearization of the right-hand side:

$$\beta E_0\left[\left(1 + \left(\frac{1-\alpha}{\alpha}\right) b \left(\frac{1}{x_{t+1}} - \gamma_{t+1}\right)\right) x_{t+1}\right] \simeq \beta E_0\left[\left(\frac{1-\alpha}{\alpha}\right) b + \gamma_{t+1}\left[\frac{\partial x_{t+1}}{\partial \gamma_{t+1}}\right]_{ss}\left(1 + \left(\frac{1-\alpha}{\alpha}\right) b\frac{1}{x_{ss}}\right)\right.$$

$$\left. -\gamma_{t+1}\left(\frac{1-\alpha}{\alpha}\right) b \left(\frac{1}{x_{ss}}\left[\frac{\partial x_{t+1}}{\partial \gamma_{t+1}}\right]_{ss} + x_{ss}\right) + x_{t+1}\right]$$

$$= \beta\left(\frac{1-\alpha}{\alpha}\right) b + \underbrace{\beta E_0\left[\gamma_{t+1}\left[\frac{\partial x_{t+1}}{\partial \gamma_{t+1}}\right]_{ss}\left(1 + \left(\frac{1-\alpha}{\alpha}\right) b\frac{1}{x_{ss}}\right)\right]}_{=0 \text{ since } E_0\left[\gamma_{t+1}\right]=0}$$

$$\underbrace{-\beta E_0\left[\gamma_{t+1}\left(\frac{1-\alpha}{\alpha}\right) b \left(\frac{1}{x_{ss}}\left[\frac{\partial x_{t+1}}{\partial \gamma_{t+1}}\right]_{ss} + x_{ss}\right)\right]}_{=0 \text{ since } E_0\left[\gamma_{t+1}\right]=0} + \beta E_0\left[x_{t+1}\right]$$

$$= \beta\left(\left(\frac{1-\alpha}{\alpha}\right) b + E_0\left[x_{t+1}\right]\right)$$

Equating the left and right hand sides, after some simple algebra, we finally get the linearly approximated differential equation on x_t:

$$x_t = \frac{1}{1+\gamma_t+\frac{1}{b}} + \frac{\beta}{b\left(1+\gamma_t+\frac{1}{b}\right)}E_0\left[x_{t+1}\right] + \frac{\beta(1-\alpha)}{\alpha\left(1+\gamma_t+\frac{1}{b}\right)}$$

B

Next we provide proof that the substitution effect is greater than the income effect, as defined in Sandmo (1970).

For this purpose, let us adapt out infinite horizon model to the two period model suggested in Sandmo (1970). We use the same approach that this author uses in his discussion of the results in Levhari, Srinivasan (1969), who also present an infinite horizon problem. In particular, the two-period equivalent of our utility function is:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

The first period budget constraint is given by:

$$a_2 = \underbrace{(1+r_1)a_1 + w_1}_{given} - c_1$$

The second period budget constraint is:

$$\underbrace{a_3}_{0} + c_2 = (1+r_2)a_2 + w_2 = (w_1 + (1+r_1)a_1 - c_1)(1+r_2) + w_2$$

w_2 is a function of γ_1 and it is therefore known when decisions are made in period 1, since we assume that these are taken after uncertainty has been realized.

On the other hand, r_2 is a function of γ_2 and it is therefore unknown. Hence, we have that the only source of risk in this case is in the return to R&D and it is thus analogous to the *capital risk* section in Sandmo (1970).

The expected utility maximization problem solved by the representative household is:

$$\max_{\{c_1\}} E\left[U(c_1, c_2)\right] = \int U(c_1, (1+r_2)(w_1 + (1+r_1)a_1 - c_1) + w_2)g(r_2)dr_2$$

where

$$U(c_1, c_2) = \ln(c_1) + \beta \ln(c_2)$$

and $g(r_2)$ stands for the probability density function of the random variable r_2.

The necessary and sufficient conditions for a maximum are:

FOC:

$$E\left[\frac{1}{c_1} - \beta\frac{(1+r_2)}{c_2}\right] = 0 \qquad \Leftrightarrow \qquad E\left[U_1 - \beta(1+r_2)U_2\right] = 0$$

SOC:

$$E\left[-\frac{1}{c_1^2} - \beta\frac{(1+r_2)^2}{c_2^2}\right] < 0 \qquad \Leftrightarrow \qquad H = E\left[U_{11} + \beta(1+r_2)^2 U_{22} - \underbrace{2(1+r_2)U_{12}}_{0}\right] < 0$$

where the cross-derivative is equal to zero due to time separability of preferences.

Consider the following transformation on tomorrow's return in R&D: $\pi r_2 + \theta$. Here, π is the multiplicative shift parameter and θ is the additive one. For it to be a true mean-preserving spread in risk, it must be the case that

$$E\left[\pi r_2 + \theta\right] = E\left[r_2\right] \quad \Rightarrow \quad \theta = (1-\pi)E\left[r_2\right] \quad \Leftrightarrow \quad \frac{d\theta}{d\pi} = -E\left[r_2\right] = -\mu$$

where we let $E\left[r_2\right] = \mu$.

Given these conditions, we proceed with the formal proof that an increase in risk, captured by an upward change in parameter π, unambiguously leads to higher first period consumption and, hence, a lower a_2.

Proof. Differentiating the FOC with respect to π after writing the return to R&D as $\pi r_2 + \theta$ gives:

$$E\left[\frac{\partial c_1(\pi)}{\partial \pi}\left(-\frac{1}{c_1^2(\pi)}\right) - \beta\left(\frac{\frac{\partial c_1(\pi)}{\partial \pi}(1+\pi r_2+\theta)^2+\left(\frac{\partial \theta}{\partial \pi}+r_2\right)w_2}{[(1+\pi r_2+\theta)(w_1-c_1(\pi)+(1+r_1)a_1)+w_2]^2}\right)\right] = 0$$

Using $\frac{d\theta}{d\pi} = -\mu$ and evaluating at $(\pi = 1; \theta = 0)$, we get:

$$E\left[\frac{\partial c_1(\pi)}{\partial \pi}\left(U_{11} + \beta(1+r_2)^2 U_{22}\right) + \beta\left(r_2 - \mu\right)w_2 U_{22}\right] = 0$$

Then,

$$\frac{\partial c_1(\pi)}{\partial \pi} = -\frac{\beta w_2 E[U_{22}(r_2-\mu)]}{E\left[U_{11}+\beta(1+r_2)^2 U_{22}\right]} = -\frac{1}{H}\beta w_2 E\left[U_{22}\left(r_2 - \mu\right)\right]$$

since w_2 is deterministic.

In order to prove that the sign of the latter expression is positive, it suffices to show that $E\left[U_{22}\left(r_2 - \mu\right)\right] > 0$ given that $H < 0$:

Let $\bar{c}_2 = \left(w_1 + (1+r_1)a_1 - c_1\right)(1+\mu) + w_2$ denote a given deterministic level of period-two consumption. Then, any level of period-two consumption can be written as $c_2 = \bar{c}_2 + (r_2 - \mu)\left(w_1 + (1+r_1)a_1 - c_1\right)$.

Now,

$$U_{22} = u''(c_2) = -\frac{1}{c_2^2} \leq 0 \qquad \forall c_2 \geq 0$$

$$u'''(c_2) = \frac{2}{c_2^3} \geq 0 \qquad \forall c_2 \geq 0$$

Then, since $u'''(c_2) \geq 0$,

$$u''(c_2) \geq u''(\bar{c}_2) \text{ if } (r_2 - \mu)\left(w_1 + (1+r_1)a_1 - c_1\right) \geq 0$$

$$\Longleftrightarrow$$

$$u''(c_2)\left[(r_2 - \mu)\left(w_1 + (1+r_1)a_1 - c_1\right)\right] \geq$$
$$\underbrace{u''(\bar{c}_2)}_{deterministic}\left[(r_2 - \mu)\left(w_1 + (1+r_1)a_1 - c_1\right)\right] \qquad \forall r_2 \geq 0$$

Taking expectations on both sides of the latter inequality,

$$E\left[u''(c_2)\left[(r_2 - \mu)(w_1 + (1 + r_1)a_1 - c_1)\right]\right] \geq u''(\bar{c}_2)(w_1 + (1 + r_1)a_1 - c_1)E\left[r_2 - \mu\right] = 0$$

where the latter equality comes from $E\left[r_2\right] = \mu$

Thus,

$$\underbrace{(w_1 + (1 + r_1)a_1 - c_1)}_{=a_2 \geq 0}E\left[u''(c_2)(r_2 - \mu)\right] \geq 0 \quad \Leftrightarrow \quad E\left[u''(c_2)(r_2 - \mu)\right] \geq 0$$

since households cannot hold a negative asset position.

Hence, given $H < 0$, $w_2 > 0$, $\beta > 0$, $E\left[u''(c_2)(r_2 - \mu)\right] \geq 0$,

$$\frac{\partial c_1(\pi)}{\partial \pi} = -\frac{1}{H}\beta w_2 E\left[U_{22}(r_2 - \mu)\right] \geq 0$$

with strict inequality if either $r_2 > \mu$ or $r_2 < \mu$. $\qquad\square$

It can therefore be concluded that the substitution effect dominates the income effect, implying that the representative agent will always increase today's consumption at the expense of investment when faced with an increase in risk, captured by an increase in the variance of the shock γ.

C

Next it is shown how to obtain the expression in Section 4 that relates total welfare in FMA and total welfare in FMI. This equation is the result of, first, substituting the policy functions for consumption obtained in the two different scenarios directly into the expression for lifetime discounted utility and, second, subtracting the so obtained object in FMI from the one in FMA. To derive the latter expression, a second-order Taylor approximation is employed to proceed with the calculation just described:

Welfare under FMA: $\qquad E_0 \sum\limits_{t=0}^{\infty} \beta^t \ln(c_t(n_t, \gamma_t)) = E_0 \sum\limits_{t=0}^{\infty} \beta^t \ln\left(\frac{n_t \alpha(1+b)(1+b-\beta)}{b(\alpha+\beta(1-\alpha))(1+b(1-\gamma_t))}\right)$

Welfare under FMI:
$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t(n_t)) = E_0 \sum_{t=0}^{\infty} \beta^t \ln \left(\frac{n_t(1+b-\beta)\alpha}{b(\alpha+\beta(1-\alpha))} \right)$$

Subtracting one from the other:

$$WFMA - WFMI =$$

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(n_t) - \sum_{t=0}^{\infty} \beta^t \ln(n_t) - E_0 \sum_{t=0}^{\infty} \beta^t \ln(1 + b(1 - \gamma_t)) + \frac{1}{1-\beta} \ln(1+b)$$

Using the (given) initial stock of varieties, n_0, and the technology of production of these varieties in the R&D sector, we have:

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln(n_t) = \frac{1}{1-\beta} \ln(n_0) + A \frac{1}{(\beta-1)^2}$$

where

$$A = \beta E_0 \ln \left(\frac{(\alpha+\beta(1-\alpha))(1+b(1-\gamma_t))(1+b(1+\gamma_t)) - \alpha(1+b)(1+b-\beta)}{(\alpha+\beta(1-\alpha))(1+b(1-\gamma_t))} \right)$$

and

$$\frac{1}{(\beta-1)^2} = \left(1 + 2\beta + 3\beta^2 + ... + \lim_{T \to \infty} (1+T) \beta^T \right)$$

Similarly for the analogous expression in FMI:

$$\sum_{t=0}^{\infty} \beta^t \ln(n_t) = \frac{1}{1-\beta} \ln(n_0) + \frac{\beta}{(\beta-1)^2} \left[\ln(\beta(1 + b(1 - \alpha))) - \ln(\alpha + \beta(1 - \alpha)) \right]$$

After some algebraic manipulation, the welfare difference reads:

$$WFMA - WFMI = \frac{\beta}{(\beta-1)^2} \cdot$$

$$(E_0 \left[\ln(\alpha + \beta(1 - \alpha))(1 + b(1 - \gamma_t))(1 + b(1 + \gamma_t)) - \alpha(1+b)(1+b-\beta) \right) - \ln(\alpha + \beta(1 - \alpha))$$

$$\text{-}\ln(1 + b(1 - \gamma_t))$$

$$- \ln(\beta(1 + b(1 - \alpha))) + \ln(\alpha + \beta(1 - \alpha)))$$

$$- E_0 \sum_{t=0}^{\infty} \beta^t \ln(1 + b(1 - \gamma_t)) + \frac{1}{1-\beta} \ln(1+b)$$

A second-order Taylor expansion around the deterministic steady state of the terms involving γ_t inside the expectations operator is conducted in order to be able to proceed with the calculations:

$$\ln\left(\alpha + \beta\left(1-\alpha\right)\right)\left(1 + b\left(1-\gamma_t\right)\right)\left(1 + b\left(1+\gamma_t\right)\right) - \alpha\left(1+b\right)\left(1+b-\beta\right)) \simeq$$
$$\ln\left(1+b\right) + \ln\left(\beta\left(1+b\left(1-\alpha\right)\right)\right) - \gamma_t^2\frac{b^2\left(\alpha+\beta\left(1-\alpha\right)\right)}{\beta\left(1+b\right)\left(1+b\left(1-\alpha\right)\right)}$$

$$\Longleftrightarrow$$

$$\ln\left(1 + b\left(1-\gamma_t\right)\right) \simeq \ln\left(1+b\right) - \gamma_t\frac{b}{1+b} - \gamma_t^2\frac{b^2}{2\left(1+b\right)^2}$$

Substituting the approximated versions of these objects into the expression for the difference in total welfare, and after simplifying, it finally stays:

$$WFMA - WFMI = var(\gamma)\frac{b^2}{2\left(1-\beta\right)\left(1+b\right)^2} - \frac{\beta b^2}{\left(\beta-1\right)^2}var(\gamma)\left[\frac{2\alpha\left(1+b-\beta\right)+\beta\left(1+b\left(1-\alpha\right)\right)}{2\beta\left(1+b\right)^2\left(1+b\left(1-\alpha\right)\right)}\right]$$

$$= b^2 var\left(\gamma\right)\left[\frac{\left(1+b\left(1-\alpha\right)\right)\left(2\beta-1\right)+2\alpha\left(1+b-\beta\right)}{2\left(\beta-1\right)\left(1-\beta\right)\left(1+b\right)^2\left(1+b\left(1-\alpha\right)\right)}\right]$$

D

In this section of the Appendix, we prove the result that it is not possible to find a vector of parameters that simultaneously satisfies the condition that socially optimal growth is higher than decentralized equilibrium growth but does not satisfy $WFMA < WFMI$.

Proof. As already mentioned in Section 3, the set of restrictions on the parameter vector (α, β, b) for which the Pareto optimal growth rate is higher than the competitive equilibrium one is:

$$\begin{cases} 0 < \alpha \leq \frac{1}{2} \quad \cup \quad \frac{2\alpha-1}{\alpha-1} < \beta < 1 \quad \cup \quad b > \frac{\left(\alpha-1\right)\left(1-\beta\right)}{1-\beta+\alpha\left(\beta-2\right)} \\ \frac{1}{2} < \alpha < 1 \quad \cup \quad 0 < \beta < 1 \cup \quad b > \frac{\alpha\left(\beta-1\right)}{\beta\left(\alpha-1\right)} \end{cases}$$

And the set of conditions for which $WFMA < WFMI$ is:

$$\begin{cases} 0 < \alpha \le \frac{1}{3} \begin{cases} \frac{3\alpha-1}{2(\alpha-1)} < \beta \le \frac{3\alpha-1}{2(2\alpha-1)} & \cup & b > \frac{2\alpha-1+2\beta(1-\alpha)}{1-3\alpha+2\beta(\alpha-1)} \\ \frac{3\alpha-1}{2(2\alpha-1)} < \beta < 1 & \cup & b \ge \frac{\alpha(\beta-1)}{\beta(\alpha-1)} \end{cases} \\ \frac{1}{3} < \alpha < 1 \quad \cup \quad 0 < \beta < 1 \quad \cup \quad b \ge \frac{\alpha(\beta-1)}{\beta(\alpha-1)} \end{cases}$$

Let A be the set of three-dimensional parameter vectors defined by the first group of restrictions and let B stand for the analogous set defined by the second group of restrictions. What we will show is: $x \in A \implies x \in B$, for all $x \in A$.

For any $x = (\alpha, \beta, b) \in \{x \in A \mid 0 < \alpha \le \frac{1}{3}\}$, it must be the case that $\frac{2\alpha-1}{\alpha-1} < \beta < 1$ and $b > \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)}$ hold. In order for $x \in B$, we need to show that both $\frac{3\alpha-1}{2(2\alpha-1)} \le \frac{2\alpha-1}{\alpha-1}$ and $\frac{\alpha(\beta-1)}{\beta(\alpha-1)} \le \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)}$ hold at the same time. The first of the latter two inequalities is equivalent to $0 \le 3 - 4\alpha$, which is always satisfied, since $0 < \alpha \le \frac{1}{3}$ holds. To prove that the second of the two inequalities holds, let us first look at the limiting case when $\alpha \to 0$. Given the restriction on α, this is the same as saying that $\alpha = 0 + \varepsilon$ with $\varepsilon > 0$ and arbitrarily small. Then, $\beta = 1 - \varepsilon$, and the second of the two inequalities is equivalent to $\varepsilon^3 \le (1-\varepsilon)^3$, which always holds. Now, the denominator of the right hand-side of the second of the two inequalities is negative: since $\frac{2\alpha-1}{\alpha-1} < \beta < 1$, the lowest possible value that β may take is $\frac{2\alpha-1}{\alpha-1} + \varepsilon$ with $\varepsilon > 0$ and arbitrarily small. Hence, the maximum value that $1 - \beta + \alpha(\beta - 2)$ can take is $\varepsilon(\alpha-1) < 0$ for any $\alpha \in (0, \frac{1}{3}]$. This, in turn, implies that the second of the two inequalities is equivalent to $\beta(\alpha(3-\alpha)-1) + \alpha(\alpha-1) \le 0$, which always holds. This proves that $x \in \{x \in A \mid 0 < \alpha \le \frac{1}{3}\} \implies x \in B$.

For any $x = (\alpha, \beta, b) \in \{x \in A \mid \frac{1}{3} < \alpha \le \frac{1}{2}\}$, it must be the case that $\frac{2\alpha-1}{\alpha-1} < \beta < 1$ and $b > \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)}$ hold. In order for $x \in B$, we need to show that both $0 \le \frac{2\alpha-1}{\alpha-1}$ and $\frac{\alpha(\beta-1)}{\beta(\alpha-1)} \le \frac{(\alpha-1)(1-\beta)}{1-\beta+\alpha(\beta-2)}$ hold at the same time. The first of the latter two inequalities can be seen to hold directly from the fact that $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Exactly the same argument used in the

last paragraph carries over to the case when $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ to prove that the second of the two inequalities holds. This proves that $x \in \left\{ x \in A \mid \frac{1}{3} < \alpha \leq \frac{1}{2} \right\} \implies x \in B$.

Finally, for any $x = (\alpha, \beta, b) \in \left\{ x \in A \mid \frac{1}{2} < \alpha < 1 \right\}$, it must be the case that $0 < \beta < 1$ and $b \geq \frac{\alpha(\beta-1)}{\beta(\alpha-1)}$. In order for $x \in B$, both $0 \leq 0$ and $\frac{\alpha(\beta-1)}{\beta(\alpha-1)} \leq \frac{\alpha(\beta-1)}{\beta(\alpha-1)}$ must hold at the same time. The latter two follow trivially. This proves that $x \in \left\{ x \in A \mid \frac{1}{2} < \alpha < 1 \right\} \implies x \in B$.

This provides proof that whenever the socially optimal growth rate is higher than the competitive one, the representative agent is strictly better-off in the financially integrated scenario. $\square$

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Chapter 2

Population and environmental quality in a neoclassical model with pollution irreversibilities

JEL codes: Q01, Q50, Q54, Q56, Q59, E20.

Keywords: Population, Pollution, Consumption, Irreversibilities, Economic-Environment trap.

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2.1 Introduction

Population growth, and the higher demand for goods and services that comes with it, is acknowledged as one of the key factors lying at the root of the current concern about climate change and the depletion of non-renewable resources and energy. Even though the exact relationship and the possible two-way effects are not yet flawlessly understood, many agree nowadays that demographic and environmental factors are intimately connected (United Nations Population Fund, 2001, Ehrlich and Ehrlich, 1990).

Although some have argued that a switch to cleaner technologies may offset any negative pollution effects of population growth as economies move along their environmental Kuznet's curves, the evidence collected so far seems to deny that such a process is actually happening. In a cross-country study, Huesemann and Huesemann (2008) find evidence for the existence of *relative decoupling* (i.e., reductions in energy and resource intensity per unit of output) but fail to find it for *absolute decoupling* (i.e., reductions in total energy and resource use). They suggest that the reason for the lack of absolute decoupling observed in the data may lie in the presence of a *rebound effect*, by which any new income generated by the deployment of more efficient technologies leads to a rise in aggregate demand that outweighs the improvements in the environmental impact of the production processes[1]

Despite the need for a sound theoretical framework for the analysis of the linkages between population and the environment, contributions in this respect are yet relatively scant and devoid of the new insights accruing from the latest research in the different disciplines involved. In the following, a brief overview of the relevant literature is provided. At the end of this section, motivation and justification for the questions raised and the extensions proposed are discussed.

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[1]For recent positive evidence on existence of such a rebound effect, see Hanley et al (2009). For a thorough review of the empirical evidence, see Sorrell, S. (2007).

Literature review

A number of contributions stand out as closest in objectives and methodology to ours. Cronshaw and Requate (1997) are arguably among the first to analyze the impact of population on emissions and the pollution stock in a static general equilibrium framework with exogenous population. They aim at obtaining the optimal Pigouvian tax rate that should be levied on firms to attain the first-best allocation. Instead, we will be concerned with the optimal development of a dynamic economy for different population levels. Also, they do not consider the implications that the inclusion of an irreversibility threshold may have on their results.

Harford (1997, 1998) tackles the same optimal emissions taxation problem as in Cronshaw and Requate (1997) redefined in a dynamic model instead. Pollution is a by-product of output whose dynamics are characterized by a constant fraction of its stock being naturally sequestrated in every period. Optimal taxes on child-bearing are calculated. One would expect that the presence of irreversibility effects may change their conclusions. In particular, introducing non-monotonic dynamics may imply higher optimal taxes in order to avoid catastrophic outcomes. Moreover, analyses involving the concept of the optimal number of children are subject to a host of criticisms in the endogenous fertility literature (Golosov, Jones and Tertilt (2007), Farhi and Werning (2005)).

In the same optimal taxation vein, Frank Jöst and Martin F. Quaas (2010) consider the externalities arising from both births and consumption on the degradation of the environment. Their goal differs from ours in that they do not investigate the socially optimal path of the economy, population and environment modules over time, focusing rather on the attainment of optimal allocation decentralization. Furthermore, the use of Pareto criteria to evaluate the optimal birth rate (e.g., problem of the "non-born", Golosov, Jones and Tertilt (2007)) is a controversial topic for which they do not provide a discussion.

The optimal management problem of an integrated economy-environment-population system

is studied in Jöst et al (2004). Their main conclusion is that, given optimistic assumptions, the first-best allocation does not necessarily give rise to environmentally-harming population growth. That means that, in their model, preserving the environment is compatible with the number of people growing indefinitely. This runs counter to the general agreement and the indirect empirical evidence of a negative relationship between the two (Cohen, J. (2010), Shi, A. (2003), Dietz, T. and Rosa, A.E (1997)). In addition, it also suffers from the issues regarding welfare criteria in optimal growth models with endogenous population mentioned before.

The latter is the closest paper in scope to ours. We shall attempt to gain further insights on the optimal growth problem by considering both the possibility of exhaustion of the regeneration capacity of nature and the possibility of abatement (*aka* maintenance). At the same time, avoid the theoretical concerns surrounding the concept of optimal population by assuming exogeneity.

Motivation

The aim of this paper is twofold. First, we seek to cast new light on the sign and importance of the effects of population growth on pollution and income. For this purpose, we employ a standard model-economy exhibiting environmental externalities and (exogenous) population shocks and analyze its optimal management from a utilitarian, welfare-maximizing point of view. Second, we wish to analyze the critical role played by irreversible damages to the natural regeneration process for environmental quality. In particular, the focus is put on the implications of irreversibility for the relationship between population size and the environment and their differences with respect to the no-irreversibility benchmark case commonly found in the literature.

The specific questions raised are:

1. How should an economy which is both economically and environmentally poor respond

to demographic changes? could it be optimal for a society to choose to follow a path leading to low levels of wealth coupled with high levels of pollution? If so, under which conditions?

2. What are the consequences for economic development of the existence of a threshold level of pollution after which nature stops providing its *free* maintenance services to humanity? what can we learn from this in terms of the exertion of caution when evaluating policies affecting aggregate population?

As it will become apparent, the more sophisticated dynamic structure of pollution suggested in this paper gives rise to new potential long-term scenarios. These scenarios are characterized by drastically different levels for the capital and environmental quality stocks. The fact that the widely-used, constant natural absorption rate of pollution fails to encompass these possible outcomes means that the policy message concerning the consequences of a growing world population on people's well-being is rather disparate from one paradigm to the other.

As the understanding of the dynamics governing crucial environmental processes, such as greenhouse gas atmospheric concentrations, has improved over time, it now seems appropriate to incorporate the latest insights to economic modeling and depart from the general overly strong simplifying assumptions. Irreversibility is present in local pollutant problems such as the eutrophication of lakes, the salinification of soils or the loss of biodiversity because of land use. There is growing evidence and it is commonly agreed that global environmental threats, like global warming, also share this feature (Prieur (2009)). A number of important points on the implications of increasing emission levels and concentrations of greenhouse gases (GHG) for the regeneration capacity of natural ecosystems have already been gathered. As an example, oceans, that form the largest carbon sink on the planet, display a buffering stock that is slowly saturating. At the same time, the assimilation ability of terrestrial ecosystems (mainly lands and forests) is estimated to reach a peak by mid-century and then decline to become a net source of carbon by the end of the century. In fact, the

role these play at present may be reversed in the future: there is the chance that not only will the regeneration capacity be exhausted, but ecosystems may actually turn from sinks to net emitters of carbon (!).

In this sense, the present paper is in line with those views that stress the finiteness of the human carrying capacity of the Earth and the incompatibility of perpetual population and material growth, which is supported by the lack of evidence showing any sign of absolute decoupling of the production process from its resource and energy base in recent times. Although these concepts generally refer to the inescapable boundedness of resources, they may as well be applied to the concentration of GHGs. Ultimately, the Earth is as uninhabitable under an excessively polluted atmosphere as it is under fully exhausted energy and resource inputs.

For the purposes outlined above, we begin by analyzing the impact of population size on the relevant economic and environmental variables in the context of a dynamic general equilibrium model with neoclassical technology. One of the important results stemming from this analysis is that, contrary to the predictions of other theoretical models, population has a detrimental effect on environmental quality. Following this analysis, we proceed to extend the scope of the related literature by relaxing the assumption of smooth regeneration, from which we draw a number of new and interesting conclusions.

The chapter is organized as follows. In section 2.2, technology, preferences and the optimal allocation problem are introduced and solved for the benchmark case of linear, smooth regeneration of environmental quality. A brief discussion of the first-best allocation decentralization is also provided. In section 2.3, the optimal allocation problem is reformulated by incorporating an irreversibility threshold in the stock of environmental quality. We distinguish between the two possible zones where this stock may lie. The implications of the new set of conditions for the relationship between population and the environment-economy are analyzed and the results are compared to the benchmark model. Finally, section 2.4

contains concluding remarks.

2.2 An optimal management problem with constant assimilation capacity

The baseline model with proportional and smooth regeneration is outlined in this section. Let us start by describing the environment and economic sectors first and then characterize the optimality problem. At the end of the section, we comment briefly on the appropriate tax policy needed for decentralized markets to achieve the optimal allocation.

Environmental quality, Q_t is defined as a stock variable featuring autonomous dynamics in discrete time. In particular, it is characterized by the ability to reproduce part of its stock period after period according to a constant proportional regeneration coefficient. In the absence of human activity, the law of motion governing its behavior over time is simply given by:

$$Q_{t+1} - Q_t = \eta Q_t$$

where η denotes the fraction of the environment stock that regenerates itself every period. This ongoing regeneration process can persist up until an exogenously given, maximum attainable level of environmental quality, $\bar{Q} > 0$. Adding the non-negativity constraint, quality must satisfy $0 < Q_t \leq \bar{Q} \quad \forall t$. This description of the evolution of environmental quality will be modified to account for the impact of economic activity later.

At any time t, output per capita is produced according to a Cobb-Douglas production function defined on capital and labor, $y_t = k_t^\alpha$. The N infinitely-lived, dynastic households share identical preferences that are defined over both consumption and environmental quality. The instantaneous utility function assumed for the representative household is:

$$U(c_t, Q_t) = \ln c_t + \frac{1}{N} \ln Q_t$$

where c_t denotes consumption per capita. We follow Harford's (1998) way of modeling population as exerting a congestion effect on environmental assets by which the utility derived from environmental quality decreases as its quantity per capita falls. The following properties apply:

$$U_c, U_Q > 0, \quad U_{cc}, U_{QQ} < 0, \quad U_{cQ} = U_{Qc} = 0$$

Individuals enjoy both per capita consumption and environmental quality at a decreasing rate, given falling marginal utility. Also, we assume separability in both arguments. This set of assumptions, including the use of logarithmic utility, is the most prevalent in the literature[2]

The link between the economy and the environment is in the form of polluting emissions that originate every period as a by-product of output. These emissions have a negative effect on the quality of the environment, since they are a source of pollution to the atmosphere. In particular, a fraction $\mu \in (0,1)$ of units of output is assumed to harm the environment each period. Emissions (net of abatement) along with natural regeneration determine the sign and size of the change in environmental quality:

$$Q_{t+1} - Q_t = \eta Q_t + N\left(m_t - \mu k_t^\alpha\right)$$

where m_t represents the units of the final good that are sacrificed for abatement at the expense of consumption and investment. Abatement enters linearly as a flow variable in the law of motion for quality and is able, in principle, to more than compensate the effect of *gross* emissions and turn *net* emissions positive. Per capita emissions need to be multiplied by N to express them in aggregate form.

Finally, assuming a zero depreciation rate of capital and given the three possible allocations for final output, the capital accumulation equation reads:

$$k_{t+1} - k_t = k_t^\alpha - c_t - m_t$$

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[2]Adding cross-effects would only complicate the analysis with little increase in insight.

Optimal allocation problem

The optimization program solved by the benevolent social planner consists of choosing paths for consumption and maintenance, $\{c_t\}_{t=0}^{\infty}$, $\{m_t\}_{t=0}^{\infty}$, to maximize the present discounted value of the infinite stream of aggregate utility subject to the non-negativity constraints, the resource constraint and the equation of motion for quality:

$$\max_{\{c_t, m_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(N \ln c_t + \ln Q_t \right)$$

s.t.

$$k_{t+1} - k_t = k_t^{\alpha} - c_t - m_t$$

$$Q_{t+1} - Q_t = \eta Q_t + N \left(m_t - \mu k_t^{\alpha} \right)$$

Note that the welfare function in the latter problem differs from the individual utility function presented before because the social planner considers the sum of per capita utility across individuals (preferences are homogeneous and, thus, everyone exhibits the same consumption patterns). Consumption, as opposed to environmental quality, is a private good. This means that the more people there are, the higher competition for one unit of consumption is. An increase in population gives rise to a potential welfare loss or *dilution* effect in consumption that the social planner will optimally trade off. Thus, population acts as a weight placed in consumption that captures its rival nature. In contrast, the quantity of environmental services available is unaffected by an increase in the number of households.

The Lagrangian associated with the problem above reads:

$$L =$$

$$\sum_{t=0}^{\infty} \beta^t \left[N \ln c_t + \ln Q_t - \lambda_t^k \left(k_{t+1} - k_t + c_t - k_t^{\alpha} + m_t \right) - \lambda_t^Q \left(Q_{t+1} - Q_t - \eta Q_t - N \left(m_t - \mu k_t^{\alpha} \right) \right) \right]$$

where λ_t^k and λ_t^Q stand for the co-state variables for capital and environmental quality, respectively.

The first order necessary[3] conditions for an interior optimum are[4]:

$$\frac{N}{\lambda_t^k} = c_t \tag{1}$$

$$\lambda_t^k = N\lambda_t^Q \tag{2}$$

$$\lambda_t^k = \beta\left[(1 + \alpha k_{t+1}^{\alpha-1})\lambda_{t+1}^k - N\alpha\mu k_{t+1}^{\alpha-1}\lambda_{t+1}^Q\right] \tag{3}$$

$$\lambda_t^Q = \beta\left[\frac{1}{Q_{t+1}} + (1+\eta)\lambda_{t+1}^Q\right] \tag{4}$$

The first relationship imposes that the marginal utility of consumption be equal to the shadow price of capital in every period. As discussed before, a higher population raises, *ceteris paribus*, the importance of capital, since it raises the welfare weight assigned by the social planner to per capita consumption. Equation (3) requires the shadow price of per capita capital and environmental quality to evolve one-for-one in equilibrium. Again, a higher household count means a higher importance of consumption (higher λ_t^k) relative to environmental quality in welfare terms. Equation (4) states that the marginal welfare value of per capita capital must decrease over time (i.e., capitalization occurs) as long as the marginal product of capital, corrected for its adverse effect on emissions, is higher than the level of individuals' impatience. The lower (higher) the number of units of the final good that turn into emissions (captured by parameter μ), the lower (higher) the social planner's degree of concern regarding this negative externality and, thus, the more (less) likely it is that capital deepening is optimal. Finally, equation (5) relates the marginal utility of

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[3]Sufficiency is established in the Appendix.

[4]We leave out the analysis of the $m_t = 0$ corner case, where the non-negativity constraint on abatement is binding. This case is not interesting for our purposes and not relevant to the current environmental problems faced by modern societies, where the need for some sort of maintenance technology is widely acknowledged. Details about the corner case are available upon request.

quality to its shadow price at all points in time and dictates that an improvement in the environment should obtain ($\lambda_{t+1}^Q - \lambda_t^Q < 0$) if the marginal utility of quality is higher than the time discount rate net of the natural regeneration coefficient. The more *productive* the environment is in sequestrating its pollution stock, all things being equal, the higher chances are that it pays off for society to give up consumption in favor of enhanced environmental quality.

The transversality conditions for per capita capital and environmental quality read, respectively:

$$\lim_{t \to \infty} \lambda_t^k k_t = 0$$

$$\lim_{t \to \infty} \lambda_t^Q Q_t = 0$$

The first order conditions, together with these boundary conditions and the initial (given) values for the two states pin down the optimal paths followed by the variables in the model.

Combination of equations (1)-(5) above along with the resource constraint and the law of motion for environmental quality form the following dynamic system:

$$\frac{c_{t+1}}{c_t} = \beta \left(1 + (1 - \mu)\alpha k_{t+1}^{\alpha-1} \right) \tag{5}$$

$$\alpha k_t^{\alpha-1}(1 - \mu)Q_t = c_t + \eta Q_t \tag{6}$$

$$k_{t+1} = k_t + k_t^\alpha - c_t - m_t \tag{7}$$

$$Q_{t+1} = (1 + \eta)Q_t + N\left(m_t - \mu k_t^\alpha \right) \tag{8}$$

The first equation is the standard Euler equation for consumption. According to it, consumption will grow over time if the marginal productivity of capital, duly corrected for the harmful externality of production, is sufficiently high relative to the exogenous rate of time preference. The second equation follows from the optimality condition for abatement, which

states that the change over time in the marginal social cost of abatement should always be proportional to the change over time in its marginal benefit. The former is given by the corrected marginal productivity of capital, which measures the units of the final good that are foregone by allocating one unit to maintenance. The variation across periods in the marginal benefit of abatement depends positively on consumption, the marginal utility of environmental quality and the natural regeneration rate, since higher levels of these objects make an additional unit of maintenance more valuable.

Steady state analysis

Let us refer to a steady state as the long-run situation where all four variables in the model converge to a constant value. In the present context, the steady state may well be interpreted as *sustainable development* in the sense that achieving a constant positive level of environmental quality is compatible with the economic system landing an equilibrium with strictly positive wealth. More precisely, the convergence of income per capita to a constant level does not involve a continuous deterioration of the natural environment.

Removing all time indices from the variables in the previous system of equations and reshuffling, the steady state values of the four objects in the system read:

$$k_{bss} = \left(\frac{\beta\alpha(1-\mu)}{1-\beta} \right)^{1/1-\alpha} \tag{9}$$

$$Q_{bss} = \frac{N\beta(1-\mu)k_{bss}^{\alpha}}{N\left(1-\beta(1+\eta)\right)-\beta\eta} \tag{10}$$

$$c_{bss} = \frac{N\left(1-\beta(1+\eta)\right)(1-\mu)k_{bss}^{\alpha/1-\alpha}}{N\left(1-\beta(1+\eta)\right)-\beta\eta} \tag{11}$$

$$m_{bss} = \frac{\left(\beta\alpha(1-\mu)/(1-\beta)\right)^{\alpha/1-\alpha}\left(N\mu(1-\beta(1+\eta))-\beta\eta\right)}{N\left(1-\beta(1+\eta)\right)-\beta\eta} \tag{12}$$

The only parameter restriction needed for existence of this steady state is: $\beta < \frac{N\mu}{N\mu(1+\eta)+\eta}$.

It can easily be shown that, consistently with the indirect empirical evidence, a higher population lowers the steady state level of environmental quality. On the other hand, the effect on the level of capital per capita is nil while abatement increases at the expense of consumption. Thus, a demographic shock that increases the world's household count will cause a transition (analyzed in the next subsection) to a new steady state after which the representative individual will be permanently worse off in welfare terms since both the consumption level and the quality of the environment she enjoys are lower. The following table summarizes the long-run effects on the model's variables of positive changes in the different fundamental parameters:

	Q_{bss}	k_{bss}	c_{bss}	m_{bss}
N	< 0	0	< 0	> 0
β	> 0	> 0	> 0	> 0
η	> 0	0	> 0	< 0
μ	< 0	< 0	< 0	$??$
α	$??$	> 0	$??$	$??$

Table 1: comparative static results for preference and technology parameters. A double question mark stands for an indeterminate effect.

It is worth noting the signs of the effects of a positive variation in the natural regeneration rate, μ. As with an increase in population, the long-run level of per capita capital is left unchanged, but the contrary effects to an increase in the number of people occur for the other variables; namely, quality and consumption rise whereas abatement is permanently lower. Thus, society re-allocates resources spent on maintenance and uses them for consumption instead, while simultaneously enjoying the value of a cleaner atmosphere. Therefore, all else equal, individuals will undoubtedly be better off in the long term equilibrium if nature is endowed with a higher assimilation capacity.

The coefficient that captures the income share of capital, α, has an ambiguous effect on all variables except for capital. An increase in the output share of capital means that a higher percentage of total income accrues to this input and, thus, an increment in investment is obtained that generates a higher capitalization level of the economy in the new equilibrium[5].

The representative agent's long-run welfare diminishes after a reduction in β. A higher degree of impatience (i.e., decrease in β) shifts the consumption profile to the present at the cost of sacrificing investment in both future physical capital and future environmental quality. Both consumption and environmental quality are thus negatively affected in the long run.

Lastly, a less pollutant-efficient output technology (i.e., increase in μ) also has the social planner reacting by choosing a path leading to both lower quality and lower consumption. An exogenous higher rate of output transformed into pollution in the atmosphere always comes as a harmful event, since society is unable to compensate its negative utility impact by increasing either steady state consumption or quality, even if it is at the expense of one other. Therefore, the welfare level unequivocally falls.[6,7]

Next, we provide a discussion of the system's transitional patterns that result as a consequence of both temporary and permanent shocks to population.

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[5]Whether this rise happens at the expense of consumption and abatement hinges on the range of values considered for α. When $\alpha < 1/2$, capital and output expand partly due to a reduction in consumption and maintenance. These changes necessarily lead to a drop in environmental quality. However, when $\alpha > 1/2$, the increase in the marginal productivity of capital allows increments in the steady state value of consumption and abatement as well. The relative sizes of theses changes also permit an improvement in environmental quality.

[6]All the comparative statics analysis of welfare here have to be interpreted considering an infinite time approach. Thus, we can safely bypass a welfare counting exercise during the transition, since it becomes negligible in this time frame.

[7]The sign of the effect on abatement is indefinite. For relatively low values of μ ($\mu < \rho(1-\alpha)/\alpha^2$), the social planner partially offsets the fall in quality by increasing maintenance. For higher values, abatement decreases for in that case society does not find it worthwhile to cushion the drop in quality since it is at a very high cost.

Transitional dynamics

The solution to the model is computed numerically using Dynare (Villemot et al (2011)). We comment on the shape and timing of the response of the economic and environment modules to both temporary and permanent shocks in population over a certain time horizon. Stability of the steady state hinges on the value for some of the parameters. We set the values of the well-known parameters capturing the income share of capital and the time discount rate to their most widespread values in the literature. The remaining two and population are set to values that are consistent with both existence and saddle-path stability of the unique steady state equilibrium[8,9]:

$$\alpha = 0.3 \ \beta = 0.9 \ ; \ \eta = 0.04 \ ; \ \mu = 0.8 \ ; \ N = 1$$

The steady state values for the variables corresponding to this parametrization are:

$$k_{bss} = 0.4147 \ ; \ Q_{bss} = 4.937 \ ; \ c_{bss} = 0.3510 \ ; \ m_{bss} = 0.4169$$

Let us examine the effects of a 10% temporary increase in population starting at the steady state of the system. The following plot shows the transitional dynamics for each variable back to its original steady state:

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[8]Stability is rather robust for a wide range of possible values of the vector of parameters.

[9]The values of the three eigenvalues obtained for this choice of parameters are: 1.47e-015, 0.9847, 1.128. Hence, according to the Blanchard-Kahn conditions, the equilibrium is a saddle-point.

Figure 1: Transition paths after a 10% temporary increase in the steady state value of population (assumed equal to 1). Population increases in the first period and returns to 1 thereafter.
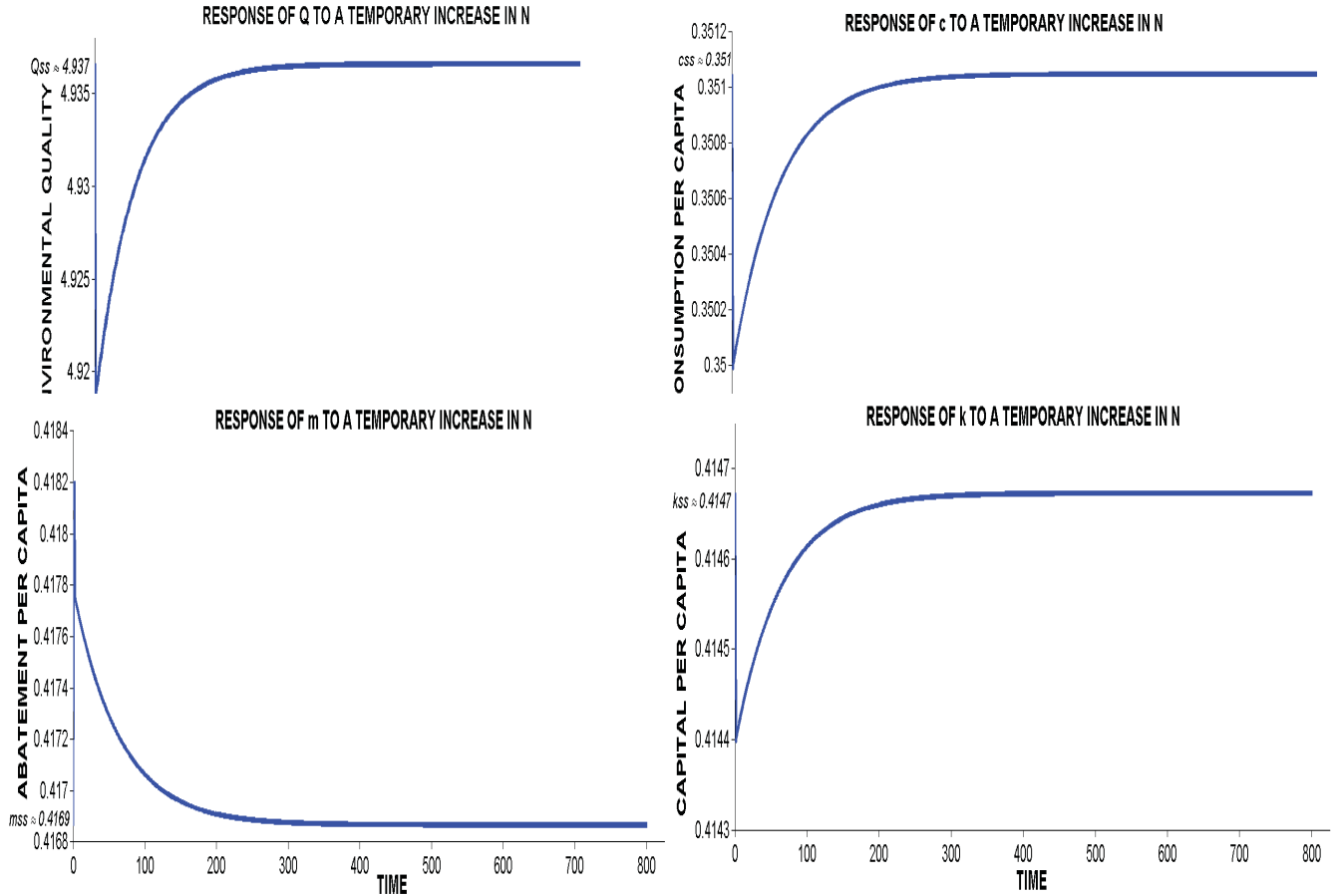
The demographic change is an unexpected event, since reactions to it begin in the same period in which it takes place (i.e., period zero). This means that, at first, the variables exhibit jumps in their trajectories. Given logarithmic preferences, the transition of consumption back to its original steady state value must be as smooth as possible. Abatement acts as the auxiliary control for consumption to achieve this objective. It also partly controls the non-autonomous term of the dynamics for environmental quality.[10]

The sudden increase in population at the initial period brings aggregate emissions up and causes environmental quality to drop. This exogenous shock is immediately followed by the

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[10]Although utility is also logarithmic with respect to quality, the social planner does not have complete control over it given that it belongs to the state vector of the system.

decision of the social planner to partly ameliorate the negative effect on quality by allocating more resources to maintenance at the expense of consumption. Since the fall in the latter is lower than the increase in the former, the capital stock shrinks to a smaller value at the end of the period in which the shock is realized. From then onwards, the focus is put on restoring consumption back to its original value in a smooth way. For that purpose, abatement decreases over time at a rate that, first, allows for both positive capital investment and consumption growth and, second, is low enough for improvement in environmental quality to occur. The time change rates diminish gradually until all variables converge back to their original values.

As opposed to the case just analyzed, an unexpected, permanent increase in the number of households leads the economy to a new steady state. The following chart pictures the optimal transitions after a permanent increment in population of the same magnitude as before[11]:

Figure 2: Transition paths after a 10% permanent increase in the steady state value of population (assumed equal to 1). Population increases in the first period to 1.1 and stays constant thereafter.

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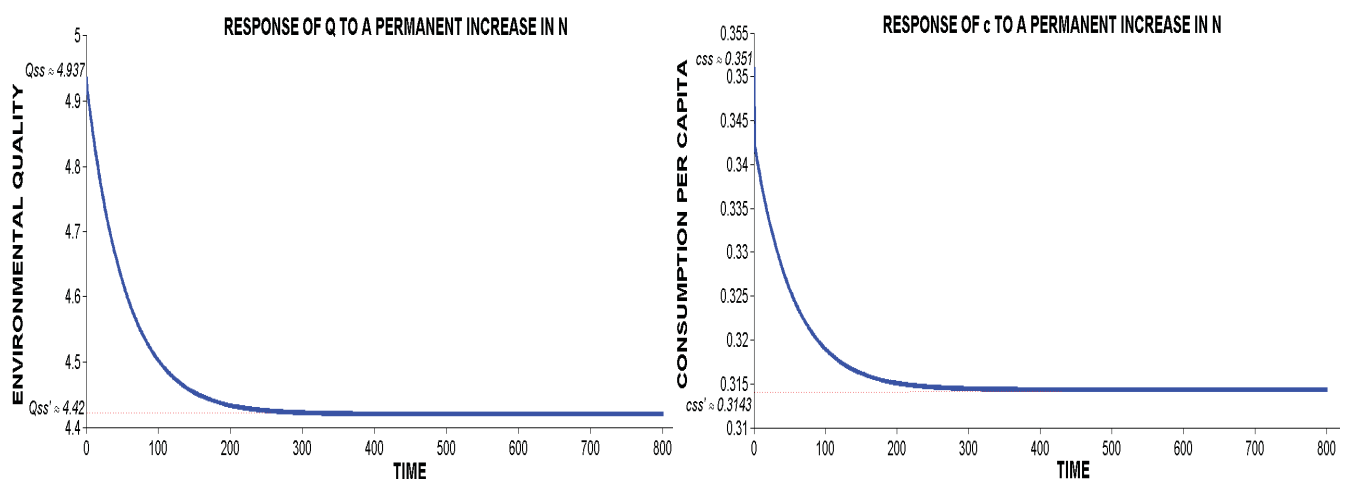
[11]The new steady state values are: $k_{bss} = 0.4147$, $Q_{bss} = 4.42$, $c_{bss} = 0.3143$, $m_{bss} = 0.4536$, $N = 1.1$
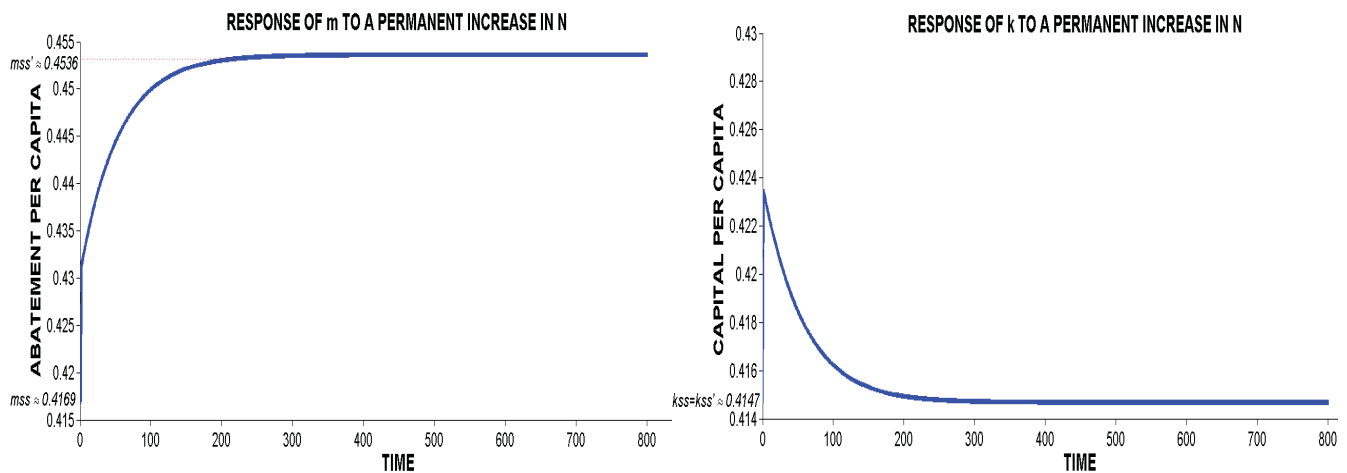
Both environmental quality and per capita consumption converge to a lower steady state value whereas abatement increases and capital stays the same. As before, aggregate net emissions shoot up at first, making environmental quality move to a lower level. In order to compensate for augmented pollution due to the (permanent) presence of more individuals, the social planner needs to allocate more resources to maintenance in order to mitigate the overall impact on quality. The rationale for this can be found in the law of motion for quality. First, the autonomous part of the dynamics for quality is such that lower levels of environmental quality translate into proportionally lower total natural regeneration and, second, overall emissions associated with human economic activity are greater after the population increase. Thus, left to its own devices, the state of the environment would suffer even more if more abatement effort was not made. In order to increase maintenance, either consumption or capital investment must be sacrificed. However, if it were decided to use part of the capital stock, it would not be possible to feed the ulterior increase in abatement needed for it to reach its new higher value. Furthermore, a decrease in capital would cause an increase in its marginal productivity that would entail a positive consumption growth rate, which is infeasible since consumption must converge to a lower new steady state value. Therefore, all in all, the price paid by society to ameliorate the negative effects of population on the environment is in the form of foregone consumption. Capital adjusts in accordance

with the responses of abatement and consumption: a sizeable increase in capital investment ensues right after the shock hits the economy, allowing a subsequent decapitalization process that flows to maintenance until converge back to its steady state level is attained.

Optimal allocation decentralization

The decentralized market equilibrium and the optimal fiscal policy mix are explored next. Let us start by analyzing the firms' and households' decisions which follow from the solution to their respective optimization problems.

Firm's problem   Assume there is a continuum of measure one of identical firms. The dynamic profit maximization problem faced by them can be interpreted as an infinite series of equivalent static programs. Given this symmetry, it thus suffices to obtain the short-run capital and labor rental decisions. The period profit maximization problem in aggregate terms is:

$$\max_{K_t,L_t} \quad \pi_t = (1 - \tau_t) K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t$$

where r_t and w_t are the market prices for capital and labor, respectively. Their value is set in equilibrium and it is hence taken as given by the representative firm. K_t represents the aggregate capital stock and L_t is the fraction of population available for work, assumed for simplicity equal to total population, N_t. Finally, τ_t stands for a proportional tax that is levied by the government on production of the final good and whose purpose is to correct for the negative externality going from output to environmental quality. Profit is maximized when the following two conditions, written in per capita terms, are met:

$$\alpha \tau_t k_t^{\alpha-1} = r_t \tag{13}$$

$$\tau_t (1 - \alpha) k_t^\alpha = w_t \tag{14}$$

These equations state that firms hire capital and labor until the point where their marginal productivity (net of taxes) equals their rental prices.

Household's problem   The infinite horizon utility maximization problem solved by the representative household reads:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\ln c_t + \ln Q_t \right)$$

s.t.

$$k_{t+1} - k_t = r_t k_t + w_t + (T_t - m_t) - c_t$$

with k_0 given. The sources of household income consist of the returns to capital and labor plus a net government transfer, which the household takes as given. The latter is the result of subtracting the part that is confiscated by the government to finance abatement effort, m_t, from a subsidy that equals the revenue accruing from firms' profit taxation, $T_t = (1 - \tau_t) k_t^\alpha$. The resulting amount can then be used for savings or consumption. The first order necessary conditions of this problem are:

$$\frac{1}{c_t} = \lambda_t^k \tag{15}$$

$$\lambda_t^k = \lambda_{t+1}^k \beta \left(1 + r_{t+1} \right) \tag{16}$$

where λ_t^k represents the shadow price of the assets owned by the household. Combining the optimality conditions of the firm problem with the latter two, we obtain the market's Euler equation for consumption:

$$\frac{c_{t+1}}{c_t} = \beta \left(1 + \alpha \tau_{t+1} k_{t+1}^{\alpha-1} \right) \tag{17}$$

Equating (5) and (17) requires the social planner to set $\tau_t = 1 - \mu$ for all t. With this optimal tax in place, the pollution externality vanishes and individual actions achieve the second-best consumption-savings allocation. Note that the higher the emissions rate of output,

the lower firms' profits and the higher government revenue. This is because increasing μ is,*ceteris paribus*, the same as decreasing the social marginal productivity of capital.

Given linearity in abatement technology, the government converts the resources taken from the household directly into pollution abatement in a way such that the capital and environmental quality stocks accumulate optimally across time. Hence, the optimal fiscal policy consists of, first, a constant profit tax, second, a transfer to households that equals raised profit taxes, and, third, a lump-sum tax on household income equal to the socially optimal abatement level calculated by the government in every period.

2.3 Optimal control with a pollution irreversibility threshold

In this section, a non-convex structure for the independent dynamics of environmental quality is postulated in the context of the previous economy-environment model with exogenous population shocks. Firstly, we briefly introduce and discuss the type of threshold effects used. Secondly, the optimal growth problem is solved and analyzed in the same fashion as in the benchmark. Finally, results are compared to the baseline case, delivering a number of important policy conclusions.

Pollution and the environment

Let Q_t denote the stock of environmental quality in period t. In the absence of human activity, let the transition equation for the stock of environmental quality (for non-negative values) be denoted by:

$$Q_{t+1} - Q_t = \Gamma(Q_t)$$

The key assumption regarding the evolution of greenhouse gas (GHG) concentration levels is that nature's ability to absorb pollution hinges on the already existing pollutant concentration in the atmosphere. In particular, it is widely agreed by now among environmental

scientists that the decay function, $\Gamma(.)$, exhibits some type of non-linearities. This functional form is said to approximate relatively accurately the assimilation process of pollutants in ecosystems such as shallow lakes and there are reasons to believe that it is also a good representation of global environmental problems such as global warming (Prieur (2009)). Concordantly, many authors have adopted this view and assumed $\Gamma(.)$ to have an inverted-U shape. However, for the needs and purpose of this paper, the use of such a quadratic specification does not yield any additional insights to the ones obtained with a simpler, linear function. Therefore, for the sake of tractability, the decay function will be assumed linear in the level of environmental quality. Its properties, summarized below, convey the idea that, beyond a certain point, higher (lower) levels of pollution (quality) alter the natural regeneration capacity in an irreversible manner:

Assumption. $\Gamma(Q) : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and satisfies* $\Gamma(0) = 0$, $\exists\ \underline{Q} > 0$ *such that* $\Gamma(Q) = 0\ \forall Q \leq \underline{Q}$, $\Gamma(Q) > 0\ \forall Q \in \left(\underline{Q}, \overline{Q}\right)$. $\Gamma'(Q) > 0\ \forall Q \in \left[\underline{Q}, \overline{Q}\right)$. $\Gamma''(Q) = 0$ $\forall Q \in \left[\underline{Q}, \overline{Q}\right)$.

As before, $\overline{Q} < \infty$ denotes an exogenously given upper bound for environmental quality, which can be interpreted as the state of nature that prevailed prior to the advent of human civilization. Departing from a level of environmental quality close to this upper physical limit, the volume of pollution absorbed autonomously by nature is relatively high at first and starts declining linearly as greenhouse gases accumulate, just as in the benchmark scenario. Once the quality stock reaches a critical threshold level $\underline{Q}$, the assimilation capacity of nature is exhausted to the point that it ceases to perform its *cleaning* role, rendering it unable to sequestrate any amount of harmful pollutants. The graph below illustrates the assumptions just made about the shape of this function:

Figure 3: The assimilation function
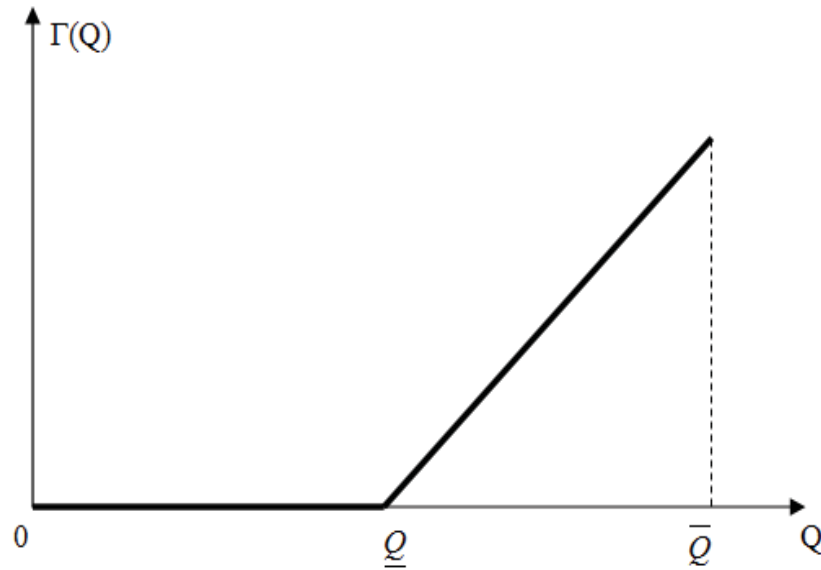
Although uncertainty remains about the exact level of GHG concentration that would trigger such a dramatic change in the absorption capacity of the environment, it is still commonly accepted that such a turning point exists and, thus, considering that the assimilation potential of nature depends on the pollution stock and could collapse at some point in time is an acceptable assumption, even if the exact form of the assimilation function is not perfectly known yet.

Given the previous discussion, the time path followed by environmental quality, after factoring in the effect of economic activity, is defined by parts as follows:

$$Q_{t+1} - Q_t = \begin{cases} N\left(m_t - \mu k_t^\alpha\right) & if \quad 0 \leq Q_t \leq \underline{Q} \\ \Gamma(Q_t) + N\left(m_t - \mu k_t^\alpha\right) & \forall Q_t \in \left(\underline{Q}, \bar{Q}\right] \end{cases}$$

The optimal allocation problem redefined

With the new dynamics for quality, the infinite horizon optimization problem solved by the social planner can be restated as:

$$\max_{\{c_t, m_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(N \ln c_t + \ln Q_t \right)$$

s.t.

$$k_{t+1} - k_t = k_t^\alpha - c_t - m_t$$

$$Q_{t+1} - Q_t = \begin{cases} N \left(m_t - \mu k_t^\alpha \right) & if \quad 0 \leq Q_t \leq \underline{Q} \\ \Gamma(Q_t) + N \left(m_t - \mu k_t^\alpha \right) & \forall \quad Q_t \in \left(\underline{Q}, \bar{Q} \right] \end{cases}$$

For $Q_t \in \left(\underline{Q}, \bar{Q} \right]$, the Lagrangian associated with this problem reads:

$$L =$$

$$\sum_{t=0}^{\infty} \beta^t \left[N \ln c_t + \ln Q_t - \lambda_t^k \left(k_{t+1} - k_t + c_t - k_t^\alpha + m_t \right) - \lambda_t^Q \left(Q_{t+1} - Q_t - \Gamma(Q_t) - N \left(m_t - \mu k_t^\alpha \right) \right) \right]$$

The first order necessary[12] conditions for an interior optimum[13] are:

$$\frac{N}{\lambda_t^k} = c_t \tag{18}$$

$$\lambda_t^k = N \lambda_t^Q \tag{19}$$

$$\lambda_t^k = \beta \left[(1 + \alpha k_{t+1}^{\alpha-1}) \lambda_{t+1}^k - N \alpha \mu k_{t+1}^{\alpha-1} \lambda_{t+1}^Q \right] \tag{20}$$

$$\lambda_t^Q = \beta \left[\frac{1}{Q_{t+1}} + (1 + \Gamma'(Q_{t+1})) \lambda_{t+1}^Q \right] \tag{21}$$

where $\Gamma'(Q_{t+1})$ is the first-order derivative of the regeneration function with respect to environmental quality at $t+1$. What was discussed before as regards the economic meaning

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[12]Given the properties described for $\Gamma(.)$, the concavity of the problem is maintained and therefore the conditions are also sufficient.

[13]As before, the $m_t = 0$ case is left out of the analysis.

of each of these equations in the baseline case of smooth natural dynamics applies here equally. The transversality conditions for each of the two states are also exactly the same as in the previous section.

In the remainder of the analysis, we will use the following functional form for the natural regeneration process: $\Gamma(Q_t) = \eta Q_t > 0$. This specification coincides with the proportional regeneration benchmark scenario previously studied, which exhibits all the properties assumed before regarding its shape. Thus, the difference with respect to the dynamic system of the baseline case lies only in the existence of a break-point $(\underline{Q})$ in the regeneration process of environmental quality captured by function $\Gamma(Q_t)$. As in the last section, after some simple algebraic manipulation of the first order conditions, the four equations governing the behavior of the system are:

$$\frac{c_{t+1}}{c_t} = \beta \left(1 + (1-\mu)\alpha k_{t+1}^{\alpha-1} \right) \tag{22}$$

$$\frac{1}{c_t} = \beta \left(\frac{1}{Q_{t+1}} + \frac{1+\eta}{c_{t+1}} \right) \tag{23}$$

$$k_{t+1} = k_t + k_t^\alpha - c_t - m_t \tag{24}$$

$$Q_{t+1} = (1+\eta)Q_t + N\left(m_t - \mu k_t^\alpha\right) \tag{25}$$

Given the equivalence between this equilibrium and the benchmark's, the interpretation of these relationships stays the same.

Steady State Analysis

The fact that the dynamics for environmental quality are piecewise defined implies that multiple steady states can emerge. We first characterize the long-run equilibrium for the case where the regeneration capacity is destroyed (i.e., $Q_t \leq \underline{Q}$) and then proceed to analyze the steady state equilibria for the case where natural pollution assimilation is operative (i.e., $\underline{Q} < Q_t \leq \bar{Q}$).

The irreversible steady state   In this case, the state of the environment is such that the history of damages to it causes natural regeneration to stop. As a matter of fact, this scenario coincides with the benchmark for the particular case where $\eta = 0$. We will refer to the long-run equilibrium in this situation as the *irreversible steady state*:

Definition. *The **irreversible** steady state (IRSS) is defined as the long run equilibrium of the system where the constant value for the stock of environmental quality is below the critical irreversibility threshold: $Q_{irss} \leq \underline{Q}$.*

Removing all time indices from the variables in the previous system of equations and setting $\eta = 0$, the irreversible steady state values for the four objects obtain:

$$k_{irss} = \left(\frac{\beta \alpha (1 - \mu)}{1 - \beta} \right)^{\frac{1}{1-\alpha}} \tag{26}$$

$$Q_{irss} = \left(\frac{\beta (1 - \mu) \alpha^{\alpha}}{1 - \beta} \right)^{\frac{1}{1-\alpha}} \tag{27}$$

$$c_{irss} = \left((1 - \mu) \left(\frac{\beta \alpha}{1 - \beta} \right)^{\alpha} \right)^{\frac{1}{1-\alpha}} \tag{28}$$

$$m_{irss} = \mu \left(\frac{\beta \alpha (1 - \mu)}{1 - \beta} \right)^{\frac{\alpha}{1-\alpha}} \tag{29}$$

A quick comparison of these with the ones obtained for the benchmark linear assimilation case reveals that the value for capital per capita coincides, whereas the other three variables take different levels and are not affected by population. The reason for this is straightforward: as opposed to the benchmark case where aggregate net emissions are strictly positive at the steady state, irreversibility implies that these emissions must equal zero for environmental quality to be constant. Since gross emissions (given by μk_t^{α}) do not change inasmuch as capital per capita is not influenced by population in the steady state, there is no allocation

shift between consumption and maintenance after a population shock. Given that abatement stays the same, net emissions are unaltered. Consequently, population shocks have no long-run effects on any of the variables.

The following table summarizes the comparative static effects of changes in the fundamental parameters on the variable's steady state values for the irreversible case:

	Q_{bss}	k_{bss}	c_{bss}	m_{bss}
N	0	0	0	0
β	> 0	> 0	> 0	> 0
μ	< 0	< 0	< 0	$??$
α	$??$	> 0	$??$	$??$

Table 2: comparative static results for preference and technology parameters. A double question mark stands for an indeterminate effect.

The economic intuition behind the signs of these steady state effects are exactly the same as the ones discussed for the benchmark case for all parameters except population, whose effects were just explained.

The reversible steady state   In this instance, nature's pollution assimilation is strictly positive and proportional. We will refer to the long-run equilibrium in this case as the *reversible steady state*:

Definition. *A **reversible** steady state (RSS) is defined as the long run equilibrium of the system where the constant value for the stock of environmental quality is above the critical irreversibility threshold:* $\underline{Q} < Q_{rss} \leq \bar{Q}$.

The only object that coincides in the IRSS and RSS equilibria is the wealth level achieved, measured by the value of per capita capital[14]. The following proposition establishes the

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[14]Recall that the benchmark and reversible scenarios are exactly the same. Hence, all comparisons between the IRSS and RSS apply equally to the benchmark steady state. We omit the duplication of the set of equations (9)-(12) here, but encourage the reader to keep them in mind in what follows

signs of the differences in levels among the rest of the variables between each scenario and calls attention to the joint economic and environmental trap property of the IRSS:

Proposition 1. *The IRSS possesses the characteristics of an environmental and economic poverty trap, for a higher abatement effort (and, thus, less consumption) is needed to attain a level of environmental quality that is lower than that in the RSS: $m_{rss} < m_{irss}$, $Q_{rss} > Q_{irss}$, $c_{rss} > c_{irss}$, $k_{rss} = k_{irss}$.*[15]

According to this proposition, the representative agent is better off in the RSS since welfare is unambiguously higher in this long-run equilibrium. Moreover, the IRSS represents a truly inescapable trap in the sense that, unless technological change or exogenous wealth shocks are considered, the system can never move away from this grievous situation and is thus *doomed* to remain permanently at it. This result is partially supportive of the strand of the development literature (e.g., Rostow's stages of growth model) that postulates the need for a direct capital injection into developing (in this case, both economically and environmentally poor) countries to enable them to *take off*.

Transitional dynamics

As previously discussed, an innovation in population does not have any bearing on the environment-economic system when it is located in the irreversible steady state. Outside the steady state, population amplifies the effect of net emissions on the evolution of quality and, hence, affects the speed and shape of the transition. However, when the system is located at the IRSS, all variables remain unaltered at their steady state values upon a shock in population. As a result, we are only left to analyze the transitional dynamics for the reversible case. Given equivalence to the benchmark case, the analysis in section 2.3 applies here equally.

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[15]Proof in the Appendix.

<u>**The path to an environmental and economic trap**</u>   Next we provide an example whereby an ever increasing population irrevocably leads the economy to an environmental-economic trap. The following graph illustrates the transitional dynamics in (Q_t, k_t) space occurring after a permanent positive shock in population[16]. This demographic change entails a negative variation in the steady state value for environmental quality so large that it lies below the irreversibility threshold. Thus, a RSS as such no longer exists after the population increase and the environmental system starts moving towards the irreversibility frontier from the very period in which the shock takes place. The dynamics in the irreversibility region take over once the $\underline{Q}$ threshold is reached. From then onwards, the stock of environmental quality follows the optimal path that leads to the IRSS that corresponds to the given parameter values.

Figure 4: Transition path for environmental quality and capital per capita after a permanent change in N.

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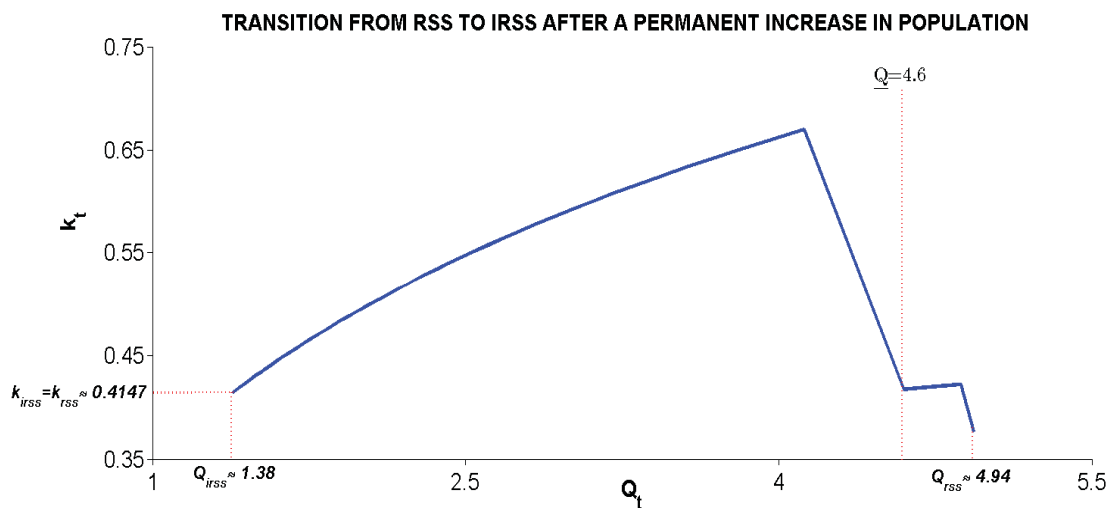
[16]Population increases from an original value of 1 to 1.1, the same 10% change as in the previous simulations. The size of the increase suffices to take the system to the irreversible region departing from a point close enough to the threshold. The rest of the parameters stay constant during the simulation. The values assigned to them are: $\alpha = 0.3$, $\beta = 0.9$, $\eta = 0.04$, $\mu = 0.8$, $\underline{Q} = 4.6$.

A number of important points can be gathered in light of this graph. First, as previously shown, demographic variations have no impact on the steady state level of per capita capital. This lack of influence in the long-run notwithstanding, population does have an impact on the short-run dynamics for this variable.

Second, the factors behind the type of movements observed for the two stocks at the beginning of the simulation are the same as the ones discussed in the transitional dynamics of section 2 for a permanent shock to population. For the reasons explained there, quality decreases from the beginning while the economy deepens its capital at first until a smooth downward sloping path for both states is achieved[17].

Third, the most remarkable result stemming from this simulation exercise is that the possibility exists that a rise in the number of households (or a positive population growth rate for that matter) can destroy nature's ability to regenerate environmental quality and, more importantly, make society sink into an environmental and economic poverty trap. Without exogenous aid in the form of technology or capital improvements, this trap is inescapable. One might argue that the social planner may try to avoid this outcome by investing in abatement prior to crossing the irreversibility frontier. However, even if the social planner could contemplate this way out, if population continues to rise indefinitely, the cost of maintenance would get increasingly higher and cause society to have no alternative but to let the economy be absorbed by the trap. This is so because the opportunity cost in terms of foregone consumption would become unbearable as more and more resources are allocated to abatement. Once the threshold is crossed from the right, reverting back to the reversibility zone is more burdensome than before, for in this case nature no longer plays its regeneration role that formerly helped agents save resources for consumption and capital

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[17]It is worth noting that these variations in capital and environmental quality would be less abrupt if we considered period-by-period infinitesimal changes in population. The population shock assumed produces nevertheless a stable transition to a new steady state. Since Dynare is based on a linear approximation around the steady state, this implies that the change in population is still small enough for it to be considered inside the neighbourhood of the original steady state.

accumulation. Thus, in the context of the model, an ever increasing population always leads to a welfare-costly trap situation.

Conclusions

In the introduction, we posited two policy-relevant questions for which a simple economy-environment model has been proposed. First, concerning population and development issues, according to our specification, an economy with both poor environmental and economic states may find itself in a trap which is unaffected by changes in population. In this gloomy scenario, demographic policies aimed at reducing population via immigration or fertility measures are ineffective in an attempt at escaping or avoiding a costly development trap. These economies therefore need exogenous aid in the form of, for example, an improved maintenance technology or a capital inflow for them to be able to achieve long-run prosperity. The possibility exists, however, for relatively wealthy societies whose environment is damaged to the point of the extinction of natural regeneration, to reverse this situation by sacrificing present consumption in return for higher long-run levels of environmental quality and consumption[18]. Yet, an economy endowed with a low starting level of wealth, a high time discount rate, a low level of natural regeneration productivity (captured by the functional form of $\Gamma(Q)$) or a large population (which acts as a utility weight for consumption) will tend to find it optimal to choose to follow a path leading to the economic-environment trap.

Regarding the question whether considering potential irreversibility thresholds matters, the results derived indicate that they are indeed important in shaping demographic policies, provided that it is still the case that the state of nature has not been degraded to the point that it has stopped performing its regeneration functions. Specifically, irreversibilities make the case for the exertion of caution favoring demographic control measures to, if not halt

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[18]A welfare comparison analysis of the different trajectories must be carried out to investigate this. What we can firmly assert, nevertheless, is that the lower the capital level the economy is endowed with, the less likely it is that making an effort to take the system to the RSS equilibrium is worthwhile

population growth altogether, at least reduce its rate considerably until strong evidence of absolute decoupling in production is found.

The reason for why the conclusions reached here differ strikingly from the smooth regeneration conventional models is that in the economic-environmental trap that the IRSS is characterized by, it is impossible to "'undo"' the damages of an increasing population by reversing its trend as it can be done in the case where no threshold turning points are present. Returning to higher environmental quality and consumption levels in the benchmark case only requires the reduction of the population size to smaller levels. In the irreversible scenario, however, this is infeasible since population changes do not have any bearing on the steady state and only other types of shocks can save the economy. Thus, we conclude that considering zero or proportional regeneration can make a big difference when it comes to the evaluation of population impacts. Overlooking the possibility of irreversible changes in the stock of pollution in the atmosphere misses hence an important point.

As argued before, in the context of the model, a constantly increasing population could only be made compatible with not incurring into irreversibility problems if aggregate net emissions are kept at bay by devoting an increasing amount of resources to abatement over time in order to prevent further deterioration of the environment. Nonetheless, if population grows unbound, the consumption costs of abatement will eventually outweigh the benefits in terms of a cleaner environment, which renders this potential trajectory unsustainable in the long run. All in all it can be concluded that population growth leads irrevocably to the trap unless other ingredients such as labor or abatement productivity-enhancing technological progress are considered. The addition of technological change is thus a possible fruitful avenue for future research.

Appendix

A: Sufficiency conditions

The second-order sufficient conditions for a local maximum of the optimal dynamic allocation program require that the Hessian matrix associated with the maximized Lagrangian of the corresponding static problem at any point in time be negative semidefinite (Arrow and Kurz (1970), Proposition 6). We show that our problem satisfies this condition in the following lines.

The maximized Lagrangian of the static problem, L_t^0, is obtained by substituting in the first order necessary conditions (1)-(5):

$$L_t^0 = -N \ln \lambda_t^Q + \ln Q_t - N\lambda_t^Q \left(k_{t+1} - k_t - k_t^\alpha\right) - N - \lambda_t^Q \left(Q_{t+1} - Q_t(1+\eta) + \mu N k_t^\alpha\right)$$

To establish sufficiency, we need to check that

$$\begin{pmatrix} \dfrac{\partial^2 L_t^0}{\partial k_t^2} & \dfrac{\partial^2 L_t^0}{\partial k_t \partial Q_t} \\ \dfrac{\partial^2 L_t^0}{\partial Q_t \partial k_t} & \dfrac{\partial^2 L_t^0}{\partial Q_t^2} \end{pmatrix}$$

is negative semidefinite.

The second-order partial and cross derivatives read:

$$\frac{\partial^2 L_t^0}{\partial k_t^2} = (1-\alpha)\alpha\lambda_t^Q N k_t^{\alpha-2}(\mu-1) < 0$$

$$\frac{\partial^2 L_t^0}{\partial k_t \partial Q_t} = \frac{\partial^2 L_t^0}{\partial Q_t \partial k_t} = 0$$

$$\frac{\partial^2 L_t^0}{\partial Q_t^2} = -\frac{1}{Q_t^2}$$

In this case, it suffices to investigate the sign of the determinants of the Hessian matrix's leading principal minors:

$$D_1 = \frac{\partial^2 L_t^0}{\partial k_t^2} = -\frac{1}{Q_t^2} < 0$$

$$D_2 = \begin{vmatrix} \dfrac{\partial^2 L_t^0}{\partial k_t^2} & 0 \\[2ex] 0 & \dfrac{\partial^2 L_t^0}{\partial Q_t^2} \end{vmatrix} > 0$$

where the last inequality follows from the fact that $\dfrac{\partial^2 L_t^0}{\partial k_t^2} < 0$ and $\dfrac{\partial^2 L_t^0}{\partial Q_t^2} < 0$.

Hence, given that the determinants of the principal minors are of alternating sign, the Hessian matrix is negative definite. Thus, we conclude that the first order conditions are not only necessary but sufficient for a local maximum of the optimization problem.

B: Proof of proposition 1

Proof.

Recall that μ, η, $\beta \in (0, 1)$.

Rewrite Q_{rss} as:

$$Q_{rss} = \frac{N \left(\beta(1-\mu)\alpha^\alpha \right)^{1/1-\alpha}}{(1-\beta)^{\alpha/(1-\alpha)} \left(N(1-\beta(1+\eta)) - \beta\eta \right)}$$

Then, a direct comparison between this value and the one for Q_{irss} reveals that it is greater than the latter if and only if the following condition is satisfied:

$$\frac{N(1-\beta)}{N(1-\beta(1+\eta)) - \beta\eta} > 1$$

This is always the case given that $N > -1$. Thus, $Q_{rss} > Q_{irss}$ for any parameter values.

Rewrite c_{rss} as:

$$c_{rss} = \frac{N(1-\beta(1+\eta)) \left((1-\mu)(\alpha\beta)^\alpha \right)}{(1-\beta)^\alpha N(1-\beta(1+\eta)) - \beta\eta}$$

Then, a direct comparison between this value and the one for c_{irss}, reveals that it is greater than the latter if and only if the following condition is satisfied:

$$\frac{N(1 - \beta(1 + \eta))}{N(1 - \beta(1 + \eta)) - \beta\eta} > 1$$

This is always the case given that $-\beta\eta < 0$. Thus, $c_{rss} > c_{irss}$ for any parameter values.

Lastly, rewrite m_{rss} as:

$$m_{rss} = \frac{(N\mu(1 - \beta(1 + \eta)) - \beta\eta)(\alpha\beta(1 - \mu))^{\alpha/1-\alpha}}{(1 - \beta)^{\alpha/1-\alpha}(N(1 - \beta(1 + \eta)) - \beta\eta)}$$

Then, a direct comparison between this value and the one for m_{irss}, reveals that it is greater than the latter if and only if the following condition is satisfied:

$$\frac{N\mu(1 - \beta(1 + \eta)) - \beta\eta}{\mu(N(1 - \beta(1 + \eta)) - \beta\eta)} > 1$$

This is never the case given that $\eta\mu < \eta$. Thus, $m_{rss} < m_{irss}$ for any parameter values.

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Chapter 3

The implications of habit and congestion for optimal consumption taxation[1]

JEL codes: E62, H21, H23.

Keywords: Habits, Congestion, Endogenous growth, Externality, Second-best.

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[1]This chapter draws heavily from Alonso-Carrera, Caballé and Raurich (2001) and is intended as a baseline for future joint work with these authors.

3.1 Introduction

The important consequences of habit formation for the design of fiscal policy aimed at correcting average-consumption based congestion externalities analyzed in this paper represent a new contribution to the strands of literature nearest in scope to ours. On the one hand, the work on habits has mainly focused on their role as a potential resolution to some of the empirical *puzzles* that arise under more conventional utility specifications. On the other hand, studies including congestion externalities are mainly circumscribed to the analysis of the optimal provision of public infrastructure goods that are subject to capacity limits. Thus, to our knowledge, the problem of optimal taxation of a congestionable and habitual consumption good is, to date, an unexamined issue to which this paper sheds some light on. We proceed first by briefly summarizing earlier contributions. Then, a discussion of the way this paper is related to the literature and motivation are provided.

With respect to the existence of habitual patterns in consumption, there is by now a large body of empirical evidence showing that the satisfaction derived from material consumption is subject to a sizeable habituation component (Layard (2005)). The habituation effect has been shown to be present in all forms of consumption. It refers to the fact that individuals tend to become accustomed to an ever increasing consumption flow of goods and services. Put another way, departing from any level, marginal increases in consumption have a low impact on utility relative to marginal reductions in it. A number of microeconometric studies provide evidence of such an effect. Clark (1999) finds that the main determinant of job satisfaction in a panel of UK workers is the change in wages over time. In a cross-country study, Di Tella et al (2002) showed that lagged income was almost as good a predictor of reduced happiness as current income. This evidence highlights the importance of the constant attempt of individuals at improving their standard of living due to acquired habits in materialistic well-being.

On top of the empirical literature on the existence of habits, several authors have incorpo-

rated them in theoretical models as a mechanism for the explanation of important challenges such as the equity premium puzzle. In the presence of habits, the equilibrium intertemporal elasticity of substitution is lower and, hence, the coefficient of relative risk aversion higher due to the lower willingness to experience changes in consumption, ceteris paribus. This higher risk aversion in turn generates higher risk premia in equilibrium that are in line with the findings in Mehra and Prescott (1985)[2].

Regarding the treatment of congestion, as mentioned before, the bulk of the literature revolves around the analysis of its impact on the services derived from public infrastructure, which influence either firms' productivity and/or individuals' utility. A typical example is road congestion: the use of vehicles during certain times of the day creates congested traffic which greatly affects the time efficiency of, for instance, transportation services[3]. Given the inconveniences it causes, it might also negatively affect individual's utility.

Fisher and Turnovsky (1998) falls into the first category of analyses of the supply-side consequences of congested public capital goods. In particular, it explores the long and short run effects of public investment on the behavior of private capital accumulation. Public capital enters into the production function in such a way that the amount of public services obtained by private firms depends on their individual to aggregate capital ratio[4].

An inquiry on congestion externalities affecting not only technology but also preferences is made in Ghosh and Chatterjee(2009). Identically to the manner in which congestion damages productivity, it is assumed that the size of the congestion effect on utility depends on the amount of capital an individual owns relative to the aggregate. The aim of the paper is to disentangle the optimal fiscal policy mix given that the external effect stems from private capital accumulation rather than individuals' consumption. The latter is, in

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[2]See, e.g., Abel (1990) and Boldrin et al. (2001).

[3]Mayeres and Proost (1997).

[4]This specification of congestion is based on the median voter demand model and is ubiquitous in the literature. Other examples of productivity-harming congestion in public infrastructure include Eicher and Turnovsky (2000), Ghosh and Nolan (2007) and Gómez (2010)

contrast, the source of the externality in our analysis.

All in all, on the topic of congestion, it seems that most authors assume that the cause of the negative externality is in the individual accumulation of some factor relative to the aggregate; to our knowledge, it has not been proposed yet to model congestion in the consumption of a public good subject to habits in the context of a simple endogenous growth model[5]. Aggregate consumption externalities have been used in studies such as Galí(1994) as an additional element to solve the equity premium puzzle. However, this common "keeping up with the Joneses" feature has a different interpretation than that of congestion. The former is based on the idea of envy and group reference effects, whereas from a congestion perspective, others' consumption is viewed as generating rivalry for a limited quantity of a certain public facility.

We will adopt the latter view of congestion and posit the question of the optimal correction of a consumption externality from a new angle[6]. This approach is fruitful in delivering a number of valuable insights for a number of reasons. First, as will be shown, the introduction of either habits or consumption spillovers reduces the model's predicted rate of convergence to the stationary equilibrium to a value more consistent with the empirical evidence in, for example, Ortigueira and Santos (1997). Thus, using either of these two elements in any growth model seems to be a more adequate strategy than using the conventional framework. Second, habits are necessary to generate transitional dynamics in an endogenous model such as the one featuring AK technology that we use. It is lagged consumption in preferences that makes the economy converge gradually to the balanced growth path. This is a fundamental property for the study of the optimal taxation path of a growing economy as the next point

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[5]some studies suggest the accumulation of workers or output instead as the source of congestion, but not consumption

[6]Note that the congestion externality can alternatively be thought of as an environmental consumption-based negative externality. Examples include the generation of household waste and greenhouse gas emissions entirely due to direct consumption (i.e., not embedded in production). According to Hertwich (2011), as much as 46% of global average total household greenhouse gas releases due to consumption can be attributed to the shelter and mobility categories, which are mainly comprised of fuel and flow energy use and, hence, primarily constitute direct rather than indirect environmental impact.

highlights. Third, it can be shown that in the absence of habit formation in consumption, if the centralized and competitive equilibria exist, the decentralized intertemporal marginal rate of substitution coincides with that of the social planner, meaning that the competitive equilibrium is efficient and there is no role for taxation. Even though some technical conditions need to be met, adding a process of habit formation gives rise to inefficiencies due to the congestion externality, thus calling for the design of an optimal fiscal policy to eliminate them.

Equipped with these ingredients, we characterize the consumption tax rate that implements the socially optimal solution. The optimal gross consumption growth rate is counter-cyclical off the steady growth equilibrium and tends to zero in the long run since the competitive balanced growth path exhibits no inefficiencies. Furthermore, the stronger the congestion effect on utility from aggregate consumption, the higher the taxation path profile. As will become apparent, the qualitative impact of an increase in the strength of congestion on the speed of convergence depends on where the state of the economy lies with respect to its balanced growth path.

This chapter is organized as follows. Section 3.2 presents the endogenous growth model with consumption habit formation and congestion in the utility and derives the equations describing the dynamic competitive equilibrium. In section 3.3, the local behavior of the competitive equilibrium around its steady state is studied. Furthermore, we establish a number of results on the short and long run effects of variations in the tax growth rate and preference parameters. Section 3.4 replicates the contents of section 3.3 for the centrally planned economy case. In section 3.5, the optimal path for the consumption tax gross growth rate is characterized and its relationship with the parameters capturing the importance of habits and congestion analyzed. Finally, section 3.6 contains concluding remarks.

3.2 The Model

In this section, the system of equations governing the decentralized market equilibrium is computed by solving the representative consumer problem, given the available savings technology. We start by describing the instantaneous utility function of individuals, whose special structure is at the core of the majority of the results that will be obtained.

Let $u(h_t, \bar{c}_t)$ denote the felicity function of the representative dynasty, where $\bar{c}_t$ represents average consumption in the economy at period t. The first argument, $h_t = c_t - \gamma c_{t-1}$, with $\gamma \in (0, 1)$ as the inverse of the depreciation rate of past consumption, reflects the fact that the utility derived from consumption at any date does not hinge on the current level only, but, rather, on this level in comparison with the one in the previous period, which acts as a reference or standard level of consumption. In this sense, for technical reasons, the process of habit formation is introduced additively[7]. Nevertheless, all the results derived would apply equally in a multiplicative specification.

We draw from Galí's (1994) choice (which features consumption spillovers only) in specifying the following function for the ordering of preferences[8]:

$$u(h_t, \bar{c}_t) = \frac{h_t^{1-\sigma} \bar{c}_t^{-\theta\sigma}}{1 - \sigma} \tag{1}$$

where $\sigma > 0$ is the coefficient of relative risk aversion with respect to habit-adjusted consumption and $\theta \in [0, 1)$ captures the strength of the negative congestion effect of others' consumption on utility[9]. An important restriction imposed on these two parameters is:

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[7]as opposed to recent contributions such as Abel (1990,1991), Carroll et al. (1997,2000) and Carroll (2000).

[8]Two important assumptions imposed on the shape of this function are homogeneity of the partial derivatives with respect to its two arguments and a positiveness degree of complementarity between the two. The former assumption is needed to guarantee existence of balanced growth paths, while the latter is imposed so as to ensure inefficiency of the competitive equilibrium

[9]Hence, we assume that negative congestion externalities occur at any positive aggregate consumption level. The idea behind this is that, for instance, even though a few cars on the roads do not generally cause traffic congestion, they still diminish the quantity (or quality) of services we derive from this publicly provided infrastructure. We also assume that the harm done in utility terms is proportional to the number of cars on the road.

$\frac{\theta}{\sigma-1} \geq 0$. This is the same as requiring $\sigma > 1$, which entails a stationary intertemporal elasticity of substitution greater than one. It can be shown that this restriction is needed to ensure both monotonicity and joint concavity of the objective function with respect to c_t and c_{t-1} in both the competitive and social planner problems[10]. The following properties of the utility function hold:

$$u_h(h, \bar{c}) > 0$$

$$u_{\bar{c}}(h, \bar{c}) < 0$$

$$-\frac{u_{hh}(h, \bar{c})h}{u_h(h, \bar{c})} = \sigma$$

$$u_{h\bar{c}}(h, \bar{c}) < 0$$

The first three properties are standard. The fourth identity implies that the marginal utility of habit-adjusted consumption decreases with average consumption, which reflects the negative congestion externality by which the consumption of a public service (e.g., highway) by others lowers the marginal utility of own consumption.

Household's Optimization Problem

Let us consider an economy in discrete time populated by a constant continuum of infinitely-lived, identical households of measure 1 that maximize the discounted sum of the flow of instantaneous utilities. The representative individual is endowed with a production technology per capita in every period,

$$y_t = Ak_t$$

where $A > 0$ is the constant marginal productivity of capital and where k_t and y_t denote capital and output per capita, respectively.

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[10]A proof of this point is available upon request.

The government levies a proportional tax, τ_t, on household's consumption in every period. The proceeds, S_t, are transferred back to the individuals in a lump-sum fashion. Given that the units produced of the final good can only be allocated to either savings or consumption, the budget constraint of the representative dynasty then reads:

$$(1 + \tau_t)c_t = (1 + A)k_t + S_t - k_{t+1} \tag{2}$$

Let $\beta \in (0, 1)$ capture individuals' time preference. Taking the initial capital per capita, k_0, the *inherited* initial consumption level, c_{-1}, and the sequence of economy-wide consumption, $\{\bar{c}_t\}_{t=0}^{\infty}$, as given, the representative household solves the following dynamic maximization program:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(h_t, \bar{c}_t)$$

s.t.

$$(1 + \tau_t)c_t = (1 + A)k_t + S_t - k_{t+1}$$

where the objective function is given by (1). The Lagrangian associated with the problem above is:

$$L(c, k, \lambda) = \sum_{t=0}^{\infty} \beta^t \left[u(h_t, \bar{c}_t) + \lambda_t \left((1 + A)k_t - (1 + \tau_t)c_t + S_t - k_{t+1} \right) \right]$$

where $\lambda = \{\lambda_t\}_{t=0}^{\infty}$ is the infinite sequence of current-value Lagrange multipliers. The first order conditions of this problem read:

$$\frac{\partial L}{\partial c_t} = (1 + \tau_t)\lambda_t - u_h(t) + \beta\gamma u_h(t+1) = 0 \tag{3}$$

$$\frac{\partial L}{\partial k_{t+1}} = \lambda_t - \beta\lambda_{t+1}(1 + A) = 0 \tag{4}$$

where $u_h(t) = (c_t - \gamma c_{t-1})^{-\sigma} \bar{c}_t^{-\theta\sigma}$. The transversality conditions attached to this optimization problem are:

$$\lim_{t \to \infty} \beta^t \lambda_t^k k_{t+1} = 0 \tag{5}$$

$$\lim_{t \to \infty} \beta^t u_h(t) c_t = 0 \tag{6}$$

With the restrictions on the parameters assumed before in place, these boundary conditions coupled with the first order conditions, the budget constraint and the initial conditions are not only necessary but sufficient for characterizing the solution to the equilibrium paths for consumption and capital.

Combining (3) and (4), we obtain a version of the standard Euler equation modified to account for the time-dependent nature of consumption in utility:

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{1 + \tau_{t+1}}{(1 + \tau_t)\beta(1 + A)} = \frac{u_h(t+1) - \beta\gamma u_h(t+2)}{u_h(t) - \beta\gamma u_h(t+1)}$$

Letting $\psi_{t+1} = \frac{(1+\tau_t)(1+A)}{1+\tau_{t+1}}$, we can rewrite the latter equation as:

$$\frac{1}{\beta\psi_{t+1}} = \left(\frac{u_h(t+1)}{u_h(t)}\right) \left(\frac{1 - \beta\gamma(\frac{u_h(t+2)}{u_h(t+1)})}{1 - \beta\gamma(\frac{u_h(t+1)}{u_h(t)})}\right) \tag{7}$$

Noting that, in equilibrium, $c_t = \bar{c}_t$ for all t. Thus, the marginal utility of habit-adjusted consumption stays: $u_h(t) = (c_t - \gamma c_{t-1})^{-\sigma} c_t^{-\theta\sigma}$. Define the gross growth rate of the marginal utility of habit-adjusted consumption as:

$$f_t = \frac{u_h(t+1)}{u_h(t)}$$

Then, (7) simplifies to:

$$\frac{1}{\beta\psi_{t+1}} = f_t\left(\frac{1-\beta\gamma f_{t+1}}{1-\beta\gamma f_t}\right)$$

Isolating f_{t+1} from the latter equation, one obtains:

$$f_{t+1} = \frac{1}{\beta\psi_{t+1}}\left(1-\frac{1}{\beta\gamma f_t}\right) + \frac{1}{\beta\gamma} \qquad (8)$$

Let $x_{t+1} = \frac{c_{t+1}}{c_t}$ denote the gross growth rate of consumption. From the definition of f_t, using the explicit $u(t)$ specified before, we can write:

$$f_t = \left(\frac{u(t+1)}{u(t)}\right)\left(\frac{c_t - \gamma c_{t-1}}{c_{t+1} - \gamma c_t}\right) = x_t^{-\sigma}\left(\frac{x_{t+1}-\gamma}{x_t-\gamma}\right)x_{t+1}^{-\theta\sigma}$$

The latter can be expressed as an equation on the gross growth rates of consumption and marginal utility:

$$g(x_{t+1}, x_t, f_t) = \left(\frac{x_t - \gamma}{x_{t+1}-\gamma}\right)(x_{t+1})^{-\theta} - x_t f_t^{1/\sigma} = 0 \qquad (9)$$

Assume that the government runs a balanced budget in every period. Then, the government budget constraint is:

$$\tau_t c_t = S_t \quad \forall t$$

Substituting the latter identity into the household's budget constraint, (2), the economy's resource constraint obtains:

$$k_{t+1} = (1+A)k_t - c_t$$

Dividing this equation by c_t and defining the capital-consumption ratio, $z_{t+1} = k_{t+1}/c_t$, as the only one state of the economy, we get the following difference equation:

$$z_{t+1} = \left(\frac{z_t}{x_t}\right)(1+A) - 1 \qquad (10)$$

The system of difference equations (8), (9) and (10) plus the initial condition $z_0 = k_0/c_{-1}$ and the transversality conditions, (5) and (6), fully describe the decentralized market equilibrium. We proceed to analyze its properties in the next section.

3.3 The competitive equilibrium

Let us first discuss the long-run stationary equilibrium and then continue by characterizing the short-run dynamic behavior of the economy. In both cases, the implications of changes in the tax policy and in the fundamental preference parameters are examined.

We assume throughout the analysis in this section that the government chooses a constant consumption tax growth rate: $\eta = \dfrac{1 + \tau_{t+1}}{1 + \tau_t} \quad \forall t$. Then,

$$\psi_t = \frac{1 + A}{\eta} \equiv \psi$$

Along the balanced growth path (BGP), all variables in per capita terms grow at a constant rate. From the resource constraint, it follows that all variables must share the same constant growth rate in equilibrium as well. From the definition of both z_t and f_t, it is clear that they must also be constant in the long-run. Thus, making $x_t = x$, $f_t = f$ and $z_t = z$ in the system (8)-(10), the steady state values of these three variables read[11]:

$$f = \frac{1}{\beta\psi} \tag{11}$$

$$x = f^{-1/\sigma(1+\theta)} \tag{12}$$

$$z = \frac{x}{(1 + A) - x} \tag{13}$$

---

[11]The difference equation (8) has in fact two steady state roots for f: the one in (11) and another equal to $1/\beta\gamma$. However, the latter can be ruled out as it violates the positiveness condition on the Lagrange multiplier discussed further below.

The next lemma establishes parametric restrictions for the existence of a well-defined BGP exhibiting strictly positive growth:

Lemma 1. *If a competitive BGP with $x > 1$ exists, then the following three inequalities must hold:*

$$\beta\psi > 1 \tag{14}$$

$$\beta\psi^{1-\sigma(1+\theta)} < 1 \tag{15}$$

$$1 + A > (\beta\psi)^{1/\sigma(1+\theta)} \tag{16}$$

Proof. The first condition is needed for $x > 1$, which is the same as requiring that $f < 1$, as follows from (11) and (12), which in turn equivalent to condition (14). The fact that $x > 1$ ensures that the arguments of the utility function are always positive in the steady state. Using the definitions of f and x, note that satisfaction of transversality condition (6) at a BGP requires that $\beta f x < 1$. Given (11) and (12), the latter is the same as requiring that restriction (15) be always in place. Finally, for $z > 0$, we simply need that $1 + A > x$, which is equivalent to (16). ■

As a final note on the necessary conditions for a well-defined equilibrium consider the BGP expression for the Lagrange multiplier, λ_t, that follows from the first order condition (3):

$$(1 + \tau_t)\lambda_t = \beta^t u_h(t)(1 - \beta\gamma f)$$

From this we can see that $\lambda_t > 0$ if and only if $\beta\gamma f < 1$. Given (11), this inequality is equivalent to $\psi > \gamma$. The latter always holds since $\beta < 1$ and inequality (14) imply $\psi > 1$ and $\gamma < 1$ by assumption[12].

---

[12]Last but not least, the transversality condition on capital is also satisfied. To see this, we make use of the BGP expression for λ_t in (5) and note that (15) is equivalent to $\beta f x < 1$

From inspection of (11)-(13), the long-run effects of changes in the consumption tax growth rate, the size of the congestion externality and the persistence of habits can be summarized in the following proposition:

Proposition 1. *The signs of the effects of different parameter changes on the stationary values of the system's variables are:*

$$\frac{\partial x}{\partial \eta} < 0 \quad ; \quad \frac{\partial f}{\partial \eta} > 0 \quad ; \quad \frac{\partial z}{\partial \eta} < 0$$

$$\frac{\partial x}{\partial \theta} < 0 \quad ; \quad \frac{\partial f}{\partial \theta} = 0 \quad ; \quad \frac{\partial z}{\partial \theta} < 0$$

$$\frac{\partial x}{\partial \gamma} = 0 \quad ; \quad \frac{\partial f}{\partial \gamma} = 0 \quad ; \quad \frac{\partial z}{\partial \gamma} = 0$$

Proof.   Trivial from considering the set of restrictions (14)-(16) and using them in (11)-(13). In particular, the sign of the effect of θ on x is due to $f_t < 1$. ∎

An increase in the stationary consumption tax growth rate promotes current consumption at the expense of savings and, thus, future growth. Given decreasing marginal utility of habit-adjusted consumption, this results in a higher ratio of future to current marginal utility, closer to one. Finally, a lower capital investment flow due to higher *de facto* taxes on the return to capital also leads to a lower capital-consumption ratio in the long-run.

In turn, a rise in the importance of habit formation for utility leaves all steady state values unchanged. Although a positive variation in the persistence of habits does reduce the willingness of individuals to experience swings in their consumption profile and, thus, makes short-run transitions more sluggish, it has no impact in the long-run growth rate. The reason for this is that the intertemporal marginal rate of substitution of consumption is unaffected in the BGP[13]. Thus, the optimal consumption-savings decision of the representative individual remains the same.

---

[13]To see this, note that, in the stationary equilibrium, parameter γ changes the marginal utilities of habit-adjusted consumption homogeneously; for any two consecutive periods, taking into account the constant consumption gross growth rate, the marginal rate of substitution can be written as the ratio of marginal utilities in each period:

As regards the effects of an increase in the strength of the congestion externality, θ, we observe that both the consumption growth rate and the capital-consumption ratio diminish, while the ratio of marginal utilities remains the same. As can be seen shortly below, a positive change in θ has the same qualitative effects as a positive change in σ: both reduce the intertemporal elasticity of substitution, which slows down growth in the long run for a fixed marginal product of capital. The reduction in the intertemporal elasticity tends to decrease the marginal rate of substitution, whereas slower growth tends to increase it[14]. These two countervailing effects render the BGP gross growth rate of the marginal utility of habit-adjusted consumption, f, invariant. Put differently, individuals, faced with a smaller exogenous term due to consumption spillovers in the utility function, decide to offset the negative effect on the marginal utility of habit-adjusted consumption by decreasing the distance in consumption levels across periods (i.e., by effectively decreasing habit-adjusted consumption).

The expression for the stationary elasticity of intertemporal substitution, defined as the elasticity of the stationary consumption growth rate with respect to the return to capital, is:

$$\frac{\partial lnx}{\partial ln((1+A)/\eta)} = \frac{1}{\sigma(1+\theta)}$$

As it is intuitive, an increment in the return to capital promotes savings and leads to a higher BGP consumption growth rate. Also, as usual, an increase in the individual's relative risk aversion (i.e., increase in σ) makes her more unwilling to substitute consumption across time. Las but not least, a higher importance of congestion for the utility of individuals (i.e., higher θ) implies a lower intertemporal elasticity of substitution[15].

$$MRS_{t,t+1} \equiv \frac{u_h(t+1)}{u_h(t)} = \left(\left(\frac{x-\gamma}{x-\gamma}\right)\left(\frac{c_t}{c_{t-1}}\right)\right)^{-\sigma}\left(\frac{c_t}{c_{t-1}}\right)^{-\sigma\theta} = \left(\frac{c_t}{c_{t-1}}\right)^{-\sigma(1+\theta)}$$

Hence, γ cancels out and has a zero net impact on the MRS.

[14]The relationship between the MRS and the IES in the BGP is: $MRS = IES\sigma(1+\theta)x^{-\sigma(1+\theta)}$.

[15]In Galí (1994), this result is also shown. The author leans on this point to argue that a negative consumption externality can be the cause for a higher equity premium.

Assuming the set of restrictions (14)-(16) in place throughout the rest of the analysis, let us investigate now the short-run behavior of the economy.

The dynamics around the competitive steady state

In order to analyze the short-run equilibrium dynamics, let us first-order approximate the system formed by (8)-(10) around the steady state. Inspection of these equations reveals that the linearized system will be of the following form:

$$
\begin{bmatrix} f_{t+1} - f_t \\ x_{t+1} - x_t \\ z_{t+1} - z_t \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} f_t - f_{t-1} \\ x_t - x_{t-1} \\ z_t - z_{t-1} \end{bmatrix}
$$

The lower triangular nature of the matrix of partial derivatives implies that the elements along the diagonal coincide with its eigenvalues. From differentiation of (8)-(10), we get:

$$
\lambda_{11} = \frac{\partial f_{t+1}}{\partial f_t} = \frac{\psi}{\gamma} > 1
$$

$$
\lambda_{22} = \frac{\partial x_{t+1}}{\partial x_t} = -\frac{\frac{\partial g}{\partial x_t}}{\frac{\partial g}{\partial x_{t+1}}} = \frac{\gamma}{x + \theta(x - \gamma)}
$$

$$
\lambda_{33} = \frac{\partial z_{t+1}}{\partial z_t} = \frac{1 + A}{x} > 1
$$

The first inequality comes from the requirement that $\psi > \gamma$. Moreover, given $0 \leq \theta < 1$, $\gamma \in (0, 1)$ and $x > 1$, we have that $\lambda_{22} \in (0, 1)$. Finally, the last inequality follows from imposing $1 + A > x$. Given these restrictions for the eigenvalues, we conclude that the steady state of the system is locally saddle-path stable.

Equation (8) governs the dynamic behavior of the control f autonomously. Given its associated eigenvalue, its steady state is unstable. This means that f_t jumps immediately to its stationary value, thus exhibiting no transition. We can then analyze the dynamic behavior

of the sub-system formed by (9) and (10) on (x_t, z_t) with $f_t = f$. The saddle-path of the linearized sub-system reads:

$$x_t = v(z_t - z) + x \qquad (17)$$

where the scalar v is such that the $(v, 1)$ is an eigenvector associated with the eigenvalue λ_{22} of the following sub-matrix:

$$\begin{bmatrix} \lambda_{22} & 0 \\ \lambda_{32} & \lambda_{33} \end{bmatrix}$$

Therefore, $(v, 1)$ must be orthogonal to vector $(\lambda_{32}, \lambda_{33} - \lambda_{22})$, which implies that $v = \frac{\lambda_{22} - \lambda_{33}}{\lambda_{32}}$. Evaluating the derivative of the right-hand side of (10) with respect to x_t at the steady state and using the expression for λ_{33}, one gets that $\lambda_{32} = -\lambda_{33} \left(\frac{z}{x} \right)$. Thus,

$$v = \left(1 - \frac{\lambda_{22}}{\lambda_{33}} \right) \left(\frac{x}{z} \right) > 0$$

where the inequality is due to $\lambda_{22} \in (0, 1)$ and $\lambda_{33} > 1$. Hence, it is confirmed that the saddle path in the plane (x_t, z_t) is increasing around the steady state. This path is the policy function that assings to each value of the state variable z_t the corresponding optimal value of the control variable x_t.

The short-run dynamic effects of variations in taxes and in preference parameters are analyzed next in light of equation (17). Concerning the effects on f_t, note that they coincide with the those in the long-run, for this variable displays no transition. The following proposition characterizes the short-run dynamic behavior of the other control variable in the economy, x_t:

Proposition 2. *Assume that the decentralized economy is initially at its steady state. Suppose that the government chooses a constant consumption tax gross growth rate, η. If η suffers an unexpected and permanent marginal increase, then the following effects occur:*

i) The control variable f_t jumps instantaneously to its new higher steady state value.

ii) The consumption growth rate, x_t, increases in the short-run and converges monotonically to a new lower steady state value. iii) The capital-consumption ratio, z_t, moves smoothly and monotonically towards its new lower steady state value.

Proof. The effect on f_t is straightforward. The effect on z_t is also intuitive since this variable does not jump at the time of the shock, but, rather, starts changing one period after the initial change in x, as its equation of motion, (10), reflects. To compute the instantaneous effect on x_t when the economy is at its steady state, we need to perform the differentiation of (17) at $t = 0$ and evaluate it at $z_0 = z$ so as to get:

$$\frac{\partial x_0}{\partial \eta} = -v \left(\frac{\partial z}{\partial \eta} \right) + \left(\frac{\partial x}{\partial \eta} \right)$$

After some algebra, it follows from (13) and the expression for λ_{33} that $\frac{\partial z}{\partial \eta} = (1+A) \left(\frac{z}{x} \right)^2 \left(\frac{\partial x}{\partial \eta} \right)$. Also, by isolating x as a function of z in (13), one obtains $\lambda_{33} = 1 + \frac{1}{z}$. Finally, substituting in the expression for v, we get:

$$\frac{\partial x_0}{\partial \eta} = z(\lambda_{22} - 1) \left(\frac{\partial x}{\partial \eta} \right) \tag{18}$$

Therefore, we conclude that $\frac{\partial x_0}{\partial \eta} > 0$ since $\frac{\partial x}{\partial \eta} < 0$ and $\lambda_{22} \in (0, 1)$ ∎

According to the previous result, the short-run and long-run effects on the gross growth rate of changes in the consumption tax policy have opposite signs[16]¡0and$\lambda_{22} \in (0, 1)$..

It is worth emphasizing at this point that if habit formation was absent, (i.e., $\gamma = 0$), then no dynamic transition would take place after a change in the tax rate. This can be seen simply by noticing that the expression for the eigenvalue λ_{22} becomes equal to zero in this

---

[16]By an analogous argument, the same applies to the case of an increase in the size of the congestion externality, θ: $\frac{\partial x_0}{\partial \theta} > 0$ since $\frac{\partial x}{\partial \theta}$

case. As a matter of fact, equation (9) becomes $x_{t+1} = f_t^{-1/\sigma(1+\theta)}$; the lack of transitional dynamics for f_t is inherited by the consumption growth rate, x_t.

Another interesting question is how the speed of convergence of the economic variables depends on the values of the structural parameters of the model. To undergo this analysis, the inverse of the intertemporal elasticity of substitution in the BGP, $\sigma(1+\theta)$, will be held constant by combining changes in the spillover parameter θ with changes in σ that leave unchanged the steady state values of the three stationary variables. The local behavior of the variable z_t around its steady state value, z, is controlled by:

$$z_t = (z_0 - z)(\lambda_{22})^t + z$$

Thus, the speed of convergence is inversely related to the eigenvalue λ_{22} of the matrix associated with the approximated linear dynamic system. The following proposition summarizes the speed of convergence effects of changes in the preference parameters:

Proposition 3. *Suppose that the stationary growth rate, x, is fixed. The speed of convergence is decreasing both in the parameter measuring the intensity of habit formation, γ, and in the parameter measuring the intensity of congestion externalities, θ.*

Proof. Directly from the expression for λ_{22}, we can calculate the following two partial derivatives:

$$\frac{\partial ln\lambda_{22}}{\partial \gamma} = \frac{(1+\theta)\lambda_{22}x}{\gamma^2} > 0$$

$$\frac{\partial ln\lambda_{22}}{\partial \theta} = \frac{\lambda_{22}(\gamma - x)}{\gamma} < 0$$

Therefore, the speed of convergence decreases with both γ and θ. ∎

The fact that the speed of convergence decreases as the value of γ increases is a quite intuitive result, for, as past consumption becomes more important, individuals face a more concave utility function in the short-run. To see this, note that the index of relative risk aversion reads:

$$-\frac{c_t u_{c_t c_t}(c_t - \gamma c_{t-1}, \bar{c}_t)}{u_{c_t}(c_t - \gamma c_{t-1}, \bar{c}_t)} = -\frac{c_t u_{h_t h_t}(h_t, \bar{c}_t)}{u_{h_t}(h_t, \bar{c}_t)} = \frac{\sigma c_t}{c_t - \gamma c_{t-1}}$$

This means that individuals become more risk averse as γ rises. This is the same as saying that households dislike to a greater extent to experience changes in consumption along the equilibrium path, which results in a lower speed of adjustment to the BGP. As can be seen from the expression for v, this object is decreasing in γ, which in turn means that the policy function (17) becomes *flatter* as habit formation gains importance. The consumption growth rate being less sensitive to changes in the state variable directly causes a lower rate of convergence.

In contrast, the speed of convergence increases with θ. For a constant rate of long-run growth, as the value of parameter θ increases, parameter σ needs to move in the opposite direction so as to keep $\sigma(1+\theta)$. Since the concavity of the felicity function decreases with the reduction in σ, consumers are less willing to substitute consumption across periods, making the duration of the transition to the steady state longer.

Leaning on Proposition 3, we can characterize the impact of the preference parameters on the short-run response of the growth rate to changes in the tax growth rate. From (18), we can compute the following derivatives:

$$\frac{\partial x_0}{\partial \eta \partial \gamma} = z \left(\frac{\partial x}{\partial \eta} \right) \left(\frac{\partial \lambda_{22}}{\partial \gamma} \right) < 0$$

$$\frac{\partial x_0}{\partial \eta \partial \theta} = z \left(\frac{\partial x}{\partial \eta} \right) \left(\frac{\partial \lambda_{22}}{\partial \theta} \right) > 0$$

Hence, the growth rate of consumption is less sensitive (more sensitive) to unanticipated permanent variations in the consumption tax growth rate when parameter γ (parameter θ) is higher. Once again, this is the immediate consequence of the increasing (decreasing) sluggishness of the consumption policy triggered by stronger habits (congestion spillovers).

3.4 The Social Planner Problem

In this section, the solution that a benevolent, time-consistent social planner would implement is analyzed. The difference with respect to the optimization problem in the decentralized market economy is that, in this case, the planner maximizes the same objective function but taking into account (i.e., internalizing) the congestion externality stemming from average consumption. Thus, the only dissimilar aspect is that the average consumption term in the utility function is no longer treated as exogenous. Let us define this function as follows:

$$\hat{u}_t \equiv \hat{u}(c_t, c_{t-1}) = u(c_t - \gamma c_{t-1}, c_t) = \frac{(c_t - \gamma c_{t-1})^{1-\sigma} c_t^{-\theta\sigma}}{1 - \sigma}$$

Noticing that the planner faces only the aggregate resource constraint per capita, we can write the Lagrangian for the centralized problem as follows:

$$\hat{L}(c, k, \hat{\lambda}) = \sum_{t=0}^{\infty} \beta^t \left[u(c_t - \gamma c_{t-1}, c_t) + \hat{\lambda}_t^k \left((1 + A)k_t - c_t - k_{t+1} \right) \right]$$

where $\hat{\lambda}^k = \left\{ \hat{\lambda}_t^k \right\}_{t=0}^{\infty}$ is the infinite sequence of positive Lagrange multipliers. Letting $\hat{u}_1(t) = \frac{\partial \hat{u}(c_t, c_{t-1})}{\partial c_t}$ and $\hat{u}_2(t) = \frac{\partial \hat{u}(c_t, c_{t-1})}{\partial c_{t-1}}$, the first order conditions read:

$$\frac{\partial \hat{L}}{\partial c_t} = \hat{u}_1(t) + \beta \hat{u}_2(t+1) - \hat{\lambda}_t^k = 0 \tag{19}$$

$$\frac{\partial \hat{L}}{\partial k_{t+1}} = \beta \hat{\lambda}_{t+1}^k (1 + A) - \hat{\lambda}_t^k = 0 \tag{20}$$

with associated transversality conditions:

$$\lim_{t \to \infty} \beta^t \hat{\lambda}_t^k k_{t+1} = 0 \tag{21}$$

$$\lim_{t \to \infty} \beta^t \hat{u}_1(t) c_t = 0 \tag{22}$$

The previous boundary conditions combined with the first order conditions, the resource constraint and the initial conditions k_0 and c_{-1} fully characterize the solution paths of c_t, k_t and $\hat{\lambda}_t$.

Let $\hat{f}_t$ denote the gross growth rate of the average utility of habit-adjusted consumption[17]. Then,

$$\hat{f}_t = \frac{\hat{u}(t+1)/h_{t+1}}{\hat{u}(t)/h_t} = \left(\frac{\hat{u}(t+1)}{\hat{u}(t)}\right)\left(\frac{c_t - \gamma c_{t-1}}{c_{t+1} - \gamma c_t}\right) \tag{23}$$

Define $\hat{x}_t = \frac{c_t}{c_{t-1}}$ as the gross rate of consumption growth of the social planner's solution. Plugging this and the parameterized utility function into (23), we get:

$$\hat{f}_t = \hat{x}_t^{-\sigma}\left(\frac{\hat{x}_t - \gamma}{\hat{x}_{t+1} - \gamma}\right)^\sigma \hat{x}_{t+1}^{-\theta\sigma}$$

which can be rewritten as:

$$g(\hat{x}_{t+1}, \hat{x}_t, \hat{f}_t) \equiv \left(\frac{\hat{x}_t - \gamma}{\hat{x}_{t+1} - \gamma}\right)\hat{x}_{t+1}^{-\theta} - \hat{x}_t(\hat{f}_t)^{1/\sigma} = 0 \tag{24}$$

Define $\hat{z}_t = \frac{k_t}{c_{t-1}}$ as the only state variable of the system. Also, let $\hat{\psi} = 1 + A$, $\epsilon = \frac{1-\sigma-\theta\sigma}{\beta(1-\sigma)\gamma}$ and $\rho = \frac{\theta\sigma}{\beta(1-\sigma)}$. Then, the difference equation (24) along with the following two[18]

$$\hat{f}_{t+1} = \epsilon + \frac{\rho}{\hat{x}_{t+1}} + \frac{1}{\beta\hat{\psi}} - \frac{1}{\hat{f}_t\beta\hat{\psi}_t}\left(\frac{\rho}{\hat{x}_t} + \epsilon\right) \equiv M(\hat{x}_{t+1}, \hat{x}_t, \hat{f}_t) \tag{25}$$

$$\hat{z}_{t+1} = \left(\frac{\hat{z}_t}{\hat{x}_t}\right)(1 + A) - 1 \tag{26}$$

---

[17]Note that the definition of $\hat{f}_t$ coincides with that of $\hat{f}_t$ for the competitive economy. However, while f_t is also the gross growth rate of the marginal utility of consumption in the competitive economy, the corresponding $\hat{f}_t$ in the socially planned economy is not necessary equal to this object. We will see that f_t and $\hat{f}_t$ only coincide at the BGP.

[18]See Appendix for the derivation of (25). Equation (26) follows trivially from dividing the resource constraint up by c_t. ϵ, $\hat{\psi}$ and ρ denote the following parametric expressions: $\epsilon = \frac{1-\sigma-\theta\sigma}{\beta(1-\sigma)\gamma}$; $\rho = \frac{\theta\sigma}{\beta(1-\sigma)}$; $\hat{\psi} = 1 + A$.

constitute a system that, together with the initial condition $z_0 = \frac{z_0}{c_{-1}}$ and the transversality conditions (21) and (22), fully describes the dynamics of the socially optimal equilibrium, $\left\{ \hat{f}_t, \hat{x}_t, \hat{z}_t \right\}_{t=0}^{\infty}$.

Evaluating (24)-(26) at $\hat{x}_t = \hat{x}$, $\hat{f}_t = \hat{f}$ and $\hat{z}_t = \hat{z}$ for all t and solve the resulting system of equations, we obtain the stationary values for the two controls and the state variable:

$$\hat{f} = \frac{1}{\beta \hat{\psi}} \tag{27}$$

$$\hat{x} = \hat{f}^{-1/\sigma(1+\theta)} \tag{28}$$

$$\hat{z} = \frac{\hat{x}}{(1+A) - \hat{x}} \tag{29}$$

By looking at their competitive steady state counterparts in (11)-(13), we can see that they coincide whenever $\eta = 1$. That is, any level of the proportional consumption tax ensures that the competitive solution at the steady state is efficient as long as it is kept constant. In particular, $\tau_t = 0 \quad \forall t$ at the BGP would suffice. This is because the decentralized market long-run equilibrium is optimal and, thus, there is no need to distort the intertemporal allocation chosen by individuals.

As in the competitive case, we introduce a number of additional assumptions necessary for the existence of a socially planned BGP displaying strictly positive growth. The following lemma shows that they are the exact counterparts of the decentralized case:

Lemma 2. *If a BGP for the socially planned economy with $\hat{x} > 1$ exists, then the following two inequalities must hold:*

$$\beta \hat{\psi} > 1 \tag{30}$$

$$\beta\hat{\psi}^{1-\sigma(1+\theta)} < 1 \qquad (31)$$

Proof. As apparent from (27)-(28), $\hat{x} > 1$ if and only if (30) holds. The fact that $\hat{x} > 1$ ensures that the arguments of the utility function are always positive in the steady state. In order to check whether the transversality condition (21) holds, note first that the ratio of marginal utilities with respect to today's consumption is:

$$\frac{\hat{u}_1(t+1)}{\hat{u}_1(t)} = \left(\frac{\hat{u}_1(t+1)}{\hat{u}_t}\right)\left(\frac{1}{\hat{x}}\right) = \hat{f}$$

where the last equality follows from (23). Hence, we conclude that the satisfaction of the boundary condition (21) at a BGP requires $\beta\hat{f}\hat{x} < 1$. As follows from (27)-(28), this condition is equivalent to (31). ∎

It can be shown that conditions (30) and (31) are also sufficient for $\hat{z} > 0$, $\hat{\lambda}_t^k > 0$ and for the transversality condition (21) to be satisfied[19]. We assume these two conditions to be in place throughout the rest of the analysis.

The dynamics around the socially optimal steady state

In order to study the local dynamics of the optimal path selected by the benevolent planner, let us linearize the system of difference equations, (24)-(26), around its steady state. Such approximated dynamic system has the following form:

$$\begin{bmatrix} \hat{f}_{t+1} - \hat{f}_t \\ \hat{x}_{t+1} - \hat{x}_t \\ \hat{z}_{t+1} - \hat{z}_t \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_{11} & \hat{\lambda}_{12} & 0 \\ \hat{\lambda}_{21} & \hat{\lambda}_{22} & 0 \\ 0 & \hat{\lambda}_{32} & \hat{\lambda}_{33} \end{bmatrix} = \begin{bmatrix} \hat{f}_t - \hat{f}_{t-1} \\ \hat{x}_t - \hat{x}_{t-1} \\ \hat{z}_t - \hat{z}_{t-1} \end{bmatrix}$$

Lemma 3. *The steady state equilibrium of the previous dynamic system exhibits saddle-path stability.*

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[19]See Appendix.

Proof. Provided in the Appendix. ∎

As shown in the proof of this lemma, the previous dynamic system has only one stable root (namely $\hat{\lambda}_2 \in (0,1)$). Hence, we can write the stable solution path of the approximated system as a linear equation on this eigenvalue and the steady state value of each variable:

$$\hat{f}_t = A_1 \hat{\lambda}_2^t + \hat{f}$$

$$\hat{x}_t = A_2 \hat{\lambda}_2^t + \hat{x}$$

$$\hat{z}_t = A_3 \hat{\lambda}_2^t + \hat{z} \tag{32}$$

with $A_3 = (\hat{z}_0 - \hat{z})$. Let $A_1 = k_1 A_3$ and $A_2 = k_2 A_3$. Then,

$$\hat{f}_t = k_1 (\hat{z}_0 - \hat{z}) \hat{\lambda}_2^t + \hat{f} \tag{33}$$

$$\hat{x}_t = k_2 (\hat{z}_0 - \hat{z}) \hat{\lambda}_2^t + \hat{x} \tag{34}$$

where the vector $(k_1, k_2, 1)$ is the eigenvector associated with eigenvalue $\hat{\lambda}_2$ of the matrix of partial derivatives. Thus,

$$k_2 = \frac{\hat{\lambda}_2 - \hat{\lambda}_{33}}{\hat{\lambda}_{32}} > 0 \tag{35}$$

where the inequality is due to $\hat{\lambda}_{32} < 0$ and $\hat{\lambda}_{33} > 1$ (see (51) and (53) in the Appendix).

Now, we can make $k_1 = k_2 \hat{v}$, where $(\hat{v}, 1)$ is an eigenvector associated with eigenvalue $\hat{\lambda}_2$ of the sub-system formed by the linearization of (24) and (26). Then,

$$\hat{v} = \frac{\hat{\lambda}_2 - \hat{\lambda}_{22}}{\hat{\lambda}_{21}} = \frac{\hat{\lambda}_2 - \left[\frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)} \right]}{\hat{\lambda}_{21}} \tag{36}$$

In order to investigate the sign of this object, observe first that the denominator is negative (see (54) in the Appendix). Second, to determine the sign of the numerator, we need to compute the following:

$$\left(\hat{\lambda}_1 - \frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)}\right)\left(\hat{\lambda}_2 - \frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)}\right) =$$

$$\hat{\lambda}_1\hat{\lambda}_2 + \left(\frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)}\right)\left(\frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)} - (\hat{\lambda}_1 + \hat{\lambda}_2)\right) = \left(\frac{\hat{\psi}}{\hat{x}}\right)\left(\frac{\theta(1-\theta)(\hat{x} - \gamma)^2}{(1-\sigma)(\hat{x} + \theta(\hat{x} - \gamma))^2}\right) < 0$$

The second equality comes from algebraic simplification of expressions $\hat{\lambda}_1\hat{\lambda}_2$ and $\hat{\lambda}_1 + \hat{\lambda}_2$ (see equations (55) and (55) of the Appendix). The final sign follows from the concavity condition, $\sigma > 1$, and the restrictions $\hat{x} - \gamma > 0$ and $\theta < 1$. Thus, we conclude that the eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ satisfy:

$$\hat{\lambda}_1 > \frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)} > \hat{\lambda}_2 \tag{37}$$

This means, then, that $\hat{v} > 0$ and, hence, that $k_1 = k_2\hat{v} > 0$, as follows from (35) and (36). Therefore, we have shown that there is a positive relationship between the state variable, $\hat{z}_t$, and the control variables $\hat{f}_t$ and $\hat{x}_t$ along the saddle path in the vicinity of the stationary solution to the planner's problem. In particular, the following policy function for the gross growth rate of consumption obtains from (32) and (34):

$$\hat{x}_t = k_2(\hat{z}_t - \hat{z}) + \hat{x} \tag{38}$$

with $k_2 > 0$. Note also that f_t did not display transition in the competitive equilibrium, whereas it does in the planner's solution as apparent from (33).

In the next proposition, the convergence rates of the competitive and socially planned economies are compared:

Proposition 4. *The rate of convergence to the steady state of the decentralized economy with constant consumption tax rates is lower than that of the socially planned economy.*

Proof. Recalling the discussion of the competitive equilibrium's system, we know that $\lambda_{22} = \frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)}$, since $\hat{x} = x$ whenever $\eta = 1$. It follows from (37) that $\lambda_{22} > \hat{\lambda}_2$. Since

these objects reflect the inverse of the speed of convergence for each case, we conclude that the decentralized market economy converges towards its BGP at a slower speed than the centralized one. ■

The intuition behind this result is that, if the growth rate lies initially above the its steady state value, then the marginal rate of substitution (MRS) in the socially planned equilibrium is smaller than the MRS faced by individuals along the competitive solution[20]. Since the MRS of the social planner is lower than that of the consumers in the decentralized equilibrium, if the initial rate of growth is higher than the stationary one, then the speed of convergence in the socially planned economy is higher than in the competitive one because the social planner is more willing to bring consumption to the present and reduce immediately the rate of consumption growth. The symmetric argument applies to the case where the initial growth rate is below its steady state level.

3.5 Optimal consumption taxation

To characterize the socially optimal path for the tax rate on consumption is a relatively simple task in our case, since, given an initial value for the state variable, $_0$, we only need to calculate the sequence $\{\tau_t\}_{t=1}^{\infty}$ that makes the path of the control variable f_t equal to that of the variable $\hat{f}_t$ of the socially planned solution, for all $t \geq 0$. Considering the fact that the control variable f_t in the competitive equilibrium does not exhibit transitional dynamics (meaning that it always lies at the steady state) and recalling that $\psi_{t+1} = \frac{(1+\tau_t)(1+A)}{1+\tau_{t+1}}$, the competitive path of f_t is given by the solution to the difference equation (8):

$$f_t = \frac{1}{\beta \psi_{t+1}} = \frac{1 + \tau_{t+1}}{(1 + \tau_t)\beta(1 + A)} \tag{39}$$

as follows from the independence of f_t of the state variable z_t. The Pareto optimal path for $\hat{f}_t$ is derived from solving the system of equations (24)-(26) for a given initial value the

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[20]Specific details and a formal proof of this point are available upon request.

state, z_0. By inserting the efficient sequence $\left\{\hat{f}_t\right\}_{t=0}^{\infty}$ in (39) and isolating $\eta_t = \frac{1+\tau_{t+1}}{1+\tau_t}$, we obtain the optimal path for the gross growth rate of consumption taxes:

$$\eta_t = \hat{f}_t \beta(1 + A) \tag{40}$$

It follows from the combination of (32) and (33) that the centrally planned path of $\hat{f}_t$ around its stationary level satisfies:

$$\hat{f}_t = k_1(\hat{z}_t - \hat{z}) + \hat{f} \tag{41}$$

where $\hat{f}$ and $\hat{z}$ are the steady state values of the control variables f_t and $\hat{f}_t$ and the state variables z_t and $\hat{z}_t$ for both the competitive case with constant taxes and the socially efficient case, respectively. Plugging (41) in (40), we get that the behavior of the optimal consumption tax growth rate around the BGP is governed by:

$$\eta_t = \beta(1 + A)\left[k_1(\hat{z}_t - \hat{z}) + \hat{f}\right] = 1 + k_1(\hat{z}_t - \hat{z})\beta(1 + A) \tag{42}$$

where the last equality comes from $\hat{f} = \frac{1}{\beta\hat{\psi}} = \frac{1}{\beta(1+A)}$. Since $k_1 > 0$, the sign of the tax growth rate around the steady state hinges on the sign of $\hat{z}_t - \hat{z}$ (with $\hat{z}_t$ sufficiently close to its stationary value $\hat{z}$). The following proposition establishes more precisely this relationship between the optimal tax growth rate and the current state of the economy:

Proposition 5. *Assume that $\theta \neq 0$. Then, the optimal consumption tax growth rate around a steady state satisfies:*

$$\frac{\tau_{t+1}-\tau_t}{1+\tau_t} > 0 \quad if \quad z_t > z$$

$$\frac{\tau_{t+1}-\tau_t}{1+\tau_t} < 0 \quad if \quad z_t < z$$

and

$$\lim_{t\to\infty} \eta_t = 1$$

Proof. First, note that the assumption $\theta \neq 0$ is needed, for otherwise the externality arising from congestion in consumption would be absent and, thus, there would be no *raison d'etre* for taxation since in this case the decentralized and socially optimal allocations would coincide. The result on the short-run dynamics for the tax growth rate is straightforward by noting that, first, (42) can be written as:

$$\tfrac{\tau_{t+1}-\tau_t}{1+\tau_t} = k_1(\hat{z}_t - \hat{z})\beta(1 + A)$$

and, second, $\hat{z} = z$. Furthermore, we can make $\hat{z}_t = z_t$ as both objects are predetermined at date t. The asymptotic result is trivially obtained by using $\lim\limits_{t\to\infty} z_t = z$ in (42). ∎

The fiscal policy prescription stemming from the previous proposition has a countercyclical flavor. As can be seen from (17), when the variable z_t lies below its stationary value, the consumption growth rate x_t is also below its steady state value. Thus, in this case, since the competitive economy converges towards is BGP at a lower speed than the centrally planned one (i.e., the market delivers *too* little growth), the optimal policy should consist of accelerating the rate of convergence, which is attained by promoting the rate of capital accumulation via a more burdensome tax on today's consumption relative to tomorrow's (i.e., $\eta_t < 1$). Similarly, for economies that are growing faster than their (optimal) BGP level, the tax policy should be designed such that individuals intertemporally reallocate their consumption from the future to the present, which is achieved by taxing future consumption at higher rates (i.e., $\eta_t > 0$).

As regards the implications of congestion, whose importance is captured by parameter θ, for the optimal consumption tax growth rate, we can analyze the impact it has on the coefficient k_1 that measures the sensitivity of η_t to the distance $z_t - z$. Irrespective of the latter difference between the state variable at t and its BGP value, it can be shown that the greater the size of the congestion effect arising from aggregate consumption on utility, the more sensitive the optimal tax growth rate to changes in $z_t - z$[21]. Thus, starting from

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[21]The numerical procedure used to evaluate $\frac{dk_1}{d\theta}$ is outlined in the Appendix.

$\theta = 0$, where the decentralized intertemporal allocation does not need to be modified since it coincides with the efficient one, further increments in this parameter lead to gradually higher tax consumption profiles over time, ceteris paribus; intuitively, a larger θ means higher disutility from congestion which causes the social planner to penalize aggregate decentralized consumption in equilibrium to a greater extent than for lower values of this parameter. This implies that a higher tax path, $\{\tau_t\}_{t=1}^{\infty}$, will be chosen by the benevolent government for all t. As a consequence, whenever $z_t > z$ and an increase in θ occurs, the rate of convergence to the steady state will tend to decline given that the resulting rise in consumption taxes will make individuals favor savings over investment, thereby deepening the capital stock and spurring growth. The opposite outcome holds whenever $z_t < z$: an increase in θ leads to higher consumption taxes which are succeeded by higher capital investment rates that make z_t grow faster.

3.6 Conclusions

A simple endogenous growth model featuring habits and externalities, both arising from consumption, has been presented and analyzed. The source of the harmful aggregate consumption spillover assumed is the common use of a public facility/infrastructure that is subject to a degree of congestion. A process of habit formation in own households' consumption is also introduced in order to generate interesting transitional dynamics that would otherwise be absent. A proportional consumption tax rate is levied in every period in order to align the decentralized market equilibrium paths with the socially optimal ones. In this fiscal policy context, we believe that our contribution helps bridge the two strands of literature involved. The main insights gathered from the analysis in this paper are summarized next.

First, in our framework, the presence of a negative external effect does not necessary call for corrective public intervention; in the model's long-run competitive equilibrium, all variables grow at their efficient rates and there is no distortion in the intertemporal marginal rate of

substitution. Thus, in this case, the policy prescription is to disregard levels and simply set a constant consumption tax rate over time that does not alter the incentives that individuals take into account when facing their savings-consumption decision.

Second, even if no distortions arise at the competitive steady state, the short-run dynamics differ between the decentralized and centrally planned economies. Specifically, we have shown that the efficient path converges faster towards its steady state than the competitive one. Thus, in this case, a benevolent government is indeed required to employ fiscal instruments aimed at raising the competitive rate of convergence so as to achieve second-best optimality. As we argued, a countercyclical policy for the consumption tax growth rate serves this purpose, for it accelerates the rate of capital accumulation when the economy is growing *too* slowly, whereas it discourages savings when the economy is growing *too* fast.

Third, the importance of the congestion externality in preferences matters for the design of optimal fiscal policy. In particular, when congestion is very burdensome in utility terms, consumption tax rates should be raised in every period so as to bring the competitive equilibrium's aggregate consumption path down to equalize the socially optimal one, which lowers as the harmful effects of, for instance, overcrowded facilities become higher.

Fourth, both habit formation and congestion externalities make the speed of convergence more sluggish, given fixed stationary growth rates. In the corner case where both of these effects are absent, we know that, in fact, the economy does not exhibit any transition but, rather, jumps to the steady state instantly. When habits and congestion are introduced, agents dislike large swings in their consumption profile over time and try to smooth out variations in it as much as possible. The reason behind this result is that a greater persistence of habits, captured by a higher γ, or a greater disutility from congestion, captured by a higher θ, diminish the marginal rate of substitution of consumption across periods, making the individual increasingly unwilling to experience change in her consumption levels.

Our investigation has just focused on the interaction between congestion externalities and

additive habits, where it is the difference between current and past consumption that generates utility. As pointed out before, however, all our results could be extended to the case with multiplicative habits, where the ratio of present to former consumption is, instead, the relevant argument in the felicity function. A possibly fruitful extension of our model would be to consider *external* habits, whereby, as opposed to *internal* habits, it is the economy-wide average past consumption that sets the reference standard of living that is used by individuals to evaluate the utility accruing from present consumption.

Appendix

A: Derivation of equation (25)

We provide the algebra that leads to the difference equation (25) here. First, from the combination of the first order conditions (19) and (20), one can write:

$$\frac{\hat{u}_1(t+1) + \beta\hat{u}_2(t+2)}{\hat{u}_1(t) + \beta\hat{u}_2(t+1)} = \frac{1}{\beta\hat{\psi}} \tag{43}$$

where $\hat{\psi} = 1 + A$. After some algebraic manipulation of the utility function, $\hat{u}_t$, specified in the text, it can be shown that:

$$\hat{u}_1(t) = \hat{u}(t) \left(\frac{(1-\sigma-\theta\sigma)c_t + \gamma\theta\sigma c_{t-1}}{c_t(c_t - \gamma c_{t-1})} \right) \tag{44}$$

and

$$\hat{u}_2(t) = -\hat{u}(t) \left(\frac{(1-\sigma)\gamma}{(c_t - \gamma c_{t-1})} \right) \tag{45}$$

Plugging (44) and (45) in (43), we obtain:

$$\frac{\hat{u}(t+1)\left[\frac{(1-\sigma-\theta\sigma)c_{t+1}+\gamma\theta\sigma c_t}{c_{t+1}(c_{t+1}-\gamma c_t)}\right] - \beta\hat{u}(t+2)\left[\frac{(1-\sigma)\gamma}{c_{t+2}-\gamma c_{t+1}}\right]}{\hat{u}(t)\left[\frac{(1-\sigma-\theta\sigma)c_t+\gamma\theta\sigma c_{t-1}}{c_t(c_t-\gamma c_{t-1})}\right] - \beta\hat{u}(t+1)\left[\frac{(1-\sigma)\gamma}{c_{t+1}-\gamma c_t}\right]} = \frac{1}{\beta\hat{\psi}}$$

The latter can be rewritten after reshuffling as:

$$\left(\frac{\frac{1-\sigma-\theta\sigma}{\beta(1-\sigma)\gamma} + \frac{\theta\sigma c_t}{\beta(1-\sigma)c_{t+1}} - \frac{\hat{u}(t+2)}{\hat{u}(t+1)}\left(\frac{c_{t+1}-\gamma c_t}{c_{t+2}-\gamma c_{t+1}}\right)}{\frac{1-\sigma-\theta\sigma}{\beta(1-\sigma)\gamma} + \frac{\theta\sigma c_{t-1}}{\beta(1-\sigma)c_t} - \frac{\hat{u}(t+1)}{\hat{u}(t)}\left(\frac{c_t-\gamma c_{t-1}}{c_{t+1}-\gamma c_t}\right)} \right) x \left(\frac{\hat{u}(t+1)}{\hat{u}(t)} \right) \left(\frac{c_t - \gamma c_{t-1}}{c_{t+1} - \gamma c_t} \right) = \frac{1}{\beta\hat{\psi}} \tag{46}$$

Using the definitions of $\hat{f}_t$ and $\hat{x}_t$, (46) becomes:

$$\left(\frac{\epsilon + \frac{\rho}{\hat{x}_{t+1}} - \hat{f}_{t+1}}{\epsilon + \frac{\rho}{\hat{x}_t} - \hat{f}_t} \right) \hat{f}_t = \frac{1}{\beta\hat{\psi}} \tag{47}$$

where $\epsilon = \frac{1-\sigma-\theta\sigma}{\beta(1-\sigma)\gamma}$ and $\rho = \frac{\theta\sigma}{\beta(1-\sigma)}$. Note that $\epsilon > 0$ and $\rho < 0$ given $\sigma > 1$ and $\theta \in [0,1)$. Finally, (25) obtains after rewriting (47):

$$\hat{f}_{t+1} = \epsilon + \frac{\rho}{\hat{x}_{t+1}} + \frac{1}{\beta\hat{\psi}} - \frac{1}{\hat{f}_t\beta\hat{\psi}_t}\left(\frac{\rho}{\hat{x}_t} + \epsilon\right) \equiv M(\hat{x}_{t+1}, \hat{x}_t, \hat{f}_t)$$

B: Conditions for a well-defined steady state in the SP solution

We show next that conditions (30) and (31) of Lemma 2 are also sufficient for the satisfaction of the rest of the conditions needed for a well-defined efficient BGP. First, let us begin by noticing that a strictly positive value for the state variable at the BGP, $\hat{z} > 0$, requires:

$$1 + A > \hat{x}$$

which follows trivially from inspection of (29). Using (27) and (28), the latter inequality becomes:

$$1 + A > (\beta\hat{\psi})^{1/\sigma(1+\theta)}$$

It is straightforward to check that the latter holds whenever (31) is assumed.

Second, we can see from the first order condition (19) that strict positivity of the Lagrange multiplier at the steady state, $\hat{\lambda}_t^k > 0$ requires:

$$\hat{\lambda}_t^k = \hat{u}_1(t) + \beta\hat{u}_2(t+1) > 0 \tag{48}$$

Making use of (44) and (45) from the last Appendix's section and the definition of $\hat{f}$, we obtain:

$$\frac{\hat{u}_2(t+1)}{\hat{u}_1(t)} = -\hat{f}\left(\frac{(1-\sigma)\gamma c_t}{(1-\sigma-\theta\sigma)c_t + \gamma\theta\sigma c_{t-1}}\right)$$

which at a BGP becomes:

$$\frac{\hat{u}_2(t+1)}{\hat{u}_1(t)} = -\frac{\hat{f}}{\beta}\left(\frac{\hat{x}}{\epsilon\hat{x}+\rho}\right)$$

Plugging the latter into (48), we get:

$$\hat{\lambda}_t^k = \beta\hat{u}_1(t)\left[1 - \hat{f}\left(\frac{\hat{x}}{\epsilon\hat{x}+\rho}\right)\right] \tag{49}$$

Note from (44) that $\hat{u}_1(t) > 0$ since

$$\left(1 - \frac{\theta\sigma}{1-\sigma}\right)c_t > \frac{-\gamma\theta\sigma c_{t-1}}{1-\sigma}$$

is satisfied given that a concave and well-defined $\hat{u}(t)$ requires $-\frac{\theta}{1-\sigma} \geq 0$ and $c_t - \gamma c_{t-1} > 0$, respectively.

Thus, $\hat{\lambda}_t^k > 0$ if and only if:

$$\hat{f}\left(\frac{\hat{x}}{\epsilon\hat{x}+\rho}\right) < 1$$

From (27) and the definitions of ϵ and ρ, the latter inequality becomes:

$$\frac{\hat{x}\gamma}{\hat{x} + \left[\frac{\theta\sigma(\gamma-\hat{x})}{1-\sigma}\right]} < \hat{\psi}$$

Given that $\gamma - \hat{x} < 0$ and $\frac{\theta}{1-\sigma} < 0$, we conclude that the denominator on the left-hand side of the previous inequality is positive. Let us rewrite it as:

$$\hat{x}(\hat{\psi} - \gamma) + \frac{\hat{\psi}\theta\sigma(\gamma - \hat{x})}{1-\sigma} > 0 \tag{50}$$

The latter always applies under the set of assumptions we are making. To see this, note

from (27) and (28) that $\hat{x} > 1$ is equivalent to $\beta\hat{\psi} > 1$, which in turn implies $\hat{\psi} > 1$ due to $0 < \beta < 1$. Hence, $\hat{\psi} - \gamma > 0$ and $\hat{x} - \gamma > 0$. Again, because of the concavity of $\hat{u}$, we have $\frac{\theta}{1-\sigma} < 0$.

Finally, substituting equation (49) in the transversality condition (21) and using the fact that (31) is equivalent to $\beta\hat{f}\hat{x} < 1$, we get that this condition is actually satisfied at the BGP.

C: Proof of Lemma 3

Here we show that the steady state of the dynamic system resulting from the solutions to the social planner problem is saddle-path stable.

The elements of the 3x3 matrix of partial derivatives associated with the first-order approximated dynamic system in subsection 3.1 are:

$$\hat{\lambda}_{11} = \frac{\partial \hat{f}_{t+1}}{\partial \hat{f}_t} = \frac{\partial M}{\partial \hat{f}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}$$

$$\hat{\lambda}_{12} = \frac{\partial \hat{f}_{t+1}}{\partial \hat{x}_t} = \frac{\partial M}{\partial \hat{x}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}$$

$$\hat{\lambda}_{21} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}$$

$$\hat{\lambda}_{22} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}$$

$$\hat{\lambda}_{32} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{x}_t} = -\left(\frac{\hat{z}}{\hat{x}^2}\right)(1 + A) < 0 \tag{51}$$

$$\hat{\lambda}_{33} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{z}_t} \tag{52}$$

Let $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ denote the eigenvalues of that matrix. Because of the lower triangular nature of the 2x2 matrix of the linearized sub-system composed of equations (24) and (26), it must be the case that one of the eigenvalues satisfies:

$$\hat{\lambda}_3 = \hat{\lambda}_{33} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{z}_t} = \left(\frac{1}{\hat{x}}\right)(1 + A) > 1 \tag{53}$$

as follows from differentiation of (26) and from the fact that $\hat{z} > 0$ requires $1 + A > x$. Given the algebraic properties (namely, trace and determinant) of matrices, from the sub-system that consists of the linearized versions of equations (24) and (25), we obtain that the remaining two eigenvalues of the original 3x3 matrix must satisfy:

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \hat{\lambda}_{11} + \hat{\lambda}_{22} = \frac{\partial M}{\partial \hat{f}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t} + \frac{\partial \hat{x}_{t+1}}{\hat{x}_t}$$

$$\hat{\lambda}_1\hat{\lambda}_2 = \hat{\lambda}_{11}\hat{\lambda}_{22} - \hat{\lambda}_{12}\hat{\lambda}_{21} = \left(\frac{\partial M}{\partial \hat{f}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}\right)\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t} - \left(\frac{\partial M}{\partial \hat{x}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}\right)\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}\frac{\partial M}{\partial \hat{f}_t} - \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}\frac{\partial M}{\partial \hat{x}_t}$$

Let us compute the partial derivatives (evaluated at the BGP) appearing in the previous expressions using (24) and (25):

$$\frac{\partial M}{\partial \hat{f}_t} = \left(\epsilon + \frac{\rho}{\hat{x}}\right)\left(\frac{1}{\hat{\psi}\beta\hat{f}^2}\right) = \left(\frac{1}{\beta\hat{f}(1-\sigma)}\right)\left(\frac{1-\sigma(1+\theta)}{\gamma} + \frac{\theta\sigma}{\hat{x}}\right)$$

$$\frac{\partial M}{\partial \hat{x}_t} = \frac{\rho}{\hat{x}^2} = \frac{\theta\sigma}{\beta(1-\sigma)\hat{x}^2}$$

$$\frac{\partial M}{\partial \hat{x}_{t+1}} = -\left(\frac{\rho}{\hat{x}^2}\right) = -\frac{\theta\sigma}{\beta(1-\sigma)\hat{x}^2}$$

$$\frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t} = -\frac{\partial g/\partial \hat{x}_t}{\partial g/\partial \hat{x}_{t+1}} = \frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)}$$

$$\hat{\lambda}_{21} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t} = -\frac{\partial g/\partial \hat{f}_t}{\partial g/\partial \hat{x}_{t+1}} = -\left(\frac{\hat{x}}{\sigma \hat{f}}\right)\left(\frac{\hat{x}-\gamma}{\hat{x}+\theta(\hat{x}-\gamma)}\right) < 0 \qquad (54)$$

With this information, we can now calculate the following product:

$$\hat{\lambda}_1 \hat{\lambda}_2 = \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t}\frac{\partial M}{\partial \hat{f}_t} - \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t}\frac{\partial M}{\partial \hat{x}_t} =$$

$$\left(\frac{\gamma}{\hat{x}+\theta(\hat{x}-\gamma)}\right)\left(\frac{1}{\beta\hat{f}(1-\sigma)}\right)\left(\frac{1-\sigma(1+\theta)}{\gamma}+\frac{\theta\sigma}{\hat{x}}\right)+\left(\frac{\hat{x}}{\sigma\hat{f}}\right)\left(\frac{\hat{x}-\gamma}{\hat{x}+\theta(\hat{x}-\gamma)}\right)\frac{\theta\sigma}{\beta(1-\sigma)\hat{x}^2}$$

Making use of (27) and (28), the previous cumbersome expression boils down to:

$$\hat{\lambda}_1 \hat{\lambda}_2 = \left(\frac{\hat{\psi}}{\hat{x}}\right) > 0 \qquad (55)$$

Regarding the sign of the sum of eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$, one gets after some tedious algebraic manipulation:

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\partial M}{\partial \hat{f}_t} + \frac{\partial M}{\partial \hat{x}_{t+1}}\frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t} + \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t} =$$

$$\frac{\gamma}{\hat{x}+\theta(\hat{x}-\gamma)}+\left(\frac{\hat{\psi}}{\gamma}\right)\left[\frac{\hat{x}+\theta(\hat{x}-\gamma)}{\hat{x}}-\frac{\theta(1-\theta)(\hat{x}-\gamma)^2}{(1-\sigma)\hat{x}(\hat{x}+\theta(\hat{x}-\gamma))}\right] > 0 \qquad (56)$$

The inequality follows from $\hat{x} - \gamma > 0$, $\theta < 1$ and $\frac{\theta}{(\sigma-1)} \geq 0$. Finally, we also need the following condition in order to determine the signs of the two eigenvalues:

$$(1-\hat{\lambda}_1)(1-\hat{\lambda}_2) = 1 + \hat{\lambda}_1\hat{\lambda}_2 - (\hat{\lambda}_1+\hat{\lambda}_2) = -\left(\frac{(\hat{x}-\gamma)(1+\theta)}{\gamma\hat{x}(\hat{x}+\theta(\hat{x}-\gamma))}\right)\left(\hat{x}(\hat{\psi}-\gamma) - \frac{\hat{\psi}\theta(\hat{x}-\gamma)\sigma}{(1-\sigma)}\right) < 0$$

where the second equality comes from simplification of (55) and (56). The sign of the latter product follows from $\theta < 1$, $\hat{x} - \gamma > 0$ and the fact that the second term in the multiplication is equivalent to (50), whose positivity always holds under our parametric restrictions, as shown in the last section of the Appendix.

Thus, since $\hat{\lambda}_1\hat{\lambda}_2 > 0$, $\hat{\lambda}_1 + \hat{\lambda}_2 > 0$ and $(1 - \hat{\lambda}_1)(1 - \hat{\lambda}_2) < 0$, we conclude that $\hat{\lambda}_1 > 1$ and $\hat{\lambda}_2 \in (0, 1)$. Recalling that $\hat{\lambda}_3 > 1$, it follows that the steady state of the linearized system exhibits saddle-path stability.

D: Derivation of the effect of congestion externalities on optimal taxation

The sign of the impact of the importance of congestion, θ, on the parameter measuring the sensitivity of η_t to $z_t - z$, k_1, is numerically examined given the analytical intractability of the algebraic expressions involved.

Recalling from section 3.1 that we can make $k_1 = k_2\hat{v}$ and using equations (35) and (36), we can write an expression for k_1 on the elements of the matrix of partial derivatives associated with the linearized dynamic system of the social planner problem:

$$k_1 = k_2\hat{v} = \left(\frac{\hat{\lambda}_2 - \hat{\lambda}_{33}}{\hat{\lambda}_{32}}\right)\left(\frac{\hat{\lambda}_2 - \hat{\lambda}_{22}}{\hat{\lambda}_{21}}\right) \tag{57}$$

We know from (51) and (53) in the previous section of the Appendix that:

$$\hat{\lambda}_{32}\frac{\partial \hat{z}_{t+1}}{\partial \hat{x}_t} = -\left(\frac{\hat{z}}{\hat{x}^2}\right)(1 + A)$$

$$\hat{\lambda}_{33} = \frac{\partial \hat{z}_{t+1}}{\partial \hat{z}_t} = \left(\frac{1}{\hat{x}}\right)(1 + A)$$

Also, making use of (24), we can easily derive:

$$\hat{\lambda}_{21} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{f}_t} = \left(\frac{\hat{x}}{\sigma\hat{f}}\right)\left(\frac{\gamma - \hat{x}}{\hat{x} + \theta(\hat{x} - \gamma)}\right)$$

$$\hat{\lambda}_{22} = \frac{\partial \hat{x}_{t+1}}{\partial \hat{x}_t} = \frac{\gamma}{\hat{x} + \theta(\hat{x} - \gamma)}$$

where, as follows from (27)-(29),

$$\hat{f} = \frac{1}{\beta(1+A)} \quad ; \quad \hat{x} = \left(\frac{1}{\beta(1+A)}\right)^{-1/\sigma(1+\theta)} \quad ; \quad \hat{z} = \frac{\hat{x}}{(1+A) - \hat{x}}$$

Thus, the only object left to be computed in order to find out k_1 is $\hat{\lambda}_2$. Now, using (36) we can solve for k_2 in $k_1 = k_2 \hat{v}$ to get:

$$k_2 = k_1 \left(\frac{\hat{\lambda}_{21}}{\hat{\lambda}_2 - \hat{\lambda}_{22}} \right)$$

which can be rewritten after some reshuffling as:

$$k_1(\hat{\lambda}_{12} + (\hat{\lambda}_{11} - \hat{\lambda}_2)(\hat{\lambda}_2 - \hat{\lambda}_{22})) = 0 \tag{58}$$

Since $k_1 > 0$ as shown in section 4, we have that, according to Appendix C, there are two roots to the latter second-order equation, the smaller one corresponding to $\hat{\lambda}_2 \in (0, 1)$. In order to find out the value of the latter we proceed by assigning the following starting values to the parameters of the model:

$$\beta = 0.9 \quad ; \quad A = 2 \quad ; \quad \sigma = 1.5 \quad ; \quad \theta = 0.8 \quad ; \quad \gamma = 0.8$$

These values satisfy all the restrictions imposed on the parameters throughout the analysis.

Given these parametrization, the numerical solution of (58) is substituted in (57). This process is repeated for several values of θ, keeping the rest of the parameters constant. The following table shows the different values of k_1 obtained for different values of θ:

θ	k_1
0.99	0.8492
0.9	0.7193
0.8	0.6112
0.7	0.5309
0.5	0.4211
0.3	0.3438
0.1	0.262
0.01	0.2186

We can induce in light of the previous table that k_1 and θ bear a non-linear, positive relationship:

$$\frac{\partial k_1}{\partial \theta} > 0 \quad ; \quad \frac{\partial^2 k_1}{\partial \theta^2} < 0$$

Recalling that,

$$\eta_t = 1 + k_1(\hat{z}_t - \hat{z})\beta(1 + A)$$

we conclude that the sensitivity of η_t to the distance $z_t - z$ increases with the importance of the negative congestion effect on utility, while becoming exponentially less responsive to changes in θ as the latter rises.

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