EUROPEAN UNIVERSITY INSTITUTE
Department of Economics

Essays on R&D-Races and Cournot Oligopoly

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Introduction

This thesis was written out of the desire to gain a deeper understanding of competition in imperfect markets, in particular oligopolies. Firms have many possible strategies at their disposal that may effect present and future payoffs, and the structure of their markets. In the first two chapters we study R&D and innovation, that have received an ever increasing amount of interest, for a variety of market frameworks, including Cournot and Bertrand oligopoly, and repeated patent races. Here firms compete by innovating; we study how they do it, and in which direction competition will evolve.

Still, many questions remain unanswered even in now classic static models of oligopoly, as the Cournot oligopoly, which he have used as a building block in the first two chapters. In chapter 3, we set out to examine the equally classical questions of existence and uniqueness of equilibrium, and the reaction of the market to the entry of new firms. Our analysis is innovative since it deals with heterogeneous goods and makes use of a very new set of methods.

R&D races

There are at least two reasons for studying R&D-races: First, because they help understand how firms compete, and second, because now they are being used as building blocks in macroeconomic models of endogenous growth. Firms have a large range of strategic variables available, of which to include all but a few quickly renders modelling infeasible. Most models of innovation, like patent races or step-by-step R&D-races, pay attention to only one strategic variable: research effort. While some important insights can be won, for example about the incentives of market leaders to innovate, or about persistence of leadership in markets, this restriction certainly neglects other important aspects of firms' strategies.

Our research examines a second important strategic dimension of R&D: the targets. Certainly firms will decide whether they want to make small improvements (and maybe many of them), or go for the big innovation that will give them a lasting advantage over their competitors. The type of targets that firms adopt may be very different depending on whether a firm is a market leader or is trailing behind.

In chapter 1 we will discuss a very general framework for R&D-races which allows for explicit choice of research effort and innovation targets. This research is motivated by chapter 2, a note on step-by-step innovation races, which therefore we will introduce first.

In the note in chapter 2, we analyze a standard step-by-step innovation model (of Aghion, Harris and Vickers 1997), where firms by assumption cannot leapfrog each other, and where innovation targets are fixed. We show that these assumptions do constrain equilibrium strategies, in the sense that if we introduce the possibility of leapfrogging then firms would make use of it in most circumstances. This result means that even though this model is a very useful building block for macroeconomic models, it must be changed to allow for more complex behaviour if it is to be used as a microeconomic model of competition.

We show that the unique symmetric equilibrium in the step-by-step race that is commonly examined is unstable for certain values of the exogenous parameters, and that at the same time asymmetric equilibria arise. If this happens then the predictive power of these symmetric equilibria is diminished, because the market will have a tendency to evolve away from the symmetric equilibria and towards the asymmetric ones. This is particularly interesting since in the unique symmetric equilibrium economic growth is higher when competition in the product market is intense; we show that in the accompanying asymmetric equilibria, which occur precisely when product market competition is high, economic growth may be lower than in the unique equilibrium under less intense product market competition, in some sense reversing the previous results.

Our general model of R&D-races is set out in chapter 1. It incorporates many special cases, among them competition in process innovations, in products or product qualities, and repeated patent races. We assume a basic trade-off between the size of an innovation target and the ease to reach it: The more ambitious a target, the more difficult it is to reach it. Since research targets can have strategically quite different functions as defending a leading position, or gaining an advantage, or catching up with the competition, there is no reason why in all these situations firms should even have similar targets, or always the same targets as it is assumed in the step-by-step models. We prove that it is possible that all firms will have the same fixed innovation targets in equilibrium, but also show that this case is very exceptional. That is, in general firms have different innovation targets depending on the state of competition. We show that for industry leaders it is optimal to approximately move in steps, while followers adopt either one of two equilibrium strategies: Either they also move in steps (but of generally different size), or they try to make one big jump. Which of these two possibilities occurs depends on an intriguingly simple condition, which determines whether innovation is 'difficult' or 'easy'. Lastly, we show that persistence of leadership does not depend on how followers optimally catch up, but rather on the well-known replacement and efficiency effects that determine whether the incumbent has more incentives to innovate than the follower.

Our results therefore show that it is important to look at innovation targets as a decision variable to obtain a fuller understanding of how firms compete.

Cournot Oligopoly

The Cournot model is one of the most widely accepted oligopoly models, and is increasingly used to analyze markets under product differentiation. For applied work it is desirable to make use of as few and as weak *a priori* restrictions as possible, and our framework is general in two senses: We allow for nonlinear demand and cost functions, and for heterogeneous goods.

We set out to analyze the questions of existence of equilibria, uniqueness, and comparative statics with identical firms, but where products may differ from each other. There are relatively few general results on existence of equilibrium or comparative statics for heterogenous goods, and they either apply to special classes of demand functions (Spence 1976), or to cases where firms react to an increase in competitors' output by either raising or decreasing their own output (reaction functions are either increasing (Vives 1990) or decreasing (Kukushkin 1994 and Corchón 1994, 1996)).

For our analysis we employ a new set of tools, lattice theory and monotone comparative statics, which allow to isolate the economic assumptions that drive the results, without relying on non-essential assumptions as differentiability, stability, or convexity. In particular, we assume that goods are substitutes (homogeneous goods are a special case), and identify a weak additional condition on the firms' demand and cost functions that guarantees that Cournot equilibrium exists:

Condition A: Each firm reacts to an increase in competitors' output in such a way that its market price does not rise.

This condition is not related to whether goods are strategic substitutes or complements, therefore reaction functions may be increasing or decreasing or both. Condition A means the following: If firm i's competitors raise their outputs, and firm i does not react by changing its output, then its market price will decrease anyway because goods are substitutes. If firm i increases its output, market price decreases even further. But when firm i restricts it output, raising its price, condition A says that firm i will not restrict its quantity so much that market price is higher than before. In particular, condition A rules out strongly increasing returns to scale in production, which might cause higher prices because producing less raises average costs, leading to a further cut in production.

We show that under condition A and some standard regularity conditions pure symmetric Cournot equilibria exist. Asymmetric equilibria can be ruled out if we add an additional natural assumption:

Condition B: Each firm's price reacts stronger to changes in its own output than in competitors' outputs.

This means that in its own market firm i has more control over its market price than other firms. It can be shown that this condition can be derived from utility maximization by a representative consumer. The possibility of multiple symmetric equilibria can only be ruled out under much stronger assumptions involving quasi-concavity of payoffs (see Kolstad and Mathiesen 1987 for homogeneous goods).

In the second half of chapter 3, we present results on the effects of entry of new competitors. The inherited intuition, acquired under homogeneous goods and convex production cost, says that prices will decrease and total output will rise, while individual outputs will decrease. It has been stressed recently that prices may go up, and total quantity go down, if there are significant increasing returns to scale in production, even with homogeneous goods (Amir and Lambson 1998). It is also well-known that equilibrium prices and quantities move in the 'wrong' direction in *unstable* equilibria. However, if we assume in addition to condition A that competitors' outputs can be aggregated, we reinstate the typical scenario. Here aggregation may mean that outputs are simply added up, although we treat this more generally.

However, we exhibit a counter-example that shows that if aggregation of competitors' outputs is not possible, then equilibrium prices may go up after entry even if there are no increasing returns to scale (we assume costless production) and the equilibrium is stable. This remarkable result is entirely due to the effect of non-aggregation.

Total quantity and price do not necessarily move in lockstep, since market prices may depend on each individual quantity instead of their sum. If we evoke condition B, then total equilibrium output goes up with entry while equilibrium price goes down. The change in individual production quantities is characterized as with homogeneous goods: they fall (rise) if goods are strategic substitutes (complements).

In the light of these results it becomes clear that in the previous literature for homogeneous goods quite often too many conditions were imposed at the same time. For example, sometimes it is assumed that goods are strategic substitutes, that profits are concave in own output, and that costs are convex. Of these conditions, the first one yields the conclusions that equilibrium exists if goods can be aggregated (Kukushkin 1994), and that individual quantities decrease after entry; the second one yields existence of equilibrium; the third one, being sufficient for condition A to hold with homogeneous goods, yields existence of symmetric equilibria (McManus 1964), and equilibrium prices decreasing (total quantity increasing) in the number of firms if goods are homogeneous. Equally, it is a strong assumption that reactions functions are either increasing or decreasing. If an oligopoly model with only a subset of these comparative statics properties is needed, one also only needs a subset of the different conditions that have been imposed on demand and costs in the past. Knowing the exact consequences of each single condition, and not imposing unnecessary assumptions, is very useful since it helps to alleviate the trade-off between very specific assumptions and the general applicability of the results of a model.

Part I

R&D-races

Chapter 1

Dynamic R&D competition with endogenous targets

1.1 Introduction

It is now common knowledge in economics that 'competition' in the marketplace is a dynamic phenomenon. Firms seek their advantage through continuous adjustments in prices, quality, cost, variety, and organization. These changes are often supported by technological progress in form of product and process innovations. Indeed, a large percentage of economic growth has been attributed to technological improvements, and Schumpeter's idea of a 'process of creative destruction' has recently been revived, and has entered explicitly into endogenous growth theory, see e.g. Aghion and Howitt (1992), Amable (1996), Aghion and Howitt (1998).

A plethora of models has been developed in the last two decades to analyse the mechanics of dynamic competition. One strand is the literature on *patent races*, where two firms compete to make a randomly occurring innovation first, after which the race ends. These models focus on the optimal path of expenditure or *research effort*, while the innovation *target*, winning the patent, is exogenous and fixed. Classical references are Loury (1979), Lee and Wilde (1980), Reinganum (1981), Reinganum (1985), and surveys are contained in Reinganum (1989), Tirole (1988, ch. 10), and Martin (1993). Nevertheless, these models are not concerned with *repeated* strategic interaction between competitors, and are therefore essentially static.

A set of related models studies truly dynamic competition: Firms do not win in one step, but need many to do so (e.g. Harris and Vickers 1987), or competition may go on forever, as in Budd, Harris and Vickers (1993, BHV), and Aghion, Harris and Vickers (1997, AHV). In these models the different behaviors of firms that are close (neck-to-neck) or far apart (leader and follower) are studied, but again only in relation to research efforts, while firms' innovation targets are steps of fixed size in a fixed order (Harris and Vickers 1987, or AHV) or the state of competition moves continuously (BHV). AHV show that the intensity of competition is highest when firms are neck-to-neck, and that leaders compete harder than followers. BHV show that the state of competition moves into the direction of highest joint payoff, leading either to persistence or frequent change of leadership.

The research quoted above concentrates on the optimal allocation of research efforts while not allowing firms to choose *where* they want to go, for example: leapfrog the leader, or catch up slowly; gain a large lead, or stay just a little ahead of the follower. Some recent work has focussed on the selection of innovation targets: Cabral (1997) considers the *optimal choice of variance and covariance* of motion while expected progress is held constant. He gives sufficient conditions that leaders choose safe strategies, and followers risky ones, but also shows that the opposite may happen. *Leapfrogging* has been the focus of some recent literature on R&D under vertical product differentiation (Rosenkranz 1996 and, applied to international trade, Motta, Thisse and Cabrales 1997), where in two period-models firms or countries reposition themselves after an intervening shock (change in technology or opening of global markets), obtaining conditions for persistence of leadership, or leapfrogging. These two models analyse the change from one static (interpreted as long-run) equilibrium to another, and are therefore not concerned with ongoing competition with action and counter-action, in particular not mutual leapfrogging.

The empirical literature on R&D has largely concentrated on the aggregated aspects of innovation, or on verifying various Schumpeterian hypotheses. Comparatively little work has been done studying general patterns of competition between sets of firms, apart from case studies. One set of these, Scherer (1992, ch. 3), shows the various instruments of competition: Improvements in quality (razors, tires, photographic film), learning curve effects (airliners), production cost (calculators), patents and standards (TV's and VCR's). Some firms were able to defend their leadership by increasing R&D efforts, such as Gillette (razors), Eastman Kodak (films), General Electric (diagnostic imaging), others ceded their markets to competitors, as for TV's, VCR's, and fax machines. Some firms were able to persist but with substantially reduced market shares, as calculator and tire manufacturers, and Boeing (airliners).

In our research we concentrate on the optimal repeated choice of innovation targets, and its interplay with the issues raised in previous work: optimal allocation of research efforts, occurrence of leapfrogging, optimal catching up by followers, optimal defence of leadership, and persistence of leadership. We let firms choose both research efforts and targets. The fact that firms' strategies are two- instead of one-dimensional significantly enriches the firms' choice set, and leads to a more interesting mix of strategies. The state of competition may be anything that differentiates firms, as differences in productive efficiency, product quality, or accumulated knowledge towards obtaining a patent *et cetera*. Each firm determines where it will try to move this state of competition, with a larger move less likely to achieve than a small one, and how much it wants to spend to do so.

Letting firms choose their innovation targets necessitates an enlargement of the state space to a continuum, therefore essentially transforming the model into a differential game with stochastic jumps. Literature on this type of games is scarce, with some notable exceptions such as Wernerfelt (1988), where existence of Nash equilibrium is studied, or Malliaris and Brock (1982), where the resulting individual dynamic programming problems are discussed. In general, models of this type cannot be solved analytically. We have been able to characterize the optimal strategies asymptotically, and completely in some cases, and have solved the resulting games numerically.

Our results are as follows: In a first step, we characterize the equilibrium firm value, and the optimal innovation targets and efforts, using asymptotic expansions for large discounting. This analysis draws heavily on the techniques developed in Budd, Harris and Vickers (1993).

Secondly, we show that optimal innovation targets are the more ambitious the further a firm is ahead, i.e. no firm will try to reach a better state of competition than if it were further ahead. Also, we characterize the class of ow profit functions that give rise to optimal innovation targets that are steps of constant size. This result is of interest on its own insofar as it shows when the step-wise structure of races like Harris and Vickers (1987) and AHV, which is imposed by assumption, may arise endogenously in equilibrium.

Thirdly, we analyse the optimal strategies of firms that are very far ahead, or behind, respectively. In equilibrium, the leader will move on in a step-by-step fashion (e.g. "reduce unit costs by 20%"), while the follower will adopt one of two different equilibrium strategies: Either it tries to catch up in a step-by-step fashion, or it attempts to match a multiple of the leader's level of progress (e.g. "reduce unit costs to 120% of the leader's unit cost").

Which of these strategies is chosen depends on a simple asymptotic condition relating the probability (more precisely, hazard rate) of making the innovation to ow profits: If after making an innovation ow profits increase at a rate sufficiently high compared to the innovation hazard rate, that is, if their elasticity with respect to the state of competition is higher than the elasticity of the hazard rate with respect to the innovation target, then a follower will use the matching strategy.

Leapfrogging will occur when firms are sufficiently close, but whether a matching strategy involves leapfrogging depends on the hazard rate and ow profits (and value) for firms that are close. Therefore there are cases where firms that are behind will *always* attempt to leapfrog, while in others they will first try to catch up and then leapfrog making smaller steps.

The fourth and last focus of our analysis is the relation between the optimal innovation targets and the phenomenon of persistence of leadership. We find that leadership may be persistent (or not) no matter whether followers will try to catch up in a step-by-step or matching fashion. Rather, persistence will depend on the well-known relationship between the replacement and efficiency effects, which determine the relative levels of research effort expended by leaders and followers.

The rest of the paper is structured as follows: Section 1.2 states the model formally, introduces the firms' problem, and discusses the equilibrium concept. Section 1.3 provides a description of equilibrium firm value and strategies under large discounting. Section 1.4 contains general results on global and asymptotic properties of the optimal innovation targets. Section 1.5 contains some closed-form and numeric equilibrium solutions. Section 1.6 concludes, and indicates some directions for future research.

1.2 The Model

We consider a duopoly of two firms that are competing over time. These firms differ in the level of technical progress or knowledge achieved, and both firms conduct R&D to improve their competitiveness. Both firms have access to the same R&D technology.

Time is continuous, and profits are discounted at rate $r > 0$. At each point in time, both firms sell their output in the market, creating ow profits $\pi(\delta_i)$. We assume that $\pi : \mathbb{R}_{++} \to \mathbb{R}_+$ is nonincreasing and depends only on the state of competition $\delta_i \in \mathbb{R}_{++}$ (for the other firm, $\delta_j = 1/\delta_i$), which is the technology gap between follower and leader, where firm i is leading if $\delta_i < 1$. Three examples of possible definitions of the state of competition and ow profits are the following:

- Productive efficiency: $\delta_i = c_i/c_j$ (ratio of unit production costs). With unit-elasticity demand ($Quantity = 1/price$), for quantity competition the resulting Cournot equilibrium yields ow profits of $\pi(\delta) = 1/(1+\delta)^2$, and for price competition Bertrand ow profits are $\pi(\delta) = \mathrm{ma}\ \{0, 1-\delta\}$. These are the two cases treated in Aghion, Harris and Vickers (1997).

- Product quality: $\delta_i = \mathrm{e\ p}(u_j - u_i)$, where u_i is firm i's quality level. Let there be a mass S of consumers with utility $U = u_k - p_k$, $k \in \{i, j\}$, each buying one unit per period, and where p_k is the price of the good chosen. If firms have constant marginal costs of production c and, given product quality, compete in prices, ow profits are

$$\pi(\delta_i) = \mathrm{ma}\ \{(u_i - u_j)S, 0\} = \mathrm{ma}\ \{-S \log \delta_i, 0\}.$$

- Repeated patent race: $\delta_i = k_j/k_i$, where k_i is firm i's accumulated knowledge. If a firm makes an innovation that leaves it sufficiently ahead of the other firm, $\delta_i \leq \bar{\delta} \leq 1$, then it receives a patent on its technology and consequently monopoly profits π^m in the product market, otherwise profits are zero:

$$\pi(\delta_i) = \begin{cases} \pi^m & if \quad \delta_i \leq \bar{\delta} \\ 0 & if \quad \delta_i > \bar{\delta} \end{cases}.$$

Over time, both firms aim to innovate, choosing as innovation target to move the state of competition to $\Delta_i \in (0, \delta_i]$, and $\Delta_j \in (0, \delta_j]$, respectively, and expend research efforts z_i, $z_j \geq 0$ to reach these targets. These are attained randomly and independently over time and between firms, at Poisson hazard rates determined by the R&D-efforts and the sizes of the targets. By exerting a research effort z_i, firm i in state δ_i reaches the innovation target Δ_i with Poisson hazard rate $z_i\psi(\Delta_i, \delta_i)$. Let $D_\psi = \{(\delta_1, \delta_2) \in \mathbb{R}^2_{++} | \delta_1 \leq \delta_2\}$. The function $\psi : D_\psi \to \mathbb{R}_+$ is assumed to have the following properties: It is continuously differentiable, increasing in its first argument and decreasing in its second. In other words, the more ambitious the target, the longer is the expected time to reach it.[1] We will mostly assume that ψ is of the

[1]For a model using a similar idea see Aghion and Howitt (1992).

form $\psi(\Delta_i, \delta_i) = \Delta_i^{\eta}/\delta_i^{\mu}$, $\eta, \mu > 0$. For $\mu = \eta$ this functional form incorporates the assumption that the probability of achieving an innovation only depends on one's own progress, and not on the state of competition. Research is costly, with ow cost $c(z)$ at each point in time for the research effort level $z \geq 0$, where $c(.)$ is an increasing, convex, and differentiable function with $c(0) = c'(0) = 0$. We will mostly assume that $c(z) = z^2/2$.

We will characterize subgame-perfect equilibria in feedback or Markov strategies $(\Delta_i(.), z_i(.))$ (also called Markov-perfect equilibria, see Maskin and Tirole (1997)), i.e. strategies that at each point in time only depend on the state of competition, taking into account that at later stages play is optimal given the state of competition. The space of Markov strategies is a natural choice in this model, since occurrences of innovations, following a memoryless Poisson process, are not history-dependent.[2][3]

Formally, a *strategy* for player i is a pair (Δ_i, z_i) of functions $\Delta_i : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ and $z_i : \mathbb{R}_{++} \rightarrow \mathbb{R}_{+}$, such that if the state of competition is δ_i, then player i selects the innovation target $\Delta_i(\delta_i)$ and exerts research effort $z_i(\delta_i)$. A *Markov-perfect equilibrium* is a pair of strategies (Δ_i, z_i) and (Δ_j, z_j) that are best responses to each other.

Each firm i maximizes its value $V_i(\delta_i)$, subject to the law of motion of the state of competition ($j \neq i$):

$$V_i(\delta_i) = \underset{\Delta_i(.), z_i(.)}{\mathrm{ma}} \mathsf{E}\left[\int_{t=0}^{\infty} (\pi(\delta_i) - c(z_i))\, e^{-rt} dt\right] \tag{1.1}$$

$$s.t. \quad d\delta_i = (\Delta_i - \delta_i)\, dq_i + (1/\Delta_j - \delta_i)\, dq_j, \tag{1.2}$$

where q_i and q_j follow independent Poisson processes with hazard rates $z_i\psi(\Delta_i, \delta_i)$ and $z_j\psi(\Delta_j, \delta_j)$, respectively. Note that $1/\Delta_j$ is the new state for firm i if firm j makes an innovation. Taking expectations as in appendix 1.A.1, or directly invoking

[2] Also, equilibria in Markov strategies are subgame perfect equilibria in the space of all closed-loop strategies.

[3] Our research concentrates on characterizing equilibrium behavior of firms if a symmetric equilibrium exists, and we do not treat the questions of existence or uniqueness. In fact, it may well be possible that multiple equilibria exist as in BHV, at least for small discounting.

the techniques of stochastic dynamic programming with Poisson processes (see e.g. Malliaris and Brock 1982), leads to the following Hamilton-Jacobi-Bellman equation characterizing the value function $V_i(.)$ of firm i, given strategy (Δ_j, z_j) of firm j:

$$\begin{aligned} rV_i(\delta_i) &= \max_{\Delta_i(.),z_i(.)} \{\pi(\delta_i) - c(z_i) + z_i\psi(\Delta_i,\delta_i)[V_i(\Delta_i) - V_i(\delta_i)] \\ &\quad -z_j\psi(\Delta_j,\delta_j)[V_i(\delta_i) - V_i(1/\Delta_j)]\}. \end{aligned} \tag{1.3}$$

Interpreting this condition is straightforward: The value flow $rV_i(\delta_i)$, at the optimum, is equal to flow profits $\pi(\delta_i)$ minus costs of research $c(z_i)$, plus the expected gains from achieving a cost-reducing innovation $z_i\psi(\Delta_i,\delta_i)[V(\Delta_i) - V(\delta_i)]$, minus the expected losses caused by innovations that the other firm may make, $z_j\psi(\Delta_j,\delta_j) \times [V(\delta_i) - V(1/\Delta_j)]$.

Interior optimal strategies are characterized by the first-order conditions (assuming that the value function is differentiable at the relevant points)

$$\frac{\partial\psi}{\partial\Delta_i}[V_i(\Delta_i) - V_i(\delta_i)] + \psi V_i'(\Delta_i) = 0 \tag{1.4}$$

$$\psi(\Delta_i,\delta_i)[V(\Delta_i) - V(\delta_i)] = c'(z_i). \tag{1.5}$$

Condition (1.4) determines the optimal innovation size such that a marginal increase in expected value due to a higher target is exactly offset by the marginally lower probability of reaching it. The second-order condition

$$\frac{d^2}{d\Delta_i^2}\{\psi(\Delta_i,\delta_i)[V(\Delta_i) - V(\delta_i)]\} \leq 0$$

must also be satisfied. Condition (1.5) states that marginal effort cost should be equal to its marginal return, which is the expected increase in firm value after an innovation. Because of the convexity of $c(.)$ the second-order condition is satisfied.

Generally, this model cannot be solved analytically, as is usual for differential games. Nevertheless, we have been able to identify several classes of closed-form solutions which will be discussed later in sections 1.4 and 1.5. In section 1.5 we will also exhibit some numerical solutions to the examples given above.

1.3 Firm Value and Strategies in Equilibrium

In general, the value function V, which summarizes a firm's possibilities in dynamic equilibrium for all possible values of the state of competition, does not have the same functional form as the ow profit function, or may even not be obtainable in closed form. Therefore a precise analytical characterization of the optimal strategies and competitive effects involved in equilibrium is hindered by the problem that the main tool of analysis, the value function, is not known. There are two ways out of this problem: Either one analyses optimal innovation targets for extreme values of the state ($\delta \to 0$ or $\delta \to \infty$), as will be done in the following sections, or one uses the techniques of asymptotic expansions introduced in Budd, Harris and Vickers (1993, BHV) to characterize the value function and strategies for large discounting ($r \to \infty$). Here we will state and discuss the results of this approach, assuming that the respective expansions exist. See appendix 1.A.2 for the computations underlying the following results.

We assume that the following expansions exist:[4]

$$\begin{aligned} rV(\delta) &= \Sigma_{n=0}^{\infty} r^{-n} v_n(\delta), \\ \Delta(\delta) &= \Sigma_{n=0}^{\infty} r^{-n} \Delta_n(\delta) \\ z(\delta) &= \Sigma_{n=0}^{\infty} r^{-n} z_n(\delta) \end{aligned}$$

where at least for $0 \leq n \leq 2$ the functions $v_n(.)$ are twice continuously differentiable. Furthermore, assume that the profit function $\pi(.)$ and the innovation technology $\psi(., \delta)$ are twice continuously differentiable, and costs of research effort are $c(z) = z^2/2$.

Let

$$\Delta_0(\delta) = \arg\max_{\Delta} \; \psi(\Delta, \delta)[\pi(\Delta) - \pi(\delta)] \tag{1.6}$$

[4] We have dropped the index i for clarity, while still indicating variables pertaining to the other firm by the index j.

be the innovation target that maximizes the expected increase in ow profits. Then the equilibrium value function can be expressed as

$$rV(\delta) = \pi(\delta) + r^{-2}v_2(\delta) + O\left(r^{-3}\right), \tag{1.7}$$

where

$$v_2(\delta) = \left(\psi(\Delta_0, \delta)\left[\pi(\Delta_0) - \pi(\delta)\right]\right)^2/2 + z_{1j}\psi_{0j}\left[\pi(1/\Delta_{0j}) - \pi(\delta)\right] \tag{1.8}$$

and z_{1j}, $\psi_{0j} = \psi(\Delta_{0j}, 1/\delta)$, and Δ_{0j} are the first terms of the respective expansions of the other firm.

The optimal innovation target can be written as

$$\Delta(\delta) = \Delta_0(\delta) - r^{-2}\frac{\partial(\psi(\Delta,\delta)[v_2(\Delta) - v_2(\delta)])/\partial\Delta}{\partial^2(\psi(\Delta,\delta)[\pi(\Delta) - \pi(\delta)])/\partial\Delta^2}\Big|_{\Delta=\Delta_0(\delta)} + O\left(r^{-3}\right), \tag{1.9}$$

and optimal research effort is given by

$$\begin{aligned} z(\delta) &= r^{-1}\psi(\Delta_0, \delta)\left[\pi(\Delta_0) - \pi(\delta)\right] \\ &\quad + r^{-3}\psi(\Delta_0, \delta)\left[v_2(\Delta_0) - v_2(\delta)\right] + O\left(r^{-4}\right). \end{aligned} \tag{1.10}$$

Effort can be partitioned into two terms involved in the *replacement* and *efficiency effects*, respectively. Although these terms were originally coined to explain the incentives of monopolists to innovate, either to raise his profits, or to fend off a possible entrant, they readily extend to duopoly.

The replacement effect describes the "pure" incentives of the leader (the "monopolist") to innovate, given that by making the innovation he only replaces himself as the market leader, and leaving aside the threat of being overtaken by a competitor. This effect therefore favours the emergence of frequent changes in leadership if the leader cannot increase his ow profits, as e.g. in patent races. In our model this effect will be determined by the first term in (1.10),

$$z^r = r^{-1}\psi(\Delta_0, \delta)\left[\pi(\Delta_0) - \pi(\delta)\right], \tag{1.11}$$

since this is the share of effort that is motivated by the expected increase in own ow profits.

The efficiency effect, on the other hand, describes the incentives of the leader to fend off the follower: In the classical example, the value of a nondrastic innovation is higher to the monopolist than to the entrant because the ensuing competition in a duopoly dissipates monopoly rents. Therefore, if joint value after an innovation of the leader is higher than after an innovation of the follower, this effect favours persistence of leadership. The part of research effort attributable to the efficiency effect, apart from higher-order effects, is

$$z^e = r^{-3}\psi(\Delta_0, \delta)\left[v_2(\Delta_0) - v_2(\delta)\right]. \tag{1.12}$$

In general, both effects will be at work simultaneously, and the one that dominates will determine whether leader or follower expend more effort, and therefore whether persistence or change of leadership will follow. Depending on the type of ow profits, the replacement effect may actually work *in favour* of the leader, for example when ow profits of the leader are not constant (as in a patent race) but increase with his advantage; in this case the leader may have a strong incentive to innovate even further.

Using the above partition of efforts, value function and strategies can be rewritten as

$$\begin{aligned}
rV(\delta) &= \pi(\delta) + r^{-2}v_2(\delta) + O\left(r^{-3}\right), && (1.13)\\
v_2(\delta) &= -c(rz^r) + (rz^r)\psi(\Delta_0, \delta)\left[\pi(\Delta_0) - \pi(\delta)\right] \\
&\quad + \left(rz_j^r\right)\psi_{0j}\left[\pi(1/\Delta_{0j}) - \pi(\delta)\right] && (1.14)\\
\Delta(\delta) &= \Delta_0(\delta) - \tfrac{\partial}{\partial\Delta}z^e / \tfrac{\partial^2}{\partial\Delta^2}z^r|_{\Delta=\Delta_0} + O\left(r^{-3}\right), && (1.15)\\
z(\delta) &= z^r + z^e + O\left(r^{-4}\right). && (1.16)
\end{aligned}$$

The interpretation of these results is straightforward. If the discount rate is large, research efforts will be close to zero since firms are very myopic and future profits count little; firm value will then mainly be determined by extending present ow profits into the indefinite future, $V(\delta) \approx \pi(\delta)/r$. To second order in r, competitive effects stemming from effort cost, expected gains from making in innovation, and expected losses from being preempted by the other firm, enter in the value function through the term $v_2(\delta)$.

The optimal innovation targets can also be partitioned into two terms: For large discounting the first term dominates, and the innovation target Δ_0 is chosen as to maximize the expected increase in ow profits $\psi(\Delta, \delta)[\pi(\Delta) - \pi(\delta)]$. This value determines the size of the *replacement effect.* To second order in r, competitive effects enter through

$$\Delta(\delta) \approx \Delta_0(\delta) - \tfrac{d}{d\Delta_0} z^e / \tfrac{d^2}{d\Delta_0^2} z^r.$$

Assuming that Δ_0 is an internal maximizer, and therefore $d^2 z^r / d\Delta_0^2 < 0$, this second order effect is determined by the effect of the choice of the innovation target on the *efficiency effect*: A firm will raise or lower its innovation target as compared to Δ_0 if doing so increases the joint effect of expected gains from innovation minus effort costs and expected losses from an innovation of the other firm.

As approximations, the above results will hold true not only for $r \to \infty$, but by continuity also for large but finite r. Even though we do not analyze this issue here, as in BHV for large r very likely there is a unique and symmetric equilibrium, with value function, innovation targets, and research effort levels determined by the lowest order effects, and therefore directly by the form of the profit function and the innovation technology.

Nevertheless, as in BHV, for small r, i.e. very patient firms, additional "self-enforcing" effects may enter that are not determined by the profit function or innovation technology, and can even lead to multiple equilibria.[5]

The preceding discussion also sheds some light on the possibility of equilibria where the value function is "of the same functional form" as, or an affine linear transformation of, the profit function. A standard technique in dynamic programming consists of exploiting this feature by employing as candidate value function an

[5] We have not yet been able to identify such a case, which is not surprising given the difficulties described in BHV, p. 560.

affine linear transformation of ow profits and determining the unknown constants through the Hamilton-Jacobi-Bellman equation (see e.g. Stokey and Lucas 1989). From (1.7), in our model a necessary condition is that $v_2(\delta)$ as defined in (1.14) is an affine transformation of the profit function, in particular possibly a constant. Below we exhibit such a case, but in general this condition is hard to meet.

1.4 Optimal Innovation Targets

In this section we will discuss several properties of the optimal innovation targets. We will tackle the following questions:

- Are innovation targets of better-placed firms always more ambitious?
- When are optimal targets given by constant steps, i.e. $\Delta(\delta) = \rho\delta$ with $0 < \rho < 1$?
- What are the innovation targets of firms that are far ahead?
- What are the innovation targets of firms that are far behind?

For the rest of this section we will assume that the innovation technology is given by $\psi(\Delta, \delta) = \Delta^\eta / \phi(\delta)$, where $\eta > 0$ and ϕ is increasing in δ. For this specification the elasticity of the hazard rate with respect to the innovation target Δ is constant and equal to η. Sometimes we will assume that $\phi(\delta) = \delta^\mu$, $\mu > 0$.

1.4.1 Monotonicity of Innovation Targets

First we will show that innovation targets $\Delta(\delta)$ are nondecreasing in the state if the value function is decreasing. Let $\delta_1 > \delta_0$; then $v_1 = V(\delta_1) < v_0 = V(\delta_0)$, and assume that $\Delta_1 = \Delta(\delta_1) < \Delta_0 = \Delta(\delta_0)$. Since Δ_1 and Δ_0 are solutions of the problem

$$\operatorname*{ma}_{\Delta} \frac{\Delta^\eta}{\phi(\delta)} \left(V(\Delta) - V(\delta)\right) \tag{1.17}$$

for $\delta = \delta_1$ or $\delta = \delta_0$, the following inequalities hold:

$$\begin{aligned}\frac{\Delta_0^\eta}{\phi(\delta_0)}\left(V\left(\Delta_0\right)-v_0\right) &\geq \frac{\Delta_1^\eta}{\phi(\delta_0)}\left(V\left(\Delta_1\right)-v_0\right), \\ \frac{\Delta_1^\eta}{\phi(\delta_1)}\left(V\left(\Delta_1\right)-v_1\right) &\geq \frac{\Delta_0^\eta}{\phi(\delta_1)}\left(V\left(\Delta_0\right)-v_1\right).\end{aligned}$$

These can be reformulated to

$$\begin{aligned}\Delta_0^\eta V\left(\Delta_0\right)-\Delta_1^\eta V\left(\Delta_1\right) &\geq \left(\Delta_0^\eta-\Delta_1^\eta\right) v_0, \\ \Delta_0^\eta V\left(\Delta_0\right)-\Delta_1^\eta V\left(\Delta_1\right) &\leq \left(\Delta_0^\eta-\Delta_1^\eta\right) v_1.\end{aligned}$$

Now we arrive at a contradiction to $\Delta_1 < \Delta_0$ since $v_0 > v_1$ and $\eta > 0$. Therefore the conclusion is that:

Remark 1 *If the value function is strictly decreasing, innovation targets Δ are nondecreasing.*

At this level of generality, the above reasoning is not able to demonstrate that Δ may be (strictly) increasing: If $\Delta_1 = \Delta_0$ then the conditions describing the optimum do not contradict each other, rather they show that in this case Δ is constant in the range $\delta_0 \leq \delta \leq \delta_1$.

If on the other hand we assume that problem (1.17) has exactly one solution for each δ, then the above weak inequalities sharpen to strict ones, and this time we arrive at a contradiction if $\Delta_1 \leq \Delta_0$. Also, we can weaken the requirement on the value function:

Remark 2 *If the value function is nonincreasing, and if problem (1.17) has exactly one solution for each $\delta > 0$, then innovation targets Δ are (strictly) increasing.*

Of course, the additional condition may be difficult to check without solving the problem in the first place. Applications include cases where first-order conditions are hard or impossible to solve explicitly but the shape of the value function is known.

1.4.2 Step-by-step Targets

In some part of the literature on R&D races, for example in AHV, firms are by assumption constrained to making innovations that reduce production costs by some exogenously given constant percentage $\rho \in (0,1)$, or more generally make innovations of a fixed size, which we will call *step-by-step innovation targets.* We will show later through various examples that stepping may indeed arise exactly or approximately in equilibrium, but as well may be far off track. In this section we tackle this subject explicitly and, under some mild technical assumptions, characterize the value and profit functions that give rise to optimal innovation targets $\Delta(\delta) = \rho\delta$ locally or globally.

Under step-by-step targets, the set of states of competition that can be reached is very simple: If the initial state of competition is δ_0, then the states that can be reached are of the form $\delta = \delta_0 \rho^n$, where $n \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $\rho > 0$ is some constant. Then 'catching up' to being neck-to-neck as in AHV occurs if and only if $\delta_0 = \rho^m$ for some $m \in \mathbb{Z}$, otherwise firms will always leapfrog each other when they 'meet'.

To begin the analysis, let for $\delta \in (0, \bar{\delta})$, with $\bar{\delta} > 0$,

$$V(\delta) = a_0 + a_1\delta^\alpha + a_2\delta^\beta, \tag{1.18}$$

with $\beta > \alpha \geq -\eta$, $a_1\alpha < 0$ and $a_2\beta < 0$. Assume that the optimal target is $\Delta(\delta)$ $= \rho\delta$, $0 < \rho < 1$, and insert this expression into the first order condition of the maximization problem

$$\underset{\Delta}{\mathrm{ma}} \; \frac{\Delta^\eta}{\phi(\delta)} (V(\Delta) - V(\delta))$$

leading to

$$a_1 ((\alpha + \eta)\rho^\alpha - \eta)\delta^\alpha + a_2 ((\beta + \eta)\rho^\beta - \eta)\delta^\beta = 0.$$

Since this relation must hold for all $\delta \in (0, \bar{\delta})$, the terms in the brackets must be identically zero, or

$$\rho = (1+\alpha/\eta)^{-1/\alpha} = (1+\beta/\eta)^{-1/\beta}. \tag{1.19}$$

Since these terms are strictly increasing in α and β (with upper limit equal to 1), equality can only hold if $\alpha = \beta$, contradicting the assumption $\beta > \alpha$. The same argument holds if any countable number of terms were included in the value function V, and the immediate conclusion therefore is:

Remark 3 *If the value function on $\delta \in (0,\bar{\delta})$, is of the form*

$$V(\delta) = a_0 + \Sigma_{i=1}^{\infty} a_i \delta^{\alpha_i}, \tag{1.20}$$

where $-\eta < \alpha_1 < ... < \alpha_n < ...$, $a_i\alpha_i < 0$, then $\Delta(\delta) = \rho\delta$ for $\delta \in (0,\bar{\delta})$ and for some $\rho \in (0,1)$ if and only if there is exactly one $i \in \mathbb{N}$ such that $a_i \neq 0$. In this case the step size is given by

$$\rho = (1+\alpha_i/\eta)^{-1/\alpha_i}. \tag{1.21}$$

Note that (1.20) does not cover the case $\alpha_i = 0$, which corresponds to $V(\delta) = -\ln\delta$, in which case the optimal innovation target is step-by-step with

$$\begin{aligned} \Delta(\delta) &= \arg\max_{\Delta} \tfrac{\Delta^{\eta}}{\delta^{\mu}}(-\ln\Delta + \ln\delta) \\ &= e^{-1/\eta}\delta = \lim_{\alpha\to 0}(1+\alpha/\eta)^{-1/\alpha}. \end{aligned}$$

Rather than an exception to the above conclusions, this case turns out to be a border case for which the elasticity of the value function goes to zero, a value for which our result on the step size is still valid in the limit. We will discuss logarithmic value functions further in the examples section.

If the value function has a Taylor expansion around zero, then some straightforward conclusions follow:

Remark 4 *Let V be defined and equal to its Taylor series on $[0,\bar{\delta})$, and let the optimal innovation target be given by $\Delta(\delta) = \rho\delta$ for some $\rho \in (0,1)$ for all $\delta \in [0,\bar{\delta})$.*

If the slope of V is negative at $\delta = 0$ then V is linear on $[0, \bar{\delta})$, and

$$\rho = \tfrac{\eta}{\eta+1}. \tag{1.22}$$

If the slope of V is zero at $\delta = 0$ then there is a unique integer $m > 1$ such that $V(\delta) = a_0 + a_m \delta^m$, and

$$\rho = (1 + m/\eta)^{-1/m}. \tag{1.23}$$

The first statement is proved by setting $\alpha_1 = 1$ and $a_1 < 0$, which is necessary for a negative slope at $\delta = 0$, noting that the Taylor expansion of V around $\delta = 0$ will be

$$V(\delta) = a_0 + a_1\delta + a_2\delta^2 + ...$$

If the slope at $\delta = 0$ is zero, there will be a unique smallest integer $m > 1$ such that

$$V(\delta) = a_0 + a_m\delta^m + a_{m+1}\delta^{m+1} + ...,$$

with $a_m < 0$.

In both cases, the value function will be of the form $V(\delta) = a\delta^\beta + b$ ($\beta > -\eta$, $a\beta < 0$). The innovation target will be

$$\Delta(\delta) = (1 + \beta/\eta)^{-\frac{1}{\beta}}\delta, \tag{1.24}$$

and inserting these expressions and $\Delta_j(1/\delta) = (1 + \beta/\eta)^{-\frac{1}{\beta}}/\delta$ in the equation describing the value function (1.3) with quadratic effort costs $c(z) = z^2/2$, the corresponding 'candidate' profit functions are found to be

$$\pi(\delta) = \alpha_1\delta^{2(\eta+\beta-\mu)} + \alpha_2\delta^\beta + \alpha_3\delta^{2(\mu-\eta)} + \alpha_4, \tag{1.25}$$

with

$$\begin{aligned}
\alpha_1 &= -\tfrac{1}{2}(a\beta/\eta)^2\rho^{2\eta+2\beta} < 0,\ \alpha_2 = ra, \\
\alpha_3 &= (a\beta/\eta)^2\rho^{2\eta+\beta} > 0,\ \alpha_4 = rb.
\end{aligned}$$

Note that $\pi(\delta) = a\delta^{\beta} + b$ if either $\beta = 2(\mu - \eta)$, or $\mu = \eta$.

Depending on the parameters, this *candidate* profit function may be increasing over some range, which means that for this set of parameters the innovation target cannot be steps of the given size. A *sufficient* condition (in addition to $a\beta < 0$) for a negative derivative of π is

$$\eta - \mu \geq \text{ma}\ \ \{0, -\beta\},$$

as can be easily seen from $\alpha_1 < 0$ and $\alpha_3 > 0$. Thus, the above family of profit functions is very special, and intimately linked to the parameters of the innovation technology. Moreover, it is only *one-dimensional* given r, η, and μ.

To sum up, the preceding discussion shows that, if we require that the value function has an expansion around zero, the family of profit functions that give rise to step-by-step innovation targets is a one-dimensional family in the space of all profit functions (therefore a null-set), and is intimately related to the parameters of the innovation technology.

1.4.3 Leaders' Innovation Targets

The analysis of the last section can be applied in an approximate manner to determine the evolution of the innovation targets for firms that are far ahead, i.e. when $\delta \approx 0$. To this end, assume that for δ close to zero the value function can be represented as

$$V(\delta) = v_0 + v_1\delta^{\alpha} + o(\delta^{\alpha}), \tag{1.26}$$

where the last term converges faster to zero than δ^{α}, and $\alpha \geq -\eta$ is the asymptotic elasticity of the value function. This approximation may stem from a Taylor expansion about 0 (in that case α is a positive integer), or may simply describe asymptotic behavior as e.g. in $1/\left(\delta + \delta^2\right) \approx \delta^{-1}$ for δ close to zero. Also, assume that for $\Delta \to 0$ the innovation technology may be approximated as

$$\psi(\Delta, \delta) = \tfrac{\Delta^{\gamma}}{\phi(\delta)} + o(\Delta^{\gamma}), \tag{1.27}$$

for some $\eta > 0$, taking on in the limit the form we have used before. The maximization problem is then

$$\underset{0\leq\Delta\leq\delta}{\text{ma}}\left(\frac{\Delta^\eta}{\phi(\delta)}+o\left(\Delta^\eta\right)\right)\left(v_1\Delta^\alpha - v_1\delta^\alpha + o\left(\delta^\alpha\right)\right),$$

with first order condition

$$\left(\eta+\alpha+\frac{o\left(\Delta^{\eta-1}\right)}{\Delta^{\eta-1}}+\frac{o\left(\Delta^{\eta}\right)}{\Delta^{\eta}}\right)\Delta^\alpha=\left(\eta+\frac{o\left(\Delta^{\eta-1}\right)}{\Delta^{\eta-1}}\right)\left(1+\frac{o\left(\delta^{\alpha}\right)}{\delta^{\alpha}}\right)\delta^\alpha.$$

Since for $\delta \to 0$ also $\Delta \to 0$, the solution in the limit is

$$\Delta/\delta \to (1+\alpha/\eta)^{-1/\alpha} \text{ as } \delta \to 0. \tag{1.28}$$

Let us summarize:

Remark 5 *If the value function V and the innovation technology ψ can be approximated as in (1.26) and (1.27), then for $\delta \to 0$ the relative innovation target converges to a constant depending on the elasticities of the innovation technology and the value function: $\Delta/\delta \to (1+\alpha/\eta)^{-1/\alpha}$, i.e. the leader pursues step-by-step innovation targets.*

If the value function is of the types described in the previous section where the step size ρ is constant, then necessarily $\rho = (1+\alpha/\eta)^{-1/\alpha}$.

We will now apply this finding to some value functions, some of which will appear in the examples treated below. For most of them the first order conditions determining the innovation targets cannot be solved explicitly.

- $V(\delta) = (1+\delta)^{-\alpha} = 1-\alpha\delta+o(\delta) : \Delta/\delta \to \eta/(\eta+1)$, $(0<\alpha)$.
- $V(\delta) = \text{e p}(-\lambda\delta^m) = 1-\lambda\delta^m+o(\delta) : \Delta/\delta \to (1+\alpha/m)^{-1/m}$, $(\lambda > 0$, $m \in \mathbb{N})$.
- $V(\delta) = -\ln(a+\delta) = -\ln a - \frac{1}{a}\delta + o(\delta) : \Delta/\delta \to \eta/(\eta+1)$ $(a>0)$. Note that the innovation target does not depend on a, which carries over to the limit case $a=0$.

1.4.4 Followers' Innovation Targets

In this section we will characterize the innovation targets for a firm that is very far behind, $\delta \to \infty$. Also, this analysis will give important insights into the occurrence of leapfrogging: A firm that leapfrogs when it is very far behind will leapfrog also when it is closer. On the other hand, a firm may try to catch up with the leader in a step-by-step fashion and leapfrog only when it is close. We show in the following that the resulting behavior depends on the interplay between the profit function π and the innovation technology ψ.

Assume that the value function V has an asymptotic expansion of the form

$$V(\delta) = b_0 + b_1\delta^{-\gamma} + o\left(\delta^{-\gamma}\right), \quad (\gamma b_1 > 0), \tag{1.29}$$

as well as the innovation technology,

$$\psi(\Delta, \delta) = \frac{\Delta^\eta}{\phi(\delta)} + o(\Delta^\eta), \quad (\eta > 0), \tag{1.30}$$

where $\lim_{x\to\infty} o(x)/x = 0$. The following analysis is technically very similar to the one in the previous section, apart from one vital difference: We have to allow for the possibility that the innovation target Δ does not rise proportionally with the state δ, in particular that Δ may remain bounded from above while the state δ goes to infinity. If this is the case we must use the exact functional form of $V(\Delta)$ instead of approximating it asymptotically.

Let us first treat the case where Δ goes to infinity with δ. Then the maximization problem is

$$\max_{0\leq\Delta\leq\delta} \left(\frac{\Delta^\eta}{\phi(\delta)} + o(\Delta^\eta)\right)\left(b_1\Delta^{-\gamma} - b_1\delta^{-\gamma} + o\left(\delta^{-\gamma}\right)\right),$$

with approximate first order constraint

$$\left(\eta - \gamma + \frac{o(\Delta^{\eta-1})}{\Delta^{\eta-1}} + \frac{o(\Delta^\eta)}{\Delta^\eta}\right)\Delta^{-\gamma} = \left(\eta + \frac{o(\Delta^{\eta-1})}{\Delta^{\eta-1}}\right)\left(1 + \frac{o(\delta^{-\gamma})}{\delta^{-\gamma}}\right)\delta^{-\gamma}.$$

In the limit $\delta, \Delta \to \infty$ this can be written as

$$\eta\left(\Delta/\delta\right)^\gamma = \eta - \gamma. \tag{1.31}$$

Contrary to the case $\delta \to 0$, this first order condition only has a well-defined solution in the limit if $\gamma < \eta$, with

$$\Delta/\delta \to (1 - \gamma/\eta)^{1/\gamma}. \tag{1.32}$$

For $\gamma = \eta$ in the limit it follows that $\Delta/\delta \to 0$ even though Δ may still go to infinity. Finally, for $\gamma > \eta$ we arrive at an outright contradiction (to the form of the expansion of V) since the term on the left of (1.31) is positive. Therefore in this case Δ must have a finite limit.

Let us now treat the case $\gamma = \eta$: Assume that the asymptotic expansion of V can be refined to

$$V(\delta) = b_0 + b_1\delta^{-\eta} + b_2\delta^{-\nu} + o\left(\delta^{-\nu}\right), \quad (\nu > \eta). \tag{1.33}$$

The first order condition then becomes in the limit (using $\delta^{\eta-\nu} \to 0$)

$$\eta b_1 \Delta^{\nu} = b_2 (\nu - \eta)\, \delta^{\eta},$$

which has a solution if and only if $b_2 < 0$ with

$$\Delta = \left(b_2 (1 - \nu/\eta)/b_1\right)^{1/\nu} \delta^{\eta/\nu}, \tag{1.34}$$

i.e. Δ still converges to infinity, but slower than δ. If on the other hand asymptotically $V(\delta) = b_0 + b_1\delta^{-\eta}$, then the optimal innovation may be infinite or finite, depending on the exact form of the value function.

Finally we treat the case of $\gamma > \eta$ where the innovation target remains finite. For large δ the maximization problem can be written as

$$\max_{0 \le \Delta \le \delta} \psi(\Delta, \delta) \left(V(\Delta) - b_0 - b_1\delta^{-\gamma} + o\left(\delta^{-\gamma}\right)\right),$$

and knowing that $\Delta(\delta)$ remains finite, $\Delta(\delta)$ will be close to

$$\tilde{\Delta}(\delta) = \arg \max_{0 \le \Delta \le \delta} \psi(\Delta, \delta) \left(V(\Delta) - b_0\right), \tag{1.35}$$

for large δ, at least if this maximizer is unique. $\tilde{\Delta}(\delta)$ is bounded from above since $\psi V \sim \Delta^{\eta-\gamma}$ tends to zero for large Δ. Let

$$\bar{\Delta} = \lim_{\delta \to \infty} \tilde{\Delta}(\delta) < \infty \tag{1.36}$$

if this limit is defined, then $\Delta(\delta) \to \bar{\Delta}$ as $\delta \to \infty$. Thus, the decision which type of strategy to adopt depends only on the comparison of the elasticities of the value function (γ) and the innovation hazard (η).

Remark 6 *If the value function V and innovation technology ψ can be approximated as in (1.29) and (1.30), then for $\delta \to \infty$ the innovation target converges as follows:*

- $\gamma < \eta$: $\Delta \to \infty$ *and* $\Delta/\delta \to (1 - \gamma/\eta)^{1/\gamma}$, *the follower makes step-by-step innovations;*
- $\gamma = \eta$: $\Delta \to \infty$ *and* $\Delta/\delta^{\eta/\nu} \to (b_2 (1 - \nu/\eta)/b_1)^{1/\nu}$ *(under condition (1.33));*
- $\gamma > \eta$: $\Delta \to \bar{\Delta} < \infty$, *i.e. the follower matches a certain multiple of the opponent's level of progress.*

In other words, if the value function increases slower with an improvement in the state than the probability of making the innovation decreases, then it is optimal to aim for step-by-step innovations. If on the other hand the value function increases fast, then the follower should aim for a big innovation leaving him close to the leader. The probabilities of reaching the targets are rather different: Step-by-step (small) innovations are 'easy' because the probability of success if rather high, whereas matching (big) innovations are 'difficult' and have a low probability of success.

We apply the above results to some value functions and compute the asymptotic innovation targets:

- $V(\delta) = (1+\delta)^{-\alpha} = \delta^{-\alpha} - \alpha\delta^{-(\alpha+1)} + o\left(\delta^{-(\alpha+1)}\right)$, $(\alpha > 0)$

  $\alpha < \eta : \Delta/\delta \to (1 - \alpha/\eta)^{1/\alpha}$

  $\alpha = \eta : \Delta/\delta^{\eta/(\eta+1)} \to 1.$

$\alpha > \eta : \Delta \to \eta/(\alpha - \eta)$ which is the maximizer of $\Delta^\eta/(1+\Delta)^\alpha$. In the case of "Cournot competition" with $\alpha = 2$, and $\eta = 1$, the result is $\Delta \to 1/(2-1) = 1$, close to the numerical solution shown below.[6]

- $V(\delta) = \exp(-\lambda\delta^m)$, $(\lambda > 0, m \in \mathbb{N})$. Since for this value function "$\gamma = \infty$" (it decays faster than any $\delta^{-\alpha}$ with $\alpha < \infty$), Δ will have a finite limit equal to $\arg\max \Delta^\eta \exp(-\lambda\Delta^m)$, i.e. $\Delta \to (\eta/\lambda m)^{\frac{1}{m}}$.

- $V(\delta) = -\ln(a+\delta) = -\ln\delta + o(\ln\delta)$, $(a \geq 0)$. Since this value function decreases slower than any $\delta^{-\alpha}$ $(\alpha > 0)$, a good guess given the above observations is that Δ converges to infinity with

$$\Delta/\delta \to \lim_{\gamma\to 0}(1-\gamma/\eta)^{1/\gamma} = e^{-\frac{1}{\eta}},$$

which is exactly what we have determine analytically for $a = 0$ in section 1.4.2.

The above results were concerned with determining the nature of optimal innovation targets when the *value function* is known asymptotically. In general, what is known is the profit function π, and the value function has to be determined. We will now use our findings from above to classify optimal innovation targets according to the asymptotic behavior of the respective *profit* functions in relation to the innovation technology.

Specialize the innovation technology to $\psi(\Delta,\delta) = \Delta^\eta/\delta^\mu$, and assume that the profit function can be approximated as $\pi(\delta) \approx p\delta^{-\alpha}$ for large δ, with $p\alpha > 0$. Therefore α is (the modulo of) the asymptotic elasticity of profits with respect to the state of competition, while η is the constant elasticity of the innovation hazard with respect to the innovation target. We obtain the following classification:

Remark 7 *Let $\eta < 2\mu$ and $\pi(\delta) = p\delta^{-\alpha} + o(\delta^{-\alpha})$ for large δ ($p\alpha > 0$).*[7] *Then*

[6] Note that while here we specify a form of the *value* function, in the Cournot example the resulting value function is only approximately of this form, given the form of the *profit* function.

[7] For $\eta > 2\mu$ we have not been able to derive similar results. It is possible that V has no asymptotic expansion in this case. Also, the results can be generalized easily to effort cost functions $c(z) = z^v/v$ $(v > 1)$, with $\eta \leq 2\mu$ becoming $\eta \leq v\mu/(v-1)$.

almost always[8] *for* $\delta \to \infty$ *(the firm is far behind) the value function and innovation target behave asymptotically as follows:*

- $\alpha < \eta$: *The value function is of the same nature as the profit function,* $V(\delta) = O(\delta^{-\alpha})$, *and the asymptotic innovation targets are fixed steps,* $\Delta/\delta \to (1 - \alpha/\eta)^{1/\alpha}$.

- $\alpha = \eta$: *The value function is of the same nature as the profit function,* $V(\delta) = O(\delta^{-\alpha})$, *and the asymptotic innovation targets are increasing steps, e.g.* $\Delta/\delta^{\eta/\nu} \to const$, *while* $\Delta/\delta \to 0$.

- $\eta < \alpha \leq 2\mu$: *The value function is of the same nature as the profit function,* $V(\delta) = O(\delta^{-\alpha})$, *and the asymptotic innovation targets are to match a multiple of the opponent's progress,* $\Delta \to \bar{\Delta}$.

- $\alpha > 2\mu$: *The value function declines slower than the profit function,* $V(\delta) = O(\delta^{-2\mu})$, *and the asymptotic innovation targets are to match a multiple of the opponent's progress,* $\Delta \to \bar{\Delta}$.

Proof. (Outline) Substitute the asymptotic innovation targets derived above into equation (1.3) defining the value function. Use asymptotic expansions when variables go to infinity, and collect terms by magnitude. Then compare the exponents according to the cases above. ■

To summarize the above discussion: If profits are less elastic than the innovation hazard ($\alpha < \eta$), followers will try to catch up in a step-by-step manner, with a relatively high probability of success; but if profits are more elastic ($\alpha > \eta$), followers will try to reduce their disadvantage by matching a fixed multiple of the opponent's level of progress, but the probability of success is low.

[8] This is, apart from a null-set of profit functions where the term in the profit function with the highest exponent cancels out in (1.3), the equation defining the value function. In fact, the profit functions described at the end of section 1.4.2 are of this type.

1.4.5 Leapfrogging and Persistence of Leadership

We will now turn to the discussion of the two remaining points, leapfrogging and persistence of leadership. It is important to remember that in this model *leapfrogging* only confers a temporary advantage which may be immediately reversed by competition, while in other models, e.g. Rosenkranz (1996) and Motta *et al.* (1997), this new state of competition is stable. Therefore, in equilibrium the decision whether to leapfrog or not is subordinate to the deeper questions of how the race will go on in the future, and how to catch up in the first place before leapfrogging can take place. The result are as following: Under the matching strategy, if the target multiple is less than 100% then followers will *always* try to leapfrog, no matter how far they are behind. In all other cases, followers will first try to get close, either step-by-step or matching, and then leapfrog with a small step. Thus, even a firm that when far behind chooses the 'risky' matching strategy, may attempt to leapfrog 'safely', i.e. to first come close with a big innovation and then leapfrog making a small one.

Persistence of leadership depends on a comparison of the properties of the innovation hazard and the value function (and ultimately the ow profits) between leader and follower, while the decision between step-by-step or matching targets depends on the properties of the same functions for *large* values of the state of competition δ alone. In the former case, the replacement and efficiency effects determine which of the two will exert more effort, and ultimately have a higher probability of making the innovation, while in the latter case it is only the prospect of higher expected value that counts. It is therefore not surprising that numerical simulations show that persistence of, or frequent changes in, leadership may each go together with both step-by-step or matching targets. Some examples are the following:

- Cournot competition, $\pi(\delta) = (1+\delta)^{-2}$, and $\psi(\Delta,\delta) = (\Delta/\delta)^3$: Step-by-step innovation target ($\Delta/\delta \approx 1/\sqrt{3}$), persistence of leadership;

- Bertrand competition, $\pi(\delta) = \text{ma }\{1-\delta, 0\}$, and $\psi(\Delta,\delta) = (\Delta/\delta)^7$ for any

$\eta > 0$: Matching innovation target[9] with leapfrogging ($\Delta \approx \eta/(\eta+1)$), persistence of leadership;

- Patent race, with $\pi(\delta) = \pi^m > 0$ if $\delta \leq \bar{\delta} \leq 1$, and $\pi(\delta) = 0$ otherwise; $\psi(\Delta, \delta) = (\Delta/\delta)^\eta$ for any $\eta > 0$: Matching innovation target ($\Delta \approx \bar{\delta}$), frequent changes in leadership;

- Variant of patent race, with $\pi(\delta) = 1$ if $\delta \leq 1$, and $\pi(\delta) = 1/\delta$ otherwise; $\psi(\Delta, \delta) = (\Delta/\delta)^2$: Step-by-step innovation target ($\Delta/\delta \approx 1/2$), frequent changes in leadership.

Summing up, we can conclude that persistence of leadership is not related to optimal catching-up behaviour of followers, but rather has to do with the classical replacement and efficiency effects.

1.5 Examples

In this section we will exhibit two short examples of closed-form and numerical solutions. As before, we will concentrate on symmetric equilibria where $(\Delta_i, z_i) = (\Delta_j, z_j)$, and assume that $c(z) = z^2/2$.

Example 1 (Logarithmic ow profits): $\pi(\delta) = \alpha_1 \ln(\delta) + \alpha_2$ ($\alpha_1 < 0$), with candidate value function $V(\delta) = a \ln(\delta) + b$, with $a < 0$. This is one of the few cases that allows for a closed-form solution, and as stated above, is a limit case of the family of value functions that result in innovation targets of fixed step size.

Assume that $\psi(\Delta, \delta) = (\Delta/\delta)^\eta$. Then the optimal innovation target is step-by-step, $\Delta(\delta) = e^{-1/\eta}\delta$. The expected gain from making an innovation, and thus also the optimal effort level, is constant with $v_+ = z_i = -a/\eta e$, and given identical strategies for the other firm, the expected loss from an innovation by the other firm

[9] In this example, as in the next one, the elasticity of ow profits for followers can be interpreted as $\alpha = \infty$.

is $v_- = a/\eta e$. The value function fulfills the HJB equation

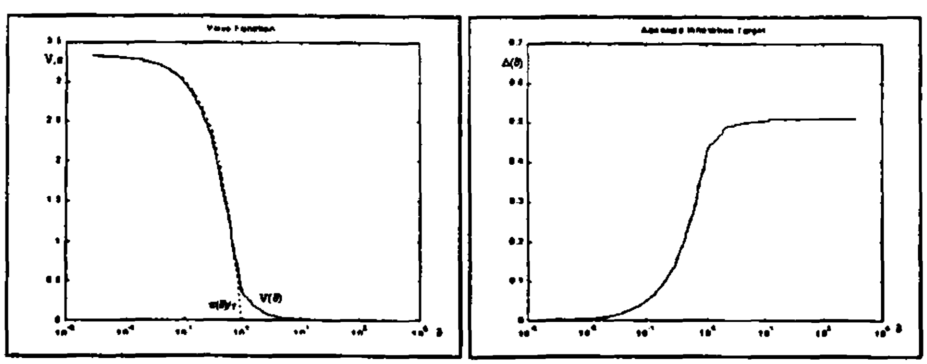
$$\begin{aligned} r\left(a\ln(\delta)+b\right) &= \pi(\delta) - z_i^2/2 + z_i v_+ + z_j v_- \\ &= \pi(\delta) + \left(-a/\eta e\right)^2/2 + \left(-a/\eta e\right) a/\eta e, \end{aligned}$$

which, comparing parameters, can be solved for

$$V(\delta) = \tfrac{\alpha_1}{r}\ln(\delta) + \left(\tfrac{\alpha_2}{r} - \tfrac{\alpha_1^2}{2r^3\eta^2 e^2}\right).$$

In this case, since research efforts are equal and constant, and also the hazard rate of making innovations is constant, the state of competition follows a random walk, with leadership changing often between intermittent periods of persistent leadership. The expected value of the state, conditional on the initial value, is identical to this initial value (the same is true of any future state, though). On the other hand, the variance of the state conditional on the initial state becomes arbitrarily large. In this sense initial differences in technical progress do not matter in the long run.

Example 2: (Bertrand competition with unit-elasticity demand): The profit function is $\pi(\delta) = \text{ma}\ \ \{1-\delta, 0\}$. Let $\psi(\Delta,\delta) = \Delta/\delta$. Since ow profits are constant (and equal to zero) for $\delta \geq 1$, but the value of followers is positive and decreasing with δ because of the prospect of catching up, the functional forms of the profit and value functions are rather different in this case, as can also be seen from the following numerical results ($r = 0.3$):



The value function remains very close to $\pi(\delta)/r$ for small δ, but differs significantly for δ around 1. Optimal innovation targets of followers are to always leapfrog the leader, and match about 50% of the leader's level, i.e. achieve the double of the leader's progress (since $\Delta(\delta) \approx \eta/(\eta+1)$ with $\eta = 1$), while leaders attempt to roughly double their knowledge with each innovation (or $\Delta(\delta)/\delta \approx \eta/(\eta+1)$). Research efforts (not shown here) take on their maximum exactly at $\delta = 1$ where firms are neck-to-neck, whereafter they fall fast for rising δ. Nevertheless, the leader's research efforts are always higher than the follower's, and therefore persistence of leadership emerges.

1.6 Conclusions

In this research we have introduced an R&D race with endogenous innovation targets and research effort levels. We have indicated the economic effects that determine the optimal strategies of leaders and followers, respectively. Leaders will follow a (safe) step-by-step innovation strategy, while followers in equilibrium either optimally adopt step-by-step innovations or choose the risky strategy of matching a certain level of the leader's progress. Followers choose the latter option if the elasticity of ow profits (with respect to the state of competition) is higher than the elasticity of the innovation hazard (with respect to the innovation target). That is, if innovation is difficult then step-by-step targets are optimal, and if innovation is relatively easy then aiming for a big jump is best.

Under the matching strategy, the follower may always try to leapfrog; but it is also possible that the follower first catches up and leapfrogs when close. Persistence of leadership is independent of the choice of the follower's strategy, and is solely determined by the replacement and efficiency effects.

Further research should tackle the following points: First, the above solutions are valid only under the assumption that no firm exits even if firm value is negative. If firms can exit the industry, which should be determined endogenously as part of the equilibrium solution, then this would give the industry leader an additional incentive to move even further ahead in order to push the competitor out of the market. On the other hand, if firms can reenter (the same firm or a new one), then the conditions under which it enters will be relevant for the incentives of the leader. In particular, a market leader may not push out a little efficient competitor to prevent entry of more efficient newcomers.

Second, we have only treated symmetric equilibria, where two firms that are identical apart from the initial state of competition apply the same strategies in they find themselves in comparable circumstances. We are able to show that some of the value functions treated in the text may only be the result of asymmetric equilibria if firms have differing profit functions or innovation technologies. This result is in contrast with the existence of asymmetric equilibria in the Aghion *et al.* (1997) model (see chapter 2) and Budd *et al.* (1993). In particular, Budd *et al.* show that for large discounting no asymmetric equilibria arise, while for small discounting they are possible. Therefore, an analysis of our model for small discounting may be of interest.

1.A Appendix

1.A.1 The Derivation of the Value Function

As noted in Aghion *et al.* (1997), the value function can be derived heuristically from e.g. (notation adapted)

$$\begin{aligned} V_i(\delta_i) &= [\pi - c(z_i)]dt \\ &+e^{-rdt}\{z_i\psi(\Delta_i,\delta_i)\,V_i(\Delta_i)\,dt + z_j\psi(\Delta_j,\delta_j)\,V_i(1/\Delta_j)\,dt \\ &+[1-(z_i\psi(\Delta_i,\delta_i)+z_j\psi(\Delta_j,\delta_j))dt]V_i(\delta_i)\}\,, \end{aligned}$$

by approximating e^{-rdt} by $1-rdt$ and dropping all terms containing dt^2 or higher powers of dt.

Alternatively, it can be derived analytically in the following way: Let the random variables t_1 and t_2 denote the points in time where the state switches ('exits') from

δ_i to $1/\Delta_j$ or Δ_i, respectively. Both t_1 and t_2 are independently exponentially distributed with hazard rates $p_1 = z_j\psi(\Delta_j, \delta_j)$ and $p_2 = z_i\psi(\Delta_i, \delta_i)$, respectively. The value $V_i(\delta_i)$ of being in state δ_i is the sum of three parts: Expected instantaneous profits until exit, plus expected value of exiting to any of the other two states. Let $\tau = \min\{t_1, t_2\}$ be the time of exit. The distribution function of τ is

$$\begin{aligned} G(t) &= \Pr(\tau \le t) = \Pr(\min\{t_1, t_2\} \le t) \\ &= \Pr(t_1 \le t \text{ or } t_2 \le t) \\ &= 1 - \Pr(t_1 > t \text{ and } t_2 > t) \\ &= 1 - e^{-p_1 t} e^{-p_2 t} \\ &= 1 - e^{-(p_1+p_2)t}, \end{aligned}$$

with density $g(t) = (p_1+p_2)e^{-(p_1+p_2)t}$, i.e. τ is exponentially distributed with hazard rate (p_1+p_2). The present discounted value of instantaneous payoffs until exit time τ is

$$\int_0^\tau e^{-rt}(\pi(\delta_i) - c(z_i))\,dt = \frac{1-e^{-r\tau}}{r}(\pi(\delta_i) - c(z_i)),$$

and its expectation over τ is

$$\begin{aligned} &\mathsf{E}_\tau\left[\frac{1-e^{-r\tau}}{r}(\pi(\delta_i) - c(z_i))\right] \\ &= (\pi(\delta_i) - c(z_i)) \int_0^\infty \frac{1-e^{-r\tau}}{r}(p_1+p_2)\,e^{-(p_1+p_2)\tau} d\tau \\ &= \frac{\pi(\delta_i) - c(z_i)}{r+p_1+p_2}. \end{aligned}$$

If exit at τ is to state $1/\Delta_j$, then $t_1 = \tau < t_2$. Therefore the expected value of exiting to state $1/\Delta_j$ is

$$\begin{aligned} &\int_0^\infty \left(e^{-r\tau} V_i(1/\Delta_j)\right) \Pr(t_2 > \tau)\, p_1 e^{-p_1\tau} d\tau \\ &= p_1 V_i(1/\Delta_j) \int_0^\infty e^{-(r+p_1+p_2)\tau} d\tau \\ &= \frac{p_1 V_i(1/\Delta_j)}{r+p_1+p_2}. \end{aligned}$$

Similarly, the expected value of exiting to state Δ_i can be found as

$$\frac{p_2 V_i(\Delta_i)}{r + p_1 + p_2}.$$

The sum of all three terms is

$$V_i(\delta_i) = \frac{\pi(\delta_i) - c(z_i) + p_1 V_i(1/\Delta_j) + p_2 V_i(\Delta_i)}{r + p_1 + p_2},$$

and rearranging leads to

$$\begin{aligned} rV_i(\delta_i) &= \pi(\delta_i) - c(z_i) + p_1 (V_i(1/\Delta_j) - V_i(\delta_i)) \\ &\quad + p_2 (V_i(\Delta_i) - V_i(\delta_i)). \end{aligned}$$

1.A.2 Asymptotic Expansions

In this appendix we will perform an analysis of the competitive effects for large discounting, using asymptotic expansions as $r \to \infty$ as in Budd *et al.* (1993, BHV). Our analysis will be heuristic, for an exact exposition of this method we refer the reader to the appendix of BHV. We assume that the following expansions exist:

$$\begin{aligned} v(\delta) &= rV(\delta) = \Sigma_{n=0}^{\infty} r^{-n} v_n(\delta), \\ \Delta(\delta) &= \Sigma_{n=0}^{\infty} r^{-n} \Delta_n(\delta) \\ z(\delta) &= \Sigma_{n=0}^{\infty} r^{-n} z_n(\delta) \end{aligned}$$

where for all $0 \leq n \leq 2$ the functions $v_n(.)$, $\Delta_n(.)$, $z_n(.)$ are sufficiently often continuously differentiable. Also assume that the profit function $\pi(.)$ and the innovation technology $\psi(., \delta)$ are twice continuously differentiable. In the following, we will suppress the argument δ whenever this does not lead to ambiguities.

Step 1: Expand $v(\Delta) = \Sigma_{n=0}^{\infty} r^{-n} w_n(\delta)$ and $v'(\Delta) = \Sigma_{n=0}^{\infty} r^{-n} u_n(\delta)$:
Consider $0 \leq m \leq 2$. Using the expansion of Δ, we find that

$$\begin{aligned} v_m(\Delta) &= v_m(\Delta_0) + r^{-1} v_m'(\Delta_0)\Delta_1 \\ &\quad + r^{-2}\left(\tfrac{1}{2} v_m''(\Delta_0)\Delta_1^2 + v_m'(\Delta_0)\Delta_2\right) + O\left(r^{-3}\right) \end{aligned}$$

if the derivatives exist. Collecting terms with the same powers of r,

$$\begin{aligned} v(\Delta) &= v_0(\Delta) + r^{-1}v_1(\Delta) + r^{-2}v_2(\Delta) + O\left(r^{-3}\right) \\ &= w_0(\delta) + r^{-1}w_1(\delta) + r^{-2}w_2(\delta) + O\left(r^{-3}\right), \end{aligned}$$

with

$$\begin{aligned} w_0(\delta) &= v_0(\Delta_0), \\ w_1(\delta) &= v_0'(\Delta_0)\Delta_1 + v_1(\Delta_0), \\ w_2(\delta) &= \tfrac{1}{2}v_0''(\Delta_0)\Delta_1^2 + v_0'(\Delta_0)\Delta_2 + v_1'(\Delta_0)\Delta_1 + v_2(\Delta_0). \end{aligned}$$

Proceeding similarly, we obtain

$$\begin{aligned} v'(\Delta) &= u_0(\delta) + r^{-1}u_1(\delta) + r^{-2}u_2(\delta) + O\left(r^{-3}\right), \\ u_0(\delta) &= v_0'(\Delta_0), \\ u_1(\delta) &= v_0''(\Delta_0)\Delta_1 + v_1'(\Delta_0), \\ u_2(\delta) &= \tfrac{1}{2}v_0'''(\Delta_0)\Delta_1^2 + v_0''(\Delta_0)\Delta_2 + v_1''(\Delta_0)\Delta_1 + v_2'(\Delta_0). \end{aligned}$$

Step 2: Analyze the first order necessary constraint on effort

$$\tfrac{1}{r}\psi(\Delta,\delta)\left(v(\Delta) - v(\delta)\right) = c'\left(z(\delta)\right), \tag{1.37}$$

using that $c(z) = z^2/2$ and

$$\psi(\Delta,\delta) = \zeta_0(\delta) + r^{-1}\zeta_1(\delta) + r^{-2}\zeta_2(\delta) + O\left(r^{-3}\right),$$

with coefficients ζ_n to be determined later. In

$$\begin{aligned} z(\delta) &= \Sigma_{n=0}^{\infty} r^{-n} z_n(\delta) \\ &= \left(\Sigma_{n=0}^{\infty} r^{-n}\zeta_n\right)\left[\Sigma_{n=0}^{\infty} r^{-(n+1)}\left(w_n - v_n\right)\right] \end{aligned}$$

compare terms of order 0 and 1 in r to obtain

$$\begin{aligned} z_0(\delta) &= 0 \\ z_1(\delta) &= \zeta_0\left[v_0(\Delta_0) - v_0(\delta)\right]. \end{aligned}$$

Step 3: Apply (1.37) to the Hamilton-Jacobi-Bellman equation (HJB) defining the value function,

$$v(\delta) = \pi(\delta) + z^2/2 + \tfrac{1}{r} z_j \psi_j \left(v(1/\Delta_j) - v(\delta)\right),$$

with $z_j\psi_j$ as the opponent's hazard rate, and Δ_j its innovation target. Then use $z_0(\delta) = 0$ and various series expansion to restate the HJB as

$$\begin{aligned}&\Sigma_{n=0}^{\infty} r^{-n} v_n = \pi + \left(\Sigma_{n=1}^{\infty} r^{-n} z_n\right)^2/2 \\ &+ \left(\Sigma_{n=1}^{\infty} r^{-n} z_{nj}\right) \psi_j \Sigma_{n=0}^{\infty} r^{-(n+1)} \left(v_n(1/\Delta_j) - v_n(\delta)\right).\end{aligned}$$

Comparing terms of order 0 and 1 in leads to

$$v_0(\delta) = \pi(\delta),\; v_1(\delta) = 0.$$

Step 4: Simplify $v(\Delta)$ and $v'(\Delta)$, using the expressions just derived:

$$\begin{aligned} w_0(\delta) &= \pi(\Delta_0) \\ w_1(\delta) &= \pi'(\Delta_0)\Delta_1 \\ w_2(\delta) &= \tfrac{1}{2}\pi''(\Delta_0)\Delta_1^2 + \pi'(\Delta_0)\Delta_2 + v_2(\Delta_0), \end{aligned}$$

and

$$\begin{aligned} u_0(\delta) &= \pi'(\Delta_0) \\ u_1(\delta) &= \pi''(\Delta_0)\Delta_1 \\ u_2(\delta) &= \tfrac{1}{2}\pi'''(\Delta_0)\Delta_1^2 + \pi''(\Delta_0)\Delta_2 + v_2'(\Delta_0). \end{aligned}$$

Step 5: Now we will apply the results obtained so far to the first order constraint defining the optimal innovation targets,

$$\psi'\left[v(\Delta) - v(\delta)\right] + \psi v'(\Delta) = 0, \tag{1.38}$$

where we used the shorthand $\psi' = \partial\psi/\partial\Delta$ ($\psi'' = \partial^2\psi/\partial\Delta^2$). Expanding $\psi(\Delta,\delta)$ and $\psi'(\Delta,\delta)$ about $r^{-1} = 0$, we obtain (suppressing the argument δ in ψ and ψ' for conciseness)

$$\begin{aligned}\psi(\Delta) &= \zeta_0(\delta)+r^{-1}\zeta_1(\delta)+r^{-2}\zeta_2(\delta)+O\left(r^{-3}\right)\\ &= \psi(\Delta_0)+r^{-1}\psi'(\Delta_0)\Delta_1\\ &\quad +r^{-2}\left(\tfrac{1}{2}\psi''(\Delta_0)\Delta_1^2+\psi'(\Delta_0)\Delta_2\right)+O\left(r^{-3}\right),\end{aligned}$$

and

$$\begin{aligned}\psi'(\Delta) &= \xi_0(\delta)+r^{-1}\xi_1(\delta)+r^{-2}\xi_2(\delta)+O\left(r^{-3}\right)\\ &= \psi'(\Delta_0)+r^{-1}\psi''(\Delta_0)\Delta_1\\ &\quad +r^{-2}\left(\tfrac{1}{2}\psi'''(\Delta_0)\Delta_1^2+\psi''(\Delta_0)\Delta_2\right)+O\left(r^{-3}\right).\end{aligned}$$

Substituting the expansions into (1.38),

$$\left(\Sigma_{n=0}^{\infty}r^{-n}\xi_n\right)\left[\Sigma_{n=0}^{\infty}r^{-n}\left(w_n-v_n\right)\right]+\left(\Sigma_{n=0}^{\infty}r^{-n}\zeta_n\right)\left(\Sigma_{n=0}^{\infty}r^{-n}u_n\right)=0,$$

and collecting the terms of order 0 in r^{-1} leads to

$$\begin{gathered}\xi_0(w_0-v_0)+\zeta_0u_0=0\\ \Leftrightarrow \psi'(\Delta_0)\left[\pi(\Delta_0)-\pi(\delta)\right]+\psi(\Delta_0)\pi'(\Delta_0)=0,\end{gathered}$$

or, assuming that the second order constraint is satisfied,

$$\Delta_0(\delta)=\arg\max_{\Delta}\ \psi(\Delta,\delta)\left[\pi(\Delta)-\pi(\delta)\right].$$

Collecting terms of order 1 in r^{-1},

$$\begin{gathered}\xi_0(w_1-v_1)+\xi_1(w_0-v_0)+\zeta_0u_1+\zeta_1u_0=0,\\ \Leftrightarrow \psi(\Delta_0)\pi''(\Delta_0)\Delta_1+2\psi'(\Delta_0)\pi'(\Delta_0)\Delta_1\\ +\psi''(\Delta_0)\left[\pi(\Delta_0)-\pi(\delta)\right]\Delta_1=0,\\ \Leftrightarrow \Delta_1\tfrac{\partial^2}{\partial\Delta^2}\left(\psi(\Delta,\delta)\left[\pi(\Delta)-\pi(\delta)\right]\right)|_{\Delta=\Delta_0}=0,\end{gathered}$$

and assuming that generically the second order sufficient condition holds ($\partial^2(.)/\partial\Delta^2 < 0$), leads to

$$\Delta_1(\delta)=0.$$

Step 6: Simplify $v(\Delta)$, $v'(\Delta)$, $\psi(\Delta)$, and $\psi'(\Delta)$ using $\Delta_1 = 0$:

$$\begin{aligned}
w_0 &= \pi(\Delta_0), \; w_1 = 0, \; w_2 = \pi'(\Delta_0)\Delta_2 + v_2(\Delta_0), \\
u_0 &= \pi'(\Delta_0), \; u_1 = 0, \; u_2 = \pi''(\Delta_0)\Delta_2 + v_2'(\Delta_0), \\
\zeta_0 &= \psi(\Delta_0), \; \zeta_1 = 0, \; \zeta_2 = \psi'(\Delta_0)\Delta_2, \\
\xi_0 &= \psi'(\Delta_0), \; \xi_1 = 0, \; \xi_2 = \psi''(\Delta_0)\Delta_2.
\end{aligned}$$

Step 7: Find $z_2(\delta)$: Simplify the result for z_1, and compare terms of order 2 in r^{-1} in

$$\begin{aligned}
z(\delta) &= \Sigma_{n=1}^{\infty} r^{-n} z_n(\delta) \\
&= \left(\Sigma_{n=0}^{\infty} r^{-n}\zeta_n\right)\left[\Sigma_{n=0}^{\infty} r^{-(n+1)}(w_n - v_n)\right],
\end{aligned}$$

to obtain

$$\begin{aligned}
z_0(\delta) &= 0, \; z_1(\delta) = \psi(\Delta_0)\left[\pi(\Delta_0) - \pi(\delta)\right] \\
z_2(\delta) &= \zeta_0(w_1(\delta) - v_1(\delta)) + \zeta_1(w_0(\delta) - v_0(\delta)) \\
&= 0.
\end{aligned}$$

Step 8: Using the last result, identify the terms of order 2 in r^{-1} in the HJB equation

$$\Sigma_{n=0}^{\infty} r^{-n} v_n = \pi + \left(\Sigma_{n=1}^{\infty} r^{-n} z_n\right)^2/2 + \left(\Sigma_{n=1}^{\infty} r^{-n} z_{nj}\right) \times$$
$$\left(\Sigma_{n=0}^{\infty} r^{-n}\psi_{nj}\right)\left(\Sigma_{n=0}^{\infty} r^{-(n+1)}\left[v_n(1/\Delta_j) - v_n(\delta)\right]\right),$$

to obtain

$$\begin{aligned}
v_2 &= z_1^2/2 + z_{1j}\psi_{0j}\left[v_0(1/\Delta_j) - v_0(\delta)\right] \\
&= \left(\psi(\Delta_0)\left[\pi(\Delta_0) - \pi(\delta)\right]\right)^2/2 + z_{1j}\psi_{0j}\left[\pi(1/\Delta_{0j}) - \pi(\delta)\right],
\end{aligned}$$

where $z_{1j}, \psi_{0j}, \Delta_{0j}$ are the first terms of the respective expansions of the other firm.

Step 9: Find Δ_2, the next term in the expansion of the optimal effort level by collection all terms of order 2 in r^{-1} in (1.38):

$$\xi_0(w_2 - v_2) + \xi_1(w_1 - v_1) + \xi_2(w_0 - v_0) + \zeta_0 u_2 + \zeta_1 u_1 + \zeta_2 u_0 = 0,$$
$$\Leftrightarrow \psi'(\Delta_0)\left[\pi'(\Delta_0)\Delta_2 + v_2(\Delta_0) - v_2(\delta)\right] + \psi''(\Delta_0)\Delta_2\left[\pi(\Delta_0) - \pi(\delta)\right]$$
$$+\psi(\Delta_0)\left[\pi''(\Delta_0)\Delta_2 + v_2'(\Delta_0)\right] + \psi'(\Delta_0)\Delta_2\pi'(\Delta_0) = 0,$$

which can be simplified to

$$\Delta_2(\delta) = -\frac{\partial(\psi(\Delta)[v_2(\Delta)-v_2(\delta)])/\partial\Delta}{\partial^2(\psi(\Delta)[\pi(\Delta)-\pi(\delta)])/\partial\Delta^2}\Big|_{\Delta=\Delta_0}.$$

Step 10: Apply previous results to (1.37) and collect terms of order 3 in r^{-1} to find the next term in the expansion of $z(\delta)$:

$$\begin{aligned}
w_0 &= \pi(\Delta_0),\ w_1 = 0,\ w_2 = \pi'(\Delta_0)\Delta_2 + v_2(\Delta_0),\\
u_0 &= \pi'(\Delta_0),\ u_1 = 0,\ u_2 = \pi''(\Delta_0)\Delta_2 + v_2'(\Delta_0),\\
\zeta_0 &= \psi(\Delta_0),\ \zeta_1 = 0,\ \zeta_2 = \psi'(\Delta_0)\Delta_2,\\
\xi_0 &= \psi'(\Delta_0),\ \xi_1 = 0,\ \xi_2 = \psi''(\Delta_0)\Delta_2.
\end{aligned}$$

$$\begin{aligned}
z_3(\delta) &= \zeta_0[w_2 - v_2] + \zeta_1[w_1 - v_1] + \zeta_2[w_0 - v_0]\\
&= \psi(\Delta_0)[\pi'(\Delta_0)\Delta_2 + v_2(\Delta_0) - v_2(\delta)] + \psi'(\Delta_0)\Delta_2[\pi(\Delta_0) - \pi(\delta)]\\
&= \Delta_2\frac{\partial}{\partial\Delta}(\psi(\Delta)[\pi(\Delta) - \pi(\delta)])\,|_{\Delta=\Delta_0} + \psi(\Delta_0)[v_2(\Delta_0) - v_2(\delta)]\\
&= \psi(\Delta_0,\delta)[v_2(\Delta_0) - v_2(\delta)],
\end{aligned}$$

since the derivative is zero.

As in BHV these expansions could be continued to find higher-order effects, but these will be hard to interpret since they will be a mixture of the other effects and involve higher-order derivatives of the underlying ow profits and innovation technology.

Chapter 2

A Note on Step-by-step races

2.1 Introduction

In this note we will examine a relatively new model of technical progress that has been used as a building block in the 'Schumpeterian' or endogenous growth literature, e.g. in Aghion, Harris and Vickers (AHV, 1997). It models non-drastic innovation by strongly restricting the dynamics of competition: Innovations occur 'step-by-step', which means that a firm that has fallen behind must first catch up and equalize with the leading firm before moving ahead. In particular, 'leapfrogging' is ruled out by assumption. AHV analyze this model to give an example of a model of dynamic innovation competition where contrary to the results on drastic innovations "more intense product market competition and/or imitations may be growth-enhancing". They do this by comparing Bertrand and Cournot competition in the product market.

Our note elaborates on this example concerning two points: First, the authors only analyzed symmetric equilibria, showing that there is a unique one. This analysis does not reveal whether there are asymmetric equilibria as well, and whether the symmetric one is stable. In fact, these two points are connected, since in the presence of asymmetric equilibria the symmetric one is mostly unstable. Stability of equilibrium is a desirable property for predictions or empirical applications, since the market will move away fast from unstable equilibria if there is the slightest amount of noise. We show that a pair of asymmetric equilibria may arise under Bertrand competition in the output market, leaving the symmetric equilibrium unstable, and that economic growth is lower in these asymmetric equilibria.

Second, we will discuss the assumption that firms cannot leapfrog the leaders and delineate the circumstances where firms would leapfrog in equilibrium when they have the choice to do so. Leapfrogging in equilibrium would occur under Bertrand competition, but may not occur under Cournot competition. Our analysis provides hints that traditional Schumpeterian leapfrogging models of drastic innovation are more appropriate if market competition is high, while the step-by-step assumption is justified when competition is less intense.

2.2 The Model

We will first describe the setup of the AHV model. Two duopolists with constant marginal cost compete in a market for a homogeneous product either in quantities or prices (Cournot or Bertrand competition, respectively). Market demand has unit-elasticity, i.e. it is of the form $p = 1/Q$, where Q is total output. Therefore if firms' marginal costs are c_i and c_j, flow profits of firm i under Cournot competition are $\pi^i = 1/(1 + c_i/c_j)^2$, while with Bertrand competition they are $\pi^i = \max\{1 - c_i/c_j, 0\}$.

Both firms conduct research to make cost-reducing innovations. These innovations reduce unit costs by a fixed factor $\gamma > 1$, i.e. $c_i' = c_i/\gamma$, and arrive randomly and independently in continuous time with Poisson hazard rates determined by the research efforts of the two firms. A fundamental assumption of this model is that any firm can be maximally one step ahead: After a firm moves ahead it has to wait until the other one has caught up. This assumption can be justified by assuming that any further innovation would immediately disclose the last innovation to the other firm, so that in practice a new innovation does not change the cost gap. The possible states of competition are therefore $S = \{-1, 0, 1\}$, meaning that either firm i is behind, or firms are neck-to-neck, or firm i is in front.

Let firm i's research efforts be $(x, y) \in \mathbb{R}_+^2$, and firm j's $(\bar{x}, \bar{y}) \in \mathbb{R}_+^2$. When firms are neck-to-neck their research efforts are x and $\bar{x}$, and when they fall behind the efforts are y and $\bar{y}$. Research is costly, with cost $c(z) = z^2/2$. Catching up is easier than moving ahead: The hazard rate of a firm that has fallen behind is $y + h$, where $h \geq 0$ measures the ease of imitation. Market profit ows are given by π_1, π_0, and π_{-1}, for a firm that is leading, neck-to-neck, or behind, respectively; $r > 0$ is the common discount factor. As in AHV, we will only be concerned with pure strategy equilibria in perfect Markov strategies, i.e. pure strategies that form a subgame-perfect Nash equilibrium of the dynamic game, and only depend on the state in $\{-1, 0, 1\}$ of the game.

Expected equilibrium payoffs are then characterized by the value functions

$$\begin{aligned} rV_1 &= \pi_1 - (\bar{y} + h)(V_1 - V_0), \\ rV_0 &= \pi_0 - c(x) + x(V_1 - V_0) - \bar{x}(V_0 - V_{-1}), \\ rV_{-1} &= \pi_{-1} - c(y) + (y + h)(V_0 - V_{-1}), \end{aligned} \tag{2.1}$$

where V_1, V_0, and V_{-1} are the values of being ahead, neck-to-neck, and behind, respectively. For example, rV_0, the discounted value of being neck-to-neck, is determined as ow profits π_0 minus costs of research $c(x)$, plus the expected gain from innovating $x(V_1 - V_0)$, minus the expected loss caused by an innovation of the other firm, $\bar{x}(V_0 - V_{-1})$. Given the strategy pair $(\bar{x}, \bar{y})$ of the other firm, the optimal strategies x and y of a firm that is neck-to-neck or behind, respectively, have to satisfy the necessary first order conditions

$$\begin{aligned} c'(x) &= (V_1 - V_0) \\ c'(y) &= (V_0 - V_{-1}). \end{aligned} \tag{2.2}$$

Taking pair-wise differences between the value functions (2.1) and inserting the first order conditions (2.2) we obtain the following system of equations characterizing the best responses $(x, y) \in b(\bar{x}, \bar{y})$ (only the non-negative solutions are relevant), where $b : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is the best response correspondence (also, $p_1 = \pi_1 - \pi_0$, $p_0 = \pi_0 - \pi_{-1}$, and $s = h + r$):

$$\begin{aligned} \tfrac{1}{2}x^2 + (\bar{y} + s)\,x - \bar{x}y &= p_1, \\ \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + (\bar{x} + s)\,y &= p_0. \end{aligned} \tag{2.3}$$

Corresponding equations characterize $\bar{x}$ and $\bar{y}$. Note that we explicitly allow asymmetric choices for all effort levels. For symmetric strategy pairs, $x = \bar{x}$ and $y = \bar{y}$, (2.3) are solved by AHV to yield the equilibrium strategies

$$\begin{aligned} x &= \sqrt{s^2 + 2p_1} - s, \\ y &= \sqrt{s^2 + x^2 + 2\,(p_1 + p_0)} - \sqrt{s^2 + 2p_1}. \end{aligned}$$

They also show that the growth rate of the economy if there are many identical sectors is given by

$$g = 2x\frac{y + h}{y + h + 2x}\ln\gamma.$$

It can be shown that in asymmetric equilibria the average growth rate is given by

$$g = \frac{(x_i + x_j)\,(y_i + h)\,(y_j + h)}{(y_i + h + x_j)\,(y_j + h + x_i) - x_i x_j}\ln\gamma.$$

2.3 Asymmetric Equilibria

In theory, (2.3) could be solved explicitly for (x, y) and $(\bar{x}, \bar{y})$, since it can be shown that these four equations can be 'reduced' to four independent polynomial equations of order four, which still have analytical solutions. Instead, we will use the inverse response map $(\bar{x}, \bar{y}) = b^{-1}(x, y)$, similar to Harris and Vickers (1987).

2.3.1 The Inverse Best Response

We will move in two steps. First, as we show in appendix 2.A.1, note that the strategies that are used in some equilibrium are exactly the solutions to the fixed point equation $(x,y) \in b(b(x,y))$, while strategies in symmetric equilibria make up the subset for which also $(x,y) \in b(x,y)$ holds. Second, if b^{-1} is the inverse of the best response, $(x,y) \in b(b(x,y))$ if and only if $(x,y) \in b^{-1}(b^{1}(x,y))$, as shown in appendix 2.A.2. Therefore we can work with the inverse reaction map which in this case is more straightforward, and have the following: $(x,y) \in \mathbb{R}^2_{++}$ is a strategy played in some pure strategy Markov perfect equilibrium (symmetric or asymmetric) if and only if $(x,y) \in (b^{-1})^2(x,y)$. It is part of some *symmetric* equilibrium if and only if $(x,y) \in b^{-1}(x,y) \subset (b^{-1})^2(x,y)$.

Best responses $(x,y) = b(\bar{x},\bar{y})$ in pure Markov strategies for the R&D-race are given by the equations (2.3). Under the natural assumptions that $\pi_1 > \pi_0$ (profits of a leader are strictly higher than those of a firm that is neck-to-neck), and $\pi_0 \geq \pi_{-1}$, for any best response $(x,y) \in \mathbb{R}^2_+$ to $(\bar{x},\bar{y}) \in \mathbb{R}^2_+$ we must have $x > 0$ and $y > 0$, i.e. $(x,y) \in \mathbb{R}^2_{++}$. The inverse response map $b^{-1} : \mathbb{R}^2_{++} \to \mathbb{R}^2$ is then given by solving (2.3) for $\bar{x}$ and $\bar{y}$,

$$\begin{aligned} \bar{x} &= \left(p_0 + x^2/2 - y^2/2\right)/y - s, \\ \bar{y} &= \left(p_1 + p_0 - y^2/2 - ys\right)/x - s. \end{aligned} \tag{2.4}$$

Note that for large x or y the images $\bar{x}$ or $\bar{y}$ may be negative. This simply means that this (x,y) is not a best response to any feasible (i.e. non-negative) strategy of the other firm. Therefore, (x,y) is a best response to some feasible strategy by the other firm if and only if $b^{-1}(x,y) \geq 0$.

2.3.2 The Graphical Solution

The fixed point condition on pure Markov perfect equilibrium strategies, $(x,y) \in (b^{-1})^2(x,y)$, is still difficult to visualize, since $(b^{-1})^2$ is a map with a four-dimensional graph. The same is true for the fixed point condition on symmetric equilibria, $(x,y) \in b^{-1}(x,y)$. They can be 'solved' numerically, but this is little intuitive, and also there is no guarantee that all solutions are found. We therefore propose a graphical solution where the fixed points can be visualized in two-dimensional space. Let F be either one of the maps $(b^{-1})^2$ or b^{-1}, and let $F = (F_x, F_y)$, where $F_x, F_y : \mathbb{R}_+^2 \to \mathbb{R}_+$. That is, if $F(x,y) = (\bar{x}, \bar{y})$, then $F_x(x,y) = \bar{x}$, and $F_y(x,y) = \bar{y}$. The fixed point condition $(x,y) = F(x,y)$ can then be expressed equivalently by the two conditions $x = F_x(x,y)$ and $y = F_y(x,y)$. These describe two curves in $\mathbb{R}_+^2$, and equilibrium strategies can be found at their intersections. Denote by A_x^1 and A_y^1 the curves pertaining to $F = b^{-1}$ (for symmetric equilibria), and by A_x^2 and A_y^2 the curves pertaining to $F = (b^{-1})^2$ (all equilibria).

Let us consider the cases of Cournot and Bertrand competition. Under Cournot competition and unit-elasticity demand the profit function is $\pi_i = 1/(1+\gamma^{-i})^2$; with $s = 0.02$ and $\gamma = 2$ we obtain the following figure:

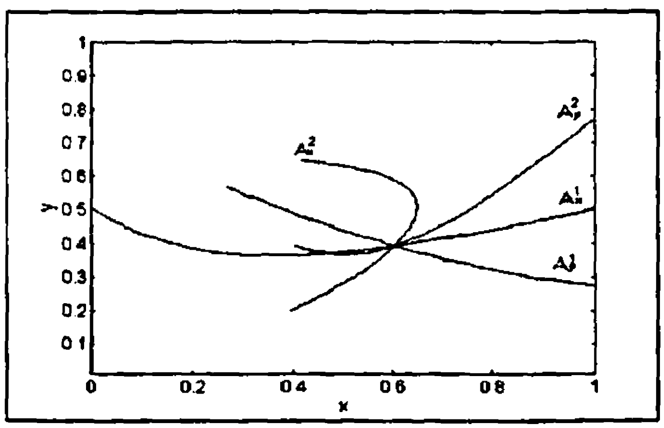


Figure 1: Equilibria under Cournot competition ($s = 0.02$, $\gamma = 2$).

We can see that A_x^2 and A_y^2 meet at the same point as A_x^1 and A_y^1, since A_x^1 and A_y^1 describe the symmetric equilibria. Since A_x^2 and A_y^2, and also A_x^1 and A_y^1, do not meet anywhere else, there is exactly one equilibrium, and it is symmetric (at $x \approx 0.604$ and $y \approx 0.392$, with average growth rate $g \approx 0.209$). This is the generic result under Cournot competition as we will argue below.

However, under Bertrand competition ($\pi_i = \mathrm{ma}\ \{0, 1-\gamma^{-i}\}$), also with $s = 0.02$ and $\gamma = 2$, we obtain Figure 2:

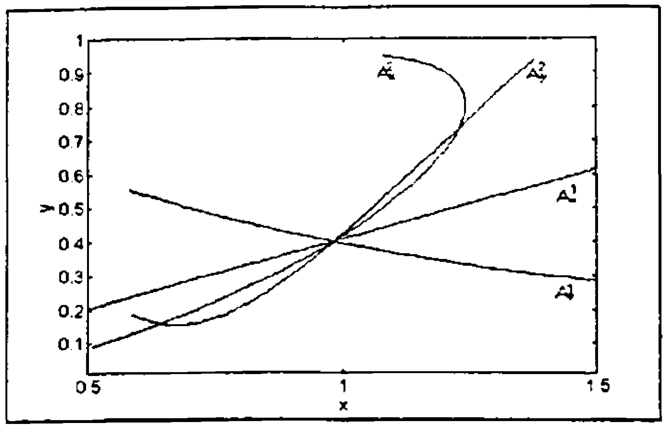


Figure 2: Equilibria under Bertrand competition ($s = 0.02$, $\gamma = 2$).

Here there is a pair of asymmetric equilibria (involving the strategy pairs $(x_1, y_1) \approx (1.229, 0.734)$ and $(x_2, y_2) \approx (0.643, 0.156)$, and growth rate $g \approx 0.140$) and one symmetric equilibrium, with $(x, y) \approx (0.980, 0.400)$ and growth rate $g \approx 0.235$. We can see that the existence of asymmetric equilibria depends on the relative slopes of the loci A_x^2 and A_y^2 around the symmetric equilibrium: The example in Figure 3 ($s = 0.15$, $\gamma = 2$) shows that asymmetric equilibria exist if and only if the slope of A_y^2 is steeper than the slope of A_x^2 (in coordinates (x, y)):

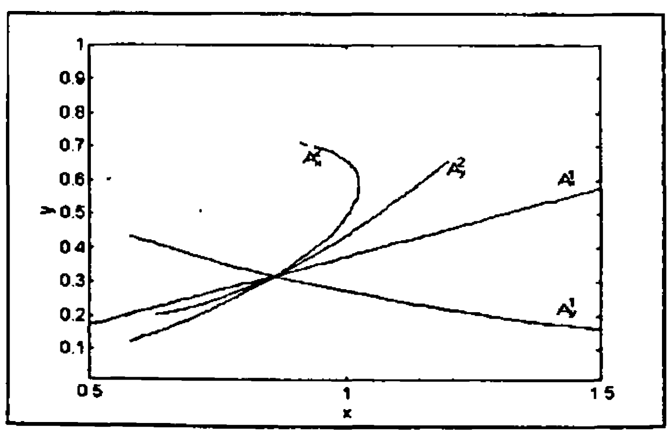


Figure 3: Equilibria under Bertrand competition ($s = 0.15$, $\gamma = 2$).

Simulations show that these relative slopes vary monotonically with s and γ: Decreasing "discount rate + ease of imitation" s and increasing innovation size γ makes A_y^2 steeper relative to A_x^2 around the symmetric equilibrium, i.e. make asymmetric equilibria more likely. Therefore, under Cournot competition there are no asymmetric equilibria since it can be shown that in the extreme case $s \to 0$ and $\gamma \to \infty$ no asymmetric equilibria exist. For Bertrand competition, given s (γ) there is γ_0 (s_0) such that there are no asymmetric equilibria for $\gamma < \gamma_0$ ($s > s_0$).

2.3.3 Welfare Properties of Asymmetric Equilibria

The existence of asymmetric equilibria under Bertrand competition has a straightforward interpretation: More relative advantage for the leader, either because the product market is more competitive (Bertrand instead of Cournot competition), or because cost reduction through innovations are bigger (γ higher), or because imitation is more difficult (lower h), or because players are patient and care about long-term advantage (lower r), may result in endogenous asymmetry. *Ex ante* identical firms choose different strategies because 'the market is too small', and one of them emerges as a 'natural leader', whereas the other becomes a 'natural follower'. Beliefs about each others' strategies then reinforce the asymmetry even though there are times when both firms are neck-to-neck and have the *same* cost of production, because they follow different investment strategies.

As argued above, considering the symmetric equilibrium as the "legitimate solution" or even prediction of the game is questionable if it is unstable. Using the best-response maps, we can show in our examples that in the presence of a pair of asymmetric equilibria the symmetric equilibrium is unstable in the sense of Seade (1980), i.e. not all eigenvalues of the matrix $M = I_2 + B$ are non-positive, where I_2 is the 2×2 unity matrix, and B is the matrix of derivatives of the best-response function.[1] Here the most reasonable prediction would be that the market ends up in one of the asymmetric equilibria, even if this is subject to the equilibrium selection problem.

The comparison of payoffs and growth rates between symmetric and asymmet-

[1] This is essentially an index-theoretic result, as was applied to Cournot oligopoly by Kolstand and Mathiesen (1987).

ric equilibria is also of interest: It turns out that the (neck-to-neck) payoffs for the Bertrand example in Figure 2 are 28.39 and 1.525, whereas in the symmetric equilibrium they are 8.813 for each firm. Therefore, in the asymmetric equilibrium the 'follower' is much worse off, but *joint payoffs* are higher than in the symmetric equilibrium. We were not able to prove this analytically, but suspect that this may hold generally: Since for the disadvantaged firm it is rational to hold back its efforts, there will be less dissipation of monopoly rents than in a symmetric duopoly.

On the other hand, as noted above, average growth rates are higher in symmetric equilibria. This can be explained by the same factor: Even though the 'leader' credibly exerts very high research efforts, since the disadvantaged firm invests less in research, catching up or overtaking will occur less often, which lowers the long-run growth rate.

This divergence between joint payoffs and growth, together with the instability of the symmetric equilibrium, may make asymmetry a welfare issue: Too much advantage for the leader (even if the competitors are ex ante on equal footing) may slow down growth while keeping industry profits at a higher level. Therefore, higher competition in the product market will only certainly raise growth if the equilibrium will not give rise to asymmetric equilibria.

2.4 To leapfrog or not to leapfrog

AHV assume that cost-reducing innovations are of a fixed size γ, and that a firm that has fallen behind first has to catch up with the leader (make an innovation of size γ) instead of leapfrogging him (making an innovation of size γ^2). In this section we will discuss the equilibrium outcomes if leapfrogging is possible.

We will analyze whether in the present model 'no leapfrogging' is an optimal choice if leapfrogging to the leader's position is possible. In this case the assumption of 'no leapfrogging' imposes no restriction on the equilibrium strategies.

We will assume that 'no leapfrogging' is an equilibrium, with value functions as in (2.1) given by

$$\begin{aligned} r\bar{W} &= \pi_1 - (y_j + h)\left(\bar{W} - \bar{V}\right), \\ r\bar{V} &= \pi_0 - c(x_i) + x_i\left(\bar{W} - \bar{V}\right) - x_j\left(\bar{V} - \bar{U}\right), \\ r\bar{U} &= \pi_{-1} - c(y_i) + (y_j + h)\left(\bar{V} - \bar{U}\right), \end{aligned}$$

and first order conditions for optimal effort levels given as in (2.2) by

$$\begin{aligned} c'(x_i) &= \left(\bar{W} - \bar{V}\right), \\ c'(y_i) &= \left(\bar{V} - \bar{U}\right). \end{aligned}$$

A follower who is deliberating to leap-frog faces the following value of leapfrogging (assuming that afterwards the equilibrium without leapfrogging is played):

$$rU_l = \pi_{-1} - c(z) + (z + h_l)\left(\bar{W} - U_l\right),$$

where we assume that imitation is more difficult than for just catching up: $0 \leq h_l < h$. Research effort is $z \geq 0$, and at the optimum is characterized by the usual first order condition $c'(z) = \left(\bar{W} - U_l\right)$. Using this first order conditions and solving[2] for U_l leads to (assuming quadratic cost of research as above)

$$\begin{aligned} \bar{U} &= \tfrac{1}{r}\left(\pi_{-1} + \tfrac{1}{2}y^2 + hy\right), \\ U_l &= \bar{W} - \sqrt{(h_l + r)^2 + 2\left(r\bar{W} - \pi_{-1}\right)} + (h_l + r). \end{aligned}$$

The follower prefers catching up over leapfrogging if $\bar{U} \geq U_l$. After some manipulations using the first order conditions, and because in equilibrium $\bar{W} = x + y + \bar{U}$, this condition can be written as

$$x^2 + 2xy + 2xh_l + 2yh_l \leq 2yh.$$

This can only hold if, in equilibrium, effort levels are very small and h_l is small enough. A necessary condition for preferring catching up over leapfrogging is $x^2 +$

[2] The second root of the equation for U_l is excluded by $\bar{W} - U_l \geq 0$.

$2xy \leq 2yh$, which is independent of the value of h_l. It can be shown that this condition is more likely to be satisfied if innovation size γ is small or discount rate/ease of imitation s is large (and therefore x and y are small), and an example for Cournot competition where catching up is preferred to leapfrogging is the equilibrium under the parameters $\gamma = 1.1$, $h = 0.2$, and $r = 0.01$. On the other hand, under Bertrand competition this condition never holds for any $\gamma > 1$, and therefore firms prefer leapfrogging even if imitation is difficult, $h_l = 0$, and therefore the 'equilibrium' with catching up is never an equilibrium under Bertrand competition if we allow firms to leapfrog.

To sum up, if we relax the assumption that firms *cannot* leap-frog, then no leapfrogging arises as an *equilibrium outcome* only when the intensity of competition in the product market is small (Cournot competition), or innovation size is small, or the discount rate and ease of imitation are large. That is, if the advantages of being a leader are high then leapfrogging is more attractive.

2.5 Conclusion

For a simple model of step-by-step innovation competition we have shown that the unique symmetric equilibrium may be unstable if product market competition is high, innovations large, and discount rate and ease of innovation small. In this case asymmetric equilibria exist as well, possibly altering the empirical predictions and welfare properties of this model.

We have also shown that the assumption that firms cannot leapfrog each other does restrict equilibrium strategies in the sense that firms would prefer to directly leapfrog each other unless market competition is low, innovations are small, and discount rate and ease of innovation are large.

2.A Appendix

2.A.1 Symmetric and Asymmetric Equilibria

In a game with 2 players, strategy spaces S_i $(i = 1, 2)$ and given pure strategy best response maps $b_i : S_j \to S_i$ $(j \neq i)$, the pure strategy equilibria $s = (s_1, s_2) \in S_1 \times S_2$ are characterized by the fixed point equations $s_i \in b_i(s_j)$ $(j \neq i)$. This leads to the necessary condition on an equilibrium strategy s_i

$$s_i \in b_i(b_j(s_i)) \quad (j \neq i), \tag{2.5}$$

which does not depend on the strategy of firm j. For identical best response maps $b_1 = b_2 = b$ on $S_1 = S_2$, this condition becomes $s_i \in b(b(s_i)) = b^2(s_i)$ $(i = 1, 2)$. Any s_i with $s_i \in b^2(s_i)$ is *part* of some pure strategy Nash equilibrium $(s_i, \hat{s}_j)$ since there is $\hat{s}_j \in b(s_i)$ with $s_i \in b(\hat{s}_j)$, but not all combinations of (s_1, s_2) with $s_i \in b^2(s_i)$ $(i = 1, 2)$ is a Nash equilibrium.

Also, s_i with $s_i \in b(s_i)$ are part of *symmetric* pure strategy equilibria (s_1, s_2) with $s_1 = s_2$. It is obvious that

$$\{s_i \in S_i | s_i \in b(s_i)\} \subset \{s_i \in S_i | s_i \in b^2(s_i)\},$$

i.e. symmetric equilibria are trivial 'asymmetric' equilibria. Strategies appearing as part of (non-trivial) *asymmetric* pure strategy equilibria are therefore given by

$$\{s_i \in S_i | s_i \in b^2(s_i) \wedge s_i \notin b(s_i)\}.$$

These conditions are illustrated in the context of the following static game: There are two players with strategy sets $S_1 = S_2 = [0, 1]$, and payoffs are $s_i(s_j - 1)^2 - s_i^2/2$ $(i = 1, 2; i \neq j)$. Best responses are found to be single-valued with $b(s_j) = (s_j - 1)^2$, and the pure strategies in equilibrium are given as the solutions (over $[0, 1]$) of the following equations

$$\begin{aligned} \text{symmetric} &: \quad s_i = (s_i - 1)^2 \Rightarrow s_i = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}, \\ \text{(a)symmetric} &: \quad s_i = \left((s_i - 1)^2 - 1\right)^2 \Rightarrow s_i \in \left\{0, \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}, 1\right\}. \end{aligned}$$

The pure strategy Nash equilibria are therefore $(0, 1)$, $(1, 0)$, and $(\frac{3}{2} - \frac{1}{2}\sqrt{5}, \frac{3}{2} - \frac{1}{2}\sqrt{5})$.

2.A.2 The inverse best-response map

Above we argued that to find the set of all pure equilibrium strategies we can work with the inverse response map instead of the response map itself. To this end we prove the following lemma:

Lemma 1 *The sets of fixed points of* $b^2 : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ *and of* $\left(b^{-1}\right)^2 : \mathbb{R}^2_{++} \to \mathbb{R}^2$ *are identical:*

$$\left\{(x,y) \mid (x,y) \in b^2(x,y)\right\} = \left\{(x,y) \mid (x,y) \in \left(b^{-1}\right)^2 (x,y)\right\}.$$

Proof. Extend b^{-1} to the whole of $\mathbb{R}^2$ by defining $b^{-1}(x,y) = \emptyset$ if $(x,y) \in \mathbb{R}^2 \backslash \mathbb{R}^2_{++}$, and define $b^{-1}(\emptyset) = \emptyset$. Let (x,y) be such that $(x,y) \in b^2(x,y)$; then there is $(\bar{x},\bar{y}) \in \mathbb{R}^2_{++}$ such that $(\bar{x},\bar{y}) \in b(x,y)$, $(x,y) \in b(\bar{x},\bar{y})$, and therefore $(x,y) \in \mathbb{R}^2_{++}$. Then it is obvious that $(\bar{x},\bar{y}) \in b^{-1}(x,y)$ and $(x,y) \in b^{-1}(\bar{x},\bar{y})$, therefore $(x,y) \in \left(b^{-1}\right)^2 (x,y)$.

For the converse, let $(x,y) \in \left(b^{-1}\right)^2 (x,y)$. Then there is $(\bar{x},\bar{y}) \in b^{-1}(x,y) \cap \mathbb{R}^2_{++}$ such that $(x,y) \in b^{-1}(\bar{x},\bar{y})$, otherwise $\left(b^{-1}\right)^2 (x,y)$ would be empty. Therefore $(x,y) \in b(\bar{x},\bar{y})$ and $(\bar{x},\bar{y}) \in b(x,y)$, i.e. $(x,y) \in b^2(x,y)$. ■

Part II

Cournot Oligopoly

Chapter 3

Existence and Comparative Statics in Heterogeneous Cournot Oligopolies

3.1 Introduction

Since Cournot's early contribution his model of oligopoly has received more and more attention, and nowadays is a basic building block of applied work on a wide range of topics involving imperfect competition. Its usefulness depends on two features: First, existence and uniqueness of equilibria at the market stage must be easily established, and second, comparative statics results should be readily available. In the context of homogeneous goods both these aspects have been treated extensively, whereas for heterogeneous goods there are much fewer results available.

It is possible to ascertain the existence of pure Cournot equilibria under the assumption that profits are concave, using the general result that games with concave payoffs possess pure equilibria (see Friedman 1991). This condition is not easily translatable into assumptions about demand and production costs, therefore there have been many attempts to identify those features that guarantee the existence of equilibria. The first strand of the literature identified conditions on demand that could be fruitfully exploited: Novshek (1985), Kukushkin (1994) and Corchón (1994, 1996) assume that goods are strategic substitutes, while Vives (1990) assumes that goods are strategic complements. Therefore it is assumed that firms' reaction functions are either decreasing or increasing. The second strand initially imposed assumptions only on costs and proved the existence of symmetric equilibria: McManus (1962, 1964), and Roberts and Sonnenschein (1976) assumed that costs were linear or convex, i.e. had nonincreasing returns to scale. Amir and Lambson (1998) showed for homogeneous goods that it was possible to allow for limited increasing returns to scale in production, resulting in a condition that combines both the demand and cost functions. It is interesting to note that it is not by chance that these two strands of the literature exist, since fundamentally each strand uses one of the two stability conditions by Hahn (1962), which impose different kinds of regularity on the model.

Most of the above authors have only covered the case of homogeneous goods. Kukushkin (1994) and Corchón (1994, 1996) deal with additive aggregation, i.e. where the *sum* of competitors' outputs is relevant, but assume strategic substitutes; Vives (1990) allows for general non-homogeneous goods, but under strategic complements. Spence (1976) indicates how to prove existence of Nash equilibria for a special class of inverse demand functions with heterogeneous goods. Our work is the first to address the question of existence of equilibria with heterogenous goods in a general context that does make not use of the assumption of strategic substitutes or complements. Rather, it is based on the second strand of literature and directly generalizes Amir and Lambson's (1998) work to heterogeneous goods.

We impose the condition that firms react to a rise in competitors' quantities by adjusting their own production in such a way that their market price does not rise (condition A). Doing so, output may increase or decrease, but must not decrease too strongly. This condition is formulated without making use of differentiability or convexity assumptions, rather it is formulated in lattice-theoretic terms as a single-crossing condition. If goods are substitutes, and under standard regularity conditions, we show that this condition implies the existence of symmetric pure Cournot equilibria even when outputs are heterogeneous.

Concerning uniqueness, asymmetric equilibria can be ruled out if we add the additional weak assumption that own market price reacts more to changes in own output than in competitors' outputs (condition B). Multiple symmetric equilibria can be excluded only under much stronger assumptions.

Comparative statics on demand or cost variables for Cournot oligopoly have been analyzed by many authors, among them Frank (1965), Dixit (1986), Corchón (1994, 1996), while comparative statics with respect to the number of firms have been discussed by Frank (1965), Ruffin (1971), Seade (1980), Szidarovsky and Yakowitz (1982), Corchón (1994, 1996), and recently by Amir and Lambson (1998). The case of heterogeneous goods has been treated by Dixit (1986) for two firms, and by Corchón (1994, 1996) for additive aggregation, but they as most authors have imposed the assumption of strategic substitutes from the outset, which is irrelevant for most comparative-statics conclusions. In fact, Amir and Lambson have shown for homogeneous goods that the only relevant condition for decreasing equilibrium prices and increasing equilibrium total quantity is that there are no strong increasing returns to scale.

One of the fundamental conclusions of this literature is that stability of equilibrium is closely connected to "non-paradoxical" comparative statics results. Our analysis for heterogeneous goods, which is based on lattice-theoretic monotone comparative statics methods, makes precise predictions for maximal and minimal equilibria that do not rely on stability, while we show that comparative statics results for arbitrary equilibria continue to depend decisively on the stability of equilibrium.

In this work we will concentrate exclusively on the comparative statics of entry. Our main result is that if competitors' quantities enter inverse demand in some aggregated form, then equilibrium prices do not increase as more firms enter the industry. We also show by means of an example that this result is not extendable to general heterogenous goods, i.e. equilibrium prices may rise even if there are no increasing returns to scale and equilibrium is stable.

Total equilibrium output may rise or fall even if prices are decreasing, but will rise under the same condition that we already used to rule out asymmetric equilibria. Individual output rises or falls depending on whether goods are strategic complements or substitutes, while profits always decrease.

The rest of our paper continues as follows: Section 3.2 sets out the model, and section 3.3 introduces the main condition. Existence results are presented in section 1, and related conditions and some examples are discussed in sections 3.5 and 3.6. Section 3.7 presents our comparative statics results, and section 3.8 concludes. All proofs are in the appendix.

3.2 The Model

There are n firms with identical finite production capacities[1] $0 < K < \infty$ and identical production cost functions $c : [0, K] \to \mathbb{R}_+$, which are assumed to be lower semi-continuous.

Denote firm i's output quantity by x_i, and by x_{-i} the vector of outputs of the other firms. Inverse demand of firm i ($i = 1..n$) is given by a continuous function $p : [0, K]^n \to \mathbb{R}_+$, with $p_i = p(x_i, x_{-i})$,[2] which is symmetric in the other firms' outputs: Let $\tilde{x}_{-i}$ be any permutation of x_{-i}, then $p(x_i, \tilde{x}_{-i}) = p(x_i, x_{-i})$ for all $(x_i, x_{-i}) \in [0, K]^n$. That is, firms are completely symmetric in that, apart from identical production technologies, all demand functions have the same functional form and all competitors' goods enter each demand function symmetrically. Assume that p is nonincreasing in x_i and x_{-i}, i.e. in particular goods are substitutes, and

[1] Alternatively, as is often done, one may assume that inverse demand falls below marginal cost (given any output of rivals) or even becomes zero for outputs larger than a certain limit. All these assumptions ensure that firms' outputs are bounded.

[2] By $p(x_i, x_{-i})$ we do mean that x_i is the first argument of p, i.e. that for all i own quantity x_i enters firm i's inverse demand differently from other firms' quantities x_j, $j \neq i$.

strictly decreasing in x_i where inverse demand is positive.

Firm i's profits are, for $x_i \in [0,K]$ and given $x_{-i} \in [0,K]^{n-1}$,

$$\Pi(x_i, x_{-i}) = x_i p(x_i, x_{-i}) - c(x_i), \; i = 1 \ldots n. \tag{3.1}$$

We will now reformulate profits Π as a function of firm i's *market price* p_i and the other firms' outputs[3]. As shown in the appendix, there is a set $X \subset \mathbb{R}_+ \times [0,K]^{n-1}$ such that we can express firm i's output as a function $\chi : X \longrightarrow [0,K]$ of own market price p_i and the others' output that is continuous and nonincreasing in $(p_i, x_{-i}) \in X$, and strictly decreasing in p_i. Also, there is a new constraint set $\pi(x_{-i})$ that is non-empty, closed, compact, and nonincreasing in x_{-i}. Firm i's maximization problem can then be expressed as

$$\max_{p_i \in \pi(x_{-i})} \tilde{\Pi}(p_i, x_{-i}) = \chi(p_i, x_{-i}) p_i - c(\chi(p_i, x_{-i})), \tag{3.2}$$

resulting in the price best response[4] $P(x_{-i})$.

If we consider profits $\tilde{\Pi}(p_i, x_{-i})$ "on the diagonal" where all competitors produce the same amount $y \in [0,K]$, we can define

$$\hat{\Pi}(p_i, y) = \tilde{\Pi}(p_i, y, \ldots, y). \tag{3.3}$$

Some special cases are, in order of increasing specialization, what we will call "*Competitor aggregation*", "*industry aggregation*",[5] and *homogeneous goods.* Under competitor aggregation there are functions $\hat{p} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $f : [0,K]^{n-1} \rightarrow \mathbb{R}_+$, where f is strictly increasing, such that

$$p(x_i, x_{-i}) = \hat{p}(x_i, f(x_{-i})), \tag{3.4}$$

where the competitors' quantities are aggregated into one number. One example is *additive aggregation* with $f(x_{-i}) = Y_i = \Sigma_{j \neq i} x_j$.

[3]Note that the variable to be maximized over (output or price) is irrelevant as long as afterwards each firm 'commits' to a fixed production quantity or at least competitors believe that it is so.

[4]We adopt this formulation to avoid confusion with the standard (quantity) best response or reaction function $x_i = r(x_{-i})$.

[5]I would like to thank Karl Schlag for proposing these terms.

Under industry aggregation there exists a function $\bar{p} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$ such that

$$p(x_i, x_{-i}) = \bar{p}(x_i + \delta Y_i), \qquad (3.5)$$

and if goods are homogeneous then $\delta = 1$.

For industry aggregation it is easy to see that $\chi(p_i, x_{-i}) = D(p_i) - \delta Y_i$, where $D = \bar{p}^{-1}$ is the demand function, and the profit maximization problem becomes

$$\max_{p_i \in [\bar{p}(K+\delta Y_i), \bar{p}(\delta Y_i)]} \tilde{\Pi}(p_i, x_{-i}) = (D(p_i) - \delta Y_i) p_i - c(D(p_i) - \delta Y_i),$$

where for identical outputs by firm i's competitors,

$$\hat{\Pi}(p_i, y) = (D(p_i) - \delta(n-1)y) p_i - c(D(p_i) - \delta(n-1)y).$$

3.3 The Condition

Our main condition on profits is of a type that has recently been shown to be of central importance in any exercise of comparative statics: In a lattice-theoretic context, Milgrom and Shannon (1994) have shown that the set of maximizers of the parametric maximization problem $\max_{x \in S} f(x,t)$, where $S \subset \mathbb{R}$, is nondecreasing in (t, S) *if and only if f satisfies the weak single crossing property in* (x,t): for all $x' > x$ and $t' > t$ we have that

$$f(x',t) - f(x,t) \geq (>) 0 \Rightarrow f(x',t') - f(x,t') \geq (>) 0.$$

In the context of game theory this result can be applied to best response maps, and we do so after our change of variables from own quantity to own price described above. Underlying our results is the following condition:

Condition A: $\hat{\Pi}(p_i, y)$ satisfies the dual[6] *weak single crossing property in (p_i, y), i.e. for all $p_i' > p_i$ and $y' > y$ we have that*

$$\hat{\Pi}(p_i', y) - \hat{\Pi}(p_i, y) \leq (<) 0 \Rightarrow \hat{\Pi}(p_i', y') - \hat{\Pi}(p_i, y') \leq (<) 0. \qquad (3.6)$$

[6]This is a "dual" single-crossing property because the inequality signs in the definition are reversed.

Even though this condition seems to be extremely abstract, its interpretation is very simple and economically intuitive: Condition A means that, *starting from a situation where all other firms produce identical quantities, if the other firms raise their outputs by the same amount, it will be advantageous for firm i to adjust its output only in such a way that the resulting market price is not higher than before.* Doing so, own output may increase or decrease, depending on whether goods are strategic substitutes or complements. This condition is a very natural condition to consider when one is interested in how equilibrium prices changes with entry of new firms in a setting where all firms are equal; here we will show that it even implies existence of equilibrium, subject to some regularity conditions.

Note that condition A imposes the dual single crossing property only on the "diagonal", i.e. where competitors all produce the same quantity. This is sufficient for the following existence result since we are only interested in symmetric equilibria, while we will have to state a condition covering the whole space of outputs to deal with asymmetric equilibria.

In addition, condition A is formulated for identical increases in output for all competitors. This is equivalent to formulating the corresponding condition in terms of an increase in just one competitor's quantity as long as inverse demand is symmetric in competitors' outputs, while it is more general if inverse demand is not symmetric. Condition A therefore even applies to cases where firms are identical but inverse demands are not symmetric in all competitors' outputs. One example of this is a situation where each firm only has two neighbors, as in a "circular city" model. In this paper we will concentrate on the symmetric case.

It is important to note that condition A rules out the existence of *avoidable fixed cost*, i.e. fixed costs that are not incurred if nothing is produced: If they were present, firm i might prefer to stop producing at all (in effect raising own price), instead of lowering its own price, if the other firms raise their outputs. Any other upward jump in production cost is similarly excluded.

Condition A applies no matter whether inverse demand and production costs are differentiable or not. Since it is an ordinal condition, it is not surprising that there is no *equivalent* condition in terms of derivatives even if demand or cost are differentiable. Using the *method of dissection* discussed in Milgrom and Shannon (1994, p. 167), we can find a sufficient differential condition that is slightly stronger than condition A (see appendix 3.A.2). If Π^{ii} and Π^{ij} are the second partial derivatives of the profit function of firm i with respect to outputs, and p^i and p^j the partial derivatives of the inverse demand of firm i with respect to x_i and x_j, condition A is implied by

Condition AD: For all i, $x_i \in [0, K]$, and $x_{-i} = (y, ..., y) \in [0, K]^{n-1}$,

$$\Pi^{ii}(x_i, x_{-i}) - \frac{p^i(x_i, x_{-i})}{p^j(x_i, x_{-i})}\Pi^{ij}(x_i, x_{-i}) \leq 0. \tag{3.7}$$

In the cases of industry aggregates or homogeneous goods, condition AD reduces to the condition (as we discuss in section 3.6) $p' - c'' \leq 0$. Here condition AD has the following interpretation: There are *at most weakly increasing returns to scale*, or *profit margins $p - c'$ are falling in own output.*[7] The relation between conditions AD and A is as follows: If output by the other firms increases marginally, and if firm i *reduces* output such that its market price remains constant, then firm i's profits decrease by the profit margin $(p - c')$, which is a first-order effect; since by condition AD profit margins are decreasing in own output, this decrease in profits can only be counterbalanced by an *increase* in own output and a resulting lower market price, therefore firm i will not want to drive prices up. Best response output may go up or down since there are two opposite movements in output involved.

An interesting implication of condition AD is that it implies a bound on the

[7] An equivalent interpretation, due to Amir and Lambson (1998) for homogenous goods is that, "inverse demand or price decreases faster (...) at any given output level than does marginal cost at all lower output levels."

slopes of quantity best responses $r_i(x_{-i})$,

$$\frac{\partial}{\partial x_j} r_i(x_{-i}) = -\frac{\Pi^{ij}}{\Pi^{ii}} \geq -\frac{p^j}{p^i}.$$

Apart from condition AD there are various other conditions that imply condition A and that may sometimes be easier to verify. Some we will discuss in the next section, but the one most easily verified is the following:[8]

Condition D: $\tilde{\Pi}(p_i, x_{-i})$ *has (weakly) decreasing differences in* (p_i, x_{-i}), *i.e.*

$$\tilde{\Pi}(p_i', x_{-i}') - \tilde{\Pi}(p_i, x_{-i}') \leq \tilde{\Pi}(p_i', x_{-i}) - \tilde{\Pi}(p_i, x_{-i}) \tag{3.8}$$

for all $p_i' \geq p_i \geq 0$ *and* $x_{-i}' \geq x_{-i} \in [0, K]^{n-1}$.

This condition is strictly stronger than condition A (Milgrom and Shannon 1994). If inverse demand and production costs are twice continuously differentiable, this is equivalent to (see appendix 3.A.3), for all $j \neq i$,

$$\frac{\partial^2}{\partial p_i \partial x_j} \tilde{\Pi}(p_i, x_{-i}) \leq 0,$$

or

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} + \left(p_i + xp^i - c'\right) \frac{p^j p^{ii} - p^i p^{ij}}{p^i p^j} \leq 0. \tag{3.9}$$

The last term in (3.9) disappears if goods are industry aggregates ($p^j p^{ii} - p^i p^{ij} = 0$), or at interior best responses ($p_i + xp^i - c' = 0$), leading to condition AD.

3.4 Existence of Equilibria

We will now state our main result on the existence of symmetric pure Cournot equilibria. In addition to condition A stated above, we need some regularity conditions to ensure that the decision problem of each firm has an optimal solution:

Condition R (Regularity): 1. Production capacity K is limited: $0 < K < \infty$;
2. production cost $c(x_i)$ *is lower semi-continuous;*

[8] Here $x_{-i}' \geq x_{-i}$ means that $x_j' \geq x_j$ for all $j \neq i$.

3. inverse demand $p(x_i, x_{-i})$ is continuous in (x_i, x_{-i}).

Multiple symmetric equilibria can be ranked according to equilibrium quantities (or prices). If there is a symmetric equilibrium where quantities are smaller (higher) than in any other symmetric equilibrium, this equilibrium is called *minimal* (*maximal*).

Theorem 1 *Assume that inverse demand is nonincreasing in all arguments (goods are substitutes), and strictly decreasing in own output while inverse demand is positive. Under conditions R and A there exist maximal and minimal symmetric pure Cournot equilibria.*

From a technical point of view, at the heart of theorem 1 lies the fact that under condition A the price best response $P(x_{-i})$ has *nonincreasing* maximal and minimal selections, which allows for the construction of *nondecreasing* maps from the space of prices into itself. Tarsky's (1955) theorem, which states that any nondecreasing map from an interval into itself has a fixed point, can then be applied to show that maximal and minimal fixed points exist. These result in maximal and minimal symmetric pure strategy Cournot equilibria.

The equilibrium is unique if and only if the maximal and minimal equilibria are identical, but this cannot be established without further assumptions. Under the strong assumption that profits are quasiconcave, *multiple symmetric* equilibria can be excluded if one assumes that the slopes of quantity best reactions are smaller than $1/(n-1)$, which follows in particular if all equilibria are stable (see appendix 3.A.4). On the other hand, using stronger versions of condition A and a weak additional condition B, one can exclude the existence of *asymmetric* equilibria.

Let us state two conditions related to condition A, both of which are strictly stronger and involve the whole space of competitors' outputs $[0, K]^{n-1}$: For all $i = 1...n$,

Condition AS: $\tilde{\Pi}(p_i, x_{-i})$ *satisfies the dual strict single crossing property in*

(p_i, x_{-i}) *for all* $x_{-i} \in [0, K]^{n-1}$*: For all* $p_i' > p_i$ *and* $x_{-i}' > x_{-i}$.

$$\tilde{\Pi}\left(p_i', x_{-i}\right) - \tilde{\Pi}\left(p_i, x_{-i}\right) \leq 0 \Rightarrow \tilde{\Pi}\left(p_i', x_{-i}'\right) - \tilde{\Pi}\left(p_i, x_{-i}'\right) < 0. \tag{3.10}$$

Condition ASD: $\partial\tilde{\Pi}/\partial p_i$ *exists, and is strictly decreasing in* x_j *for all* $j \neq i$, *where* $p(x_i, x_{-i})$ *is positive and for all* $x_{-i} \in [0, K]^{n-1}$.

Condition AS means that a firm will not raise *any* best response market price as a reaction to an increase in competitors' outputs (as opposed to just maximum and minimum best response prices under condition A). Condition ASD implies that a firm will *strictly decrease* its market price as a reaction to an increase in competitors' outputs and is stronger than conditions A and AS, and even stronger than weakly or strictly decreasing differences of profits (see Edlin and Shannon 1998a).

Let x_{-ij} be the vector of outputs of firms $k \neq i, j$. The additional conditions on inverse demands are:

Condition BW (weak): For all $j \neq i$, *all* $(x_i, x_j, x_{-ij}) \in [0, K]^n$, *and all* $\varepsilon > 0$, $p(x_i + \varepsilon, x_j, x_{-ij}) \leq p(x_i, x_j + \varepsilon, x_{-ij})$.

Condition BS (strict): For all $j \neq i$, *all* $(x_i, x_j, x_{-ij}) \in [0, K]^n$, *and all* $\varepsilon > 0$, $p(x_i + \varepsilon, x_j, x_{-ij}) \leq p(x_i, x_j + \varepsilon, x_{-ij})$, *where the inequality is strict when* $p(x_i, x_j + \varepsilon, x_{-ij}) > 0$.

If inverse demand is differentiable these conditions correspond to $p^i \leq p^j$ and $p^i < p^j$, respectively. Conditions BW and BS mean that each firm's changes in quantity in uence its own market price more than the same changes in other firms' quantities, which is a very reasonable assumption as firms are symmetric. Note that the case of homogeneous goods, where $p^i = p^j = p'$, falls under condition BW. In fact, both these conditions follow from utility maximization of a representative consumer: If inverse demands are derived from maximizing a (strictly) concave utility function U, where at the optimum $p_i = \partial U/\partial x_i$, then the condition $p^i \leq p^j$ ($p^i < p^j$) follows from the (strict) definiteness of the Hessian and the symmetry of the demand functions.[9]

[9]Note that $p^i = \partial^2 U/\partial x_i^2$, $p^j = \partial^2 U/\partial x_i \partial x_j$, and that the determinant of every 2x2 minor of

With these conditions, we have the following proposition:

Proposition 2 *Asymmetric equilibria do not exist if either 1. or 2. holds:*

1. *Conditions AS and BS hold.*
2. *Conditions ASD and BW hold.*

For homogeneous goods we must assume condition ASD (including the assumption that profits are differentiable) to rule out asymmetric equilibria, while for heterogeneous goods the weaker condition AS is enough. On the other hand, condition AS must be accompanied with the slightly stricter condition BS.

3.5 Related conditions

In the following we will discuss the conditions which have been used so far to establish existence of Cournot equilibrium and their relation to condition A. In general, strong conditions on payoffs, like concavity or quasiconcavity, yield existence in arbitrary games, see Friedman (1977, 1991), but are difficult to translate into economically meaningful statements about demand or cost.

Spence (1976) presents a class of demand functions with a special functional structure where Cournot equilibria can be found maximizing a certain 'wrong' surplus function. Here the question of existence of Nash equilibria reduces to the question of existence of maxima of this function. Slade (1994) finds a necessary and sufficient condition for this relation between equilibria and maxima to exist, and shows that for homogeneous goods such functions exist if and only if demand is linear, while there are more general cases for heterogeneous goods.

Most work has concentrated on economically meaningful conditions on demand or cost, or both. Unsurprisingly, practically all are related with either one or the

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the Hessian must be non-negative (positive):

$$(p^i)^2 - (p^j)^2 \geq (>)\ 0.$$

other of Hahn's (1962) pair of stability conditions,

$$p' - c'' \leq 0, \quad p' + xp'' \leq 0, \tag{3.11}$$

or in our notation

$$\Pi^{ii} \leq \Pi^{ij}, \quad \Pi^{ij} \leq 0, \quad j \neq i. \tag{3.12}$$

For homogeneous goods, McManus (1962, 1964) and Roberts and Sonnenschein (1976) prove existence of symmetric pure Cournot equilibrium assuming that production costs were convex, while Szidarovsky and Yakowitz (1977) additionally assume that inverse demand is concave. Kukushkin (1993), assuming convex cost, shows existence of pure symmetric equilibria if outputs are discrete variables.[10] If demand is nonincreasing, from the assumption of convex costs follows that $p' - c'' < 0$ or $\Pi^{ii} < \Pi^{ij}$ $(i \neq j)$, i.e. the first of Hahn's stability conditions. Amir and Lambson (1998), again for homogeneous goods, directly assume this, and prove existence of pure symmetric Cournot equilibria. Their work is important in several respects: It shows that the assumption of convex costs can be relaxed, and that the relevant condition $p' - c'' < 0$ is lattice-theoretic in nature. As one can see from condition AD, our condition A is a direct generalization to heterogenous goods of this condition.

The second of Hahn's stability conditions, $p' + xp'' \leq 0$ or $\Pi^{ij} \leq 0$, means that marginal revenue does not increase if competitors raise their outputs, i.e. that goods are *strategic substitutes* in the terminology of Bulow *et al.* (1985). More generally, goods being strategic substitutes is equivalent to profits $\Pi(x_i, x_{-i})$ having weakly decreasing differences in outputs (x_i, x_j) for $j \neq i$. This condition is not related to our condition A.

For homogeneous goods, Novshek (1985) shows that (possibly non-symmetric) Cournot equilibria exist if goods are strategic substitutes. Van Long and Soubeyran (1999) prove existence and uniqueness of Cournot equilibria under strategic substitutes *and* convex cost. For general aggregative games, i.e. "competitor aggregation",

[10] *Mixed* equilibria always exist if outputs are discrete.

Corchón (1994 , 1996) imposes a generalization of Hahn's conditions, which can be written as $\Pi^{ii} < \Pi^{ij} < 0$, and proves existence through the concavity of payoffs, while Kukushkin (1994) only assumes strategic substitutes.

For *strategic complements*, i.e. weakly increasing differences of $\Pi(x_i, x_{-i})$ in outputs (x_i, x_j) for $j \neq i$, or $\Pi^{ij} \geq 0$ $(j \neq i)$ under differentiability, Vives (1990), shows existence of pure Cournot equilibria for heterogeneous goods (symmetric with symmetric firms) in the general context of supermodular games.

The most intuitive way to characterize both strands of literature is to express all conditions used in terms of slopes of (quantity) reaction functions $r(x_{-i})$: Assuming that profits are twice differentiable, we obtain $r'(x_{-i}) = -\Pi^{ij}/\Pi^{ii}$. The first strand of literature effectively assumes that this slope is bounded below by -1, while strategic substitutes (complements) imply that $r' \leq (\geq) 0$. Our condition AD in general implies $r' \geq -p^j/p^i$, which is equal to -1 under homogeneous goods, and larger than -1 under condition BS.

While condition A generalizes the first part of the literature, its relation with the second group is not straightforward. For homogeneous goods the assumption of strategic complements implies condition A if profits are (at least locally) concave. Since profits are locally concave at interior best responses, this result captures the fact that reaction functions certainly have slope larger than -1 if they are nondecreasing. For heterogeneous goods this relationship is not clear.

Figures 1-3 summarize the relations between the various conditions mentioned above according to their implications on the slopes of best responses, which for simplicity are assumed to be differentiable. Most conditions only apply to homogeneous goods or "competitor aggregation".

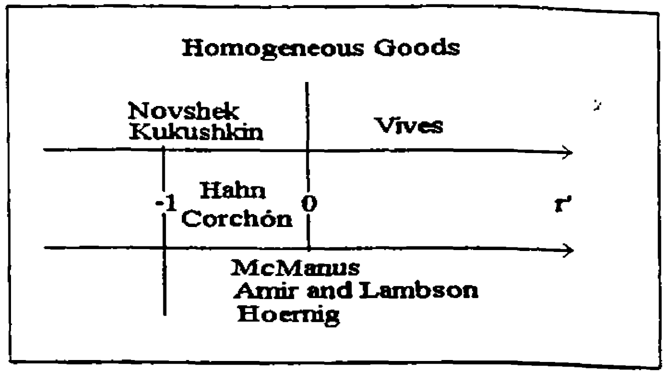


Figure 1

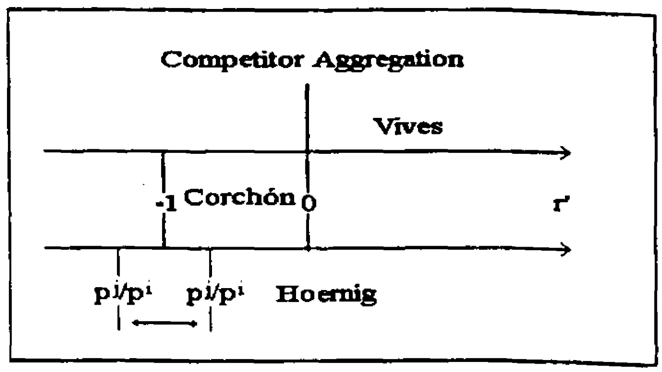


Figure 2

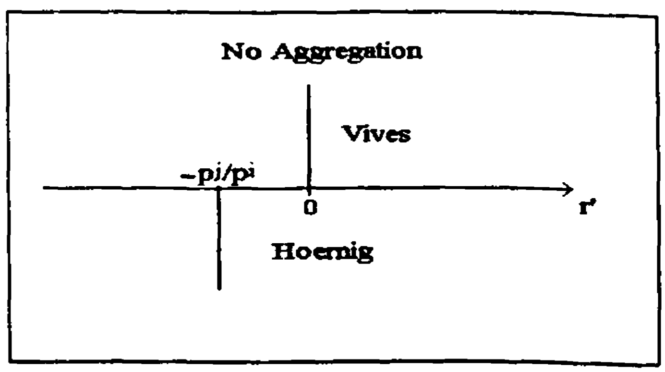


Figure 3

3.6 Examples

3.6.1 Linear Cournot Oligopoly

Consider the class of Cournot models with heterogeneous linear demand and linear cost functions, with $p_i = A - Bx_i - CY_i$, where $Y_i = \Sigma_{j \neq i} x_j$, and $B > 0$. Goods are substitutes for $C > 0$, and homogeneous if $C = B$. Condition AD is always satisfied since $\Pi^{ii} = -2B$, $\Pi^{ij} = -C$, and $\Pi^{ii} - (p^i/p^j)\,\Pi^{ij} = -B < 0$. On the other hand, condition D is equivalent to strategic substitutes if goods are substitutes: $\Pi^{ij} = -C$, and $\frac{\partial^2}{\partial p_i \partial Y_i}\tilde{\Pi} = -C/B$, and both expressions are negative. Assuming w.l.o.g. that marginal costs are zero, equilibrium prices $p_{(n)} = AB/\left(2B + (n-1)\,C\right)$ are decreasing in the number of firms if goods are substitutes.

3.6.2 Industry Aggregation

Here the demand function is of the type $\tilde{p}(x + \delta Y)$, and under differentiability condition AD becomes

$$\Pi^{ii} - \frac{1}{\delta}\Pi^{ij} = \tilde{p}' - c'' \leq 0,$$

which is equal to the condition $\tilde{p}' - c'' \leq 0$ under homogeneous goods. Note that the lower bound on the slope of the quantity reaction function becomes $r'(Y) \geq -\delta$. If $\delta < 1$ there are no asymmetric equilibria since for their existence it is necessary that the slope is -1 or smaller.

3.6.3 Non-aggregative demand

Here we give an example that shows that there are reasonable assumptions on consumers' preferences that give rise to inverse demand functions that do not allow aggregation of competitors' quantities (demand is *non-aggregative*).

Let the utility of a representative consumer be quasi-linear, and depend on a numeraire good y and n other goods $x_1, ..., x_n$ in the following form:

$$U(y, x_1, .., x_n) = y + \Sigma_{i=1}^n \left(x_i - x_i^2/2\right) + \Sigma_{i=1}^n \Sigma_{j>i} \ln(1 - x_i x_j),$$

$U(.)$ is a *generalized quadratic* utility function (Spence 1976).

At the consumer's optimum we have $\partial U / \partial x_i = p_i$ for $i = 1 \ldots n$, therefore the inverse demand functions are defined on $x \in [0,1]^n$, with

$$p(x_i, x_{-i}) = \frac{\partial U}{\partial x_i} = 1 - x_i - \Sigma_{j \neq i} \frac{x_j}{1 - x_i x_j}$$

while $\partial U/\partial x_i > 0$, and zero otherwise. With zero production cost, condition AD is satisfied "on the diagonal" $x_j = \bar{x}$, $j \neq i$ since p is twice differentiable and it can be shown that

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} = -\frac{(n-1-x_i^2)\,\bar{x}^2 + (1-2x_i\bar{x})^2}{(1-x_i\bar{x})^2} < 0.$$

In the next section we give an example of a non-aggregative inverse demand function that does not fall in Spence's class.

3.7 Effects of Entry

A long-standing point of interest has been the question whether Cournot equilibrium approaches a competitive equilibrium as more firms enter the market. It has become common to call a Cournot equilibrium *quasi-competitive* if equilibrium total quantity is increasing or price is decreasing in the number of firms. It it easy to see that for heterogeneous goods there is not necessarily a strict inverse relation between total quantity and market prices even if goods are symmetric. Since the sum of outputs makes less sense as a measure of quasi-competitiveness for heterogeneous goods precisely because outputs are not of the same good, we argue that the more useful measure is whether market prices are decreasing.

One should note that with heterogeneous goods the entry of a new competitor raises the number of goods (and welfare if consumers value variety), which in general may have surprising effects. As we will see in the following, under competitor aggregation the conventional wisdom (equilibrium prices decrease after entry) prevails, while for more general forms of heterogeneity this need no longer be true.

At first we will restrict attention to competitor aggregation. Assume there is a countable number of identical firms that may enter in the market[11]. Let $f : [0, K]^\infty \to F \subset \mathbb{R} \cup \{\infty\}$ be continuous, strictly increasing, and symmetric in its

[11] Since we are not interested in determining a free entry equilibrium, fixed cost of entry are irrelevant.

arguments: Let $\tilde{x}_{-i}$ be a permutation of $x_{-i} \in [0,K]^{\infty}$, then $f(\tilde{x}_{-i}) = f(x_{-i})$. Let inverse demand be given by

$$p = p(x_i, f(x_{-i})), \tag{3.13}$$

where $x_{-i} \in [0,K]^{\infty}$ and $p : [0,K] \times F \rightarrow \mathbb{R}_+$ is continuous and nondecreasing in (x_i, f_i), and strictly decreasing in its first argument while inverse demand is positive. For competitor aggregation, industry aggregation and homogeneous goods the simplest definition of f is $f(x_{-i}) = \Sigma_{j \neq i} x_j$.

Under condition A, the existence of symmetric Cournot equilibria follows from theorem 1, therefore the following comparative statics conclusions are not empty.

Theorem 3 *Under competitor aggregation the following holds:*

1. Under condition A maximal and minimal equilibrium prices are nonincreasing in the number of firms n.

2. Under condition ASD, and if $p(x_i, x_{-i})$ is strictly decreasing in x_j for all $j \neq i$ while $p_i > 0$, then maximal and minimal equilibrium prices are strictly decreasing in the number of firms n as long as they are positive.

Some remarks are in order: As noted above, even if $\tilde{\Pi}$ is differentiable, condition ASD, the condition that $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j for all $j \neq i$, is strictly stronger than condition A or even strictly decreasing differences of $\tilde{\Pi}$ in (p_i, x_{-i}) (see Edlin and Shannon 1998a).

Second, our method also can say something about the comparative statics of symmetric equilibria that are *interior* , i.e. characterized by first-order conditions, if they are *stable* equilibria in the usual definition as asymptotically stable equilibria under some classes of adjustment mechanisms. Hahn (1962) and Seade (1980) gave sufficient conditions for stability, while Seade also gave sufficient conditions for instability of Cournot equilibria: Equilibria with n firms are stable if the slopes of best reactions lie between the following bounds:

$$-1 < \tfrac{\partial}{\partial x_j} r(x_{-i}) = -\Pi^{ij}/\Pi^{ii} < 1/(n-1), \; j \neq i, \tag{3.14}$$

and instable if

$$\frac{\partial}{\partial x_j} r(x_{-i}) = -\Pi^{ij}/\Pi^{ii} > 1/(n-1), \; j \neq i. \tag{3.15}$$

Define $g(x,n) = f(x,...,x,0,...)$, where n firms all produce x, and the others nothing.

Corollary 4 *Under competitor aggregation, let inverse demand and production cost be twice continuously differentiable, and let $g(x,n)$ be differentiable in n with partial derivative $g_n > 0$. If condition AD holds then at interior equilibria the equilibrium price is nonincreasing in the number of firms if the equilibrium under consideration is stable. If equilibrium price is increasing then the equilibrium is unstable.*

In the appendix we show, making explicit use of the aggregation, that

$$\frac{dp_{(n)}}{dn} = \frac{g_n p_2}{\Pi^{ii} + (n-1)\Pi^{ij}} \left(\Pi^{ii} - \frac{p^i}{p^j} \Pi^{ij} \right), \tag{3.16}$$

where the second factor is non-positive by condition AD, and the first one is positive if the equilibrium is stable. Therefore $dp_{(n)}/dn$ is non-positive under condition AD and stability.

In the following schematic portrait of the fixed point map $\psi(p)$ determining equilibrium prices, which was used in the proof of theorem 1, the maximal and minimal equilibria (fixed points) are stable, and equilibrium prices decrease when we shift the map downwards to the dotted curve; the interior fixed point corresponds to an unstable equilibrium and indeed equilibrium prices increase.[12]

[12] We show in appendix 3.A.4 that equilibria are unstable if the fixed point map cuts the diagonal from below.

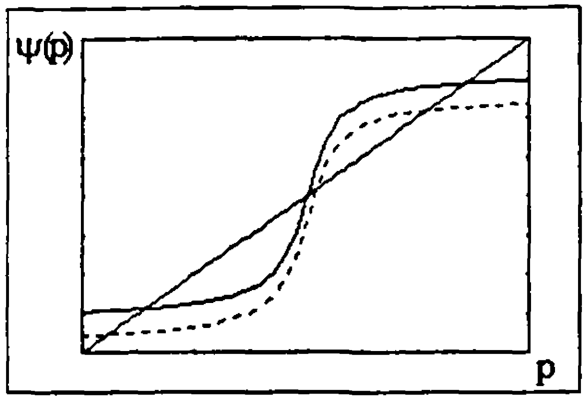


Figure 4

It can be shown that if $\hat{\Pi}(p_i, y)$ satisfies the (*non-dual*) weak single-crossing property in (p_i, y), then extremal equilibrium prices are *nondecreasing*, which demonstrates that condition A is critical for establishing quasi-competitiveness (see the proof of theorem 3).

Fourth, theorem 3 and corollary 4 do not extend to the case of non-aggregative demands, as the following example shows: Assume there are n firms with production capacity $K \geq 1/2$ and zero production cost. Inverse demands are $p(x_i, x_{-i}) = 1 - x_i - \Sigma_{j \neq i}(1 - e^{-x_i x_j})$ where this expression is positive, and otherwise $p(x_i, x_{-i}) = 0$. Symmetric equilibrium outputs are given by the first-order constraint (sufficient second-order conditions are satisfied)

$$2 - 2x - n + (n-1)\left(1 - x^2\right)e^{-x^2} = 0,$$

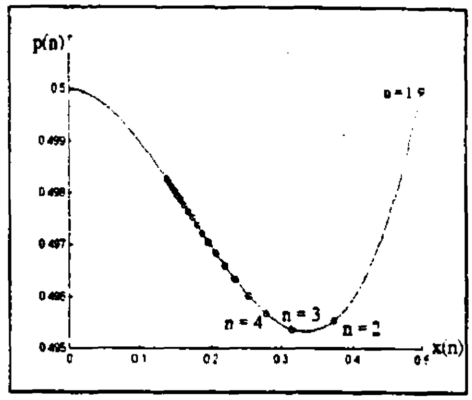
which for each value of $n \geq 1$ has exactly one solution $x_{(n)} \leq 1/2$. Therefore for each n there is exactly one symmetric equilibrium which at the same time is minimal, maximal and interior. On the diagonal $x_j = \bar{x}$ $(j \neq i)$ condition AD is fulfilled since

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} = -x\bar{x} < 0,$$

and the equilibrium is stable according to the above definition since

$$\Pi^{ii} + (n-1)\Pi^{ij} = -2 - 2(n-1)x\left(2 - x^2\right)e^{-x^2} < 0.$$

Still, equilibrium prices fall until $n = 3$, and then rise:



The reason for this maybe perplexing result is that the fixed-point maps defining successive equilibria are shifted *upwards* instead of downwards for $n > 3$ since the reduction in individual output best responses caused by the entrant is outweighed by a corresponding reduction in *competitors'* outputs (see the argument in appendix 3.A.4). Note that the traditional comparative statics analysis based on the implicit function theorem, which in appendix 3.A.4 we extend to the case of competitor aggregation, is not applicable here since inverse demand cannot be written as a *differentiable* function of the number of firms.

There are three other variables of interest whose equilibrium values vary with the number of firms: Total output, individual outputs, and firm profits. In supermodular games, i.e. games with strategic complements, equilibrium strategies and individual payoffs rise with the number of players, see Topkis (1998), theorem 4.2.3. In Cournot oligopoly the comparative statics of individual quantities, total quantities and profits each depend on a different condition.

We state the comparative statics results about quantities for competitor aggregation. Let us assume that inverse demand and production cost are twice continuously differentiable, and concentrate on interior equilibria, i.e. equilibria characterized by first order conditions. This is sufficient for our purposes since we want to make the simple point that conditions A or AD do not drive the results.

Corollary 5 *Under condition AD, consider any interior equilibrium where equilibrium prices* $p_{(n)}$ *are decreasing in the number of firms n.*

1. *Under condition BS total equilibrium output $Q_{(n)}$ is strictly increasing in n.*

2. *Individual equilibrium quantities $x_{(n)}$ are decreasing (increasing) in n if goods are strategic substitutes (complements).*

Without proof we note that to both comparative statics results there corresponds an own differencing condition on profits: For total quantity, it is that

$$\hat{\Pi}(Q, x_{-i}) = \Pi(Q - \Sigma_{j \neq i} x_j, x_{-i})$$

has nondecreasing differences in (Q, x_j) for all $j \neq i$, and for individual quantities that $\Pi(x_i, x_{-i})$ has nonincreasing (nondecreasing) differences in (x_i, x_j) for all $j \neq i$. These could of course be generalized to single-crossing conditions.

On the other hand, condition A is sufficient to show that individual profits are decreasing:

Corollary 6 *Under condition A individual profits $\Pi_{(n)}$ in maximal and minimal equilibria are nonincreasing in the number of firms n, and strictly decreasing if inverse demand $p(x_i, f_i)$ is strictly decreasing in f_i.*

Total firm profits, i.e. the sum of profits of all firms in the industry, may be increasing or decreasing.

The different effects of an increase in the number of firms on prices and quantities are summarized in the following two figures, where '+' ('-') means that the variable is going up (down):

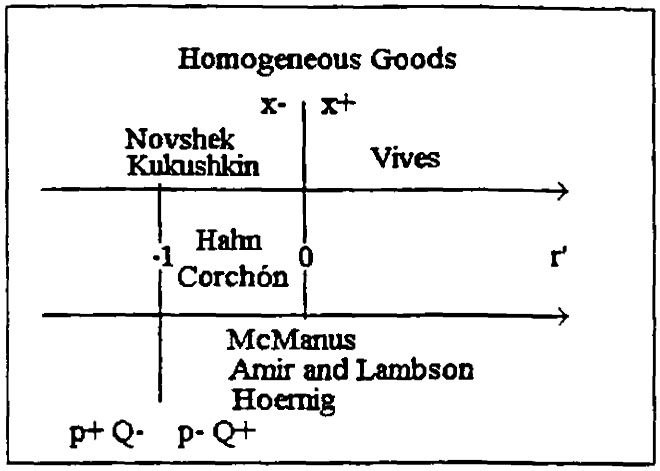


Figure 5

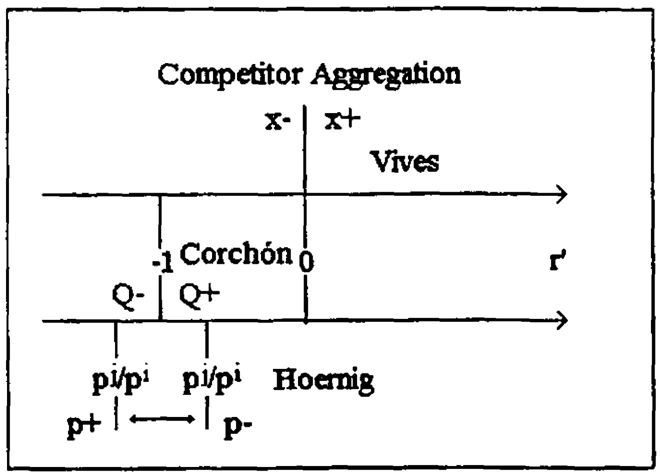


Figure 6

3.7.1 Related conditions

Most of the early literature on quasi-competitiveness, i.e. Frank (1965), Ruffin (1971), Seade (1980), Szidarovsky and Yakowitz (1982), all for homogeneous goods, assume both $\Pi^{ij} = p' + xp'' < 0$ (strategic substitutes) and $p' - c'' < 0$ (condition AD for homogeneous goods), and show that equilibrium market prices decrease as more firms enter. Corchón (1994, 1996) generalizes these conditions and results to general aggregative games; in the special case of Cournot competition his conditions are

$$\Pi^{ii} < \Pi^{ij} < 0,$$

where the first inequality corresponds to $p' - c'' < 0$ for homogeneous goods.

Not until in Amir and Lambson (1998) it became clear that the only condition relevant for quasi-competitiveness with homogeneous goods is $p' - c'' < 0$. As their work is based on lattice theory they can identify the conditions that are critical for their conclusions, and avoid unnecessary ones like strategic substitutes and concavity.[13] Our condition AD is a generalization to heterogeneous goods of the classical condition $p' - c'' < 0$, and condition A applies in more general contexts.[14]

Now we will give a simple example under homogeneous goods that shows that under condition A equilibrium prices go down, while under its 'opposite' they go up. Let inverse demand be given by $p(Q) = 3 - 2Q$, therefore goods are strategic substitutes.

First assume that marginal cost is constant with $c(x) = \frac{1}{2}x$. Then equilibrium price is $p_n = (6+n)/2(1+n)$ which is strictly decreasing and converges to $c'(0) = \min_x c(x)/x = 1/2$, and therefore to the competitive outcome. Condition A is satisfied since $p' - c'' = -2 < 0$.

For strongly increasing returns to scale, for example $c(x) = \frac{1}{2}x - \frac{22}{20}x^2$ (for $x \leq 1/2$) equilibrium price $p_n = (5n-3)/(10n-1)$ is strictly *increasing* in the number of firms and converges to $c'(0) = 1/2 > \min c(x)/x = 0$. Condition A rules this case out since $p' - c'' = -2 + 44/20 = 1/5 > 0$.

[13] De Meza (1985) and Villanova, Paradis and Viader (1999) exhibit examples where n-firm oligopoly prices are decreasing (therefore quasi-competitive according to the definition used here) but are higher than the monopoly price. This outcome is due to the assumption of increasing returns to scale in production that only set in for large output quantities.

[14] An issue that is related but different from quasi-competitiveness is the issue of *convergence* of equilibrium to the 'competitive price'. Ruffin (1971), following McManus (1964) and Frank (1965), shows that the Cournot equilibrium price converges to the competitive equilibrium price if and only if there are no increasing returns to scale.

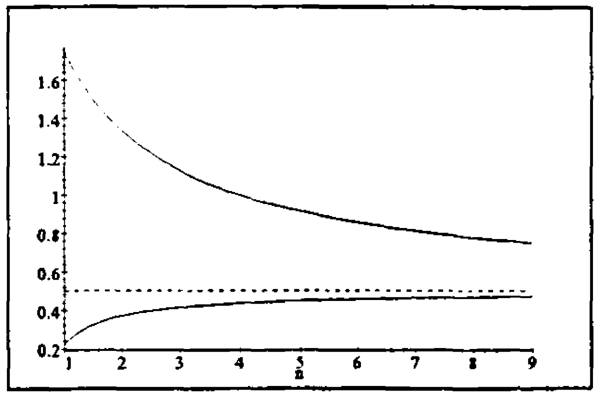


Figure 7

3.8 Conclusions

Using a lattice theory-based approach, we established the existence of pure symmetric Cournot equilibria for homogeneous or heterogeneous goods under a simple condition that generalizes the condition of weakly increasing returns used in the literature for the case of homogeneous goods. We were able to rule out asymmetric equilibria using a weak additional condition.

Under our main condition maximal and minimal equilibrium prices are decreasing in the number of firms if competitors' quantities enter inverse demand as an aggregate, but may be increasing if inverse demand is non-aggregative. We obtain the same result for stable interior equilibria. Total quantity increases with the number of firms under the same additional condition as above, while individual quantities increase (decrease) if goods are strategic complements (substitutes). Individual firm profits are decreasing after entry. These results show quite clearly that each comparative statics result depends on a different critical condition, and therefore model builders striving for generality should attempt to only include the assumptions that drive the comparative statics results that they really need.

One topic for further research is to extend our methods (and maybe some results) to cases where product heterogeneity is not symmetric, as e.g. in Hotelling models, or to models with exogenous or endogenous quality differences.

Second, similar results will certainly be obtained by applying corresponding conditions to models of heterogenous price competition, which to some extent already have been treated.

3.A Appendix

3.A.1 Change of variable in the profit function

In this appendix we discuss thoroughly the properties of the function $\chi(p_i, x_{-i})$ that is used to change the decision variable from own quantity to market price in the profit function. The important points are: monotonicity and continuity of $\chi(p_i, x_{-i})$, and convexity, closedness and monotonicity of the constraint set $\pi(x_{-i})$.

Let minimum and maximum prices be $p_K = p(K, .., K)$ and $p_0 = (0, .., 0)$, and the interval of possible prices with outputs by the other firms fixed

$$\pi(x_{-i}) = [p(K, x_{-i}), p(0, x_{-i})], \; x_{-i} \in [0, K]^{n-1}. \tag{3.17}$$

The set $\pi(x_{-i})$ is the new constraint set for the maximization over p_i, obviously non-empty, compact and convex, and is descending (nonincreasing) in x_{-i} (in the strong set order, see Milgrom and Shannon (1994)) since $p(., x_{-i})$ is nonincreasing in x_{-i}.

The range of possible combinations between market price and the others' outputs is

$$X = \left\{(p_i, x_{-i}) \in [p_K, p_0] \times [0, K]^{n-1} \,|\, p_i \in \pi(x_{-i})\right\}. \tag{3.18}$$

Let $\bar{x}(x_{-i})$ be the maximum output that firm i will produce given that the other firms are already producing x_{-i}, either because this output is equal to capacity, or because inverse demand becomes zero:

$$\bar{x}(x_{-i}) = \min\{K, \min\{x \in [0, K] \,|\, p(x, x_{-i}) = 0\}\} \geq 0. \tag{3.19}$$

Then since p is strictly decreasing and continuous on $x_i \in [0, \bar{x}(x_{-i})]$, we can express firm i's output as a function of market price and the others' output $\chi : X \to [0, K]$ that is continuous and nonincreasing in $(p_i, x_{-i}) \in X$, and strictly decreasing in p_i with image $[0, \bar{x}(x_{-i})]$ for each $x_{-i} \in [0, K]^{n-1}$.[15]

3.A.2 Dissection condition

In this appendix we derive a differential condition that is sufficient for condition A to hold. Assume that inverse demand and production costs are twice continuously differentiable. We apply the *method of dissection* described in Milgrom and Shannon (1994, p. 167). This method works as follows: The effect of an increase in own price p_i on profits is "dissected" into two parts, a *beneficial* effect due to a price increase (higher revenue per unit and lower total production cost due to the associated decrease in demand), and a *costly* effect due to the effect of the decrease in demand on revenue. To these effects are associated the price variables p^+ and p^-, respectively. Profits are written as

$$U(p^-, p^+, y) = p^+ \chi(p^-, y, .., y) - c(\chi(p^+, y, .., y)),$$

where

$$\frac{\partial U}{\partial p^-} = p^+ \chi^p < 0, \quad \frac{\partial U}{\partial p^+} = \chi - c' \chi^p > 0,$$

where $\chi^p = \partial \chi / \partial p_i = 1/p^i < 0$ (superscripts denote partial derivatives). Then $\hat{\Pi}(p_i, y)$ satisfies the dual weak single crossing property in (p_i, y), i.e. condition A, if $(\partial U/\partial p^-) / |\partial U/\partial p^+|$ is nonincreasing in y. We have, replacing p^+ and p^- by p_i,

$$\begin{aligned} \frac{d}{dy}\frac{\partial U/\partial p^-}{|\partial U/\partial p^+|} &= \frac{d}{dy}\frac{p_i \chi^p}{\chi - c'\chi^p} \\ &= (n-1)\frac{p_i \chi^{pj}(\chi - c'\chi^p) - p_i \chi^p (\chi^j - c'' \chi^j \chi^p - c' \chi^{pj})}{(\chi - c'\chi^p)^2} \\ &= -\frac{(n-1) p_i (\chi^p)^2 \chi^j}{(\chi - c'\chi^p)^2}\left(\frac{1}{\chi^p} - c'' - \frac{\chi}{\chi^p}\frac{\chi^{pj}}{\chi^p \chi^j}\right) \le 0, \end{aligned}$$

[15] This is an application of a continuous version of the implicit function theorem.

where $\chi^j = \partial\chi/\partial x_j = -p^j/p^i \leq 0$, $\chi^{pj} = \partial^2\chi/\partial p_i \partial x_j = \left(p^j p^{ii} - p^i p^{ij}\right)/\left(p^i\right)^3$, and we have used the symmetry of inverse demand with respect to opponents' outputs. Therefore condition A holds if

$$\begin{aligned}
\frac{1}{\chi^p} - c'' - \frac{\chi}{\chi^p}\frac{\chi^{pj}}{\chi^p\chi^j} &= p^i - c'' + x_i\frac{p^j p^{ii} - p^i p^{ij}}{p^j} \\
&= \left(2p^i + x_i p^{ii} - c''\right) - \frac{p^i}{p^j}\left(p^j + x_i p^{ij}\right) \\
&= \Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} \leq 0,
\end{aligned}$$

where

$$\begin{aligned}
\Pi^{ii} &= \tfrac{\partial^2}{\partial x_i^2}\Pi\left(x_i, x_{-i}\right) = 2p^i\left(x_i, x_{-i}\right) + x_i p^{ii}\left(x_i, x_{-i}\right) - c''\left(x_i\right), \\
\Pi^{ij} &= \tfrac{\partial^2}{\partial x_i \partial x_j}\Pi\left(x_i, x_{-i}\right) = p^j\left(x_i, x_{-i}\right) + x_i p^{ij}\left(x_i, x_{-i}\right).
\end{aligned}$$

3.A.3 The differential version of Condition D

Condition D holds if and only if

$$\begin{aligned}
\tfrac{\partial^2}{\partial p_i \partial x_j}\tilde{\Pi}\left(p_i, x_{-i}\right) &= \left(1 - c''\chi^p\right)\chi^j + \left(p_i - c'\right)\chi^{pj} \\
&= -\frac{p^j}{\left(p^i\right)^2}\left(p^i - c''\right) + \left(p_i - c'\right)\chi^{pj} \\
&= -\frac{p^j}{\left(p^i\right)^2}\left(\Pi^{ii} - \tfrac{p^i}{p^j}\Pi^{ij} + \left(p_i + xp^i - c'\right)\frac{\chi^{pj}}{\chi^p\chi^j}\right) \\
&\leq 0.
\end{aligned}$$

In the special cases of industry aggregation or homogeneous goods we have $\chi^{pj} = 0$, as can easily be seen:

$$\chi^{pj} = \left[\left(\delta\bar{p}'\right)\bar{p}'' - \bar{p}'\left(\delta\bar{p}''\right)\right]/\left(\bar{p}'\right)^3 = 0.$$

3.A.4 Proofs

Existence of Equilibrium

Proof. (Theorem 1).

1. First we show that price best responses are well-defined given the regularity conditions. Given any vector of outputs $x_{-i} \in [0, K]^{n-1}$ of the other firms, firm i's maximization problem is

$$\max_{p_i \in \pi(x_{-i})} \tilde{\Pi}(p_i, x_{-i}) = \chi(p_i, x_{-i}) p_i - c(\chi(p_i, x_{-i})),$$

where χ is continuous in p_i, c is lower semi-continuous, and

$$\pi(x_{-i}) = [p(K, x_{-i}), p(0, x_{-i})]$$

is a non-empty compact set. Then $\tilde{\Pi}$ is an upper semi-continuous function of p_i on the compact set $\pi(x_{-i})$ and therefore attains its maximum. Thus the price best response $P(x_{-i})$ exists, where $P : [0, K]^{n-1} \to [p_K, p_0]$ is a correspondence that is symmetric in x_{-i} since $p(x_i, x_{-i})$ is symmetric in x_{-i}. Now restrict P to the 'diagonal' $x_{-i} = (y, ..., y)$, $y \in [0, K]$, and define

$$\bar{P}(y) = P(y, .., y), \; y \in [0, K].$$

2. Maximal and minimal price best responses in $\bar{P}(y)$ are nonincreasing in y: The constraint set $\pi(x_{-i})$ is descending, or decreasing in the strong set order, since both $p(K, x_{-i})$ and $p(0, x_{-i})$ are nonincreasing in x_{-i}. This follows from the assumptions that goods are substitutes and that $p(x_i, x_{-i})$ is continuous in x_{-i}. Invoking this fact and condition A, by Milgrom and Shannon' s (1994) monotonicity theorem the set of maximizers $\bar{P}(y)$ is decreasing in y in the strong set order. This implies in particular that maximum and minimum selections of $\bar{P}$ exist and are nonincreasing in y. Let $\tilde{P} : [0, K] \to [p_K, p_0]$ be a maximum or minimum selection, then $\tilde{P}$ is a nonincreasing function.

3. Continuation 1 (Fixed point in prices): Consider prices at identical outputs for *all* firms: Let

$$\tilde{p}(x) = p(x, x, .., x), \; x \in [0, K].$$

Then $\tilde{p}$ is nonincreasing since p is nonincreasing in own output and because goods are substitutes. It is strictly decreasing while positive, and maps $[0, K]$ onto $[p_K, p_0]$.

Let $\bar{x}$ be the largest symmetric output that all firms might produce in equilibrium (for all larger outputs less than capacity market price is zero),

$$\bar{x} = \min\{K, \min\{x \in [0, K] \,|\, \tilde{p}(x) = 0\}\},$$

then the restricted $\tilde{p} : [0, \bar{x}] \to [p_K, p_0]$ is strictly decreasing and one-to-one, and has a strictly decreasing inverse $\tilde{\chi} : [p_K, p_0] \to [0, \bar{x}] \subset [0, K]$.

4. Construct a fixed point map: The map

$$\psi(p) = \tilde{P}(\tilde{\chi}(p)), \ p \in [p_K, p_0] \tag{3.20}$$

is a nondecreasing function from $[p_K, p_0]$ into itself. By Tarsky's (1955) theorem there is a fixed point $p^* = \psi(p^*)$.

5. The market price p^* is attained in the market of firm i if all firms produce $x^* = \tilde{\chi}(p^*)$. On the other hand, if all of firm i's competitors produce x^*, then firm i adjusts production such that its best response market price is p^*, with best response quantity $\chi(p^*, x^*, .., x^*) = x^*$ because χ is strictly decreasing in p. Therefore a symmetric equilibrium exists where all firms produce x^* and market price is p^* in all markets.

3'. Continuation 2 (Fixed point in quantities): Given identical outputs $y \in [0, K]$ for all competitors, quantity best responses $\tilde{r} : [0, K] \to [0, K]$ are given by $\tilde{r}(y) = \chi\left(\tilde{P}(y), y, ..., y\right)$. Then $\tilde{r}$ is continuous but for upward jumps, since χ is continuous in all arguments, and decreasing in its first, while $P(x_{-i})$ is nonincreasing and therefore has no upward jumps, only downward jumps. Therefore $\tilde{r}$ has a fixed point (Milgrom and Roberts 1994b, cor. 1), which is an equilibrium output. ∎

Stability and Multiple Symmetric Equilibria

We will now prove that instability of an equilibrium point corresponds to a slope larger than 1 of the fixed point map defining this equilibrium point, i.e. it cuts the diagonal from below. Note that since the fixed point map starts above the diagonal, multiple symmetric equilibria will exist if and only if it crosses the diagonal from below at least once. In particular, if the map jumps upwards over the diagonal (which cannot happen if profits are quasiconcave) then multiple symmetric equilibria will exist and all of them may be stable. Kolstad and Mathiesen (1987) give necessary and sufficient conditions for uniqueness of equilibrium with homogenous goods which boil down to $p' - c'' < 0$ and $\Pi_{11} + (n-1)\Pi_{12} < 0$ (stability). They do assume quasiconcavity of profits, and the above heuristic argument shows that this assumption is indispensable.

Remember the fixed point map used in the proof of theorem 1, $\psi(p) = \tilde{P}(\tilde{\chi}(p))$ on $p \in [p_K, p_0]$. From maximizing profits over prices we find that

$$\begin{aligned} \frac{d}{dy}\tilde{P}(y) &= (n-1)\left(-\tilde{\Pi}^{pj}/\tilde{\Pi}^{pp}\right), \\ \frac{d}{dp}\tilde{\chi}(p) &= \frac{1}{p^i + (n-1)p^j}, \end{aligned}$$

with

$$\begin{aligned} \tilde{\Pi}^{pp} &= 2\chi^p + (p - c')\chi^{pp} - c''(\chi)(\chi^p)^2 \\ &= \Pi^{ii}/(p^i)^2 \end{aligned}$$

where we used $\chi^{pp} = -p^{ii}/(p^i)^3$, and

$$\begin{aligned} \tilde{\Pi}^{pj} &= \chi^j + (p - c')\chi^{pj} - c''(\chi)\chi^p\chi^j \\ &\quad -\left(\Pi^{ii} - \tfrac{p^i}{p^j}\Pi^{ij}\right)p^j/(p^i)^2 \end{aligned}$$

if the first order condition $\chi + p\chi^p - c'(\chi)\chi^p = 0$ holds. Then the slope of the fixed-point map is

$$\begin{aligned} \tfrac{d}{dp}\psi(p) &= \tfrac{d}{dy}\tilde{P}(\tilde{\chi}(p))\tfrac{d}{dp}\tilde{\chi}(p) \\ &= (n-1)\frac{p^j\Pi^{ii} - p^i\Pi^{ij}}{(p^i + (n-1)p^j)\Pi^{ii}}, \end{aligned}$$

which is larger than 1, i.e. the fixed point map cuts the diagonal from below, if and only if

$$\Pi^{ii} + (n-1)\Pi^{ij} > 0 \Leftrightarrow \tfrac{\partial}{\partial x_j}r(x_{-i}) = -\Pi^{ij}/\Pi^{ii} > 1/(n-1),$$

i.e. if the equilibrium is unstable according to Seade (1980).

Asymmetric equilibria

Proof. (Proposition 2)

Consider an asymmetric equilibrium $x = (x_1, \ldots, x_n)$, where w.l.o.g. $x_1 = y+\varepsilon > x_2 = y$. Then $x'=(x_2, x_1, x_3, \ldots, x_n)$ is also an equilibrium. For conciseness we now suppress the arguments $(x_3, \ldots, x_n)$. Equilibrium prices for firm 1 are $p = p\,(y+\varepsilon, y)$ and $p'\,(y, y+\varepsilon)$, with $p' > p$ by condition BS or $p' \geq p$ by condition BW.

First impose condition BS, leading to $p' > p$. Under condition AS, i.e. under the dual strict single-crossing property of firm i's profit $\tilde{\Pi}$ in (p_i, y) for all $i = 1..n$, all selections of best price responses are nonincreasing, i.e. $p' \leq p$ since $y + \varepsilon > y$, which is a contradiction to $p' > p$.

For the second statement impose condition BW, leading to $p' \geq p$. Since under condition ASD for all $i = 1..n$ the partial derivative $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j $(j \neq i)$, price best responses are strictly decreasing in x_j (Theorem 2.8.5 in Topkis 1998). Therefore, since p is a best response at y and p' at $y + \varepsilon > y$, we must have $p' < p$, and again arrive at a contradiction. ■

Entry: Prices

Before giving the proof of theorem 3, we will shortly discuss why it is not possible to give a corresponding proof for the non-aggregative case. Let us pay attention to the dependence on the number of firms in the fixed point map (3.20) used in the proof of theorem 1:

$$\psi\,(p, n) = \tilde{P}\,(\tilde{\chi}\,(p, n)\,, n) \tag{3.21}$$

Then there are two opposing effects of an increase in n: First, best price responses are lower since $\tilde{P}\,(x, n)$ is nonincreasing in n; second, the market price p can only be sustained if all firms produce less, since $\tilde{\chi}\,(p, n)$ is decreasing in n. The first effect moves ψ downwards, while the second effect moves it upwards. Under aggregation the first effect is stronger, but this is hard to show here. In the following proof we avoid this difficulty by constructing a different fixed point map making strong use of the assumption of aggregation.

Proof. (Theorem 3)

1. Price best responses: Because competitors' outputs are aggregated by $f(x_{-i})$, price best response is a correspondence $P : F \to [p_\infty, p_0]$, $p = P(f(x_{-i}))$, where $p_\infty = p(K, K, ...)$, $p_0 = p(0, 0, ...)$, and maximal and minimal selections exist. Under condition A these selections are nonincreasing in f.

2. Symmetric outputs (this is the hard part, where the aggregation is really used): If the first $n-1$ competitors of firm i are active and the others produce zero, let

$$\tilde{f}(x, n) = f(x, .., x, 0, ...), \ x \in [0, K],$$

with image $\phi(n) = \left[\tilde{f}(0, n), \tilde{f}(K, n)\right] = [\underline{f}, \bar{f}(n)]$, where $\underline{f} = f(0, ...)$ and $\bar{f}(n) = f(K, ..., K, 0, ...)$. Then $\tilde{f}$ is strictly increasing and continuous in x, and strictly increasing in n. Let $\Phi = \{(f, n) \in F \times \mathbb{N} | f \in \phi(n)\}$, then we can express every firm's output x by the value of the aggregator and the number of firms through a function $\tilde{x} : \Phi \to [0, K]$ such that $\tilde{x}$ is strictly increasing and continuous in f, and strictly decreasing in n.

Consider market price at a given value of the aggregator, if all firms produce the same amount, even firm i (this is the basic trick):

$$\hat{p}(f, n) = p(\tilde{x}(f, n), f),$$

where $\hat{p} : F \times \mathbb{N} \to [p_\infty, p_0]$ is strictly decreasing and continuous in f, and strictly increasing in n. For fixed n, its image is $\hat{\pi}(n) = \left[p\left(K, \bar{f}(n)\right), p\left(0, \underline{f}\right)\right]$, where the upper limit is fixed, and the lower limit is nonincreasing in n. Let $\hat{\Pi} = \{(p, n) \in [p_\infty, p_0] \times \mathbb{N} | p \in \hat{\pi}(n)\}$. Invert $\hat{p}$ with respect to f, to obtain a function $\hat{f} : \hat{\Pi} \to \mathbb{R}$ which is strictly decreasing (and continuous) in p and strictly increasing in n. The interpretation of $\hat{f}$ is: given price p and number of active firms n, value of aggregator if all n firms produce the same amount (even firm i), resulting in price p.

3. Symmetric equilibria: Let $\bar{P} : F \rightarrow [p_\infty, p_0]$ be a maximal or minimal selection of the price best response map P. Consider the family of maps $\psi_n : \hat{\pi}(n) \rightarrow [p_\infty, p_0]$ defined by

$$\psi_n(p) = \bar{P}\left(\hat{f}(p,n)\right),$$

then a maximum or minimum fixed point $p_{(n)}$ of this map is maximal or minimal equilibrium price. Under condition A, ψ_n is nonincreasing in n since $\bar{P}$ is nonincreasing, so that extremal equilibrium prices are nonincreasing in n (taking into account that $\pi(n) \subset \pi(n+1)$) by corollary 2.5.2 of Topkis (1998).

Under the (*non-dual*) weak single-crossing property of profits in (p_i, y), $\bar{P}$ is nondecreasing, and ψ_n is nondecreasing in n. Thus extremal equilibrium prices are *nondecreasing* in n if equilibria exist.

4. If $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j, then by theorem 2.8.5 of Topkis (1998), which follows Amir (1996) and Edlin and Shannon (1998b) extremal price best responses $\bar{P}(x_{-i})$ are strictly decreasing while interior, i.e. positive. Since $\hat{f}$ is strictly increasing in n, the map $\psi_n(p)$ is strictly decreasing in n for each p, and therefore by corollary 2.5.2 of Topkis (1998) the extremal fixed points of ψ_n are strictly decreasing in n (Note that a subtle point of this proof is that we can only say something about the extremal fixed points and not about the others). ■

Entry: Interior equilibrium prices

Proof. (Corollary 4)

Denote the partial derivatives of inverse demand $p(x_i, f(x_{-i}))$ with respect to x_i and f as $p_1 < 0$ and $p_2 < 0$, of $f(x_{-i})$ with respect to x_j ($j \neq i$) as $f_j > 0$ (therefore $p^i = p_1$ and $p^j = p_2 f_j$), and of $g(x,n)$ with respect to x and n as $g_x = (n-1) f_j > 0$ and $g_n > 0$, respectively. The first-order necessary condition for an interior best response in terms of quantity, and the second derivatives of profits, are

$$\begin{aligned}
\Pi^i &= \tfrac{\partial}{\partial x_i}\Pi(x_i, x_{-i}) = p(x_i, f(x_{-i})) + x_i p_1(x_i, f(x_{-i})) - c'(x_i) = 0, \\
\Pi^{ii} &= \tfrac{\partial^2}{\partial x_i^2}\Pi(x_i, x_{-i}) = 2p_1 + x_i p_{11} - c'', \\
\Pi^{ij} &= \tfrac{\partial^2}{\partial x_i \partial x_j}\Pi(x_i, x_{-i}) = (p_2 + x_i p_{12}) f_j.
\end{aligned}$$

At a symmetric equilibrium with n firms,

$$\begin{aligned}
\tilde{\Pi}^i\left(x_{(n)}, n\right) = \Pi^i\left(x_{(n)}, .., x_{(n)}\right) &= p\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) \\
&+ x_{(n)} p_1\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) - c'\left(x_{(n)}\right) = 0,
\end{aligned}$$

we find the following second derivatives with respect to $x_{(n)}$ and n, respectively:

$$\begin{aligned}
\tilde{\Pi}^{ix} &= \tfrac{\partial}{\partial x_{(n)}}\tilde{\Pi}^i = \Pi^{ii} + \left(p_2 + x_{(n)} p_{12}\right) g_x = \Pi^{ii} + (n-1)\Pi^{ij}, \\
\tilde{\Pi}^{in} &= \tfrac{\partial}{\partial n}\tilde{\Pi}^i = \left(p_2 + x_{(n)} p_{12}\right) g_n = \Pi^{ij} g_n / f_j.
\end{aligned}$$

Equilibrium quantities evolve with

$$\frac{dx_{(n)}}{dn} = -\frac{\tilde{\Pi}^{in}}{\tilde{\Pi}^{ix}} = -\frac{\Pi^{ij} g_n / f_j}{\Pi^{ii} + (n-1)\Pi^{ij}},$$

where by Seade's stability condition the denominator $\tilde{\Pi}^{ix}$ is negative. The total derivative of equilibrium prices is

$$\begin{aligned}
\frac{dp_{(n)}}{dn} &= \frac{d}{dn} p\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) = (p_1 + p_2 g_x)\frac{dx_{(n)}}{dn} + p_2 g_n \\
&= \frac{g_n p_2}{\Pi^{ii} + (n-1)\Pi^{ij}}\left(\Pi^{ii} - \frac{p_1}{p_2 f_2}\Pi^{ij}\right).
\end{aligned}$$

Since by condition AD the second term on the right-hand side is non-positive, the sign of $dp_{(n)}/dn$ depends entirely on whether the equilibrium is stable: Prices are decreasing (increasing) if the equilibrium is stable (instable), i.e. $\Pi^{ii} + (n-1)\Pi^{ij} < (>) 0$. ■

Entry: Quantities

Proof. (Corollary 5)

From the proof of corollary 4 we already know that

$$\frac{dx_{(n)}}{dn} = -\frac{x_{(n)}\Pi^{ij}}{\Pi^{ii} + (n-1)\,\Pi^{ij}}.$$

If condition AD holds and equilibrium price is decreasing, then from (3.16) we can conclude that $\Pi^{ii} + (n-1)\,\Pi^{ij} < 0$. Therefore $dx_{(n)}/dn < (>)\,0$ if $\Pi^{ij} < (>)\,0$, i.e. if goods are strategic substitutes (complements). Similarly, we can find from the first order condition $\Pi^{i}\left(Q_{(n)}/n, (n-1)\,Q_{(n)}/n\right) = 0$ that

$$\frac{dQ_{(n)}}{dn} = \frac{Q_{(n)}}{n}\frac{\Pi^{ii} - \Pi^{ij}}{\Pi^{ii} + (n-1)\,\Pi^{ij}},$$

i.e. $dQ_{(n)}/dn > 0$ if $\Pi^{ii} - \Pi^{ij} < 0$. Now from condition AD and condition BS, whose differential form is $p^{i} < p^{j} < 0$,

$$0 \geq \frac{p^{j}}{p^{i}}\Pi^{ii} - \Pi^{ij} > \Pi^{ii} - \Pi^{ij}$$

since at interior equilibria $\Pi^{ii} \leq 0$ and $0 \leq p^{j}/p^{i} < 1$. ∎

Entry: Profits

Proof. (Corollary 6)

From the proof of theorem 3 it is easy to see that under condition A $f_{(n)} = \hat{f}\left(p_{(n)}, n\right)$ is strictly increasing in n, since $\hat{f}$ is strictly decreasing in $p_{(n)}$ and strictly increasing in n, and $p_{(n)}$ is nonincreasing in n.

Since goods are substitutes, profits $\Pi\left(x, f\right)$ are nonincreasing in f. As $f_{(n)}$ is increasing in n,

$$\Pi\left(x_{(n)}, f_{(n)}\right) \geq \Pi\left(x_{(n+1)}, f_{(n)}\right) \geq \Pi\left(x_{(n+1)}, f_{(n+1)}\right),$$

where $x_{(n)}$ and $x_{(n+1)}$ are the corresponding equilibrium outputs, and the first inequality expresses the fact that $x_{(n)}$ maximizes profits. If p is strictly decreasing in f then the second inequality is strict. ∎

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