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# Non-Cooperative Network Formation with Network Maintenance Costs

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# Non-cooperative network formation with network maintenance costs

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**Abstract**

This paper presents a model of non-cooperative network formation similar to Bala and Goyal (2000), except that marginal costs in the size of the network is increasing. Agents link among each other to gain information and update their links according to a better reply dynamics. In the long run the system settles in a state that consists of starred-wheel networks. This is reminiscent of some real world features. Collections of smaller disjoint networks connecting few agents are more common than global networks connecting all agents of the community. Differences within a connected component such as core and periphery are established.

Keywords: Networks, coordination, learning dynamics, non-cooperative games.

JEL Classification: C72, D83.

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# 1 Introduction

A network formation game is presented in this paper. This is a non-cooperative game among agents who individually decide whether or not to build a link with some other agent. Economic investigation in network formation and interaction among networked agents can be divided in two branches. The first one considers the network structure exogenous and studies the interaction of linked agents given the network. Local interaction and peer pressure are examples of interactions of agents within given networks. This research agenda on "static networks" was developed both theoretically and empirically.[1] The second branch of investigation focuses on changing networks and adopts a game-theoretic approach to analyse the dynamic process leading to the formation of the actual network. Jackson [15] provides a clear and up-to-date overview of this literature.

Game-theorists initially focused on cooperative network formation. The cooperative feature is the fact that if one agent wants to link to another one then the former needs the agreement of the latter: i.e. the two must cooperatively agree on being linked. This literature[2] comprises Jackson, Watts ([30], [16] and [17]), Dutta, van den Nouweland, Slikker and Tijs ([6], [25], [26] and [27]) among others. The cooperative approach is helpful in many contexts in which is not a limitation to assume that the agent who receives a link may veto it.

The stream of literature to which this paper belongs was opened by Bala and Goyal [1] (henceforth BG) who focus on the importance of non-cooperative incentives for network formation. BG innovate the literature by modelling self-interested boundedly-rational agents who can unilaterally decide whether to build or sever a link. Their predictions depend on the relative cost of a link and on whether information flows in one direction only or in both directions through the links. In every time period BG's agents select the best response given the current network. It is noteworthy that both with one-way and with two-way

[1] Within this literature see for example Ellison [7] and Tesfatsion [28] which are theoretical papers and Bertrand *et al.* [2], Case and Katz [4] and Ichino and Maggi [14] which are empirical works.

[2] For a detailed account of this literature see Jackson [15] and references therein.

information flow the BG non-empty steady-state networks connect all the agents in the population.

Goyal, Galeotti, Joshi, Moraga, Vega-Redondo ([10], [11], [12] and [13]), Larrosa and Tohmé ([18]) present applications and extensions of this framework. Falk and Kosfeld [8] experimentally test BG's model both with one-way and with two-way information flow. The predictions of the one-way flow model can be replicated in the laboratory while those of the two-way flow one cannot.[3] Currarini and Morelli [5] and Mutuswami and Winter [21] develop a mechanism-design approach to characterise the mechanism achieving efficient networks. Networks, local and group interactions are of interest in all the social sciences. Historians for example use networks to analyse behavioural and power relationships among agents in order to have a better understanding of micro-determinants of historical events.[4]

The observation that it is extremely rare that real-world networks connect all the individuals in a society motivates this research. In fact there are two main features that arise in the real world. Agents are usually connected locally and not globally with the whole community. Often they are also partitioned in a core-periphery dichotomy where agents in the core are usually better off then those in the periphery.

This paper develops an extension of BG's with one-way information flow. The homogenous agents in the network bear the cost of the links they sponsor. In addition each agent pays the network maintenance cost. This is modelled as an increasing and convex function of the number of observed agents, but it does not directly depend on the number of sponsored links of each agent. In every period the agents play the network formation game, each of them with a probability of maintaining the strategy implemented in the previous period. Active agents

[3] Falk and Kosfeld [8] observe that fairness considerations (Fehr and Schmidt [9]) may explain these results. Fairness is defined as inequality aversion, hence BG's two-way information flow equilibria –which are (pay-off) asymmetric– are not fairness compatible while equilibria with one-way information flow are symmetric and hence fairness compatible.

[4] See Padgett and Ansell [23] and Lipp and Krempel [20] among others.

change their current strategy only if they switch to another strategy that improves their current payoff. So the dynamic analysis is based on better response.

The essential difference between this paper and BG is the presence of the network maintenance cost. Such a cost makes the marginal cost of every additional link an increasing function of the total number of observed agents, while the marginal benefit is constant. Hence there exists an optimal number of agents that each player wants to observe. The network maintenance costs implies that there are decreasing returns to linking and it can also be interpreted as costs due to congestion in the network.

The main results are as follows. The dynamics converges in finite time. The basic component of the absorbing state architecture is a *starred wheel* where some agents form a wheel[5] and others are linked to the wheel "from outside". Limit networks consist of disjoint components each of which is characterised by the fact that some agents (who are in the wheel at the centre of the network) enjoy a higher payoff -in fact the maximum payoff attainable- than the peripheral ones. While all the outside agents observe the central wheel, they do not observe each other. The results of this paper are in line with the real world features mentioned above: limit networks are local rather than global and in the absorbing state agents are partitioned between a centre and a periphery. All the starred wheels have the same dimension. This means that the number of agents involved in the central wheel is the same among the components of the limit network. The number of peripheral agents connected to each of this wheels may however vary. Social welfare –as measured by the sum of agents' payoffs– increases with the number of disconnected components in the limit network.

Simple comparison between the limit networks obtained here and those of BG shows the impact of the introduction of the network maintenance cost. As we are considering the one-way information flow, let us recall the two networks that are absorbing states in this case of BG analysis: the (global) wheel and

[5] Given a (sub)set of agents the wheel is a network that connects all of them, each of whom has one link to another one and is linked by a third (different) one only.

the empty network.[6] The BG dynamics settles to a wheel for low values of the unitary cost of link and to the empty network for high values of the cost, while both networks are absorbing states for intermediate values of the cost of a link. BG analysis is a special case of the model presented in this paper: there are no network maintenance costs. For a small population size BG results hold even in presence of network maintenance costs. For any given level of the maintenance cost, the bigger the population the larger the number of disconnected components in the limit networks. In addition in BG's limit networks all agents receive the same payoff while central agents in a starred wheel are better off than peripheral ones.

Of particular interest is also a comparison of the results of this paper with those of Galeotti and Goyal [10] since they reach some results which have a similar flavour building on different assumptions to those used here. Galeotti and Goyal [10] restrict the analysis to the case of two-way information flow and assume that agents are heterogeneous and that their heterogeneity is an observable characteristic. Their findings are that limit networks may be either collections of disconnected components or characterised by an insider-outsider dichotomy. Galeotti and Goyal [10]'s results reinforce rather than contradict those presented here. Total connectedness of the limit network as in BG can be broken in two manners. With one-way information flow it is sufficient to assume a convex cost for network maintenance maintaining the original assumption of homogeneous agents as is done in this paper. With two-way information flow one can introduce exogenous heterogeneity among the individuals as in Galeotti and Goyal [10].

The paper is organised as follows. The next section outlines the model. Section 3 provides the preliminary analysis. Section 4 characterises the absorbing state networks and section 5 concludes. The appendices collect some of the proofs.

[6] This network is obviously characterised by the absence of any link between any two agents.

# 2 The Model

There is a population of $P$ agents. With a slight abuse of notation let us indicate with $P$ both the population and its size. Every agent plays the network formation game. A strategy of each agent $i$ indicates for all agents $j$ $j \neq i$ whether $i$ has a direct link to $j$. It is represented with a vector $g_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,P})$ of dimension $P-1$, where each element of the vector takes value 1 if $i$ has one (direct) link to $j$ and 0 otherwise. We say that agent $i$ *observes* agent $j$ $j \neq i$ if either $i$ built a link to $j$, i.e. $g_{i,j} = 1$, or there exists a path in the network that goes from $i$ to $j$, i.e. there exists a set of agents $\{k_1, .., k_n\} \subset P$ such that $g_{i,k_1} = g_{k_1,k_2} = ... = g_{k_m,k_{m+1}} = ... = g_{k_n,j} = 1$. We adopt the convention that every agent always observes himself.

The set of all strategies of each player is $\mathcal{G}_i = \{0,1\}^{P-1}$ for $i = 1, ..., P$. Every strategy profile translates into a (directed) network. Both a network and the strategy profile that generates it are indicated with $g \in \mathcal{G}_1 \times \ldots \times \mathcal{G}_P$. We write $g = g_i \oplus g_{-i}$ to stress that the network $g$ is made by the composition of the strategy of $i$ with those of his opponents.

Consider the following example: $P = 4$, $g_1 = (0,0,0)$, $g_2 = (0,1,0)$, $g_3 = (1,1,0)$ and $g_4 = (0,0,1)$. Agent 1 has no links. Agents 2 and 4 each have one link only to agent 3. Agent 3 has two links, one with agent 1 and the other with agent 2. This network is depicted in the figure below. The arrows indicate the direction of the information flow.

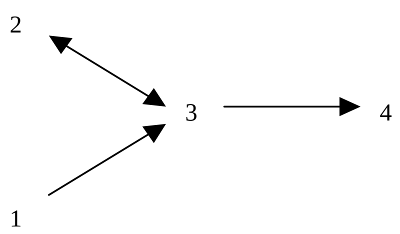


**Figure 1: Example of a network**

This is the one-way information flow model of BG.

Let us indicate with $c$ the cost building one link, with $\mu_i(g)$ the number of agents $i$ is directly or indirectly linked with through the network $g$ (including agent $i$ himself) and with $\mu_i^d(g)$ the number of links set up by agent $i$. In the example of Figure 1 above $\mu_1 = 1$, $\mu_1^d = 0$, $\mu_2 = 2$, $\mu_2^d = 1$, $\mu_3 = 3$, $\mu_3^d = 2$ and $\mu_4 = 4$, $\mu_4^d = 1$. Note that the arrows indicate the identity of the agent who is bearing the cost of the link.

The individual payoff function is a function of the network $g$ which player $i$ belongs to:

$$\pi_i(g) = \mu_i(g) - c\,\mu_i^d(g) - \alpha\,[\mu_i(g)]^2 \tag{1}$$

The term $-\alpha\,[\mu_i(g)]^2$ in the payoff function represents the network maintenance cost. This implies that there are decreasing returns to linking, as the total cost of a network $c\,\mu_i^d(g) + \alpha\,[\mu_i(g)]^2$ is increasing and convex in the number of observed agents. Parameter values belong to the set $\mathcal{R}$

$$\mathcal{R} = \left\{ (\alpha, c) \in \mathbb{R}^2 \;\middle|\; 0 < \alpha \leq \tfrac{1}{4},\;\; 0 \leq c \leq \tfrac{1}{4\alpha} - 1 \right\} \tag{2}$$

which is needed to guarantee that agents have incentives to connect. Notice that in the limit case $\alpha = 0$ the model is the same as that in BG.

Define $N(i, g)$ as the set of agents observed by $i$ through the network $g$. So $\mu_i(g) = \|N(i, g)\|$ i.e. $\mu_i(g)$ is the cardinality of the set $N(i, g)$. Define the geodesic distance between agents $i$ and $j$ in a network $g$ $d(i, j; g)$ as the number of links on the shortest path from $j$ to $i$. If $j \notin N(i, g)$ set $d(i, j; g) = +\infty$. Given an agent $i$ in any network $g$ the agent who is furthest away from $i$ among those he observes is $j := \arg\max_{\ell \in N(i,g)} d(i, \ell; g)$. Note that $i$ does not need any links of $j$ to observe any one in $N(i, g)$, otherwise $j$ would not be furthest away from $i$. For all $g$ in $\mathcal{G}$ the set of all networks that are the same as $g$ up to a permutation of the indexes are called the set of *architectures* equivalent to $g$. Finally define a *network component* as a subgraph consisting only of agents that have links to agents belonging to the same component and who are not observed by any other agent.

Let us now give the definition of some special network components.

**Definition 1** *A network component is called a* <u>*wheel*</u> *of dimension* $n$ *if there exists* $k_1, .., k_n$ *with* $\{k_1, .., k_n\} \subset P$ *such that* $g_{k_i k_{i+1}} = 1$ *for* $i = 1, ..., n-1$, $g_{k_n, k_1} = 1$ *and* $g_{r,s} = 0$ *otherwise.*
*The set of wheels of dimension* $n$ *is denoted by* $W(n)$.

The figure below depicts a wheel with 6 agents.

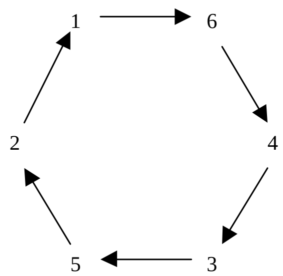


**Figure 2: A wheel of dimension 6**

In a wheel of dimension $n$ the payoff of each agent belonging to this wheel equals $n - c - \alpha n^2$.

Another important network component for the analysis of this paper is the starred wheel. A starred wheel of dimensions $n$ and $m$ consists of $n + m$ agents, such that $n$ agents are connected in a wheel, with all the further $m$ agents being directly connected to the central wheel. The $n$ agents who form the wheel are called the *central agents* and the other $m$ are the *peripheral agents*.

**Definition 2** *A* <u>*starred wheel*</u> *of dimensions* $n$ *and* $m$ *is a network component connecting* $n + m$ *agents characterised by the following conditions:*
*1. each agent only sponsors one link, i.e.* $\mu_i^d(g) = 1$ *for all* $i$*;*
*2. there exists a permutation of* $n$ *agents* $k_1, ..., k_n$ *such that* $g_{k_i k_{i+1}} = 1$ *for* $i = 1, ..., n-1$ *and* $g_{k_n, k_1} = 1$*;*
*3. for each* $j \notin \{k_1, ..., k_n\}$ *there exists* $i \in \{1, ..., n\}$ *such that* $g_{j,k_i} = 1$.
*The* <u>*set of starred wheels*</u> *of dimensions* $n$ *and* $m$ *is denoted by* $SW(n, m)$.

We say a *starred wheel of dimension* $n$ (omitting the number of peripheral agents) when it only is important to stress the number of agents forming the central wheel.

**Definition 3** *A constellation of starred wheels of dimension* $m$ *is a network which can be partitioned into components each of which is a (starred) wheel of dimension* $m$*.*

Recall that the *floor* of $x$ is indicated with $\lfloor x \rfloor$ and is defined as the smallest integer greater than or equal to $x$, i.e. $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$ for all $x \in \mathbb{R}$. Note that a constellation of starred wheels of dimension $m$ can be made of a number of starred wheels ranging from 1 to $\left\lfloor \frac{P}{m} \right\rfloor$ and is obtainable from any permutation of the agents provided that: i) in the network there are only starred or simple wheels; ii) each wheel has dimension $m$; iii) all the $P$ agents are linked.

Figure 3 represents a starred wheel with 4 central agents and 3 peripheral ones.

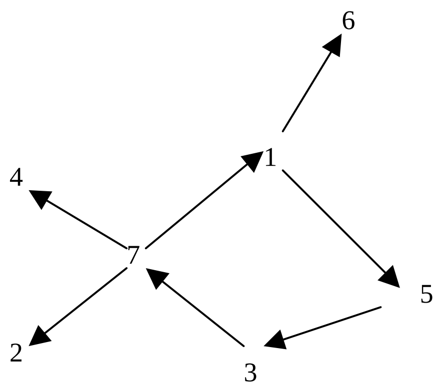


**Figure 3: A starred wheel of dimensions 4 and 3**

Let us finally define two special roles that an agent can play in a network $g$: the stand-alone and the terminal. A stand-alone is an agent who does not have any links and is not observed by anyone in the network. A terminal does not have any links, yet he is observed by someone else in the network.

**Definition 4** *Agent $i$ is a stand-alone if $N(i,g)=\{i\}$ and $i \notin N(j,g)$ for all $j \in P\backslash\{i\}$.*
*Agent $i$ is a terminal if $N(i,g)=\{i\}$ but $i$ is not a stand-alone, i.e. there exists $k \in P\backslash\{i\}$ such that $i \in N(k,g)$.*

Both stand-alones and terminals receive the same payoff: $1-\alpha$.

# 3 Preliminary Properties

Let us exclude the (zero-measure) case that there exists an integer $\ell$ such that $\alpha = \frac{1}{2\ell+1}$, i.e. $\alpha$ is the inverse of an even number. Define then $n^*$ as the integer that is closest to $\frac{1}{2\alpha}$. Formally $n^* \in \mathbb{N}$ such that

$$\left|n^* - \tfrac{1}{2\alpha}\right| < \tfrac{1}{2}$$

The restriction on the values of the parameter $\alpha$ guarantees that $n^*$ is unique. Further define

$$\underline{n} := \left\lceil \tfrac{1-\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} \right\rceil$$

where $\lceil x \rceil$ is the *ceiling* of $x$ that is $\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$ for all $x \in \mathbb{R}$ and

$$\overline{n} := \left\lfloor \tfrac{1+\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} \right\rfloor$$

where $\lfloor x \rfloor$ is the *floor* of $x$ defined earlier.

Lemma 4 in Appendix A demonstrates that $\underline{n}$ and $\overline{n}$ are well defined and $0 < \underline{n} < \overline{n}$, $0 < \underline{n} \leq n^*$ and $n^* \leq \overline{n}$ for all $\alpha$ and $c$ in $\mathcal{R}$. The following lemma compares the payoff of a stand-alone with that of an agent connected in a network establishing the incentives to link. The proof is in Appendix A.

**Lemma 1** *If $(\alpha, c) \in \mathcal{R}$ then there exist a network $g$ such that $\pi_i(g) > 1-\alpha$ for some $i \in P$.*

Lemma 1 shows that agents have incentives to connect for parameter values in the set $\mathcal{R}$. Otherwise stand-alones may receive a higher payoff than connected agents.

**Lemma 2** *Let* $(\alpha, c) \in \mathcal{R}$ *and* $m \in \mathbb{N}$*. Then* $m - c - \alpha m^2 \geq 1 - \alpha$ *if and only if* $m = \{\underline{n}, ..., \overline{n}\}$*.*

The proof is given in Appendix A. Lemma 2 also offers an intuitive interpretation of the thresholds $\underline{n}$ and $\overline{n}$: $\underline{n}$ and $\overline{n}$ represent respectively the dimension of the smallest (profitable) wheel and of the largest (profitable) wheel which any member has no incentives to break.

**Lemma 3** *The payoff of an individual* $i \in P$ *is maximal if he belongs to a wheel of dimension* $n^*$ *formally* $W(n^*) \subset \{g | \arg\max_g \pi_i(g)\}$*.*

**Proof.** Note that the payoff of each agent is a decreasing function of the number of links he builds. Consider agent $i$. It is always payoff improving to observe the same number of agents with less links as $\pi_i(g) \leq \|N(i,g)\| - c - \alpha \|N(i,g)\|^2$. If agent $i$ has only one link and observes $m$ agents then $i$'s payoff is the one he would get in a wheel of dimension $m$, i.e. $\pi_i(g) = m - c - \alpha m^2$. So we can restrict attention to wheel network components. Let us extend the (wheel) payoff function to the real line: i.e. $\varphi(\gamma) := \gamma - c - \alpha\gamma^2$. It can be easily shown that the maximum of $\varphi(\gamma)$ is attained for $\gamma = \gamma^* := \frac{1}{2\alpha}$. The function $\varphi(\gamma)$ is symmetric about the axis $\gamma = \gamma^*$, as $\varphi(\gamma) = \frac{1}{4\alpha} - c - \alpha(\gamma - \gamma^*)^2$. Hence the payoff function (1) is maximised in a wheel of dimension $n^*$ agents where $n^* \in \mathbb{N}$ solves $|n^* - \gamma^*| \leq \frac{1}{2}$. ∎

Notice that the central agents of a starred wheel of dimension $n^*$ enjoy the maximum payoff. The peripheral ones observe $n^* + 1$ agents with one single link: they receive the payoff of an agent who belongs to a $W(n^* + 1)$. Lemma 3 also shows that the decreasing returns to linking equal the marginal benefit of observing one additional agent for $\mu_i(g) = n^*$.

Let us now define a Nash network and a strict Nash network. These are the networks generated by strategy profiles that respectively constitutes a Nash equilibrium and a strict Nash equilibrium of the linking game. Formally:

**Definition 5** *A network $g^*$ is a* <u>*Nash network*</u> *if*

$$\pi_i\left(g^*\right) \geq \pi\left(g_i' \oplus g_{-i}^*\right) \tag{3}$$

*for all $g_i' \in \mathcal{G}_i$ and all $i \in P$.*
*A Nash network $g^*$ is a* <u>*strict Nash network*</u> *if equality in equation (3) implies $g_i' = g_i^*$ for any agent $i$ in the population.*

To illustrate the above definitions consider a population of 9 agents. Set $\alpha = \frac{1}{10}$ and $c = \frac{3}{5}$, so that $(\alpha, c) \in \mathcal{R}$. Then we obtain that $n^* = 5$, $\underline{n} = 2$ and $\overline{n} = 8$. The following figure depicts a Nash equilibrium network which is not a strict Nash network.

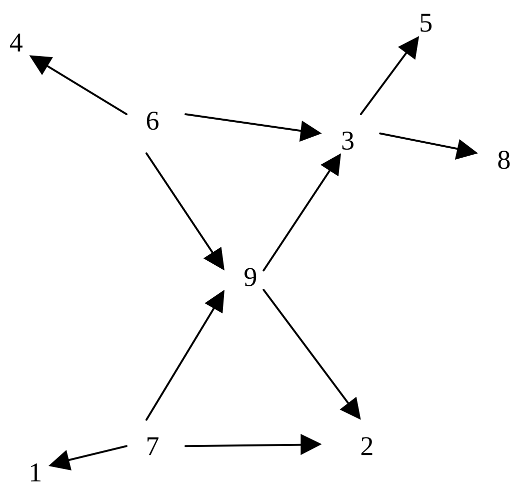


**Figure 4: A Nash equilibrium network**
**for $P = 9$, $\alpha = \frac{1}{10}$ and $c = \frac{3}{5}$**

For instance agent 2 is indifferent between connecting to 9 (as depicted) or to 6 and player 3 is indifferent between having a link to 9 (as in the figure) or to 7.

In the rest of the paper we use the following set-wise solution concept.

**Definition 6** *A non-empty set of pure strategy combinations $\mathcal{B} \subseteq \mathcal{G}_1 \times \ldots \times \mathcal{G}_P$ is a pure-strategy strict Nash equilibrium set (PSNES) if for every strategy profile $\sigma \in \mathcal{B}$, for all $i \in P$ and every $g_i \in \mathcal{G}_i$*

$$\pi_i(\sigma) \geq \pi_i(g_i \oplus \sigma_{-i})$$

*where equality implies $(g_i \oplus \sigma_{-i}) \in \mathcal{B}$.*

This is the restriction to pure strategies of the concept of strict equilibrium set (Balkenborg [3]).

# 4 Dynamic Analysis

Building on BG and on Ritzberger and Weibull [24], we now consider the dynamics induced when in each round a single random agent is selected who then chooses among the strategies that make him better off given that the others do not change their strategy. Formally,

**Definition 7** *In a network $g = g_i \oplus g_{-i}$ $g_i'$ is a better response to $g_{-i}$ than $g_i$ for $i$ if $\pi_i(g_i' \oplus g_{-i}) \geq \pi_i(g)$.*
*The set of all agent $i$'s better responses to $g_{-i}$ is defined as $\beta_i(g_{-i})$.*

In any time period agents observe the network built in the previous periods. With positive independent probability $\gamma_i > 0$ each agent will exhibit "inertia", i.e. will maintain the strategy played in the previous period. With the complementary probability $1-\gamma_i > 0$ the agent will play a better response to the current network. This induces the better reply dynamics introduced by Ritzberger and Weibull [24] and defined below

$$g_i^{t+1} = \begin{cases} g_i' \in \beta_i\left(g_{-i}^t\right) & \text{with probability} \quad 1-\gamma_i \\ g_i^t & \text{with probability} \quad \gamma_i \end{cases} \tag{4}$$

for all agents in the population.[7]

[7] See also Maynard Smith and Price [22].

A limit network of the better reply dynamics (4) can be a steady state or belong to an absorbing set.

**Definition 8** *A network $\hat{g}$ is steady state of the better reply dynamics (4) if $g_i^t = \hat{g}_i$ implies that $g_i^{t+1} = \hat{g}_i$ for all $i \in P$.*

**Definition 9** *A subset $\mathcal{A} \subset \mathcal{G}_1 \times \ldots \times \mathcal{G}_P$ is an absorbing set of the better reply dynamics (4) if $g^t \in \mathcal{A}$ implies $g^{t+1} \in \mathcal{A}$.*

For the sake of simplicity we restrict to the case in which agents face strong incentives to connect. Specifically this restriction is used in the following remark.

**ASSUMPTION:** *For $(\alpha, c) \in \mathcal{R}$, assume further that $0 < c < 1 - 3\alpha$.*

The parameter space used in the rest of the paper is thus given by

$$\begin{aligned} \mathcal{P} \;\; &= \;\; \Big\{(\alpha, c) \in \mathbb{R}^2 \,\Big|\, 0 < c \leq 1 - 3\alpha \text{ for } \alpha \in \big(0, \tfrac{1}{6}\big] \\ &\qquad \text{and } 0 < c \leq \tfrac{1}{4\alpha} - 1 \text{ for } \alpha \in \big(\tfrac{1}{6}, \tfrac{1}{4}\big)\Big\} \end{aligned}$$

**Remark 1** *If $(\alpha, c) \in \mathcal{P}$ then $\underline{n} = 2$ which means that no agent has an incentive to cut all his links in a network in which he observes 2 agents.*

The proof is given in Appendix A.

Combining the previous remark with Lemma 2 we observe that it is never payoff maximising to have no links at all if it is possible to observe no more than $\overline{n}$ agents with one link.

In the following we first establish results for the case $n^* \leq \overline{n} - 1$ and a large and non-pathological population size. Proposition 3 deals with these special cases.

**Proposition 1** *Assume that* $P > n^*$*, that* $(\alpha, c) \in \mathcal{P} \setminus \left\{(\alpha, c) : \frac{1}{6} \leq \alpha < \frac{1}{5},\ 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1\right\}$ *and in addition if* $n^* > \frac{1}{2\alpha}$ *that there exists no integer* $k$ *such that* $P = (k+1)\, n^* - 1$*. Then a constellation of starred wheels of dimension* $n^*$ *is a PSNES.*

**Proof.** Consider a constellation of starred wheels of dimension $n^*$. Given the $n^* + m$ agents who form one $SW(n^*, m)$, none of them wants to individually deviate in a way that alters the starred wheel architecture. Consider first the $n^*$ agents who form the $W(n^*)$. They obtain the maximum payoff since they observe $n^*$ agents and only pay for one link. So they have no incentive to deviate. Let us now consider the $m$ peripheral agents who are linked to the wheel. None of them can improve his payoff: if one of them cuts his link and links somewhere else to the wheel neither his payoff nor the architecture change. If he links to someone else who is directly linked to the $W(n^*)$ his payoff reduces since now this agent observes $n^* + 2$. If a peripheral agent links somewhere outside the starred wheel then the starred wheel still exists. He can only increase his payoff linking (with only one link) to someone who observes $n^* - 1$. This is impossible since the original network was a constellation of starred wheels of dimension $n^*$. So a constellation of starred wheels of dimension $n^*$ is a PSNES because every time agents deviate in a way that the resulting architecture is not a constellation of $SN(n^*, m)$ these agents are worse off and agents are indifferent only among strategies that do not alter the architecture. ■

Let us now analyse the better reply dynamics. The following proposition proves that the dynamics always converges to a constellation of starred wheels of dimension $n^*$ in finite time.

**Proposition 2** *Assume that* $P > n^*$*, that* $(\alpha, c) \in \mathcal{P} \setminus \left\{(\alpha, c) : \frac{1}{6} \leq \alpha < \frac{1}{5},\ 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1\right\}$ *and in addition if* $n^* > \frac{1}{2\alpha}$ *that there exists no integer* $k$ *such that* $P = (k+1)\, n^* - 1$*. Then in finite time each agent belongs to a starred wheel of dimension* $n^*$*. From then on, while the network might change, it remains a constellation of starred-wheels of dimension* $n^*$ *in each period.*

The proof is given in the Appendix B and contains 6 steps. Starting from an arbitrary network we first show that in finite time there will be no connected agents who observe less than $\underline{n}$ or more than $\overline{n}$ agents (Step 1). Secondly we prove that all the agents in the network either observe someone or they are stand-alones, so in finite time terminals connect to someone (Step 2). Thirdly also stand-alones have an incentive to join in the network (Step 3). Hence in finite time the network is such that all the agents observe a number of agents between $\underline{n}$ and $\overline{n}$. We then show that starting from such a network in finite time (at least) one agent gets to observe $n^*$ agents (Steps 4). Each time someone observes $n^*$ a starred wheel of dimension $n^*$ arises (Step 5). The final step of proof shows that the absorbing set of the better reply dynamics is a constellation of starred wheels of dimension $n^*$.

Note that the actual number of starred wheels of dimension $n^*$ that arise in the limit state of the dynamics is indeterminate. During the dynamic process a peripheral agent of a starred wheel might sever his link and join another subset of agents if by so doing he gets to observe $n^*$ agents. Then by Proposition 2 the process that leads to the formation of a new starred wheel begins. This implies that the better reply dynamics (4) does not converge to a steady state but that a constellation of starred wheels of dimension $n^*$ is an absorbing state of the better reply dynamics (4). Notice that BG assume that $\alpha = 0$ and $c \in [0, 1]$.[8] In our set-up this implies that $\underline{n} = 1$ and $n^* = +\infty$. Thus they obtain a global wheel which is a limit case of the absorbing states found in the above proposition.

Let us now consider the three cases of a pathological population size, of $n^* = \overline{n}$ and of a small population. If $n^* > \frac{1}{2\alpha}$ and there exists an integer $k$ such that $P = (k+1)\, n^* - 1$ or if $(\alpha, c) \in \left\{(\alpha, c) : 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1, \frac{1}{6} \leq \alpha < \frac{1}{5}\right\}$ or if $P < n^*$ then the steady state architectures are as shown by the following proposition.

**Proposition 3** *i) Let $P$ be such that there exist an integer $k$ such that $P = (k+1)\, n^* - 1$ and $n^* > \frac{1}{2\alpha}$. Then with positive probability the better reply dynam-*

[8] BG, Theorem 3.1, Part a), p. 1197.

*ics settles in finite time to a network consisting of $k$ wheels of dimension $n^*$ and one wheel of dimension $n^* - 1$, which is a steady state.*

*ii) Let $(\alpha, c) \in \left\{(\alpha, c) : 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1, \frac{1}{6} \leq \alpha < \frac{1}{5}\right\}$. If there exist an integer $h$ such that $P = 3h + 1$ then the unique steady state of the better reply dynamics is a set of $h$ wheels $W(3)$ and the remaining agent is a stand-alone; if $P = 3h + 2$ the unique steady state is a set of $h$ wheels $W(3)$ and the remaining 2 agents form a $W(2)$; and if $P = 3h$ the unique steady state is a set of $h$ wheels $W(3)$.*

*iii) Let $P < n^*$ then in finite time the agents form one wheel of dimension $P$ which is a steady state of the better reply dynamics.*

**Proof.** Part i): assume there are $k$ players who observe $k$ disjoint groups of $n^*$ agents. This is an event that happens with positive probability. Then $k$ wheels of dimension $n^*$ surely arise (applying Step 5.a of the proof of Proposition 2). Assume that the remaining $(n^* - 1)$ agents are linked to each other only. Then they have the choice between forming their own wheel of dimension $(n^* - 1)$ or linking (from outside) to the existing wheels of dimension $n^*$. Since $n^* > \frac{1}{2\alpha}$, the payoff of a wheel of dimension $(n^* - 1)$ is higher than that one can get by linking to wheel of dimension $n^*$. Hence they form a wheel $W(n^* - 1)$.

Part ii) It can be easily verified that the assumptions on $\alpha$ and $c$ are equivalent to the following values for $n^*$, $\overline{n}$ and $\underline{n}$: $n^* = \overline{n} = 3$ and $\underline{n} = 2$ which is the only possible case in which $n^* = \overline{n}$ for $(\alpha, c) \in \mathcal{P}$ . We can apply Steps 1, 2, 4 and 5 a. of the proof of Proposition 2 to show that at least one $W(3)$ forms. Once all the $W(3)$ formed, there will be either 1 or 2 or no remainders. If there are 2 remainders they will form a $W(2)$ as $\underline{n} = 2$.

Part iii): Take the agent who is observing the largest number of agents and call him $i_1$. Now either $i_1$ observes the whole population or there exists someone who is not observed by $i_1$. In the first case apply replacing $n^*$ with $P$, the argument of Step 5.a of Proposition 2 to show that one simple wheel of dimension $P$ forms. Otherwise take among the agents not observed by $i_1$ the one who observes the most agents and call him $j$. For him it is payoff improving to cut all his link(s) and link to $i_1$ directly. So $j$ does and now $j$ is the one who observes most agents in $P$. Repeat the argument until one agent observes all the $P$ agents. Then apply the argument of Step 5.a. of Proposition 2 (replacing $n^*$ with $P$). ■

The case of Part iii) is not surprising since it is intuitive that every agent is connected and observes everyone else when the population is small. This means that a small population size prevents the decreasing returns to linking to prevail on the marginal benefit of observing one additional agent.

Let us concentrate on the absorbing architecture found in Proposition 2. Define now $r, q \in \mathbb{N}$ such that

$$P = q\,n^* + r \qquad \text{for} \quad 0 \leq r < n^* \tag{5}$$

so $q = \left\lfloor \frac{P}{n^*} \right\rfloor$, i.e. $q$ is the maximum number of starred wheel that can arise. Define the *aggregate payoff of the population* as the sum of the payoff of each agent. A network $g$ is called a *Pareto efficient* architecture if it is impossible to increase the payoff of any agent without reducing the payoffs of others.

**Corollary 1** *Assume that* $P > n^*$, *that* $(\alpha, c) \in \mathcal{P} \backslash \left\{(\alpha, c) : \frac{1}{6} \leq \alpha < \frac{1}{5},\right.$ $\left. 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1\right\}$ *and in addition if* $n^* > \frac{1}{2\alpha}$ *that there exists no integer* $k$ *such that* $P = (k+1)\,n^* - 1$. *Consider the absorbing set found in Proposition 2: the aggregate payoff of the population increases with the number of starred wheels in the limit architecture. The* Pareto efficient *architecture consists of* $q$ *starred wheels of dimension* $n^*$ *with the remaining* $r$ *agents being linked from outside to these wheels, where* $q$ *and* $r$ *are defined in equation (5).*

**Proof.** The first part of the statement is easily verified. The aggregate payoff of the population increases with the number of starred wheels as central agents in a starred wheel of dimension $n^*$ enjoy the maximum payoff and their number increases of with the number of starred wheels in the limit architecture.

Let us now prove the second part. Given a constellation of $q$ starred wheels of dimension $n^*$ it is impossible to increase the payoff of any agent without reducing that of another one. The $n^* \times q$ central agents enjoy the maximum payoff attainable (by Lemma 3) so there is no way to improve it. Peripheral agents observe $n^* + 1$ agents with one single link and receive a lower payoff, that of the members of a $W(n^* + 1)$. The only way they can improve the payoff is to reduce by one unit the number of agents observed. This is impossible because the peripheral players are too few ($r < n^*$ by equation 5) to set up a $W(n^*)$ on their own without reducing the payoff of some central agents. ■

Notice that the aggregate payoff of the population can be interpreted as the *Social Welfare*.

# 5 Conclusion

In this paper the effects of network maintenance in the process of network formation are studied as a variant of the model by Bala and Goyal [1]. Network maintenance which is related to network size affects the payoffs of the agents in a way that dramatically changes the steady state predictions with respect to the original. The presence of the maintenance cost implies that there are decreasing returns to linking. The agents who are assumed homogenous have to trade off not only the number of links they sponsor with the benefit of observing the others. They also have to consider that larger networks are proportionally more expensive to maintain than smaller ones. Strategy revision occurs by better response.

In this model the dynamics converges in finite time. Absorbing states are a constellation of disjoint starred wheels, where core agents are linked in the optimally-sized wheel and peripheral agents link to the wheel from outside. Similar architectures of links among economic agents are found in Taiwanese industrial districts (Lee [19]).

Examples of disjoint networks are much more common in the real world than global networks. As an illustration consider the furniture industry in Italy: it is localised in nine major industrial districts which are not located close to each other, rather than being agglomerated in one single location. Similarly there are eight major industrial districts in Italy in the textile and apparel sector and nine in the leather goods sector[9]. The result that social welfare increases with network fragmentation may explain this observation.

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# A Miscellaneous Preliminary Results

**Lemma 4** *For all $(\alpha, c)$ in the set $\mathcal{R}$ we find that $\underline{n} < \overline{n}$, $\underline{n} \leq n^*$ and $n^* \leq \overline{n}$.*

**Proof.** Recall that $n^*$ is defined as the integer that is closest to $\frac{1}{2\alpha}$. Formally, $n^* \in \mathbb{N}$ such that

$$\left|n^* - \tfrac{1}{2\alpha}\right| < \tfrac{1}{2}$$

which is positive and well-defined for all positive $\alpha$ as we excluded the zero-measure case that there exists an integer $\ell$ such that $\alpha = \frac{1}{2\ell+1}$, i.e. that $\alpha$ is the inverse of an even number. Moreover recall that

$$\underline{n} := \left\lceil \tfrac{1-\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} \right\rceil$$

and

$$\overline{n} := \left\lfloor \tfrac{1+\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} \right\rfloor$$

The thresholds $\underline{n}$ and $\overline{n}$ are well-defined for $\alpha > 0$ and $c \in \left[0, \frac{1}{4\alpha} + \alpha - 1\right]$. Also note that both $\underline{n}$ and $\overline{n}$ are non-negative, as $\frac{1-\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} > 0$ if and only if $c + 1 > \alpha$ -which is always true- and also $\frac{1+\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} > 0$.

Finally $\underline{n} < \overline{n}$, $\underline{n} \leq n^*$ and $n^* \leq \overline{n}$ is guaranteed by

$$\tfrac{1+\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} - \tfrac{1-\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} \geq 2$$

which holds as $(\alpha, c) \in \mathcal{R}$. This concludes the proof. ■

**Lemma 1** *If $(\alpha, c) \in \mathcal{R}$ then there exist a network $g$ such that $\pi_i(g) > 1 - \alpha$ for some $i \in P$.*

**Proof.** Let a network $g$ contain a wheel of dimension $m$, i.e. in $g$ there exists a subset $M$ of the population such that the agent in $M$ form a wheel $W(m)$. Consider an agent $i$ who belongs to this wheel. Agent $i$ receives a payoff $\pi_i(g) = m - c - \alpha m^2$ in the network $g$. Recall that stand-alones receive a payoff equal to $1 - \alpha$.

It can be shown that $-m^2\alpha + m + \alpha - c - 1 > 0$ for some $m$ if $(\alpha, c) \in \mathcal{R}$. Let $\phi(\gamma) := -\alpha\gamma^2 + \gamma + \alpha - 1 - c$. Consider the equation $\phi(\gamma) = 0$. Its roots are $\gamma_{1,2} := \frac{1 \mp \sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha}$. The roots $\gamma_1$ and $\gamma_2$ are real number only if $1 - 4\alpha(c + 1 - \alpha) \geq 0$, i.e. $c \leq \frac{1}{4\alpha} - 1 + \alpha$. If $1 - 4\alpha(c + 1 - \alpha) = 0$ then $\phi(\gamma) = 0$ for $\gamma = \frac{1}{2\alpha}$ and $\phi(\gamma) < 0$ otherwise. So $1 - 4\alpha(c + 1 - \alpha) > 0$ which means that the parameters belong to the set $\mathcal{R}$ defined in equation (2) above. The statement follows as $\alpha > 0$ and hence $-m^2\alpha + m - c > 1 - \alpha$ if and only if $\gamma_1 < m < \gamma_2$. Notice that Lemma 4 above guarantees the existence of such $m$. ■

**Lemma 2** *Let* $(\alpha, c) \in \mathcal{R}$ and $m \in \mathbb{N}$. *Then* $m - c - \alpha m^2 \geq 1 - \alpha$ *if and only if* $m \in \{\underline{n}, ..., \overline{n}\}$.

**Proof.** Recall that $\underline{n} = \lceil \gamma_1 \rceil$ and $\overline{n} = \lfloor \gamma_2 \rfloor$ and that $\gamma_1$ and $\gamma_2$ are the solutions of the equation $-\alpha\gamma^2 + \gamma + \alpha - 1 - c = 0$. So $m - c + \alpha m^2 \geq 1 - \alpha$ for all $m = \underline{n}, ..., \overline{n}$ and $m - c + \alpha m^2 < 1 - \alpha$ for $m < \underline{n}$ and $m > \overline{n}$. ■

**Remark 1** *If* $(\alpha, c) \in \mathcal{P}$ *then* $\underline{n} = 2$ *which means that no agent has an incentive to cut all his links in a network in which he observes 2 agents.*

**Proof.** Recall the definition of the set $\mathcal{P}$

$$\begin{aligned} \mathcal{P} &= \Big\{(\alpha, c) \in \mathbb{R}^2 \,\Big|\, 0 < c \leq 1 - 3\alpha \text{ for } \alpha \in \left(0, \tfrac{1}{6}\right] \\ &\quad \text{and } 0 < c \leq \tfrac{1}{4\alpha} - 1 \text{ for } \alpha \in \left(\tfrac{1}{6}, \tfrac{1}{4}\right)\Big\} \end{aligned}$$

Notice that $\frac{1}{4\alpha} - 1 < 1 - 3\alpha$ for $\alpha \in \left(\frac{1}{6}, \frac{1}{4}\right)$.

Figure A1 below plots the set $\mathcal{P}$.

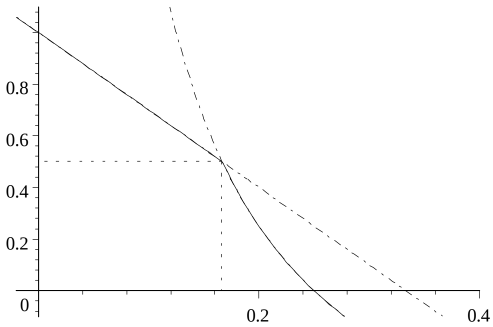


**Figure A1: The parameter space $\mathcal{P}$**

Recall that $\underline{n} = \lceil \gamma_1 \rceil$. So it is enough to verify that $1 < \gamma_1 \leq 2$. Note that

$$\begin{gathered} \gamma_1 = \tfrac{1-\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} > 1 \\ \text{as } (1-2\alpha)^2 > 1 - 4\alpha + 4\alpha^2 - 4\alpha c \\ \text{as } 0 > -c \end{gathered}$$

Now verify that $\gamma_1$ is smaller than 2:

$$\begin{gathered} \gamma_1 = \tfrac{1-\sqrt{1-4\alpha(c+1-\alpha)}}{2\alpha} \leq 2 \\ \text{if and only if } 1 - 4\alpha < \sqrt{1 - 4\alpha\,(c + 1 - \alpha)} \end{gathered}$$

As $\alpha < \frac{1}{4}$ then $1 - 4\alpha > 0$. So $\gamma_1 \leq 2$ if and only if $(1-4\alpha)^2 \leq 1 - 4\alpha\,(c + 1 - \alpha)$, i.e. $c \leq 1 - 3\alpha$ which holds for all $\alpha, c \in \mathcal{P}$. So $\underline{n} = 2$ for all $\alpha, c \in \mathcal{P}$ ■

# B Proof of the Main Results

To prove Proposition 2 we need to establish some preliminary results. The Lemmas in the next subsection present some relevant cases in which the better response dynamics (4) leads to the emergence of a wheel or of an agent who observes $n^*$ players. These are building blocks for the proof of the proposition as we will show that every time an agent observes $n^*$ players a starred wheel of dimension $n^*$ arises.

## B.1 Preparatory Lemmas

Lemma 5 is of some independent interest as it deals with the case of (simple) wheel formation. It is sufficient that an agent is better off than all those he observes for a wheel to emerge among these agents.

**Lemma 5** *Let agent $i_1$ be such that $\pi_{i_1} \geq \pi_j$ for all $j \in N(i_1, g)$, i.e. $i_1$ is best off among those he observes. Let $m := \|N(i_1, g)\|$ so $m$ is the number of agents observed by $i_1$. If $m > 1$ then in finite time a wheel $W(m)$ arises among the agents originally in $N(i_1, g)$.*

**Proof.** Let $M := N(i_1, g)$. Consider $i_2 \in \arg\max_{\ell \in M} d(i_1, \ell; g)$ so $i_2$ is furthest away from $i_1$ among those who are observed by $i_1$. Note that $i_1$ does not use any of $i_2$'s links to observe anyone else. Agent $i_2$ can improve his payoff by cutting all his links and linking to $i_1$ directly since

$$\pi_{i_2}(g) \leq \pi_{i_1}(g) \leq m - c - \alpha m^2$$

as $i_2 \in N(i_1, g)$. So he does. Call this new network $g^{(2)}$.

Note that now $i_2$ observes all the agents in $M$ (so $m$) with one single link and his payoff is exactly $m - c - \alpha m^2$ which is the new maximum payoff in $M$. Moreover all the agents who observe $i_2$ now observe all the agents in $M$.

We proceed by induction. Assume we have done the first $\ell - 1$ iterations. Call agent $i_\ell$ the player who moved in iteration $\ell - 1$ for $\ell > 1$. Let $g^{(\ell)}$ be the network formed at the end of iteration $\ell$, i.e. after $i_\ell$ moved. Notice that all the agents who moved in the previous round are consecutively linked to each other, i.e. $i_s$ has one single link to agent $i_{s-1}$ for $s = 2, \ldots, \ell$. So the distance between $i_\ell$ and $i_1$ in the network $g^{(\ell)}$ is $\ell - 1$ for $\ell > 1$. Moreover $\left\| N\left(i_s, g^{(\ell)}\right) \right\| = m$ for $s = 1, \ldots, \ell$ and $\ell > 1$. Consider now $i_{\ell+1} \in \arg\max_{j \in M} d\left(i_\ell, j; g^{(\ell)}\right)$ so $i_{\ell+1}$ is furthest away from $i_\ell$. By the reasoning applied to $i_2$ agent $i_{\ell+1}$ has a better reply of cutting all his links and linking to $i_\ell$ directly.

We now show that eventually $i_1$ is selected, so that a wheel $W(m)$ forms. Consider the iterative procedure above. Since $i_h \in M$ and $\left\| N\left(i_h, g^{(h)}\right) \right\| = m$ for all $h > 2$ eventually $i_1$ is selected as $M$ is finite and the distance between $i_h$ and $i_1$ increases by 1 in each step. Let $i_k$ be the last agent who moves before $i_1$ is selected. Notice that $i_k$ observes $i_1$ through a path consisting of all the agents who moved before him. As $i_k$ observes $m$ agents and $i_1$ is the furthest away from $i_k$ it follows that $k = m$.

Notice now that $i_1$ is the only agent in $M$ who can possibly have more than one link. If so then let $i_1$ play. By the same reasoning as $i_2$ agent $i_1$ has a better response to cut all his links and link to $i_m$. Now $i_1$ observes $m$ agents with one single link and closes a wheel of dimension $m$ among the agents $i_1, \ldots, i_m$ with possibly other players observing them. ■

In the proof of Proposition 2 we find a path of the better reply dynamics that leads to the formation of a constellation of starred wheels once at least one agent in the population observes exactly $n^*$ players. The next Lemmas consider first the case of a highly fragmented network and then of a highly connected one. In particular Lemma 6 assumes that there exist more than $n^*$ agents who observe less than $n^*$ and shows that one of them will observe $n^*$ agents in finite time.

**Lemma 6** *If there exist more than $n^*$ agents each of whom observes between $\underline{n}$ and $n^* - 1$ agents then in finite time at least one agent who observes exactly $n^*$ agents will arise, i.e. the better response dynamics $g^t$ is such that for some finite $t'$ there exist an agent $j$ for whom $\left\| N\left(j, g^{t'}\right) \right\| = n^*$.*

**Proof.** Consider the assumptions of the Lemma. Call $H$ the set of agents each of whom observes between $\underline{n}$ and $n^* - 1$ agents in the network $g$. Notice that $\|H\| \geq n^* + 1$. Define $\hat{P}$ as the subset of the population consisting of all the agents observed by some $h \in H$, formally $\hat{P} = \bigcup_{h \in H} N(h, g)$. Among the agents belonging to $\hat{P}$ let $i_1$ be an agent who observes the most agents, i.e. $i_1 \in \arg\max_{j \in \hat{P}} \|N(j, g)\|$. Let $m_1 := \|N(i_1, g)\|$. Note that $m_1 \leq n^* - 1$.

Consider $i_2 \in \arg\max_{\ell \in N(i_1, g)} d(i_1, \ell; g)$ so $i_2$ is furthest away from $i_1$ among those who are observed by $i_1$. Recall that $i_1$ does not use any of $i_2$'s links to observe anyone else. Agent $i_2$ can improve his payoff by cutting all his links and linking to $i_1$ directly since

$$\pi_{i_2}(g) \leq m_1 - c - \alpha m_1^2$$

as $i_2 \in N(i_1, g)$. So he does and receives a payoff exactly equal to $m_1 - c - \alpha m_1^2$.

Call this new network $g^0$. Note that $N(i_2, g^0) = N(i_1, g)$ and that $i_2$ is the best off among the agents he observes. So by Lemma 5 the agents observed by $i_2$ form a $W(m_1)$.

Call this new network $g'$. Assume there exist a $k \in \hat{P} \backslash N(i_j, g')$ for some agent $i_j$ in the $W(m_1)$ such that $d(k, i_j; g') = 1$, i.e. agent $k$ is directly linked the wheel without belonging to it. As $m_1 \leq n^* - 1$, agent $i_{j+1}$ in the $W(m_1)$ whose only link is to $i_j$ has a better response to cut his link and link to $k$. This makes a wheel of dimension $m_1 + 1$. Repeat until an agent gets to observe $n^*$ (which completes the proof) or no such $k$ outside the wheel exists anymore.

In the letter case call the new network $\tilde{g}$ and $m_2$ the dimension of the (enlarged) wheel. So $m_2 \geq m_1$. Note that all the agents in $\hat{P}$ either belong to the wheel $W(m_2)$ or are not in the wheel and hence observe less than or equal to $m_1$ agents. Call $L$ the subset of $\hat{P}$ of agents who do not belong to the wheel

So $\hat{P}$ is partitioned into the agents belonging to the wheel $W(m_2)$ and those in $L$. Each agent $\ell \in L$ has a better response to cut all his links and link directly to someone who belongs to the wheel as $\|N(\ell, \tilde{g})\| \leq m_2 \leq n^* - 1$. By so doing all the agents in $L$ directly link to the wheel without belonging to it. So we re-apply the argument developed above for an agent $k$ who is linked the wheel directly without belonging to it. Note that at some point an agent observes $n^*$ players as $\left\|\hat{P}\right\| > n^*$ by assumption of the Lemma. This completes the proof. ■

In the next Lemma we consider the case of a network which is so highly connected that there exists a subset of the population such that all the agent in this subset observe more than $n^*$ agents and all the agents observed by them also observe more than $n^*$. We then show that in this case an agent observing exactly $n^*$ will emerge.

**Lemma 7** *Assume there exist an agent $k$ such that $n^* + 1 \leq \|N(\ell, g)\| \leq \overline{n}$ for all $\ell \in N(k, g)$, i.e. such that all the agents observed by him observe between $n^* + 1$ and $\overline{n}$ agents. Then in finite time at least one agent who observes exactly $n^*$ agents will arise.*

**Proof.** Let $\hat{P} = \bigcup_{\ell \in N(k,g)} N(\ell, g)$. Note that $\left\|\hat{P}\right\| \geq n^* + 1$ since $N(i, g) \geq n^* + 1$ for all $i \in \hat{P}$. Consider now the agent who is best off in $\hat{P}$ and call him $i_1$, so $\pi_{i'}(g) \leq \pi_{i_1}(g)$ for all $i' \in N(i_1, g)$. Let $m_1 = \|N(i_1, g)\|$ so $m_1 \geq n^* + 1$. By Lemma 5 the agents in $N(i, g)$ form a wheel $W(m_1)$.

Call this new network $g'$. Take now an agent $j$ belonging to this wheel $W(m_1)$. Agent $j$ has a better response (in fact it is his best response) to cut his link and link to agent $r$ in the wheel such that $d(r, j; g') = n^*$. So he does and observes exactly $n^*$ agents improving his payoff. This completes the proof ■

The following lemma considers another situation in which there is an agent $j_1$ who observes more than $n^*$ agent. It assume further that among the agents he observes all those who observe less than $n^*$ players observe so few that they have a better response to cut all their links and link to $j_1$ directly. Once all of them linked to $j_1$ no-one observed by $j_1$ observes less than $n^*$. Then by Lemma 7 one agent who observes $n^*$ players arises in finite time.

**Lemma 8** *Assume there exist an agent $j_1$ such that $n^* + 1 \leq \|N(j_1, g)\| \leq \overline{n}$ and for each agent $\ell \in N(j_1, g)$ such that $\|N(\ell, g)\| \leq n^* - 1$ we have that $\|N(j_1, g)\| + \|N(\ell, g)\| + 1 < \frac{1}{\alpha}$. Then in finite time at least one agent who observes exactly $n^*$ agents will arise.*

**Proof.** Let $m_1 := \|N(j_1, g)\|$, so $m_1$ is the number of agents observed by $j_1$. Consider $j_2 \in \arg\max_{\ell \in \{\ell \in N(j_1,g) \colon \|N(\ell,g)\| < n^*\}} \|N(\ell, g)\|$ so $j_2$ is the agent who observes the maximum number of agents among those observed by $j_1$ who observe less than $n^*$. Agent $j_2$ exists by assumption of the Lemma. Consider an agent $\hat{\jmath} \in \arg\max_{\ell \in N(j_2,g)} d(j_2, \ell; g)$ so $\hat{\jmath}$ is furthest away from $j_2$ among those who are observed by $j_2$. So $\|N(\hat{\jmath}, g)\| \leq \|N(j_2, g)\| \leq n^* - 1$ and $\pi_{\hat{\jmath}}(g) \leq \|N(\hat{\jmath}, g)\| - c - \alpha \|N(\hat{\jmath}, g)\|^2$. Notice that $m_1 + \|N(\hat{\jmath}, g)\| + 1 < \frac{1}{\alpha}$ implies that $\|N(\hat{\jmath}, g)\| - c - \alpha \|N(\hat{\jmath}, g)\|^2 < m_1 - c - \alpha m_1^2$. Hence it is a better response for $\hat{\jmath}$ to cut all his links and link to $j_1$ getting to observe $m_1$ agents. So he does and receives a payoff equal to $m_1 - c - \alpha m_1^2$.

Call this new network $g'$. Notice that $\hat{\jmath}$ and all the agents in $N(j_1, g)$ who observed $\hat{\jmath}$ in $g$ now observe exactly $m_1 \geq n^* + 1$ agents in $g'$ as they were all observed by $j_1$ in $g$. However nothing changed for the other agents observed by $j_1$. So $g'$ is identical to $g$ with the exception of the move made by $\hat{\jmath}$. In particular for each agent $\ell \in N(j_1, g')$ such that $\|N(\ell, g')\| \leq n^* - 1$ we still have that $\|N(j_1, g')\| + \|N(\ell, g')\| + 1 < \frac{1}{\alpha}$. We can then replicate this argument until no agent observes less than $n^*$. Now by Lemma 7 at least one player who observes exactly $n^*$ agents emerges. This completes the proof. ■

We are now ready to prove the main result of the paper.

## B.2 Proof of Proposition 2

**Proposition** 2 *Assume that $P > n^*$, that $(\alpha, c) \in \mathcal{P} \backslash \left\{(\alpha, c) : \frac{1}{6} \leq \alpha < \frac{1}{5}, 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1\right\}$ and in addition if $n^* > \frac{1}{2\alpha}$ that there exists no integer $k$ such that $P = (k+1)n^* - 1$. Then in finite time each agent belongs to a starred wheel of dimension $n^*$. From then on, while the network might change, it remains a constellation of starred-wheels of dimension $n^*$ in each period.*

**Proof.** We find a path of the better reply dynamics (4) such that starting from an arbitrary network it leads in finite time to a constellation of starred wheels of dimension $n^*$ through the following steps:

1. From an arbitrary network in finite time we eliminate all agents whose

payoff is less that $1-\alpha$. In the resulting network there are either stand-alones or terminals or agents who do not observe less than $\underline{n}$ or more than $\overline{n}$ agents.

**Proof.** Take an arbitrary network $g$. Consider the set of the agents $Q(g) = \{i \in P | \pi_i(g) < 1-\alpha\}$. Take the agent with the highest index in $Q(g)$ and call this agent $j$. So $j$ has a better response is to cut all his links. Thus $j$ becomes a stand-alone (or a terminal in the case he was observed by someone else) and receives $1-\alpha$.

Call this new network $g'$. For every agent $i \notin Q(g)$ such that $j \in N(i,g)$ if $j$ cuts all his links then $\pi_i(g') \geq 1-\alpha$ since $\underline{n} = 2$. So $\|Q(g')\| \leq \|Q(g)\| - 1$. Replicate this argument until all the agents receive a payoff greater or equal to $1-\alpha$. Then by Lemma 2 all the agents who are neither stand-alones nor terminals observe between $\underline{n}$ and $\overline{n}$. ■

2. Let us eliminate from the network the terminals in finite time.

   **Proof.** Consider a terminal agent $i$. By definition there exists an agent $j \in P\backslash\{i\}$ who observes $i$, i.e. $i \in N(j,g)$. We show that if $i$ links to $j$ then $i$ is better off. Call the new network $g'$. When $i$ links to $j$ then $N(i,g') = N(j,g)$ as $i$ was a terminal in $g$. Since $i$ has only one link in $g'$, $\|N(i,g')\| = \|N(j,g)\| \leq \overline{n}$ and $\pi_j(g) \geq 1-\alpha$ we find $\pi_i(g') \geq 1-\alpha$. Replicate this argument until all terminals connect. ■

3. We now show that in finite time stand-alones connect.

   (a) *If there exist an agent $i \in g$ such that $\underline{n} \leq \|N(i,g)\| \leq \overline{n}-1$, i.e. if there exists an agent $i$ who observes no more than $\overline{n}-1$ and no less than $\underline{n}$ then we eliminate all the stand-alone in finite time.*

   **Proof.** Any stand-alone has a better response to link to $i$. By so doing he observes $\|N(i,g)\|+1 \leq \overline{n}$ and receives a payoff greater than or equal to $1-\alpha$. Notice that agent $i$ still observes the same agents as in $g$. ■

(b) *If* $\|N(j,g)\| = \overline{n}$ *holds for all* $j \in g$ *such that* $\|N(j,g)\| > 1$ *we eliminate all the stand-alone in finite time.*

**Proof.** The assumption $(\alpha, c) \in \mathcal{P} \setminus \left\{(\alpha, c) : \frac{1}{6} \leq \alpha < \frac{1}{5}, 3 - 15\alpha < c \leq \frac{1}{4\alpha} - 1\right\}$ is equivalent to $n^* \leq \overline{n} - 1$ and $\underline{n} = 2$.

Let us divide the proof in two cases.

First assume that there exist an agent $i$ such that $\mu_i^d(g) = 1$, i.e. $i$ has only one link. Then $i$ is best off among all the agents he observes. So by Lemma 5 a wheel $W(\overline{n})$ forms among all the agents in $N(i,g)$. Call this new network $g'$. Take now an agent $j$ belonging to this wheel $W(\overline{n})$. As $n^* \leq \overline{n} - 1$ agent $j$ in the wheel has a better response to cut his link and link to agent $r$ in the wheel such that $d(r, j; g') = \overline{n} - 1$. So he does and observes exactly $\overline{n} - 1$ agents improving his payoff. Now we are back to the case considered in part a) of this Step and the proof is complete.

Second assume that $\mu_i^d(g) \neq 1$ for all $i$. Fix one agent and call him $i_1$. Consider an agent $i_2 \in \arg\max_{\ell \in N(i_1, g)} d(i_1, \ell; g)$ so $i_2$ is furthest away from $i_1$ among those who are observed by $i_1$. Note that $i_1$ does not use any of $i_2$'s links to observe anyone else. Agent $i_2$ improves his payoff by cutting all his links (which are more than one) and linking to $i_1$ directly. Now agent $i_2$ has one link only and we are back in the previous case of part b). ■

4. Now all the players in the network only observe a number of agents between $\underline{n}$ and $\overline{n}$. From any such network in finite time a player that observes $n^*$ agents emerges.

   **Proof.** Take the agent who observes the maximum number of agents $i_1 \in \arg\max_{j \in P} \|N(j,g)\|$. Take $i_2 \in \arg\max_{j \in \{\ell \in P : \|N(\ell, g)\| < n^*\}} \|N(j,g)\|$, so $i_2$ is the agent who observes the maximum number of agents among those who observe less than $n^*$. If $i_2$ does not exist then by Lemma 7 at least one player who observes exactly $n^*$ agents will arise. Let $m_1 := \|N(i_1, g)\|$ and $m_2 := \|N(i_2, g)\|$ whenever it exists.

Let us consider the following 3 cases.

(a) *Assume that* $m_1 = m_2$*. In finite time one player observing* $n^*$ *emerges.*

**Proof.** If $m_1 = m_2$ then all the players observe less than $n^*$ agents and hence by Lemma 6 at least one player observing exactly $n^*$ agents emerges. ■

(b) *Assume that* $m_1 + m_2 + 1 < \frac{1}{\alpha}$ *and* $m_1 \geq n^* + 1$*. Then in finite time one agent observes exactly* $n^*$ *agents.*

**Proof.** Assume first that some agents who observe less than $n^*$ are observed by $i_1$. Note that for all agents $j$ with $\|N(j, g)\| \leq n^* - 1$ we have $\|N(j, g)\| \leq m_2$. In particular for all $\hat{\jmath} \in N(i_1, g)$ such that $\|N(\hat{\jmath}, g)\| \leq n^* - 1$ we have that $m_1 + \|N(\hat{\jmath}, g)\| + 1 < \frac{1}{\alpha}$. So by Lemma 8 in finite time at least one player observing exactly $n^*$ agents emerges.

If instead for all $j$ such that $\|N(j, g)\| \leq n^* - 1$ we have $j \notin N(i_1, g)$ then $\|N(k, g)\| \geq n^* + 1$ for all agents $k \in N(i_1, g)$ and hence by Lemma 7 at least one player who observes exactly $n^*$ agents will arise. ■

(c) *Assume* $m_1 + m_2 + 1 \geq \frac{1}{\alpha}$ *and* $m_1 \geq n^* + 1$*. To show: one agent that observes* $n^*$ *players will arise in finite time.*

**Proof.** Let $\hat{P} := N(i_1, g) \cup N(i_2, g)$. Note that $\left\|\hat{P}\right\| \geq m_1 > n^*$. Also note that $m_1 + m_2 + 1 \geq \frac{1}{\alpha}$ implies $\pi_{i_1}(g) \leq m_2 + 1 - c - \alpha (m_2 + 1)^2$. So agent $i_1$ has a better response of linking to $i_2$ cutting all his original links. Let him do it and call the new network $g'$.

Note that $N(i_1, g') = \{i_1\} \cup N(i_2, g)$ and $\|N(i_1, g')\| = m_2 + 1 \leq n^*$. If $\|N(i_1, g')\| = n^*$ then the proof is complete. If $\|N(i_1, g')\| \leq n^* - 1$ then $i_1$ is best off among the agents in $N(i_1, g')$. So by Lemma 5 a wheel $W(m_2 + 1)$ forms with possibly other agents observing it. Call this new network $g''$.

We now show that the wheel enlarges so that no agent in $\hat{P}$ can be directly linked to the (enlarged) wheel and observe less than $n^*$ overall. Call this agent $i$. By definition of $i$ there exists an agent $i'$ in the wheel such that $d(i, i'; g) = 1$. As $m_2+1 \leq n^*-1$ agent $i''$ in the wheel whose only link is to $i'$ has a better response to cut his link and link to $i$. So the wheel enlarges by 1. Call this new network $g'''$. Assume further that $i$ observes some agents who do not belong to the $W(m_2+1)$. As $\|N(i, g'')\| \leq n^* - 1$ also $\|N(i, g''')\| \leq n^* - 1$. Notice that for all $j$ in the wheel we have that $N(j, g''') = N(i, g''')$. In particular agent $i'$ who belongs to the wheel $W(m_2+1)$ is best off among the agents in $N(i', g''')$ so by Lemma 5 a wheel of dimension $\|N(i', g''')\|$ forms. Repeat until an agent gets to observe $n^*$ (which completes the proof) or no such $i$ linked to the wheel and observing less than $n^*$ agents overall exists anymore. Call this new network $\check{g}$ and the wheel dimension $\check{m} := \|N(i, g''')\|$.

We now enlarge the wheel further so to partition the agents in $\hat{P}$ into those who belong to the wheel and observe less than $n^*$players and those who observe more than $n^*$ players. So we eliminate all the agents observing less than $n^*$ agents who do not belong to the wheel. Note that in $\check{g}$ if agent $j$ does not belong to the $W(\check{m})$ then either $\|N(j, \check{g})\| \geq n^* + 1$ or $\|N(j, \check{g})\| \leq m_2$ by definition of $m_2$. So $\|N(j, \check{g})\| \leq \check{m}$ whenever $\|N(j, \check{g})\| \leq n^* - 1$ for all $j \in \hat{P}$. Take agent $h_1$ who does not belong to the $W(\check{m})$ such that $\|N(h_1, \check{g})\| \leq n^* - 1$. As $\|N(h_1, \check{g})\| < \check{m}$ and $\check{m} \leq n^* - 1$ then $h_1$ has a better response to cut all his links and link to the $W(\check{m})$ (from outside). So he does and observes $\check{m} + 1$ agents.

If $\check{m}+1 = n^*$ then the proof is complete. Otherwise take $h_2$ among the agents not observed by $h_1$ in the new network such that $h_2$ observes less than $n^*$. By the same reasoning applied to $h_1$ agent $h_2$ improves his payoff cutting all his links and linking to $h_1$. Repeat this argument until either some agent observes $n^*$ or there exist no agent in the network who does not belong to the wheel and observes less than

$n^*$ players. If some agent observes $n^*$ then the proof is complete. Otherwise call $\check{h}$ the last agent who moved and note that he is best off among all those he observes as he observes the most and still observes less than $n^*$ players with one link only. So by Lemma 5 a wheel forms among the agents observed by $\check{h}$. Call this new network $\hat{g}$ and the (new) wheel dimension $\hat{m} \leq n^* - 1$.

Note that in $\hat{g}$ all the agents in $\hat{P}$ either observe more than $n^*$ agents or belong to the wheel $W(\hat{m})$ and thus observe $\hat{m} \leq n^* - 1$. Take $j_0 = \arg\min_{h \in \left\{\ell \in \hat{P} :\, \|N(\ell,\hat{g})\| > n^*\right\}} \|N(h,\hat{g})\|$. Let $m_0 := \|N(j_0,\hat{g})\|$.

Assume first $\hat{m} + m_0 + 1 < \frac{1}{\alpha}$ then $i_1$ who belongs to the wheel $W(\hat{m})$ has a better response to cut his only link and link to $j_0$. By so doing $i_1$ breaks the wheel. As in $\hat{g}$ everyone who observed less than $n^*$ agents belonged to the wheel $W(\hat{m})$ now there exists no agent $\ell \in \hat{P}$ such that $\|N(\ell,\hat{g})\| < n^*$. So by Lemma 7 one agent observing $n^*$ agents surely emerges.

Assume instead $\hat{m} + m_0 + 1 \geq \frac{1}{\alpha}$ then $j_0$ cuts all his links and links to the wheel. By definition of $m_0$ for all $j' \in \hat{P}$ with $\|N(j',\hat{g})\| \geq n^*$ we have that $\hat{m} + \|N(j',\hat{g})\| + 1 \geq \frac{1}{\alpha}$. So all $j' \in \hat{P}$ with $\|N(j',\hat{g})\| \geq n^*$ have the same better response and link to the wheel. As $\hat{m} \leq n^* - 1$ and $\left\|\hat{P}\right\| > n^*$ the wheel enlarges to $n^* - \hat{m} - 1$ (peripheral) agents and an agent observing $n^*$ players will arise. ■

5. Now in the network at least one agent observes $n^*$ players. In finite time all the agents in the network observing $n^*$ players belong to a $SW(n^*, m)$.

   **Proof.** The proof is divided in two parts.

   (a) *If a player observes $n^*$ agents then this player belongs to a $W(n^*)$ in finite time.*

   **Proof.** By assumption of this Step a player who observes $n^*$ agents exists. Take an agent who observes $n^*$ agents and call him $i_1^*$, that is

$\|N(i_1^*, g)\| = n^*$. Consider $i_2^* \in \arg\max_{\ell \in N\left(i_1^*, g\right)} d(i_1^*, \ell; g)$, i.e. consider an agent who is furthest away from $i_1^*$. We already know that then $i_1^*$ does not use any of $i_2^*$'s links to observe anyone else. Agent $i_2^*$ improves his payoff (in fact it is his best response) by cutting all his links and linking to $i_1^*$ directly. So he does.

Call this new network $g'$. Now $i_2^*$ enjoys the maximum payoff attainable: he observes $n^*$ agents paying the cost of one single link and the maintenance cost of a network of $n^*$ agents. Notice that $i_2^*$ is best off in $N(i_2^*, g')$ and that $\|N(i_2^*, g')\| = n^*$ so by Lemma 5 a wheel $W(n^*)$ forms. Replicate this argument until there exist no agents observing $n^*$ players who do not belong to a wheel $W(n^*)$. ■

(b) *If all the agents who observe $n^*$ players belong to a wheel $W(n^*)$ and there exists an agent who does not belong to a $W(n^*)$ then a starred wheel of dimension $n^*$ emerges in finite time.*

**Proof.** Call $i$ the agent who does not belong to a $W(n^*)$. Let $m_i := \|N(i, g)\|$. Note that $m_i \neq n^*$.

We claim that agent $i$ has a better response to cut all his links and link to the wheel from outside since by assumption of the Proposition if $n^* > \frac{1}{2\alpha}$ then there exists no integer $d$ such that $P = (d+1)n^* - 1$. Assume first that either $m_i > n^*$ or $m_i < n^* - 1$ then $\pi_i(g) \leq n^* + 1 - c - \alpha(n^*+1)^2$. So $i$ has a better response to cut all his links and to link to a wheel of dimension $n^*$ forming a starred wheel of the same dimension.

Assume instead that $m_i = n^* - 1$. Note that $n^* - 1 - c - \alpha(n^* - 1)^2 \gtreqless n^* + 1 - c - \alpha(n^*+1)^2$ is equivalent to $n^* \gtreqless \frac{1}{2\alpha}$. So if $n^* \leq \frac{1}{2\alpha}$ any agent observing $n^* - 1$ has a better response to link to a $W(n^*)$ forming a starred wheel of the same dimension. Assume now that $n^* > \frac{1}{2\alpha}$. If there exists another agent $j$ who does not observe $n^*$ then $j$ has a better response to cut all the links and link to $i$ getting to observe $n^*$ agents. Then a new wheel will arise by part a) of this Step. If no such agent $j$ exists then exists an integer $d$ such that $P = (d+1)\, n^* - 1$ which is a contradiction. ■

6. In finite time the better reply dynamics converges to a constellation of $h$ starred wheels of dimension $n^*$, $h = 1, ..., \left\lfloor \frac{P}{n^*} \right\rfloor$.

   **Proof.** By Proposition 1 a constellation of starred wheels of dimension $n^*$ is a PSNES which is absorbing for the better reply dynamics (4). ■

This concludes the proof of the Proposition. ■