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Continuous-Time Contracting With Ambiguous
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Abstract

This study examines the contracting problem in continuous time with ambiguous information. A problem of this nature arises, for example, in an employment relationship where there is limited knowledge, or ambiguity, about the technology that governs the performance. To address this problem, we connect the models of contracting problem in continuous time with the models of decision making under ambiguity in continuous time. The connection uses the continuous-time techniques and preserves the tractability in analysis. By means of computed examples we show that the consideration of ambiguity results in compensation schemes that are less sensitive to performance relative to the classical case. This answers to a criticism leveled at the extant theories of contracts that predicted compensation schemes that are unrealistically too sensitive to performance. Our work provides one possible rationale for simpler contracts through ambiguity.

Keywords

Ambiguity, contracts, continuous-time stochastic methods.

JEL classification: D82, D86, J33

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1. INTRODUCTION

In contractual relationships parties often do not have precise information about the economic environment. The classic approach formulates the contracting problem between a principal and an agent under the assumption that parties have good understanding of the productive process. In particular, unobservable effort by the agent is assumed to impact the distribution over outcomes in a precise and commonly known fashion. This framework has been applied to many economic problems including managerial compensation schemes, financial contracts, insurance contracts, and mortgage design. In many cases, especially with new technologies and newly formed interactions, the relationship mapping effort into outcomes is imprecisely known. This raises the question how ambiguity affects contractual relationships.

We address this problem in the context of a continuous-time contracting model where a principal and an agent engage in a contractual relationship over unobservable effort that generates output with ambiguity. In contrast to the classical case in Sannikov [36] we consider a model in which efforts maps to *sets* of probability distributions over outputs. In particular, technology for output is characterized by a diffusion process and its drift belongs to a *set* instead of a point as in the classical case associated with each effort level. This assumption reflects the distinction made by Knight [29] between risk and uncertainty (or ambiguity). Risk as in the classical case refers to the situation where there is a known probability distribution associated with each action. Ambiguity, on the other hand, as in our case, refers to the situation where the information is too imprecise to summarize likelihoods into a single probability distribution and instead there is a *set* of probability distributions associated with each action. The consideration of ambiguity as representing a richer description of informational possibilities poses challenges for the formulation and the analysis of a contracting problem and offers a broader set of economic interpretations.

The aim of this paper is to examine the design of optimal contracts with unobservable effort and ambiguity in a dynamic environment. To analyze this problem, we connect the models of contracting problem in continuous time in Sannikov [36] with the models of decision making under ambiguity in continuous time in Chen and Epstein [9]. We construct this connection using the continuous-time framework while preserving the tractability in analysis. More specifically, our model is based on Sannikov. It is flexible to allow us to incorporate ambiguity for several reasons. First, using the Girsanov Theorem in analogous way as in Chen and Epstein we represent ambiguity as the set of drift terms on the diffusion process for productivity for a given effort level.¹ Second, using this representation for ambiguity, and the related recursive representation for ambiguity averse preferences as a solution to backward stochastic differential equations (BSDE), we can extend the martingale formulation to the dynamic contracting problem.² Third, the contracting problem reduces to solving ordinary differential equations (ODEs)

¹See, for example, Karatzas and Shreve [28].

²See El Karoui *et al.* [21] for the role of BSDEs in finance.

as in Sannikov, which can be numerically solved. We then solve and characterize the optimal contract with ambiguous information.

In our formulation of the contracting problem, we make several important modeling choices. We assume that both the principal and the agent have common ambiguous beliefs about output distribution. Besides that we do not place restrictions on the nature of ambiguity. This assumption allows us to conduct analysis in a tractable fashion and gain economic insights in a setting that provides a building block for contracting problems with a more general ambiguity structure. In particular, parties do not have differential information about the technology. That raises the question of updating with ambiguous beliefs. This modeling choice, which abstracts from learning, is made due to a lack of analytical framework in continuous-time literature that could formulate updating ambiguous beliefs. Our work in progress attempts such a formulation incorporating adverse selection.

To model decision making under ambiguity we follow the approach formulated by Chen and Epstein [9]. This approach formulates in continuous time the maxmin expected utility model of Gilboa and Schmeidler [22]. This formulation allows analyzing both the constraint on the set of priors and the incentive-compatibility constraint separately which highlights the trade-off the consideration of ambiguity introduces on incentives and on the optimal contract in clear and simple terms.

In the contracting-problem, the principal faces ambiguity regarding an agent's productivity. The principal offers a contract that maximizes his expected discounted utility according to his subjective worst-case scenario under the contract while respecting the agent's incentive compatibility and participation constraint. The agent has the same ambiguous belief about the technology and chooses an effort level that maximizes his expected utility under his subjective worst-case scenario, which a priori can be different than that of the principal. Utility functions are modeled as recursive ambiguity averse preferences proposed by Epstein and Schmeidler [9]. In this approach, utility function is represented recursively using continuation value as a state variable, similar to in the characterization of dynamic contracts in discrete-time with imperfect public monitoring (Spear and Srivastava [39]). Additionally, the concern for ambiguity is reflected in one additional term added to the martingale representation in the classical case and that term is determined by the worst case. Since the worst-case scenario depends on the contract and unobserved effort choice by the agent, parties to the contract can disagree on the worst case.

Our new results are two folds. First, we construct a tractable representation of incentive compatibility constraint and use it to highlight the key trade-off with ambiguity aversion. Second, we develop the characterization of optimal contract under ambiguity. The former mainly follows from an application of Girsanov Theorem which transforms a set of probability measures for output under one action choice to another and enables the key comparison to verify incentive compatibility of the effort. More specifically, we construct an analogous set of probabilities with a reference measure as in Chen and Epstein by letting the set and the reference measure vary with different effort processes and show that this construction

preserves the regularity properties. This result does not depend on the particular structure of the contracting problem and hence can be applied independently where there is a concern for trend change, for example, as in asset pricing models. For the contracting problem of our main interest this representation of multiple priors together with their time-consistency as in Chen and Epstein [9] then enables us to write the agent’s utility under alternative deviations recursively *and* find that ambiguity reduces the expected continuation utility by a term proportional to the worst-case scenario. This added term highlights the effect of ambiguity on incentives. In the standard case, to induce higher effort the compensation scheme need vary with performance. Higher ambiguity penalizes the continuation value more and therefore the agent prefers compensation scheme that varies less with the performance. Therefore, ambiguity aversion acts like a effort cost. However, unlike the latter the effect of ambiguity aversion is forward looking as it is jointly determined by the worst-case scenario *and* the continuation value of the contract.

Our second contribution is the characterization of optimal contract with ambiguity. This characterization mainly follows from the tractable incentive compatibility condition we have identified and the analogous stochastic analysis in Sannikov [36]. We find that unlike the classic case in Sannikov concern for ambiguity introduces two terms in to the principals optimization problem. The first is the penalty in output due to the worst case and the second is the cost of providing incentives to the agent through variation in the continuation utility. The former has a first-order direct effect on the profit and reduces it, while the latter has an indirect effect through compensation to the agent for variability in payments. We find that the direct effect dominates the indirect one and hence the principal implements higher effort and lower variability in the continuation value. This compensation scheme is not optimal in the classical case and the difference arises from the fact that, with ambiguity, the principal is more tolerant in that the contract is terminated at a lower continuation value. In sum, the consideration of ambiguity results in compensation schemes that are less sensitive to performance relative to the classical case of Sannikov and that the behavioral implications of ambiguity aversion in this contracting problem differ from those of risk aversion. Our results provide a possible resolution to a criticism leveled at the extant theories of contracts that predicted compensation schemes that are unrealistically too sensitive to performance. Therefore, our work suggests that ambiguity aversion provides one possible rationale for simpler contracts.

Our paper is related to a growing literature on dynamic contracting problems in continuous time.³ Our paper is most closely related to the seminal contributions of Sannikov [36] and Chen and Epstein [9]. Our main contribution is

³Holmstrom and Milgrom [27], Schaettler and Sung [37], Ou-Yang [31], DeMarzo and Sannikov [16], Biais et al. [6], Biais et al. [7], Sannikov [36], He [25], He [26], Williams [42], Zhang [43], Piskorski and Tchisty [33], Prat and Jovanovic [34], De Marzo et al. [15], Cvitanic and Zhang [12], Zhu [44], and Szydlowski [40]. This literature complements and extends the vast literature on dynamics contracts in discrete-time including Spear and Srivastava [39], Thomas and Worrall [41], Atkeson and Lucas [3], Albuquerque and Hopenhayn [1], Clementi and Hopenhayn [10], Quadrini [35], DeMarzo and Fishman [14], and DeMarzo and Fishman [13].

to extend the latter and introduce ambiguity into the former model and examine the optimal contract and compensation schemes. Our work is also related to the microeconomic literature that studies contracting problems and mechanism design in static settings (see Bergemann and Schlag [5], Bodoh-Creed [8], Bergemann and Morris [4] and the references therein.) These work typically uses static models with adverse selection rather than moral hazard. Miao and Rivera [30] introduces robustness considerations into a dynamic contracting problem in continuous time. They focus on the principal's concern for robustness and their modeling of ambiguity builds on a model of multiplier preferences proposed by Anderson et al. [2] and Hansen et al. [24], while it differs from the model of Chen and Epstein [9] we adopted. Szydlowski [40] examines dynamic contracting problem in continuous time with ambiguity. His model assumes ambiguity regarding the agent's effort cost and can be interpreted as a behavioral approach to ambiguity. His preference representation differs from the model of Chen and Epstein.

The remainder of the paper proceeds as follows. Section 2 specifies the contracting problem in continuous time with ambiguous information. Section 3 formalizes the ambiguous information. Sections 4 specifies utility values associated with a contract. Section 5 derives the optimal contracts and characterizes its properties through parametric examples. Section 6 concludes. Technical details are relegated to appendices.

2. THE CONTRACTING PROBLEM WITH AMBIGUOUS INFORMATION

We present a model of a continuous-time principal/agent problem with ambiguous information. The agent chooses effort at each instant of time a_t from a compact set A_t . Following Sannikov [36] the choice of action process (a_t) determines the realization of output $\{X_t\}$ over time in a stochastic manner. Formally, we assume that the total output X_t produced up to time t evolves according to a diffusion process

$$dX_t = \mu_t(a_t)dt + \sigma dB_t, \tag{1}$$

where $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion under a reference measure P ; as in standard moral hazard problems, the agent's choice of effort level a_t is privately observed; and unlike the classical case productivity of actions $\mu(a_t)$ are not perfectly known, rather both parties only know that it belongs to a set $\mu_t(a_t) \in \Theta_t^a$. The latter is our main departure from the literature that studies dynamic contracting problems and we refer to it as ambiguous information. It formulates a more general information structure. Taking a singleton drift term specializes to the contracting problem studied by Sannikov [36] : $\Theta_t^a = a_t$ for all t . The problem we address here is the design of a contract by the principal when there is ambiguous information, in the sense as defined here regarding the contracting environment. Moreover, we assume that the agent and the principal have the same knowledge of the technology.⁴ Our aim is to solve and characterize the optimal contract problem in this environment.

⁴We discuss briefly the role of this modeling choice in the conclusion.

In the rest of this section we formulate the optimal choice of the contract by the principle as an optimization problem. First we represent the ambiguous information as a set of probabilities. Then using this set we specify an criterion for the evaluation of the contract by the parties.

Following Epstein and Chen [9] ambiguous information is equivalently formulated as a set of priors. The key observation is that a drift process (θ_t) is a density generator: it induces a probability measure Q^θ under which $\theta_t dt + \sigma dZ_t$ is a Brownian motion. This probability measure Q^θ is determined by its density with respect to the reference measure P using Girsanov exponential as follows

$$\frac{dQ^\theta}{dP} |_{\mathcal{F}_t} = \exp \left\{ - \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right\} \quad (2)$$

Each effort process (a_t) induces a set of density generators Θ^a and the corresponding set of priors is

$$\mathcal{P}^a = \{Q^\theta : \theta \in \Theta^a \text{ and } Q^\theta \text{ is defined by (2)}\} \quad (3)$$

In other words, ambiguity concerns the drift of the diffusion process for output.

The principal offers a contract to the agent, which specifies a stream of consumption (C_t) contingent on the realized output and an incentive-compatible advice of effort (a_t) . Effort process induces output with ambiguity so that the set of priors \mathcal{P}^a determines output realizations in a stochastic manner. We assume that both parties evaluate the contract using the worst-case criterion. Accordingly, the principal contract offer maximizes his expected profit under his worst-case criterion

$$\mathcal{E}(F) = \min_{Q \in \mathcal{P}^a} E^Q \left[r \int_0^\infty e^{-rt} dX_t - r \int_0^\infty e^{-rt} C_t dt \right] \quad (4)$$

subject to delivering the agent a required initial value of at least \hat{W}

$$\mathcal{E}(V(C, a)) = \min_{Q \in \mathcal{P}^a} E^Q \left[r \int_0^\infty e^{-rt} (u(C_t) - h(a_t)) dt \right] \geq \hat{W}$$

The interest is in contracts that generate non-negative expected profits for the principal. To gain tractability by exploiting the power of continuous-time formulation the next section sets up keys features of the set of multiple priors.

3. THE SET OF PRIORS

The contracting problem as posed in a general form in (25) is difficult to solve. It allows for arbitrary dependence of uncertainty on the history of actions and outcomes. However, little is known about solution methods in this case. The tractability of the analysis of relies on representing it in a recursive manner. This in turn depends on the nature of multiple priors \mathcal{P}_a induced by any effort process a . Chen and Epstein [9] formalizes a notion of ambiguity for choice problems that yield recursive representation. In particular, at each instant of time, the increments in the diffusion process θ_t are independently drawn from a family Θ_t . We follow this modeling for ambiguous information and generalize it to cases in which ambiguity varies with action choices. The latter is important for incentive

compatibility as, in general, different efforts induce different multiple priors. More specifically, following Chen and Epstein, we model the sets of one-step-ahead densities for any effort process (a_t) via a process (Θ_t^a) of correspondences from Ω into its range $R^A \subset R$, that is, for each t

$$\Theta_t^A : \Omega \rightsquigarrow R^A.$$

The set of all measures that can be constructed by some selection from these sets of one-step-ahead densities is defined using the following set of density generators:

$$\Theta^a = \{(\theta_t) : \theta_t \in \Theta_t^a(\omega) dt \otimes dP_a \text{ a.e.}\}. \quad (5)$$

Fixing an effort process to a constant, say zero, for each time t and ω specializes to the formulation in Chen and Epstein. The generalization to an arbitrary effort process is due to the concern for the incentive compatibility of the effort process. The related changes are in: the base-line measure P_a and the variable interval size Θ_t^a . The main results of this section show, using mainly Girsanov's theorem for changes of measures, these generalizations are possible while preserving the key properties of the set of priors, namely regularity that guarantees that the contracting problem is well-defined, and "dynamic consistency" that enables recursive representation. For its simplicity we first consider fixing base-line measure P and changing the interval size. For the sake of easing the illustration we start with a particular case in which the set of drift terms are time and state invariant, is centered around zero, and depends on the effort: $\Theta_t^a(\omega) = [-\kappa_a, \kappa_a]$ for each t and ω . Following the terminology introduced by Chen and Epstein we denote this case as κ -ignorance with variable interval.

3.1. kappa-ignorance with variable interval. On the standard Wiener space (Ω, \mathcal{F}, P) the process (X_t) governing the agent's output is a Brownian motion. The agent's technology is described by the set of drifts induced by his choices. In a particular case examined by Chen and Epstein [9], the technology is characterized by κ -ignorance. In particular, the base-line measure is augmented by a family of measures using a process $\theta = (\theta_t)$ that determines the size of the interval which in the present sense captures (interpreted as) ambiguity associated with each choice of action. This is the simplest case that generalizes Chen and Epstein [9] while connecting with Sannikov [36]'s contracting problem. Consider first a base-line ambiguity with base-line action of no effort. That is, under P the process $dX_t = \sigma dW_t$ is a Brownian motion. Uncertainty is modeled as a family of Brownian motions following analogous ideas in Chen and Epstein [9]. In particular, drift terms belong in a time-invariant set and are represented by a process $\theta = (\theta_t)$ with $\theta_t \in [-\kappa, \kappa] = \mu(0) + [-\kappa, +\kappa]$. Incentive-compatibility consideration of a contract requires a comparison in a one-stage deviation sense and accordingly we consider a more specialized set for the drift terms that: $\Theta_0 := (\Theta_t)$ with $\Theta_\tau = [-\kappa, +\kappa]$ for $\tau \leq t$ and $\Theta_\tau = [-\tilde{\kappa}, +\tilde{\kappa}]$ for $\tau < t \leq T$. That is, up to a fixed time τ drift term is in set $[-\kappa, +\kappa]$ and after then in $[-\tilde{\kappa}, \tilde{\kappa}]$. This interpretation is based on the following characterization. Taking the supermartingale $Z_t = \exp\left(-\int_0^t \theta_s dW_s\right)$ and noticing that $\int_0^T \|\theta_t\|^2 dt < \infty$ and that $E[Z_T] = 1 < \infty$

so that the supermartingale is actually a martingale (under the measure P), by Girsanov's the change of variable [23] give that

$$\widetilde{W}_t = W_t - \int_0^t \theta_s ds, \quad \mathcal{F}_t, \quad 0 \leq t \leq T$$

is a Brownian motion that possibly has different drift after τ .

The set of probability measures $\mathcal{P}^{\Theta_0} := \{\widetilde{P}^\theta : \theta \in \Theta_0\}$ is equivalent to the base-line measure P , that is absolutely continuous with respect to it. Conversely, adapting arguments in Duffie (1996, pg. 289) any set of equivalent probabilities can be constructed in this fashion. The interest is in showing that the set of probability measures \mathcal{P}^{Θ_0} satisfies rectangularity or "time-consistency." More formally, let $\Theta_t : \Omega \rightsquigarrow \mathcal{R}_{t,A,\tilde{A}}$ be the progressively measurable correspondence that maps paths to the drift terms where $\mathcal{R}_{t,A,\tilde{A}} = \Theta_\tau(\Omega) = [-\kappa, +\kappa]$ for $\tau < t$ and $\mathcal{R}_{t,A,\tilde{A}} = \Theta(\Omega) = [-\tilde{\kappa}, +\tilde{\kappa}]$ and take Θ to be the collection of all progressively measurable selections from Θ_t $0 \leq t \leq T$, that is, $\Theta = \{(\theta_t) : \theta_t(\omega) \in \mathcal{R}_{t,A,\tilde{A}} dt \otimes dP_A \text{ w.p.1}\}$. The set \mathcal{P}^Θ therefore contains all the probability measures equivalent to P constructed using (2) for all $\theta \in \Theta_0$.

When the base-line measure is fixed, Girsanov's theorem therefore shows that the change of actions transforms the set probability measures equivalent (in the sense of absolute continuity) to a base-line measure into another set of measures equivalent to the same base-line measure. Since the set of all measures are constructed by some selection from the set Θ_0 of one-step-ahead densities, adopting the terminology from Chen and Epstein this establishes the rectangularity of multiple priors

Lemma 1. The set \mathcal{P}^{Θ_0} of probability measures, under which the progressive measurable processes $X_t = \theta dt + \sigma dB_t$ for $\theta \in [-\kappa, +\kappa]$ are Brownian Motions with drift θ , is rectangular.

This property ensures that the probability measures on the sample paths induced by any change in action process at a later date is recognized from the current period's perspective. Its significance lies in representation of utility functions recursively and thereby making the analysis tractable without having to impose any restrictions as to which probabilistic models are more relevant for the decision maker. In the latter sense, it corresponds to a situation where the agents have learned "everything" relevant for the contractual relationship.

Next, we address a normalization assumed by Chen and Epstein [9], which in our formalization of multiple priors corresponds to a particular choice of effort process $a_t = 0$ for all t and ω . We show that the normalization can be replaced by an arbitrary alternative probability measure while preserving time consistency of the multiple priors.

3.2. kappa-ignorance with variable base measure. An important aspect of the contracting problem is that different choices of effort by the agent not only affect the set of drifts through the change in the size of interval but also through a change in the base measure. The main result of this section in Proposition 1 shows that the normalization used by Chen and Epstein to a base measure P

with respect to which the output process is a Brownian motion without a drift and can be made to an arbitrary base measure \tilde{P} .

To fully formalize the structure of the set of multiple priors as the base measure varies, first consider the case that different actions induce different base-line measures but they have the same interval. Formally, let us assume that changing the effort process from (A_t) to (\tilde{A}_t) changes the family of probability measures that gives Brownian motions $(A_t + \theta_t)dt + \sigma dB_t$ with drift in $\theta_t + [-\kappa, +\kappa]$ under each of P_A^θ measures corresponding to Z_A^θ to the family that gives Brownian Motions $(\tilde{A}_t + \theta_t)dt + \sigma dB_t$ with drift in $\theta_t + [-\kappa, +\kappa]$ under each of $P_{\tilde{A}}^\theta$ measures corresponding to $Z_{\tilde{A}}^\theta$ by transforming A to \tilde{A} through Girsanov theorem. Thus the family obtained has measures each equivalent to P_A . The family hereby is rectangular set of probability measures. The rest of the section fills in the formal details.

Take Girsanov exponential

$$Z_t^\theta = \exp \left(- \int_0^t \theta_s dB_s^A - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds \right) \quad (6)$$

and denote Θ_A as the set of Girsanov exponentials Z_t^θ associated with the processes (θ_t) : $\theta_\tau \in [-\kappa, +\kappa]$ for $0 \leq \tau \leq t$ and $\theta_\tau \in (\tilde{A}_t - A_t) + [-\kappa, +\kappa]$. The corresponding set of probability measures \mathcal{P}^{Θ_A} contains all possible measures that can be generated using the probability densities. As in the case with variable interval in kappa-ignorance Sect. 3.1, since the set of all measures are constructed by some selection from the set Θ_0 of one-step-ahead densities, we establish:

Lemma 2. The set of probability measures \mathcal{P}^{Θ_A} is a rectangular set of probability distributions with a base-line probability measure P_A .

Nothing in the previous line of reasoning depends on the processes (θ_t) except that it satisfies weak regularity conditions in applying Girsanov and that it has a bounded second moment, namely that $E_P(\int_0^T \|\theta_t\|^2 dt) < \infty$ $P - a.e.$ By construction, these conditions hold in the form of the ambiguity the contracting problem deals with. In particular, taking a function for the drift term μ , which is measurable and has a bounded second moment, in the role of θ the previous analysis goes through and therefore:

Proposition 1. The set of probability measures $\mathcal{P}^{\Theta_A, \tilde{A}, \kappa}$ is a rectangular set of probability distributions with a base-line probability measure P_A .

Notice that in this case the base-line measure P_A is the one that makes the process $dX_t = \mu(A_t)dt + \sigma dB_t$ Brownian motion. Note also that the base-line probability measure P_A is the center of the ambiguous set of probability distributions associated with the action A . It captures the notion of ambiguity formalized by Dumav and Stinchcombe [20] and Siniscalchi [38] in which a set of multiple priors is represented as the sum of its center and a set centered at zero.

In summary, the analysis in this section verifies that various sets of probability distributions \mathcal{P}^a that correspond to the set of drift terms Θ^a 5 and hence arise

in formulating the contracting problems in continuous time satisfy time consistency. This property will play an important role in the recursive representation of the contracting problem below. Before this, we turn to examine the regularity properties of the multiple priors.

3.3. Regularity Properties of the Set of Priors. The sets of priors that arise in the contracting problem are ones that vary in base-line measures and in the interval around the base-line by different choices of effort process. We examine in this section whether these extensions preserve regularity properties (defined below) so that the contracting problem is well-posed and admits a solution. We show that the contracting problems with ambiguity satisfy required regularity properties.

Chen and Epstein [9]’s formulation for decision problems uses $0 \in \Theta_t(\omega) dt \otimes dP$ a.e. In our case, this corresponds to taking $\kappa_{A_t} = 0$ and setting base-line measure to P : for each $t \in (0, T]$ so that $\mu(A_t) \in \Theta_t(\omega) dt \otimes dP_A$ a.e. Intuitively, the agents consider the base-line measure to be the one that corresponds to the center of the interval for the values of drift. Our main departure is to allow dependence of the drift on the effort process. By Girsanov’s theorem [23] for changes of measures, our base-line measure is equivalent to the base-line. Therefore, this difference is but in looking at the processes equivalently. Second, the measurability follows from the fact that the correspondence $(t, \omega) \mapsto \Theta_t(\omega)$ which defines the set of priors through (3) when restricted to $[0, s] \times \Omega$ is $\mathcal{B}([0, s]) \otimes \mathcal{F}_s$ -measurable for any $0 < s \leq T$. The remaining regularity properties of the set are its compactness and convexity, and follow from standard arguments. We collect these in the following result and its proof fills in the remaining details.

Proposition 2. The set of priors $\mathcal{P}^{\tilde{\Theta}}$ satisfies:

- (a) $P_A \in \mathcal{P}^{\tilde{\Theta}}$.
- (b) $\mathcal{P}^{\tilde{\Theta}}$ is absolutely continuous with respect to P and each measure in $\mathcal{P}^{\tilde{\Theta}}$ is equivalent to P .
- (c) $\mathcal{P}^{\tilde{\Theta}}$ is convex.
- (d) $\mathcal{P}^{\tilde{\Theta}} \subset ca_+^1(\Omega, \mathcal{F}_T)$ is compact in the weak topology.
- (e) For every $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, there exists $Q^* \in \mathcal{P}^{\tilde{\Theta}}$ such that

$$E_{Q^*}[\xi|\mathcal{F}_t] = \min_{Q \in \mathcal{P}^{\tilde{\Theta}}} E_Q[\xi|\mathcal{F}_t], \quad 0 \leq t \leq T$$

Parts (a)-(c) are self-explanatory. By (d), min exists for any $\xi \in L^1(\Omega, \mathcal{F}_T, P)$, a fortiori in $L^2(\Omega, \mathcal{F}_T, P)$. Part (e) extends the existence of a minimum to the process of conditional expectations.

Proof. (b) The process $dW^A = \mu(A_t)dt + \sigma dB_t$ is a Brownian motion under the base-line measure $Q^A = Z^A P$.

Fix $B \in \mathcal{F}_t$ and $Q_A^\theta \in \mathcal{P}_A^\Theta$. By Girsanov’s Theorem, $Q_A^\theta(B|\mathcal{F}_t) = y_t$, where (y_t, σ_t) is the unique solution to

$$dy_t = \sigma_t(\mu(A_t)dB_t), \quad y_T = 1_B$$

By the bounding inequality in El Karoui, Peng, and Quenez [21] and Uniform Boundedness, there exists $k > 0$ such that

$$(Q_A^\theta(B))^2 \leq kE_{Q_A}(1_B) = kQ_A(B),$$

where k is independent of θ . This delivers uniform absolute continuity. Equivalence obtains because $Z_T^\theta > 0$ for each θ .

(c) Follows from replacing P with $Q_A^{\theta=0}$ in the proof by CE. For $i = 1, 2$, let Q^i be the measure corresponding to $\theta^i \in \Theta_A$ and the martingale $Z_A^{\theta^i}$ as in (6). Define $\theta = (\theta_t)$ by

$$\theta_t = \frac{(\theta_t^1 + \theta_t^2)}{z_t^1 + z_t^2}$$

It thus follows that $\theta \in \Theta_A$ and $d(z_t^1 + z_t^2) = -(z_t^1 + z_t^2)\theta_t \cdot dW_t^A$, which implies that $(z_T^1 + z_T^2)/2$ is the density for $(Q^1 + Q^2)/2$. This shows that the latter measure lies in \mathcal{P}_A^Θ .

(d) By the analogous arguments in Cuoco and Cvitanic [11], using the weak compactness of Θ_A by Lemma 3, $Z^\Theta = \{z_T^\theta : \theta \in \Theta_A\}$ is norm closed in $L^1(\Omega, \mathcal{F}_T, P)$. Moreover, because Z is convex, it is also weakly closed. Since $E_A(|z_T^\theta|) = 1$ for all θ , Z is norm-bounded. Therefore, by the Alaoglu Theorem, Z is weakly compact. Finally, Z^Θ is homeomorphic to \mathcal{P}^Θ when weak topologies are used in both cases.

(e) follows from the properties of Θ established in Lemma 4. \square

Having established rectangularity and regularity of the set of multiple priors that arise we next move to give a recursive representation of the contracting problem.

4. RECURSIVE UTILITY IN THE CONTRACTING PROBLEM

The tractability of analysis in contracting problems in continuous time relies on recursive representation of values to the contracting parties. This section shows that the recursive utility formulation of Chen and Epstein [9] for decision problem under ambiguity generalizes to the contracting problem. The key observation that allows for the generalization is that the contract variables consumption and effort processes takes an analogous role of consumption processes in the analysis of Chen and Epstein. The main difference is that the effort choice made in the current period is imperfectly observed. The latter concern is not present in Chen and Epstein. Insights from Sannikov's [36] formulation relevant for the contracting problem further our analysis in this case and we represent the utility from each consumption and effort process specified in a recursive manner.

The main elements in our analysis builds on Chen and Epstein's recursive formulation which in turn uses recursive utility formulation in Duffie and Epstein [18, 19]. Duffie and Epstein showed that, under suitable Lipschitz conditions on contemporaneous utility function f , the recursive utility solves a Backward Stochastic Differential Equation (BSDE) and satisfies the usual properties of standard utilities (e.g., concavity with respect to consumption if the BSDE is concave). Their analysis makes powerful use of the Martingale Representation

theorem and Girsanov's theorem for change of measures. Our construction of recursive formulation rests on these ideas.

The main result of this section shows that the value processes in the contracting problem (25) under the minmax criterion has an equivalent recursive representation. As a preliminary step that specializes to Duffie and Epstein [19]'s formulation, fix a contract (c_t) , take $a_t = 0$ and assume no ambiguity. In this case the consumption process is measurable only with respect to the standard Brownian motion under the reference measure P . Following Duffie and Epstein [19] the expected utility process of any given consumption process (c_t) is then defined by

$$V_t^P = E_P \left[\int_t^T f(c_s, V_s^P) ds | \mathcal{F}_t \right]. \quad (7)$$

where f is an aggregator function that in general allows for non-separability over temporal composition of utility flow. In the special case of our main interest we assume the standard expected discounted utility $f(c, a, v) = u(c) - h(a) - \beta v$. In this case, the value process is given by

$$V_t^P = E_P \left[\int_t^T e^{-\beta(s-t)} (u(c_s) - h(a_s)) ds | \mathcal{F}_t \right].$$

Under ambiguity given any action process $a = (a_t)$ there is a set of priors \mathcal{P}^{Θ^a} associated with set of Girsanov exponentials Θ^a induced by the action process and the minmax criterion implies the following value to the agent

$$V_t^a = \min_{\theta \in \Theta^a} E_{Q^\theta} \left[\int_t^T e^{-\beta(s-t)} u(c_s) - h(a_s) ds | \mathcal{F}_t \right]. \quad (8)$$

Our goal is to represent this value process recursively in a tractable manner. To develop the analysis consider first with Duffie and Epstein [19] that the process in the standard expected utility specification is rewritten in a simpler recursive form:

$$V_t^P + \int_0^t f(c_s, V_s^P) ds = E_P \left[\int_0^T f(c_s, V_s^P) ds | \mathcal{F}_t \right]$$

which is a martingale under P . The recursive formulation of value in this case follows from the martingale representation theorem:

$$dV_t^P = -f(c_t, a_t, V_t^P) dt + \sigma_t^P \cdot dB_t, \quad V_T^P = 0 \quad (9)$$

with the unique solution for the value process (V_t^P) and the volatility process (σ_t^P) , where the dependence on the reference measure is noted by superscript P . Using the fact that $\int_0^t \sigma_s^P dW_s$ is a martingale and reversing the arguments establish that the solution to the BSDE (9) for (V_t^P) is the expected utility process for (c_t) in (7).

Using Girsanov Theorem one can change the measure from P to Q^θ for each θ and hence obtains the analogous representation of the utility process. By the

representation in Chen and Epstein [9] the value process solves the following BSDE

$$dV_t^\theta = [-f(c_t, V_t^\theta) + \theta_t \cdot \sigma_t^\theta] dt + \sigma_t^\theta \cdot dB_t, \quad V_T^\theta = 0 \quad (10)$$

The additional additive term in drift relative to that in (9) accounts for the change in measure.⁵ In the contracting problem by Sannikov [36], there is no ambiguity and a contract induces a costly effort process (a_t) that generates an outcome process with a drift $\mu(a_t) = a_t$. Taking the latter in the role of (θ_t) in (10) and using the standard aggregator, $f(c, a, V) = u(c) - h(a) + \beta V$, gives the recursive representation for the agent's utility process in Sannikov [36] as a (weakly) unique solution to the following BSDE

$$dV_t^a = [-f(c_t, a_t, V_t^a) + \mu(a_t) \cdot \sigma_t^a] dt + \sigma_t^a \cdot dB_t, \quad V_T^a = 0 \quad (11)$$

Building on this representation we introduce the notion of ambiguity (IID and symmetric) for any effort process (a_t) which gives rise to a set of drift terms $\mu(a_t) + \Theta^a$. Our main representation result is that under the minmax criterion the expected utility process can similarly be represented as a diffusion process by generalizing the representation of recursive utility in Chen and Epstein [9] to allow for a family of drift terms that depends on the action process:

Proposition 3. Fix a contract $(c_t, a_t) \in D$ and let Θ_a be the corresponding set of measures. Then:

- (a) There exists unique processes (V_t^a) and (σ_t^a) solving the BSDE

$$dV_t^a = \left[-f(c_t, a_t, V_t^a) + \mu(a_t) \cdot \sigma_t^a - \min_{\theta \in \Theta^a} \theta_t \cdot \sigma_t^a \right] dt + \sigma_t^a \cdot dB_t, \quad V_T^a = 0. \quad (12)$$

- (b) For each $Q_a \in \mathcal{P}^{\Theta^a}$, let $(V_t^{Q_a})$ be the unique solution to (11). Then V_t^a defined in (a) is the unique solution to (8) and there exists $Q^* \in \mathcal{P}^{\Theta^a}$ such that

$$V_t^a = V_t^{Q^*}, \quad 0 \leq t \leq T. \quad (13)$$

- (c) The process (V_t^a) is the unique solution to $V_T^a = 0$ and

$$V_t^a = \min_{Q \in \mathcal{P}^{\Theta^a}} E_Q \left[\int_t^\tau f(c_s, a_s, V_s) ds + V_\tau | \mathcal{F}_t \right], \quad 0 \leq t < \tau \leq T. \quad (14)$$

The formulation of the recursive utility is related to Chen and Epstein [9] due to ambiguity and to Sannikov [36] due to the contracting problem. The main difference from the former is in generalizing the recursive utility formulation given by Chen and Epstein to a family of drift terms that varies with the effort process chosen by the agent motivated by Sannikov. The generalization follows from using Girsanov's Theorem that allows the changes of measures. and specializes to the case examined by Sannikov by choosing $\Theta_t^a(\omega) = \mu(a_t)$ for each t and ω . We have shown earlier in Proposition 2 that the notion of ambiguity that is modeled as

⁵For details on the use of the Martingale representation theorem and Girsanov theorem, see the manuscript Duffie [17].

IID and symmetric between the principal and agent gives rise to the set of priors which satisfies time-consistency and regularity conditions as defined in Chen and Epstein. Accordingly, the regularity ensures that various value processes that arise in contracting problem are well-defined and the rectangularity of the set of priors allows us to replace the agent's optimization under the entire contract with a sequence of temporal optimization problems. This, together with two powerful results from stochastic analysis, namely the martingale representation theorem and Girsanov's theorem for changes of measures, yield a recursive formulation for the agent's expected utility in a similar manner as in Sannikov. The interpretation with maxmin criterion is that the agent evaluates a given contract under the worst-case scenario which corresponds to the lowest drift induced by her effort choice.

The last piece among the analytical results represents the primitive set (Θ_t) in an equivalent functional form, using its support functions and this form is more convenient in the theoretical development. Because each correspondence Θ_t is convex-valued, its structure, by Hahn-Banach theorem in its supporting functions form, can be represented by its support functions defined by

$$e_t(x)(\omega) = \max_{y \in \Theta_t^a(\omega)} y \cdot x, \quad x \in R^d. \quad (15)$$

The difference from Chen and Epstein [9] is the renormalization to the base-line drift to $\mu(a_t)$ under the action process (a_t) . Under this renormalization, the support function is still Lipschitz continuous, convex and linear; and the joint measurability holds: the map $(t, \omega) \rightarrow e_t(x)(\omega)$ is $\mathcal{B}([0, s]) \times \mathcal{F}_s$ -measurable on $[0, s] \times \Omega$ on $(0, T] \times R^d$. However, unlike in Chen and Epstein it need not be non-negative as the normalization is not the origin for each effort choice but a principal does not implement such an effort as an outside option that yields non-negative value is always feasible. With these elements in place the proposition follows from the following observations.

Proof. (of Proposition 3)

- (a) Since the support function e and the utility function are Lipschitz continuous and satisfy progressive measurability, the unique existence of the solution to (11) and (12) follows from Pardoux and Peng [32, Theorem 4.1.]
- (b) With the renormalization, the set of density generators Θ_a to the base-line measure $\mu(a_t)$ under the action process (a_t) , as we show in Propositions 1 and 2, satisfy dynamic consistency and regularity defined in Chen and Epstein. Furthermore, the Comparison Theorem applies the same way as it does not depend on the structure of the set of density generators. It therefore follows from analogous arguments as in Chen and Epstein [9, Theorem 2.2 (b)].
- (c) The analogous arguments from Chen and Epstein go through as they do not depend on the particular choice of the normalization used as we established in Propositions 1 and 2.

□

The analysis thus far has used a fixed terminal time T . This is mainly done to bring forth the key elements in the analysis in a simple way. The contracting problem, however, does not necessarily have a relationship for a predetermined period of time. In particular, the continuation of a contract depends on the performance within the relationship and there can be termination following sufficiently many observations of poor performance or retirement when the continuation of a contract becomes costly after good performances. The extension of the results to allowing a stopping-time instead of a deterministic time horizon follows from virtually the same way as it is done in Duffie and Epstein [18] for stochastic differential utility.

Having established the recursive representation of value induced by any effort process (a_t) we move to derive a tractable incentive-compatibility condition and using it characterize the optimal contract. In the next section we formulate a “one-shot deviation” principle from discrete-time dynamic games to verify incentive compatibility of effort process given a contract.

5. INCENTIVE COMPATIBILITY UNDER AMBIGUITY

An effort process (a_t) is implementable if there is a contract that specifies transfers (c_t) to the agent given observable output realizations and that (a_t) is compatible with the agent’s incentives, that is he chooses effort (a_t) . We use this standard definition for implementability of effort to determine the feasible set of implementable contracts for the principal. We specialize the implementability to the ambiguity regarding the drift term. Assume that for each effort process (A_t) the associated multiple set of priors \mathcal{P}^A is equivalently characterized by the set of drift terms Θ^A using the formulation in (2) and (3). A useful characterization for implementability follows below from representing agent’s value from a contract as a diffusion process.

Proposition 4. (Representation of the agent’s value as a diffusion process) For any contract (C_t) and any effort process (A_t) with its associated set of drift terms Θ^A there exists a progressively measurable process (Z_t) such that

$$W_t = W_0 + \int_0^t r \left(W_s - u(C_s) + h(A_s) + \min_{\theta_s \in \Theta_s^A} \theta_s |Z_s| \right) ds + \int_0^t r Z_t (dX_s - \mu(A_s) ds) \quad (16)$$

for every $t \in [0, \infty)$.

Proof. For a given pair of processes (C_t) and (A_t) for transfers to the agent and effort, respectively, define the valuation process V by

$$V_t = r \int_0^t e^{-rs} (u(c_s) - h(A_s)) ds + e^{-rt} W_t(C, A) \quad (17)$$

where $W_t(C, A)$ is the continuation value defined by

$$W_t = \min_{Q \in \mathcal{P}^{\Theta^A}} E^Q \left[\int_t^\infty e^{-rs} (u(C_s) - h(A_s)) ds \mid \mathcal{F}_t \right]$$

By rectangularity of the multiple-priors, the valuation process (V_t) is a g -martingale. Using the g -martingale representation theorem in Chen and Epstein [9], there exists a measurable process Z_t such that

$$-dV_t = -\kappa_t^* r e^{-rt} |Z_t| dt - \sigma r e^{-rt} Z_t dB_t^A \quad (18)$$

where B_t is a Brownian motion under the reference measure P ; $\kappa_t^* \min_{\theta_t \in \Theta_t^A} \theta_t |Z_t|$ is the worst-case drift; and the factor $r e^{-rt} \sigma$ is a convenient rescaling. On the other hand, differentiating (17) with respect to t one finds that

$$dV_t = r e^{-rt} (u(C_t) - h(A_t)) dt - r e^{-rt} W_t dt + e^{-rt} dW_t \quad (19)$$

Together (18) and (19) imply that

$$\begin{aligned} r e^{-rt} (u(C_t) - h(A_t)) dt - r e^{-rt} W_t dt + e^{-rt} dW_t &= \kappa_t^* r e^{-rt} \kappa |Z_t| dt + \sigma r e^{-rt} Z_t dB_t^A \\ \implies dW_t &= r \kappa_t^* |Z_t| dt + \sigma r Z_t dB_t^A - r (u(C_t) - h(A_t)) dt + r W_t dt \end{aligned}$$

This further implies

$$W_t = W_0 + \int_0^t r \left(W_s - u(C_s) + h(A_s) + \min_{\theta_s \in \Theta_s^A} \theta_s |Z_s| \right) ds + \int_0^t r Z_s dB_s^A$$

□

The analysis here is closely related to Sannikov's representation. Compared to the formulation in Sannikov [36] the analysis with ambiguous information introduces a term $\kappa_{A_s}^* |Z_s|$ which is interpreted as capturing the effect introduced by ambiguity. The agent uses the worst case to evaluate a contract. Here the worst case corresponds to the drift terms that yield the minimum value to the agent. Using this observation we next present a tractable incentive compatibility condition that characterizes the agent's effort choice for a given contract in an environment with ambiguity.

Proposition 5. (The Agent's incentives) For a given strategy $A = (A_t)$, let (Z_t) be the volatility process from Proposition 4. Then A is optimal if and only if

$$\forall a \in \mathcal{A} \quad Z_t \mu(A_t) - h(A_t) + \min_{\theta_t \in \Theta^A} \theta_t |Z_t| \geq Z_t \mu(a_t) - h(a_t) + \min_{\theta_t \in \Theta^a} \theta_t |Z_t| \quad dt \otimes dQ^A \text{ a.e.} \quad (20)$$

Remark: Since the Q^A is equivalent to P by Girsanov's Theorem, and hence has the same zero-sets, without any loss in generality $dt \otimes dP$ a.e. replaces $dt \otimes dQ^A$ a.e. in Sannikov [36].

The characterization uses analogous ideas from Sannikov, generalizes to ambiguous information using the representation we develop earlier and finally applies a version of the Comparison Theorem that has been helpful in establishing principle of optimality in stochastic analysis. The following fills in the details.

Proof. Consider an arbitrary alternative strategy A' that follows possibly different actions A'_τ up to t and afterwards continues with A_t . The effort process A' induces a set of densities Θ^A satisfying the regularity conditions as specified earlier. The

corresponding set of multiple priors $\mathcal{P}^{\mathcal{Q}_A}$ is rectangular. The agent's expected payoff from this action process is well defined by

$$V'_t = \min_{Q \in \mathcal{P}^{\mathcal{Q}_{A'}}} V_t^Q,$$

where V_t^Q is unique solution (ensured by Duffie and Epstein [19]) to BSDE

$$V_t^Q = E^Q \left[\int_t^\infty f(C_s, A'_s, V_s^Q) \right],$$

where in our formulation we use the standard aggregator, i.e., $f = u(C) - h(A) - \beta V$. By [9, Theorem 2.2], V' is equivalently uniquely characterized as follows:

$$dV'_t = \left[-f(C_t, A'_t, V'_t) + \max_{\theta \in \Theta^{A'}} \theta_t Z'_t \right] dt + Z'_t dB'_t$$

for a unique volatility process Z' .

More generally, V' and Z' uniquely solves a BSDE of the following form

$$dV_t = g'(V_t, Z_t, \omega, t) dt + Z_t dB'_t, \quad (21)$$

with terminal condition ξ . In the special case relevant for our analysis, we have

$$g'(V, Z, \omega, t) = -f(C_t(\omega), A'_t(\omega), V) + \max_{\theta \in \Theta^{A'}} \theta(\omega) Z \quad (22)$$

Under the action process A_t the value process V_t and volatility Z_t solve (21) for A and $g(\cdot)$.

Suppose that the condition (20) holds. Since the terminal conditions are the same under A and A' , by the Comparison Theorem [21, Theorem 2.2]

$$g(V, Z, \omega, t) \leq g'(V, Z, \omega, t) \quad dt \otimes dP \text{ a.e.} \quad (23)$$

or equivalently the condition (20) in our model

$$\mu(A_t) Z_t - h(A_t) - \max_{\theta \in \Theta^A} \theta(\omega) Z_t \geq \mu(A'_t) Z_t - h(A'_t) - \max_{\theta \in \Theta^{A'}} \theta(\omega) Z_t \quad dt \otimes dP \text{ a.e.}$$

implies that $V \geq V'$ for almost every t .

Suppose now that the condition (20) fails on a set of positive measures, choose A' that maximizes $\mu(A') Z_t - h(A') - \max_{\theta \in \Theta^{A'}} \theta(\omega) Z$ for all $t \geq 0$. Then $g(V, Z, \omega, t) \leq g'(V, Z, \omega, t) \quad dt \otimes dP \text{ a.e.}$. Since A' specifies the same action as A after t , by the Comparison theorem $V' > V$. Therefore, A is suboptimal. \square

If the volatility process is written as $-Z_t$ the minimum replaces the maximum in (23). Notice that the Proposition 5 is formulated for any generating process Θ^A . To illustrate the intuition that the presence of ambiguity introduces into contract design we specialize the formulation to a simpler case. Taking $\Theta^A := \{(\theta)_t : \mu(A_t) + |\theta_t| \leq \kappa(A_t)\}$ in the set up of Proposition 4, Proposition 5 specializes the result to κ -ignorance model (that features symmetry around the base-line drift) and the corresponding necessary and sufficient incentive compatibility condition is given by

Lemma 3. For a given strategy A , let (Z_t) be the volatility process from Proposition 4 for κ_A -ignorance. Then A is optimal if and only if

$$\forall a \in \mathcal{A} \quad Z_t \mu(A_t) - \kappa(A_t) |Z_t| - h(A_t) \geq Z_t \mu(a_t) - \kappa(a_t) |Z_t| - h(a_t) \quad dt \otimes dP \text{ a.e.} \quad (24)$$

Notice that setting $\kappa \equiv 0$ removes ambiguity and specializes the condition to the incentive compatibility condition in the classical case formulated in Sannikov [36] without ambiguity. Compared to this case the presence of ambiguity introduces added additive terms in the middle of both sides of the incentive compatibility comparison. These additional terms have negative signs and discounts for the worst case using the minimum drift relative to the continuation value.

In particular, since the process (Z_t) reflects from (16) how the agent values the variation in the continuation value, we see that higher values of κ , which is interpreted as higher ambiguity, reduces the value of process more drastically. In the standard contracting problem, high level of effort is incentivized through variation in the continuation value that is sensitive to output realizations. This effect is still present in the first terms as (Z_t) measures the utility consequence to the agent of this variation. However, the incentive effect of variation in the continuation value is now tempered by the presence of ambiguity, which acts as a cost and penalizes high variations in the continuation value. The extent of this effect is directly reflected by the additional terms in the incentive-compatibility condition. Therefore, everything else being equal, the presence of ambiguity limits the incentive effects of variation in the continuation value through output realizations. Note also that, as in the classic contracting problem, the variation (Z_t) in continuation value is an endogenous object and depends on the contract offered. Therefore, in the contract design the principal optimally resolves the trade-off between high effort and high variation. Using the tractable incentive compatibility condition presented in this section, the next section formulates the optimal contracting problem and analyzes it.

6. THE OPTIMAL CONTRACT

Trading off the benefit of higher effort against its effort cost and ambiguity aversion the principal designs the optimal contract. The principal offers a contract to the agent that specifies a stream of consumption (C_t) contingent on the realized output and an incentive-compatible advice of effort (A_t) that maximizes the principal's expected profit under minmax criterion

$$\mathcal{E}(F) = \min_{Q \in \mathcal{P}^{\Theta^A}} E^Q \left[r \int_0^\infty e^{-rt} dX_t - r \int_0^\infty e^{-rt} C_t dt \right] \quad (25)$$

subject to delivering the agent a required initial value of at least \hat{W}

$$\mathcal{E}(V(C, A)) = \min_{Q \in \mathcal{P}^{\Theta^A}} E^Q \left[r \int_0^\infty e^{-rt} (u(C_t) - h(A_t)) dt \right] \geq \hat{W}$$

Implicit in this formulation are the termination and retirement clauses of a contract. These events are explicitly characterized below within the set of consumption streams. To illustrate briefly, termination is captured as follows. After a sufficiently long period of low enough output realizations consumption a contract is terminated and the consumption stream is set to a low level. The interest is in contracts that generate non-negative expected profits for the principal. Derivation of the optimal contract uses the techniques of Sannikov [36] in a continuous-time moral hazard problem while introducing ambiguity similar to Chen and Epstein [9].

One possible option for the principal is to retire the agent with any value $W \in [0, u(\infty)$, where $u(\infty) = \lim_{c \rightarrow \infty} u(c)$. To retire the agent with value $u(c)$, the principal offers him constant consumption c and allows him to choose zero effort. Denote the principal's profit from retiring the agent by

$$F_0(u(c)) = -c.$$

Since the agent can always guarantee himself non-negative utility by taking effort 0, the principal cannot deliver any value less than 0. The only way to deliver value 0 is through retirement. To see this, notice that the future payments to the agent are not always 0, the agent can guarantee himself a strictly positive value by putting effort 0. We call F_0 the principal's *retirement profit*.

Given the agent's consumption $c(W)$ and recommended effort $a(W)$, the evolution of the agent's continuation value W_t can be written as

$$dW_t = r(W_t - u(c(W_t)) + h(a(W_t)) + \kappa(a(W_t))|Z(W_t)|) dt + rZ(W_t)\sigma dY_t$$

where $\sigma dY_t := (dX_t - \mu(a(W_t))dt)$ and $rZ(W)$ is the sensitivity of the agent's continuation value to output and follows from the representation given in the previous section. When the agent takes the recommended effort, the second term $dX_t - \mu(a(W_t))dt$ has mean 0, and so drift of the agent's expected continuation value is given by the first term $r(W_t - u(c(W_t)) + h(a(W_t)) + \kappa(a(W_t))|Z(W_t)|)$. To account for the value that the principal owes to the agent, W_t grows at the interest rate r and falls due to the flow of repayments $r(u(c(W_t)) - h(a(W_t)))$ and *additionally* due to aversion to ambiguity it is reduced by $\kappa(a(W_t))|Z(W_t)|$ to account for the worst case. The latter is the main effect that ambiguity aversion introduces to the design of dynamic contracts.

The sensitivity $rZ(W_t)$ of the agent's value to output affects the agent's incentives. If the agent deviates to a different effort level, his actual effort affects only the drift of X_t and his incentive compatible choice is characterized by (24).

The optimal contract offered by the principal describes the choice of payments $c(W)$ and effort recommendations $a(W)$. Let $F(W)$ be the highest profit that the principal can obtain when he delivers the agent value W . Function $F(W)$ together with the optimal choices of $a(W)$ and $c(W)$ satisfy the Hamiltonian-Jacobi-Bellman (HJB) equation

$$rF(W) = \max_{a>0,c} r[\mu(a) - \kappa(a) - c] + F'(W)r[W - u(c) + h(a) + \kappa(a)|Z(a)|] + \frac{F''(W)}{2}r^2\sigma^2Z(a)^2 \quad (26)$$

In this formulation, the principal is maximizing the expected current flow of profit $r(\mu(a) - \kappa(a) - c)$ discounted according to the worst-case drift plus the expected change of future profit due to the drift and volatility of the agent's continuation value that reflects the agent's ambiguity aversion.

The equation (26) is rewritten in the following form suitable for computation

$$F''(W) = \min_{(a>0,c)} \frac{F(W) - a + c + \kappa(a) - F'(W)(W - u(c) + h(a) + \kappa(a)|Z(a)|)}{r\sigma^2 Z^2(a)/2} \quad (27)$$

The optimal contract is characterized as a solution to this differential equation by setting

$$F(0) = 0 \quad (28)$$

and choosing the largest slope $F'(0) \geq F'_0(0)$ such that the solution F satisfies

$$F(W_{gp}) = F_0(W_{gp}) \quad \text{and} \quad F'(W_{gp}) = F'_0(W_{gp}) \quad (29)$$

at some point $W_{gp} \geq 0$, where $F'(W_{gp}) = F'_0(W_{gp})$ is called the *smooth-pasting condition*. Let functions $c : (0, W_{gp}) \rightarrow [0, \infty)$ and $a : (0, W_{gp}) \rightarrow \mathcal{A}$ be the minimizers in equation (27). A typical form of the value function $F(0)$ together with $a(W)$, $c(W)$ and the drift of the agent's continuation value is shown in Figure 2.

Theorem 1, which is proved formally in the Appendix, characterizes the optimal contracts.

Theorem 1. *The unique concave function $F \geq F_0$ that satisfies (27), (28), and (29) characterizes any optimal contract with positive profit to the principal. For the agent's starting value of $W_0 > W_{gp}$, $F(W_0) < 0$ is an upper bound on the principal's profit. If $W_0 \in [0, W_{gp}]$, then the optimal contract attains profit $F(W_0)$. Such a contract is based on the agent's continuation value as a state variable, which starts at W_0 and evolves according to*

$$dW_t = r(W_t - u(C_t) + h(A_t) + \kappa(A_t)|Z(A_t)|) dt + rZ(W_t)\sigma dY_t \quad (30)$$

where $\sigma dY_t := dX_t - (\mu(A_t) - \kappa(A_t))dt$ under payments $C_t = c(W_t)$ and effort $A_t = a(W_t)$, until the retirement time τ . Retirement occurs when W_τ hits 0 or W_{gp} for the first time. After retirement the agent gets constant consumption of $-F_0(W_\tau)$ and puts effort 0.

As in discrete time continuation-value W_t summarizes the past history in the optimal contract. Replacing the continuation contract, while leaving the continuation value the same, does not affect the incentives governing the choice of effort in the current period. Therefore, to maximize the principal's profit after any history, the continuation contract must be optimal given W_t . It follows that the agent's continuation value W_t completely determines the continuation contract. This logic does not necessarily follow when there are additional state variables, for example, when hidden savings by the agent are allowed. We abstract from the latter to focus on the implication of ambiguous information.

Turning to the discussion of optimal effort and consumption using (26) notice that the optimal effort maximizes

$$r(\mu(a) - \kappa(a)) + r(h(a) + \kappa(a)|Z(a)|) F'^2(W)\sigma^2 Z(a)^2 \frac{F''(W)}{2} \quad (31)$$

where $r(\mu(a) - \kappa(a))$ is the expected flow of output according to the worst-case scenario, $rF'(W)(h(a) + \kappa(a)|Z(a)|)$ is the cost of compensating the agent for his effort, and $r^2\sigma^2\frac{\gamma(a)^2}{2}F''(W)$ is the cost of exposing the agent to income uncertainty to provide incentives. The presence of ambiguity introduces a worst-case scenario κ_a and changes the sensitivity of the continuation value $Z(a)$ relative to the case without ambiguity. These two costs typically work in opposite directions, creating a complex effort profile (see Figure 2). While $F'(W)$ decreases in W because F is concave, $F''(W)$ increase over some ranges of W . It turns out that in the optimal contract the introduction of ambiguity in the contracting problem reduces the sensitivity of the optimal incentive scheme to output realizations.

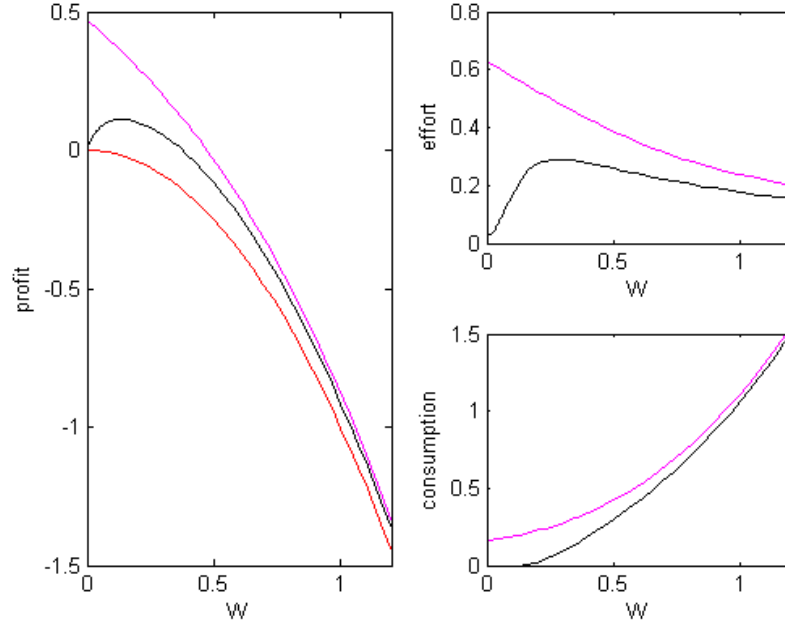


FIGURE 1. The Classical contracting problem in Sannikov

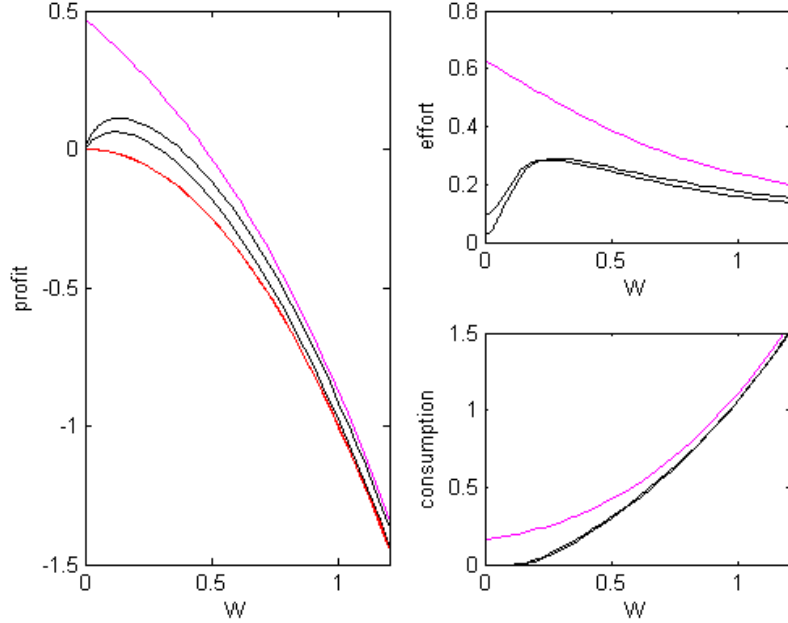


FIGURE 2. Kappa model of ambiguity in the classical contracting

The optimal choice of consumption maximizes

$$-c - F'(W)u(c)$$

Thus the agent's consumption is 0 when $F'(W) \geq -1/u'(0)$ in the probationary interval $[0, W^{**}]$, and it is increasing in W according to $F'(W) = -1/u'(c)$ above W^{**} . Intuitively, $1/u'(c)$ and $-F'(W)$ are the marginal costs of giving the agent value through current consumption and through his continuation payoff, respectively. Those marginal costs must be equal under the optimal contract, except in the probationary interval. There, consumption zero is optimal because it maximizes the drift of W_t away from the inefficient low retirement point.

An important feature of the optimal contract regards the termination of the contract. Similarly as in Sannikov low continuation values lead to termination of the contract. With ambiguity continuation value at which contract is terminated and at which the agent is retired remain the same but the sensitivity of continuation value is lower with ambiguity. One implication of these observations is that the contractual relationship on average lasts longer. Formally, this result follows from the analysis of HJB equation as in Sannikov. In particular, the profit function with ambiguity satisfies $F(W) > F_0(W)$ for all $W \in [0, \infty)$ and lies below the profit function without ambiguity, i.e. $\kappa(\cdot) = 0$. Since the retirement value F_0 does not depend on the nature of ambiguity, F is tangent to F_0 at W_{gp} as in Sannikov. A longer contract makes the termination less likely and a longer stream of payments to the agent and in effect yields back-loaded benefits to agent. One interpretation is that the presence of ambiguity leads to more tolerance on the part of the principal following low realization of output that is attributed to lower values of drifts attributed to the worst case. The optimality of such delay also

means longer contractual relations for the agent to receive flow payments and this helps with incentives to provide higher effort levels as shown in Figure 2.

7. CONCLUDING REMARKS

Contracting parties interact with imprecise information about the environment they interact in. In this paper, we focused on a particular form of information imprecision: both contracting parties have common ambiguity about the productive technology. This assumption has allowed us to: (1) apply and extend the decision-theoretic model of Chen and Epstein [9] to continuous-time setting relevant for the dynamic contracting problem; and (2) tractably generalize the principal-agent problem proposed by Sannikov [36] to incorporate richer uncertainty. Pursuing the latter we have found that our model of ambiguity illustrates a new trade-off between effort and variation of compensation and that the optimal resolution of the trade-off favors simple contract structures.

For tractability we have abstracted our analysis from differential information between the contracting parties on the technology. It is left to future research to extend our model to incorporate the richer nature of ambiguous information that, for instance, could allow for learning and experimenting in the design of contracts.

8. APPENDIX FOR THE PROOFS

The analysis follows analogous steps as in Sannikov [36]. Using the HJB we formulate a conjecture for an optimal contract. We show that the HJB satisfies appropriate regularity properties and that it has a unique solution. From that solution we form a conjecture for an optimal contract and then verify its optimality.

For regularity we consider a version of HJB

$$F''(W) = \min_{(a,Z) \in \Gamma, c \in [0, \bar{C})} \frac{F(W) - a + c + \kappa_a - F'(W)(W - u(c) + h(a) + \kappa_a |Z(a)|)}{r\sigma^2 Z^2(a)/2} \quad (32)$$

where the sensitivity parameter Z is bounded from below by γ_0 , and consumption is bounded from above by the level \bar{C} such that $u'(\bar{C}) = \gamma_0$. The existence and uniqueness of solutions to the HJB equation (27) satisfying the boundary conditions (29) follows from analogous arguments made by Sannikov [36] since the right-hand side of (27) is Lipschitz continuous in all of its arguments.

8.1. Conjecture of a contract. We conjecture an optimal contract from the solution of equation HJB just constructed.

Proposition 6. Consider the unique solution $F(W) \geq F_0(W)$ that satisfies boundary conditions (29) for some $W_{gp} \in [0, W_{gp}^*]$. Let $a : [0, W_{gp}^*] \rightarrow \mathcal{A}$, $Y : [0, W_{gp}^*] \rightarrow [\gamma_0, \gamma_1]$ and $c : [0, W_{gp}^*] \rightarrow [0, \bar{C}]$ be the minimizers in (27). For any

starting condition $W_0 \in [0, W_{gp}]$ there is a unique solution, in the sense of weak probability law, to the following equation

$$dW_t = r(W_t - u(c(W_t)) + h(a(W_t)) + \kappa_a(W_t)|Z(W_t)|)dt + rZ(W_t)(dX_t - [a(W_t) - \kappa_a(W_t)]dt)$$

where the last term is a Brownian Motion: $\sigma dB_t^{a(W)} = dX_t - a(W_t)Z(W_t)dt$ until the time τ . The contract (C, A) defined by

$$\begin{aligned} C_t &= c(W_t), \text{ and } A_t = a(W_t), \text{ for } t \in [0, \tau] \\ C_t &= -F_0(W_\tau), \text{ and } A_t = 0, \text{ for } t \geq \tau \end{aligned}$$

is incentive-compatible, and it has a value W_0 to the agent and profit $F(W_0)$ to the principal.

Proof. From the representation of $W_t(C, A)$ in Proposition 4, we have

$$d(W_t(C, A) - W_t) = r(W_t(C, A) - W_t)dt + r(Y_t - Y(W_t))\sigma dB_t^A + r\kappa_A(|Z_t| - |Z(W_t)|)dt$$

where the changes of measures are conducted under the worst-case measures. This implies that

$$E_t[W_{t+s}(C, A) - W_{t+s}] = e^{rs}(W_t(C, A) - W_t) + e^{rs}E_t\kappa_A(|Z_t| - |Z(W_t)|)$$

Notice that the left hand side must remain bounded, because both W and $W(A, C)$ (since C_t is bounded) are bounded, and the processes Z_t and $Z(W_t)$ are bounded by the representation theorem. It follows that $W_t = W_t(C, A)$ for all $t \geq 0$, and in particular, the agent gets value $W_0 = W_0(C, A)$ from the entire contract. Also, the contract (C, A) is incentive compatible, since $(A_t, Z_t) \in \Gamma$ for all t .

To see that the principal gets profit $F(W_0)$, consider

$$G_t = r \int_0^t e^{-rs}(A_s - \kappa_{A_s} - C_s)ds + e^{-rt}F(W_t).$$

By Ito's lemma, the drift of G_t is

$$re^{-rt} \left((A_t - \kappa_{A_t} - C_t - F(W_t)) + F'(W_t)(W_t - u(C_t) + h(A_t) + \kappa_{A_t}|Z_t|) + r\sigma^2 Z_t^2 \frac{F''(W_t)}{2} \right).$$

The value of this expression is 0 before time τ by the HJB equation. Therefore, G_t is a bounded martingale until τ and the principal's profit from the entire contract is

$$\begin{aligned} & \min_{Q^A \in \mathcal{P}^A} E^{Q^A} \left[r \int_0^\tau e^{-rs}(A_s - C_s)ds + e^{-r\tau}F_0(W_\tau) \right] \\ &= E \left[e \int_0^\tau e^{-rs}(A_s - \kappa_{A_s} - C_s)ds + e^{-r\tau}F_0(W_\tau) \right] = E[G_\tau] = G_0 = F(W_0), \end{aligned}$$

since $F(W_\tau) = F_0(W_\tau)$. □

8.2. Verification. Our last step is to verify that the contract presented in Proposition 6 is optimal. We start with a lemma that bounds from above the principal's profit from contracts that give the agent a value higher than W_{gp}^* .

Lemma 4. *The profit from any contract (C, A) with the agent's value $W_0 \geq W_{gp}^*$ is at most $F_0(W_0)$*

Proof. Define c by $u(c) = W_0$. Then $W_0 \geq W_{gp}^*$ implies that $u'(c) \leq \gamma_0$. We have For any Q we have

$$\begin{aligned} E^Q \left[r \int_0^\infty e^{-rt} (u(C_t) - h(A_t)) \right] &\leq E^Q \left[r \int_0^\infty e^{-rt} (u(c) + (C_t - c)u'(c) - \gamma_0 A_t) dt \right] \\ &\leq u(c) - u'(c) \left(E^Q \left[r \int_0^\infty e^{-rt} (A_t - C_t) dt \right] + c \right), \end{aligned}$$

In particular,

$$\begin{aligned} W_0 &= \min_Q E^Q \left[r \int_0^\infty e^{-rt} (u(C_t) - h(A_t)) \right] \\ &\leq u(c) - u'(c) \left(E \left[r \int_0^\infty e^{-rt} (A_t - \kappa_{A_t} - C_t) dt \right] + c \right) \end{aligned}$$

where $u(c) = W_0$ and $c = -F_0(W)$. It follows that the profit from this contract is at most $F_0(W)$. \square

Next, note that function F from which the contract is constructed satisfies

$$\min_{W' \in [0, \infty)} F(W) - F_0(W') - F'(W)(W - W') = \min_{c \in [0, \infty)} F(W) + c - F'(W)(W + u(c)) \geq 0 \quad (33)$$

for all $W \geq 0$. For any such solution, the optimizers in the HJB equation satisfy $a(W) > 0$ and $c(W) < \bar{C}$. If either of these conditions failed, (33) would imply that $F''(W) \geq 0$. Also we have that $Z(W) = \gamma(a(W))$.

Proposition 7. Consider a concave solution F to the HJB equation that satisfies (33). Any incentive-compatible contract (C, A) achieves profit of at most $F(W_0(C, A))$.

Proof. Denote the agent's continuation value by $W_t = W_t(C, A)$, which is represented by (16) using the process Z_t . By the Lemma, the profit is at most $F_0(W_0) \leq F(W_0)$ if $W_0 \geq W_{gp}^*$. If $W_0 \in [0, W_{gp}^*]$, define

$$G_t = r \int_0^t r e^{-rs} (A_s - \kappa_{A_s} - C_s) ds + e^{-rt} F(W_t)$$

as in Proposition 3. By Ito's lemma, the drift of G_t is

$$r e^{-rt} \left((A_t - \kappa_{A_t} - C_t - F(W_t)) + F'(W_t)(W_t - u(C_t) + h(A_t) + \kappa_{A_t} |Z_t|) + r \sigma^2 Z_t^2 \frac{F''(W_t)}{2} \right)$$

which is computed under the worst-case scenario.

Let us show that the drift of G_t is always non-positive. If $A_t > 0$ then Proposition 2 and the definition of γ imply that $Y_t \geq \gamma(A_t)$. Then equation HJB and

together with $F''(W_t) \leq 0$ imply that the drift of G is non-positive. If $A_t = 0$, then $F''(W) < 0$ and (33) imply that the drift of G_t is non-positive.

It follows that G_t is a bounded supermartingale until the stopping time τ' (possibly ∞) when W_t reaches W_{gp}^* . At time τ' the principal's future profit is less than or equal to $F_0(W_{gp}^*) \leq F(W_{gp}^*)$ by Lemma 4. Therefore, the principal's expected profit at time 0 is less than or equal to

$$E^A \left[\int_0^{\tau'} e^{-rt} (dX_t - C_t dt) + e^{-r\tau'} F(W_{\tau'}) \right] = E^A[G_{\tau'}] \leq G_0 = F(W_0).$$

□

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