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(In)determinacy and Time-to-Build

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Abstract

This paper generalizes Benhabib and Farmer [5], by allowing for a strictly positive time-to-build of capital. The introduction of a time-to-build delay yields a system of mixed functional differential equations. We develop an efficient strategy to fully describe the dynamic properties of our economy; in the simpler case of no externalities, the dynamic behavior of the economy around the steady state is of "saddle-path" type while, in the Behnabib Farmer model, it is possible to show that the presence of local indeterminacy rises for sufficiently large value of the time to build parameter even for small externalities.

Keywords: Indeterminacy; Time-to-Build; Mixed Functional Differential Equations

JEL Classification: E00, E3, O40.

1   Introduction

This paper is an extension of Benhabib and Farmer [5] under the time-to-build assumption that new capital goods become productive with some delay. The main concern is to understand how the dynamic properties of a neoclassical growth economy with production externalities change by the introduction of a time-to-build delay, as well as variations on its magnitude. In particular, we are interested in capturing the influence of time-to-build on the existence of local indeterminacy.

The implications of time-to-build has long been analyzed by economists (s.a. Bohm-Bawerk [8]), who have conjectured that production lags may induce cycles in output (see also Kalecki [18]) and account for the persistence of output fluctuations. In their seminal paper, Kydland and Prescott [22] argue that time-to-build, in the sense that investment projects need more than one period to be completed, strongly contributes to the persistence of the business cycle. Asea and Zak [1] propose a continuum time optimal growth model with a time-to-build delay and show that the optimal path may converge to the steady state, eventually by oscillations, or even (Hopf) cycle around it. Consequently, they show that the dynamics can be intrinsically oscillatory due (entirely) to the time-to-build technology.[1]

Local indeterminacy is a concept strictly related to the dynamics, and in particular to the stability properties of the equilibrium in an infinite horizon economy. In a two dimension dynamic general equilibrium model, with one control and one state, there is local indeterminacy when a steady state is not

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[1]Asea and Zak [1] use delayed differential equations to rigorously analyze the implications of time-to-build delays. See also Collard et al [11]. A rigorous proof of the existence of cycles in an optimal growth model with time-to-build was done by Rustichini [24].

(locally) a saddle path, as usual, but a stable node or a stable focus.[2] In these cases, the equilibrium is said to be locally indeterminate since for any given initial condition for the state variable there exists a continuum of initial levels of the control (or co-state), each of which associated to a different equilibrium path. Kehoe and Levine [19] argue that in pure exchange economies with infinitely lived consumers, equilibria are generically determinate. However, from the beginning of the nineties, infinitely lived agent models with some degree of increasing returns have been shown to exhibit multiple equilibria, indeterminacy, and the possibility of sunspots. Benhabib and Farmer [5] (hereafter BF) add increasing returns to the one sector neoclassical growth model and show that the equilibrium may be locally indeterminate[3].

In a discrete time Benhabib-Farmer framework, Hintermaier [16] analyses the existence of indeterminacy for different time frequencies. He shows that the conditions for the existence of indeterminacy are stronger the lower is the time frequency. At the limit, when the time frequency goes to infinite, or the period length goes to zero, he obtains the same conditions than in BF. As it is standard in discrete general equilibrium models, Hintermaier assumes that capital produced at time t becomes productive at time $t+1$. This is a one period time-to-build assumption. Consequently, by reducing the frequency of the economy the time-to-build becomes longer and longer.

The introduction of adjustment costs in the BF model, has been shown by Kim [20] to increase the required degree of increasing returns for indeterminacy to rise; Herrendorf and Valentinyi [15], starting with a two sector model characterized by mild sector-specific externalities, extend this result both in the case of total and of sector's specific capital adjustment costs.

In this paper, we extend BF by assuming that capital produced at time t becomes productive at time $t+\tau$, where $\tau > 0$ is a time-to-build delay. The analysis focuses, first, on the effect of the time to build in a Ramsey model with endogenous labour supply and then in a Behnabib Farmer model. It is possible to show that under some parametrization, local indeterminacy rises even when the necessary condition for indeterminacy of the "original" model doesn't hold. In other terms, the presence of any level of externalities are able to produce local indeterminacy of the steady state when a sufficiently large level of the time to build parameter is chosen.

The paper is organized as follows. Section 2 describes the time-to-build economy. In section 3 we analyze the dynamics of the model and we present the major theoretical results; section 4 concludes.

2   Time-to-Build

We model time-to-build in the simplest possible way by assuming, as suggested by Kalecki [18], that capital goods produced at time t become operative at time $t+\tau$, the time-to-build delay τ being strictly positive[4]. This assumption is appended to the dynamic general equilibrium model with externalities

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[2]In continuous time, the eigenvalues lie respectively, in $\mathbb{R}^{-}/\{0\}$, and in the left of the imaginary axis. In discrete time, the eigenvalues are real and inside the unit circle, and complex and inside the unit circle, respecively.

[3]The empirically plausibility of the BF model has been extensively discussed in the literature, since an implausible high level of externalities are required to the equilibrium be indeterminate. Benhabib and Nashimura [7] and Benhabib and Perli [6] propose more general models where the conditions for indeterminacy are plausible.

[4]Kalecki refers to the parameter τ as "gestation period" of any investment. This period starts with the investment orders and finished with the deliveries of finished industrial equipments.

proposed by Benhabib and Farmer [5].

2.1 Firm's Problem

Markets are perfectly competitive and there is a continuum of measure one of identical firms using a Cobb-Douglas technology that transforms labor N and capital K into output Y:

$$Y(t) = A(t)K(t-\tau)^a N(t)^b.$$

As said before, the time-to-build assumption imposes that at time t firms use capital goods produced at time $t - \tau$. The state of technology is $A(t) = \bar{K}(t-\tau)^{\alpha-a}\bar{N}(t)^{\beta-b}$, where $1 > \alpha > a > 0$, and $\beta > b > 0$. As in BF, no-tradeable externalities come from the economy-wide capital average $\bar{K}$, and the economy-wide labor average $\bar{N}$. *Constant returns to scale* at the firm level requires $a + b = 1$. There are, however, *increasing returns to scale* at the aggregate level, since $\alpha + \beta > 1$. The aggregate technology, after substitution of $\bar{K}$ by K and $\bar{N}$ by N, can be written as

$$Y(t) = K(t-\tau)^\alpha N(t)^\beta. \tag{1}$$

Under the time-to-build assumption, the representative firm faces the following static profit maximization problem:

$$\max_{N(t),K(t)} A(t)K(t-\tau)^a N(t)^b - w(t)N(t) - [r(t)+\delta]K(t-\tau).$$

where $w(t)$ is the wage rate, $\delta > 0$ is the depreciation rate and $r(t) + \delta$ is the rental rate of capital.

From the first order conditions, we get

$$bY(t) = w(t)N(t) \tag{2}$$

$$aY(t) = [r(t)+\delta]K(t-\tau). \tag{3}$$

Constant private returns to scale imply that factors of production receive a fixed share of output and profits are zero, which is consistent with perfect competition.

2.2 Consumer's Problem

The economy is inhabited by a continuum of measure one of infinitely lived households, with preferences depending positively on consumption C and negatively on employment N. Households are assumed to own the capital stock. The representative household faces the following infinite horizon problem:

$$\max \int_0^\infty \left\{ \log C(t) - \frac{N(t)^{1-\chi}}{1-\chi} \right\} e^{-\rho t} dt,$$

$$\text{s.t.} \quad \dot{K}(t) = r(t)K(t-\tau) + w(t)N(t) - C(t), \tag{4}$$

given initial conditions $K(t) = \xi(t)$, for $t \in [-\tau, 0]$. Parameter $\chi \le 0$ while $\rho > 0$. This dynamic optimization problem differs from the standard consumers problem mainly because the budget constraint (4) is not an ordinary differential equation but a delayed differential equation. From the time to build

assumption, consumers rent at time t the capital stock produced at $t - \tau$ and they build new capital which will be available at $t + \tau$. Consequently, initial conditions $\xi(t)$ need to be specified in order to identify the relevant history of the state variable K.

Following Kolimanovskii and Myshkis [21], the Hamiltonian associated to this problem is

$$\mathcal{H}(t) = \left\{ \log C(t) - \frac{N(t)^{1-\chi}}{1-\chi} \right\} e^{-\rho t} + \lambda(t) \left[r(t) K(t-\tau) + w(t) N(t) - C(t) \right],$$

and the associated optimal conditions are

$$\frac{1}{C(t)} e^{-\rho t} = \lambda(t) \tag{5}$$

$$\frac{1}{N(t)^\chi} e^{-\rho t} = \lambda(t) w(t) \tag{6}$$

$$\lambda(t+\tau) r(t+\tau) = -\dot{\lambda}(t) \tag{7}$$

and, as shown by Boucekkine *et al* [10], the standard transversality conditions

$$\lim_{t \to \infty} \lambda(t) \geq 0 \quad \text{and} \quad \lim_{t \to \infty} \lambda(t) K(t) = 0$$

holds. The main difference with respect to a standard optimal control problem is in equation (7). The fundamental trade off is between consuming today, whose marginal value is given by $\lambda(t)$, and consuming at $t + \tau$, with marginal value $\lambda(t+\tau)$. From (5) and (6) we get the standard *intratemporal substitution condition between consumption and labor*

$$\frac{C(t)}{N(t)^\chi} = w(t). \tag{8}$$

From (5) and (7), we get the *forward-looking Euler-type condition*:

$$\frac{\dot{C}(t)}{C(t)} = \frac{C(t)}{C(t+\tau)} e^{-\rho \tau} r(t+\tau) - \rho, \tag{9}$$

where the real interest rate, which the household get at time $t+\tau$ by investing in capital today, is weighted by the marginal elasticity of substitution between consumption at t and consumption at $t+\tau$. It reflects the fact that investment allows households to substitute current consumption by consumption at time $t + \tau$.

3 Analysis of the Dynamics

In order to reduce the problem to a nonlinear functional differential equations (FDEs) system,[5] we proceed in the following way. Firstly, we use equations (2) and (3) to substitute w and r into (4), (8) and (9). Secondly, we substitute N from (8). Finally, we substitute Y from (1) in (4) and (9). After making a logarithmic transformation of K and C, we get a delayed differential equation (DDE) for capital

$$\dot{k}(t) = e^{k(t-\tau)-k(t)} \left\{ e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} - \delta \right\} - e^{c(t)-k(t)}, \tag{10}$$

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[5] See Hale and Lunel [14].

and an advanced differential equation (ADE) for consumption

$$\dot{c}(t) = e^{-\rho\tau + c(t) - c(t+\tau)} \left\{ a \, e^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t+\tau)} - \delta \right\} - \rho, \tag{11}$$

where

$$\lambda_0 = -\frac{\beta \log b}{\beta + \chi - 1}, \quad \lambda_1 = \frac{(\alpha - 1)(\chi - 1) - \beta}{\beta + \chi - 1}, \quad \text{and} \quad \lambda_2 = \frac{\beta}{\beta + \chi - 1}.$$

Small capital letters refer to variables in logarithms. We can immediately observe the following:

Remark 1 *The FDEs system (10)-(11) becomes the differential system in Behnabib and Farmer [5]*

$$
\begin{aligned}
\dot{k}(t) &= e^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t)} - \delta - e^{c(t) - k(t)} \\
\dot{c}(t) &= a \, e^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t)} - \delta - \rho.
\end{aligned}
$$

when the time-to-build assumption is ruled out, i.e. $\tau \to 0$.

Moreover, we can prove some relevant relations between the signs of λ_2, $\lambda_1 + \lambda_2$, and $1 + \lambda_1$.

Lemma 1 *The following sign relations holds: $sign(\lambda_2) = sign(\lambda_1 + \lambda_2) = -\, sign(1 + \lambda_1)$.*

Proof. See Appendix A.1. ■

Finally let us give the following definition of an equilibrium path in a functional differential equation context.

Definition 1 *An equilibrium path is any trajectory $\varphi(t) = \{c(t), k(t)\}$ that solves the two autonomous mixed differential equations (10)-(11) subject to the boundary condition $k(t) = \log(\xi(t))$, for $t \in [-\tau, 0]$, and the transversality conditions*

$$\lim_{t \to \infty} e^{-c(t)} e^{-\rho t} \geq 0 \quad \text{and} \quad \lim_{t \to \infty} e^{k(t) - c(t)} e^{-\rho t} = 0.$$

3.1 Steady State Analysis

Under the usual assumption that at steady state $\dot{k}(t) = \dot{c}(t) = 0$, implying $c(t) = c(t + \tau) = c_s$ and $k(t) = k(t - \tau) = k_s$, from (10) and (11), we get

$$k_s = \frac{1}{\lambda_1 + \lambda_2} \left(\log[A] - \lambda_2 \log[A - \delta] - \lambda_0 \right) \tag{12}$$

$$c_s = \log[A - \delta] + k_s, \tag{13}$$

where $A \equiv \dfrac{\delta + \rho \, e^{\rho\tau}}{a}$.

Since k_s and c_s are natural logarithms, they may have either positive or negative sign.

Remark 2 *Equations (12)-(13) are identical to those obtained by Behnabib and Farmer [5], when $\tau = 0$.*

Moreover, as expected the following result holds:

Proposition 1 *The time-to-build delay τ affects negatively both k_s and c_s.*

Proof. See Appendix A.2. ■

The economy is more inefficient the larger the time-to-build delay is, implying that the steady state values of capital and consumption are smaller.

3.2 Stability Analysis

Let first linearize the system (10)-(11) around its steady state and compute the characteristic equation. As shown in Bellman and Cooke [3] (page 337-339), the solution of the linearized system will have the same properties of the nonlinearized one for sufficiently small perturbations. After some algebra,[6] the characteristic equation of the linearized system is

$$h(z) = \det(\Delta(z)) = \begin{vmatrix} z - (A - \delta + \lambda_1 A) \ \mathrm{e}^{-z\tau} & -\lambda_2 A + A - \delta \\ -a\lambda_1 A \ \mathrm{e}^{-\rho\tau} & z - \rho + (aA - \delta - a\lambda_2 A) \ \mathrm{e}^{-\rho\tau} \ \mathrm{e}^{z\tau} \end{vmatrix}. \tag{14}$$

The Jacobian is not symmetric due to the presence of externalities.[7] Moreover, introducing $\varepsilon = -\lambda_2 A + A - \delta$, with $\varepsilon \in \mathbb{R}$, using the definitions of λ_1 and λ_2, we can rewrite (14) as follows:

$$\tilde{h}_\varepsilon(z) = \det(\tilde{\Delta}_\varepsilon(z)) = \begin{vmatrix} z - [(\alpha - 1)\delta + \alpha\varepsilon] \ \mathrm{e}^{-z\tau} & \varepsilon \\ -a\left[A - \alpha\left(\delta + \varepsilon\right)\right]\mathrm{e}^{-\rho\tau} & z - \rho + [(a - 1)\delta + a\varepsilon] \ \mathrm{e}^{z\tau}\mathrm{e}^{-\rho\tau} \end{vmatrix}. \tag{15}$$

This characteristic equation describes completely the spectrum of the eigenvalues $Z_\infty = \{z_r\}_r$ associated to the FDEs system. Let us call $\mathrm{Re}(Z_\infty)$ the set of the real parts of the eigenvalues; and with Z_∞^k and Z_∞^c the sets of all the eigenvalues coming, respectively, from the characteristic equation of the linearized law of motions of capital and consumption.

In the following, we present, first, some preliminar results on the theory of FDEs; then we prove a general statement which ensures necessary and sufficient condition for local determinacy. This is a new crucial result which let us to fully characterized the dynamic behavior of the economy both in the case of no externalities and in the case of externalities. We show the main propositions in these two cases for the case $\varepsilon = 0$, and when possible we analyze the general case $\varepsilon \neq 0$.

3.2.1 Some Preliminary Results

Before proceeding, let us evoke some theoretical results. Consider the general linear delay differential equation with forcing term $f(t)$:

$$a_0\dot{u}(t) + b_0 u(t) + b_1 u(t - \tau) = f(t) \tag{16}$$

subject to the initial condition

$$u(t) = \xi(t) \quad \text{with} \quad t \in [-\tau, 0]. \tag{17}$$

Theorem 1 (Existence and Uniqueness) *Suppose that f is of class C^1 on $[0, \infty)$ and that ξ is of class C^0 on $[-\tau, 0]$. Then there exists one and only one continuous function $u(t)$ which satisfies (17), and (16) for $t \geq 0$. Moreover, this function u is of class C^1 on (τ, ∞) and of class C^2 on $(2\tau, \infty)$. If ξ is of class C^1 on $[-\tau, 0]$, $\dot{u}$ is continuous at τ if and only if*

$$a_0\dot{\xi}(\tau) + b_0\xi(\tau) + b_1\xi(0) = f(\tau) \tag{18}$$

If ξ is of class C^2 on $[-\tau, 0]$, $\ddot{u}$ is continuous at 2τ if either (18) holds or else $b_1 = 0$, and only in these cases.

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[6] See Appendix A.3 for technical details.

[7] In particular the no-symmetry of the Jacobian derives from the presence of increasing return to scale in the production function; in fact it can be easily shown that if $\alpha = a$ the Jacobian's symmetric properties hold.

Proof. See Bellman and Cooke [3], page 50-1. ∎

The function u singled out in this theorem is called the continuous solution of (16) and (17). Then in order to see the shape of this continuous solution the following theorem is useful:

Theorem 2 *Let $u(t)$ be the continuous solution of (16) which satisfies the boundary condition (17). If ξ is C^0 on $[-\tau, 0]$ and f is C^0 on $[0, \infty)$, then for $t > 0$,*

$$u(t) = \sum_r p_r e^{z_r t} + \int_\tau^t f(s) \sum_r p_r e^{z_r(t-s)} ds$$

where $\{z_r\}_r$ and $\{p_r\}_r$ are respectively the roots and the residue coming from the characteristic equation, $h(z)$, of the homogeneous delay differential equation

$$a_0 \dot{u}(t) + b_0 u(t) + b_1 u(t - \tau) = 0$$

Note: $\{p_r\}_r$ are the residue of $e^{zt} h(z)^{-1} p(z)$ where

$$p(z) = \xi(\tau) e^{-zt} - b_1 \int_{-\tau}^0 \xi(s) e^{-z\tau} ds$$

and for Theorem 1 are unique for given initial condition.

Proof. See Bellman and Cooke [3], page 75. ∎

An important theorem on stability of functional differential equation is

Theorem 3 (Hayes Theorem) *All the roots of $pe^z + q - ze^z = 0$, where $p, q \in \mathbb{R}$, have negative real parts if and only if*

(a) $p < 1$ and

(b) $p < -q < \sqrt{a_1^2 + p^2}$

where a_1 is the root of $a = p \tan a$ such that $a \in (0, \pi)$. If $p = 0$, we take $a_1 = \frac{\pi}{2}$.

Proof. see Bellman and Cooke [3], page 444. ∎

In the next section, the concept of local determinacy is reformulated in the context of FDEs.

3.2.2 The determinacy and indeteminacy of solution in FDEs system

Before stating the main proposition we recall the following two mathematical concepts:

Definition 2 *A section (X, Y) is an ordered pair of non-empty sets of $Z \subset \mathbb{R}$ such that any elements of Z lies in one of them and $x \in X$ and $y \in Y$ implies $x < y$.*

Definition 3 *A cutting element of a section (X, Y) of Z, is any real number L with the property that $x \leq L \leq y$ for all $x \in X$ and $y \in Y$.*

Given these definition we are ready to present the following crucial and general result

Theorem 4 *If exists a parametrization for τ and ε such that $\left(\text{Re}\left(Z_\infty^k\right), \text{Re}\left(Z_\infty^c\right)\right)$ is a section of $\text{Re}\left(Z_\infty\right)$ with cutting element $L = 0$ then the dynamic behavior of the economy is locally determinate.*

Proof. Suppose such a parametrization exists. and $L = 0$ is the cutting element, then all the eigenvalues having negative real part come necessarily from the linearized law of motion of capital while all the positive from the linearized law of motion of consumption. Now following Boucekkine et al. [10],[8] we can rule out all the eigenvalues coming from the linearize law of motion of consumption which are all the eigenvalues with positive real part. Taking into account Theorem 2, we can write the solution of the system which for Theorem 1 exists and it is unique for any specification of the boundary condition $\xi(t)$.[9]

We can also observe that if such a parametrization exists but the cutting element is on the left of the origin then the dynamic behavior of the economy is locally indeterminate since it exists at least one eigenvalue coming from the characteristic equation of the linearized law of motion of consumption having negative real part, call it $\hat{z}_l$. In this case the solution of consumption, after ruling out the roots having positive real part, writes

$$c(t) = p_l e^{z_r t} + p_l \int_\tau^t k(s) e^{z_r(t-s)} ds$$

As we know in the term p_l we have to specify initial condition for consumption. However we choose it, we have that

$$\lim_{t \to \infty} c(t) = 0$$

and then it exists more than one trajectory which converges to the equilibrium and then locally indeterminacy rises. ∎

Now we show several exemples in which we fully characterized the dynamic behaviour of the economy using Theorem 4.

3.2.3 The Ramsey problem with time to build and endogenous labor supply

We first study the particular and simplest case of $\varepsilon = 0^+$, remembering that for Rouche's theorem, the results obtained under this assumption are invariant to small enough perturbation of ε.[10] Under $\alpha = a$ and $\varepsilon = 0^+$ the characteristic equation (15) becomes:

$$\tilde{h}_0(z) = \det(\tilde{\Delta}_0(z)) = \left[z - (a-1)\delta \, e^{-z\tau} \right] \left[z - \rho + (a-1)\delta \, e^{z\tau} e^{-\rho\tau} \right]. \tag{19}$$

Taking into account Theorem 4, it is possible to prove the following Proposition

Proposition 2 *Under $\varepsilon = 0^+$, an equilibrium exists and is unique iff $\tau \in \left(0, \frac{\pi}{2(1-a)\delta}\right)$. Moreover, capital dynamics exhibits oscillatory convergence.*

Proof. In order to prove the existence and uniqueness of the solution we have to show the existence of a parametrization for τ such that $\left(\text{Re}\left(Z_\infty^k\right), \text{Re}\left(Z_\infty^c\right)\right)$ is a section of $\text{Re}\left(Z_\infty\right)$ with cutting element $L = 0$. It's immediate to observe that the first parenthesis in the right hand side of (19) is exactly the characteristic equation for the law of motion of capital. Moreover it's possible to rewrite it as follows

$$h_k(w) = -w e^w + \tau (a-1) \delta \tag{20}$$

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[8]This is simply an application of Lemma 14.
[9]For a more detailed discussion of the solution of system of functional differential equations see Bellman and Cooke [3], pag.186.
[10]An identical approach is used in Rustichini [24].

with $w = z\tau$. From Hayes theorem, all the roots have negative real part if $\tau \in \left(0, \frac{\pi}{2(1-a)\delta}\right)$. Under this parametrization, the second parenthesis in (19), call it $h_c(w)$, has exactly the same eigenvalues of $h_k(w)$ but with real parts of opposite sign. In order to see this we can rewrite $h_c(w)$ as follows

$$h_k(w) = -we^w + \tau (a-1) \delta$$

where this time $w = (-z + \rho)\tau$. From Hayes theorem all the roots have negative real part if $\tau \in \left(0, \frac{\pi}{2(1-a)\delta}\right)$ and since $z = -\frac{w}{\tau} + \rho$ then all the roots before the transformation have positive real part. Finally, zero is not a root. Then, $\left(\mathrm{Re}\left(Z_\infty^k\right), \mathrm{Re}\left(Z_\infty^c\right)\right)$ is a section of $\mathrm{Re}\left(Z_\infty\right)$ with cutting element $L = 0$. From Theorem 2, the continuous and unique solution for capital is

$$k(t) = \sum_r p_r e^{z_r t} \tag{21}$$

with p_r residue of $e^{zt}h(z)^{-1}p(z)$ where

$$p(z) = k(\tau)e^{-z\tau} + (a-1)\delta e^{-z\tau} \int_{-\tau}^0 k(t)e^{-zt}dt.$$

Finally let (x_r, y_r) such that $z_r = x_r + iy_r$. Then the converging solution of (21) can be written as:

$$k(t) = \sum_{x_r \in \mathrm{Re}^-(Z_\infty)} p_r e^{x_r t} \cos(y_r t)$$

and since $x_r < 0$, the capital dynamics exhibits oscillations that decrease in magnitude as $t \to \infty$. ∎

Now we analyze the more general case, $\varepsilon \neq 0$. This case is of particular interest, since it corresponds to a general Ramsey model with time-to-build and endogenous labor supply. Under $\alpha = a$, equation (15) becomes

$$\tilde{h}_\varepsilon(z) = f_\varepsilon(z) + \tilde{h}_\varepsilon, \tag{22}$$

where $f_\varepsilon(z) = (z - [(a-1)\delta + a\varepsilon] e^{-z\tau})(z - \rho + [(a-1)\delta + a\varepsilon] e^{z\tau}e^{-\rho\tau})$ and $\tilde{h}_\varepsilon = a\varepsilon [A - a(\delta + \varepsilon)]$, which is positive iff $\varepsilon \in \left(0, \frac{A}{a} - \delta\right)$. We want to study $\tilde{h}_\varepsilon(z) = 0$ and derive the dyanmic behavior of the economy.

Proposition 3 *The equilibrium of a Ramsey model with time to build and endogenous labour supply exists and it is unique. The dynamics of capital exhibits oscillatory convergence.*

Proof. The prove is divided in two steps. In the first step, we prove that function $f_\varepsilon(z)$ is even and then any shift of the x-axis, call it $\tilde{h}_\varepsilon$, identifies new zeros with the property that to any root having positive real part corresponds another root having a real part of the same magnitude but opposite sign. By using this result we prove, in the second step, that $\left(\mathrm{Re}\left(Z_\infty^k\right), \mathrm{Re}\left(Z_\infty^c\right)\right)$ is a section of $\mathrm{Re}\left(Z_\infty\right)$ with cutting element $L = 0$.

Step 1: For ρ sufficiently small, $f_\varepsilon(z)$ is "almost" an even function, since

$$f_\varepsilon(-z) = \left\{-z - [(a-1)\delta + a\varepsilon] e^{z\tau}\right\}\left\{-\tilde{z} + [(a-1)\delta + a\varepsilon] e^{-\tilde{z}\tau}\right\} \simeq f_\varepsilon(z).$$

where $\tilde{z} = z - \rho \simeq z$. Then, for any shift of the x-axis, call it $\tilde{h}_\varepsilon$, the zeros of function $\tilde{h}_\varepsilon(z)$ verify that $\mathrm{Re}^-(Z_\infty) = \left\{\tilde{x}_j^1\right\}_j$ and $\mathrm{Re}^+(Z_\infty) \simeq \left\{-\tilde{x}_j^1\right\}_j$.

Step 2: In this step we show that all the eigenvalues coming from the law of motion of capital and consumption have respectively negative and positive real parts. In order to prove that, we start with the equations of the linearized law of motion of capital and consumption

$$\dot{k}(t) = g_\varepsilon k(t-\tau) - \varepsilon c(t) \tag{23}$$

$$\dot{c}(t) = -g_\varepsilon c(t+\tau) + d_\varepsilon k(t) \tag{24}$$

where for ρ sufficiently small, $g_\varepsilon \simeq [(a-1)\delta + a\varepsilon]$, which is greater than zero for any feasible value of ε, and $d_\varepsilon \simeq a[A - a(\delta + \varepsilon)]$. Let us assume that $k(t) = e^{\bar{z}t}$ is a solution of (23) and $c(t) \simeq e^{-\bar{z}t}$ is the associated solution of (24). Given these solutions, we have that the characteristic equations of (23) and (24) are:

$$h_k = \bar{z} - g_\varepsilon e^{-\bar{z}\tau} + \varepsilon e^{-2\bar{z}t} \tag{25}$$

$$h_c = -\bar{z} + g_\varepsilon e^{-\bar{z}\tau} - d_\varepsilon e^{2\bar{z}t}. \tag{26}$$

From (26), $h_c = 0$ implies

$$e^{2\bar{z}t} = \frac{-\bar{z} + g_\varepsilon e^{-\bar{z}\tau}}{d_\varepsilon}$$

and substituting it in (25) we get

$$h_k = \frac{[\bar{z} - g_\varepsilon e^{-\bar{z}\tau}]^2 - \tilde{h}_\varepsilon}{\bar{z} - g_\varepsilon e^{-\bar{z}\tau}} \tag{27}$$

where $\bar{z} - g_\varepsilon e^{-\bar{z}\tau} \neq 0$, $\varepsilon > \frac{1-a}{a}\delta > 0$ and (25) has to be equal to zero. By proving that, it exists ε and τ such that all the roots of the numerator of (27) have negative real parts, then all the considerations in Lemma 2 apply also to this case. In order to do it, we observe that $\varepsilon \in \left(\frac{1-a}{a}\delta, \frac{1-a^2}{a^2}\delta\right)$ as shown in Appendix A.4. We can rewrite $h_k = 0$ as follows

$$-we^w - \tau\sqrt{\tilde{h}_\varepsilon}e^w + \tau g_\varepsilon = 0 \tag{28}$$

since $\tilde{h}_\epsilon$ is positive and $w = z\tau$. By applying the Hayes Theorem we find that the condition *(a)* is always satisfied since $-\tau\sqrt{\tilde{h}_\varepsilon} < 1$. Moreover condition *(b)* is satisfied, too. In fact the inequality $-\tau g(\varepsilon) < \sqrt{a_1^2 + \tau^2 \tilde{h}_\varepsilon}$ is obviously satisfied, while the inequality $g_\varepsilon < \sqrt{\tilde{h}_\varepsilon}$ is respected for any value of $\varepsilon \in \left(\frac{1-a}{a}\delta, \tilde{\varepsilon}\right)$ with $\tilde{\varepsilon}$ very close to $\frac{1-a^2}{a^2}\delta$. In particular it's possible to show that the last inequality always holds for the usual calibration of the parameters[11]. Then all the eigenvalues of (28) to have negative real part. Finally, since for Hayes theorem the condition is necessary and sufficient, we have that our guessed solutions for (23) and (24) were exact. This is sufficient to prove that $\left(\text{Re}\left(Z_\infty^k\right), \text{Re}\left(Z_\infty^c\right)\right)$ is a section of $\text{Re}\left(Z_\infty\right)$ with cutting element $L = 0$ and then from Theorem 4 the equilibrium exists and is unique. Finally we can rewrite the solution in its trigonometric form

$$k(t) = \sum_{x_r \in \text{Re}^-(Z_\infty)} p_{1r}(t)e^{x_r t}\cos(y_r t)$$

$$c(t) = \sum_{x_r \in \text{Re}^-(Z_\infty)} p_{2r}(t)e^{x_r t}\cos(y_r t)$$

---

[11] For example under the parametrization $a = \frac{1}{3}$, $b = \frac{2}{3}$, $\chi = -0.25$ and $\delta = 0.1$ we have $\sqrt{h_\varepsilon} = 0.125 > 0.114 = g_\varepsilon$.

and since $x_r < 0$, capital and consumption dynamics exhibit at the beginning oscillations that decrease in magnitude and finally disappear. ∎

Now we introduce externalities and we study the new economy.

3.2.4 A simplified version of the B.-F. model with time to build

We now turn to the Behnabib-Farmer model with time-to-build. In this case, the graph of the function $f_\varepsilon(z)$ is no more symmetric respect the vertical axis and we choose to derive in a first moment some results only for $\varepsilon = 0^+$ which makes the Jacobian matrix triangular. In this case the BF condition for indeterminacy is not satisfied as shown in Appendix A.5. Under this assumption, the characteristic equation (15) becomes

$$\tilde{h}_0(z) = \det(\tilde{\Delta}_0(z)) = \underbrace{\left[z - [(\alpha-1)\delta]\ e^{-z\tau}\right]}_{\tilde{h}_0^1(z)} \underbrace{\left[z - \rho + [(a-1)\delta]\ e^{z\tau}e^{-\rho\tau}\right]}_{\tilde{h}_0^2(z)}$$

The following result holds:

Proposition 4 When $\tau \in \left(0, \frac{\pi}{2(1-a)\delta}\right)$ the equilibrium exists and it is unique; however when $\tau \in \left(\frac{\pi}{2(1-a)\delta}, \frac{\pi}{2(1-\alpha)\delta}\right)$, the equilibrium exists but it is not more unique. In both cases capital and consumption dynamics exhibits at the beginning oscillations that decrease in magnitude and finally disappear.

Proof. From Hayes theorem we can easily show that $\tilde{h}_0^1(z)$ has all the roots with negative real part if $\tau \in \left(0, \frac{\pi}{2(1-\alpha)\delta}\right)$ while $\tilde{h}_0^2(z)$ has all the roots with positive real part if $\tau \in \left(0, \frac{\pi}{2(1-a)\delta}\right)$. Now since $\alpha \in (a, 1)$, it follows immediately that $\frac{\pi}{2(1-\alpha)\delta} > \frac{\pi}{2(1-a)\delta}$; but then in the interval $\tau \in \left(0, \frac{\pi}{2(1-a)\delta}\right)$, $\left(\text{Re}\left(Z_\infty^k\right), \text{Re}\left(Z_\infty^c\right)\right)$ is a section of $\text{Re}\left(Z_\infty\right)$ with cutting element $L = 0$ and then a unique equilibrium exists. On the other hand in the interval $\tau \in \left(\frac{\pi}{2(1-a)\delta}, \frac{\pi}{2(1-\alpha)\delta}\right)$ this cannot more be done since it exists at least one root of $\tilde{h}_0^2(z)$, call it $\hat{z}_l$, having no-positive real part[12]. It's immediate to verify that the values of the parameter τ under which the real part of $\hat{z}_l$ is equal to zero is a zero measure set[13]. Then for τ in this interval of values, as shown also in the proof of Theorem 4, the solution for consumption writes:

$$c(t) = p_l e^{\hat{z}_l t} + \tilde{d}_0 \int_\tau^t \left(\sum_r p_r e^{z_r s}\right) p_l e^{\hat{z}_l (t-s)} ds \qquad (29)$$

where $\tilde{d}_0 = -a\left[A - \alpha\delta\right]$. As we know in the term p_l we have to specify initial condition for consumption. However we choose it, we have that

$$\lim_{t \to \infty} c(t) = 0$$

and then it exists more than one trajectory which converges to the equilibrium and then locally indeterminacy rises. Finally let $(\hat{x}, \hat{y})$ such that $\hat{z} = \hat{x} + i\hat{y}$. Then the solution of consumption (29) can be re-written in term of cosine and sine and since $\hat{x}_l < 0$, also the consumption dynamics exhibits oscillatory convergence. ∎

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[12] To be precise this is always true when $\rho \in \left(0, \frac{1}{\tau} \text{Re}\left(\hat{w}_l\right)\right)$ with $\tau \in \left(\frac{\pi}{2(1-a)\delta}, \frac{\pi}{2(1-\alpha)\delta}\right)$.

[13] This restrict the possibility of cycle equilibrium and Hopf bifurcation to zero measure parameter set.

Then when the time to build assumption is introduced in a Behnabib Farmer model, we have that if we choose values for τ in the identified interval, the steady state of the economy will be indeterminate even if the necessary condition for indeterminacy of the original model is not satisfy.

It should be of extreme interest to study the general case of $\varepsilon \neq 0$. The analysis of the general case is really complex and it seems extremely difficult to reach a clear analytical result. If the effect of consider $\varepsilon \neq 0$ is similar to that obtained for the Ramsey model with endogenous labor supply, we can guess that the dynamic behavior of the economy remains the same except for the absence of the region of divergence.

4   Conclusions

We have studied a Behnabib Farmer model in order to analyze the effect of the time to build assumption on the dynamic behavior of the economy. In a first moment, we have focused on a simpler Ramsey model with endogenous labor supply, and we have proved that the dynamic behavior of the economy around the steady state remains of *"saddle-path"* type. This result has been obtained by proving that all the eigenvalues having negative and positive real part come respectively from the law of motion of capital and consumption. Using the transversality conditions all the positive have been ruled out and an explicit and perfectly determinate solution for capital and consumption has been got from the boundary condition of capital. Then we have tried to use the same strategy in the case of presence of externalities. The analysis becomes very complicated in this case, however we have been able to prove evidence of presence of local indeterminacy for particular choice of the time to build parameter, independently by the magnitude of the externalities. To be more precise the presence of local indeterminacy can be always obtained when a sufficiently large value of the time to build parameter is chosen in a model with externalities independently by their magnitude.

A   Appendix

A.1   Proof of Lemma1.

We start with the case $\lambda_2 > 0$. We can observe immediately that

$$\lambda_2 > 0 \Longleftrightarrow \beta > 1 - \chi.$$

But then given the assumptions $\alpha \in (0,1)$ and $\chi \leq 0$, follows immediately that

$$\lambda_2 > 0 \Longleftrightarrow \lambda_1 + \lambda_2 = \frac{(\alpha-1)(\chi-1)}{\beta+\chi-1} > 0,$$

$$\lambda_2 > 0 \Longrightarrow 1 + \lambda_1 = \frac{\alpha(\chi-1)}{\beta+\chi-1} < 0.$$

Now we analyze the case $\lambda_2 < 0$. We can observe immediately that

$$\lambda_2 < 0 \Longleftrightarrow \beta < 1 - \chi.$$

But then given the assumptions $\alpha \in (0,1)$ and $\chi \leq 0$, follows immediately that

$$\lambda_2 < 0 \Longleftrightarrow \lambda_1 + \lambda_2 = \frac{(\alpha-1)(\chi-1)}{\beta+\chi-1} < 0,$$

$$\lambda_2 < 0 \Longrightarrow 1 + \lambda_1 = \frac{\alpha(\chi-1)}{\beta+\chi-1} > 0.$$

and then we have proven all the relations between $\lambda_2, \lambda_1 + \lambda_2$ and $1 + \lambda_1$. Moreover since we can write λ_1 as follows:

$$\lambda_1 = \frac{\alpha(\chi-1)}{\beta+\chi-1} - 1 = \lambda_1 = \frac{\alpha(\chi-1+\beta-\beta)}{\beta+\chi-1} - 1 = -\alpha\lambda_2 + \alpha - 1,$$

then we'll have that if

$$\lambda_2 \in \left[\tfrac{\alpha-1}{\alpha}, +\infty\right) \Longrightarrow \lambda_1 \leq 0,$$

$$\lambda_2 \in \left(-\infty, \tfrac{\alpha-1}{\alpha}\right) \Longrightarrow \lambda_1 > 0,$$

A.2   Proof of Proposition1.

We need to prove that both $\frac{dk_s}{d\tau}$ and $\frac{dc_s}{d\tau}$ are negative. First of all we'll have that:

$$\frac{dk_s}{d\tau} = \frac{A'(\tau)}{\lambda_1 + \lambda_2} \left\{ \frac{(1-\lambda_2)A(\tau) - \delta}{A(\tau)[A(\tau) - \delta]} \right\}$$

now since $A(\tau) > 0$, $A'(\tau) = \frac{\rho^2}{a}e^{\rho\tau} > 0$ and $A(\tau) - \delta \overset{(13)}{>} 0$ then $sign\left(\frac{dk_s}{d\tau}\right)$ depends exclusively on λ_2. If $\lambda_2 < 0$ then $(1-\lambda_2)A(\tau) - \delta > 0$ but for Lemma1, $\lambda_1 + \lambda_2 < 0$ and then $\frac{dk_s}{d\tau} < 0$. On the other hand if $\lambda_2 > 0$, since $1 - \lambda_2 < 0$, we'll have that $(1-\lambda_2)A(\tau) - \delta < 0$ but this time $\lambda_1 + \lambda_2 > 0$ and then $\frac{dk_s}{d\tau} < 0$.

Now we'll study the $sign\left(\frac{dc_s}{d\tau}\right)$ in order to do that we put (12) into (13) and then we take the derivative respect to τ :

$$\frac{dc_s}{d\tau} = \frac{A'(\tau)}{\lambda_1 + \lambda_2} \left\{ \frac{(1+\lambda_1)A(\tau) - \delta}{A(\tau)[A(\tau) - \delta]} \right\}$$

as before the $sign\left(\frac{dc_s}{d\tau}\right)$ depends exclusively on λ_2. In fact if $\lambda_2 > 0$, since $1 + \lambda_1 \overset{L1}{<} 0$, we'll have $(1 + \lambda_1) A(\tau) - \delta < 0$ but $\lambda_1 + \lambda_2 \overset{L1}{>} 0$ and then $\frac{dc_s}{d\tau} < 0$. On the other hand suppose that $\lambda_2 < 0$, if we prove that $(1 + \lambda_1) A(\tau) - \delta > 0$ since $\lambda_1 + \lambda_2 \overset{L1}{<} 0$ then $\frac{dc_s}{d\tau} < 0$. In order to prove that $(1 + \lambda_1) A(\tau) - \delta > 0$ we distinguish the following two cases:

$$\lambda_2 \in \left(-\infty, \frac{\alpha-1}{\alpha}\right) \overset{L1}{\Longrightarrow} \lambda_1 > 0 \Longrightarrow A(1+\lambda_1) - \delta > A - \delta \overset{(13)}{>} 0$$

$$\lambda_2 \in \left(\frac{\alpha-1}{\alpha}, 0\right) \overset{L1}{\Longrightarrow} A(1+\lambda_1) - \delta > 0$$

where the last relation is obtained by studying the limit case $\lambda_2 \to 0^-$. In fact if

$$\lambda_2 \to 0 \overset{L1}{\Longrightarrow} 1 + \lambda_1 \to \alpha \Longrightarrow \Pi_2 \to \alpha A - \delta > aA - \delta = \rho e^{\rho\tau} > 0.$$

A.3   Linearization of the FDEs System.

We show how to obtain the Jacobian starting from the *DDE for capital* and the *ADE for consumption*. In order to simplify the algebra we rewrite the two functional differential equations as follows:

$$\dot{k}(t) = e^{f(k(t),k(t-\tau))}\left\{e^{g(k(t-\tau),c(t))} - \delta\right\} - e^{h(k(t),c(t))},$$

$$\dot{c}(t) = e^{v(c(t),c(t+\tau))}\left\{ae^{\tilde{g}(k(t),c(t+\tau))} - \delta\right\} - \rho,$$

and we'll use the following notation:

$$e^{\lambda_0 + \lambda_1 k_s + \lambda_2 c_s} = \frac{\delta + \rho e^{\rho\tau}}{a} \equiv A, \qquad (30)$$

$$e^{c_s - k_s} = \frac{\delta + \rho e^{\rho\tau}}{a} - \delta \equiv A - \delta. \qquad (31)$$

Now we calculate the following derivative[14]:

$$\frac{\partial \dot{k}(t)}{\partial k(t)} \equiv \left[\frac{\partial}{\partial k(t)}f(k(t), k(t-\tau))\right]e^{f(k(t),k(t-\tau))}\left\{e^{g(k(t-\tau),c(t))} - \delta\right\}$$

$$+ e^{f(k(t),k(t-\tau))}\left[\frac{\partial}{\partial k(t)}g(k(t-\tau), c(t))\right]e^{g(k(t-\tau),c(t))}$$

$$- \left[\frac{\partial}{\partial k(t)}h(k(t), c(t))\right]e^{h(k(t),c(t))}$$

$$= \left(e^{-z\tau} - 1\right)e^{k(t-\tau)-k(t)}\left\{e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} - \delta\right\} +$$

$$e^{k(t-\tau)-k(t)}\lambda_1 e^{-z\tau}e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} + e^{c(t)-k(t)},$$

and then

$$\left.\frac{\partial \dot{k}(t)}{\partial k(t)}\right|_{s.s.} = \left(e^{-z\tau} - 1\right)\left(e^{\lambda_0 + \lambda_1 k_s + \lambda_2 c_s} - \delta\right) + \lambda_1 e^{-z\tau}e^{\lambda_0 + \lambda_1 k_s + \lambda_2 c_s} + e^{c_s - k_s},$$

and taking into account the relations (30) and (31) we'll have finally:

$$\left.\frac{\partial \dot{k}(t)}{\partial k(t)}\right|_{s.s.} = e^{-z\tau}\left(A - \delta + \lambda_1 A\right). \qquad (32)$$

---

[14]We search for a solution of type $c(t) = k(t) = e^{zt}$ and then we have the following relations $k(t - \tau) = e^{z(t-\tau)}$ and $c(t + \tau) = e^{z(t+\tau)}$

Now we search for

$$\frac{\partial \dot{k}(t)}{\partial c(t)} \equiv e^{f(k(t),k(t-\tau))}\left[\frac{\partial}{\partial c(t)}g(k(t-\tau),c(t))\right]e^{g(k(t-\tau),c(t))} - \left[\frac{\partial}{\partial c(t)}h(k(t),c(t))\right]e^{h(k(t),c(t))}$$

$$= \lambda_2 e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} - e^{c(t)-k(t)},$$

and then in steady state we get:

$$\left.\frac{\partial \dot{k}(t)}{\partial c(t)}\right|_{s.s.} = \lambda_2 A - A + \delta. \tag{33}$$

Now we pass to find

$$\frac{\partial \dot{c}(t)}{\partial k(t)} \equiv e^{v(c(t),c(t+\tau))}\left[\frac{\partial}{\partial k(t)}\tilde{g}(k(t),c(t+\tau))\right]ae^{\tilde{g}(k(t),c(t+\tau))}$$

$$= e^{-\rho\tau + c(t) - c(t+\tau)}a\lambda_1 e^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t+\tau)},$$

and then in steady state we get:

$$\left.\frac{\partial \dot{c}(t)}{\partial k(t)}\right|_{s.s.} = e^{-\rho\tau}a\lambda_1 A. \tag{34}$$

At last we calculate:

$$\frac{\partial \dot{c}(t)}{\partial c(t)} \equiv \left[\frac{\partial}{\partial c(t)}v(c(t),c(t+\tau))\right]e^{v(c(t),c(t+\tau))}\left\{ae^{\tilde{g}(k(t),c(t+\tau))} - \delta\right\} +$$

$$e^{v(c(t),c(t+\tau))}\left[\frac{\partial}{\partial c(t)}\tilde{g}(k(t),c(t+\tau))\right]ae^{\tilde{g}(k(t),c(t+\tau))}$$

$$= (1-e^{z\tau})e^{-\rho\tau + c(t)-c(t+\tau)}\left\{ae^{\lambda_0+\lambda_1 k(t)+\lambda_2 c(t+\tau)} - \delta\right\} +$$

$$e^{-\rho\tau+c(t)-c(t+\tau)}a\lambda_2 e^{\lambda_0+\lambda_1 k(t)+\lambda_2 c(t+\tau)},$$

and then in steady state we get

$$\left.\frac{\partial \dot{c}(t)}{\partial c(t)}\right|_{s.s.} = -(aA - \delta - a\lambda_2 A)e^{-\rho\tau}e^{z\tau} + (aA - \delta)e^{-\rho\tau}, \tag{35}$$

and then taking into account (32),(33),(34), and (35) we can construct the Jacobian (14).

The trace and the determinant of (14) are given by[15]:

$$Tr(J) = (A - \delta + \lambda_1 A)e^{-z\tau} - (aA - \delta - a\lambda_2 A)e^{-\rho\tau}e^{z\tau} + (aA - \delta)e^{-\rho\tau}, \tag{36}$$

$$Det(J) = (A - \delta + \lambda_1 A)(aA - \delta)e^{-\rho\tau}e^{-z\tau} - (A - \delta)(aA - \delta - a\lambda_2 A)e^{-\rho\tau} \tag{37}$$

$$+\lambda_1 A(1-a)\delta e^{-\rho\tau}.$$

A.4 Feasible values of ε in a Ramsey model with endogenous L.

In a model without externalities, the set of the feasible values of ε are in the interval $\left(\frac{1-a}{a}\delta, \frac{1-a^2}{a^2}\delta\right)$. In order to prove this we proceed as follows: starting from the expression of ε we can easily get:

$$\varepsilon = -\lambda_2 A + A - \delta = \frac{\chi - a - 1}{\chi - a}\cdot\frac{1-a}{a}\delta.$$

Now taking into account that $\chi \in (-\infty, 0)$ we deduce from the previous expression that ε lies necessarily in the interval $\left(\frac{1-a}{a}\delta, \frac{1-a^2}{a^2}\delta\right)$ as we want to show.

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[15] As we expected, we can obtain the same BF results for trace and determinant just assuming the delay equal to zero.

A.5 Relation between the B.-F. condition for indeterminacy and ε.

The Benhabib Farmer condition for indeterminacy holds if and only if $\varepsilon \in (-\infty, -\delta)$ .In order to prove this we proceed as follows: from the definition of ε, $\lambda_2 = \frac{A-\delta-\varepsilon}{A} = 1 - \frac{\delta+\varepsilon}{A}$, which is positive if and only if $\varepsilon < A - \delta$. Suppose for example $\varepsilon = A - \delta - \xi$ with $\xi \in \mathbb{R}^+$ and remembering that $\lambda_2 = \frac{\beta}{\beta+\chi-1}$ will have that

$$\frac{1-\chi}{\beta+\chi-1} = -\frac{A-\xi}{A}$$

and then, since $\chi \in \mathbb{R}^-$ while β and $\lambda_2 \in \mathbb{R}^+$ it must be $\beta > 1 - \chi$ which is satisfy only when $\xi > A$ and then $\varepsilon < -\delta$.

On the other hand λ_2 is negative if and only if $\varepsilon > A - \delta > 0$. Suppose for example $\varepsilon = A - \delta + \xi$ with $\xi \in \mathbb{R}^+$ and remembering that $\lambda_2 = \frac{\beta}{\beta+\chi-1}$ will have that

$$\frac{1-\chi}{\beta+\chi-1} = -\frac{A+\xi}{A} < 0$$

and then follows necessary that $\beta < 1 - \chi$ for any $\xi \in \mathbb{R}^+$. It's important to note that the interval $\varepsilon \in (-\delta, 0]$ is not feasible since it would imply $\beta < 0$.

References

[1] Asea, P.K. and Zak, P.J. (1999), "Time-to-build and cycles", Journal of Economic Dynamics & Control 23,1155-1175.

[2] Augeraud-Veron, E. and D'Albis, H. (2004), "Competitive growth in a life-cycle model: existence and dynamics." Univerité Paris-I Working paper.

[3] Bellman, R. and Cooke, K. (1963), "Differential-difference equations" New York Academic Press.

[4] Benhabib, J. and Rustichini, A. (1994), "Introduction to the symposium on growth, fluctuations, and sunspots: confronting the data", Journal of Economic Theory 63,1-18.

[5] Benhabib, J. and Farmer, R. (1994), "Indeterminacy and increasing returns", Journal of Economic Theory 63,19-41.

[6] Benhabib, J. and Perli, R. (1994), "Uniqueness and indeterminacy: on the dynamics of endogenous growth", Journal of Economic Theory 63,113-142.

[7] Benhabib, J. and Nishimura, K. (1998), "Indeterminacy and sunspots with constant returns", Journal of Economic Theory 81,58-96.

[8] Bohm-Bawerk, Eugen von (1884-9), "Capital and interest", South Holland, Libertarian Press.

[9] Boldrin, M. and Rustichini, A. (1994), "Growth and indeterminacy in dynamic models with externalities", Econometrica, vol.62 no.2, 323-342.

[10] Boucekkine, R., Licandro, O., Puch, L., del Rio, F. (2005), "Vintage capital and the dynamics of the AK model", Journal of Economic Theory, 120, 39-72.

[11] Collard, F., Licandro, O. and Puch, L. (2004), "The Short-Run Dynamics of Optimal Growth Models with Delays," European University Institute ECO 2004/04.

[12] Gyori, I. and Ladas, G. (1991), "Oscillation theory of delay differential equation", Claredon Press - Oxford.

[13] Hale, J. and Koçak, H. (1991) "Dynamics and bifurcations", Springer-Verlag.

[14] Hale, J and Lunel, S. M. V. (1993) "Introduction to functional differential equation", Springer-Verlag.

[15] Herrendorf, B. and Valentinyi, A. (2003) "Determinacy Through Intertemporal Capital Adjustment Costs", Review of Economic Dynamics 6, 483-497.

[16] Hintermaier, T. (2004) "A Sunspot Paradox", Institute for Advanced Studies (IHS), Economic Series ISO.

[17] Ioannides, Y., M., and Taub, B. (1992), "On dynamics with time-to-build investment technology and non-time-separable leisure", Journal of Economic Dynamics and Control 16, 225-241.

[18] Kalecki, M. (1935), "A macroeconomic theory of the business cycle", Econometrica, Vol.3, 327-344.

[19] Kehoe, T. J. and Levine, D. K. (1985), "Comparative statistics and perfect foresight in infinite horizon economies", Econometrica, vol.53 no.2, 433-454.

[20] Kim, J. (2003), "Indeterminacy and investment adjustment costs: an analytic result", Macroeconomic Dynamics 7,394-406.

[21] Kolimanovskii, V. and Myshkis, A. (1992), "Applied theory of functional differential equations", Dordrecht academic publishers.

[22] Kydland, F. E. and Prescott, E. C. (1982), "Time-to-build and aggregate fluctuations", Econometrica, vol. 50 no.6, 1345-1370.

[23] Ramsey, F. (1928). "A mathematical theory of saving". *Economic Journal* 38, 543-559.

[24] Rustichini, A. (1989), "Hopf bifurcation for functional differential equation of mixed type", Journal of Dynamics and Differential Equation, 1, (20), 145-177.