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Abstract

This paper provides a complete characterization of equilibria in a game-theoretic version of Rothschild and Stiglitz (1976)'s model of competitive insurance. I allow for stochastic contract offers by insurance firms and show that a unique symmetric equilibrium always exists. Exact conditions under which the equilibrium involves mixed strategies are provided. The mixed equilibrium features: (i) cross-subsidization across risk levels, (ii) dependence of offers on the risk distribution and (iii) price dispersion generated by firm randomization over offers.

Keywords

Asymmetric and Private Information; Mechanism Design; Oligopoly; Economics of Contracts; Insurance.

JEL Classification: C72, D43, D82, D86, G22

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1 Introduction

This paper provides a complete characterization of equilibria in a game-theoretic version of Rothschild and Stiglitz (1976)'s (henceforth, RS) model of competitive insurance with private information. I allow for stochastic contract offers by insurance firms and show that a unique symmetric equilibrium always exists, extending the classical result of RS to mixed strategies. The unique equilibrium is explicitly presented and its comparative static results discussed. The equilibrium simultaneously features: (i) cross-subsidization across risk levels, (ii) dependence of offers on the risk distribution and (iii) price dispersion.

The literature on competitive insurance mostly restricts attention to equilibria with deterministic contract offers.[1] This restriction is problematic, as it rules out important economic phenomena present in insurance markets.

First, the focus on deterministic contract offers implies that an equilibrium cannot feature cross-subsidization.[2] Cross-subsidization means that firms may make profits form low-risk agents in order to subsidize high-risk agents. In a deterministic equilibrium, any such set of contracts is vulnerable to cream-skimming deviations by one of the competing firms, which only attract low-risk agents and leave the high-risks to its competitors. However, the construction of such cream-skimming deviations hinges on firms knowing exactly which offer they are competing against, which is not true when firms use mixed strategies. This observation is relevant for policy analysis. The fact that cross-subsidization might be welfare improving has been used as a justification for government intervention (see Bisin and Gottardi (2006)). In the model considered in this paper, cross-subsidization may arise in equilibrium without governmental intervention.

Second, the absence of cross-subsidization means that the contract consumed by each risk type is priced correctly. Hence equilibrium contracts are independent of the relative share of each risk in the market. However, the dependence of market outcomes on risk distribution is a central theme in the policy arena.[3]

We consider a competitive market where firms offer contract menus to an agent that is privately informed about his own risk level. I follow Dasgupta and Maskin (1986) in modeling competition as a simultaneous offers game with a finite number of firms. The consumer (or agent) has private information about having high or low risk of an accident. Dasgupta and Maskin (86, Theorem

[1] See Rothschild and Stiglitz (1976); Dubey and Geanakoplos (2002); Dubey, Geanakoplos, and Shubik (2005); Bisin and Gottardi (2006); Guerrieri, Shimer, and Wright (2010).

[2] This claim refers exclusively to static models of competitive insurance. In seminal papers, Wilson (1977), Miyazaki (1977) and Riley (1979) obtain equilibria with cross-subsidization while considering non-standard equilibrium notions incorporating anticipatory behavior typical of dynamic models.

[3] During the implementation of the health care exchanges following the approval of Affordable Care Act In the United States, the presence of young adults with lower risk level was considered a necessary condition for the successful rollout and "stability" of the program (for example, see Levitt, Claxton, and Damico (2013)). In regulated markets such as the exchanges, observable conditions such as age and previous diagnostics are treated as private information since they can affect the coverage choice of consumers while not being used (or having limited use) explicitly in pricing contracts.

5) proved existence of equilibria for this game, but provided only a partial characterization and present no results regarding multiplicity of equilibria. The main contributions of this paper are: (i) to establish uniqueness of symmetric equilibria, (ii) to solve explicitly for this equilibrium and (iii) to derive properties and comparative statics of the equilibrium.

In section 4 I explicitly describe an equilibrium for all prior distributions. Equilibrium offers lie on a critical set of separating offers that generate zero expected profits in the market as a whole, referred to as *cross-subsidizing offers*. The equilibrium offers coincide with the zero cross-subsidization offers described in RS, whenever a pure strategy equilibrium exists. An equilibrium in pure strategies exists whenever cross-subsidization cannot lead to pareto improvements. This occurs whenever the probability of high-risks is sufficiently high[4] (Corollary 1).

Equilibria necessarily involve mixed strategies whenever the RS menu of contracts cannot be sustained as an equilibrium. If the RS separating contracts fail to be an equilibrium outcome, the equilibrium involves each firm offering cross-subsidizing offers, with a random amount of cross-subsidization between zero and a pareto efficient (positive) level. Offers in the support of equilibrium strategies have the following properties: (i) high-risk agents always receive a full insurance contract; (ii) low-risk agents always receive partial insurance, which leaves the high-risk agent indifferent between this contract and his own; (iii) all the menus of contracts in the support of the equilibrium strategy are ordered by attractiveness. The firm that delivers the most attractive menu of contracts attracts the customer, no matter what his type is. Moreover, firms always earn zero expected profits.[5]

The equilibrium distribution over the possible levels of cross-subsidization comes from a local condition that guarantees that, for any menu offer in the support of the equilibrium strategy, there is no local profitable deviation. I show that this condition implies there is no global profitable deviation by a firm.

In section 5 I show that the equilibrium described is the unique symmetric equilibrium. Equilibrium offers can be described by the utility vector they generate to both possible risk types. Describing offers in terms of utility profiles means that the offer space is essentially two-dimensional. The main challenge in the analysis lies in showing that equilibrium offers necessarily lie in a one-dimensional subset of the feasible utility space. The crucial step uses properties of the equilibrium utility distribution to show that expected profits are supermodular in the utility vector offered to the consumer, i.e., there is a complementarity in making more attractive offers to both risk types. This

[4] The competitive equilibrium concept considered in RS is different from the (game-theoretic) equilibrium concept considered here. Nevertheless, the RS pair of contracts is an equilibrium outcome of my game if and only if it is an equilibrium of their model, provided that entering firms are allowed to propose pair of contracts. In their main definition of competitive equilibrium (Section I.4), outside firms are only allowed to offer a single contract, while it is acknowledged that a new pair of contracts might be more profitable than a deviating pooling contract (Section II.3). As a consequence, the exact condition for existence of a pure strategy equilibrium is related to separating, and not pooling, offers.

[5] In fact, each firm earns zero expected profits for any realization of its opponents' randomization (but in expectation with respect to the agent's type).

property is used to show that equilibrium offers are necessarily ordered in terms of attractiveness, i.e., a more attractive offer provides higher utility to both risk types.

Since firms make zero profits, this ordering of offers implies they generate zero profits even if they are accepted by both risk types with probability one. Hence, offers can be indexed by the amount of subsidization that occurs across different risk types. The use of supermodularity and the zero profits condition to reduce the dimensionality of the equilibrium support is non-standard in the literature.

Our uniqueness result allows us to discuss comparative static exercises in a meaningful way. Exploiting the explicit characterization of the equilibrium, I analyze two relevant comparative statics exercises: with respect to the prior distribution and the number of firms. With respect to the prior distribution over types, equilibrium offers have monotone comparative statics. If the probability of low-risk agents increases, firms make more attractive offers, in the sense of first order stochastic dominance. Both agent types are better off. This means that, in a large market, a higher prior probability of low-risk is beneficial to both types of agents.

Equilibrium strategies are continuous with respect to the prior belief. More specifically, the equilibrium outcome converges to the complete information outcome as the prior converges to the extreme points. When the probability of low risk agents converges to one, the distribution of offers converges to a mass point at the actuarially fair full insurance allocation of the low-risk agent. When the probability of low risk agent is sufficiently small, the RS pair of contracts are an equilibrium and the vast majority of high-risk agents consume their actuarially fair full insurance contract.

Equilibrium strategies also feature monotone comparative statics with respect to the number of firms, $N \geq 2$. The support of the equilibrium strategies does not change with the number of firms, but the distribution does. Surprisingly, the welfare of both types decreases with the number of firms. Each firm's offers converge to the worst pair of offers in the support, the pair of RS separating contracts. The distribution of the best offer in the market converges, as $N \to \infty$, to the equilibrium offer of a single firm in a duopoly. This result clarifies the impossibility of construction of mixed equilibrium when there are infinitely many firms and sheds light on the non-existence results for the competitive equilibrium concept considered in RS. All comparative statics results hold with strict inequalities whenever the equilibrium involves mixed strategies.

As mentioned before, the presence of cross-subsidization in equilibrium is possible because firms face uncertainty about competing offers. The link between uncertainty over competing offers and cross-subsidization can be illustrated in other models, without the presence of mixed strategies. The outcome presented in this paper can be obtained in a model with infinitely many firms and consumers with limited search capacity. In this case uncertainty about competing offers is generated by sampling made by the consumer from the set of available contracts, rather than mixed strategies.

The paper is organized as follows. The next section discusses the related literature. Section 3 formally describes the model. Section 4 describes a specific symmetric strategy profile and shows that it is an equilibrium. Section 5 shows that the constructed equilibrium is the unique symmetric one. Section 6 presents the comparative static results discussed above. Finally, Section 7 concludes.

2 Related Literature

Several papers have considered alternative models or equilibrium concepts that deal with the non-existence problem in the RS model. Maskin and Tirole (1992) consider two alternative models: the model of an informed principal and a competitive model in which many uninformed firms offer mechanisms to the agent. In the informed principal model, the agent proposes a mechanism to the (uninformed) firm. The equilibrium set consists of all incentive compatible allocations that Pareto dominate the RS allocation. The equilibrium outcome in my model is contained in the equilibrium set of the informed principal model.

Maskin and Tirole (1992) also consider a competitive screening model in which firms simultaneously offer mechanisms to a privately informed agent. A mechanism is a game form, in which both the chosen firm and the agent choose actions. The equilibrium set of this model is always large: it contains any allocations that are incentive compatible and satisfy individual rationality for the agent and firms. Hence the equilibrium set also contains the unique equilibrium outcome of my model. [6]

More recently, several models of adverse selection with price taking firms have been studied. Bisin and Gottardi (2006) consider a general equilibrium model with adverse selection that always has a unique equilibrium, which has the same outcome as RS. Dubey and Geanakoplos (2002) and Dubey, Geanakoplos, and Shubik (2005) consider a general equilibrium model in which agents trade shares of pools that combine the endowment of many agents. In their model, equilibrium always exists and it coincides with the separating contracts presented in RS. Guerrieri, Shimer, and Wright (2010) consider a competitive search model, in which the chance of an agent getting a given insurance contract depends on the ratio of insurance firms offering and agents demanding it. They also show that equilibrium always exists and reduces to the Rothschild and Stiglitz pair of contracts in the two-types case. Both models have uniqueness results that depend on different sets of belief refinements (on pools that are never traded or contract options that are not offered in equilibrium, respectively).

A separate strand of the literature uses notions of anticipatory equilibria, incorporating dynamic responses to a deviation by one firm, which guarantee existence. Wilson (1977) and Miyazaki (1977) restrict deviating contracts to be attractive, even after the incumbent firms are allowed to remove some of their contracts from the market. Both papers obtain equilibrium allocations

[6] The distinguishing feature of their model is the richness of the strategy set. A firm can react to moves by its opponents by offering a mechanism that contains a subsequent move by it. In equilibrium, a firm can respond to a "cream skimming" attempt by an opponent by choosing to offer no insurance, if the mechanism allows for such move by the firm.

that feature cross-subsidization and depend on the type distribution. These equilibrium notions have been justified as equilibria of specific extensive forms with multiple stages by Hellwig (1987), Mimra and Wambach (2011) and Netzer and Scheuer (2014). Alternatively, Riley (1979)'s model allows incumbent firms to propose new contracts after a deviation and obtains the RS contracts as the unique outcome, for any interior prior distribution. This outcome is obtained as equilibria in dynamic games by Engers and Fernandez (1987) and Mimra and Wambach (2014).

Rosenthal and Weiss (1984) present an analysis of a competitive version of the Spence model that shares several common feature with ours. They characterize a mixed equilibrium of the model, whenever a pure equilibrium does not exist. They have no results regarding uniqueness and dependence on the prior distribution. The effect of the number of firms on the constructed equilibrium is discussed, and is very similar the one presented here. Chari, Shourideh, and Zetlin-Jones (2014) characterizes a mixed strategy equilibrium in a linear competitive screening model where firms are privately informed about their asset qualities.[7]

An important feature of the equilibrium characterized is that the distribution of offers and welfare depend on the prior probability of types. Offers get strictly better as the probability of the good type becomes large. This phenomenon is absent in models of static competition with deterministic contracts. In Rothschild and Stiglitz (1976), the equilibrium is prior independent whenever it exists. I show that the restriction to prior probabilities for which competitive equilibrium exists is meaningful, since it is also the region for which the prior is not important for outcomes. Dubey and Geanakoplos (2002), Guerrieri, Shimer, and Wright (2010) and Bisin and Gottardi (2006) obtain the RS separating contracts as the unique equilibrium outcome for any distribution. This leads to a discontinuity of the equilibrium at the perfect information case in which all agents have low risk and there is efficient provision of insurance.

3 Model

A single agent faces uncertainty regarding his future income. There are two possible states $\{0, 1\}$ and his income in state 0 (1) is $y_0 = 0$ ($y_1 = 1$). The agent has private information regarding his risk type, which determines the probability of each state. For an agent of type $t \in \{h, l\}$, the probability of state 0 is p_t. Assume that $0 < p_l < p_h < 1$. This means that the l-type (low-risk) agent has higher expected income than h-type agents (high-risk). The prior probability of type t is denoted μ_t. Define $\overline{p} \equiv \mu_l p_l + \mu_h p_h$. There are N identical firms $i = 1, \ldots, N$ which compete in offering menus of contracts.

I assume that the realization of the state is contractible. A contract is a vector $c = (c_0, c_1) \in \mathbb{R}^2_+$ that denotes the final consumption available to the agent in case any of the states is realized.

[7] They obtain a partial uniqueness result under while assuming that contract offers are ordered in terms of attractiveness. In my paper, this property is a central part of my analysis as it allows one to move from a two-dimensional strategy space to a one-dimensional subset.

Contracts are exclusive. A menu of contracts is a compact subset of $\mathbb{R}_+^2$ denoted $\mathcal{M}^i$. The set of all compact subsets of $\mathbb{R}_+^2$ is defined as $\boldsymbol{M} \subseteq 2^{\mathbb{R}_+^2}$. A special case of a menu of contracts is a pair of contracts. I show later that one can focus without loss on pairs of contracts, with each one of them targeted for one specific type.

Timing is as follows. All firms simultaneously offer menus of contracts $\mathcal{M}^i \in \boldsymbol{M}$. Nature draws the agent's type according to probabilities μ_l and μ_h. After observing his own type t and the complete set of contracts $\mathcal{M}^1, \ldots, \mathcal{M}^N$, the agent announces a choice $a \in \bigcup_i \left(\mathcal{M}^i \times \{i\}\right) \cup \{\emptyset\}$. A choice $a = (c, i)$ indicates that contract $c \in \mathcal{M}^i$ is chosen from firm i, while choice $a = \emptyset$ means that the agent chooses to get no contract (and will maintain his own income).

A final outcome of the game is $\left(\mathcal{M}^1, ..., \mathcal{M}^N, t, a\right)$ (everything is evaluated before the income realization is revealed). Given outcome $\left(\mathcal{M}^1, ..., \mathcal{M}^N, t, (c, i)\right)$, the realized profit by firm j is zero, if $j \neq i$, and otherwise is

$$\Pi\left(c \mid t\right) \equiv \left(1 - p_t\right)\left(1 - c_1\right) - p_t c_0.$$

The agents have instantaneous utility function $u\left(\cdot\right)$, which is strictly concave, increasing and continuously differentiable. Finally, the utility achieved by the agent is

$$U\left(c \mid t\right) \equiv \left(1 - p_t\right) u\left(c_1\right) + p_t u\left(c_0\right).$$

Given outcome $\left(\mathcal{M}^1, ..., \mathcal{M}^N, t, \emptyset\right)$ the realized profit by all firms is zero and the utility achieved by the agent is $U\left(y \mid t\right)$.

A (pure) strategy profile is a menu of contracts for each firm $\left(\mathcal{M}^i\right)_i$ and a choice strategy for the agent, which is a measurable function $s : \{h, l\} \times \left(\times_i \boldsymbol{M}\right) \to \left(\mathbb{R}_+^2 \times \{1, .., N\}\right) \cup \emptyset$ such that $s\left(t, \left(\mathcal{M}^i\right)_i\right) \in \bigcup_i \left(\mathcal{M}^i \times \{i\}\right) \cup \{\emptyset\}$.

A mixed acceptance rule is a Markov kernel[8]

$$s : \{h, l\} \times \left(\times_i \boldsymbol{M}\right) \to \Delta\left[\left(\mathbb{R}_+^2 \times \{1, \ldots, N\}\right) \cup \emptyset\right]$$

with the restriction $s\left(\bigcup_i \left(\mathcal{M}^i \times \{i\}\right) \cup \{\emptyset\} \mid t, \left(\mathcal{M}^i\right)_i\right) = 1$ (with abuse of notation)

A mixed strategy profile is a probability measure over menus of contracts Φ_i for each firm i and a mixed acceptance rule s.[9] A mixed strategy profile defines a probability distribution over outcomes in the natural way, expected profits are defined by integrating realized profits across outcomes according to this probability distribution.

The equilibrium concept is subgame perfect equilibrium.[10] This means that (i) each firm i

[8]The Markov kernel definition includes the requirement that, for any measurable set $A \subseteq \mathbb{R}_+^2 \times \{1, \ldots, N\}$, the functions $\left(t, \mathcal{M}^1, ..., \mathcal{M}^N\right) \longmapsto s\left(A \mid t, \mathcal{M}^1, ..., \mathcal{M}^N\right)$ is measurable.

[9]I endow $\boldsymbol{M}$ with the Borel sigma algebra induced by the open balls in the Hausdorff metric. I will only use two properties from this sigma algebra: (i) it contains any single contract and (ii) the function that leads to the best available utility to any fixed risk type must be measurable.

[10]In this game, perfect Bayesian equilibrium is outcome equivalent to subgame perfection. Considering a game

maximizes expected profits, given the strategies used by its opponents and the acceptance rule by the agent and (ii) the agent only chooses contracts that maximize his (interim) utility, i.e.,

$$(c, i) \in supp\left(s\left(t, \left(\mathcal{M}^i\right)_i\right)\right) \Rightarrow c \in \arg \max_{c \in \bigcup_i \mathcal{M}^i \cup \{y\}} U\left(c \mid t\right),$$

and

$$\emptyset \in supp\left(s\left(t, \left(\mathcal{M}^i\right)_i\right)\right) \Rightarrow U\left(y \mid t\right) \geq \max_{c \in \bigcup_i \mathcal{M}^i} U\left(c \mid t\right).$$

The optimization problem faced by the agent always has a solution because the set of available contracts, $\bigcup_i \mathcal{M}^i \cup \{y\}$, is compact.

4 Equilibrium construction

In this section I construct an equilibrium of the described model. The existence issue raised in Rothschild and Stiglitz (1976) is overcome by the use of mixed strategies by insurance firms. This is the first characterization of a mixed strategies equilibrium in an insurance setting. The novel feature of this equilibrium is the potential presence of cross-subsidization, which generates a dependence of the equilibrium allocation on the risk distribution in this market. In section 5, this equilibrium is shown to be unique. In the first part of this section I assume that $N = 2$. I show, in the end of this section, how to adjust the equilibrium to the case $N > 2$.

4.1 Equilibrium offers

In equilibrium firms "compete away" profit opportunities. However, zero expected profits are consistent with cross-subsidization from low-risk to high-risk agents: the presence of losses generated from high-risk individuals, which in turn get subsidized by profits from low-risk individuals. In what follows, I construct a family of offers, indexed by the amount of cross-subsidization across types, and show that an equilibrium using these offers always exists.

Low-risk agents have higher expected income and as a consequence receive more attractive offers from firms. The only way to respect incentive constraints is by offering partial insurance contracts (i.e., with $c_1 > c_0$) to low-risk agents. High-risk agents, on the other hand, receive less attractive contracts that do not conflict with incentive constraints. As a consequence they receive full insurance contracts (i.e., $c_1 = c_0$). This implies that the set of contracts that arise in equilibrium lies in a restricted locus, which is described in the following.

tree in which the firms act sequentially, each subgame perfect equilibrium has a corresponding PBE with the same strategy profiles and firms' beliefs, about the earlier firms' play, given directly by equilibrium strategies. Notice that the agent, moving last, has perfect information because he knows all the offers and his type as well.

In this game, the concept of Nash equilibrium allows the agent to behave "irrationally" to menu offers off the equilibrium path. This enables many additional "collusive" equilibria. In fact, I can sustain any individually rational allocation as a Nash equilibrium outcome.

For a level $k \in [0, p_h - \overline{p}]$ of subsidies received by high-risk individuals, the full insurance contract received by high-risk agents has consumption $c = 1 - p_h + k$, which is above their actuarily fair consumption level by k. Also, define $\gamma(k) = (\gamma_1(k), \gamma_0(k))$ to be the partial insurance contracts that can be offered to the low-risk agent together with subsidy k to the high-risk agent. These contracts leave the high-risk agent indifferent between partial and full insurance, which provides incentives efficiently, and generate zero expected profits.

Formally, I define the following (set-valued) function $\gamma : [0, p_h - \overline{p}] \to 2^{\mathbb{R}^2_+}$ by

$$\gamma(k) \equiv \left\{ c \in \mathbb{R}^2_{++} \;\middle|\; \begin{array}{c} U(c \mid h) = u(1 - p_h + k); \\ \mu_l \Pi(c \mid l) + \mu_h(-k) = 0; \\ c_1 \geq c_0. \end{array} \right\}.$$

Lemma 1. *For any $k \in [0, p_h - \overline{p}]$, $\gamma(k)$ is a singleton, i.e., there exists a unique $c \in \mathbb{R}^2_+$ such that $c \in \gamma(k)$.*

Proof. Let us define $\zeta = \sup\{c_1 | \exists c_0 \text{ such that } U(c \mid H) = u(1 - p_h + k)\}$. The strict concavity of u implies that $\zeta > 1$ ($\zeta = \infty$ is possible). Consider the path $\iota : I = [0, w] \to \mathbb{R}^2$ that starts at $(1 - p_h + k)(1, 1)$ and moves along the indifference curve of $U(\cdot \mid h)$ by increasing c_1, i.e., $\iota_1(t) = k + t$ ($w = \infty$ if $\zeta = \infty$). Let total profit generated by point t in the path, when the high-risk agent consumes $(1 - p_h + k)(1, 1)$ and the low-risk agent consumes $\iota(t)$, be denoted as $\pi(t)$. We know that $\pi(0) \geq 0$ because $k \leq 1 - \overline{p}$. If $\zeta < \infty$, continuity implies that

$$\pi(w) \leq \mu_l (1 - p_l)(1 - \zeta) < 0.$$

If $\zeta = \infty$, it follows that $\lim_{t \to \infty} \pi(t) = -\infty$. Therefore in both cases continuity of $\pi(t)$ implies that there is t_0 such that $\pi(t_0) = 0$. It also follows from concavity of $u(\cdot)$ that $\pi'(t) > 0$ for all $t > 0$, which means that $\pi(t_0) = 0$ for at most one point t_0. □

From now on, I refer to $\gamma(\cdot)$ as a single-valued function. Figure 1 illustrates the locus of $\{\gamma(k) \mid k \in [p_l, \overline{p}]\}$. From now on, I refer to offers

$$\mathcal{M}^k \equiv \{(1 - p_h + k, 1 - p_h + k), \gamma(k)\},$$

for $k \in [0, p_h - \overline{p}]$, as *cross-subsidizing offers.* And also define the utilities obtained from cross-subsidization level k as

$$U_l(k) \equiv U(\gamma(k) \mid l),$$

$$U_h(k) \equiv u(1 - p_h + k).$$

The pair of contracts with zero cross-subsidization, coincide with the unique equilibrium allocation in RS. Given their importance on the analysis of this model, we introduce notation to refer to these contracts.

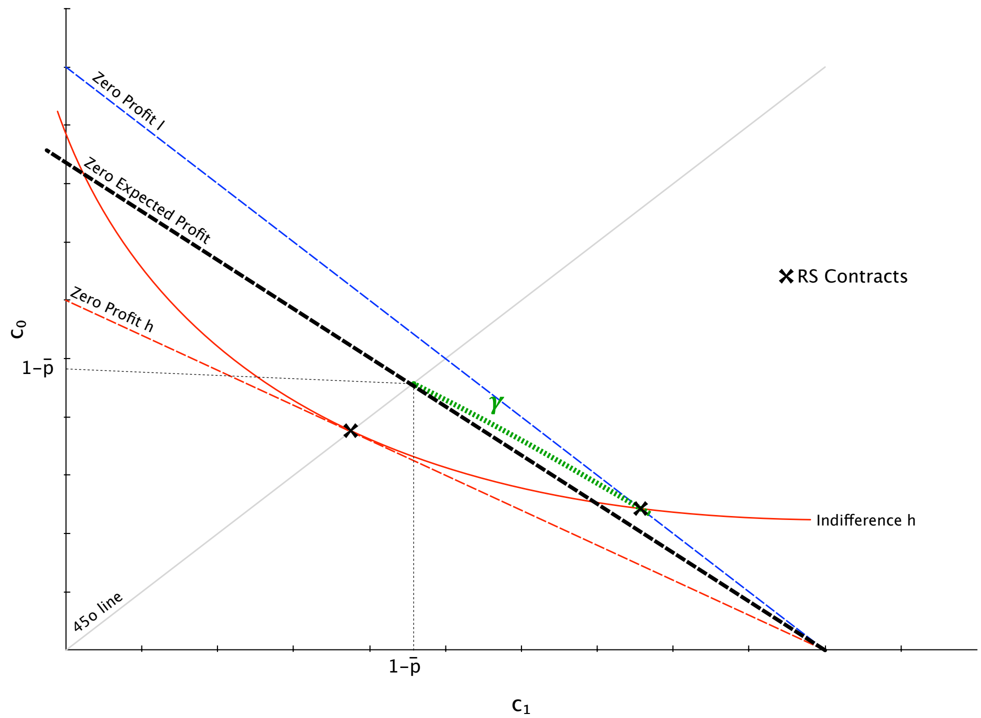

Figure 1: The contract space and the image of the $\gamma(\cdot)$ function. Notice that the RS partial insurance contract, c_l^{RS}, is equal to $\gamma(0)$ and coincides with the lowest level of cross-subsidization. The other extreme point in the image of $\gamma(\cdot)$ is $\gamma(p_h - \overline{p})$, which features full insurance at the correct price for the average population risk.

Definition 1. The Rothschild-Stiglitz (RS) contracts are the pair

$$\left\{c_l^{RS}, c_h^{RS}\right\} \equiv \mathcal{M}^0,$$

we also define $u_t^{RS} \equiv U_t(0)$, for $t = l, h$ and $u^{RS} \equiv \left(u_l^{RS}, u_h^{RS}\right)$.

The pareto efficiency of cross-subsidization plays a crucial role in equilibrium analysis. Cross-subsidization always benefits high-risk agents, since their complete coverage comes at lower prices. What is more surprising is that low-risk agents can also benefit from cross-subsidization when the prior probability of high-risk is sufficiently low. The reason for that it is that subsidizing high-risk is cheap when the probability of such state is small. In the following lemma, we show that the gains from cross-subsidization are negative for large subsidization levels, and potentially positive for low subsidization levels.

Lemma 2. *There exists $\overline{k} \in [0, p_h - \overline{p})$ such that $U_l(\cdot)$ is strictly increasing for $k < \overline{k}$ and strictly decreasing for $k > \overline{k}$. More specifically, $\overline{k}$ is the unique peak of $U_l(\cdot)$ in $[0, p_h - \overline{p}]$.*

Proof. The implicit function theorem implies that γ is continuously differentiable and satisfies

$$\gamma_1' = -\frac{u'\left(1-p_h+k\right)+\frac{p_h}{p_l}u'\left(\gamma_0\left(k\right)\right)\frac{\mu_h}{\mu_l}}{p_h\left[\frac{(1-p_l)}{p_l}u'\left(\gamma_0\left(k\right)\right)-\frac{(1-p_h)}{p_h}u'\left(\gamma_1\left(k\right)\right)\right]},$$

$$\gamma_0' = \frac{\left(1-p_l\right)}{p_l}\left\{\frac{u'\left(1-p_h+k\right)+\frac{\mu_h}{\mu_l}\frac{(1-p_h)}{(1-p_l)}u'\left(\gamma_1\left(k\right)\right)}{p_h\left[\frac{(1-p_l)}{p_l}u'\left(\gamma_0\left(k\right)\right)-\frac{(1-p_h)}{p_h}u'\left(\gamma_1\left(k\right)\right)\right]}\right\}.$$

Which implies that $\gamma_1'\left(k\right)<0$ and $\gamma_0'\left(k\right)>0$. Simple differentiation gives us

$$U_l'\left(k\right)=\left\{\frac{\left[u'\left(\gamma_1\left(k\right)\right)^{-1}-u'\left(\gamma_0\left(k\right)\right)^{-1}\right]\left(1-p_l\right)u'\left(1-p_h+k\right)-\frac{\mu_h}{\mu_l}\frac{1}{p_h}\left[\frac{(1-p_l)}{p_l}-\frac{(1-p_h)}{p_h}\right]}{p_h\left[\frac{(1-p_l)}{p_l}\frac{1}{u'(\gamma_1(k))}-\frac{(1-p_h)}{p_h}\frac{1}{u'(\gamma_0(k))}\right]}\right\}.$$

The numerator in the last expression is strictly positive. The denominator is strictly decreasing for $k\in[0,p_h-\overline{p}]$ and negative for $k=p_h-\overline{p}$ (in which case, $\gamma_1\left(p_h-\overline{p}\right)=\gamma_1\left(p_h-\overline{p}\right)=1-\overline{p}$).

□

Whenever cross-subsidization leads to interim efficiency gains, which happens when $\overline{k}>0$, quasi-concavity of $U_l\left(\cdot\right)$ implies that the low-risk utility increases with the level of subsidization at any level $k\in\left[0,\overline{k}\right]$. We define this restricted set of subsidization levels, which have the property that low-risk agents benefit from it, as R:

$$R\equiv\left[0,\overline{k}\right].$$

A comment on the literature is in order. The pareto efficient cross-subsidization level $\overline{k}$ defines the separating allocation that maximizes the utility offered to the low-risk agent subject to zero profits. It coincides with the allocation described by Miyazaki (1977), who obtains this allocation as an equilibrium outcome when using a reactive equilibrium notion. As described in subsection 4.2, the equilibrium described here generates all cross-subsidization levels between zero and the efficient one.

If cross-subsidization is not optimal ($\overline{k}=0$), the no cross-subsidization contracts presented in RS are indeed an equilibrium. This follows from the fact that there is no way that a firm can attract both types of agents and make expected positive profits. However, when cross-subsidization leads to interim gains ($\overline{k}>0$) then it has to arise in equilibrium. When facing contracts $\mathcal{M}^0$ with no cross-subsidization a firm can offer a menu with optimal subsidization $\mathcal{M}^{\overline{k}}$ that will generate zero expected profits while leading to a strictly positive utility gain for both risk types. This means that a slightly less attractive offer can make positive profits.

The presence of cross-subsidization in a pure strategy equilibrium is ruled out because they are vulnerable to cream-skimming deviations. If firms make positive profits from low-risk agent and losses on high-risk agents, one firm can offer a contract with slightly less coverage that only attracts low-risk agents while leaving the losses from high-risk agents to its competitors. The construction of such deviations, however, only applies to pure strategies as a firm facing a non-degenerate distribution of competing contracts is not able to design a local deviation that attracts the low-risk agents with probability one while attracting high-risk agents with probability zero. In fact, we show here that firms will randomize continuously between a pareto efficient level of cross-subsidization $\overline{k}$ and the RS contracts, which feature zero cross-subsidization.

Formally, every firm mixes over menu offers

$$\boldsymbol{M}^R \equiv \left\{ \mathcal{M}^k \mid k \in R \right\}.$$

The set $\mathbf{M}^R$ has the property that, if all firms offer menus within this set, then all offers in this set guarantee zero profits to a firm. This occurs because the firm that makes the offer $\mathcal{M}^k$ with highest level of cross-subsidization attracts the agent, regardless of his risk type. However, the defining property of cross-subsidizing offers is that they make zero expected profits if both types consume the same menu. Firms that offer cross-subsidizing level below their opponents make zero profits as they never serve the agent.

4.2 Equilibrium distribution

The goal of section is to describe the equilibrium distribution over cross-subsidization levels $k \in R$, which is denoted as F, and to show that firms have no profitable deviations outside of $\mathbf{M}^R$.

The strategy set of firms contains all possible contract menus, and hence is very large. The first step in our analysis is to describe menu offers by the expected utility it generates to each risk type. This description is useful for the following reason. For each type, the utility generated by a menu determines the probability with which it is chosen. This is the probability that the best alternative offer is less attractive than the menu considered, which is determined in equilibrium. Also, within the set of menus that deliver a specific utility profile, firms will only offer the one that minimizes expected profits. This allows us to focus on a subset of menus that are indexed by utility profiles. The profit, or loss, that is made from each type in case he joins a firm is determined by a cost minimization problem that considers the utility vectors as constraints.

Define Υ as the set of incentive feasible utility profiles.[11] For each risk type $t \in \{l, h\}$ and utility profile $\mathbf{u} = (u_l, u_h) \in \Upsilon$, define the ex-post profit function $P_t(\mathbf{u})$ as the solution to the following

[11] The following notation is important for the proof. Let

$$\Upsilon \equiv \left\{ (u_l, u_h) \in \mathbb{R}^2 \mid \begin{array}{c} U(c_h \mid l) \leq U(c_l \mid l) = u_l \\ U(c_l \mid h) \leq U(c_h \mid h) = u_h \end{array} \text{ for some } c_l, c_h \in \mathbb{R}^2_+ \right\}.$$

problem

$$\max_{c \in \mathbb{R}_+^2} \Pi(c \mid t)$$

subject to generating utility u_t to type t:

$$U(c \mid t) = u_t,$$

and respecting incentive constraints regarding type $t' \neq t$:

$$U\left(c \mid t'\right) \leq u_{t'}.$$

Also, define as $\chi(\mathbf{u}) = (\chi_l(\mathbf{u}), \chi_h(\mathbf{u}))$ the unique solution to problems $P_l(\mathbf{u})$ and $P_h(\mathbf{u})$ respectively.

Contracts that maximize ex-post profits are the most profitable ones that deliver a specific utility profile, which in a mixed equilibrium means that the probability of attracting each type is fixed. The following properties of the optimal ex-post profit function are key to our results.

Lemma 3. *(Ex-post profit characterization) The ex-post profit function has the following properties:*

(i) it is continuously differentiable,

(ii) utility is costly: $\frac{\partial P_t(u)}{\partial u_t} < 0$,

(iii) separation is costly:

$$\frac{\partial P_t(u)}{\partial u_{t'}} > 0, \text{ if } u_t > u_{t'},$$

and

$$\frac{\partial P_t(u)}{\partial u_{t'}} = 0, \text{ if } u_t \leq u_{t'},$$

(iv) supermodularity: $\frac{\partial P_t}{\partial u_t}$ *is continuously differentiable if* $\frac{\partial^2 P_t(u)}{\partial u_t \partial u_{t'}} \geq 0$*, and the inequality is strict if* $u_t > u_{t'}$.

Proof. In the appendix. □

From the point of view of a single firm, the offers made by other firms can be treated as a stochastic type-contingent outside option to the agent. The distribution of outside options for a given firm is determined by the equilibrium contract distribution in the following way. On the support of equilibrium offers higher subsidization benefits both types, so the distribution is a monotone transformation of the distribution over cross-subsidization level $k \in R$:

$$G_l(U_l(k)) \equiv F(k),$$

and

$$G_h(U_h(k)) \equiv F(k),$$

also let $G_t(u) = 0$, for $u < U_t(1-p_h)$, and $G_t(u) = 1$ for $u > U_t(\overline{k})$.

The distributions G_l and G_h constructed in this section are absolutely continuous. In section 5, it is shown that this is necessarily the case in equilibrium. Expected profits are determined by the probability of attracting each type, which is given by distributions $(G_t)_{t=l,h}$, and the ex-post profits made from each type, which is determined by function $(P_t)_{t=l,h}$. Now we define this function formally: for any $u = (u_l, u_h) \in \Upsilon$

$$\pi(u) \equiv \mu_l G_l(u_l) P_l(u) + \mu_h G_h(u_h) P_h(u).$$

When contemplating a more attractive offer to a specific type, firms have to consider the following trade-off. When increasing the utility promised to such agent he will be attracted with higher probability, which entails a gain if the firm makes profits out of such agent. On the other hand, in order to make a more attractive offer it has to make less profits out of this agent, if indeed he ends up trading with the firm. In equilibrium, firms can only make positive profits from low-risk agents. For any utility pair $u = (u_l, u_h)$ with $u_l \geq u_h$, we define the marginal profit from attracting the low-risk agent as:

$$M(u) \equiv \frac{\partial \pi(u)}{\partial u_l} = \mu_l \left[g_l(u_l) P_l(u) + G_l(u_l) \frac{\partial P_l(u)}{\partial u_l} \right], \tag{1}$$

where $g_t(u_t) \equiv G_t'(u_t)$ is the density of the equilibrium utility distribution.

In order for cross-subsidizing offers $\mathcal{M}^k$, for $k \in R$, to arise in equilibrium it must be optimal for a firm to offer utility profile

$$U(k) = (U_l(k), U_h(k)),$$

for any $k \in R$.

This means that G_l has to satisfy the following equality

$$M(U(k)) = 0, \text{ for all } k \in R. \tag{2}$$

Hence, local deviations around any offer in the support should not be optimal. This is a necessary condition to sustain an equilibrium with this support. Using the equality $f(k) = g_l(u_l(k)) u_l'(k)$, we define F as the solution to the differential equation implied by (2).

The necessary condition for an equilibrium with support $\mathbf{M}^R$ is for F to satisfy:

$$\frac{f(k)}{F(k)} = \frac{-\frac{\partial P_l(U(k))}{\partial u_l} U_l'(k)}{P_l(U(k))}, \tag{3}$$

with final condition $F(\overline{k}) = 1$.

Lemma 4. *The differential equation (3) has a unique solution. Moreover, F is given by*

$$F(k) = \exp\left[-\int_{k}^{\overline{k}} \phi(z)\, dz\right],$$

where

$$\phi(z) = \frac{-\frac{\partial P_l(U(k))}{\partial u_l} U_l'(k)}{P_l(U(k))}.$$

Moreover, F puts no mass at zero if $\overline{k} > 0$, i.e.,

$$F(0) = 0.$$

Proof. Integration of (3) implies that

$$1 = F\left(\overline{k}\right) = F(k) \exp\left[\int_{k}^{\overline{k}} \phi(z)\, dz\right].$$

Finally notice that $P_l(U(k)) = \Pi(\gamma(k) \mid l) = \frac{\mu_h}{\mu_l} k$. This means that $\phi(z)$ is of the order of $\frac{1}{k}$ around $1 - p_h$.[12] Then it follows that

$$\lim_{k \to (1-p_h)^+} \int_{k}^{\overline{k}} \phi(z)\, dz = \infty.$$

This implies that $F(0) = \lim_{k \to (0)^+} F(k) = 0$. □

As mentioned, condition (3) implies that any offer $\mathcal{M}^k$ is locally optimal, for $k \in R$. However in equilibrium firms also consider non-local deviations. In order to rule out such deviations, we show that the expected profits are supermodular in the utility pair offered to the agent. This means that increasing the utility offered to the high-risk agent makes it more profitable, at the margin, to make a higher utility offer to the low-risk agent.

Lemma 5. *(Supermodularity of profits) For any $u = (u_l, u_h)$ such that $u_l \geq u_h$, we have that*

$$M(u_l, u_h) \text{ is non-decreasing in } u_h,$$

and it is strictly increasing if $u_l > u_h$ and $G_l(u_l) > 0$.

Proof. Follows directly form the definition of $M(\cdot)$ in (1), properties (iii) and (iv) of Lemma 3. □

The relevance of the supermodularity conditions is as follows. In equilibrium the firm must find

[12] Notice that $U_l'(0) > 0$ if $\overline{k} > 0$, $-\frac{\partial P_l(u(0))}{\partial u_l} > 0$ and both $U_l'(\cdot)$ and $\frac{\partial P_l(u(\cdot))}{\partial u_l}$ are continuous.

it optimal to offer levels of subsidization $k, k' \in R$ with $k' > k$, which generate utilities

$$U(k) = (U_l(k), U_h(k)) \ll \left(U_l\left(k'\right), U_h\left(k'\right)\right) = U\left(k'\right).$$

This mean that, in our candidate equilibrium, firms are indifferent between offering $U(k)$ or strictly increasing the utility offered to both types to $U(k')$. But then supermodularity implies that increasing only the utility offered to the low-risk agent leads to a loss:

$$\begin{aligned}\pi\left(U_l\left(k'\right), U_h(k)\right) - \pi\left(U_l(k), U_h(k)\right) &= \int_{U_l(k)}^{U_l(k')} M\left(U_l(s), U_h(k)\right) U_l'(s)\, ds \\ &< \int_{U_l(k)}^{U_l(k')} M\left(U_l(s), U_h(s)\right) ds = 0.\end{aligned}$$

In the appendix we provide the complete proof that offering cross-subsidizing contracts according to distribution F is indeed an equilibrium. The proof uses the supermodularity property of the profit function in a similar way to show that all possible deviations are unprofitable.

Proposition 1. *There exists a symmetric equilibrium such that: (i) every firm randomizes over offers in $\left\{\mathcal{M}^k \mid k \in R\right\}$, where $k \in \left[0, \overline{k}\right]$ is distributed according to $F(\cdot)$; (ii) after observing menu offers $\left(\mathcal{M}^{k_1}, \mathcal{M}^{k_2}\right)$ with $k_i > k_j$ the agent chooses according to*

$$\begin{aligned}s\left(h, \left(\mathcal{M}^{k_1}, \mathcal{M}^{k_2}\right)\right) &= \left(1 - p_h + k_i, 1 - p_h + k_i\right), \\ s\left(l, \left(\mathcal{M}^{k_1}, \mathcal{M}^{k_2}\right)\right) &= \gamma\left(k_i\right).\end{aligned}$$

After observing offers that are not of the form $\left(\mathcal{M}^{k_1}, \mathcal{M}^{k_2}\right)$, the agent chooses any arbitrary selection from his best response set.[13]

4.3 Pure strategy equilibrium

The equilibrium described in Proposition 1 coincides with the pure strategy equilibrium characterized in Rothschild and Stiglitz (1976), whenever it exists. The equilibrium involves no mixing if, and only if,

$$\overline{k} = 0,$$

in which case the set of cross-subsidization levels offered in equilibrium is $R = \{0\}$ and firms offer the zero cross-subsidization menu $\mathcal{M}^0$, that coincides with the contract pair $\left\{c_l^{RS}, c_h^{RS}\right\}$.

Lemma 2 shows that the benefits from cross-subsidization, from the low-risk agent's point of view, are quasi-concave. As a consequence zero cross-subsidization is pareto optimal if, and only if,

$$U_l'(0) \leq 0. \tag{4}$$

[13] With the restriction that s is still a mixed strategy, as defined in Section 3.

The uniqueness result, discussed in Section 5, implies that (4) is a necessary and sufficient condition for existence of a pure strategy equilibrium. The exact condition in terms of the prior distribution is presented in the following corollary.

Corollary 1. *The equilibrium described involves pure strategies if, and only if,*

$$u'\left(c_h^{RS}\right)\left[\frac{1}{u'\left(c_{l,1}^{RS}\right)}-\frac{1}{u'\left(c_{l,0}^{RS}\right)}\right]\leq\frac{\mu_h}{\mu_l}\left[\frac{p_h}{p_l}-\frac{1-p_h}{1-p_l}\right]. \tag{5}$$

If this is the case, all firms offer pair of contracts $\left\{c_l^{RS}, c_h^{RS}\right\}$ *in equilibrium.*

A pure strategy equilibrium fails to exist whenever the share of low-risk agents is sufficiently high. In this case the cost of cross-subsidization is very low since there are few high-risk agents to be subsidized.

4.4 The case $N > 2$

In the analysis of the duopoly case, I have shown that one can find a distribution over the set of menu offers $\boldsymbol{M}^R$ such that each firm finds it optimal to make any offer in this set.

This support $\boldsymbol{M}^R$ has the following property: the utility obtained by both types, $U_l(\cdot)$ and $U_h(\cdot)$, are strictly increasing in the cross-subsidization level k, for $k \in R$. This means that if firm $i = 1$ faced two firms, 2 and 3, that were choosing offers $\mathcal{M}^k$ according to continuous distributions F_2 and F_3, the relevant random variable for firm 1 would be $k_{23} = \max\{k_2, k_3\}$, which determines the only relevant threat to their offers. The distribution of this variable is given by $F(k) = F_1(k) F_2(k)$. This allows us to adapt the arguments above, by equalizing the distribution of the best among $N-1$ firms with the single firm distribution in the duopoly.

Proposition 2. *In the game with N firms, the following is an equilibrium: every firm randomizes over offer set* $\boldsymbol{M}^R$ *with distribution over cross-subsidization level* $k \in R$ *given by* $F_i(\cdot)$*, where*

$$F_i(k) = F(k)^{\frac{1}{N-1}}.$$

The equilibrium described has the following properties. First, whenever there is randomization, ties occur with zero probability: there is always a firm that offers M^{k_i} such that $k_i > \max_{j\neq i} k_j$. The agent gets a contract from this firm, independent of which type is realized. If the type is l, the agent ends up with contract (k_i, k_i). If the agent is of type h, he chooses contract $\gamma(k_i)$. Second, whenever the pure equilibrium with the RS contracts exists, $R = \{1 - p_h\}$ and the support of strategies reduces to the RS contracts.

In the next section, I show that this is the unique symmetric equilibrium. Section 6 presents monotone comparative statics results regarding the prior distribution and the number of firms.

5 Uniqueness

In this section we show that the equilibrium constructed in section 4 is the unique symmetric equilibrium. We start by showing that equilibrium offers can be fully described by the utility it generates to both possible risk types. This means that describing the equilibrium strategies used by firms reduces to describing the equilibrium distribution of utility levels generated by equilibrium offers.

Describing offers in terms of utility profiles means that the offer space is essentially two-dimensional. The main challenge in the analysis lies in showing that equilibrium offers necessarily lie in a one-dimensional subset of the feasible utility space. The crucial step uses properties of the equilibrium utility distribution and the ex-post profit functions to show that expected profits are supermodular in the utility vector offered to the consumer. There is a complementarity in making more attractive offers to both risk types. Supermodularity is used to show that equilibrium offers are necessarily ordered in terms of attractiveness, i.e., a more attractive offer provides higher utility for both risk types.

Firms in this market make zero expected profits (as shown in Proposition 3). The ordering of offers is used to show that equilibrium offers necessarily generate zero profits even if they are accepted by both risk types with probability one.[14] Hence, offers can be indexed by the amount of subsidization that occurs across different risk types. The remaining analysis follows standard steps in the literature on games with one-dimensional strategy spaces (see Lizzeri and Persico (2000) and Maskin and Riley (2003)).

For an arbitrary mixed strategy ϕ for insurance firms, we denote as G the distribution of the highest utility for each type $t \in \{l, h\}$ induced by $N-1$ offers generated according to distribution ϕ. Formally, define the utility obtained from offer $\mathcal{M} \in \mathbf{M}$ by an agent of type $t \in \{l, h\}$ as

$$u_t^{\mathcal{M}} \equiv \max \{U(c \mid t) \mid c \in \mathcal{M}\},$$

as for any $\mathbf{u} \in \mathbb{R}^2$

$$G(u_l, u_h) \equiv \left[\phi \left\{\mathcal{M} \in \mathbf{M} \mid u_t^{\mathcal{M}} \leq u_t, \text{ for } t \in \{l, h\}\right\}\right]^{N-1}.$$

In equilibrium, G is relevant because it determines the distribution of outside options that any given firm faces when trying to attract a consumer. Also, we define as G_t the marginal of G over u_t, for $t \in \{l, h\}$. The equilibrium outcome distribution is denoted as $\mathbb{P}^*$.

Proposition 3. *(Zero profits) In any symmetric equilibrium:*

(i) firms make zero expected profits.

(ii) Fix $t \in \{l, h\}$. If a firm i makes an offer $\mathcal{M}$ that generates utility profile $\mathbf{u}$, then $\chi_t(\mathbf{u}) \in \mathcal{M}$

[14] Or alternatively, if they are accepted in the market as a whole.

and this is the only possible offer accepted by type t from firm i: for any $c \in \mathcal{M} \setminus \{\chi_t(\mathbf{u})\}$,

$$\mathbb{P}^* \left[\left(\tilde{t}, s\left(\tilde{t}, \left(\mathcal{M}, \tilde{\mathcal{M}}_{-i} \right) \right) \right) = (t, c, i) \right] = 0,$$

(iii) firms make nonnegative profits from low-risk agents and non-positive profits from high-risk agents: any equilibrium utility offer $\mathbf{u}$ satisfies $\mathbf{u} \in int(\Upsilon)$, $u_l \geq u_h$ and

$$P_l(\mathbf{u}) \geq 0,$$

$$P_h(\mathbf{u}) \leq 0.$$

Proof. In the appendix. □

The previous statement contains three main results. First, it shows that firms never make positive expected profits in equilibrium. This follows from the assumption of Bertrand competition without differentiation. The main challenge in the proof is to deal with the multidimensionality of the offer space. Second, it shows that equilibrium offers can be described in terms of the utility profile they generate. If an offer $\mathcal{M}$ generates utility profile $\mathbf{u}$, then the only contracts from this set that can be consumed in equilibrium are $\{\chi_l(\mathbf{u}), \chi_h(\mathbf{u})\} \subseteq \mathcal{M}$. This comes from the fact that a firm's maximization problem can be split into choosing the attractiveness of the contract, given by the utility levels, and the minimization of costs for a fixed level of utility.[15] Finally, a consequence of zero profits is that firms necessarily make (potentially zero) losses on high-risk agent and profits on low-risk agents. The reason is that if an offer generates strictly positive profits from high-risk agents, a firm can always guarantee ex-ante expected profits since low-risk agents are always at least as profitable as high-risk agents.

Given Proposition 3, our uniqueness proof consists of showing that there exists only one possible equilibrium utility distribution generated by each firm. From part (ii) of the proposition, describing the distribution over utility profiles provides a description of equilibrium offers, namely $\chi(\mathbf{u})$.[16]

The following intermediary lemma provides a characterization of the equilibrium utility distribution G. It shows that this distribution is absolutely continuous except at one point, the utility level generated by Rothschild-Stiglitz offers. Define, for each $t = l, h$, $\underline{u}_t \equiv \inf\{u \mid G_t(u) > 0\}$ as the lowest equilibrium utility and similarly $\overline{u}_t \equiv \sup\{u \mid G_t(u) < 1\}$ as the highest equilibrium utility.

Lemma 6. *(Utility distribution) In any symmetric equilibrium, the utility distribution G satisfies the following:*

(i) lower bound on utility: $\underline{u}_t \geq u_t^{RS}$,

[15] One has to consider the possibility of the agent being indifferent between two contracts. However, this is solved in the proof of Proposition 3 by showing that a firm can always break such indifferences at infinitesimal cost.

[16] Obviously any equilibrium offer generating utility $\mathbf{u}$ can include other contracts beyond $\{\chi_l(\mathbf{u}), \chi_h(\mathbf{u})\}$, as long as they are unattractive. This means they satisfy $U(c \mid t) \leq U(\chi_t(\mathbf{u}) \mid t)$. Proposition 3 shows that if such contracts are present they are irrelevant, meaning they are never chosen on the equilibrium path.

(ii) mass points for low-risk: G_l has support $[\underline{u}_l, \overline{u}_l]$ and is absolutely continuous on $[\underline{u}_l, \overline{u}_l] \setminus \{u_l^{RS}\}$,

(iii) mass points for high-risk: the only possible mass point of G_l is u_l^{RS}.

Proof. In the appendix. □

From now on, we denote the density of G_t, whenever it exists, as g_t.

Proposition 3 shows that offers can be described in terms of the utility generated by them. Lemma 6 shows that mass points are not possible, except at the utility generated from Rothschild-Stiglitz offers. This means that, if a firm makes a utility offer $\mathbf{u} \gg (u_l^{RS}, u_h^{RS})$ in equilibrium then its profits are given by

$$\pi(\mathbf{u}) \equiv \mu_l G_l(u_l) P_l(u) + \mu_h G_h(u_h) P_h(u).$$

In what follows, we show that the function $\pi(\cdot)$ is supermodular in (u_l, u_h). This means that it satisfies an increasing differences property: the profit gains from making a more attractive offer to low-risk agent is strictly increasing with the utility offered to high-risk agents.

In order for such utility offer to be optimal, there must be no alternative utility $\mathbf{u}'$ that increases expected profits. Consider utility offer $\mathbf{u}$ satisfying $u_l > u_h$ and $u_l > u_l^{RS}$. The marginal gain from making a more attractive offer to low-risk agents is given by[17]

$$M(\mathbf{u}) \equiv \mu_l \left[g_l(u_l) P_l(u) + G_l(u_l) \frac{\partial P_l(u)}{\partial u_l} \right].$$

Where the first term captures the probability gain, which means that a more attractive offer has a higher chance of attracting low-risk individuals who generate positive profits. The second term captures the loss in profits that a firm faces in order to make a more attractive offer to low-risk agents. Using simple properties of the ex-post profit function $P_l(\cdot)$ described in Lemma 3, we show that the profit function satisfies supermodularity: the marginal profits from attracting low-risk agents is increasing in the utility offered to the high-risk agents. The following result differs from lemma 5 in that it deals with an arbitrary equilibrium distribution.

Lemma 7. *(Supermodularity) For any feasible $\mathbf{u}$ satisfying $u_l > \max\{u_l^{RS}, u_h\}$ the function $M(u_l, \cdot)$ is non-decreasing in u_h. Moreover, if $u_l \in (\underline{u}_l, \overline{u}_l)$ then it is strictly increasing.*

Supermodularity implies that there is a complementarity between how attractive an offer is to low-risk and high-risk agents. This complementarity has an important implication for equilibrium offers: more attractive offers to low-risk agents have to also be more attractive to high-risk agents. In other words: equilibrium offers can be ordered in terms of attractiveness. This is formally stated in the next result.

---

[17] Notice that $u_l > u_h$ implies that

$$\frac{\partial P_h(\mathbf{u})}{\partial u_l} = 0.$$

This is true because the cost minimizing contract to be offered to the high-risk agent, $\chi_h(\mathbf{u})$, is efficient. This means that $\chi_h(\mathbf{u})$ is equal to he full insurance contract $\overline{c} = (u^{-1}(u_h), u^{-1}(u_h))$.

Lemma 8. *(Ordering of offers) In any symmetric equilibrium, if the support of the equilibrium strategy includes an offer that generates utility profile* $\mathbf{u} = (u_l, u_h)$*, then*

$$u_h = \nu(u_l),$$

where the function $\nu : [\underline{u}_l, \overline{u}_l] \to \mathbb{R}$ *is strictly increasing.*

Proof. Part 1. We first show that all equilibrium offers generating utility $\mathbf{u} = \left(u_l^{RS}, u_h\right)$ satisfy $u_h = u_h^{RS}$, i.e.,

$$G\left(u_l^{RS}, u_h^{RS}\right) = G_l\left(u_l^{RS}\right).$$

Suppose by way of contradiction that

$$G\left(u_l^{RS}, u_h^{RS}\right) < G_l\left(u_l^{RS}\right).$$

And hence a given firm i makes with positive probability, namely $q > 0$, offers that generate utility in the set $\left\{u_l^{RS}\right\} \times \left(u_h^{RS}, \infty\right)$. For an arbitrary offer $\mathbf{u} = \left(u_l^{RS}, u_h\right)$ in this set, we know that

$$P_h(u_h) < 0.$$

So it is necessarily the case that the firm makes positive profits from low-risk agents, which means that

$$P_l(\mathbf{u}) > 0.$$

Since, at offer $\mathbf{u}$, the firm makes positive profits from the low-risk agent, it must attract the low-risk agent with probability $G_l\left(u_l^{RS}\right)$.[18] But this leads to a contradiction: if firm $j \neq i$ makes offers in $\left\{u_l^{RS}\right\} \times \left(u_h^{RS}, \infty\right)$, then it attracts the low-risk agent with probability at most

$$G_l\left(u_l^{RS}\right)^{\frac{N-2}{N-1}}\left(G_l\left(u_l^{RS}\right)^{\frac{1}{N-1}} - q\right) < G_l\left(u_l^{RS}\right),$$

since it cannot attract the low-risk agent if firm i makes offer in the set $\left\{u_l^{RS}\right\} \times \left(u_h^{RS}, \infty\right)$. Hence, offer $\mathbf{u}$ cannot be optimal to firm j, a contradiction.

We now consider $u_l > u_l^{RS}$.

Part 2. There is a unique $u_h \in \mathbb{R}$ such that an offer generating utility vector $\mathbf{u} \equiv (u_l, u_h)$ is offered. If an offer generating utility $\mathbf{u}$ is made, then optimality of $\mathbf{u}$ implies that

$$M(\mathbf{u}) = 0.$$

However, since $M(u_l, \cdot)$ is strictly increasing, by Lemma 7, there is a unique u_h that satisfies this

[18] If not, then offer $\mathbf{u}' = \mathbf{u} + (\varepsilon, 0)$ is strictly better for $\varepsilon > 0$ small since it attracts the low-risk agent with at least probability $G_l\left(U\left(c^{RS,l} \mid l\right)\right)$ while incurring an arbitrarily small extra cost (P_h is continuous).

equality. We call this utility level $\nu(u_l)$.

Part 3. The function $\nu(\cdot)$ is non-decreasing for $u_l > u_l^{RS}$. Suppose that both utility vectors $(u_l, \nu(u_l))$ and $(u_l', \nu(u_l'))$ are offered in equilibrium with $u_l' > u_l$. Optimality implies that

$$\pi\left(u_l', \nu(u_l)\right) - \pi\left(u_l, \nu(u_l)\right) \leq 0 \leq \pi\left(u_l', \nu\left(u_l'\right)\right) - \pi\left(u_l, \nu\left(u_l'\right)\right),$$

however this implies that

$$\int_{u_l}^{u_l'} M\left(s, \nu(u_l)\right) ds \leq 0 \leq \int_{u_l}^{u_l'} M\left(s, \nu\left(u_l'\right)\right) ds,$$

which implies that $\nu(u_l') \geq \nu(u_l)$.

Part 4. $\nu(u_l) > u_h^{RS}$, for $u_l > u_l^{RS}$. Suppose that an offer generating utility $\mathbf{u} = (u_l, u_h)$ with $u_l > \max\left\{u_l^{RS}, u_h\right\}$ and $u_h = u_h^{RS}$. By definition, we have that

$$P_t\left(u_t^{RS}\right) = 0, \text{ for } t \in \{l, h\}.$$

Notice that $P_h(\mathbf{u}) \leq 0$ since the high-risk is receiving the utility generated by his actuarily fair full insurance policy. Also notice that, since $P_l(\cdot, u_h)$ is strictly decreasing, $P_l(\mathbf{u}) < 0$. Hence, this means that the offering firm makes strictly negative profits: low-risk agents are attracted by this offer with probability $G_l(u_l) > 0$.

Part 5. The function $\nu(\cdot)$ is strictly increasing for $u_l > U\left(c^{RS,l} \mid l\right)$. Suppose that $u_l, u_l' \in (\underline{u}_l, \overline{u}_l)$ and that $\nu(u_l) = \nu(u_l') > u_l^{RS}$, then we know, form part 1, that $\nu(s) = \nu(u_l)$ for any $s \in (u_l, u_l')$. Hence it follows that

$$G_h\left(\nu(u_l)\right) \geq G_l\left(u_l'\right) - G_l(u_l) > 0.$$

But if $G_l(\underline{u}_l) > 0$, then lemma 6 implies $\underline{u}_l = U\left(c^{RS,l} \mid l\right)$. This contradicts Part 1. □

This result reduces the set of equilibrium utilities to a one-dimensional subset of Υ, since all offers can be indexed by the low-risk agent's utility level. The zero profits result in Proposition 3 implies that offers can be indexed as well by the amount of cross-subsidization across different types.

Corollary 2. *In any symmetric equilibrium, all equilibrium offers generate utility profile in the set*

$$\{U(k) \mid k \in R\}.$$

Proof. From Proposition 3, we know that firms make zero expected profits. Now if a firm makes offer $\mathbf{u} = (u_l, \nu(u_l))$ with $u_l > \underline{u}_l$, Lemma 8 implies that

$$G_h\left(\nu(u_l)\right) = G_l(u_l).$$

Then profits are given by

$$\pi(\mathbf{u}) \equiv G_l(u_l)[\mu_l P_l(\mathbf{u}) + \mu_h P_h(\mathbf{u})] = 0.$$

Function $P_l(\cdot, u_h)$ is strictly decreasing and $P_l(z, u_h) \to -\infty$ as $z \to \lim_{c\to\infty} u(c)$. So any equilibrium offer utility $\mathbf{u} = (u_l, u_h)$ satisfies: (i) $u_h \in \left[u_h^{RS}, U_h(p_h - \overline{p})\right]$, (ii) $\mathbf{u} = U(k)$ for $k = U_h^{-1}(u_h)$. To check (i) suppose that $U_h > u(p_h - \overline{p})$, then we have a contradiction since

$$\mu_l P_l(\mathbf{u}) + \mu_h P_h(\mathbf{u}) \leq \mu_h P_h(u_h, u_h) + \mu_l P_l(u_h, u_h) < 0.$$

In order to check (ii) notice that, for any $k \in (0, p_h - \overline{p})$, $U_l(k)$ maximizes the utility obtained by the low-risk subject to delivering utility $U_h(k)$ to the high-risk agent and generating zero expected profits. Hence $U_l(k)$ is the unique solution to

$$\mu_h P_h(u_l, U_h(k)) + \mu_l P_l(u_l, U_h(k)) = 0.$$

Finally, suppose by way of contradiction that $\overline{u}_l > U_l\left(\overline{k}\right)$. Then take equilibrium utility offers $\mathbf{u} = (U_l(k), U_h(k))$ and $\mathbf{u}' = (U_l(k'), U_h(k'))$ such that $U_l\left(\overline{k}\right) < U_l(k) < U_l(k') < \overline{u}_l$. Lemma 8 implies that

$$U_h(k) < U_h\left(k'\right) \Rightarrow k' > k.$$

But the fact that $k' > k > \overline{k}$ implies that

$$U_h\left(k'\right) < U_h(k) < U_h\left(\overline{k}\right),$$

since $u_h(\cdot)$ is concave and $\overline{k}$ is its peak, a contradiction. □

The importance in the previous lemma is in reducing the set of possible utility profiles to a one-dimensional set, where offers are indexed by the amount of cross-subsidization occurring between different risk types. This allows us to use standard tools from games with one-dimensional strategy spaces in order to describe the unique possible equilibrium distribution over this feasible set.

Proposition 4. *(Uniqueness) Consider any symmetric equilibrium. Then the set of utility profiles generated by equilibrium offers by any single firm is:*

$$\{\mathbf{u}(k) \mid k \in R\},$$

where variable $k \in R$ is distributed according to $F^* = F^{\frac{1}{N-1}}$, where F presented in Lemma 4.

Proof. First we show that $supp(G) = \{U(k) \mid k \in R\}$. Suppose that $\overline{u}_l = U_l(k)$ such that $k < \overline{k}$.

Then a firm can make utility offer $\left(U_l\left(\frac{k+\overline{k}}{2}\right)-\varepsilon, U_h\left(\frac{k+\overline{k}}{2}\right)\right)$ for $\varepsilon>0$ satisfying

$$U_l\left(\frac{k+\overline{k}}{2}\right)-\varepsilon>U_l\left(k\right).$$

This would attract both risk types with probability one and make positive expected profits.

Now suppose that $\underline{u}_l=U_l\left(\underline{k}\right)>U_l\left(0\right)$. Then one firm can make utility offer

$$\left(U_l\left(\underline{k}\right)+\varepsilon, U_h\left(\underline{k}\right)-\varepsilon\right)$$

with $\varepsilon>0$ sufficiently small so that

$$P_l\left(U_l\left(\underline{k}\right)+\varepsilon, U_h\left(\underline{k}\right)-\varepsilon\right)>0.$$

This would attract the high-risk agent with zero probability, attract the low-risk agent with positive probability and make positive expected profits.

Finally, let the distribution over $k\in R$ be F^* and define $F=\left(F^*\right)^{N-1}$ as the distribution of the highest draw from $N-1$ levels of cross-subsidization. Notice that the distribution of the best competing for a low-risk agent a firm faces satisfies $G_l\left(U_l\left(k\right)\right)=F\left(k\right)$.

Since any offer in $\{\mathbf{u}\left(k\right)\mid k\in R\}$ must be optimal, the distribution over cross-subsidization $k\in R$ must satisfy

$$\frac{f\left(k\right)}{u_l'\left(k\right)}P_l\left(U\left(k\right)\right)+F\left(k\right)\frac{\partial P_l\left(U\left(k\right)\right)}{\partial u_l}=0,$$

and $F\left(\overline{k}\right)=1$. But the unique solution to these conditions is given by lemma 4. □

6 Comparative statics

Since our uniqueness result is valid for all possible number of firms and prior distributions over types, an analysis of the the comparative statics with respect to these primitives of the model is possible and informative.

6.1 Prior Distribution

One of the main advantages of considering an extensive form version of Rothschild and Stiglitz's model is to obtain equilibrium existence for all prior distributions. Here I consider how equilibrium strategies change with the prior probability of the low-risk agent. The equilibrium distribution of offers continuously changes with the prior distribution, converging to the RS contracts for μ_l sufficiently small and converging to the full information full insurance offer $(1-p_l, 1-p_l)$ as μ_l converges to one. More specifically, this means that both agents benefit from a better pool of agents. This result is in conflict with the ones obtained by Bisin and Gottardi (2006), Dubey and Geanakoplos (2002) and Guerrieri, Shimer, and Wright (2010), who consider extensions of the original Rothschild-Stiglitz model.[19] In these papers, equilibria generate the RS pair of contracts as final outcome for any prior distribution. I will consider a fixed number N of firms, and I will define $F^{(\mu_l)}$ as the equilibrium distribution of offers by a single firm. Also, denote as $\overline{F}^{(\mu_l)}$ the equilibrium distribution of the best offers, i.e., the distribution of $k = \max\{k_1, \ldots, k_N\}$.

Proposition 5. *If $\mu_l' > \mu_l$, then $F^{(\mu_l')}$ first-order stochastically dominates $F^{(\mu_l)}$ and $\overline{F}^{(\mu_l')}$ first-order stochastically dominates $\overline{F}^{(\mu_l)}$. When $\mu_l \to 1$, $F^{(\mu_l)}$ and $\overline{F}^{(\mu_l)}$ converge to a point mass at p_l. Moreover, the function $\mu_l \longmapsto F^{(\mu_l)}$ is continuous (weak-convergence).*

6.2 Number of firms

First, consider the number of firms in the market. These results are reminiscent of the ones presented in Rosenthal and Weiss (1984) for the Spence model. Since the distribution of the best offer among any $N-1$ firms is independent of N, it follows that as more firms are present in the market, each firm will pursue a less aggressive strategy, i.e., with offers that are less attractive to both types of the agent, in the sense of first order stochastic dominance.

Let $F^{(N)}$ denote the equilibrium distribution over $k \in R$ that defines the equilibrium distribution over contracts $\left\{\mathcal{M}^k \mid k \in R\right\}$.

Proposition 6. *The distribution $F^{(N)}$ first-order stochastically dominates $F^{(N+1)}$. In case $\overline{k} > 0$, the dominance is strict. Additionally, $F^{(N)}$ converges weakly to a Dirac measure at the zero cross-subsidization level as $N \to \infty$.*

Proof. Just notice that

$$F^{(N)} = F(k)^{\frac{1}{N-1}} \leq F(k)^{\frac{1}{N}} = F^{(N+1)}.$$

In case $\overline{k} > 1 - p_h$, there is continuous mixing over $\left[0, \overline{k}\right]$, so that the inequality is strict for any $k \in \left(0, \overline{k}\right)$.

Finally, if the distribution F is a point mass at zero, the convergence of $F^{(N)}$ is trivial. In case F is continuous on $\left[0, \overline{k}\right]$, notice that for any $k \in \left(0, \overline{k}\right)$

$$F(k) \in (0,1) \Rightarrow F(k)^{\frac{1}{N-1}} \to^{N \to \infty} 1.$$

□

Since the utility provided by offer $\mathcal{M}^k$ is increasing in k, for both types, the first part of the proposition implies the utility delivered by a single firm decreases as N increases. However, for

[19] As mentioned in the introduction, continuity with respect to the distribution is present in the dynamic extensions considered in Wilson (1977), Miyazaki (1977), Hellwig (1987), Mimra and Wambach (2011) and Netzer and Scheuer (2014).

higher number of firms, the agent is sampling a higher number of offers, so that the overall effect seems unclear. But the distribution of the best offer among any $N-1$ firms is fixed, it follows that the distribution of the best among N firms is lower (first-order stochastic dominance) and converges to F. Let $\overline{F}^{(N)}$ be the distribution of $\max_{i=1,\ldots,N} k_i$, where each k_i is distributed according to $F^{(N)}$.

Proposition 7. *The distribution $\overline{F}^{(N)}$ first-order stochastically dominates $\overline{F}^{(N+1)}$. Moreover,*

$$\overline{F}^{(N)} \to^w F^{(2)},$$

as $N \to \infty$.

Proof. Just notice that $\overline{F}^{(N)} = F^{(2)} F^{(N)}$. □

The consequence of the propositions above is that, when the RS pure equilibrium fails to exist, the model with a continuum of firms cannot be simply taken as the limit of a model with N firms, as $N \to \infty$. The problem is that as the number of firms grow, each firm provides worse offers. However, they get worse "slowly" so that the best offer among N firms converges to a nondegenerate distribution F. In the case of a continuum of firms, there is no way to obtain a nondegenerate distribution for the best offer among all firms with independent symmetric randomization across firms.

An important characteristic of the mixed equilibrium constructed is that an outside firm, facing equilibrium offers in the market, can obtain positive expected profits.

Consider the duopoly case. An outside firm (called firm 3) faces two competing offers (from firms 1 and 2) distributed according to F, so that the most attractive competing offer is distributed according to $G = F^2$. If an outside firm considers any offer $\mathcal{M}^k$, for $k \in R$, it would have zero expected profits (by definition of $\gamma(\cdot)$). However, since firms 1 and 2 have zero expected gains from the local deviation around $\mathcal{M}^k$, when facing competing offer distributed according to F, firm 3 has a strict gain from such deviation around $\gamma(k)$ that attracts low-risk agents with higher probability.

It is quite surprising that firm 3, facing two competing offers distributed according to F, can obtain higher expected profits than a firm facing a single competing offer distributed according to F. In most competitive settings, such as auctions, a player always benefits from less aggressive offer distribution from its competitors. In this model, cross-subsidization between contracts means that the relative frequency with which an offer attracts both types is the important feature.

7 Conclusion

In this paper, I consider a competitive insurance model in which a finite number of firms simultaneously offer menus of contracts to an agent with private information regarding his risk type. I show that there always exists a unique symmetric equilibrium. This equilibrium features firms offering the separating contracts analyzed in Rothschild and Stiglitz (1976), whenever they can be sustained as an equilibrium outcome. When this is not the case, which occurs if the prior probability of low-risk agents is too high, firms randomize over a set of separating pairs of offers. Firms obtain zero expected profits in equilibrium.

The equilibrium features monotone comparative statics with respect to the prior over types and the number of firms. As the probability of low-risk agents increases, firms offer more attractive menus. The equilibrium is continuous with respect to the prior. As a consequence, equilibrium outcomes converge to the perfect information allocation when the prior converges to both extremes. Regarding the number of firms, the distribution of the best offer in the market and agent's welfare decrease as it grows. The distribution of the best offer converges to the mixed strategy of a single firm in duopoly.

Due to the implicit nature of the equilibrium construction, I am not able to obtain clear comparative statics results with respect to preferences. Numerical exercises suggest that higher constant risk aversion leads to more aggressive offers by the firms and reduces the set of priors for which a pure equilibrium exists.

An interesting issue is the extension of this equilibrium for an arbitrary number of types, since the equilibrium existence problem presented by Rothschild and Stiglitz becomes more severe as the number of types increases. For the limiting case of a continuum of types, a competitive equilibrium never exists (see Riley (1979, 2001)). In the two types model considered here, every optimal pair of contracts has to satisfy one local optimality condition, which is used to characterize the equilibrium distribution. If there are n potential types, there are $n-1$ such local conditions. All of these conditions have to be simultaneously satisfied at any n-tuple offered in equilibrium. In the case of two types, the region of offers is given by γ and corresponds to the pairs of separating contracts that generate expected zero profits and is a one dimensional object. In the case of n risk types, it is a $n-1$ dimensional object, namely tuples that provide full insurance to the lowest type, leave any given type indifferent between his allocation and the next higher type and generate zero expected profits. The extra $n-2$ local optimality conditions characterize the one dimensional region in which the randomization occurs. The local condition connected to the highest type would characterize the equilibrium distribution. Given the complexity and relevance of the binary type analysis, this paper restricts attention to this case.

The analysis presented here sheds new light on the classical results on competitive insurance such as non-existence of equilibrium, uniqueness and the welfare impact of private information. However, there are issues with the interpretation of equilibrium in the insurance market when it involves mixed strategies. The outcome described here requires uncertainty with respect to the competing offers each firm faces. In actual insurance markets, this uncertainty can be generated by heterogeneity in pricing algorithms across firms or from a combination of price dispersion and limited search by consumers. In both cases, cream-skimming deviations may be deterred by uncertainty with respect to the exact offer a firm is competing against. In a scenario where all consumers and firms observe all contracts offered, the interpretation of this equilibrium offer distribution as a stable one is problematic, since the best available set of offers is known and hence subject to cream-skimming deviations. A systematic analysis of a dynamic competitive insurance model is important, and fully understanding the equilibrium properties in the static model considered here are a key step in this direction.

8 Appendix

8.1 Auxiliary lemmas

In this section we present auxiliary notation that will be used extensively in the proofs and a characterization of the feasible set of utility profiles that can be generated by incentive compatible contracts.

Denote $s_i\left(t,\left(\mathcal{M}_i\right)_i\right) \in \Delta\mathbb{R}^2_+$ as the the probability of acceptance of firm i's contracts, i.e., for any measurable set $A \subseteq \mathbb{R}^2_+$ it is defined as follows

$$s_i\left(A \mid t,\left(\mathcal{M}_i\right)_i\right) = s\left(A \times \{i\} \mid t,\left(\mathcal{M}_i\right)_i\right).$$

Expected profits for firm i, from offer $\mathcal{M}_i$, is the following

$$\pi_i\left(\mathcal{M}_i\right) = \overbrace{\mu_h \int \left[\int \Pi\left(c \mid h\right) s_i\left(dc \mid h, \mathcal{M}_i, \overline{\mathcal{M}}_{-i}\right)\right] d\left(\times_j \phi\left(\overline{\mathcal{M}}_{-i}\right)\right)}^{\equiv \pi_{i,h}\left(\mathcal{M}_i\right)} \\ + \underbrace{\mu_l \int \left[\int \Pi\left(c \mid l\right) s_i\left(dc \mid l, \mathcal{M}_i, \overline{\mathcal{M}}_{-i}\right)\right] d\left(\times_j \phi\left(\overline{\mathcal{M}}_{-i}\right)\right)}_{\equiv \pi_{i,l}\left(\mathcal{M}_i\right)}.$$

Also denote as the ex-ante expected probability of acceptance of an offer form firm i, if the consumer is of type $t = l, h$, as $s_{i,t}\left(\mathcal{M}\right)$ and the ex-ante average profit made from a type $t = l, h$ agent conditional on a contract from firm i being accepted as $\pi^E_{i,t}\left(\mathcal{M}\right)$. Define

$$u^{\mathcal{M}}_t \equiv \sup_{c \in \mathcal{M}} U\left(c \mid t\right).$$

The following are consequences of equilibrium conditions. The acceptance rule always has to satisfy the following:

$$c \in supp\left(s_i\left(t,\left(\mathcal{M}_i\right)_i\right)\right) \Rightarrow c \in \arg \max_{c \in \bigcup_i \mathcal{M}^i \cup \{Y\}} U\left(c \mid t\right),$$

which implies that

$$c \in supp\left(s_i\left(t,(\mathcal{M}_i)_i\right)\right) \Rightarrow \Pi\left(c \mid t\right) \leq P_t\left(u_l^{\mathcal{M}}, u_h^{\mathcal{M}}\right). \tag{6}$$

and

$$G_t^-\left(u_t^{\mathcal{M}}\right) \leq s_{i,t}\left(\mathcal{M}\right) \leq G_t\left(u_t^{\mathcal{M}}\right), \tag{7}$$

where $G_t^-\left(u\right) \equiv \lim_{k \nearrow u} G\left(k\right)$ is the left limit of distribution G_t.

Firms maximize profits, i.e.,

$$\pi_i\left(\mathcal{M}_i\right) \geq \pi_i\left(\mathcal{M}_i'\right),$$

for all $\mathcal{M}_i \in supp\left(\phi\right)$ and $\mathcal{M}_i' \in \mathbf{M}$.

The set of feasible utility profiles Υ has the following properties, which follow from standard convexity arguments:

(i) $\boldsymbol{\Upsilon}$ is convex,

(ii) $\left(u\left(c\right), u\left(c\right)\right) \in \Upsilon$, for all $c \in \mathbb{R}_{++}$,

(iii) $\mathbf{u} \in \Upsilon \Rightarrow u + \left(\varepsilon, \varepsilon\right) \in int\left(\Upsilon\right)$ for $\varepsilon > 0$ sufficiently small.

The following lemma is also important in the proof and eliminates the possibility of equilibrium offers that are in the frontier of the set Υ.

Lemma 9. *For any $\mathbf{u} \in \Upsilon$ such that $P_h\left(\mathbf{u}\right) \leq 0$ and $P_l\left(\mathbf{u}\right) \geq 0$, then $\mathbf{u} + \left(\varepsilon, -\varepsilon\right) \in int\left(\Upsilon\right)$ for $\varepsilon > 0$ sufficiently small.*

Proof. First, suppose that $u_h > u_l$. In this case the set[20]

$$co\left\{\mathbf{u}, \left(u\left(0\right), u\left(0\right)\right), \left(u_h, u_h\right)\right\}$$

is contained in Υ since the three generating elements are in Υ. Finally notice that $\mathbf{u} + \left(\varepsilon, -\varepsilon\right)$ is in the interior of this set for $\varepsilon > 0$ sufficiently small.

Second, if $u_h = u_l$ then $\mathbf{u}$ is clearly in the interior of Υ.

Third, now consider $u_h < u_l$. Condition $P_h\left(\mathbf{u}\right) \leq 0$ implies that $u_h > u\left(1 - p_h\right) = u_h^{RS}$. Define

$$C\left(\mathbf{u}\right) \equiv \left\{\left(c_l, c_h\right) \mid \begin{array}{c} U\left(c_h \mid l\right) \leq U\left(c_l \mid l\right) = u_l \\ U\left(c_l \mid h\right) \leq U\left(c_h \mid h\right) = u_h \end{array}\right\}.$$

If there exists $\left(c_l, c_h\right) \in C\left(\mathbf{u}\right)$ such that $c_l \in \mathbb{R}_{++}^2$, then $\mathbf{u} + \left(0, \varepsilon\right) \in \Upsilon$ since

$$\left(c_h, c_l + \left(\frac{p_h}{p_h - p_l}\varepsilon, -\frac{\left(1 - p_h\right)}{p_h - p_l}\varepsilon\right)\right) \in C\left(\mathbf{u} + \left(0, \varepsilon\right)\right).$$

[20]For any set $A \subseteq \mathbb{R}^2$, we use notation $co\left(A\right)$ to denote the convex hull of set A.

In this case we know that $\mathbf{u} \in int\left(\Upsilon\right)$ since

$$\mathbf{u} \in int\left\{co\left\{\mathbf{u} + (0,\varepsilon), (u(0), u(0)), \mathbf{u} + (\varepsilon,\varepsilon)\right\}\right\}.$$

But if $(c_l, c_h) \in C(\mathbf{u})$ implies that $c_l = (0,k)$ for some $k > 0$, then it is necessarily the case that

$$(1 - p_h)\, u(k) = u_h \geq u_h^{RS},$$

But then $k = u^{-1}\left(\frac{u_h}{1-p_h}\right) \geq k^{RS} \equiv u^{-1}\left(\frac{u_h^{RS}}{1-p_h}\right)$. But allocation $\left(0, k^{RS}\right)$ generates utility u_h^{RS} to the high-risk agent while introducing the maximal amount of risk, this implies that it generates utility strictly above u_l^{RS} to the low risk agent. Since $\left(0, k^{RS}\right)$ generates utility above $U\left(c_l^{RS} \mid l\right) = u_l^{RS}$ and has higher risk than c_l^{RS}, it follows that

$$\Pi\left(\left(0, k^{RS}\right) \mid l\right) \leq \Pi\left(c_l^{RS} \mid l\right) = 0.$$

Hence the allocation $(0,k)$ necessarily generates negative losses as it pays more than $\left(0, k^{RS}\right)$. □

And lastly, this lemma shows that weak incentive constraints can be turned into strict inequalities with arbitrarily small cost.

Lemma 10. *(Costless strict incentives) Consider* $\mathbf{u} \in int\left(\Upsilon\right)$. *For any* $\varepsilon > 0$, *there exists a pair of contracts* $\left(c_l^{\varepsilon}, c_h^{\varepsilon}\right)$ *such that*

$$U\left(c_l^{\varepsilon} \mid l\right) = u_l > U\left(c_h^{\varepsilon} \mid l\right),$$

$$U\left(c_h^{\varepsilon} \mid h\right) = u_h > U\left(c_l^{\varepsilon} \mid h\right),$$

$$\left\|\Pi\left(c_t^{\varepsilon} \mid t\right) - P_t\left(\mathbf{u}\right)\right\| \leq \varepsilon, \text{ for } t = l, h.$$

Proof. Fix $\varepsilon > 0$. Since $\mathbf{u}$ is in the interior of Υ and $P_t(\cdot)$ is convex (and hence continuous on the interior of its finite domain), for $\gamma > 0$ sufficiently small

$$\left\|P_h\left(u_l - \delta, u_h\right) - P_h\left(u_l, u_h\right)\right\| < \varepsilon,$$

$$\left\|P_l\left(u_l, u_h - \delta\right) - P_l\left(u_l, u_h\right)\right\| < \varepsilon.$$

Then the pair $\left(\chi_l\left(u_l, u_h - \delta\right), \chi_h\left(u_l - \delta, u_h\right)\right)$ satisfies all the inequalities above. □

8.2 Proof of Lemma 3

Proof. First notice that, if $\mathbf{u}$ satisfies $u_l \leq u_h$, the optimal allocation is $\chi_l\left(\mathbf{u}\right) = (z, z)$, where $u(z) = u_l$.

If $\mathbf{u}$ satisfies $u_l > u_h$, then both constraints bind in the optimal, and hence

$$\chi_l(\mathbf{u}) = \left(u^{-1}\left(\frac{(1-p_l)\,u_h - u_l\,(1-p_h)}{p_h - p_l}\right), u^{-1}\left(\frac{p_h u_l - p_l u_h}{p_h - p_l}\right)\right).$$

Hence, $\chi_l(\cdot)$ is continuous, continuous differentiable if $u_l \neq u_h$ and satisfies $\chi_l^1(\mathbf{u}) \geq \chi_l^0(\mathbf{u})$, with strict inequality if $u_l > u_h$. Function $P_l(\cdot)$ is equal to

$$P_l(\mathbf{u}) = 1 - p_l - \left[p_l \chi_l^0(\mathbf{u}) + (1-p_l)\,\chi_l^1(\mathbf{u})\right],$$

and hence it is continuous and continuously differentiable in $\Upsilon_+ \equiv \{\mathbf{u} \in \Upsilon \mid u_l > u_h\}$ and $\Upsilon_- \equiv \{\mathbf{u} \in \Upsilon \mid u_l < u_h\}$.

Direct differentiation leads to

$$\frac{dP_l(\mathbf{u})}{d(u_l, u_h)} = \begin{bmatrix} -\frac{1}{u'(u^{-1}(u_l))} \\ 0 \end{bmatrix}, \text{ if } \mathbf{u} \in \Upsilon_-,$$

and

$$\frac{dP_l(\mathbf{u})}{d(u_l, u_h)} = \begin{bmatrix} -\frac{1}{p_h - p_l}\left[\frac{(1-p_l)p_h}{u'\left(\chi_l^1(\mathbf{u})\right)} - \frac{p_l(1-p_h)}{u'\left(\chi_l^0(\mathbf{u})\right)}\right] \\ \\ \frac{p_l(1-p_l)}{p_h - p_l}\left[\frac{1}{u'\left(\chi_l^1(\mathbf{u})\right)} - \frac{1}{u'\left(\chi_l^0(\mathbf{u})\right)}\right] \end{bmatrix}, \text{ if } \mathbf{u} \in \Upsilon_+.$$

Notice that $\frac{dP_l}{d\mathbf{u}}(\mathbf{u})$ is continuous at any point $\mathbf{u}$ with $u_l = u_h$, and hence P_l is continuously differentiable.

Since $u'(\cdot)$ is continuously differentiable, P_l is twice continuously differentiable in Υ_+ and Υ_-. For any $\mathbf{u} \in \Upsilon_+$, $\frac{\partial^2 P_l}{\partial u_h \partial u_l}$ is given by

$$\frac{\partial^2 P_l}{\partial u_h \partial u_l} = -\frac{p_l(1-p_l)}{(p_h - p_l)^2}\left[\frac{(1-p_h)\,u''\left(\chi_l^0\right)}{u'\left(\chi_l^0(\mathbf{u})\right)^2} + \frac{p_h u''\left(\chi_l^1(\mathbf{u})\right)}{u'\left(\chi_l^1(\mathbf{u})\right)^2}\right] > 0,$$

and, for any $u \in \Upsilon_-$, the function $\frac{\partial P_l}{\partial u_h}$ is identically zero and hence is continuously differentiable. □

8.3 Proof of Proposition 1

Proof. Since in equilibrium, the distribution G of utility vectors generated by offers is continuous, any offer $\mathcal{M}$ is dominated by offer $\chi(\mathbf{u})$, where $\mathbf{u} = \left(u_l^{\mathcal{M}}, u_h^{\mathcal{M}}\right)$. Offer $\chi(\mathbf{u})$ attracts the agent with same probability as $\mathcal{M}$, and makes weakly more profits, conditional on attracting the agent. Henceforth, we will focus on deviations of this form.

Case A) $u_l = U_l(k)$ and $u_h = U_h(k')$, for some $k, k' \in R$.

Suppose that $k' > k$, which implies that

$$u_h = U_h\left(k'\right) > U_h\left(k\right).$$

Hence it follows from lemma 5 that for any $\tilde{u} \in \left(U_l\left(k\right), U_l\left(k'\right)\right)$

$$M\left(\tilde{u}, u_h\right) > 0.$$

And hence, offer $\chi\left(U_l\left(k'\right), U_h\left(k'\right)\right)$ strictly dominates the original offer. An analogous proof follows if $k' < k$.

Case B) $u_l < U_l\left(0\right)$.

Any offer that generates utility below $U_l\left(0\right)$ attracts a low-risk agent with zero probability. But there are no profit opportunities on high-risk agents since their equilibrium utility is above u_h^{RS}.

Case C) $u_l > U_l\left(0\right)$ and $u_h < U_h\left(0\right)$

By construction, the utility pair $\left(U_l\left(0\right), U_h\left(0\right)\right)$ satisfies $P_l\left(U_l\left(0\right), U_h\left(0\right)\right) = 0$. Since P_l is decreasing in u_l and increasing in u_h, it follows that $P_l\left(\mathbf{u}\right) < 0$. Since there are no profit opportunities from high-risk agents, a firm cannot make positive profits by offering $\mathbf{u}$.

Case D) $u_l > U_l\left(\overline{k}\right)$

Any such offer is dominated by an offer with $u_l = U_l\left(\overline{k}\right)$, which offers strictly lower utility to the low-risk agent while still attracting the agent with probability 1.

Case E) $u_h > U_h\left(\overline{k}\right)$ and $u_l = U_l\left(k\right)$, for some $k \in R$.

If $\mathbf{u} \in int\left(\Upsilon\right)$, then lemma 5 implies that

$$M\left(\mathbf{u}\right) > 0,$$

and hence offer $\mathbf{u}$ is strictly dominated by offer $\mathbf{u} + (\varepsilon, 0)$ for $\varepsilon > 0$ small. If $\mathbf{u} + (\varepsilon, 0)$ is not feasible, lemma 9 implies that $P_l\left(\mathbf{u}\right) < 0$. Hence offer $\mathbf{u}$ cannot be profitable. □

8.4 Proof of Proposition 3

I have separated the result in different three different lemmas.

Lemma 11. *(Zero profits) In any symmetric equilibrium firms make zero expected profits.*

Proof. Suppose, by way of contradiction that firm i makes expected profits $\pi_0 > 0$ and define

$$\underline{u}_t \equiv \inf\left\{u \in \mathbb{R} \mid G_t\left(u\right) > 0\right\}.$$

The proof is divided into four sub-cases.

Case 1) Suppose that $G_l\left(\underline{u}_l\right) = 0$.

There exists a sequence of on-path contracts $\mathcal{M}_n$ such that $u_{l,n} \equiv u_l^{\mathcal{M}_n} \searrow \underline{u}_l$ and $G_l(\underline{u}_l) < \frac{1}{n}$. Let $u_{h,n} \equiv u_h^{\mathcal{M}_n}$ and $\mathbf{u_n} \equiv (u_{l,n}, u_{h,n})$.

Expected profits from low-risk agents, $\pi_{i,l}(\mathcal{M}_n)$, are at most $\frac{\mu_l}{n}(1-p_l)$. It then follows that

$$\pi_0 \leq \frac{\mu_l}{n}(1-p_l) + \pi_{i,h}(\mathcal{M}_n) \Rightarrow \pi_{i,h}(\mathcal{M}_n) \geq \pi_0 - \frac{\mu_l}{n}(1-p_l),$$

and profits from high-risk agents are positive and bounded away from zero for n large. This means that $(1-p_l)\,u(1) \leq u_{h,n} < u(1-p_h) - k$ for some $k > 0$ and n large. Potentially passing to a subsequence, $u_{h,n}$ converges to $\tilde{u}_h \in [(1-p_l)\,u(1), u(1-p_h))$.

Also, for each n, the firm cannot deviate by offering $(u_{h,n}+\varepsilon, u_{h,n}+\varepsilon)$ for $\varepsilon > 0$ small, which would generate profits

$$\mu_h G_h(u_{h,n})\,P_h(u_{h,n}, u_{h,n}) + \mu_l G_l(u_{n,h})\,P_l(u_{h.n}, u_{h,n}).$$

Using the fact that profits from offer $\mathcal{M}_n$ are at most

$$\frac{\mu_l}{n}(1-p_l) + \mu_h s_{i,h}(\mathcal{M}_n)\,P_h(u_{l,n}, u_{h,n}),$$

the firm is making positive profits on high-risk agents,

$$0 < \pi_{i,h}(\mathcal{M}_n) \leq P_h(u_{l,n}, u_{h,n}) \leq P_h(u_{h,n}, u_{h,n}),$$

and acceptance probability satisfies

$$s_{i,h}(\mathcal{M}_n) \leq G_h(u_{h,n}),$$

we conclude that

$$\|P_h(u_{h,n}, u_{h,n}) - P_h(u_{h,n}, u_{l,n})\| \to_{n\to\infty} 0.$$

Since $\lim_n u_{h,n} = \tilde{u}_h$, this implies that $\lim_n u_{l,n} = \underline{u}_l \geq \tilde{u}_h$. High-risk agents must be receiving their utility level with almost no risk, otherwise a firm can profit from a deviation that offers more insurance.

Now, we show that $P_l(\underline{u}_l, \tilde{u}_h) = 0$.

First, $P_l(\underline{u}_l, \tilde{u}_h) \geq 0$. If $P_l(\underline{u}_l, \tilde{u}_h) < 0$, then by continuity $P_l\left(\underline{u}_{l,n}, u_{h,n}\right) < 0$ for n large. But then one firm can profit by offering $\chi(u_{h,n}+\varepsilon, u_{h,n}+\varepsilon)$ for $\varepsilon > 0$ small. This offer makes weakly more profits from the high-risk agents and guarantees non-negative profits from low-risk agents.

Second, notice that $P_l(\underline{u}_l, \tilde{u}_h) \leq 0$. Suppose that $P_l(\underline{u}_l, \tilde{u}_h) > 0$.

Also, since

$$\mu_h G_h(u_{h,n})(1-p_h) \geq \pi_{i,h}(\mathcal{M}_n) \geq \pi_0 - \frac{\mu_l}{n}(1-p_l),$$

we know that $G_h(u_{h,n}) \geq \frac{\pi_0 - \frac{\mu_l}{n}(1-p_l)}{(1-p_h)\mu_h} \to \frac{\pi_0}{(1-p_h)\mu_h}$ as $n \to \infty$. This implies that $G_h(\tilde{u}_h) > 0$.

Case 1.A) Suppose G_h has a mass point at $\tilde{u}_h$.
Then there exists offer $\mathcal{M}$ and a firm i, delivering $\mathbf{u} = (u_l, \tilde{u}_h)$ with $u_l \geq \underline{u}_l \geq \tilde{u}_h$ and whose acceptance probability satisfies (ties have to broken in a way that some firm gets the consumer with probability smaller than one)

$$s_{i,h}(\mathcal{M}) < G_h(\tilde{u}_h).$$

Notice that $P_h(\mathbf{u}) > 0$ since $u_l \geq \tilde{u}_h$ and $\tilde{u}_h < u(1 - p_h)$.

Also notice that for $\varepsilon > 0$ small

$$\{(\tilde{u}_h + \varepsilon, \tilde{u}_h + \varepsilon), (u_l + \varepsilon, \tilde{u}_h + \varepsilon)\} \subseteq int(\Upsilon).$$

Hence, following lemma 10, a firm can find offers that generate profits arbitrarily close to

$$\mu_h G_h(\tilde{u}_h) P_h(\tilde{u}_h, \tilde{u}_h) + \mu_l G_l(\tilde{u}_h) P_l(\tilde{u}_h, \tilde{u}_h)$$

and

$$\mu_h G_h(\tilde{u}_h) P_h(\mathbf{u}) + \mu_l G_l(u_l) P_l(\mathbf{u}).$$

The first profit level is leads to a strict improvement if $P_l(\mathbf{u}) < 0$, while the second one leads to a strict improvement if $P_l(\mathbf{u}) \geq 0$. Hence $\mathbf{u}$ is not optimal for firm i, a contradiction.

Case 1.B) Distribution G_h is continuous at $\tilde{u}_h$. This implies $\tilde{u}_h > \underline{u}_h$, since $G_h(\tilde{u}_h) > 0$. There exists an equilibrium offer $\overline{\mathcal{M}}$ generating utility $\mathbf{u} = (u_h, u_l)$ with $u_l \geq \underline{u}_l$ and $u_h < \tilde{u}_h$, which implies

$$(\underline{u}_l + \varepsilon, \tilde{u}_h) \in int[co\{\mathbf{u}, (\underline{u}_l, \underline{u}_l), (u_h, u_h)\}] \subseteq int(\Upsilon).$$

And this implies that $P_l(\underline{u}_l, \tilde{u}_h) = 0$. By way of contradiction, suppose that $P_l(\underline{u}_l, \tilde{u}_h) > 0$. Then, from lemma 10, we can find $\varepsilon > 0$ sufficiently small such that the following profit can be achieved (using the fact that $\underline{u}_l \geq \overline{u}_h$)

$$\begin{aligned}
&\mu_h G_h(\tilde{u}_h) P_h(\underline{u}_l + \varepsilon, \tilde{u}_h) + \mu_l G_l(\underline{u}_l + \varepsilon) P_l(\underline{u}_l + \varepsilon, \tilde{u}_h) \\
&= \mu_h G_h(\tilde{u}_h) P_h(\underline{u}_l, \tilde{u}_h) + \mu_l G_l(\underline{u}_l + \varepsilon) P_l(\underline{u}_l + \varepsilon, \tilde{u}_h)
\end{aligned}$$

By continuity of P_l for ε small enough we have that

$$\mu_l G_l(\underline{u}_l + \varepsilon) P_l(\tilde{u}_h, \underline{u}_l + \varepsilon) > 0.$$

But $\mu_h G_h(\tilde{u}_h) P_h(\underline{u}_l, \tilde{u}_h)$ weakly higher than the limit of $\pi_i(\mathcal{M}_n)$ as $n \to \infty$. This means that offer $\mathcal{M}_n$ is strictly dominated, a contradiction. Hence $P_l(\underline{u}_l, \tilde{u}_h) = 0$.

Given that $P_l(\underline{u}_l, \tilde{u}_h) = 0$, then there exists a firm and equilibrium offer $\mathcal{M}$ such that (i) it generates utility generating utility $\mathbf{u} = (u_h, u_l)$ with $u_l \geq \underline{u}_l$ and $u_h < \overline{u}_h$; and (ii) $s_{i,l}(\mathcal{M}) > 0$.

But notice that $\pi_{i,l}(\mathcal{M}) \leq P_l(\mathbf{u}) < 0$ since P_l is strictly increasing in u_h whenever $u_h < u_l$. This means that the offer $\mathcal{M}$ cannot be optimal as it is dominated by offer $\chi(u_h+\varepsilon, u_h+\varepsilon)$ for $\varepsilon > 0$ small. This offer same (positive) profits from high-risk agents as $\mathcal{M}$ and guarantees non-negative profits from low-risk agents.

Case 2) Suppose that $G_l(\underline{u}_l) > 0$.

Case 2.A) Suppose that there exists an equilibrium offer $\mathcal{M}$ generating utility $\mathbf{u} = (u_l, u_h) = (\underline{u}_l, \tilde{u}_h)$ and $G_h(\tilde{u}_h) > 0$.

Suppose that $P_l(\mathbf{u}), P_h(\mathbf{u}) \geq 0$, with this inequality holding strictly for type t. Then consider a firm i such that, when offering $\mathcal{M}$, has an offer accepted by a type t agent with probability strictly below $G_t(\mathbf{u}_t)$. Then $\mathbf{u} + (\varepsilon, \varepsilon) \in int(\Upsilon)$ and, using lemma 10, this firm can obtain profit

$$\mu_l G_l(\underline{u}_l+\varepsilon) P_l(\mathbf{u}+(\varepsilon,\varepsilon)) + \mu_h G_h(\tilde{u}_h+\varepsilon) P_h(\mathbf{u}+(\varepsilon,\varepsilon)) \rightarrow_{\varepsilon\to 0} \mu_l G_l(\underline{u}_l) P_l(\mathbf{u}) + \mu_h G_h(\tilde{u}_h) P_h(\mathbf{u}).$$

And the second term is strictly higher than the profits at $\mathcal{M}$ since the firm has weakly lower profits for each type and attracts both types with weakly higher probabilities (strictly for type t).

Suppose that $P_h(\mathbf{u}), P_l(\mathbf{u}) \leq 0$. This is impossible because offer $\mathcal{M}$ would generate non-positive profits.

Suppose that $P_h(\mathbf{u}) > 0$ and $P_l(\mathbf{u}) > 0$. Then consider a firm that has offer $\mathcal{M}$ accepted by the high-risk agent with probability strictly lower than $G_h(\tilde{u}_h)^{N-1}$. This firm can profitably deviate by offering $\chi(\tilde{u}_h+\varepsilon, \tilde{u}_h+\varepsilon)$ (it strictly increases profits from high-risk agents and guarantees non-negative profits from low-risk agents).

Suppose that $P_l(\mathbf{u}) > 0$ and $P_h(\mathbf{u}) < 0$. Consider a firm i that, when offering $\mathcal{M}$, is accepted by a low-risk agent with probability smaller than $G_l(\underline{u}_l)$. From lemma 9, for $\varepsilon > 0$ small enough we have that $\mathbf{u} + (\varepsilon, -\varepsilon) \in int(\Upsilon)$. This means that the firm can guarantee profits

$$\mu_l G_l(\underline{u}_l+\varepsilon) P_l(\mathbf{u}+(\varepsilon,-\varepsilon)) + \mu_h G_h(\tilde{u}_h-\varepsilon) P_h(\mathbf{u}+(\varepsilon,-\varepsilon)) \rightarrow_{\varepsilon\to 0} \mu_l G_l(\underline{u}_l) P_l(\mathbf{u}) + \mu_h \left[\lim_{u\nearrow \overline{u}_h} G_h(u)\right] P_h(\mathbf{u}).$$

Which is strictly higher than profits at $\mathcal{M}$ since it makes less profits, conditional on acceptance by both types and it attracts the low-risk agent with strictly higher probability while attracting the high-risk agent with weakly lower probability.

Case 2.B) Suppose that there are at least two utility levels $u_h^1 < u_h^2$ such that: there are infinitely many offers $\mathcal{M}(\iota)$ that generate utility $\mathbf{u} = (\underline{u}_l, \iota)$ with $\iota \in (u_h^1, u_h^2)$ and ι is not a mass point of G_h.

Consider an arbitrary such $\iota \in \left(u_h^1, u_h^2\right)$. The utility profile $(\underline{u}_l, \iota) \in int\,(\Upsilon)$ since

$$(\underline{u}_l, \iota) \in int\left[co\left\{\left(\underline{u}_l, u_h^1\right), \left(\underline{u}_l, u_h^2\right), (u(0), u(0)), (\underline{u}_l + \varepsilon, \underline{u}_l + \varepsilon)\right\}\right],$$

which is contained in $int\,(\Upsilon)$ for $\varepsilon > 0$ sufficiently small.

This means that offer $\mathcal{M}(\iota)$ has to make zero profits on the low-risk agents, i.e., $P_l(\underline{u}_l, \iota) = 0$. Suppose that $P_l(\underline{u}_l, \iota) > 0$ and $P_h(\underline{u}_l, \iota) \geq 0$. Consider a firm that, when offering $\mathcal{M}(\iota)$ has an offer accepted by the low-risk agent with probability strictly below $G_l(\underline{u}_l)$. Then $(\underline{u}_l, \iota) + (\varepsilon, \varepsilon) \in int\,(\Upsilon)$ for $\varepsilon > 0$ small enough and, from lemma 10, this firm can obtain profits

$$\begin{aligned}\mu_l G_l(\underline{u}_l + \varepsilon) P_l((\underline{u}_l, \iota) + (\varepsilon, \varepsilon)) + \mu_h G_h(\iota + \varepsilon) P_h((\underline{u}_l, \iota) + (\varepsilon, \varepsilon)) \\ \to_{\varepsilon \to 0} \mu_l G_l(\underline{u}_l) P_l(\underline{u}_l, \iota) + \mu_h G_h(\iota) P_h((\underline{u}_l, \iota)),\end{aligned}$$

which is strictly higher than obtained with offer $\mathcal{M}(\iota)$. This follows from the fact that both types are attracted with higher probability strictly for the low-risk consumer) and the firm makes weakly higher profits conditional on acceptance by each agent. The cases (i) $P_l(\underline{u}_l, \iota) > 0$ and $P_h(\underline{u}_l, \iota) < 0$, (ii) $P_l(\underline{u}_l, \iota) < 0$ and $P_h(\underline{u}_l, \iota) \geq 0$ lead to profitable deviations that exploit the interiority of utility profile $(\underline{u}_l, \iota)$. The case where $P_l(\underline{u}_l, \iota) < 0$ and $P_h(\underline{u}_l, \iota) < 0$ leads to a contradiction since any firm offering such menu would have non-positive profits. □

Lemma 12. *In a symmetric equilibrium, if a firm i makes an offer $\mathcal{M}$ that generates utility profile $\mathbf{u}$, then $\chi_t(\mathbf{u}) \in \mathcal{M}$ and this is the only possible offer accepted by type t from firm i: for any $c \neq \chi_t(\mathbf{u})$*

$$\mathbb{P}^*\left[\left(\tilde{t}, s\left(\tilde{t}, \left(\mathcal{M}, \tilde{\mathcal{M}}_{-i}\right)\right)\right) = (t, c, i)\right] = 0.$$

Proof. First remember that, for each t, problem $P_t(\mathbf{u})$ has a unique solution. This implies that

$$U(c \mid t) = u_t,$$

$$U\left(c \mid t'\right) \leq u_{t'},$$

implies

$$\Pi(c \mid t) \leq P_t(\mathbf{u})$$

with this inequality holding strictly if $c \neq \chi_t(\mathbf{u})$. This implies that for any offer $\mathcal{M}_i$ delivering utility profile $\mathbf{u}$ and any $(\mathcal{M}_j)_{j \neq i}$

$$c \in supp\left(s_i\left(t, (\mathcal{M}_k)_k\right)\right) \Rightarrow \Pi(c \mid t) \leq P_t(\mathbf{u}),$$

with strict inequality if $c \neq \chi(\mathbf{u})$. Hence, we have that

$$\pi_{i,t}(\mathcal{M}_i) \leq P_t(\mathbf{u}),$$

with this expression holding as a strict inequality if

$$\mathbb{P}^* \left\{ \left(\tilde{t}, s\left(\tilde{t}, \left(\mathcal{M}, \tilde{\mathcal{M}}_{-i} \right) \right) \right) \neq (t, \chi_t(\mathbf{u}), i) \right\} > 0. \tag{8}$$

Once again, we proceed by dividing the statement in cases.

Case 1) Assume that $\mathbf{u} \in \Upsilon$ satisfies $P_l(\mathbf{u}), P_h(\mathbf{u}) \geq 0$ and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. For $\varepsilon > 0$ sufficiently small $\mathbf{u}+(\varepsilon, \varepsilon) \in int(\Upsilon)$, then lemma 10 implies that any firm can obtain profits

$$\mu_l G_l(u_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, \varepsilon)) + \mu_h G_h(u_h + \varepsilon) P_h(\mathbf{u} + (\varepsilon, \varepsilon))$$
$$\to_{\varepsilon \to 0} \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h(u_h) P_h(\mathbf{u}).$$

If (8) holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\begin{aligned} \mu_l s_{i,l} \pi_{i,l} + \mu_h s_{i,h} \pi_{i,h} &< \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \\ &\leq \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h(u_h) P_h(\mathbf{u}). \end{aligned}$$

Hence for $\varepsilon > 0$ sufficiently small this firm has a profitable deviation.

Case 2) Assume that $\mathbf{u} \in \Upsilon$ satisfies $P_l(\mathbf{u}) \geq 0$, $P_h(\mathbf{u}) \leq 0$ and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. Lemma 9 implies that, for $\varepsilon > 0$ sufficiently small, $\mathbf{u} + (\varepsilon, -\varepsilon) \in int(\Upsilon)$, then lemma 10 implies that any firm can obtain profits

$$\mu_l G_l(u_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, -\varepsilon)) + \mu_h G_h(u_h - \varepsilon) P_h(\mathbf{u} + (\varepsilon, -\varepsilon))$$
$$\to_{\varepsilon \to 0} \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h^-(u_h) P_h(\mathbf{u}).$$

If (8) holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\begin{aligned} \mu_l s_{i,l} \pi_{i,l} + \mu_h s_{i,h} \pi_{i,h} &< \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \\ &\leq \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h^-(u_h) P_h(\mathbf{u}). \end{aligned}$$

Hence for $\varepsilon > 0$ sufficiently small this firm has a profitable deviation.

Case 3) Assume that $\mathbf{u} \in \Upsilon$ satisfies $P_l(\mathbf{u}) \leq 0$, $P_h(\mathbf{u}) \leq 0$ and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. If (8) holds, then $s_{i,t}(\mathcal{M}) > 0$ and

$\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\mu_l s_{i,l}\pi_{i,l} + \mu_h s_{i,h}\pi_{i,h} \quad < \quad \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \leq 0,$$

since $P_t(\mathbf{u}) \leq 0$ for $t = l, h$. Hence firm i would make strictly negative profits by offering $\mathcal{M}$, a contradiction.

Case 4) Assume that $\mathbf{u} \in \Upsilon$ satisfies $P_l(\mathbf{u}) \leq 0$, $P_h(\mathbf{u}) \geq 0$ and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. Utility profile $(u_h + \varepsilon, u_h + \varepsilon) \in int(\Upsilon)$ for $\varepsilon > 0$ sufficiently small. Then lemma 10 implies that any firm can obtain profits

$$\begin{aligned}&\mu_l G_l(u_h + \varepsilon) P_l(u_h + \varepsilon, u_h + \varepsilon) + \mu_h G_h(u_h + \varepsilon) P_h(u_h + \varepsilon, u_h + \varepsilon)\\ &\qquad \rightarrow_{\varepsilon \to 0} \mu_l G_l(u_h) P_l(u_h, u_h) + \mu_h G_h(u_h) P_h(u_h, u_h).\end{aligned}$$

Now notice that

$$P_h(u_h + \varepsilon, u_h + \varepsilon) \geq P_h(\mathbf{u}) \geq 0,$$

$$P_l(u_h, u_h) \geq P_h(u_h, u_h) \geq 0.$$

If 8 holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\begin{aligned}\mu_l s_{i,l}\pi_{i,l} + \mu_h s_{i,h}\pi_{i,h} \quad &< \quad \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u})\\ &\leq \quad \mu_h s_{i,h} P_h(\mathbf{u})\\ &\leq \quad \mu_h G_h^-(u_h)^{N-1} P_h(u_h, u_h)\\ &\leq \quad \mu_l G_l(u_l)^{N-1} P_l(u_h, u_h) + \mu_h G_h^-(u_h)^{N-1} P_h(u_h, u_h).\end{aligned}$$

Hence for $\varepsilon > 0$ sufficiently small this firm has a profitable deviation. □

Lemma 13. *In any symmetric equilibrium, firms make nonnegative profits from low-risk agents and non-positive profits from high-risk agents: any equilibrium utility offer* $\mathbf{u}$ *satisfies* $\mathbf{u} \in int(\Upsilon)$, $u_l \geq u_h$ *and*

$$\Pi(\chi_l(\mathbf{u})) \geq 0,$$

$$\Pi(\chi_h(\mathbf{u})) \leq 0.$$

Proof. Any equilibrium offer $\mathcal{M}$ generating utility profile $\mathbf{u} = (u_l, u_h) \equiv \left(u_l^{\mathcal{M}}, u_h^{\mathcal{M}}\right)$ satisfies

$$u_t \geq u_t^{RS}.$$

This means that the consumer always receives offers that are more attractive then the Rothschild-

Stiglitz offers. Suppose this is not the case, which implies that $G_t^-\left(u_t^{RS}\right) > 0$ for at least one type $t = l, h$.

Since $\left(u_l^{RS}, u_h^{RS}\right) \in int\,(\Upsilon)$, if $\underline{u}_t < u_t^{RS}$ for a type $t = l, h$, then firms can make profits by offering utility $\left(u_l^{RS}, u_h^{RS}\right) - (\varepsilon, \varepsilon)$. This offer guarantees positive profits on both types and attracts one of them with positive probability for $\varepsilon > 0$ sufficiently small.

Since utility u_h^{RS} is the generated by an actuarily fair contract with full insurance for the high-risk agents, it follows that $P_h(\mathbf{u}) \leq 0$ for any utility profile $\mathbf{u}$ generated in equilibrium. Also, if there is an equilibrium offer that generates utility $\mathbf{u}$ such that $P_h(\mathbf{u}) < 0$, then at least one firm making such an offer makes strictly negative profits, a contradiction.

Finally, any equilibrium offer generating utility $\mathbf{u} = (u_l, u_h)$ such that $u_h > u_l$ is strictly dominated by offer $\chi(u_l + \varepsilon, u_l + \varepsilon)$ for $\varepsilon > 0$ sufficiently small. This deviating offer attracts high-risk consumers with strictly lower probability, and makes strictly higher profits from them, while at the same time attracting low-risk agents with higher probability and loses an arbitrarily small amount of profits. To prove interiority, consider any equilibrium utility offer $\mathbf{u} \in \Upsilon$ satisfying $u_l > u_h$. From lemma 9 it follows that $\mathbf{u} + (\varepsilon, -\varepsilon) \in int\,(\Upsilon)$ for $\varepsilon > 0$ sufficiently small, which implies that then $\mathbf{u} \in co\,\{(u_l, u_l), (u_h, u_h), \mathbf{u} + (\varepsilon, -\varepsilon)\} \subseteq int\,(\Upsilon)$. □

Proposition 3 follows directly from Lemmas 11, 12 and 13 presented above.

8.5 Proof of Lemma 6

Proof. First, consider (i). If $\underline{u}_t < u_t^{RS}$ for some $t = l, h$, then firms can make profits by offering utility $\left(u_l^{RS}, u_h^{RS}\right) - (\varepsilon, \varepsilon)$. This offer guarantees positive profits on both types and attracts one of them with positive probability for $\varepsilon > 0$ sufficiently small.

Now we consider part (ii), and divide the proof in steps for clarity.

Part ii.1: Bounded derivative of π with respect to u_h: $\sup\{M(\mathbf{u}) \mid \mathbf{u} \in supp\,(\phi)\} < \infty$.

M is defined over $\{\mathbf{u} \in \Upsilon \mid u_l \geq u_h\}$ is continuous, and

$$supp\,(G) \subseteq \underline{\Upsilon} \equiv \left\{ \mathbf{u} \in \Upsilon \mid \begin{array}{c} u_l \geq u_h \\ u_h \leq u\,(1 - p_l) \end{array} \right\},$$

which is a compact set. Hence $\sup\{M(\mathbf{u}) \mid \mathbf{u} \in supp\,(G)\} \leq \overline{M} \equiv \sup\{M(\mathbf{u}) \mid \mathbf{u} \in \underline{\Upsilon}\} < \infty$.

Part ii.2: For any $n \in \mathbb{N}$, $\underline{\pi}^n \equiv \inf\left\{P_l(\mathbf{u}) \mid \mathbf{u} \in supp\,(G) \text{ and } u_l \geq \underline{u}_l + \frac{1}{n}\right\} > 0$.

Fix $n \in \mathbb{N}$ and assume, by way of contradiction that there exists a sequence of equilibrium offers $\{\mathbf{u}^n\}$ such that

$$P_l(\mathbf{u}^n) \to 0.$$

This implies that expected profits from high-risk agents also converges to zero. From lemma 10, firms can always break indifferences in order to attract high-risk agents with probability $G_h^-(u_h^n)$, hence profits from high-risk agents is

$$P_h(\mathbf{u}^n)\, G_h^-(u_h^n) \to 0.$$

Suppose that $P_h(\mathbf{u}^n) \to 0$. In this case we have that $\mathbf{u}^n \to \left(u_l^{RS}, u_h^{RS}\right)$ since $\max\{P_l(\mathbf{u}), -P_h(\mathbf{u})\}$ has $\left(u_l^{RS}, u_h^{RS}\right)$ as the only zero within the compact set $\{\mathbf{u} \in \Upsilon \mid P_h(\mathbf{u}) \le 0, P_l(\mathbf{u}) \ge 0\}$. But this is a contradiction with $u_l^n \ge \underline{u}_h + \frac{1}{n} > u_h^{RS}$.

Now suppose that $\liminf P_h(\mathbf{u}^n) > 0$ and $G_h^-(u_h^n) \to 0$. This implies that $u_h^n \to \underline{u}_h$ and $G_h(\underline{u}_h) = 0$. Considering a converging subsequence of $\{u_l^n\}$ and let its limit be $\widetilde{u}_l$. We know that $P_l(\widetilde{u}_l, \underline{u}_h) = 0$ and $\widetilde{u}_l \ge \underline{u}_l + \frac{1}{n}$ (which means that $G_l(\widetilde{u}_l) \ge G_l\left(\underline{u}_l + \frac{1}{n}\right) > 0$). Then for $\varepsilon > 0$ small firms can make offer $(\widetilde{u}_l - \varepsilon, \underline{u}_h)$ and make profit

$$\mu_l G_l^-(\widetilde{u}_l - \varepsilon)\, P_l(\widetilde{u}_l - \varepsilon, \underline{u}_h) > 0,$$

which is positive for $\varepsilon > 0$ sufficiently small, since $P_l(\cdot)$ is decreasing in u_l.

Part ii.3: For any $n \in \mathbb{N}$, G_l is Lipschitz continuous on $\left[\underline{u}_l + \frac{1}{n}, \overline{u}_l\right]$, with constant $L_n \equiv \frac{\overline{M}}{\underline{\pi}^n (N-1) G_l\left(\underline{u}_l + \frac{1}{n}\right)}$.

Consider any $u_l, u_l' \in \left[\underline{u}_l + \frac{1}{n}, \overline{u}_l\right]$ such that $u_l' - \varepsilon \le u_l \le u_l'$. Consider an equilibrium utility offer $\mathbf{u} = (u_l, u_h)$. if a firm deviates, by offering utility $\mathbf{u}' = (u_l', u_h)$, its profits must not increase:

$$\mu_l G_l(u_l)\, P_l(\mathbf{u}) + \mu_h G_h(u_h)\, P_h(\mathbf{u}) \ge \mu_l G_l\left(u_l'\right) P_l\left(\mathbf{u}'\right) + \mu_h G_h(u_h)\, P_h\left(\mathbf{u}'\right).$$

The fact that low-risk agents receive higher utility, i.e., $u_l' \ge u_l \ge u_h$ implies that $P_h(\mathbf{u}') = P_h(\mathbf{u})$. The inequality becomes

$$\left[G_l\left(u_l'\right) - G_l(u_l)\right] \le G_l\left(u_l'\right) \frac{\left[P_l(\mathbf{u}) - P_l(\mathbf{u}')\right]}{P_l(\mathbf{u})} \le \varepsilon \frac{\overline{M}}{\underline{\pi}^n},$$

using lemmas 9 and 10 as well as the fact that $G_l(u_l') \le 1$.

Part ii.4: Now, for any $n \in \mathbb{N}$, the restriction of distribution G_l to $\left[\underline{u}_l + \frac{1}{n}, \overline{u}_l\right]$ is absolutely continuous, with density g_l^n. Hence, for any bounded non-negative measurable function f we have that

$$\int_{\underline{u}_l + \frac{1}{n}}^{\infty} f\, dG_l = \int_{\underline{u}_l + \frac{1}{n}}^{\infty} f(z)\, g_l^n(z)\, dz.$$

Let g_l be the pointwise limit[21] of $\{g_l^n\}$ on $(\underline{u}_l, \infty)$ and consider any bounded non-negative measurable function f.

[21] Since $g_l^{n+1}(z) = g_l^n(z)$, for any $z \ge \underline{u}_l + \frac{1}{n}$, this pointwise limit is given by the function $g_l(u) = g_l^{n(u)}(u)$, where $n(u) \equiv \inf\left\{n' \mid \underline{u}_l + \frac{1}{n'} < u\right\}$.

Define

$$f^n(u) \equiv \mathbf{1}_{\left[\underline{u}_l, \underline{u}_l + \frac{1}{n}\right)}(u) f(\underline{u}_l) + \mathbf{1}_{\left[\underline{u}_l + \frac{1}{n}, \infty\right)}(u) f(u).$$

It follows from monotone convergence theorem that

$$\lim_{n \to \infty} \int f^n dG_l = \int f dG_l.$$

But notice that

$$\begin{aligned} \int f^n dG_l &= G_l\left(\underline{u}_l + \frac{1}{n}\right) f(\underline{u}_l) + \int_{\underline{u}_l + \frac{1}{n}}^{\infty} f dG_l \\ &= G_l\left(\underline{u}_l + \frac{1}{n}\right) f(\underline{u}_l) + \int_{\underline{u}_l + \frac{1}{n}}^{\infty} f(z) g_l^n(z) dz. \end{aligned}$$

Taking limits on both sides, and using dominated convergence theorem once again on the second term yields

$$\int f dG_l = G_l(\underline{u}_l) f(\underline{u}_l) + \int_{\underline{u}_l}^{\infty} f(z) g_l(z) dz.$$

Part ii.5: Suppose that there exists a utility level u_l in the support of G_l such that, for some $\varepsilon > 0$, $G_l(u_l) = G_l(u_l - \varepsilon) > 0$. Consider an offer that generates utility $\mathbf{u} = (u_l, u_h)$ with $u_l > u_h$. This offer is necessarily dominated. From Lemma 9 and 10 a firm can obtain, by making utility offers close to $\mathbf{u} - (0, -\delta)$ profits arbitrarily close to

$$\mu_l G_l(u_l - \delta) P_l(u_l - \delta, u_h) + \mu_h G_h^-(u_h) P_h(u_h, u_l - \delta).$$

Which are strictly higher than equilibrium profits for $\delta < \min\left\{\frac{\varepsilon}{2}, \frac{u_l - u_h}{2}\right\}$, since they attract the low-risk agent with same probability as offer $\mathbf{u}$ and make strictly more profits from them. Alternatively, any equilibrium utility offer (u_l, u_l) would be dominated by offer $\left(u_l - \frac{\varepsilon}{2}, u_l - \frac{\varepsilon}{2}\right)$.

Part (iii). Suppose that G_h has a mass point on some utility level $u_h > u_h^{RS}$. There exists a firm i and equilibrium offer $\mathcal{M}$ that deliver utility $\mathbf{u} = (u_l, u_h)$ that is attracted by the high-risk agent with probability strictly higher than $G_h^-(u_h)$. From lemmas 9 and 10, for $\varepsilon > 0$ sufficiently small, any firm can obtain profits

$$\mu_l G_l(u_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, -\varepsilon)) + \mu_h G_h(u_h - \varepsilon) P_h(\mathbf{u} + (\varepsilon, -\varepsilon)),$$

which converge, as $\varepsilon \to 0$, to

$$\mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h^-(u_h) P_h(\mathbf{u}).$$

This profit is strictly higher than the equilibrium one since it attracts high-risk agents with strictly lower probability and $P_h(\mathbf{u}) < 0$ since $u_l > u_l^{RS}$. □

8.6 Proof of Proposition 5

Proof. Since I will consider variation in μ_l explicitly, I will acknowledge dependence of each variable on μ_l as a superscript in the notation, as in $\overline{k}^{\mu_l}$.

First notice that, for each $k \in R$, $\gamma^{\mu_l}(k)$ is the solution $\gamma = (\gamma_1, \gamma_0)$ (with $\gamma_1 \geq \gamma_0$) to the following system:

$$A_1(\gamma, \mu_l, k) \equiv \begin{bmatrix} (1-\mu_l)(-k) + \mu_l \Pi(\gamma \mid l) \\ U(\gamma \mid h) - u(1 - p_h + k) \end{bmatrix} = 0.$$

Then, since $\frac{\partial A_1}{\partial \gamma}$ has full rank and $A_1(\cdot)$ is continuously differentiable, it follows that $\gamma^{\mu_l}(k)$ is continuously differentiable in (μ_l, k). It is simple to show that $\frac{\partial \gamma_1^{\mu_l}(k)}{\partial \mu_l} > 0$ and $\frac{\partial \gamma^{\mu_l}(k)}{\partial \mu_l} < 0$. From lemma 2, I know that, if $\overline{k}^{\mu_l} > 0$, $\overline{k}^{\mu_l}$ is defined implicitly as the solution to the following equation:

$$A_2(k, \mu_l) \equiv u'(1 - p_h + k)(1 - p_l)\left[\frac{1}{u'(\gamma_1^{\mu_l}(k))} - \frac{1}{u'(\gamma_0^{\mu_l}(k))}\right] - \frac{\mu_h}{\mu_l}\frac{1}{p_h}\left[\frac{1-p_l}{p_l} - \frac{1-p_h}{p_h}\right] = 0.$$

But we already showed that $\gamma^{\mu_l}(k)$ is continuously differentiable in (μ_l, k), and therefore so is A_2. Since $\frac{\partial A_2(k,\mu_l)}{\partial k} < 0$, $\overline{k}^{\mu_l}$ is also continuously differentiable. Moreover, it follows from $\frac{\partial A_2(k,\mu_l)}{\partial \mu_l} > 0$ that $\frac{\partial \overline{k}^{\mu_l}}{\partial \mu_l} > 0$. Finally, for any $\mu_l\,(0,1)$ and $k \in \left[0, p_h - \overline{p}^{\tilde{\mu}_l}\right]$, $A_2(k, \mu_l) > 0$ for μ_l sufficiently high. Hence, since $\overline{p}^{\mu_l} \to_{\mu_l \to 1} p_l$, $\overline{k}^{\mu_l} \to p_h - p_l$ as μ_l converges to one.

Also, notice that, after substitution:

$$\phi^{\mu_l}(k) \equiv \frac{\frac{p_h p_l}{p_h - p_l}\left\{u'(1 - p_h + k)(1 - p_l)\left[\frac{1}{u'\left(\gamma_1^{\mu_l}(k)\right)} - \frac{1}{u'\left(\gamma_0^{\mu_l}(k)\right)}\right] - \frac{\mu_h}{\mu_l}\frac{1}{p_h}\left[\frac{1-p_l}{p_l} - \frac{1-p_h}{p_h}\right]\right\}}{\Pi(\gamma^{\mu_l}(k) \mid l)}. \quad (9)$$

For any $(\mu_l, k) \in (0,1) \times [0, p_h - \overline{p}]$, $\Pi(\gamma^{\mu_l}(k) \mid l)$ satisfies

$$(1 - \mu_l)(-k) + \mu_l \Pi(\gamma^{\mu_l}(k) \mid l) = 0. \quad (10)$$

This implies that $\frac{\partial \Pi(\gamma^{\mu_l}(k) \mid l)}{\partial \mu_l} < 0$ and that , for any $k \in \left[0, \overline{k}^{\mu_l}\right]$,

$$\frac{\partial \phi^{\mu_l}}{\partial \mu_l} > 0.$$

Also, since the numerator in (9) is increasing in μ_l and the denominator converges to zero as $\mu_l \to 1$, we know that for all $k \in [0, p_h - p_l]$

$$\phi^{\mu_l}(k) \to \infty.$$

Now consider $\mu_l' > \mu_l$. In case $\overline{k}^{\mu_l} = 0$ the result is trivial; otherwise we have from the differential equation (3) satisfied by $\left[F^{(\mu_l)}\right]^{N-1}$, for any $k \in \left[0, \overline{k}^{\mu_l}\right)$

$$\begin{aligned}\left[\frac{F^{(\mu_l)}(k)}{F^{(\mu_l')}(k)}\right]^{N-1} &= \exp\left[\int_k^{\overline{k}^{\mu_l'}} \phi^{\mu_l'}(z)\,dz - \int_k^{\overline{k}^{\mu_l}} \phi^{\mu_l}(z)\,dz\right] \\ &= \exp\left[\int_{\overline{k}^{\mu_l}}^{\overline{k}^{\mu_l'}} \phi^{\mu_l'}(z)\,dz + \int_k^{\overline{k}^{\mu_l}} \left[\phi^{\mu_l'}(z) - \phi^{\mu_l}(z)\right] dz\right] > 1.\end{aligned}$$

Monotone convergence, together with monotonicity of $\overline{k}^{\mu_l}$ and $\phi^{\mu_l}(\cdot)$, implies that for any $k \in [0, p_h - p_l)$

$$\mu_l \longmapsto \int_k^{\overline{k}^{\mu_l}} \phi^{\mu_l}(z)\,dz, \text{ is continuous,}$$

and

$$\int_k^{\overline{k}^{\mu_l}} \phi^{\mu_l}(z)\,dz \to \infty, \text{ as } \mu_l \to 1.$$

Hence, it follows that, for any $k \in [0, p_h - p_l)$, $F^{\mu_l}(k)$ is continuous in μ_l and $F^{\mu_l}(k) \to 0$ as $\mu_l \to 1$.

The results for $\overline{F}^{\mu_l}$ follow from $\overline{F}^{\mu_l} = (F^{\mu_l})^N$. □

References

BISIN, A., AND P. GOTTARDI (2006): "Efficient competitive equilibria with adverse selection," *Journal of political Economy*, 114(3), 485–516.

CHARI, V., A. SHOURIDEH, AND A. ZETLIN-JONES (2014): "Reputation and persistence of adverse selection in secondary loan markets," *American Economic Review*, 104(12), 4027–70.

DASGUPTA, P., AND E. MASKIN (1986): "The Existence of Equilibrium in Discontinuous Economic Games, II: Applications," *The Review of Economic Studies*, 53(1), 27–41.

DUBEY, P., AND J. GEANAKOPLOS (2002): "Competitive pooling: Rothschild-Stiglitz reconsidered," *The Quarterly Journal of Economics*, 117(4), 1529–1570.

DUBEY, P., J. GEANAKOPLOS, AND M. SHUBIK (2005): "Default and Punishment in General Equilibrium," *Econometrica*, 73(1), 1–37.

ENGERS, M., AND L. FERNANDEZ (1987): "Market equilibrium with hidden knowledge and self-selection," *Econometrica: Journal of the Econometric Society*, pp. 425–439.

GUERRIERI, V., R. SHIMER, AND R. WRIGHT (2010): "Adverse selection in competitive search equilibrium," *Econometrica*, 78(6), 1823–1862.

HELLWIG, M. (1987): "Some recent developments in the theory of competition in markets with adverse selection?," *European Economic Review*, 31(1), 319–325.

LEVITT, L., G. CLAXTON, AND A. DAMICO (2013): "The Numbers Behind "Young Invincibles" and the Affordable Care Act," *Kaiser Family Foundation*, http://kff.org/health-reform/perspective/the-numbers-behind-young-invincibles-and-the-affordable-care-act/.

LIZZERI, A., AND N. PERSICO (2000): "Uniqueness and existence of equilibrium in auctions with a reserve price," *Games and Economic Behavior*, 30(1), 83–114.

MASKIN, E., AND J. RILEY (2003): "Uniqueness of equilibrium in sealed high-bid auctions," *Games and Economic Behavior*, 45(2), 395–409.

MASKIN, E., AND J. TIROLE (1992): "The Principal-Agent Relationship with an Informed Principal, II: Common Values," *Econometrica*, 60(1), pp. 1–42.

MIMRA, W., AND A. WAMBACH (2011): "A game-theoretic foundation for the wilson equilibrium in competitive insurance markets with adverse selection," .

——— (2014): "A note on uniqueness in game-theoretic foundations of the reactive equilibrium," .

MIYAZAKI, H. (1977): "The rat race and internal labor markets," *The Bell Journal of Economics*, pp. 394–418.

NETZER, N., AND F. SCHEUER (2014): "A game theoretic foundation of competitive equilibria with adverse selection," *International Economic Review*, 55(2), 399–422.

RILEY, J. (2001): "Silver signals: Twenty-five years of screening and signaling," *Journal of Economic Literature*, 39(2), 432–478.

RILEY, J. G. (1979): "Informational Equilibrium," *Econometrica*, 47(2), 331–359.

ROSENTHAL, R., AND A. WEISS (1984): "Mixed-strategy equilibrium in a market with asymmetric information," *The Review of Economic Studies*, 51(2), 333–342.

ROTHSCHILD, M., AND J. STIGLITZ (1976): "Equilibrium in competitive insurance markets: An essay on the economics of imperfect information," *The Quarterly Journal of Economics*, pp. 629–649.

WILSON, C. (1977): "A model of insurance markets with incomplete information," *Journal of Economic Theory*, 16(2), 167–207.