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ECONOMICS DEPARTMENT

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ABSTRACT

In this paper we consider the informational requirements of decentralised resource allocation mechanisms which attain both fair and efficient outcomes in pure exchange classical environments. We show that the only informationally efficient mechanism which will attain such allocations is the equal income Walrasian mechanism, in which all agents take prices as given and maximise utility subject to the average income constraint.

[1] Presented at the World Meeting of the Econometric Society, Barcelona 1990. We are grateful to Anna Cima, Peter Hammond, Carmen Herrero, Antonio Manresa and Antonio Villar for their helpful comments. We also thank Martin Hellwig and an anonymous referee, both of whose extensive comments have led to a considerable improvement in the exposition. Responsibility for remaining defects or errors is, of course, our own. This research has been supported by the CICYT grant # 87075.

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April 1991

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Xavier Calsamiglia[2] and Alan Kirman[3]

1. Introduction

Following the pioneering work of Hurwicz [1969] and Mount and Reiter [1974], there has been a significant amount of work in trying to determine the informational requirements of decentralized resource allocation mechanisms. By mechanism here we mean a system which communicates knowledge which is dispersed among agents and uses it to determine the allocation of resources. Agents send messages and these are translated into outcomes. In particular, the focus in the literature has been on the dimension of the space of messages used for communication between agents. These informational requirements depend upon two basic elements: the class of environments over which the mechanism is supposed to operate and the particular outcomes that the mechanism is required to achieve.

Most of the literature has centered on the informational requirements for obtaining Pareto optimal allocations in different environments. In this paper we discuss the information needed to obtain a more restricted class of outcomes: those which are both efficient and envy free in the context of pure exchange classical environments. The interest in such allocations is a long-standing one, and although the definition we use is that of Foley [1967], the underlying notion goes back to the ancient Egyptians and has been formally investigated in another context by Dubins and Spanier [1961][4]. Intuitively, it might seem that a great deal of information

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[4] A survey of the literature on the subject is given by Crawford [1987].

would be required to obtain such outcomes since establishing whether an allocation is fair would seem to involve each agent comparing his allocation with that of everyone else and this would require knowing the total endowments of all agents. However, we will show that in fact very little more information is needed to obtain fairness in addition to efficiency and, in particular, we will prove rigorously the idea suggested by Thomson and Varian [1985] that we can actually specify a mechanism which is the only one involving minimal informational requirements.

The design of a particular mechanism involves the specification of a message space (i.e. the set of signals used for communication), the response functions (i.e. how individuals choose the messages they send) and a rule to recognize the so called equilibrium messages, which are messages that indicate that the decisions of the independent agents have been coordinated. These equilibrium messages are then mapped by an outcome function into the allocations. The message space, response functions, equilibrium rules and outcome functions are regarded as chosen by the mechanism designer and not by the individuals. If all the outcomes of the mechanism are the ones prescribed by a social choice rule (also chosen by the designer) then it is said that the mechanism *realizes* that social choice rule. The problem of why the agent should be motivated to send the messages and behave in the way prescribed by the mechanism, that is the problem of incentive compatibility, is not directly dealt with, though useful discussions may be found in Hurwicz [1976], Reichelstein [1984] and Reichelstein and Reiter [1988]. The problem we address in this paper is of the following nature: given a social choice rule (fair allocations in our case), find a decentralized mechanism that realizes it with minimal message spaces. The question of its implementation is not considered. It is clear that the competitive mechanism which receives particular attention here has the same incentive compatibility problem as in the standard general equilibrium model. Agents' messages cannot be regarded as best responses in finite economies in a game theoretic sense. However, given the instructions received from the mechanism designer, they reflect the best choice.

There has been considerable work done on the problem of the informational requirements of an allocation mechanism which ensures efficient outcomes in classical environments (Mount and Reiter [1974], Hurwicz [1977], Osana [1978], Chander [1982] and Calsamiglia [1987]). This has been extended to environments with public goods (Sato [1981]), and those which are stochastic (Jordan [1977]), non-convex (Calsamiglia [1977,1982,1987]), discrete (Hurwicz and Marschak [1985]) and intertemporal (Hurwicz and Majumdar [1988], Brock and Majumdar [1988] and Dasgupta and Mitra [1988]).

The outcomes that are selected as socially desirable are chosen by the designer, and in our case his two criteria are efficiency and fairness. He does not require that the agents should be able to verify these properties but merely affirms that these are the desired properties of acceptable outcomes. Indeed, since the process is decentralized and informationally efficient, it is not possible for the individuals themselves with the information at their disposal to check that the outcome is fair any more than they can check on efficiency.

In the context of classical environments the competitive mechanism plays a special role and indeed Jordan [1982] has shown that, under certain assumptions, it is the unique informationally efficient way of obtaining Pareto optimality. This will be important in what follows.

We know that, in classical environments, if all agents have the same consumption sets there always exist fair and efficient allocations.[5] This was shown by considering the Walrasian outcomes obtained after dividing income equally between all agents (Kolm [1972] and Feldman and Kirman [1974]). This indicates the route to follow. Since the competitive mechanism is informationally efficient in obtaining Pareto outcomes, all that remains is to distribute income equally. The question is how much additional information it is necessary to convey in order to perform the required redistribution.

The equal income Walrasian mechanism has received considerable attention in the literature. Apart from the papers mentioned above, results by Maskin [1977] and Thompson [1979,1982] show that any Nash implementable social choice correspondence is closely related to the equal income Walrasian correspondence. Furthermore, as Varian [1976], Hammond [1979], Kleinberg [1980], Champsaur and Laroque [1981] and Mas-Colell [1983, 1985] have shown, in economies with a large number of agents with sufficiently diverse characteristics the only fair outcomes are equal income Walrasian allocations. However, if there is not enough diversity or not enough agents it is known that many other fair and efficient allocations may exist.

The three basic results of this paper also indicate the central position of the Walrasian mechanism from a different perspective. They can be summarized as follows:

First, any informationally decentralized mechanism that realizes fair allocations over the class of classical pure exchange environments has a message space of dimension greater than or equal to $n\ell$, that is the number of agents times the number of commodities.

Second, the equal income Walrasian mechanism, in which all agents take prices parametrically and maximize utility subject to the *average* income constraint, realizes fair outcomes over the class of classical pure exchange environments and has a message space of dimension $n\ell$. Besides the typical competitive message, every agent has to send a real number expressing the value of his initial endowments at going prices. Thus, the equal income Walrasian mechanism is informationally efficient.

Third, although in the class of environments considered there exist many fair allocations which are not equal income Walrasian allocations, we show that a mechanism that selects some of these necessarily has strictly larger informational

[5] If consumption sets are not identical, for example if there are non transferable commodities, then fair allocations may not exist. In particular, Pazner and Schmeidler [1974] and Tillmann [1989] have shown that, in a productive economy, with agents of differing ability there will be no such allocations.

requirements. In other words, if we insist on mechanisms with message spaces of minimal dimension, then the Walrasian mechanism from equal incomes is in fact the unique candidate.

2. Structure of the argument.

To establish the framework let us look at the classical problem of obtaining Pareto optimal outcomes in an informationally efficient way. In the Walrasian competitive mechanism only net trades and prices must be known to check that a given outcome is Pareto efficient. But this information is compatible with infinitely many different economies (represented by different Edgeworth boxes and indifference curves) as shown in figure 1. The fact that there is no need to distinguish between all of these is at the basis of the strong result that a finite dimensional message space (of dimension $n(\ell-1)$) is sufficient to select Pareto efficient allocations over an infinite dimensional class of economies.

The Walrasian mechanism achieves this in an informationally decentralized way. In a decentralized mechanism, the decision process is decomposed into two phases. In the first phase there is a communication process. Because of the assumed initial dispersion of information, messages sent by agents depend only on messages of other agents and their own characteristics. This important feature of the communication process implies that the so called "crossing condition" has to be satisfied: if two economies have the same equilibrium message, any "crossed economy" in which one agent from one of the two initial economies is "switched" with an agent from the other, must have the same equilibrium message[6].

Since for a given mechanism the translation of the equilibrium message into an action (net trade) is precisely prescribed by an *outcome function*, $z=h(m)$, if two economies have the same equilibrium message m, then the mechanism leads to the same action z for both. These two important features of any informationally decentralized mechanism have important implications that we are going to discuss.

Consider a mechanism that selects Pareto optimal outcomes and two economies which have the same equilibrium message. Then two facts are necessarily true. First, the common outcome z of the mechanism leads to an allocation which is Pareto efficient for both economies. Second, this very same trade z must also be the outcome of the mechanism for any of the "crossed" economies because of the "crossing condition". Therefore the trade z must lead to final allocations which are Pareto efficient not only in the two initial economies, but also in all the "crossed" economies.

This is illustrated in figure 1. Consider an economy represented by the Edgeworth box *ABCD* and the continuous indifference curves. The point ω represents the initial endowments and z the trade leading to the final allocation x. A second economy is represented by the Edgeworth box *EFGH* and the dotted

[6] See section 3 for a formal statement of this property.

indifference curves respectively. It is easily seen that both economies have the same equilibrium message *(p,z)* and that the trade *z* leads to an allocation *x* which is Pareto efficient for both. Now consider the crossed economy in which we take the first agent from the first economy and the second agent from the second economy. The Edgeworth box for this economy is given by *AIGK* and the relevant indifference curves are one continuous and the other dotted. It is immediately seen that for this "crossed economy" the trade *z* still leads to a Pareto efficient allocation, as was to be expected from our previous argument. It is clear that the same competitive message is compatible with Edgeworth boxes of completely different sizes. This means that the equilibrium message does not reveal the "size" of the economy since no agent has any idea about the aggregate initial endowments.

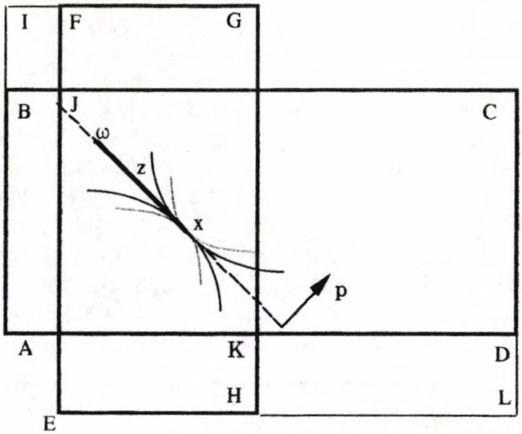

Figure 1

Now, suppose that we are interested in a mechanism whose outcomes are not only Pareto optimal, but also fair. The competitive mechanism does not guarantee such outcomes. Notice that the allocation *x* happens to be fair for the economy *EFGH*, but not for *ABCD*. It seems clear that, to check whether an allocation is fair as well as being efficient, some information concerning the "size" of the Edgeworth box is needed. Indeed, the information needed turns out to be more than that of the competitive mechanism, but not very much . At given prices let agents simply announce their incomes in addition to their information on net trades and prices, that is one extra real number per individual. Then let them maximize their utilities subject to the average income constraint. This mechanism guarantees fair and Pareto efficient allocations and has an $n\ell$-dimensional message space.

Next we show that these are the minimal informational requirements of any decentralized mechanism that selects fair and efficient outcomes. The basic argument is fairly simple. Think of a class of economies in which all consumers have the same utility function, a Cobb Douglas with unit coefficients for example.This very specific class E^*, which we shall refer to as the class of "canonical" economies, with its associated "canonical" utility function will be particularly useful in what follows. It is clear that the only fair efficient allocation in such an economy is to give each agent the same bundle, i.e., to choose the allocation at the center of the Edgeworth box. To identify an economy in this class requires only the complete description of the initial bundles of all n agents, i.e., *n* bundles of ℓ goods. Thus its dimension is $n\ell$. Now we claim that, in any informationally decentralized mechanism, two different economies in that subclass E^* must use different messages. Indeed, suppose that we have a mechanism for which two different economies share the same equilibrium message *m*. Consequently, the outcome of the mechanism in both economies must also be the same trade *z*. Consider the situation depicted in figure 2. We have two different economies, represented by the two Edgeworth boxes *ABCD* and *EFGH*, which have the same trade *z* as the outcome[7]. This outcome is fair because the final allocation point *x* obtained with the trade *z* from the initial endowment point ω is the center of the box for both economies. However, by the crossing condition, the "crossed economies" will have the same equilibrium message and consequently they must have the same outcome *z*. It is easily seen that, with the same trade, the final outcome *x* for the crossed economy given by the Edgeworth box *EJCK* is not fair because it is not in the center.The same argument holds for the other crossed economy: *x* is not at the center of *AIGL*. Therefore every economy in the subclass considered must use different messages. Hence the message space has to be at least as "big" as the $n\ell$-dimensional class of environment. If some regularity conditions are satisfied the dimension of the message space cannot be smaller than $n\ell$. This establishes the informational efficiency of the equal income Walrasian mechanism.

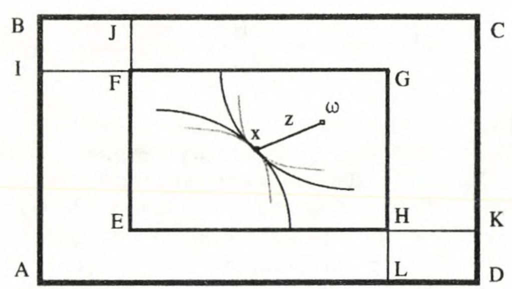

Figure 2

In the preceding argument we have been considering a very specific class of environments in which all agents have identical preferences. The information needed to attain fair efficient allocations within that restricted class is the same as that of the

[7] Since utility functions are assumed to be Cobb-Douglas, the flatter indifference curves correspond to the bigger Edgeworth box. However along a ray given by the diagonal of the boxes the normalized utility gradients are always the same.

equal income Walrasian mechanism. Of course, for this class there are other mechanisms which use the same amount of information. Think, for example, of a mechanism in which every agent simply announces his own bundle. The average bundle can then be computed and the appropriate trades assigned to every agent. Within this restricted class of economies, this mechanism is essentially equivalent to the equal income Walrasian one because it yields precisely the same outcomes. The latter mechanism has a satisfactory performance over a much larger class of environments in which agents can have different utility functions. However, over this larger class of environments the "bundle announcing" mechanism generally yields outcomes which are not even Pareto efficient, whilst the equal income Walrasian gives both efficiency and fairness.

It is important to note that the equilibrium message (prices, trades and individual incomes) for the equal income Walrasian mechanism does not reveal the particular Edgeworth box describing the economy. This must be so since, as we have seen, to specify the box completely requires $n\ell$ dimensional messages. However, to ensure efficiency in the general case, one needs to communicate supporting prices and in order to keep the message space to the same dimension, some information about the Edgeworth box must be sacrificed. This is illustrated in figure 3, where a whole class of Edgeworth boxes compatible with the same equal income Walrasian equilibrium message is shown : the upper-right corner of the box can be located at any point on the aggregate income budget line, $\overline{2A\ 2B}$, which is clearly twice that facing both individuals.

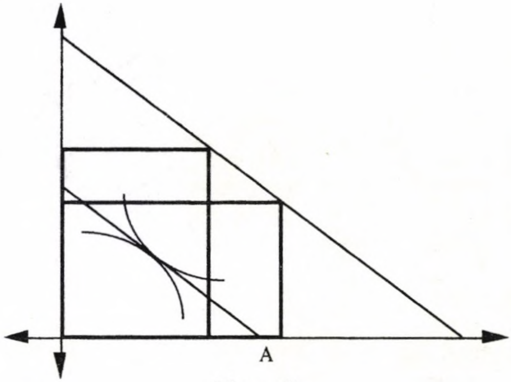

Figure 3

Finally, in section 6 and 7, we show that the only mechanism with minimal informational requirements achieving fair efficient outcomes is the equal income Walrasian one. Here we will sketch the argument used in the formal proof. Firstly, recall that if Pareto efficient outcomes are to be obtained in classical environments at least the information concerning supporting prices and competitive net trades must be conveyed. Furthermore, if fairness is required, as we have seen, some information about the total resources, (the size of the Edgeworth box) is needed. This can be obtained by eliciting agents' incomes, for example. Now our requirement of informational efficiency means that we are interested in mechanisms which use no more than this to attain fair efficient outcomes.

Our first step is to define a notion of "similarity" of an economy $\hat{e}$ to another e^*. Similarity here means that if we consider an equilibrium net trade $\hat{z}$ (for some informationally efficient mechanism) for $\hat{e}$ and the associated supporting prices and incomes at the resultant final allocation, then the same trade gives rise to the same prices and incomes in e^*. We prove that if $\hat{e}$ is similar to e^* and if, moreover e^* is in the special class of "canonical" economies to which we referred earlier, then $\hat{z}$ is also an equilibrium trade for e^*. This implies that in an informationally efficient mechanism similar economies have the same equilibrium messages[8].

We now proceed as follows: firstly, suppose that we have an informationally efficient mechanism which does not always select equal income competitive outcomes, but sometimes selects other fair and efficient ones. The latter must have the property that, at the prices implied by the common slope of the indifference curves, incomes are different. This is illustrated in figure 4 for individuals i and k of the economy $\hat{e}$ that gives rise to an equilibrium trade $\hat{z}$. and supporting price $\hat{p}$.

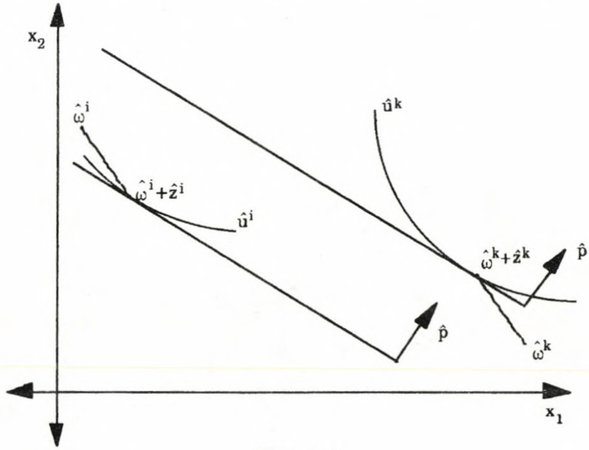

Figure 4

This allocation is fair since individual i does not envy individual k and vice versa. Now, however, we will construct a "similar" economy,e^*, that is one in which the equilibrium net trades of individuals, the slopes of their indifference curves and their incomes at those prices are all the same as in $\hat{e}$, and in particular all individuals have the "canonical" utility function u^*. Hence the final allocations of all agents will be along the same ray from the origin. To obtain our special similar economy we modify each of the agents' characteristics by bending and sliding down

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[8] See Lemma 6.2.

(or up) the indifference map along a given hyperplane as long as the tangency is preserved till we obtain the indifference curve of the "canonical" function. The initial endowments are then modified accordingly so as to preserve the same trade. The transition from the economy $\hat{e}$ to e^* is illustrated in figure 5 for agent k.

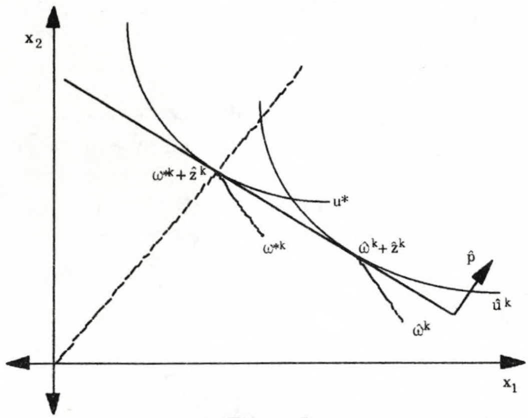

Figure 5

Now the essential point is that, in the similar economy e^* we construct, agent i is envious of agent k. Consider the situation in figure 6.

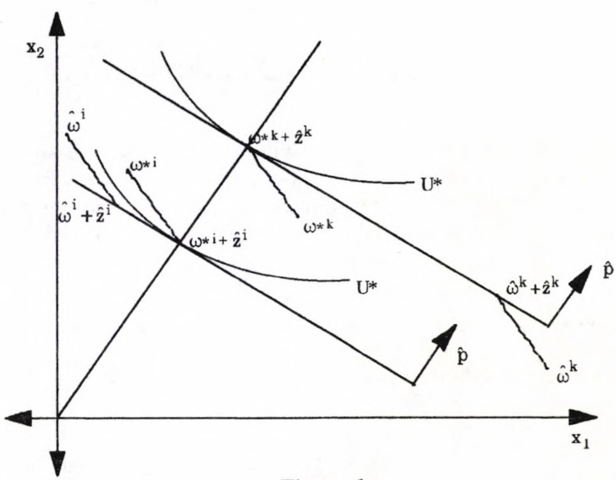

Figure 6

Each of the agents constructed in this way, has endowments ω^*, the canonical utility function u^*, and the same income as before. Clearly, however, agent i is now envious of agent k. Thus $\hat{z}$ is not an equilibrium trade for e^* which contradicts our result. The problem is clear. Any mechanism which would permit the sort of situation in figure 4 as an outcome must be able to distinguish between that and the situation in figure 6. This requires more information about the utility function and would therefore be more informationally demanding than the equal income Walrasian mechanism. This means that it cannot have been informationally efficient in the first place.

3. Informationally decentralized mechanisms.

Consider an exchange economy with ℓ commodities, and a set of agents $\mathfrak{I}=\{1,2,\ldots,n\}$. Every agent $i\in\mathfrak{I}$ is characterized by a utility function u^i, and an initial endowment ω^i. The i-th agent's characteristic is denoted by $e^i=\langle u^i,\omega^i\rangle$. An economy is denoted by the n-tuple $e=(e^1,e^2,\ldots,e^n)$. The class of possible economies, denoted by E, reflects the *a priori* knowledge available on the agents characteristics. Let $\mathcal{U}^c$ denote the set of all utility functions $u^i:\mathfrak{R}^{\ell}_{+}\to\mathfrak{R}$ such that there exists a vector $\alpha\in\mathfrak{R}^{\ell}_{++}$ with

$$u^i(x^i)=\prod_{j=1}^{\ell}(x^i_j)^{\alpha_j}.$$

The space $\mathcal{U}^c$ is topologized as $\mathfrak{R}^{\ell}_{++}$ and its generic element will be denoted either by α^i or u^i. Define the class of Cobb-Douglas environments

$$E^c=\{(e^1,e^2,\ldots,e^n)\text{: for all } i\in\mathfrak{I},\ u^i\in\mathcal{U}^c,\ \omega^i\in\mathfrak{R}^{\ell}\text{ and } \sum_{i=1}^{n}\omega^i>0\}.$$

Although most of the paper deals with the performance of decentralized mechanisms in the class of Cobb-Douglas environments, we shall specify a more general class of environments in which utility functions are not necessarily Cobb-Douglas. In order to ensure that equilibrium prices are strictly positive, we shall postulate a special type of strict monotonicity, in which monotonicity is not necessarily strict on the boundary (as is the case with Cobb-Douglas preferences). Hence, the set of possible utility functions $\mathcal{U}$ is defined as follows.

To every real valued continuous utility function u on $\mathfrak{R}^{\ell}_{+}$ we associate a set $X(u)$ defined as :

$$X(u)=\begin{cases} =\mathfrak{R}^{\ell}_{++} \text{ if for any } \bar{x} \text{ the set } \{x\in\mathfrak{R}^{\ell}_{+}:u(x)\geq u(\bar{x})\} \text{ is contained in } \mathfrak{R}^{\ell}_{++}. \\ =\mathfrak{R}^{\ell}_{+} \text{ otherwise.}\end{cases}$$

Now, let $\mathcal{U}^g$ be the set of all real valued functions such that u^i is continuous, strictly monotone on $X(u^i)$ and strictly quasi-concave. Then we can define the general class of environments E^g as follows:

$$E^g=\{(e^1,e^2,\ldots,e^n): \text{for all } i\in\Im,\ u^i\in U^g,\ \omega^i\in\Re^\ell \text{ and } \sum_{i=1}^{n}\omega^i > 0\}$$

Let x^i denote the consumption vector of the i-th agent and let $z^i=x^i-\omega^i$ denote the net trade vector. Let $x=\langle x^1,x^2,\ldots,x^n\rangle$ and $z=\langle z^1,z^2,\ldots,z^n\rangle$ denote respectively the n-tuples of consumption and net trades.

The set of possible outcomes of a resource allocation mechanism is given by the set of feasible net trades $Z=\{z\in\Re^{\ell n}: \sum_{i=1}^{n} z^i=0\}$. The Pareto optimality correspondence $P:E\rightarrow Z$ assigns to every economy $e\in E$ the set of Pareto optimal trades. For brevity we call a trade fair if the resulting allocation is both Pareto optimal and envy-free[9]. Hence a net trade z is fair for the economy $e\in E$ if $z\in P(e)$ and for all agents i and k, $u^i(x^i)\geq u^i(x^k)$, that is, there is no agent who envies other agent. Let $F:E\rightarrow Z$ denote the correspondence that assigns to every economy $e\in E$ the set of fair trades.

Following Mount and Reiter [1974], an allocation mechanism is a triple $\Pi=\langle M,\mu,h\rangle$, where M is a set of abstract messages, $\mu:E\rightarrow M$ is a message correspondence that assigns to every economy the set of equilibrium messages, and $h:M\rightarrow Z$ is the outcome function, that assigns to every equilibrium message the corresponding net trade. An allocation mechanism is *decisive* over the class of economies E if for every $e\in E$, $\mu(e)\neq\emptyset$. An allocation mechanism is *fair* over the class of economies E if it is decisive and for every $e\in E$, and every $z\in h[\mu(e)]$, $z\in F(e)$.

Given $\bar{e}$ and $\tilde{e}$ in E, the "crossed" economy $(\bar{e}^1,\bar{e}^2,\ldots,\bar{e}^{i-1},\tilde{e}^i,\bar{e}^{i+1},\ldots,\bar{e}^n)$ is denoted by $\bar{e}\otimes_i\tilde{e}$, for $i\in\Im$, while $\tilde{e}=\bar{e}\otimes_0\tilde{e}$. A resource allocation mechanism $\Pi=\langle M,\mu,h\rangle$ is privacy preserving over the class of economies E if for every $i\in\Im$ and every $\bar{e}$ and $\tilde{e}$ in E, $\mu(\bar{e})\cap\mu(\tilde{e})\neq\emptyset$ implies $\mu(\bar{e})\cap\mu(\tilde{e})=\mu(\bar{e}\otimes_i\tilde{e})\cap\mu(\tilde{e}\otimes_i\bar{e})$ whenever $\bar{e}\otimes_i\tilde{e}$ and $\tilde{e}\otimes_i\bar{e}$ belong to E. This means that if m is an equilibrium message for both $\bar{e}$ and $\tilde{e}$ in E than it must also be an equilibrium message for any "crossed" economy. If the class of economies E is a cartesian product $E=E^1\times E^2\times\ldots\times E^n$, then the privacy property can be characterized in terms of "coordinate" correspondences[10], as established by the following proposition due to Mount and Reiter.

3.1. LEMMA (Mount and Reiter [1974,Lemma 5, p.171) . Let the class of economies E be the cartesian product $E=E^1\times E^2\times\ldots\times E^n$. Then a mechanism $\Pi=\langle M,\mu,h\rangle$ is

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9 Thus we do not follow the terminology used by Schmeidler and Vind [1972].

10 See Chander [1983, p.983] for a discussion of this more general definition of the privacy property and its relation to the original definition by Mount and Reiter [1974] .

privacy preserving on E if and only if for every $i \in \Im$ there exists a correspondence $\mu^i: E^i \rightarrow M$ such that for every $e \in E$, $\mu(e) = \bigcap_{i=1}^{n} \mu^i(e^i)$

In this case, in a privacy preserving mechanism every agent can check independently whether a given message is an equilibrium message by looking at their own characteristics, $m \in \mu^i(e^i)$. The initial dispersion of information (every agent is assumed to know his own characteristics) and the privacy property (according to which the knowledge of other agents' characteristics is conveyed through formal messages) are the basic ingredients of the concept of informational decentralization. In what follows, a mechanism satisfying the privacy property will be said to be *informationally decentralized.*

The problem that we want to address is the minimal amount of information contained in M which is sufficient to guarantee that the mechanism yields fair outcomes over the class of classical economies[11]. Furthermore, we would like to know which are the properties that an informationally efficient (i.e., using minimal message spaces) mechanism must necessarily have.

4. The equal income Walrasian mechanism.

Let $\Delta = \{p \in \Re^{\ell}_{++}: \sum_{j=1}^{\ell} p_j = 1\}$ denote the ℓ-1 dimensional simplex and define the message space $M_W = \{(p,z,r) \in \Delta \times \Re^{\ell n} \times \Re^n: \sum_{i=1}^{n} z^i = 0 \text{ and } pz^k = \frac{1}{n}\sum_{i=1}^{n} r^i - r^k\}$, where p is a vector of normalized prices, z is a n-tuple of net trades and $r=(r^1, r^2, \ldots, r^n)$ is the vector of initial incomes (or values of initial endowments). Now, for every agent, define the correspondence $\mu^i_W: E^i \rightarrow M_W$ as follows: a message $(\bar{p},\bar{z},\bar{r})$ is an equilibrium message from the point of view of agent i, $(\bar{p},\bar{z},\bar{r}) \in \mu^i_W(e^i)$, if the following two conditions are satisfied:

$$(1) \qquad \begin{cases} u^i(\omega^i + \bar{z}^i) \geq u^i(\omega^i + z^i) \text{ for all } z^i \in \Re^{\ell} \text{ such that } \bar{p}(\omega^i + z^i) = \frac{1}{n}\sum_{k=1}^{n} \bar{r}^k \\ \bar{p}\omega^i = \bar{r}^i \end{cases}$$

According to the second condition, every agent checks whether the proposed initial income $\bar{r}^i$ corresponds to the value of his initial endowments at the going prices. According to the first condition, the agent maximizes utility subject to the average income constraint. This last magnitude can be computed by the agent from

[11] By classical economies, we understand the class of economies satisfying the conditions sufficient to guarantee the existence and optimality of a competitive equilibrium.

the messages sent by others. The message correspondence and outcome function are then defined as

$$\mu_W(e)=\bigcap_{i=1}^{n}\mu_W^i(e^i) \quad \text{and} \quad h_W(p,z,r)=z.$$

Hence, the equal income Walrasian mechanism is given by $\Pi_W=\langle M_W,\mu_W,h_W\rangle$. It follows from (1) that any equilibrium message (p,z,r) satisfies the following equality:

(2) $$pz^k = \frac{1}{n}\sum_{i=1}^{n} r^i - r^k.$$

It is easily verified that M_W is a smooth $n\ell$-dimensional manifold.

5. Informational requirements for fair allocations

In this section we study the minimal amount of information - as measured by the dimension of the message space - that is required to guarantee that an informationally decentralized mechanism leads to fair outcomes over the general class of environments E^g. Let us define the canonical subclass of economies E^*. An economy is a member of E^* if all agents have identical preferences which can be represented by a Cobb-Douglas utility function with unit coefficients. More specifically,

(3) $$E^*=\{e\in E^c: u^i(x^i)= \prod_{j=1}^{\ell} x_j^i\}$$

Thus, in an economy with n agents and ℓ commodities, every environment in the subclass E^* is completely specified by $n\ell$-dimensional vectors of initial endowments. Hence dim $E^*= n\cdot\ell$. Now we shall show that the fair correspondence F restricted to E^* is a single-valued function.

LEMMA 5.1. Let $\bar{e}\in E^g$ be such that all agents have identical utilities so that $\bar{u}^i=u$ for all i. Suppose further that $z\in F(\bar{e})$. Then for all agents $k\in\mathfrak{I}$ we must have

$$\omega^k+z^k=\frac{1}{n}\sum_{i=1}^{n}\omega^i.$$

PROOF : It suffices to show that for all i and k in $\mathfrak{I}$, $\omega^i+z^i=\omega^k+z^k$. It is clear that, since all agents have the same utility function u, fairness implies that $u(\omega^i+z^i)=u(\omega^k+z^k)$ for all i and k in $\mathfrak{I}$. Suppose that there exist two agents i and k such that $\omega^i+z^i\neq\omega^k+z^k$. Then we can construct a new trade $\hat{z}$ which is Pareto superior to z. Indeed, define

$$\hat{z}^i=\frac{\omega^k+z^k-\omega^i+z^i}{2} \quad \text{and} \quad \hat{z}^k=\frac{\omega^i+z^k-\omega^k+z^i}{2}$$

while $\hat{z}^j = z^j$ for all other agents. Then, the new final consumption vectors satisfy $\hat{x}^i = \hat{x}^k = \frac{1}{2}(x^i + x^k)$. By strict quasiconcavity of the common utility function u it follows that $u(\omega^i + \hat{z}^i) > u(\omega^i + z^i)$ and $u(\omega^k + \hat{z}^k) > u(\omega^k + z^k)$, while all other agents are indifferent since they receive the same consumption vector. Moreover, since $\hat{z}^i + \hat{z}^k = z^i + z^k$ it follows that $\hat{z}$ is a feasible trade. This contradiction completes the proof.

Lemma 5.1 implies that in an economy with identical individuals, the only fair allocation is the equal division of total endowments. In particular, this is true for our canonical subclass E^*. Now we shall show the basic theorem of this section.

THE INFORMATIONAL EFFICIENCY THEOREM 5.2. Suppose that $\Pi = \langle M, \mu, h \rangle$ is a resource allocation on E^g such that:

a) it is fair on E^g,

b) it is informationally decentralized,

c) the message space M is a manifold,

d) when restricted to E^* the message correspondence is locally threaded at some point $\bar{e}$ [12].

Then the dimension of the message space is at least as large as that of the equal income Walrasian mechanism defined in section 4, that is, $dim\, M \geq dim\, M_W$.

PROOF: We first show that the restriction of m to E^* is an injective correspondence. Suppose that $\bar{m} \in \mu(\bar{e}) \cap \mu(\tilde{e})$. We have to show that this implies $\bar{e} = \tilde{e}$. Since the mechanism is assumed to be informationally decentralized, it follows from the privacy property that $\bar{m} \in \mu(\bar{e} \otimes_i \hat{e}) \cap \mu(\bar{e} \otimes_i \hat{e})$ for all i. But if the mechanism is fair this implies that the outcome $\bar{z} = h(\bar{m})$ satisfies

$$\bar{z} \in F(\bar{e} \otimes_i \hat{e}) \Rightarrow \bar{\omega}^i + \bar{z}^i = \hat{\omega}^k + \bar{z}^k \text{ for all } k \in \mathfrak{I}$$

$$\bar{z} \in F(\bar{e} \otimes_i \hat{e}) \Rightarrow \hat{\omega}^i + \bar{z}^i = \hat{\omega}^k + \bar{z}^k \text{ for all } k \in \mathfrak{I}$$

From these two equalities it follows that $\bar{\omega}^i = \hat{\omega}^i$. Since the argument can be repeated for any other agent, it is clear that the initial endowments are the same in both economies and therefore $\hat{e} = \bar{e}$. Using assumption d, let U_z be an open neighborhood of $\bar{e}$ and $f: U \rightarrow M$ a continuous function such that $f(e) \in \mu(e)$ for all $e \in U$. Then f is a continuous injection from U to $f[U]$. Since U and M are manifolds, we can use Theorem 18 in Kelley [1955] to conclude that f is a homeomorphism between U and $f[U]$. Then $dim\, M_W = n \cdot \ell = dim\, U \leq dim\, M$.

[12] Let X and Y be topological spaces. A correspondence $\psi: X \twoheadrightarrow Y$ is said to be locally threaded at $x \in X$ if there exists a neighborhood U of x and a continuous function $f: U \rightarrow Y$ such that $f(x) \in \psi(x)$ for all $x \in U$.

The intuition behind this result is clear: the message space has to contain at least as much information as the subclass of environments E^* because every environment in E^* must have a different message. Therefore the message space has to contain enough information so as to distinguish between members of E^*.

Notice that in order to guarantee Pareto optimal allocations the minimal informational requirements are those of the competitive process, i.e. $n(\ell-1)$. The preceding theorem suggests the possibility of realizing a stronger optimality correspondence through mechanisms which require every agent to send just an additional real number. In the following proposition we show that the equal income Walrasian mechanism defined in section 4 is an informationally decentralized process satisfying all the assumptions of the informational efficiency theorem whose message space is of dimension $n\cdot\ell$.

COROLLARY 5.3. The equal income Walrasian mechanism is informationally efficient on E^g.

PROOF: It is clear from the construction of $\Pi_W=\langle M_W, \mu_W, h_W\rangle$ that it is informationally decentralized and that M_W is a manifold of dimension $n\cdot\ell$. It is also clear that from the conditions defining the class E^g it can be shown that the competitive equilibrium from average income exists, is Pareto optimal and envy-free. Finally, the correspondence μ_W when restricted to E^* is a continuous function and thus it is locally threaded at any $\bar{e}\in E^*$.

6. The uniqueness theorem in Cobb-Douglas economies.

In this section and the next we follow the basic strategy of the proof given by Jordan for the uniqueness of the competitive mechanism in achieving efficiency in classical environments. Our proof differs from his in several essential respects, and it is worth indicating the differences between our problem and his, and the way in which our proof is adapted to handle them. Firstly, in his problem parallel supporting hyperplanes are necessary for efficiency. In our case, the equality of incomes *and* of supporting hyperplanes is not necessary for achieving fairness and efficiency. Secondly, we construct a class of economies, those with what we have called "canonical" preferences for all individuals, where the fair outcome is unique and is also efficient. There is no equivalent in the ordinary efficiency problem. Thirdly, when considering the problem of how much information is necessary to discriminate between different economies, the relative "flatness" of indifference curves may be important. Jordan was free to modify total income in order to change the curvature of Cobb-Douglas indifference curves. We do not have this possibility, and so have to use a different construction.

We follow Jordan by first showing the uniqueness theorem for allocation mechanisms defined on the class of environments E^c. We define several binary relations on the class of Cobb-Douglas economies E^c: the similarity relation S_π induced by a mechanism π, the message relation T_π induced by a mechanism π, and the message relation T_ω induced by the equal income Walrasian mechanism. Two economies are similar relative to a mechanism π if they have the same equilibrium net trades, the same slopes of their indifference curves at these trades and the same incomes. Two economies are "message related" by a mechanism π if both have the same equilibrium message under π. The basic strategy of the proof is to show that if π is a fair mechanism with minimal informational requirements the three binary relations are equivalence relations and that the partitions induced on the class of economies E^C by the three of them are exactly the same. The proof hinges on the fact that every equivalence class contains one and only one member of the canonical class E^* so that in an informationally efficient mechanism, where the message partitions should not be finer, what happens in E^* must be the same as what happens everywhere. Now, it turns out that in the canonical class the equal income Walrasian outcome is the only fair outcome. This fact is later used to show that any informationally efficient mechanism is essentially the same as the equal income Walrasian because the outcomes are the same and the message spaces homeomorphic.

Let us start by defining the similarity relation rigorously. Since in the Cobb-Douglas class of environments E^C aggregate endowments are strictly positive and there is strict monotonicity, the fairness requirement ensures that every agents' final consumption will be contained in the strictly positive orthant. Furthermore, the corresponding supporting price will equal the unique normalized utility gradient at the final consumption point. This allows us to construct the following correspondence. Given a mechanism $\pi = \langle M,\mu,h\rangle$ define $\psi_\pi : E^C \rightarrow M_W$ that assigns a set of points in the Walrasian message space to every environment. For any given $e \in E^C$, let us define $(p, z, r) = \psi_\pi(e)$ as follows

$$(4) \qquad \begin{cases} z = h[\mu(e)] \\ p = \dfrac{Du^i(\omega^i + z^i)}{||Du^i(\omega^i + z^i)||} \\ r^i = p\omega^i \end{cases}$$

Hence, the image under ψ_π of a given environment is given by the outcome of the given mechanism (the trade z), the normalized utility gradient at the corresponding final consumption point and the vector of values of initial endowments of the n agents. If the message correspondence is a function on E^C, then ψ_π is also a function. Notice also that ψ_π depends upon the mechanism.

Two economies $\bar{e}$ and $\hat{e}$ in E^C are said to be *similar* relative to mechanism $\pi = (M,\mu,h)$ if they have the same image under ψ_π, that is, if $\psi_\pi(\bar{e}) = \psi_\pi(\hat{e})$. In this case we write $\bar{e}\, S_\pi\, \hat{e}$. It is easily verified that if the message correspondence of the mechanism, μ, is a function on E^C, S_π is an equivalence relation that induces a partition of E^C into equivalence classes. When there is no possibility of confusion, the reference to the given mechanism π will be dropped and the symbols T and S will be used.

Recall that in figure 5 we represented the k-th agent of two similar economies. It follows from the definition of similarity that we can stay within the same equivalence class if we modify the agents' characteristics as was done in that figure.

The next Lemma establishes that there exists a bijection between the quotient space of E^C/S and the subclass of environments E^*, defined in (3). This means that in every equivalence class there is one and only one element of E^*, which can be considered its canonical representation.

LEMMA 6.1. Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on E^C which is:

a) fair,

b) informationally decentralized.

Given any $\bar{e} \in E^C$ there exists a unique $e^* \in E^*$ which is similar to it.

PROOF: As was seen in Lemma 5.1, in the subclass of environments E^* the fair correspondence is a single valued function that gives equal consumption as the only fair allocation, that is, it must be the case that $\omega^i + z^i = \frac{1}{n}\sum_{k=1}^{n}\omega^k$. Therefore, when restricted to E^*, the realization of the mechanism $h \circ \mu$ and the fair correspondence F must coincide.

Let $(\bar{p},\bar{z},\bar{r}) \in \Delta \times \Re^{n\ell} \times \Re^n$ be the image of $\bar{e}$ under ψ. Since $(\bar{p},\bar{z},\bar{r}) \in M_W$ for some environment , it is the equilibrium message of some environment under the equal income Walrasian process. Hence, by (2) we must have $\bar{p}\bar{z}^k = \frac{1}{n}\sum_{i=1}^{n}\bar{r}^i - \bar{r}^k$.

Define the economy $e^* \in E^*$ by taking the initial endowments:

$$\omega_j^{*i} = \frac{\bar{r}^i + \bar{p}\bar{z}^i}{\bar{p}_j\,\ell} - \bar{z}_j^i \tag{5}$$

Let e^* be the environment in which all agents have the same Cobb-Douglas utility function with unit coefficients and the initial endowments are given by ω^* defined in (5). It is easily verified that this environment belongs to E^*. Indeed,

$$\omega_j^* = \sum_{i=1}^{n}\omega_j^{*i} = \frac{1}{\bar{p}_j\ell}\sum_{i=1}^{n}\bar{r}^i + \sum_{i=1}^{n}\sum_{j=1}^{\ell}\bar{p}_j\bar{z}_j^i - \sum_{i=1}^{n}\bar{z}_j^i = \frac{1}{\bar{p}_j\ell}\sum_{i=1}^{n}\bar{r}^i > 0 \tag{6}$$

so that aggregate initial endowments are strictly positive.

Let $z^* = h[\mu(e^*)]$ be the outcome of the mechanism. Since μ and F coincide on E^* trades must be such that final consumption is equal across individuals. Hence

$$z_j^{*i} = \frac{1}{n}\left(\sum_{k=1}^{n} \omega_j^{*k}\right) - \omega_j^{*i}$$

Substituting for the values of ω_j^{*i} given in (5) we obtain

$$z_j^{*i} = \frac{1}{n\bar{p}_j\ell}\left(\sum_{k=1}^{n} \bar{r}^k + \sum_{k=1}^{n} \bar{p}\bar{z}^k\right) - \frac{1}{n}\sum_{k=1}^{n} \bar{z}_j^k - \frac{\bar{r}^i + \bar{p}\bar{z}^i}{\bar{p}_j\ell} + \bar{z}_j^i$$

Taking into account that feasibility requires $\sum_{i=1}^{n} \bar{z}_j^i = 0$ and equality (2), we finally obtain

$$z_j^{*i} = \frac{1}{\bar{p}_j\ell}\left(\frac{1}{n}\sum_{k=1}^{n} \bar{r}^k - \bar{r}^i - \bar{p}\bar{z}^i\right) + \bar{z}_j^i = \bar{z}_j^i$$

and therefore it must be the case that $h[\mu(e^*)] = z^* = \bar{z}$. Moreover, since the parameters of the utility function equal one and final consumption of every agent equals average consumption, it is easily verified that the utility gradient of any agents' utility at $(\omega^{*i} + \bar{z}^i)$ is given by

$$Du^{*i}(\omega^{*i} + \bar{z}^i) = \left(\frac{nW}{\omega_1^*}, \frac{nW}{\omega_2^*}, \ldots, \frac{nW}{\omega_\ell^*}\right)$$

where $\omega_j^* = \sum_{i=1}^{n} \omega_j^{*i}$ and $W = \frac{1}{n^\ell}\prod_{j=1}^{\ell} \omega_j^*$. We know that for this economy equality (6) implies

$$Du^{*i}(\omega^{*i} + \bar{z}^i) = \frac{Wn\ell}{r}\bar{p}$$

Hence, for any agent k the utility gradient is proportional to the price vector. Finally, let us verify that the values of initial endowments are the same for any agent. Using (5) we get,

$$\bar{p}\omega^{*i} = \sum_{j=1}^{\ell} \bar{p}_j\left(\frac{\bar{r}^i + \bar{p}\bar{z}^i}{\bar{p}_j\,\ell} - \bar{z}_j^i\right) = \bar{r}^i = \bar{p}\bar{\omega}^i.$$

To show that e^* is unique, let $\hat{e}$ be an economy in E^* similar to $\bar{e}$. Then they have the same equilibrium trade $\bar{z}=h[\mu(\hat{e})]=h[\mu(\bar{e})]$. But by Lemma 5.1, in the subclass E^* the only fair allocation is fair division. Hence, if the mechanism is fair we must have $\hat{\omega}^i+\bar{z}^i=\omega^{*i}+\bar{z}^i$ for all i. Hence $\hat{\omega}^i=\omega^{*i}$ and the result is established.

The next step is to show that if the message space of an arbitrary mechanism $\Pi=\langle M,\mu,h\rangle$ is of minimal dimension, then all similar economies must have the same message. The proof of this lemma, that relies on the local homology of manifolds, follows closely the basic idea in Jordan [1982].

LEMMA 6.2. Suppose that $\langle M, \mu, h\rangle$ is an allocation mechanism on E^C such that:

a) it is fair,

b) it is informationally decentralized,

c) the message space is a $n\cdot\ell$-dimensional manifold, and

d) the message correspondence μ is a continuous function.

If two economies in E^C, $\bar{e}$ and $\hat{e}$ are similar, then $\mu(\bar{e})=\mu(\hat{e})$.

PROOF: The result is first shown to be true for economies in the subclass E^* with the uniqueness property defined in (1) and later extended to the Cobb-Douglas class E^C.

Step 1. The statement is true if both $\hat{e}$ and $\bar{e}$ are in E^*. Indeed, if $\hat{e}$ and $\bar{e}$ are similar, it follows from the uniqueness part of Lemma 5.1 that $\bar{e}=\hat{e}$ and, *a fortiori*, $\mu(\bar{e})=\mu(\hat{e})$.

Step 2. Let e^* be the economy in E^* similar to $\bar{e}$, which exists in view of Lemma 5.1. Let U be an open neighborhood of $e^*\in U\subseteq E^*$. Then U is homeomorphic to $V=\mu[U]$. Indeed, consider the mapping $\nu: E^*\to\Re^{n\ell}$ given by $\nu(u,\omega)=\omega$. This is clearly a homeomorphism of E^* into an open subset of $\Re^{n\ell}$ so that we shall consider E^* as an open subset of $\Re^{n\ell}$. Using theorem A.1 in Greenberg[13] we conclude that V is an open set in M which is homeomorphic to U.

Step 3. Let $q=n\ell$ and let $H_q(M,M-m^*)$ denote the q-th singular homology module of M relative to $M-m^*$ with integral coefficients. Now we claim that $H_q(M,M-m^*)=\mathbb{Z}$ where $\mathbb{Z}$ is the set of integers, and the homomorphism $i_*: H_q(V,V-v^*)\to H_q(M,M-m^*)$ induced by the inclusion map $i:(V,V-v^*)\to(M,M-m^*)$ is an isomorphism. Indeed, since the message space is a $n\ell$-dimensional manifold, it follows from theorem[14] A.2 in Greenberg [1967]

[13] see statement 118.10 in Greenberg [1967], page 82.

[14] See theorem A.2 in page 111.

that $H_q(M,M\text{-}m^*)=\mathbb{Z}$. Moreover, the subset $M\text{-}V \subseteq M\text{-}m^*$ can be excised. Indeed, since $M\text{-}V$ is closed and $M\text{-}m^*$ is open, the closure of $M\text{-}V$ is contained in the interior of $M\text{-}m^*$ and proposition A.3 of Greenberg [1967,p.60] can be applied. By definition of excision i_* is an isomorphism.

Step 4. We show that for every economy $e \in U\text{-}e^*$, paths of economies $C(e,t)$ (joining e and e^*) and $G(e,t)$ (joining e and $\bar{e}$) can be constructed in such a way that for every t, in the economies $G(e,t)$ and $C(e,t)$ all agents have the same normalized utility gradients at $\bar{z}=h[\mu(\bar{e})]$ and the same value of initial endowments at the supporting prices. Indeed, given two similar economies $\hat{e}$ and $\bar{e}$ in E^c, let e^* be the unique similar economy in E^* (which exists by Lemma 5.1) and let $(\bar{p},\bar{z},\bar{r})=\psi(\bar{e})=\psi(\hat{e})=\psi(e^*)$. By definition of ψ, $\bar{z}=h[\mu(\bar{e})]=h[\mu(\hat{e})]=h[\mu(e^*)]$ and the utility gradients at $\bar{\omega}^i+\bar{z}^i$, $\omega^{*i}+\bar{z}^i$ and $\hat{\omega}^i+\bar{z}^i$ are proportional to $\bar{p}$. Hence we have:

$$(7) \qquad \bar{\alpha}^i_j = \lambda^i \, \frac{\bar{\omega}^i_j+\bar{z}^i_j}{\omega^{*i}_j+\bar{z}^i_j}$$

where $\bar{\alpha}^i_j$ is the coefficient of the Cobb-Douglas utility function. Now define the function $G: U \times [0,1] \rightarrow E^c$ given by $\tilde{\bar{e}}=G(e,t)$, where $\tilde{\bar{e}}=(\tilde{\bar{\omega}},\tilde{\bar{\alpha}})$ is the economy given by:

$$\tilde{\bar{\omega}}^i_j = t\bar{\omega}^i_j+(1-t)\omega^i_j$$

$$\tilde{\bar{\alpha}}^i_j=[1-t(1-\lambda^i)] \, \frac{t\bar{\omega}^i_j+(1-t)\omega^i_j+\bar{z}^i_j}{t\omega^{*i}_j+(1-t)\omega^i_j+\bar{z}^i_j}$$

where λ^i is the constant given in (7). Let us check now that the required properties are satisfied:

1. $G(e,0)=e$ for all $e\in E^*$. Let $e=(\omega,\alpha)$ with $\alpha=(1,1,\ldots,1)$ be an economy in E^*. Then

$$\tilde{\bar{\omega}}^i_j = 0\cdot\bar{\omega}^i_j+(1-0)\omega^i_j = \omega^i_j$$

$$\tilde{\bar{\alpha}}^i_j = [1-0(1-\lambda^i)] \, \frac{0\bar{\omega}^i_j+\omega^i_j+\bar{z}^i_j}{0\omega^{*i}_j+\omega^i_j+\bar{z}^i_j} = \frac{\omega^i_j+\bar{z}^i_j}{\omega^i_j+\bar{z}^i_j} = 1$$

and therefore $G(e,0)=e$ for all $e\in E^*$.

2. $G(e,1)=\bar{e}$ for all $e\in E^*$. For $t=1$, using (7) we obtain

$$\tilde{\bar{\omega}}^i_j=\bar{\omega}^i_j$$

$$\tilde{\bar{\alpha}}_j^i = \lambda^i \frac{\bar{\omega}_j^i + \bar{z}_j^i}{\omega_j^{*i} + \bar{z}_j^i} = \bar{\alpha}_j^i$$

Now consider the function $C{:}U\times[0,1]\rightarrow E^C$ given by $C(e,t) = te^* + (1-t)e$ so that $C(e,t)$ is a convex combination of the economies e and e^* in E^*.

3. We claim that for any $e\in U$ the utility gradients in the economies $C(e,t)$ and $G(e,t)$ at $\bar{z}$ are proportional. Let $\tilde{\bar{e}}$ and $\bar{\bar{e}}$ be given by $\tilde{\bar{e}}=(\tilde{\bar{\omega}},\tilde{\bar{\alpha}})=G(e,t)$ and $\bar{\bar{e}}=(\bar{\bar{\omega}},\bar{\bar{\alpha}})=C(e,t)$. Then we have

$$D_j\tilde{\bar{u}}^i(\tilde{\bar{\omega}}^i+\bar{z}^i)=\frac{\tilde{\bar{\alpha}}_j^i A^i}{\tilde{\bar{\omega}}^i+\bar{z}^i}=A^i\frac{(t\lambda^i+(1-t))\dfrac{t\bar{\omega}_j^i+(1-t)\omega_j^i+\bar{z}_j^i}{t\omega_j^{*i}+(1-t)\omega_j^i+\bar{z}_j^i}}{t\bar{\omega}_j^i+(1-t)\omega_j^i+\bar{z}^i}=$$

$$=A^i\frac{(t\lambda^i+(1-t))}{t\omega_j^{*i}+(1-t)\omega_j^i+\bar{z}^i}=\kappa D_j\bar{\bar{u}}^i(\bar{\bar{\omega}}^i+\bar{z}^i)$$

where $A^i=\prod_{j=1}^{\ell}(\tilde{\bar{\omega}}_j^i+\bar{z}_j^i)^{\tilde{\bar{\alpha}}_j^i}$ and κ is a constant.

4. Finally we have to verify that $\bar{p}\bar{\bar{\omega}}^i=\bar{p}\tilde{\bar{\omega}}^i=\bar{r}^i$. First, note that the initial endowments of the unique economy e^* in E^* similar to $\bar{e}$ constructed in lemma 6.1 are such that $\bar{p}\omega^{*i}=\bar{p}\bar{\omega}^i$ for all i. Then, it follows

$$\bar{p}\bar{\bar{\omega}}^i=\bar{p}(t\bar{\omega}^i+(1-t)\omega^i)=t\bar{p}\bar{\omega}^i+(1-t)\bar{p}\omega^i=\bar{r}^i=t\bar{p}\omega^{*i}+(1-t)\bar{p}\omega^i=\bar{p}\tilde{\bar{\omega}}^i.$$

Step 5. We show that if m^* is the equilibrium message for the economy e^*, $\mu[G(e,t)]\neq m^*$ for all $e\in U-e^*$ and $0\leq t<1$. Since μ is injective on E^* we have that $\mu[C(e,t)]\neq m^*$ for all $e\in U-e^*$ and $0\leq t<1$. Now, suppose by way of contradiction that $\mu[G(e,t)]=m^*$ for some $e\in U-e^*$ and some t. Then the utility gradients of all agents in the economy $G(e,t)$ at $\bar{z}$ are proportional to $\bar{p}$. Indeed, since $m^*\in\mu[G(e,t)]\cap\mu(e^*)$, then $\bar{z}=h(m^*)$ is the outcome of the mechanism for both environments $G(e,t)$ and e^*. By the crossing condition it follows that m^* is also an equilibrium message for the "crossed" environment, $m^*\in\mu[G(e,t)\otimes_i e^*]$. But the process is fair and therefore the outcome $\bar{z}=h(m^*)$ must necessarily be a fair allocation for the economy $G(e,t)\otimes_i e^*$. Thus, the trade $\bar{z}$ gives a final allocation which is Pareto optimal for the economies $G(e,t)\otimes_i e^*$ and e^*. Hence, the normalized utility gradient of the i-th agent equals the price vector:

$$\frac{D\tilde{\bar{u}}^i(\tilde{\bar{\omega}}^i+\bar{z}^i)}{\|D\tilde{\bar{u}}^i(\tilde{\bar{\omega}}^i+\bar{z}^i)\|}=\frac{Du^{*k}(\omega^{*i}+\bar{z}^k)}{\|Du^{*k}(\omega^{*i}+\bar{z}^k)\|}=\bar{p}$$

By the result in 3 it follows that the normalized utility gradient at $\bar{z}$ of any agent in the environment $\bar{\bar{e}}=(\bar{\bar{\omega}},\bar{\bar{\alpha}})=C(e,t)$ is equal to $\bar{p}$. Moreover, by the conclusion of 4 we know that $\bar{p}\bar{\bar{\omega}}^i=\bar{r}^i$. It follows that the value of final consumption of the i-th agent after the trade $\bar{z}$ in the economy $\bar{\bar{e}}=(\bar{\bar{\omega}},\bar{\bar{\alpha}})=C(e,t)$ is given by $\bar{p}(\bar{\bar{\omega}}^i+\bar{z}^i)=\bar{r}^i+\bar{p}\bar{z}^i=\frac{1}{n}\sum_{k=1}^{n}\bar{r}^k$. Hence utility is maximized at $\bar{\bar{\omega}}^i+\bar{z}^i$ subject to the average income constraint for every agent and therefore $\bar{z}$ is the outcome of the equal income Walrasian mechanism for the economy $\bar{\bar{e}}=C(e,t)$. But by construction $C(e,t)$ is contained in the subclass E^*. By Lemma 5.1 the fair correspondence is single-valued on E^* and thus $\bar{z}$ is necessarily the outcome of any other mechanism which is fair on E^*. In particular, $\bar{z}\in h[\mu(C(e,t)]$. Then we can write

$$\begin{cases} \bar{z}\in h[\mu(C(e,t))] \\ \bar{p}=\dfrac{D\bar{\bar{u}}^i(\bar{\bar{\omega}}^i+\bar{z}^i)}{\|D\bar{\bar{u}}^i(\bar{\bar{\omega}}^i+\bar{z}^i)\|} \\ \bar{p}\bar{\bar{\omega}}^i=\bar{r}^i \end{cases}$$

for all i.By definition of ψ it follows that $\psi[C(e,t)]=(\bar{p},\bar{z},\bar{r})$, and this can only happen if $e=e^*$ because, as shown in step 1, ψ is injective on E^*.

Step 6. We finally show that $\bar{e}$ and $\tilde{e}$ have the same equilibrium message. In fact we shall show that $\mu(\bar{e})=\mu(e^*)=m^*$. A similar argument establishes that $\mu(\tilde{e})=\mu(e^*)=m^*$ and the result follows. Suppose by way of contradiction that $\mu(\bar{e})\neq m^*$. In that case the statement of step 5 could be strengthened and written as $\mu[G(e,t)]\neq m^*$ for all $e\in U-e^*$ and $t\in[0,1]$. Define the function[15] $\Phi:(V,V-m^*)\times[0,1]\rightarrow(M,M-m^*)$ given by $\Phi(m,t)=\mu(G[\mu^{-1}(m),t])$. Let us denote by $i:(V,V-m^*)\rightarrow(m,M-m^*)$ the inclusion map given by $i(m)=m$ for all $m\in V$, and by $j:(V,V-m^*)\rightarrow(m,M-m^*)$ the constant map given by $j(m)=\bar{m}=\mu(\bar{e})$ for all $m\in V$. It is easily verified that:

a) For all $t\in[0,1]$ and all $m\in V-m^*$, $\Phi(m,t)\neq m^*$. Indeed, if $m\neq m^*$ and $e\in\mu^{-1}(m)$ then $e\neq e^*$ because μ is injective on U. But $\mu[G(e,t)]\neq m^*$ for all $e\in U-e^*$ and $t\in[0,1]$.

b) $\Phi(m,0)$ is the inclusion map $i:(V,V-m^*)\rightarrow(m,M-m^*)$. Indeed, $\Phi(m,0)=\mu(G[\mu^{-1}(m),0])=\mu[\mu^{-1}(m)]=m$ for all $m\in V$.

c) $\Phi(m,1)$ is the constant map $j:(V,V-m^*)\rightarrow(m,M-m^*)$. Indeed, from the definitions of G and Φ it follows that $\Phi(m,1)=\mu[G(\mu^{-1}(m),1)]=\mu(\bar{e})=\bar{m}$. Hence, we can conclude that Φ is a homotopy between the inclusion map $\boldsymbol{i}$ and the

[15] Following the notation in algebraic topology we denote by $F:(V,V-m^*)\times[0,1]\rightarrow(M,M-m^*)$ a function $F:V\times[0,1]\rightarrow M$ such that $F[V-m^*,t]\subseteq M-m^*$.

constant map j defined above. By theorem 13.13 in Greenberg[16] it follows that the induced homomorphisms $i_*: H_n(V, V-m^*) \rightarrow H_n(m, M-m^*)$ and $j_*: H_n(V, V-m^*) \rightarrow H_n(m, M-m^*)$ on the n-th relative homology group modulo V relative to $V-m^*$ are equal. But j is the constant map, so that j_* is the zero homomorphism, and so is i_*. But this contradicts the result established in step 3, according to which if the message space is of minimal dimension i_* is an isomorphism. This completes the proof.

The following theorem establishes that if a mechanism π is informationally efficient, the partition induced by the similarity relation S_π is the same as that induced by the equal income Walrasian message relation T_ω. In particular, the outcome of the mechanism must be the equal income Walrasian one.

PROPOSITION 6.3. Let $\Pi = \langle M, \mu, h \rangle$ be an allocation mechanism on E^c such that

a) it is fair,

b) it is informationally decentralized,

c) its message space is of minimal dimension, that is, M is a $n\ell$-dimensional manifold,

d) its message correspondence μ is a continuous function on the class of Cobb-Douglas environments E^c.

Then $\mu_\omega(e) = \psi(e)$ for all $e \in E^c$.

PROOF: Suppose that the statement is not true, so that there is some $\hat{e} \in E^c$ such that $\mu_W(\hat{e}) \neq \psi(\hat{e}) = (\hat{p}, \hat{z}, \hat{r})$. Since $\langle M, \mu, h \rangle$ is a fair mechanism we know that $\hat{\omega} + \hat{z}$ is a Pareto optimal allocation in the economy $\hat{e}$. Therefore, the utility gradients of any two agents i and j at $\hat{\omega}^i + \hat{z}^i$ and $\hat{\omega}^j + \hat{z}^j$ are proportional to the price vector $\hat{p}$ and excess demands add up to zero. Hence, if $\hat{z}$ is not the equal income Walrasian trade it must be the case that for at least two agents $\hat{p}(\hat{\omega}^i + \hat{z}^i) < \hat{p}(\hat{\omega}^k + \hat{z}^k)$. This was represented in figure 4, where the final consumption vectors of agents i and k are on different hyperplanes (but parallel because of Pareto optimality).

We shall show that if this is true it is always possible to construct a similar economy e^* in which i envies k. This was done in the introduction and is illustrated in figure 6

More formally, let $e^* \in E^*$ be the unique environment similar to $\hat{e}$. Since e^* and $\hat{e}$ are similar we know that

[16] See Greenberg [1967], page 55.

$$\begin{cases} \hat{z} = h[\mu(\hat{e})] = h[\mu(e^*)] \\ \hat{p} = \dfrac{D\hat{u}^i(\hat{\omega}^i + \hat{z}^i)}{\|D\hat{u}^i(\hat{\omega}^i + \hat{z}^i)\|} = \dfrac{Du^*(\omega^{*i} + \hat{z}^i)}{\|Du^*(\omega^{*i} + \hat{z}^i)\|} \\ \hat{r}^i = \hat{p}\hat{\omega}^i = \hat{p}\omega^{*i} \end{cases}$$

In view of these equalities, we must have $\hat{p}(\omega^{*i} + \hat{z}^i) < \hat{p}(\omega^{*k} + \hat{z}^k)$ so that the values of the final consumption vectors are still different. On the other hand, since $e^* \in E^*$, all agents have the same Cobb-Douglas utility function u^* and since the normalized utility gradients are equal, all final consumption vectors must necessarily lie on the same ray and $\hat{z}$ is the equilibrium trade for the environment e^* as depicted in figure 6. But in this case, because of monotonicity, agent k envies agent i contradicting the assumption that $\Pi = \langle M, \mu, h \rangle$ is a fair mechanism. Indeed, in the subclass E^* the only fair allocation is the equal income Walrasian one. This completes the proof.

Given a fair mechanism π, consider the binary relation T_π in E^C defined as follows: $e T_\pi \bar{e} \Leftrightarrow \mu(e) = \mu(\bar{e})$, that is, two environments are related if they have the same equilibrium messages. If the message correspondence is a single valued function, then T can be shown to be an equivalence relation which induces a partition of E^C into equivalence classes. Now we shall show that, if π is an informationally efficient mechanism, the partition of the class of environments E^C induced by the message correspondence μ is finer than that induced by the similarity relation S_π.

LEMMA 6.4. Let $\Pi = \langle M, \mu, h \rangle$ be an allocation mechanism such that

a) it is fair;

b) it is informationally decentralized;

c) the message space is of minimal dimension, that is, M is a $n\ell$-dimensional manifold.

d) the message correspondence μ is a continuous function on the class of Cobb-Douglas environments E^C.

Then $\mu(\bar{e}) = \mu(\hat{e})$ implies $\psi(\bar{e}) = \psi(\hat{e})$.

PROOF: Let $\bar{m} = \mu(\bar{e}) = \mu(\hat{e})$ and $\bar{z} = h(\bar{m})$. Since the process Π is assumed to be fair it must be the case that $\bar{\omega}^i + \bar{z}^i > 0$ and $\hat{\omega}^i + \bar{z}^i > 0$ for all agents. But for the class of Cobb-Douglas economies the utility gradients are always uniquely defined at interior points of the positive orthant. Since the process is informationally decentralized, $\bar{m}$ must also be an equilibrium message for any crossed economy because $\mu(\bar{e}) \cap \mu(\hat{e}) = \mu(\hat{e} \otimes_i \bar{e}) \cap \mu(\bar{e} \otimes_i \hat{e})$. But $\bar{m} \in \mu(\hat{e} \otimes_k \bar{e})$ for any agent k implies that the trade $\bar{z}$ must be fair for any crossed economy,

$\bar{z}=h(\bar{m})\in F(\bar{e})\cap F(\hat{e})\cap F(\hat{e}\otimes_i \bar{e})\cap F(\bar{e}\otimes_i \hat{e})$. In particular, we know that $\bar{z}$ is a Pareto optimal trade for $\bar{e}$ and therefore, there exists a price vector $\bar{p}$ such that the utility gradient of any agent i, $D\bar{u}^i(\bar{\omega}^i+\bar{z}^i)$, is proportional to $\bar{p}$.

But at the same time we know that for any agent k, $\bar{z}=h(\bar{m})\in F(\hat{e}\otimes_k \bar{e})$, and if $\bar{z}$ is Pareto optimal for $\hat{e}\otimes_k \bar{e}$ it must be the case that $D\hat{u}^k(\hat{\omega}^k+\bar{z}^k)$ is proportional to the other utility gradients $D\bar{u}^i(\bar{\omega}^i+\bar{z}^i)$, for $i\neq k$, which in turn are proportional to \o(‾;p). Hence, the normalized utility gradients of any agent at the final allocations, $\bar{\omega}^i+h^i(\bar{m})$ and $\hat{\omega}^i+h^i(\bar{m})$ respectively are equal to some $\bar{p}\in\Delta$. It remains to show that $\bar{r}^i=\bar{p}\bar{\omega}^i=\bar{p}\hat{\omega}^i=\hat{r}^i$ for all i. By Proposition 6.3,

$(\bar{p},\bar{z},\bar{r})=\psi(\bar{e})$ implies $(\bar{p},\bar{z},\bar{r})=\mu_\omega(\bar{e})$. Then it follows from (2) that

$\bar{p}\bar{z}^i=\frac{1}{n}\sum_{k=1}^{n}\bar{r}^k-\bar{r}^i$ and therefore $n\bar{p}\bar{z}^i=\sum_{k\neq i}\bar{r}^k-\bar{r}^i$. By a similar argument

$(\bar{p},\bar{z},\bar{r})=\psi(\hat{e}\otimes_i \bar{e})$ implies $n\bar{p}\bar{z}^i=\sum_{k\neq i}\bar{r}^k-\hat{r}^i$. Therefore $\bar{r}^i=\hat{r}^i$. Hence $\psi(\bar{e})=\psi(\hat{e})$ and

the proof is complete.

Since in lemma 6.2 it was shown that we establish the equality of both.

Lemmas 6.2 and 6.4 taken together imply that, if π is an informationally efficient mechanism, the partitions induced by the similarity relation S_π, the mechanism's message relation T_π and the equal income Walrasian message relation T_ω are identical. Now we shall use this fact to show that for the class of Cobb-Douglas economies E^c every decentralized mechanism with a minimal message space is essentially the same as the equal income Walrasian mechanism $\Pi_W=\langle M_W,\mu_W,h_W\rangle$ in the sense that it can be transformed into it by a homeomorphism φ as shown in figure 7.

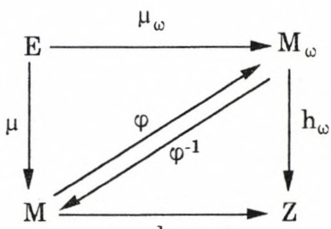

Figure 7

PROPOSITION 6.5. Let $\Pi=\langle M,\mu,h\rangle$ be an allocation mechanism such that

a) it is fair;

b) it is informationally decentralized

c) its message space is of minimal dimension, that is, M is a $n\ell$-dimensional manifold.

d) its message correspondence μ is a continuous function on the class of Cobb-Douglas environments E^c.

Then if $M=\mu[E^c]$, there exists a homeomorphism $\varphi: M \rightarrow M_W$ such that:

a) $\mu_W = \varphi \circ \mu$

b) $h_W \circ \varphi = h$.

PROOF: Let $\varphi: M \rightarrow M_W$ be the correspondence given by $\varphi(m) = \psi[\mu^{-1}(m)]$. From Lemmas 6.2 and 6.4 it follows that $\mu(\bar{e})=\mu(\hat{e})$ if and only if $\psi(\bar{e})=\psi(\hat{e})$. Since $\varphi[\mu(e)]=\psi(e)$ and $\varphi[\mu(E^c)]=M_W$, φ is a bijection.Finally, it remains to show that φ and φ^{-1} are continuous. In order to show that φ^{-1} is continuous let $\{(p^\nu, z^\nu, r^\nu)\}$ be a sequence in M_W converging to $(\bar{p}, \bar{z}, \bar{r})$. We have to demonstrate that the sequence $\{m^\nu\}$, where $m^\nu=\varphi^{-1}[(p^\nu, z^\nu, r^\nu)]$ converges to $\bar{m}=\varphi^{-1}[(\bar{p}, \bar{z}, \bar{r})]$. By taking

$$\left.\begin{aligned} \omega_j^{i\nu} &= \frac{r^{i\nu} + p^\nu z^{i\nu}}{p_j^\nu \ell} - z_j^{i\nu} \\ \alpha_j^{i\nu} &= 1 \end{aligned}\right\} \text{ for all } i, j \text{ and } \nu.$$

we define a sequence of economies $e^\nu=(\omega^\nu, \alpha^\nu)$ in the class E^* with the uniqueness property which converges to $\bar{e}=(\bar{\omega}, \bar{\alpha})$ given by

$$\left.\begin{aligned} \bar{\omega}_j^i &= \frac{\bar{r}^i + \bar{p}\bar{z}^i}{\bar{p}_j \ell} - \bar{z}_j^i \\ \bar{\alpha}_j^i &= 1 \end{aligned}\right\} \text{ for all } i, j \text{ and } \nu.$$

But by statement b of the present proposition $m^\nu=\varphi^{-1}[p^\nu, z^\nu, r^\nu]=\varphi^{-1}\left(\mu_f(e^\nu)\right)=\mu(e^\nu)$ for all ν and $m=\varphi^{-1}[\bar{p}, \bar{z}, \bar{r}]=\varphi^{-1}\left(\mu_f(\bar{e})\right)=\mu(\bar{e})$. Since μ is continuous, $\{e^\nu\}$ converges to $\bar{e}$ implies $\{m^\nu\}$ converges to $\bar{m}=\mu(\bar{e})$. Hence φ^{-1} is continuous. Since M and M_W are manifolds of the same dimension, φ^{-1} is in fact a homeomorphism[17] of M_W onto M and the proof is complete.

[17] See Greenberg [1967, 18.10, p.82].

7. The uniqueness theorem in the class of classical economies.

Now we can extend the uniqueness theorem to a wider class of environments. In particular, we shall consider a more general class of utility functions $\mathcal{U}$ which are real valued continuous, strictly monotone on $X(u^i)$ and strictly quasi-concave functions. Let E^g denote the general class of economies as defined at the beginning of section 2. The proof of assertion iii is almost immediate and ii follows easily from i. In steps 3 and 4 the proof relies on analogous techniques of finding appropriate similar economies in E^c.

UNIQUENESS THEOREM 7.1. Let $\Pi=\langle M,\mu,h\rangle$ be an allocation mechanism on E^g such that

a) it is fair;

b) it is informationally decentralized;

c) its message space is of minimal dimension, that is, M is a connected $n\ell$-dimensional manifold.

d) the restriction of μ to E^c is a continuous function .

e) $\mu[E^c]$ is a closed subset of the message space M.

Then there exists a homeomorphism $\varphi:M\rightarrow M_W$ such that:

i) $\varphi[\mu^i(e^i)]=\mu^i_W(e^i)$ for every agent i and every economy $e\in E^g$.

ii) $\varphi[\mu(e)]=\mu_W(e)$ for every economy $e\in E^g$.

iii) $h_W[\varphi(m)]=h(m)$ for all $m\in M$.

PROOF: The proof is organized in several steps.

Step 1. The Cobb-Douglas class E^c spans the whole message space, $\mu[E^c]$=M. By the same argument as in the beginning of step 2 of Lemma 6.2, given any $e\in E^c$ and any open neighborhood U_e of e, then $\mu(U_e)$ is open in M. Then $\mu[E^c]=\cup\{\mu(U_e): e\in E^c\}$ is the union of open sets and so it is open. But it is also closed by assumption. Since M is connected, it is the only open and closed set[18], so that $\mu[E^c]=M$.

Step 2. We claim that the homeomorphism $\varphi:M\rightarrow M_W$ constructed in Proposition 6.5 satisfies condition iii. But this is an immediate consequence of the result in Step 1 and the definition of the function φ.

[18] See Kelley [1955,p.53].

Step 3. We claim that given any economy $e \in E^g$ and any agent i, $\mu^i_\omega(e^i) \subseteq \varphi[\mu^i(e^i)]$. Let $(p, z, r) \in \mu^i_\omega(e^i)$. If $e^i = (u^i, \omega^i)$, define a new environment $\hat{e}$ given by $\hat{e}^k = (\hat{u}^k, \hat{\omega}^k)$, where

$$\hat{\omega}^k = \omega^i + z^i - z^k$$

$$\hat{u}^k = u^i$$

In the environment $\hat{e}$ all agents have identical utility functions and by Lemma 5.1 we know that the only fair allocation is equal division. Since by construction final consumptions are identical, $\hat{\omega}^k + z^k = \omega^i + z^i$ for all $k \in \{1,2,\ldots,n\}$, z is the equilibrium trade for the environment $\hat{e}$, that is, $z = h[\mu(\hat{e})]$. Moreover, if (p, z, r) is an equilibrium message of the equal income Walrasian mechanism Π_W, it must be true that $pz^k = \frac{1}{n}\sum_{s=1}^{n} r^s - r^k$ for all k. Using the definition of $\hat{\omega}^k$ given above we get

$$p\hat{\omega}^k = p\omega^i + pz^i - pz^k = r^i + \left(\frac{1}{n}\sum_{s=1}^{n} r^s - r^i\right) - \left(\frac{1}{n}\sum_{s=1}^{n} r^s - r^k\right) = r^k.$$

Hence we can write:

$$\begin{cases} z = h[\mu(\hat{e})] \\ p = \dfrac{D\hat{u}^k(\hat{\omega}^k + z^k)}{\|D\hat{u}^k(\hat{\omega}^k + z^k)\|} = \dfrac{Du^i(\omega^i + z^i)}{\|Du^i(\omega^i + z^i)\|} \\ r^k = p\hat{\omega}^k \text{ for all } k. \end{cases}$$

This shows that $(p,z,r) \in \psi(\hat{e})$. Let $\hat{m} \in \mu(\hat{e})$. Since the process is informationally decentralized and $e^i = \hat{e}^i$, $\hat{m} \in \mu^i(e^i)$ and $(p, z, r) = \varphi(\hat{m}) \in \varphi[\mu^i(e^i)]$.

Step 4. Given any economy $\bar{e} \in E^g$, and any agent we claim that $\varphi[\mu^i(\bar{e}^i)] \subseteq \mu^i_\omega(\bar{e}^i)$. Let $\bar{m} \in \mu^i(e^i)$ and $(p, z, r) = \varphi(\bar{m})$. We have to show that for every agent $(p, z, r) \in \mu^i_W(\bar{e}^i)$. In other words, we have to establish that

$$\begin{cases} \dfrac{D\bar{u}^i(\bar{\omega}^i + z^i)}{\|D\bar{u}^i(\bar{\omega}^i + z^i)\|} = p & (8) \\ p(\bar{\omega}^i + z^i) = \dfrac{1}{n}\sum_{k=1}^{n} r^k & (9) \\ r^i = p\bar{\omega}^i & (10) \end{cases}$$

so that $\bar{\omega}^i + z^i$ maximizes the utility function $\bar{u}^i$ subject to the average income constraint $p(\bar{\omega}^i + z^i) \leq \frac{1}{n}\sum_{k=1}^{n} r^k$. By the result in step 1 we know that there exists a Cobb-Douglas economy $\hat{e} \in E^C$ such that $\mu(\hat{e}) = \bar{m}$. By the uniqueness result of proposition 5.6 we know that $\mu_W(\hat{e}) = \varphi[\mu(\hat{e})] = \varphi(\bar{m}) = (p,z,r)$. Hence (p,z,r) is the only equilibrium message of the equal income Walrasian mechanism Π_W for the environment $\hat{e}$. By definition of the equal income Walrasian mechanism, Π_W, this implies for any agent k

$$
(11) \qquad \begin{cases} \dfrac{D\hat{u}^k(\hat{\omega}^k + z^k)}{\|D\hat{u}^k(\hat{\omega}^k + z^k)\|} = p \\[2ex] p(\hat{\omega}^k + z^k) = \dfrac{1}{n}\displaystyle\sum_{s=1}^{n} r^s \\[2ex] r^k = p\hat{\omega}^k \end{cases}
$$

For all agents (who have Cobb-Douglas utility functions) in the environment $\hat{e}$, all utility gradients at $(\hat{\omega}^k + z^k)$ are proportional to p because π is a fair mechanism and therefore z is a Pareto optimal trade for the economy $\hat{e}$. On the other hand, since $\bar{m} \in \mu^i(\bar{e}^i) \cap \mu(\hat{e})$ it follows from informational decentralization that $\bar{m} \in \mu(\bar{e} \otimes_i \hat{e})$. Again $z = h(\bar{m})$ must be a Pareto optimal trade for the environment $(\bar{e} \otimes_i \hat{e})$. Pareto optimality implies that the utility gradient of the i-th agent at $(\bar{\omega}^k + z^k)$ is also proportional to $\bar{p}$. Hence we have shown that (8) is satisfied.

In order to show (9) let us construct a Cobb-Douglas economy $\tilde{e} \in E^C$ as shown in figure 8 below. We define new Cobb-Douglas utility functions $\tilde{u}^k$ and initial endowments $\tilde{\omega}^k$ in such a way that the final consumptions $\tilde{\omega}^k + z^k$ lie on the same ray as $\bar{\omega}^i + z^i$, that the utility gradients at such points remain unchanged and that the supporting hyperplane is the same as before. Formally, define the characteristic of the k-th agent, $\tilde{e}^k = (\tilde{u}^k, \tilde{\omega}^k)$, as follows:

$$
(12) \qquad \tilde{\omega}^k = \frac{p(\hat{\omega}^k + z^k)}{p(\bar{\omega}^i + z^i)} (\bar{\omega}^i + z^i) - z^k
$$

$$
(13) \qquad \tilde{\alpha}_j^k = \frac{p(\hat{\omega}^k + z^k)}{p(\bar{\omega}^i + z^i)} \frac{\hat{W}}{\tilde{W}} \frac{\bar{\omega}_j^i + z_j^i}{\hat{\omega}_j^k + z_j^k} \hat{\alpha}_j^k
$$

where $\hat{W} = \prod_{j=1}^{\ell} (\hat{\omega}_j^k + z_j^k)$, $\tilde{W} = \prod_{j=1}^{\ell} (\tilde{\omega}_j^k + z_j^k)$ and $\hat{\alpha}_j^k$ is the corresponding parameter of the Cobb-Douglas utility function in the environment $\hat{e}$. It is easily verified that $\tilde{e} \in E^C$ since the Cobb-Douglas coefficient $\tilde{\alpha}_j^k$ is a positive real number and

$\sum_{k=1}^{n}\tilde{\omega}_j^k = n\frac{p(\hat{\omega}^k+z^k)}{p(\bar{\omega}^i+z^i)}\ (\bar{\omega}^i+z^i)>0$. From (12) and (13) it follows that the utility gradients are equal in both environments:

$$D_j\tilde{u}^k(\tilde{\omega}^k+z^k)=\frac{\tilde{\alpha}_j^k\tilde{W}}{(\tilde{\omega}^k+z^k)}=\frac{p(\hat{\omega}^k+z^k)}{p(\bar{\omega}^i+z^i)}\ \frac{\hat{W}}{\tilde{W}}\ \frac{\bar{\omega}_j^i+z_j^i}{\hat{\omega}_j^k+z_j^k}\ \frac{\hat{\alpha}_j^k\tilde{W}}{(\hat{\omega}^k+z^k)}=\frac{\hat{\alpha}_j^k\hat{W}}{(\hat{\omega}^k+z^k)}=D_j\hat{u}^k(\hat{\omega}^k+z^k)$$

Hence, using (11) we get the following equalities for all k:

$$\frac{D\tilde{u}^k(\tilde{\omega}^k+z^k)}{\|D\tilde{u}^k(\tilde{\omega}^k+z^k)\|}=p$$

$$p\tilde{\omega}^k=\frac{p(\hat{\omega}^k+z^k)}{p(\bar{\omega}^i+z^i)}\ p(\bar{\omega}^i+z^i)-pz^k=p\hat{\omega}^k=r^k.$$

$$p(\tilde{\omega}^k+z^k)=\frac{p(\hat{\omega}^k+z^k)}{p(\bar{\omega}^i+z^i)}\ p(\bar{\omega}^i+z^i)=p(\hat{\omega}^k+z^k)=\frac{1}{n}\sum_{s=1}^{n}r^s.$$

But this means that (p,z,r) is the equilibrium message of the equal income Walrasian mechanism, and using Proposition 6.5 we have that $(p,z,r)=\mu_W(\tilde{e})=\varphi[\mu(\tilde{e})]$. Hence $\hat{e}$ is similar to $\tilde{e}$ and by Proposition 6.2 $\mu(\hat{e})=\mu(\tilde{e})=\bar{m}$.

Then $\bar{m}\in\mu(\bar{e}\otimes_i\hat{e})$ and $z=h[\mu(\bar{e}\otimes_i\hat{e})]$. Now we show that it cannot be the case that $p(\bar{\omega}^i+z^i)<\frac{1}{n}\sum_{k=1}^{n}r^k$ since, in view of (11) $p(\hat{\omega}^k+z^k)=\frac{1}{n}\sum_{s=1}^{n}r^s$, and this would imply $p(\bar{\omega}^i+z^i)<p(\hat{\omega}^k+z^k)$. The situation would be analogous to that in figure 8. But then z is the equilibrium trade for the economy $\bar{e}\otimes_i\hat{e}$ and it is clear that agent i envies all other agents, contradicting the assumption that $\Pi=(M,\mu,h)$ is a fair mechanism. The only way to avoid this contradiction is that $p(\bar{\omega}^i+z^i)=\frac{1}{n}\sum_{k=1}^{n}r^k$ and this establishes (9).

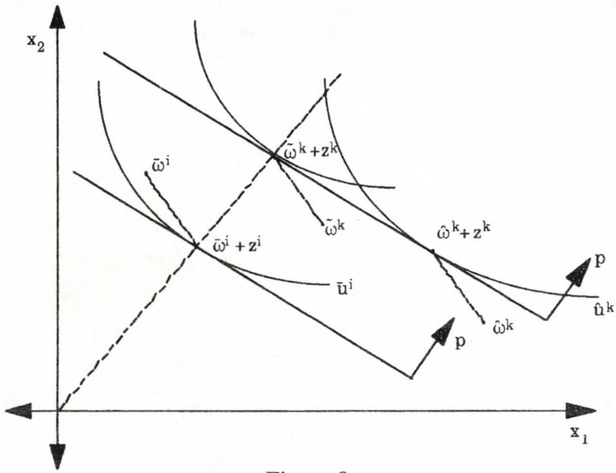

Figure 8

Finally it remains to show that (10) is satisfied. Since $\mu_W(\hat{e})=\varphi[\mu(\hat{e})]=\varphi(\bar{m})=(p,z,r)$, (p,z,r) is the only equilibrium message of the equal income Walrasian mechanism Π_W for the environment $\hat{e}$. Therefore it

satisfies $pz^i=\frac{1}{n}\sum_{s=1}^{n}r^s-r^i$.

From this it follows that $p\bar{\omega}^i=\frac{1}{n}\sum_{s=1}^{n}r^s-pz^i=\frac{1}{n}\sum_{s=1}^{n}r^s-\frac{1}{n}\sum_{s=1}^{n}r^s+r^i=r^i$ and the proof is complete.

References

Brock, W.A. and M.Majumdar (1988), "On Characterizing Optimal Competitive Programs in Terms of Decentralizable Conditions", *Journal of Economic Theory,* 45, 262-273.

Calsamiglia, X. (1977), "Decentralized Resource Allocation and Increasing Returns," *Journal of Economic Theory* , 14, 263-283.

Calsamiglia, X. (1982), "On the Size of the Message Space Under NonConvexities," *Journal of Mathematical Economics*, 10, 197-203.

Calsamiglia, X. (1987), "Informational Requirements of Parametric Resource Allocation Processes", in *Information, Incentives, and Economic Mechanisms* edited by T.Groves, R.Radner and S.Reiter, Minneapolis: University of Minnesota Press.

Champsaur, P. and G. Laroque (1981), "Fair allocations in large economies", *Journal o f Economic Theory*, 25, 269-282.

Chander, P. (1982), "On the Informational Efficiency of the Competitive Resource Allocation Process," *Journal of Economic Theory,* 31, 54-67.

Chander, P. (1983), "On the Informational Size of Message Spaces for Efficient Resource Allocation Processes", *Econometrica,* vol.51, No.4.

Crawford, V. (1987), "Fair Division", in *The New Palgrave Dictionary of Economics,* edited by J.Eatwell et al., London: Macmillan.

Dasgupta, S. and T.Mitra (1988), "Characterization of Intertemporal Optimality in Terms of Decentralizable Conditions: The Discounted Case", *Journal of Economic Theory,* 45, 262-273.

Dubins, L. and E. and Spanier (1961), "How to Cut a Cake Fairly", *American Mathematical Monthly,* 68, 1-17.

Feldman, A. and A.Kirman (1974), "Fairness, Equity and Envy", *American Economic Review,* 64, 995-1005.

Foley, D.K. (1967), "Resource Allocation and the Public Sector", *Yale Economic Essays* , 7.

Greenberg, M. (1967), *Lectures on Algebraic Topology*, New York: Benjamin.

Hammond P.J. (1979), "Straightforward individual incentive compatibility in large economies", *Review of Economic Studies*, 46, 263-282.

Hurwicz, L. (1969), "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences* , edited by Kenneth J. Arrow, Samuel Karlin, and Patrick Suppes, Stanford University Press, 1960: also in *Readings in Welfare Economics*, edited by K. J. Arrow and T. Scitovsky Irwin, 1969.

Hurwicz, L. (1977), "On the Dimensional Requirements of Informationally Decentralized Pareto-Satisfactory Processes," in *Studies in Resource Allocation Processes*, edited by K.J. Arrow and L. Hurwicz, New York: Cambridge University Press.

Hurwicz, L. (1986), "On Informational Decentralization and Efficiency in Resource Allocation Mechanisms" in *Studies in Mathematical Economics,* edited by S.Reiter, MAA Studies in Mathematics, vol. 25, The Mathematical Association of America, 1986.

Hurwicz, L. and M.Majumdar (1988), "Optimal Intertemporal Allocation Mechanisms and Decentralization of Decisions", *Journal of Economic Theory,* 45, 228-261.

Hurwicz, L. and T.Marschak (1985), “Discrete Allocation Mechanisms: Dimensional Requirements for Resource Allocation Mechanisms when Desired Outcomes are Unbounded”, *Journal of Complexity,* 1, 264-303.

Jordan, J.S. (1977), "Expectations Equilibrium and Informational Efficiency for Stochastic Environments", *Journal of Economic Theory* , 16, 354-372.

Jordan, J.S. (1982), “The competitive allocation process is informationally efficient uniquely”, *Journal of Economic Theory,* 28, 1-18.

Kelley, J.L. (1955), *General Topology,* Van Nostrand, Princeton.

Kleinberg N.L. (1980), "Fair allocations and equal income", *Journal of Economic Theory,* 23, 189-200.

Kolm, S.C. (1972), *Justice et équité,* Paris: Editions du C.N.R.S.

Mas-Colell A. (1985), *The Theory of General Economic Equilibrium: A Differential Approach,* Cambridge, Cambridge University Press.

Mas-Colell A. (1987), "On the second welfare theorem for anonymous net trades in exchange economies with many agents" in T. Groves, R. Radner and S. Reiter (eds.), *Information Incentives and Economic Mechanisms: Essays in Honor of Leonid Hurwicz,* Minneapolis, University of Minnesota Press.

Maskin, E. (1977), “Nash Equilibrium and Welfare Optimality”, *Mathematics of Operations Research,* (forthcoming).

Mount, K. and S. Reiter (1974), “The Informational Size of Message Spaces,” *Journal of Economic Theory,* 8,161-192.

Osana, H. (1978), "On the Informational Size of Message Spaces for Resource Allocation Processes," *Journal of Economic Theory* , 17, 66-78.

Pazner E. and Schmeidler D. (1974), "A difficulty in the concept of fairness", *Review of Economic Studies,* 41, 441-3.

Reichelstein, S. (1984), "Dominant strategy implementation, incentive compatibility and informational requirements", *Journal of Economic Theory,* 34 (1), October, 32-51.

Reichelstein S. and Reiter S. (1988), "Game forms with minimal strategy spaces", *Econometrica,* 56, 3, 661-692.

Sato, F. (1981), "On the Informational Size of Message Spaces for Resource Allocation Processes in Economies with Public Goods," *Journal of Economic Theory* , 24, 48-69.

Schmeidler, D. and K.Vind (1972), “Fair Net Trades”, *Econometrica* 40, 637-647.

Thomson, W. (1979), “On Allocations Attainable Through Nash Equilibria, a Comment”, in *Aggregation and revelation of Preferences,* edited by J.J.Laffont; Amsterdam: North Holland, pp. 420-431.

Thomson, W.,(1982), "The Manipulation of Mechanisms Designed to Select Equitable and Efficient Allocations", Harvard Institute of Economic Research Discussion Paper no. 909.

Thomson, W. and H.Varian (1985), "Theories of Justice Based on Symmetry" in *Social Goals and Social Organization: Essays in Honor of Elisha Pazner,* edited by L.Hurwicz et. al., New York: Oxford University Press.

Tillmann G. (1989), *Equity, Incentives, and Taxation*, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.

Varian H.R. (1976), "Two problems in the theory of fairness", *Journal of Public Economics*, 5, 249-260.

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