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The Dynamics of Learning in Mis-Specified Models

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The Dynamics of Learning in Mis-Specified Models

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THE DYNAMICS OF LEARNING IN MIS-SPECIFIED MODELS.

April 1991

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Abstract

In this paper we study an example in which agents misspecify the game that they are playing. We show that there is a class of self-sustaining equilibria which are perfectly consistent with the agents' beliefs and which are achieved by O.L.S. learning in a trivial way. We show that only the equilibria found in an earlier paper by one of us can be sustained by O.L.S. learning. Lastly, we show that if we make a continuous time approximation, then all these equilibria are unstable.

[1] I would like to thank participants in the the CORE mathematical economics seminar, in the Indian Statistical Institute conference on "Game theory and economic applications", and the University of Venice seminar, for helpful comments.

THE DYNAMICS OF LEARNING IN MIS-SPECIFIED MODELS.

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Introduction.[2]

It is a truism that agents can never fully specify the environment in which they operate. They must then function with limited and misspecified models. With some such model in mind they may try to learn from their experience about its precise specification. Learning, of course, only makes sense in a dynamic setting, and in a dynamic model two important complications can arise. Firstly the outcomes which agents observe, and from which they learn, are influenced by their actions. Secondly, these actions themselves may be influenced by the learning process. Thus two types of feed-back may interfere with the inferences that agents make about the world in which they operate. The argument made in previous papers by one of us, (Kirman (1975, 1983)) was, in some ways, analogous to that made in the "sun-spots" literature. There, by conditioning on some "irrelevant" events, agents come to believe, and, indeed, their beliefs are confirmed by their observations, that the evolution of the economy is affected by those events, (see e.g. Azariadis (1981), Cass and Shell (1983), Grandmont (1989) and Guesnerie (1986)). In Kirman (1975, 1983), agents specify a model which also does not correspond to the "true" one, but, in the long run, their forecasts are verified by their observations. Thus they have no reason to change their vision of the world. The very process of learning leads to self-fulfilling expectations. The main result was that, in the context of the simple model considered, there is a whole class of final "equilibria" which can be attained in this way, but which have no relationship, in general, to the "true" equilibria, though including the latter, as special cases.

Two questions remained open, however. Firstly the agents, who were using least squares to estimate the parameters

[1] I would like to thank participants in the the CORE mathematical economics seminar, in the Indian Statistical Institute conference on "Game theory and economic applications", and the University of Venice seminar, for helpful comments.
[2] This paper pursues the analysis started elsewhere (see Arrow and Green (1974), Gates et al. (1977, 1978, 1981), Kirman (1975, 1983)).

of their mistaken model, might have been puzzled by the behaviour of the residuals. Since the errors were generated by an omitted variable, these do not correspond to the white noise usually specified when using least squares estimation. In particular in the examples constructed all errors after the initial three periods were zero. This could be solved by adding white noise to the "true" model and after several periods the residuals behaved exactly as they were expected to do. In fact, incompatibility between observed behaviour and the model specified is common to many models with learning. (see Bray (1983) for example). Indeed it is now frequently argued that least squares learning, for example, is simply a "reasonable" learning procedure and no reference is made to the statistical properties of the estimators nor to the assumptions about the structure of the errors. However, one might ask, within the context of the particular model under consideration, if there are not final states, which could be attained without the agents making any error whatsoever. In the first part of this paper we give a positive answer to this question for a simple mis-specified duopoly model.

A second problem is that it was not proved in Kirman (1983) that there was convergence to a "rational expectations" situation from *any* initial conditions. However repeated simulations seemed to show that such convergence did, in fact, occur. In the second part of this paper we show that this is not the case and prove that, in general, the learning process does *not* converge. The apparent convergence is due to the fact that, as the memory of those learning increases, the evolution of the process is slowed down since each successive observation gets less and less weight. Thus, although the trajectory is not convergent, progress along it becomes steadily slower. We then give an example of what happens if memory is restricted to a finite number of observations. In particular it is then shown that, in this case, the behaviour of the process is qualitatively different from the infinite case since cycles can occur whilst in the latter this is impossible. Thus the work in this paper is clearly related to recent work by Nyarko (1990).

The lack of convergence, in both the finite and infinite case, casts doubt on earlier assertions of Sargent (1987) that, as a matter of practical concern, econometricians can afford to ignore the learning process that precedes the establishment of a rational expectations equilibrium. The very fact that agents are trying to learn about what they believe to be a stationary environment may generate highly non-stationary and non-convergent dynamics. Thus our results are in direct contrast with those of Bray (1981), Bray and Savin (1986), Fourgeaud, Gourieuroux, and Pradel (1986) and more recently, Marcet and Sargent (1989) all of whom show how simple learning will lead to a rational expectations equilibrium. Woodford (1990), on the other hand, shows that learning may lead to an equilibrium where originally erroneous beliefs become self-fulfilling. Our results are even less comforting however, for in our simple model, agents will, in general, never learn to believe in anything.

A simple model

We consider a symmetric duopoly in which the demand functions for firms 1 and 2 are given by the "true model":

$$d_1[p_1(t),p_2(t)] = \alpha - \beta\, p_1(t) + \gamma\, p_2(t) \tag{1}$$

$$d_2[p_1(t),p_2(t)] = \alpha - \beta\, p_2(t) + \gamma\, p_1(t) \tag{2}$$

We also assume, in the tradition of Cournot, that production is costless, and hence the goal for each of the two firms if they knew the true model would be to maximize revenue given the price of the opponent.

We suppose, as in Kirman (1983), that the two firms, through ignorance or inertia, are unaware that their demand depends on each other's actions. In a duopoly situation, such an assumption is difficult to accept, but it is more plausible in a several firm situation in which each firm feels unable to take explicit account of the behaviour of all the opponents and hence focuses on the "own-price" demand curve or on a demand curve involving only some of the prices of its opponents and adds a random term to take account of the, to him, unpredictable behaviour of the other firms. Indeed, it changes nothing in our results to consider an *n*-firm model, as do Gates and associates (1977, 1978).

The two firms will thus have the following "perceived model":

$$d_1[p_1(t)] = \alpha_1 - b_1 p_1(t) + \varepsilon_1(t) \tag{3}$$

$$d_2[p_2(t)] = \alpha_2 - b_2 p_2(t) + \varepsilon_2(t) \tag{4}$$

Each of the firms might for example assume, at least initially, in the standard econometric tradition, that the error terms are normally distributed with mean zero; that is,

$$\varepsilon_i(t) \sim N(0, \sigma_i^2)$$

If the firms have no information about their respective parameters a_i and b_i, how should or would they set about trying to establish their true values? Kirman (1975) showed that if both firms knew, or rather believed with certainty, that $b_i = \beta$, than a reasonable Bayesian learning process would lead them, despite their misspecification of the model, to the Cournot equilibrium of the "true" game.

However, as in Kirman (1983), we are concerned here with the case in which the firms have no information about the values of the parameters and try to learn from experience about them. How will the model then evolve? At each period, given the quantities observed at each price, the firms will make their estimates of the two parameters of their perceived demand curves, and we shall call these estimates $\hat{a}_i(t)$ and $\hat{b}_i(t)$. If we assume that each firm simply wishes to maximize expected profit in the next period[3], then, given the assumption that $E[\varepsilon_i(t)] = 0$, the optimal price is clearly given by

$$p_i(t) = \frac{\hat{a}_i(t)}{2\hat{b}_i(t)} \tag{5}$$

So, at each period, given its estimates, each firm will charge a price, and the demand realized as a result of these prices will, of course, be given by the true model specified by equations (1) and (2). This new observation of a price-quantity pair will lead to a revision of the estimates of the parameters and, in turn, to new prices and so forth.

As mentioned, a reasonable learning process would make the sequence of prices converge to

$$p_1^* = p_2^* = \frac{\alpha}{2\beta - \gamma}$$

[3] If firms wish to maximise the discounted sum of all future profits then an interesting problem arises. Will it not be worth charging widely dispersed prices at the outset to gain more information at the cost of a loss of short run profit? This problem is discussed in Balvers and Cosimano (1990) and Easley and Kiefer (1988) and Kiefer and Nyarko (1989) but is not solved when both players try to adopt this approach.

the Cournot solution, if for some reason the firms were to set $\hat{b}_i(t) = \beta$ for all t.

A sensible procedure for the case where there is ignorance of both the parameters is for each firm to try to fit the observed data by means of least squares[4]. The model we consider is a special case of that developed by Gates, Rickard and Wilson (1977),[5] in which they allow for n firms and variable weights for preceding observations. They were obliged, however, to confine their attention to particular cases to obtain analytic results.

In our particular model, we can specify the ordinary least square estimates for a_i and b_i as follows:

$$\hat{b}_i(t) = -\frac{\sum_{k=1}^{t-1}[d_i(k) - \bar{d}_i(t)][p_i(k) - \bar{p}_i(t)]}{\sum_{k=1}^{t-1}[p_i(k) - \bar{p}_i(t)]^2} \tag{6}$$

and

$$\hat{a}_i(t) = \hat{d}_i(t) + \hat{b}_i(t)\bar{p}_i(t) \quad (i = 1, 2) \tag{7}$$

where

$$\bar{d}_i(t) = \frac{\sum_{k=1}^{t-1} d_i(k)}{t-1} \quad \text{and} \quad \bar{p}_i(t) = \frac{\sum_{k=1}^{t-1} p_i(k)}{t-1}$$

It is helpful to observe that (6) can be rewritten for firm 1, for example, as

$$\hat{b}_i(t) = \beta - \gamma \frac{\sum_{k=1}^{t-1}[p_i(k) - \bar{p}_i(t)][p_2(k) - \bar{p}_2(t)]}{\sum_{k=1}^{t-1}[p_1(k) - \bar{p}_1(t)]^2} \tag{8}$$

and symmetrically for firm 2.

We see clearly then precisely where the "misbehaviour" in the system comes from. The second term in (8) is the covariance of the prices or the bias due to the omission of a variable which is correlated with one of the included variables and, as a result, is

[4]This behaviour on the part of firms can be justified from a Bayesian point of view; see, for example, Zellner (1971). A consideration of general forms of updating and learning that include this type of process can be found in Aoki (1976).

[5] Referred to for convenience as G.R.W.

familiar to econometricians. Indeed, it is precisely the fact that the prices are related that generates problems in the evolution of the system.

That the whole system is recursive is evident, because at each period each firm sets

$$p_i(t) = \frac{\hat{a}_i(t)}{2\hat{b}_i(t)} \tag{9}$$

Hence, from the equation for the true demand (1), we have for firm 1, for example,

$$p_1(t) = \frac{[\alpha + \gamma\,\bar{p}_2(t)\sum_{k=1}^{t-1}[p_1(k) - \bar{p}_1(t)]^2 \;\; - \gamma\,\bar{p}_1(t)\sum_{k=1}^{t-1}[p_i(k) - \bar{p}_1(t)][p_2(k) - \bar{p}_2(t)]}{2\beta\sum_{k=1}^{t-1}[p_1(k) - \bar{p}_1(t)]^2 - \gamma\sum_{k=1}^{t-1}[p_1(k) - \bar{p}_1(t)][p_2(k) - \bar{p}_2(t)]} \tag{10}$$

and symmetrically for firm 2.[6]

This recurrence relation is a special case of that given by G.R.W. (1977). It is apparent that even in this form it is not a trivial matter to establish whether convergence does or does not occur.

Self-sustaining Equilibria

We first observe that there are situations in which if the estimates $\hat{a}_i$ and $\hat{b}_i$ of the two players for their parameters take on particular values they will never move again.

Define self-sustaining equilibrium parameter values for given p_1^* and p_2^* as

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[6] Thus, the underlying data-generating process is a nonlinear dynamic model. The evolution of the estimators and in consequence the prices can also be seen by considering updating formulas and we will come back to this in studying the general dynamics of the system.

$$b_1^* = -\beta + \frac{\alpha + \gamma p_2^*}{p_1^*}$$

$$b_2^* = -\beta + \frac{\alpha + \gamma p_1^*}{p_2^*}$$

and

$$a_1^* = 2(\alpha - \beta p_1^* + \gamma p_2^*)$$

$$a_2^* = 2(\alpha - \beta p_2^* + \gamma p_1^*)$$

Clearly if at some $\bar{t}$ it is the case that

$$\hat{a}_i(\bar{t}) = a_i^* \text{ and } \hat{b}_i(\bar{t}) = b_i^*$$

then $p_i(t) = p_i^*$ for i=1, 2 and all $t \geq \bar{t}$.

Now we ask: for which of these equilibria can we find initial conditions such that the least squares learning procedure will subsequently converge to values of the parameters, in the set defined above?

This question was answered in Kirman (1983) as follows.

Theorem

If for p_1^* and p_2^*, the parameters a_1^*, a_2^*, b_1^* and b_2^* defined in (1) and (2) satisfy

$$\gamma^2 \geq (\beta - b_1^*)(\beta - b_2^*) \geq 0$$

then there exist $p_1(1), p_1(2), p_1(3)$ and $p_2(1), p_2(2), p_2(3)$ such that $\tilde{a}(4) = a_1^*$

$$\hat{b}_1(4) = b_1^* \quad \hat{a}_2(4) = a_2^* \text{ and } \hat{b}_2(4) = b_2^*$$

Hence $p_1(t) = p_1^*$ and $p_2(t) = p_2^*$ for all $t \geq 4$.

Thus there is a whole set of self-sustained equilibria which can be attained through least square learning and there is clearly dependence on initial conditions.

An objection to this result is that the alert observer would be surprised by the sudden disappearance of the error terms: This can be overcome by increasing the number of initial conditions and indeed so ordering them that the errors diminish over time. If the actor then employed generalised least squares with a decreasing variance for the error term they would then be less surprised if not fully convinced, since after some finite time the errors would still disappear. All this, of course, presupposes that the agent made specific assumptions about the nature of the stochastic error process. This is not necessary. A second observation is that it is easily shown that adding extra initial conditions does <u>not</u> enlarge the set of equilibria that can be attained.

A simpler question then is, can we find initial conditions leading to self-sustained equilibria which can be directly arrived at with no error? This would be the case if all observations of price and demand lay on a linear demand curve $a_i - b_i p_i$ and where the choice of the optimal price $p_i = \frac{a_i}{2b_i}$ gave rise to demands lying on the same curve.

The answer is that this is indeed the case, and to give a clear statement of the result let us introduce the following notation.

i) Denote by D_1 (resp. D_2) the domain of the price demand couples p_1, d_1 (resp. p_2, d_2) of the first (resp. second) firm, i.e. $D_1 = D_2 = R_{++}xR$.

ii) Denote by P the domain of the strictly positive price pairs (p_1, p_2)
i.e. $P = R^2_{++}$.

iii) Denote by N_1 (resp. N_2 resp. M) the current observation in D_1 (resp. D_2 resp. P). Thus we shall refer to $N_1(t)$ for example.

iv) It will be useful to identify three lines and two particular points in P.

Denote by p' the line

$$\alpha + \gamma y = \beta x \text{ in } P$$

and by p'' the line

$$\alpha + \gamma x = \beta y \text{ in } P$$

and by p''' the 45° line i.e. the line given by x=y and by $\overline{M}$ the point

$$\left(\frac{\alpha}{\beta-\gamma}, \frac{\alpha}{\beta-\gamma}\right)$$

Clearly M is the intersection of the three lines p', p'' and p'''

and finally by G the point

$$\left(\frac{\alpha}{2(\beta-\gamma)}, \frac{\alpha}{2(\beta-\gamma)}\right)$$

Now for any line p in P if all the observations M(1), M(2).... M(T-1) are on p then the observations $N_i(t)$ for t=1, ... T-1 lie on a line in D_i. Thus the firm perceives a demand curve of the form

$$d_i = a_i - b_i p_i.$$

Clearly we must confine our attention to the case where the parameters a_i and b_i are strictly positive.

In this case the profit functions are concave and have a unique maximum at the price $\frac{a_i}{2b_i}$.

The point M(T) will then be the point

$$\left(\frac{a_1}{2b_1}, \frac{a_2}{2b_2}\right).$$

The question is will, or can, this point lie on the line p? The answer is given by the following

Theorem 1

Assume $\beta \neq \gamma$ then for any point p in P-($p'Up''$) there is at least one line Δ(p) with positive slope $0<b<\infty$ such that the point $\left(\frac{a_1}{2b_1}, \frac{a_2}{2b_2}\right)$ is also on Δ(p).

The calculation of Δ (p) and the proof are given in the mathematical appendix with the proof of the other theorems.

The next question is which price pairs can be attained as the limit of a sequence of aligned prices? Such an equilibrium can be called a perfectly self-sustaining equilibrium.

The answer is given by

Theorem 2

Assume $\beta \neq \gamma$ then the set E of points $\left(\frac{a_1}{2b_1}, \frac{a_2}{2b_2}\right)$, where a_i, a_2, b_1, b_2 are strictly positive, which are perfect self-sustaining equilibria is a continuum of dimension 1 in P and satisfies the equation $2\beta(\gamma(x^2 + y^2) - 2\beta xy) - \alpha(\gamma - 2\beta)(x + y) - \alpha^2 = 0$

Given the constraints, the relevant part of this hyperbole is that between the lines p' and p'' defined previously and which cuts p''', the diagonal, at the point G.

Thus the joint monopoly or cooperative solution can be attained in this way whilst the Cournot solution cannot.

It is easy to check that if both firms charge the same prices i.e. if for all t

$$P_1(t) = P_2(t)$$

then for $t \geq 3$

$$P_1(t) = P_2(t) = \frac{\alpha}{2(\beta - \gamma)}$$

i.e. the cooperative solution. Thus unconscious imitation will lead to cooperation.

Least squares learning: Dynamics

Till now we have considered special sequences of prices which would lead directly, i.e. after two or three observations, to self-sustaining equilibria. The question now is would be general process of least squares learning converge from any initial conditions? It was conjectured in Kirman (1983) that this would be the case for two reasons. Firstly, the set of attainable equilibria was independent of the number of initial observations and secondly numerous simulations showed apparent convergence. In the "non-aligned" case what characterises convergence is that the empirical variance-covariance matrix converges. In the case described in Theorem 1 this matrix only changes in modulus. Thus, the process can be thought of as having a fixed direction. What we can show, however, is that the only two cases in which the process converges are those which correspond to those obtained starting from the initial conditions of the "aligned" type or from those constructed in Theorem 1. Thus somewhat informally we have

Theorem 3

If the least squares learning procedure converges to stationary values of p_1 and p_2 then these must be attained from the initial conditions constructed in Theorems 1 and 2.

Before proceeding to the proof of Theorem 3, we shall introduce some useful notation.

$$x(t_o) = \frac{1}{t}\sum_{t=1}^{t_o-1} p_1(t) \tag{11}$$

$$y(t_o) = \frac{1}{t}\sum_{t=1}^{t_o-1} p_2(t) \tag{12}$$

$$u(t_o) = \frac{1}{t}\sum_{t=1}^{t_o-1} p_1^2(t) \tag{13}$$

$$v(t_o) = \frac{1}{t}\sum_{t=1}^{t_o-1} p_1(t)p_2(t) \tag{14}$$

$$w(t_o) = \frac{1}{t}\sum_{t=1}^{t_o-1} p_2^2(t) \tag{15}$$

$$U(t_o) = u - x^2 \tag{16}$$

$$V(t_o) = v - xy \tag{17}$$

$$W(t_o) = w - y^2 \tag{18}$$

The recursive price process can then be written

$$p_1(t_o) = \frac{1}{2}\,\frac{\alpha\left(u - x^2\right) + \gamma\,(yu - xv)}{\beta\left(u - x^2\right) - \gamma\,(v - xy)} = \frac{1}{2}\,\frac{\alpha U + \gamma\,(yU - xV)}{\beta U - \gamma V} \tag{19}$$

$$p_2(t_o) = \frac{1}{2}\,\frac{\alpha\,(w - xy) + \gamma\,(xw - yv)}{\beta\left(w - y^2\right) - \gamma\,(v - xy)} = \frac{1}{2}\,\frac{\alpha W + \gamma\,(xW - yV)}{\beta W - \gamma V} \tag{20}$$

We add two further definitions:

$$\begin{aligned} g_1(t_o) = p_1(t_o) - x &= \frac{1}{2}\,\frac{(\alpha - 2\beta x)\left(u - x^2\right) + \gamma\,(yu + xv + 2x^2 y)}{\beta\left(u - x^2\right) - \gamma\,(v - xy)} \\ &= \frac{1}{2}\,\frac{(\alpha + \gamma y - 2\beta x)U + \gamma x V}{\beta U - \gamma V} \end{aligned} \tag{21}$$

$$\begin{aligned} g_2(t_o) = p_2(t_o) - y &= \frac{1}{2}\,\frac{(\alpha - 2\beta y)\left(w - y^2\right) + \gamma\,(yv + xw + 2xy^2)}{\beta\left(w - y^2\right) - \gamma\,(v - xy)} \\ &= \frac{1}{2}\,\frac{(\alpha + \gamma x - 2\beta y)W + \gamma y V}{\beta W - \gamma V} \end{aligned} \tag{22}$$

The two possible types of equilibrium

It is clearly sufficient to study the five variables x, y, U, V and W to determine the dynamics of the system. The five vector (x, y, U, V, W) at time t depends only on its values at time t-1. In fact we have

$$\begin{bmatrix} x \\ y \\ U \\ V \\ W \end{bmatrix}(t+1) = \begin{bmatrix} x \\ y \\ U \\ V \\ W \end{bmatrix}(t) + \frac{1}{t+1}\begin{bmatrix} g_1 \\ g_2 \\ g_1^2 - U \\ g_1 g_2 - V \\ g_2^2 - W \end{bmatrix}(t) \tag{23}$$

Note that two types of equilibria are possible, either those in which the five vector is unchanged or, alternatively, those which constitute an equilibrium of the dynamical system (23) in which prices do not change but the associated statistics continue to do so.

Recalling the previous notation $M = (p_1, p_2)$, we will write $\overline{M} = (x,y)$ where x,y are as defined in (11) and (12), i.e. the vector of average points at a given time.

Denote by T the empirical variance-covariance matrix of the M(t), i.e.

$$\mathrm{T} = \begin{bmatrix} U & V \\ V & W \end{bmatrix} \tag{24}$$

If T is non-null, we can think of $\mathfrak{T}$ the set of scalar multiples of T as the "direction" of T. Thus we can write

$$T = (\lambda, \mathfrak{T})$$

We observe that the functions $P = (p_1(\cdot), p_2(\cdot))$ and $G = (g_1(\cdot), g_2(\cdot))$ are homogeneous of degree 0 in T and can thus be written

$$P = P(\overline{M}, \mathfrak{T}) \text{ and } G = G(\overline{M}, \mathfrak{T})$$

Now, it is easily shown that

$$G(\overline{M}, \mathfrak{T}) = 0 \tag{25}$$

defines $\mathfrak{T}$ as an implicit function of $\overline{M}$ and hence we can write

$$\mathfrak{T} = H(\overline{M}) \tag{26}$$

Thus for any fixed $\overline{M}$ there is a unique associated direction. With this in mind we can now proceed to the

Proof of Theorem 3

Note that the system of equations (23) defines the way in which T and hence $\mathfrak{T}$ evolves. Since we have

$$T(t+1) - T(t) = \frac{1}{t+1} G^T G \tag{27}$$

where G^T is the transpose of G. $\mathfrak{J}$ will be unchanged in only two cases:

(i) if G is null or
(ii) if $G^T G$ is in $\mathfrak{J}$.

We now examine each of the cases in turn.

Equilibria corresponding to (i)

Solving the equation G=0 gives the following

$$(\alpha + \gamma\, y - 2\beta\, x)U + \gamma\, xV = 0 \tag{28}$$

$$(\alpha + \gamma\, x - 2\beta\, y)W + \gamma\, yV = 0 \tag{29}$$

These are precisely the conditions given in Kirman [1983], thus in this case the variance-covariance matrix does not degenerate. It is also interesting to note that any initial conditions M(t) which result in a variance covariance matrix satisfying (28) and (29) would result in such an equilibrium and these might be quite different from those constructed by Kirman [1983]. However, it should also be noted that increasing the number of initial conditions does not enlarge the size of the equilibrium since all the equilibria satisfying (28) and (29) can be generated from three initial conditions.

Equilibria corresponding to (ii)

In this case G is non-null and $G^T G$ defines the limit value of the direction $\mathfrak{J}$. Since $G^T G$ is of rank one, it is clear that the equilibria in question are those associated with a sequence of points on a line in D_1.

To see this, replace U, V, W by $g_1^2, g_1 g_2, g_2^2$ respectively and one can see that the direction $\mathfrak{J}$ is not changed if and only if

$$g_1 = \lambda \frac{(\alpha + \gamma y - 2\beta x)g_1 + \gamma x g_2}{2(\beta g_1 - \gamma g_2)} \tag{30}$$

$$g_2 = \lambda \frac{(\alpha + \gamma x - 2\beta y)g_2 + \gamma y g_1}{2(\beta g_2 - \gamma g_1)} \tag{31}$$

where λ is a real number. $\lambda = 0$ corresponds to the previous case. It is easy to see that the equations can be solved simultaneously if and only if the determinant of the following matrix is zero

$$\begin{vmatrix} 2g_1(\beta g_1 - \gamma y_2) & (\alpha + \gamma y - 2\beta x)g_1 + \gamma x g_2 \\ 2g_2(\beta g_2 - \gamma y_1) & (\alpha + \gamma x - 2\beta y)g_2 + \gamma y g_1 \end{vmatrix} \tag{32}$$

Writing out the determinant gives

$$\beta\gamma y g_3^1 + (\beta\alpha - \beta\gamma x - 2\beta^2 y + \gamma\alpha)g_1^2 g_2$$
$$+(\beta\gamma y - \gamma\alpha - \beta\alpha + 2\beta^2 x)g_1 g_2^2 + (-\beta\gamma\alpha)g_2^3$$

But this is, of course, the polynomial identified previously, and this completes the proof of Theorem 3.

It is important to emphasise a difference between the two cases. In the case of aligned equilibrium the variance covariance is degenerate whereas in the other case it is of rank 2 but only varies by a constant multiple. This reflects the fact that in one case the prices and statistics do not change whilst in the other, although the prices are fixed, the statistics still evolve.

<u>Behaviour of the system out of equilibrium</u>

The apparent convergence of the model in general is due simply to the fact that although prices do not converge in general the speed along the trajectory in P slows down progressively as the length of the memory increases. Thus each additional observation changes the estimates, and hence the prices, very little. However as we will see, convergence does not occur in general and this can be observed by giving geometrically declining weights to previous observations or by truncating the memory.

Given this, though, we can still ask the following question: if we perturb the initial conditions leading to a self-sustained equilibrium slightly, does the learning process still converge to that equilibrium?

The answer unfortunately is no and we have

Theorem 4

Self-sustained equilibrium prices are unstable

Before proceeding to the proof some observations are in order. Theorem 3 showed that only the "aligned" equilibrium just defined or those defined in Kirman (1983), referred to from now on as "dispersed" equilibria, can be attained. If we now use a continuous time approximation we can study the stability of these equilibria and show that the direction of the trajectory at any of the "dispersed equilibria" must be a straight line through that point. This means that in the continuous time approximation the "dispersed" equilibria cannot be attained at all, since the learning process which leads to one of these equilibria must "carry" it through that point. The "aligned" equilibria are clearly unstable, since it is always possible to perturb the initial conditions a little so as to attain a new equilibrium point on the hyperbole defined previously.

We have already mentioned the problem that the system slows down over time because of the diminishing effect of each subsequent observation. This can clearly be seen from the presence of the factor $\frac{1}{t+1}$ in equations (27).

Suppose now that we consider t large and try to calculate the solutions to the system (19)-(20) replacing t by log t. This amounts to giving less weight to earlier observations.

Solving this problem by numerical integration is equivalent to solving the system of differential equations given by

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ U \\ V \\ W \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_1^2 - U \\ g_1 g_2 - V \\ g_2^2 - W \end{bmatrix} \tag{33}$$

We will study the behaviour of this system in order to establish the

Proof of Theorem 4.

Using then (33) as an approximate model for the behaviour of the system (23), we can write

$$\frac{d}{dt}\begin{bmatrix} \overline{M} \\ T \end{bmatrix} = \begin{bmatrix} G \\ G^T G - T \end{bmatrix} = \begin{bmatrix} G \\ S(\lambda) Z(\overline{M}, \mathfrak{I}) \\ -T \end{bmatrix} \tag{34}$$

where s is a real valued function and Z for a given $\overline{M}$ is a vector field on the manifold of directional matrices. This field has a unique zero root at

$\mathfrak{I} = H(\overline{M})$.

Consider the first type of equilibrium in which G=0. In this case $\overline{M}$ and $\mathfrak{I}$ must remain constant over time. This is possible if and only if $\mathfrak{I} = H(\overline{M})$. Since the coordinate λ decreases over time, we have only a partial solution to the dynamic system.

Suppose now that we define the elements of $\mathfrak{I}$ as T/trace T. The parameter λ is therefore simply given by

λ = trace T

The system of differential equations can now be written

$$\frac{d}{dt}\begin{pmatrix}\overline{M}\\ \vartheta\\ \lambda\end{pmatrix}=\begin{pmatrix}G\\ G^{1}G-\mathfrak{I}G^{2}\\ G^{2}-\lambda\end{pmatrix} \tag{35}$$

where we recall that G maps $\overline{M}$ and $\mathfrak{I}$ into $\mathfrak{R}^2$. $G^T G$ is a 2x2 matrix and G^2 a scalar.

For a given $\overline{M}_0$ consider $\mathfrak{I}_0 = H(\overline{M}_0)$. Clearly $G(\overline{M}_0\mathfrak{I}_0) = 0$ and $(\overline{M}_0, \mathfrak{I}_0)$ is a solution of our problem.

In a neighbourhood of $\mathfrak{I}_0, \mathfrak{I}$ can be written

$$\mathfrak{I} = \mathfrak{I}_0 + M\begin{pmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} \tag{36}$$

where M is a real number and α an angle. We can also rewrite G in a neighbourhood of $G_0 = 0$ as

$$G = \begin{pmatrix}g\cos\theta\\ g\sin\theta\end{pmatrix} = g\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{37}$$

where g is a real number and θ an angle.

In fact the manifold formed by the $\mathfrak{I}$ has at $\mathfrak{I}_0$ a tangent Euclidean space TS for which the 2 matrices

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \text{ and } \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

form a base.

G is then differentiable in $\mathfrak{I}$ and is equal to 0 at $\mathfrak{I}_0$. Thus G is first order linear in $\mathfrak{I} - \mathfrak{I}_0$. This can be written

$$\text{G}= L \times (\mathfrak{I} - \mathfrak{I}_0) + \varepsilon(\mathfrak{I} - \mathfrak{I}_0) \tag{38}$$

where L is a linear mapping from the tangent space TS into the space of the G. L has rank 2, g_1 depends on U/V and g_2 on W/V and these coordinates are independent. From (38) we see that for r small when A describes a circle θ describes plus or minus a circle

(+1, or -1). Now if there is a linear bijection from one space of dimension 2 to another, then when point G makes a 360o turn in the first space its image does the same.

Turning back now to L. Consider a basis of $\mathfrak{J}_0$ where

$\mathfrak{J}_0 = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \lambda \quad \in [0,1]$. Consider θ as starting from the first eigen vector of this basis. We have then

$$\left[\frac{G^T G - \mathfrak{J}G^2}{\lambda}\right] = \frac{g^2}{2\lambda}\begin{pmatrix} \cos 2\theta + 1 - 2\lambda & \sin 2\theta \\ \sin 2\theta & -1 + 2\lambda - \cos 2\theta \end{pmatrix}$$
$$= \frac{g^2}{2\lambda}\left[((1-2\lambda) + \cos 2\theta)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (\sin 2\theta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

Since we are dealing with an equilibrium of the first type, $\mathfrak{J}_0$ is non singular, hence $\lambda \varepsilon\ ]0,\ 1[$ and $(1-2\lambda) \in\ ]-1,\ 1[$.

Thus $\left[\frac{G^T G - \mathfrak{J}G^2}{\lambda}\right]$ makes two full turns around zero.

Hence there must be a moment at which both $\left[\frac{G^T G - \mathfrak{J}G^2}{\lambda}\right]$ and $\mathfrak{J} - \mathfrak{J}_0$ have the same argument. They are thus parallel and in the same direction. In figure 1 we see what happens in a neighbourhood of $\mathfrak{J}_0$.

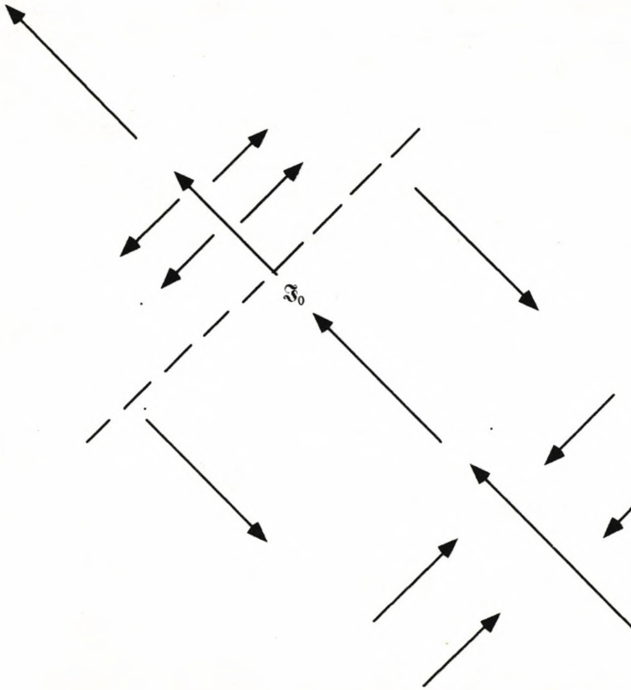

Figure 1

Thus there is always a direction leading away from $\mathfrak{I}_0$ which stabilises as r tends to 0. This completes the proof of Theorem 4.

An Example

A natural reaction to eliminate the slowing down of the process is to truncate the memory. However, this completely changes the dynamics, in the discrete case.

We simulated an example in which the parameters took the following values: $\alpha = 1$ $\beta = 3$ $\gamma = 2$ and the initial conditions were

$P_1(1) = 1 \quad P_2(1) = 2$

$P_1(2) = 3 \quad P_2(2) = 4$

$P_1(3) = 5 \quad P_2(3) = 6$

After the fifth period we restricted the memory to five periods and the process was captured by the following limit cycle which remained unchanged over 70,000 periods.

(P_1P_2) = (1.8388, 2.0160) (1.8389, 2.0162) (1.8438, 2.0178) (1.8752, 2.0069) (2.0122, 1.8724) (1.9949, 0.8652) (1.8388, 2.0160)

Thus there is a cycle of period 7 which cannot be detected by the actors.

Conclusions

Although cases may be constructed leading to a large class of self-sustained equilibria in our simple model, least squares learning is not stable. Though agents' ideas of the parameters will change more and more slowly as time goes on, they are not converging to any equilibria. The dynamics of this become very clear when agents' memories are limited. The source of the problem here is the omission of important variables, other players' strategies rather than the inclusion of initially irrelevant variables, as in the case of sunspots.

Thus our simple example illustrates not merely the problem of misspecification, but is particularly relevant to games in which players are ignorant of the identity of, or the strategies played by their opponents. An important question remains. To what extent would players be able to infer from their observations that they are dealing with strategic behaviour, and if they were able to do so, would the learning process converge? The answer is far from clear, since our example shows that agents may be misled into believing that they are converging to an understanding of the true model if they use all the observations that they have made. However, if they have a shorter memory, they may not be able to identify the sort of systematic cyclical behaviour that may then occur.

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Appendix

Proof of Theorem 1

Recall that

$d_i(p_1, p_2) = \alpha - \beta p_1 + \gamma p_2$

$d_2(p_1, p_2) = \alpha - \beta p_2 + \gamma p_1$

Consider a point $\bar{p}$ in P and consider a line Δ in P through $\bar{p}$ with parameter λ, i.e. any point $p_1\, p_2$ on this line may be written

A1 $p_1 = D_{11}\lambda + D_{12}$

A2 $p_2 = D_{21}\lambda + D_{22}$

Assume that the prices set by firms before t_o are given by a sequence of t_o -1 points $(p_i(t), p_2(t))$ all lying on Δ. Denote by $\lambda(t)$ the parameter corresponding to p(t). Since $\bar{p} \varepsilon \Delta$,

A3 $\bar{p}_1 = D_{11}\lambda_{\bar{p}} + D_{12}$

A4 $\bar{p}_2 = D_{21}\lambda_{\bar{p}} + D_{22}$

The demands at each time t can be written

A5 $d_1(p_1(t), p_2(t)) = \alpha - \beta p_1(t) + \gamma\left(D_{21}\lambda(t) + D_{22}\right)$

A6 $d_2(p_1(t), p_2(t)) = \alpha - \beta p_2(t) + \gamma\left(D_{11}\lambda(t) + D_{12}\right)$

The expressions for $\lambda(t)$ are given by

A7 $D_{11}^{-1}(p_1(t) - D_{12}) = \lambda(t)$

A8 $D_{21}^{-1}(p_2(t) - D_{22}) = \lambda(t)$

Using these in A5 and A6 we obtain

A9 $d_1(p_1, p_2) = \lambda + \left(\gamma D_{21} D_{11}^{-1} - \beta\right)p_1 - \gamma D_{21} D_{11}^{-1} D_{12} + \gamma D_{22}$

$$\text{A10}\quad d_2(p_{11},p_2) = \lambda + \left(\gamma D_{11} D_{21}^{-1} - \beta\right)p_2 - \gamma D_{11} D_{11}^{-1} D_{22} + \gamma D_{12}$$

Regrouping terms we obtain expressions for the parameters $\hat{a}_i$ and $\hat{b}_i$ of the observed demand functions $\hat{d}_i$

$$\text{A11}\quad \hat{d}_1 = \hat{a}_1 - \hat{b}_1 p_1 = \alpha + \gamma D_{11}^{-1}(D_{11}D_{22} - D_{21}D_{12}) + \left(\gamma D_{21} D_{11}^{-1} - \beta\right)p_1$$

$$\text{A12}\quad \hat{d}_2 = \hat{a}_2 - \hat{b}_2 p_2 = \alpha + \gamma D_{21}^{-1}(D_{12}D_{21} - D_{11}D_{22}) + \left(\gamma D_{11} D_{21}^{-1} - \beta\right)p_2$$

Since at the point t_o we have as the optimal choice

$$p_i(t_o) = \frac{\hat{a}_i}{2\hat{b}_i}$$

we can write

$$\text{A13}\quad p_1(t_o) = \frac{1}{2}\,\frac{\alpha D_{11} + \gamma(D_{11}D_{22} - D_{21}D_{12})}{\beta D_{11} - \gamma D_{21}}$$

$$\text{A14}\quad p_2(t_o) = \frac{1}{2}\,\frac{\alpha D_{21} + \gamma(D_{11}D_{22} - D_{21}D_{12})}{\beta D_{21} - \gamma D_{11}}$$

and since this must lie on Δ we have also

$$\text{A15}\quad p_1(t_o) = D_{11}\lambda(t_o) + D_{12}$$

$$\text{A16}\quad p_2(t_o) = D_{21}\lambda(t_o) + D_{22}$$

Using A1 and A2 to eliminate D_{12} and D_{22} we obtain

$$\text{A17}\quad \frac{\alpha D_{11} + \gamma\left(D_{11}\left(\bar{p}_2 - D_{21}\lambda_{\bar{p}}\right) - D_{21}\left(\bar{p}_1 - D_{11}\lambda\bar{p}\right)\right)}{2\left(\beta D_{11} - \gamma D_{21}\right)} + D_{11}\lambda_{\bar{p}} - \bar{p}_1 = D_{11}\lambda$$

$$\text{A18}\quad \frac{\alpha D_{21} + \gamma\left(D_{11}\left(\bar{p}_2 - D_{21}\lambda_{\bar{p}}\right) - D_{21}\left(\bar{p}_1 - D_{11}\lambda\bar{p}\right)\right)}{2\left(\beta D_{21} - \gamma D_{11}\right)} + D_{21}\lambda_{\bar{p}} - \bar{p}_2 = D_{21}\lambda$$

This can be solved for λ if the following condition is satisfied

$$\text{A19}\quad \begin{vmatrix} \dfrac{\alpha D_{11} + \gamma(D_{11}\bar{p}_2 - D_{21}\bar{p}_1)}{2\left(\beta D_{11} - \gamma D_{21}\right)} - \bar{p}_1 & D_{11} \\ \dfrac{\alpha D_{21} - \gamma(D_{11}\bar{p}_2 - D_{21}\bar{p}_1)}{2\left(\beta D_{21} - \gamma D_{11}\right)} - \bar{p}_2 & D_{21} \end{vmatrix} = 0$$

Tedious calculations lead to the following third degree polynomial

$$\text{A20} \quad \left[\bar{p}_1\beta\gamma\right] D_{21}^3 + \left[\alpha\beta+\alpha\gamma-\bar{p}_2\beta\gamma-2\bar{p}_1^2\right] D_{11}D_{21}^2 + \left[-\alpha\gamma+\bar{p}_1\beta\gamma-\alpha\beta+2\bar{p}_2\beta^2\right] D_{11}^2D_{21} - \left[\bar{p}_2\gamma\beta\right] D_{11}^3 = 0$$

Since $\alpha, \beta, \gamma, p_1, p_2$ are all positions by assumption, A20 has at least one real root.

The sum of the coefficients of A20 is given by

$$\text{A21} \quad \left(\bar{p},\beta\gamma\right) + \left(\alpha\beta+\alpha\gamma-\bar{p}_2\beta\gamma-2\bar{p}_1\beta^2\right) + \left(-\alpha\gamma+\bar{p}_1\beta\gamma-\alpha\beta+2\bar{p}_2\beta^2\right) - \left(\bar{p}_2\gamma\beta\right) = \beta(\gamma-\beta)(\bar{p}_1-\bar{p}_2)$$

Hence for every value $\left(\bar{p}_1,\bar{p}_2,\alpha,\beta,\gamma\right)$ in R^5_{++}, A20 has a positive root. Call this root ρ.

Given that $\beta\neq\gamma$, and that $\bar{p}\notin p'$ and $\bar{p}\in p''$, we now show that the expressions A13 and A14 cannot have a zero denominator when $D_{21}/D_{11}=\rho$. Thus, choosing ρ as the slope of the line $\Delta(\bar{p})$ in Theorem 1, this will conclude the proof.

Suppose the contrary, then $\beta-\gamma\rho=0$. From the development of the determinant A19 and the expression A20 it is clear that, if $\beta-\gamma\rho=0$, then

$$\text{A22} \quad \left(\alpha+\gamma(\bar{p}_2-\bar{p}_1\rho)\right)\left(\beta\rho-\gamma\right)\rho = 0.$$

We have seen that $\rho\neq 0$. Furthermore $\beta-\gamma\rho=0$ and $\beta\rho-\gamma=0$ is impossible, since $\beta = \gamma$. Hence

$$\text{A23} \quad \alpha+\gamma(\bar{p}_2-\bar{p}_1\rho) = 0$$

But since $\rho = \beta / \gamma$ we have

$$\text{A24} \quad \alpha+\gamma\bar{p}_2-\beta\bar{p}_1 = 0$$

the equation of the line p', which was excluded a priori. Thus β / γ cannot be a root of A20 and by the same argument neither can γ/β and the proof is complete.

Remark

It is easy but tedious to show that for any p on $\Delta(\bar{p})$ $\rho(p)=\rho$.

Proof of Theorem 2

We have to show which values of (p_1,p_2) can be attained as the limit of a sequence of "aligned points" $(p_1(t),p_2(t))$. Thus a point $\bar{p}$ is the limit of such a sequence if it is the "critical" point of the line $\Delta(\bar{p})$. In other words given a sequence of points p(t) t=1, ... t_o on $\Delta(\bar{p})$, $\bar{p}$ is a limit of points if and only if the sequence of optimal points p(t) is such that $p(t)=\bar{p}$ for all $t \geq t_o$.

We can clearly write

$$\text{A25} \quad p_1=\frac{\alpha+\gamma(p_2-\rho p_1)}{2(\beta-\gamma p)}$$

$$\text{A26} \quad p_2=\frac{\alpha+\gamma(p_1-\rho p_2)}{2(\beta-\gamma p^{-1})}$$

We obtain from this

$$\text{A27} \quad \frac{(2\beta p_1-\alpha-\gamma p_2)}{\gamma p_1}=\rho$$

and

$$\text{A28} \quad \frac{\gamma p_2}{(2\beta p_2-\gamma p_1-\alpha)}=\rho$$

Eliminating ρ we obtain

$$\text{A29} \quad (2\beta p_1-\alpha-\gamma p_2)(2\beta p_2-\alpha-\gamma p_1)=\gamma^2 p_1 p_2$$

Using the fact that $\beta>\gamma$, we observe that this gives a hyperbola with asymptotic slopes

$$\text{A30} \quad \Gamma=\left(\beta\pm\frac{\sqrt{\beta^2-\gamma^2}}{\gamma}\right)=\beta/\gamma\pm\sqrt{(\beta/\gamma)^2-1}$$

There are two branches to the hyperbola but clearly only that which cuts the diagonal at $\frac{\alpha}{2(\beta-\gamma)}, \frac{\alpha}{2(\beta-\gamma)}$ can be attained.

It should be noted that this branch of the hyperbola can only be attained by one of the lines through a given point p with the exception of the point G which has the three lines with respective slopes $\rho = \gamma / \beta$ and $\rho = \beta / \gamma$ lead to it.

This is easily seen, since we have $\hat{b}_i > 0$ and since

A31 $\beta - \gamma D_{21} D_{11}^{-1}$

A32 $\beta - \gamma D_{11} D_{21}^{-1}$

and

A33 $\rho = D_{21} D_{11}^{-1}$

then we have

A34 $\frac{\gamma}{\beta} < \rho < \frac{\beta}{\gamma}$

Thus the set of limit points attainable from "aligned" initial conditions is defined by the branch of A29 passing through $\frac{\alpha}{2(\beta-\gamma)}, \frac{\alpha}{2(\beta-\gamma)}$ the restriction imposed by A34, and this completes the proof.

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