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DEPARTMENT OF ECONOMICS

# Essays in International Finance and Applied Econometrics

Marek Raczko

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

Florence, 04 April 2016

European University Institute
**Department of Economics**

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I confirm that chapter one was partially the result of study I undertook at the Bank of England as a PhD intern. I also confirm that chapter three was the result of study I undertook at the Bank of England.

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I confirm that chapter one draws upon an earlier working paper I published in Bank of England Staff Working Paper series.

**Signature and Date:**
Marek Andrzej Raczko

March 1st, 2016

## Abstract

The thesis consists of three essays in the fields of international finance and applied econometrics. The first chapter analyzes the co-movement of market premia for rare adverse events, addressing the important issue of contagion. The second chapter studies the impact of rare adverse events on the estimates of the risk-aversion coefficient and on household's portfolio composition. This chapter shows that the threat of a rare disaster justifies household's positive bond holdings. Finally, the last chapter studies if the information not contained in the domestic yield curve, but contained in the foreign yield curve helps to predict future dynamics of domestic yields.

The first chapter proposes a novel approach to assessing volatility contagion across equity markets. More specifically I decompose the variance risk premia of three major stock indices into: crash and non-crash risk components and analyse their cross-market correlations. I find that crash-risk premia exhibit higher correlations than non-crash risk premia, implying the existence of volatility contagion. This suggests that investors believe that equity returns will be more highly correlated across countries during market crashes than during more normal times. The main result of the analysis holds when I apply other measures of co-movement as well as when I allow correlation to be time varying. Moreover I document that crash-premia constitute a large portion of the overall variance risk premia, highlighting the importance of crash-risks. Unlike the existing literature, my approach to testing the existence of volatility contagion does not rely on short periods of financial distress, but allows for crash-risk premia to be computed in tranquil times.

The second chapter assesses the impact of the Peso problem on the econometric estimates of the risk aversion coefficient. Rietz (1988) and subsequently Barro (2006) showed that the introduction of the crash risk allows the canonical general equilibrium framework to generate data consistent equity premia even under low risk aversion of the representative agents. They argue that the original data used to calibrate these models suffer from a Peso problem (i.e. does not encounter a crash state). To the best of my knowledge the impact of their Peso problem on the estimation of the risk aversion coefficient has not to date been evaluated. This chapter seeks to remedy this. I find that crash states that are internalized by economic agents, but are not realized in the sample, generate only a small bias in the estimates of the risk aversion coefficient. I also show that the introduction of the crash state has a strong bearing on the household's portfolio composition. In fact, under the internalized crash state scenario, households exhibit positive bond holdings even in a frictionless environment.

In the third chapter, co-authored with Andrew Meldrum and Peter Spencer, we show, using data on government bonds in Germany and the US, that 'overseas unspanned factors' - constructed from the components of overseas yields that are uncorrelated with domestic yields - have significant explanatory power for subsequent *domestic* bond returns. This result is remarkably robust, holding for different sample periods, as well as out of sample. By adding our overseas unspanned factors to simple dynamic term structure models, we show that shocks to those factors have large and persistent effects on domestic yield curves. Dynamic term structure models that omit information about foreign bond yields are therefore likely to be mis-specified.

## Acknowledgments

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# Contents

CHAPTER 1

# Volatility contagion: new evidence from market pricing of volatility risk


## Abstract

This paper proposes a novel approach to assessing volatility contagion across equity markets. I decompose the variance risk premia of three major stock indices into: crash and non-crash risk components and analyse their cross-market correlations. I find that crash-risk premia exhibit higher correlations than non-crash risk premia, implying the existence of volatility contagion. This suggests that investors believe that equity returns will be more highly correlated across countries during market crashes than during more normal times. The main result of the analysis holds when I apply other measures of co-movement as well as when I allow correlation to be time varying. Moreover I document that crash-premia constitute a large portion of the overall variance risk premia, highlighting the importance of crash-risks. Unlike the existing literature, my approach to testing the existence of volatility contagion does not rely on short periods of financial distress, but allows for crash-risk premia to be computed in tranquil times.


Keywords: Financial contagion, variance risk premium, tail-risk, equity co-movement, volatility co-movement.

JEL classification: C58; F36; G12; G13; G15

## 1.1. Introduction

The recent financial crisis highlighted the high degree of co-movement between international stock markets during crisis periods. This paper studies this comovement by decomposing international 'variance risk premia' i.e. the difference between expected market volatility and the volatility implied by equity options (for example in the case of S&P500, the VIX is the implied volatility index) - into two components: one capturing compensation for crash risk and another capturing compensation for 'non-crash' risk. More precisely, I define market crash risk as the risk of an event where the market jumps by at least -10% within one trading day and non-crash risk is defined as any market moves which are not considered to be a market crash. The analysis shows that crash risk premia exhibit higher correlations internationally than non-crash risk premia. This suggests that investors believe that equity returns will be more highly correlated across countries during market crashes than during more normal times.

This paper therefore contributes to the literature on asset price 'contagion' across countries, which - following Forbes and Rigobon (2002) – is defined as an increase in cross-market correlation[1] during times of crisis. While a number of papers have found evidence of this form of contagion (e.g. for equity returns, King and Wadhwani (1990) and Longin and Solnik (1995); for realized equity volatilities, Diebold and Yilmaz (2009); Cipollini et al. (2013); and for option-implied equity volatilities, Cipollini et al. (2013)), other studies, after correcting for estimator biases (e.g. Forbes and Rigobon (2002), Longin and Solnik (2001), and Corsetti et al. (2005)) find no evidence of contagion. Dungey and Zhumabekova (2001) point out that the primary difficulty is that periods of turmoil are usually short and consequently span only a small portion of the observed sample. Moreover the choice of dates for the financial turmoil 'regime' might also lead to inconsistent or inefficient estimates.

The novel approach developed in this paper avoids many of the drawbacks associated with distinguishing changes in correlation during short crisis periods. This is due to the fact that I look directly at market pricing of crash risk, which can be computed durign tranguil or crisis period. More precisely, I decompose variance risk premia[2] into components compensating for crash and non-crash states in the United States, the United Kingdom and euro area, by applying a modified version of the method of Bollerslev and Todorov (2011) to the S&P500, FTSE100 and Eurostoxx50, respectively. This allows me to compare the co-movement of premia that compensate for crash events with the co-movement of premia

[1] Traditionally correlation of stock market indices or asset prices were analyzed, but in this study I focus on the co-movement of volatilities of major stock market indices.

[2] Variance Risk Premium is the premium that markets require for the risk of a change of uncertainty. This premium is calculated as a difference between the statistical measure of market volatility (empirically measured by the realized volatility) and the risk neutral implied volatility (empirically measured by the options implied volatility index, ex. VIX).

for the remainder of the variance risk (i.e. 'non-crash risk premia').[3] I find that crash risk premia exhibit higher cross-country correlation than non-crash risk premia. This suggests that investors believe that the correlation of equity returns will be higher in tail events than in more normal times, which provides strong evidence for the market contagion hypothesis.

Morever, crash-risk premia correlations are elevated, relative to the correlation of non-crash risk premia, even when I account for time-varying correlation using the Dynamic Conditional Correlation model of Engle (2002). Hence the main result of the paper is robust to possible time-variation in the strength of international relationships. In fact, cross-country correlations of crash-risk premia are time-varying, yet they remain quite stable over time. I find that even though individual market crash risk premia are very sensitive to adverse market events (e.g. Russian default, LTCM collapse, Lehman Brothers bankruptcy, Sovereign default crisis, etc.), their international co-movement remains relatively stable.

Aside from providing important evidence for market contagion in times of crisis, the high correlation of tail risk premia has important implications for both financial market practitioners and policymakers. First, it shows that the potential gains from portfolio diversification are smaller than would be expected when not accounting for tail-dependency, as cross-country hedging will not be effective during times of crisis. Models that do not capture this feature seem likely to overestimate the gains from international diversification and the degree of investors' home bias.

Second, policymakers are likely to be particularly concerned with the impact of domestic monetary policy on perceptions of crash-risks. Hattori et al. (2015) studied the impact of US quantitative easing (QE) on crash risk perceptions, finding that QE resulted in a statistically significant decrease in crash premia. My analysis shows that policy that reduces crash-risk premia is likely to have a global impact. This implies that US QE might have large spillover effects on other equity markets and consequently on other economies through its impact on reducing global crash risk premia. The analysis developed in this study suggest that an interesting direction for future research is to investigate this particular global aspect of QE.

The remainder of the paper is organized as follows. Section 2 briefly describes the method and Section 3 characterizes the dataset used for the analysis. Section 4 describes the results and Section 5 concludes.

## 1.2. Methodology

The methodology of this study composes of three parts. First, I define the concept of Variance Risk Premium (VRP) and I show how it is measured using daily data on options and 5-minute frequency intra-day data on index futures prices. Second, I describe how to

[3] Bollerslev et al. (2013) or Londono (2014) show that the Variance Risk Premia are dominated by a global component, yet they do not look into the split of the VRP into the tail- and non-tail risk related premia.

decompose VRP into the part related to crash risk and the part related to non-crash risk, using techniques developed by Bollerslev and Todorov (2011). Given that my S&P500 options data differ from theirs (in that my option dataset exhibits longer average maturities) and that I am also extending their calculations to new datasets, namely FTSE100 and Eurostoxx50, I also describe my modification of the original methodology. Finally I describe the co-movement measures used in the study. Specifically, I use the Dynamic Conditional Correlation model of Engle (2002) to analyse potentially time-varying correlations between premia across equity markets.

**1.2.1. Variance Risk Premium (VRP).** Many financial studies have shown that not only equity returns, but also volatilities (risks) of those returns are time-varying. This basic fact of non-constant volatility means that this is an additional source of investment risk. The Variance Risk Premium (VRP) is the compensation that market requires for this additional risk. In fact, financial markets have already developed tools to hedge the risk of volatility increase. VRP can be traded using variance swaps (see Demeterfi et al. (1999) for details). These instruments simply swap future unknown realized variance for current option implied variance.

In technical terms, the VRP is measured as the difference between the physical expectation (the P-measure) of the realized quadratic variation of returns and the risk-neutral expectations (the Q-measure) of the quadratic variation of returns.

$$VRP_t = \frac{1}{T-t}\left(E_t^P(QV_{[t,T]}) - E_t^Q(QV_{[t,T]})\right) \tag{1.1}$$

The physical expectations (the P-measure) of the quadratic variation is simply best statistical $T-t$ periods ahead forecast. Quadratic variation under P is measured as the realized variance ($RV$) based on 5-minute frequency intra-day prices of index futures.[4] This approach has been strongly advocated by Liu et al. (2015), who showed that this is the best variance estimator. Moreover in this study, following Bollerslev et al. (2009), I use simple naïve expectations of the realized variance as a proxy for the forecast of realized variance. This approach should be effective as variance exhibits large persistence, exemplified by volatility clustering.[5]

[4]In order to adjust for the overnight price changes daily realized variance is re-scaled by the constant proportion of overnight change.

[5]More recently, however Bekaert and Hoerova (2014) or Kaminska and Roberts-Sklar (2015) show that the naïve forecast can be improved if the forecasting method models separately the continuous and the jump part of the volatility. Furthermore the forecast might be improved even more by the use of option implied volatility data. Yet, given that the focus of this study is the decomposition and cross correlation of VRPa, it seems that simple naïve expectations forecast would work well.

$$E_t^P(QV_{[t,T]}) = \sum_{i=t-(T-t)}^{t} RV_i \tag{1.2}$$

Risk-neutral expectations of the quadratic variation (the Q-measure) are measured using daily data on the panel of options. Those data enable us to calculate the model-free option-implied variance of future prices. This type of variance measure reflects the expected variance implied by option prices under the assumption of risk neutral market pricing. In more technical terms this measure is derived under the assumption that the stochastic discount factor is constant and equal to the inverse of the risk-free interest rate. This means that the Q-measure of the variance combines investors' expectations of the future variance with their risk preferences (see Figlewski (2012)).[6] The most classical example of a model-free Q-measure of volatility is the VIX index.[7]

My Q measure of the quadratic variation only slightly differs from the VIX index.[8] Both measures use approximation to calculate implied volatility for a fixed time horizon. Yet, unlike the VIX which uses only two different option maturities to calculate approximated values, I use the whole available set of different option maturities. Moreover, in contrast to the VIX methodology which approximates linearly quadratic volatility, I approximate option prices using Carr and Wu (2003) polynomial and based on theoretical option prices I calculate the implied volatility.[9] This change in the calculation method is motivated by two factors. First, the set of data used in this study, suffers from a small number of very close to maturity options, hence the VIX methodology would imply linear extrapolations from the two options with quite distant maturities. This seems inappropriate, especially when dealing with options capturing large jump probabilities. Second, I wanted to keep my measure consistent with the decomposition of the VRP presented in the next subsection.

Equation 1.3 describes the formula for the Q-measure of the quadratic variation, once the theoretical 14-day to maturity options are calculated:

[6] Simple coin flipping game might be a great example to understand the difference between Q- and P-measure of the probability distributions. Say, the game pays EUR 100 in case the flip yields heads and 0 in the other case. The P –measure would correspond to the actual distribution, hence both events have probabilities equal to 0.5. In order to determine the Q-measure of probabilities we need to know the price of the game. Say, an economic agent is willing to pay EUR 30 for that game. Under the assumption of risk-neutrality this would mean that the distribution of the probability should be 0.3 for heads and 0.7 for tails. The difference between those two measures of probabilities simply reflects agents risk aversion.

[7] To obtain implied variance, VIX index has to be divided by 100 and squared.

[8] In fact the correlation of my measures with volatility indices: VIX, VFTSE and VStoxx is very high and amounts roughly to 95%.

[9] Please refer to the Appendix A for more details on the approximation.

$$E_t^Q(QV_{[t,T]}) = \frac{2}{T-t}\sum_i \frac{\Delta K_i}{K_i^2} e^{(T-t)r} Q(K_i) - \frac{1}{T-t}\left[\frac{F}{K_0} - 1\right]^2 \tag{1.3}$$

In my calculations an option's time to maturity $T - t$ is fixed to 14 days (it is always quoted as a fraction of a year). The forward index level $F$ is calculated based on the index level at a given moment and the respective (14 day) risk-free interest rate $r$. $K_0$ denotes the first strike price below the forward index level $F$ of the panel of options. $K_i$ is the strike price of ith out-of-the-money option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$. $\Delta K_i$ is simply a mid-point between two strike prices: $K_{i-1}$ and $K_{i+1}$. The price of the option $Q(K_i)$ for a given strike price is either a price of the call option $C(K_i)$ if $K_i > K_0$ or a price of a put option $P(K_i)$ if $K_i < K_0$. The entire equation 1.3 is exactly the same as the one used to calculate the VIX index (see Chicago Board Options Exchange White Paper (2009)).

Finally, as shown in equation 1.1, VRP is measured as the difference between the two expectations, hence it reflects investors' attitude towards the risk – the so called risk appetite. The decomposition of this risk enables us to understand what drives the VRP: crash-events or more "normal" type of equity return movements. In the next section I describe the basic assumptions needed to calculate how much of the VRP is attributed to market crash risk.

**1.2.2. Tail-premia measures.** The Bollerslev and Todorov (2011a) methodology, which is applied in this paper, requires that the underlying asset price follows a very general jump-diffusion process.[10] It implies that the asset price dynamics (in case of this study price of futures for the underlying index $F_t$) follows a stochastic differential equation:

$$\frac{dF_t}{F_t} = \alpha_t dt + \sigma_t dW_t + \int_R (e^x - 1)\tilde{\mu}(dt, dx) \tag{1.4}$$

where $\alpha_t$ denotes the drift, $\sigma_t$ denotes the instantaneous volatility and $W_t$ is a standard Brownian motion. The first two elements of the sum depict the continuous part of the dynamics. The third part of the sum describes jumps or discontinuities of the asset price dynamics, where the $\tilde{\mu}(dt, dx)$ is the so-called compensated jump measure. The jump part may for example follow a Poisson process as in the Merton (1976) model. But in case of this study there is no need to limit ourselves to any parametric distribution - neither for the

[10] This type of process is very common in the financial literature, mainly due to the fact that it fits the actual data very well. Moreover, it allows prices to exhibit discontinuous patterns, which in turn, justifies the existence of markets for financial options in theoretical finance models (for some discussion of merits of jump-diffusion models please refer to Tankov and Voltchkova (2009)).

continuous, nor for the jump part. In fact, for our analysis, the most important feature of this model is the additive separability of the continuous and the jump components.

Both the diffusion and the jump part of the asset price dynamics will have their parallels in the process describing asset price variance. Consider the quadratic variation of the logs of asset prices over the [t,T] time interval:

$$QV_{[t,T]} = \int_t^T \sigma_s^2 ds + \int_t^T \int_R x^2 \mu(ds, dx) \tag{1.5}$$

where the first component $\int_t^T \sigma_s^2 ds$ is the volatility of the continuous process and the second component $\int_t^T \int_R x^2 \mu(ds, dx)$ denotes the volatility generated by the discontinuous part. In principle the first part should be responsible for the volatility generated by the "smaller" (continuous) movements in the asset prices, whereas the second part would depict volatility generated by the "larger" asset price movements (jumps).

Quadratic variation equation 1.5 implies that the VRP, defined by equation 1.1, will simply be a sum of two differences: the difference between P and Q expectations of the continuous part of the quadratic variation and the difference between P and Q expectations of the jump part of the quadratic variation:

$$\begin{aligned} VRP_t = \tfrac{1}{T-t} \left( E_t^P(\int_t^T \sigma_s^2 ds) - E_t^Q(\int_t^T \sigma_s^2 ds) \right) \\ + \tfrac{1}{T-t} \left( E_t^P \int_t^T \int_R x^2 \mu(ds, dx)) - E_t^Q(\int_t^T \int_R x^2 \mu(ds, dx)) \right) \end{aligned} \tag{1.6}$$

I need all the above presented structure to define the variance risk premium solely attributed to the market crash risk – $VRP(\tilde{k})$. This measure describes the contribution to the respective P- and Q- measures of quadratic volatility by asset price drops higher than a certain threshold k. In my study I define market crash as a state when asset prices fall by at least 10%. This implies that my threshold level $k = ln(0.9)$ and consequently $\tilde{k} = 0.9$. The price change of 10% can definitely be considered as a large move, hence it will only be reflected by the discontinuous part of the VRP. Consequently my VRP(k) measure depends only on the jump parts:

$$\begin{aligned} VRP_t(\tilde{k}) = \tfrac{1}{T-t} \left( E_t^P \left( \int_t^T \int_{x<k} x^2 v_s^P(dx) ds \right) \right) \\ - \tfrac{1}{T-t} \left( E_t^Q \left( \int_t^T \int_{x<k} x^2 v_s^Q(dx) ds \right) \right) \end{aligned} \tag{1.7}$$

Finally on the basis of the VRP(k) and the total VRP, I can also define a truncated volatility measure VRP(tr). This measure will capture the part of the variance risk premium that is atributed to the remaining non-crash risk:

$$VRP_t(tr) = VRP_t - VRP_t(0.9) \tag{1.8}$$

Having defined tail-risk premia, the next two sub-sections briefly describe how to calculate Q- and P- measures from the data.

*Risk-Neutral (Q) Measures.* The most difficult part of the Q-measure estimation is to pin down the process of the time-varying jump density $v_t^Q(dx)$. In order to construct a time-varying measure with as few assumptions regarding its structure as possible, I estimate it non-parametrically from the options data. Therefore I assume the following for jump density:

$$v_t^Q(dx) = (\varphi_t^- 1_{\{x<0\}})v^Q(x)dx \tag{1.9}$$

where $\varphi_t^-$ denotes an unspecified stochastic process of temporal variation of the jump arrivals and $v^Q(x)$ is an unspecified time-invariant density. Yet, the methodology of Bollerslev and Todorov (2011) allows us to estimate tailvolatilities $E_t^Q\left(\int_t^T \int_{x<k} x^2 v_s^Q(dx)ds\right)$ even under those very general assumptions. First of all they calculate model-free risk neutral measures from the panel of options data. Second, using the Extreme Value Theory (EVT) those measures are used to estimate Generalized Pareto Distribution (GPD) parameters (namely: scale ($\sigma$) and shape ($\xi$) parameters) and the average jump intensities $E(\frac{1}{T-t}E_t^Q(\int_t^T \varphi_s ds))$ through a just identified GMM system. This allows us to fully describe the time invariant part of the jump intensity $v^Q(x)$ for large price changes. Third, using fixed parameters for the GPD, the time varying jump intensities are backed out to fulfill exactly the moment conditions. Finally, using the estimated parameters the Q-measure of the tail-volatility is calculated for a given threshold $k$.

I describe the risk neutral jump-tail measures in detail as here I deviate slightly from the original Bollerslev and Todorov (2011) framework. They propose a model-free risk-neutral jump tail measure:

$$LT_t^Q(k) = \frac{e^r P_t(K)}{(T-t)F_t} \tag{1.10}$$

where $k = ln(\frac{K}{F})$ is the log-moneyness, $P_t(K)$ is a price of put option, $K$ is the option strike price and $F_t$ is the price of the underlying futures. This measure captures solely the jump risk as long as two conditions are fulfilled. First the options have to be deeply out of the money. Bollerslev and Todorov (2011) use moneyness levels of $\{0.9000\ 0.9125\ 0.9250\}$, which should guarantee enough distance from the underlying to capture only the jump risk. Second the option needs to be close to maturity. Bollerslev and Todorov (2011) use options that have median of 14 days to maturity. In my calculations I follow the same levels of option moneyness, but the dataset used in this study has much longer median maturity of options (see Table A.1in Appendix A). This means that my model-free risk-neutral jump tail measures might be 'contaminated' by the diffusive part of the process. In fact, when I applied the exact Bollerslev and Todorov (2011) methodology, my jump tail measure for S&P500 was substantially larger when the options had longer maturities relative to the original study.

In order to circumvent this problem I use a panel of options with different maturities for a given moneyness level to fit the polynomial describing the timedecay plot of option price. Carr and Wu (2003) show that this polynomial should approximate the time-decay of options no matter whether the underlying process contains jumps or not. This approximation allows me to calculate the theoretical price of the 14-days-to-maturity option. Appendix A provides details on the approximation method as well as some robustness checks.

Once I have the theoretical 14-days-to-maturity option price, I construct the same risk-neutral jump tail measure. In this case the pattern of my jump tail measure closely resembles the original one of Bollerslev and Todorov (2011b).

Generalized Pareto Distribution (GPD) parameters are estimated using the simple non-linear GMM procedure of Hansen and Singleton (1982). The exact moment conditions are described in the Appendix B. The basic principle is that for left tail I have 3 parameters to estimate and jump-tail measures for 3 different levels of moneyness, hence the system is just identified.

*Objective (P) Measures.* Analogous to the Q-measure estimation, the key issue in estimation of the P-measure is to pin down the time-varying jump density $v_t^P$. Unfortunately it is not possible to estimate the intensity fully non-parametrically, simply because I do not have three different points of the curve on the same day. Consequently I assume an affine model of the jump intensity. Following Bollerslev and Todorov (2011a) I assume that the temporal variation of the volatility is a function of the stochastic volatility $\sigma_t^2$ of the continuous part:

$$v_t^P(dx) = (\alpha_0^- 1_{\{x<0\}} + \alpha_1^- 1_{\{x<0\}} \sigma_t^2) v^P(x) dx \tag{1.11}$$

This implies that I have to estimate four parameters that are constant across time (namely: scale ($\sigma$) and shape ($\xi$) parameters of the GPD that characterizes $v^P(x)$, and $\alpha_0$ and $\alpha_1$). Moreover I have to get the estimate of the time-varying stochastic volatility $\sigma_t^2$. Here again, I follow closely Bollerslev and Todorov (2011) framework.

First I estimate continuous volatility using Mancini (2001) idea of truncated volatility. All intra-day asset price movements below a certain threshold contribute to the continuous volatility whereas the ones above the threshold contribute to the jump volatility. The truncation threshold is time-varying to capture the effects of the volatility clustering. The threshold is a function of the past continuous volatility. Moreover the daily pattern of volatility is also taken into account. For each index I estimate the average volatility for a given time. On that basis I calculate the time of the day volatility multiplier that either increases or decreases the threshold. For more details on the realized volatility calculations please refer to the Appendix C.

Second I select a threshold level, which is always higher than the maximum threshold used to determine continuous volatility. I select a threshold of 0.6% for all the indices. On the basis of this threshold I can mark observations that are definitely jumps in the whole sample. Then I use estimated continuous volatility along with matrices indicating jumps (the ones determined by 0.6% threshold) to estimate all four parameters in question. Again the estimation is done using the GMM framework (for details on the exact moments specification please refer to the Appendix D).

Finally, once all the parameters are calculated I calculate the tail-volatilities $E_t^P\left(\int_t^T \int_{x<k} x^2 v_s^P(dx)ds\right)$ for the threshold of $ln(0.9)$ to match the tail-volatilities for the Q-measure.

**1.2.3. Co-movement measures.** The main result of this analysis is based on the measures of co-movement of $VRP(0.9)$ as well as $VRP(tr)$ across equity markets (i.e. three indices: S&P500, FTSE100 and Eurostoxx50). In order to keep the analysis simple and yet powerful the main result is based on the simple r-Pearson correlation coefficient. The main finding is based on comparison unconditional correlations of crash risk premia to unconditional correlations of non-crash risk premia.

The correlation coefficient is known to be sensitive to outliers however, which is why I also report two non-parametric measures of co-movement: Kendall's $\tau$ and Sperman's $\rho$. Those measures are used as a robustness check of the main finding.

Market correlations are renowned to be time varying, hence as a final robustness check to my main correlation matrix I allow correlation to be time-varying. In order to capture a more complex dynamic correlation structure, I apply the Dynamic Conditional Correlation (DCC) model of Engle (2002). This model helps me not only to overcome the problem of time-varying correlation, but to control for the heterogeneity of individual shocks. The model looks at the conditional correlations of innovations, enabling me to gauge how shocks co-move across markets and is given below:

$$y_t = C + \sum_{k=1}^{K} A_k y_{t-k} + \epsilon_t \tag{1.12}$$

$$E_{t-1}(\epsilon_t \epsilon_t') = \Sigma_t \tag{1.13}$$

$$\Sigma_t = D_t R_t D_t \tag{1.14}$$

$$R_t = diag(Q_t)^{-1/2} Q_t diag(Q_t)^{-1/2} \tag{1.15}$$

$$Q_t = (1 - \lambda_1 - \lambda_2)\bar{R} + \lambda_1 \tilde{\epsilon}_{t-1}\tilde{\epsilon}'_{t-1} + \lambda_2 Q_{t-1} \tag{1.16}$$

$$\bar{R} = E[\tilde{\epsilon}_t \tilde{\epsilon}_t{}'] \tag{1.17}$$

$$\tilde{\epsilon}_t = D_t^{-1}\epsilon_t \tag{1.18}$$

The DCC model requires the level equation to be parsimonious, hence in the benchmark case I use VAR(SIC) processes to describe variables' levels (see equation 1.12), where the number of lags is selected on the basis of Schwartz information criteria.[11] The vector of variables in equation 1.12 contains either all three crash-risk VRP(0.9) or all three non-crash-risk VRP(tr).

The conditional covariance matrix (equation 1.13) is decomposed into the matrix of individual conditional standard deviations $D_t$ and conditional correlation matrix $R_t$ (see equation 1.14). Conditional standard deviation matrix $D_t$ is a diagonal matrix where each element on the diagonal simply represents a square root of individual variances which are modelled as the GARCH(1,1) process. Transformation of the conditional correlation matrix (see equation 1.15) guarantees that the matrix has ones on the diagonal. Quasi conditional correlation (see equation 1.16) is a weighted average of the unconditional sample correlation $\bar{R}$ (see equation 1.17) and the previous period cross product of 'corrected' innovations (see equation 1.18) and the previous period conditional quasi correlation. The specification of the equation 1.16 nests the Constant Conditional Correlation (CCC) model of Bollerslev (1990), hence allowing for direct testing of the time varying correlation assumption. Should $\lambda_1$ and $\lambda_2$ parameters were jointly statistically insignificant, then the correlation between innovations would be constant over time.

## 1.3. Data

The dataset used in this study allows me to replicate US results of Bollerslev and Todorov (2011) as well as to extend their calculations to the UK and euro-area. Accordingly, US calculations are based on the S&P500 index, UK on the FTSE100 index and euro-area on

[11] As an additional robustness check I have also used other models, namely: AR(1), AR(SIC) and VAR(1), but this changes did not yield qualitatively different results.

the Eurostoxx50 index. The Q-measure (implied distribution) is based on a daily panel of options, whereas the P-measure (statistical measure) is based on intra-day (5 minutes) data on traded futures, obtained from Thomson Reuters. Finally, correlation calculations are conducted on the weekly averages, as the daily data contained too much noise.

**1.3.1. Options.** I use options data collected by the Bank of England from Chicago Mercantile Exchange (CME), Eurex Exchange and London International Financial Futures and Options Exchange (LIFFE) for S&P500, Eurostoxx50 and FTSE100 index options, respectively. The data are sampled with a daily frequency. The data for S&P500 and FTSE100 options span January 1995 to December 2013. Unfortunately the data span for the Eurostoxx50 is shorter and covers only January 1999 to December 2013. This sample still allows me to cover major period of market turmoil (for US and UK only: LTCM, Russian and Asian crises and for all three markets: dotcom bubble burst, accounting scams and the great recession period for all indices).

I apply a standard set of filters on the options data before any calculations take place. The set of filters is based on programmes used by the Bank of England which are in line with the ones used in Carr and Wu (2009).

**1.3.2. Intra-day data of index futures.** I use the intra-day data provided by Thomson Reuters. The data are sampled at a 5-minutes frequency. This frequency allows me to capture price jumps limiting the impact of the microstructural noise. In fact Liu et al. (2015) show that realized variance based on 5-minute frequency data is the best estimator of the realized variance across different assets.

For S&P500 and FTSE100 I use the data spanning January 1996 to December 2012, whereas for Eurostoxx50 the data only spans January 1999 to December 2012. The range of the dataset for the S&P500 is unfortunately shorter than in the Bollerslev and Todorov (2011) paper, hence the parameter estimates might differ. In terms of trading time during each day, for each series I have tried to pick a time period for which I had data throughout all the dates. Consequently, my time windows are: for S&P500 - 81 observations (from 8.30 to 15.10), for FTSE100 - 94 observations (from 8.15 to 16.00) and for Eurostoxx50 - 81 observations (from 9.15 to 15.55).

## 1.4. Results

Before I go to the main result of the paper, i.e. the analysis of the co-movement of risk premia across equity markets, I would like to describe briefly the estimates of the GPD parameters under Q and P probability measures.

**1.4.1. Q-measure.** Table 1.1 summarizes parameter estimates for the risk-neutral Q-measure. Parameters are precisely estimated, as can be seen from standard errors. The first two parameter estimates describe the time-invariant parameters of the GPD, $\xi$- the shape parameter and $\sigma$- the scale parameter. The larger those parameters are the thicker the tails of the distributions.

It is clear that, according to my estimates, the tail of the FTSE100 index distribution is the thinnest, as all parameters are the smallest from all three indices. In case of S&P500 and Eurostoxx50 the results are more ambiguous. The scale parameter is marginally bigger for the Eurostoxx50, but the shape parameter is bigger for the S&P500. This means that, even though for smaller values Eurostoxx50 tail is thicker, for larger jumps the S&P500 tail is thicker.

The estimates of the average jump intensity parameters $\alpha v$, calculated at -7.5% price jumps, are only comparable between S&P500 and FTSE100, as Eurostoxx50 estimates were calculated on the different sample. Yet, again those estimates show that FTSE100 options exhibit the smallest tail-risk.

It is easier to interpret annualized average jump intensities presented in Table 1.2, as they swiftly summarize the impact of all three parameters on the tails. For example, the results in Table 1.2 read that we should expect about 3 jumps of -10% in four years for S&P500 index. All those numbers indicate that those probabilities are higher than the actual, even extreme, price changes observed on the futures markets. Actually, a -10% index jump has not been observed in any of the analysed samples. This is likely to be a manifestation of the fact that risk premia are embedded in the Q distribution.

Moreover it is also very interesting to note that the crash contribution (index jump of at least -10%) to the overall Q-measure of variance is 41%, 33% and 46% for S&P500, FTSE100 and Eurostoxx50, respectively. Of course those numbers are averages specific to the analysed samples.

One could also enquire how those estimates for the left tail compare to those of the right tail. Analogue calculations for the right tail can be found in the Appendix F. It is worth noting that under Q-measure tail distributions are highly skewed to the left, manifesting the so called volatility 'smile'.

**1.4.2. P-measure.** Table 1.3 summarizes estimation results for the objective, 'physical', P-measure. The first two parameters describe the time-invariant shape ($\xi$) and scale ($\sigma$) of the GPD, similarly as in case of the Q-measure. Unfortunately those estimates are not directly comparable with the ones from the Q-measure, as they were calculated at a different thresholds.

TABLE 1.1. Q-measures estimation results

| | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| $\xi$ | 0.2744 (0.0092) | 0.2693 (0.0096) | 0.2313 (0.0094) |
| $\sigma$ | 0.0563 (0.0007) | 0.0590 (0.0007) | 0.0527 (0.0006) |
| $\alpha v$ | 1.2425 (0.0152) | 1.4751 (0.0208) | 1.1012 (0.0138) |

*Notes:* The table reports estimated parameters of the generalized Pareto distribution under the risk neutral Q-measure: $\xi$ is the estimate of the shape parameter and $\sigma$ is the estimate of the scale parameter. $\alpha v$ is the estimate of the average annualized jump intensity of -7.5% jump in the price level. The estimates are based on S&P500 and FTSE100 options data from January 1996 to December 2013 and Eurostoxx50 options data from March 2002 to December 2013. The log-moneyness of options used to estimate parameters were 0.9000, 0.9125 and 0.9250. Estimates standard errors are reported in parentheses.

TABLE 1.2. Q-measure: annualized jump intensity estimates

| Jump Size | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| <-7.5% | 1.2425 | 1.4751 | 1.1012 |
| <-10% | 0.7554 | 0.9153 | 0.6445 |
| <-20% | 0.1393 | 0.1764 | 0.0999 |

*Notes:* The table reports annualized average jump intensities under the Q-measure i.e. implied by option prices. Jump sizes are in terms of percentage changes in price levels. In the case of S&P500 and FTSE100 averages are calculated from January 1996 to December 2013, and for Eurostoxx50 averages are calculated from March 2002 to December 2013. All the reported figures are based on generalized Pareto distribution estimates reported in Table *1.1.*

Estimates of those two parameters do not differ substantially across analysed markets. In contrast to estimates of the Q-measure, under the P-measure the FTSE100 tail seems to be the thickest. This might be partially explained by the fact that this market is considered to be the least liquid of the three.

The $\alpha_0$ and $\alpha_1$ parameters describe the affine process driving jump-intensities under the P-measure. The signicance of the estimates of $\alpha_1$ parameters for all three markets indicate that jump-intensities are in fact time varying and closely connected to the actual continuous volatility.

As in the case of Q-measure, it is worth looking at the average jump intensities for the P-measure. Table 1.4 shows that a single day -10% market crash is an extremely rare event. In the case of the FTSE100 index, for which the P-measure is the most leptokurtic, estimated annualized jump intensities imply that we would only observe 1 such crash in 100 years. This is even more striking when compared to roughly 55 such events in 100 years implied by the Q-distribution. This yet again underscores the impact of the risk-aversion on the Q-measure.

TABLE 1.3. P-measure estimation results

| | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| $\xi$ | 0.2500 | 0.2305 | 0.2596 |
| | (0.0766) | (0.0701) | (0.0406) |
| $100\sigma$ | 0.1594 | 0.1819 | 0.1624 |
| | (0.0155) | (0.0164) | (0.0083) |
| $\alpha_0$ | -0.0016 | -0.0016 | -0.0025 |
| | (0.0001) | (0.0001) | (0.0001) |
| $\alpha_1$ | 0.0329 | 0.0346 | 0.0406 |
| | (0.0005) | (0.0006) | (0.0007) |

*Notes:* The table reports estimated parameters of the generalized Pareto distribution under the physical P-measure: $\xi$ is the estimate of the shape parameter and $\sigma$ is the estimate of the scale parameter. $\alpha_0$ and $\alpha_1$are estimates of the parameters in equation 1.11 which links jump intensities to the time-varying continuous volatilities. The estimates are based on high-frequency 5-minute futures prices from January 1996 to December 2012 for S&P500 and FTSE100, and from January 1999 to December 2012 for Eurostoxx50. Estimates standard errors are reported in parentheses.

TABLE 1.4. P-measure: annualised jump intensity estimates

| Jump Size | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| <-7.5% | 0.0069 | 0.0082 | 0.0343 |
| <-10% | 0.0020 | 0.0022 | 0.0102 |
| <-20% | 0.0001 | 0.0001 | 0.0004 |

*Notes:* Table reports annualized average jump intensities under the P-measure i.e. based on the high frequency data estimation. Jump sizes are in terms of percentage changes in price levels. In the case of S&P500 and FTSE100 averages are calculated from January 1996 to December 2012, and for Eurostoxx50 averages are calculated from January 1999 to December 2012. All the reported figures are based on generalized Pareto distribution estimates reported in Table 1.3.

Moreover, the contribution of market crash to the total variance under the P-measure is much smaller that under the Q-measure and amounts to 0.05%, 0.01% and 0.11% for S&P500, Eurostoxx50 and FTSE100, respectively. This implies that the compensation for crash events is larger than that for the 'regular' volatility.

Finally, Appendix G contains analogous results for the right tail of the distribution. Interestingly I find that the tails under P-measure are also skewed to the left, but much less than under the Q-measure. This contrasts with the Bollerslev and Todorov (2011a) findings, which note a skew towards the right tail. It can be explained by the fact that the sample I use also covers the period of the great recession.

**1.4.3. Variance Risk Premia, Tail Risk Premia and Investor Fears Indices.** All the observed Variance Risk Premia (VRP) are on average negative (see Table 1.5). This is due to the fact that option implied variances (Q-measures) are on average larger than realized variances (P-measures). Moreover VRP are also volatile and persistent. These results show

Table 1.5. Summary statistics for Variance Risk Premia

| | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| **VRP** | | | |
| Mean | -0.0368 | -0.0537 | -0.0281 |
| Median | -0.0250 | -0.0390 | -0.0185 |
| Std dev. | 0.0373 | 0.0511 | 0.0366 |
| **VRP crash** | | | |
| Mean | -0.0336 | -0.0433 | -0.0249 |
| Median | -0.0132 | -0.0237 | -0.0114 |
| Std dev. | 0.0579 | 0.0622 | 0.0463 |
| **VRP tr** | | | |
| Mean | -0.0032 | -0.0104 | -0.0032 |
| Median | -0.0087 | -0.0136 | -0.0060 |
| Std dev. | 0.0296 | 0.0186 | 0.0139 |

*Notes:* The table reports summary statistics for Variance Risk Premia (VRP), crash-risk VRP(0.9) and non-crash-risk VRP(tr). VRP is defined as the difference between the statistical expectations (P-measure) of variance and option implied (Q-measure) variance, calculated on the basis of high frequency 5-minute futures prices data and daily option prices data, respectively. On average VRP is negative, indicating that on average implied variance is higher than the statistically expected variance, showing that market participants are risk averse. Crash-risk VRP is the part of the premia that is required solely to hedge market crash risk, defined here as a -10% jump in the underlying index. VRP(tr) is the residual premium that is required for non-crash risk. Calculations are based on weekly averages from March 2002 to December 2012.

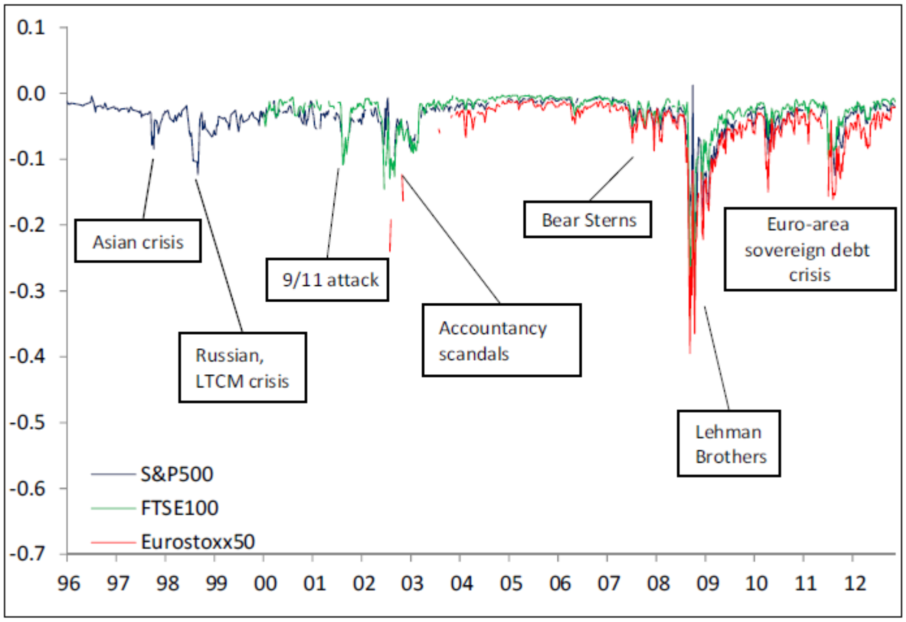
that markets are charging signicant and time-varying premia for the risk of future changes of the variance of the asset prices.

VRP seem to be quite closely co-moving across those three equity markets, hinting that the premium might be globally driven. Premia magnitudes are also very sensitive to major market events, such as accountancy scandals, Bear Sterns melt down, Lehman Brothers' bankruptcy or sovereign default crisis (see Figure 1.1).

VRP(0.9) attributed solely to the market crash seem to exhibit the same features as the total VRP. They are also negative, persistent and volatile. They also react sharply to major market events. Actually one may easily note that VRP and crash VRP(0.9) are co-moving for all three indices. This is not surprising as crash VRP(0.9) constitute large fractions of the total VRP.

In fact, on average, it captures 91%, 81% and 89% of VRP for S&P500, Eurostoxx50 and FTSE100, respectively. Those results are in line with the study of Bollerslev and Todorov (2011) who found that 88% of the S&P500 VRP is driven by the crash premium. It should be also noted that these results are driven by premia values during market turmoil times, as ratios of median crash VRP(0.9) to median total VRP are much smaller, though still substantial. More precisely, they amount to 53%, 61% and 62% for S&P500, Eurostoxx50

FIGURE 1.1. Variance Risk Premia




*Notes:* The figure shows evolution of Variance Risk Premia (VRP) over time for S&P500, Eurostoxx50 and FTSE100. Labels depict major global market events. Missing data for S&P500 and Eurostoxx50 are due to gaps in the options datasets. The figure represents weekly averages of VRP.

and FTSE100, respectively. Yet both sets of numbers clearly indicate high importance of crash premia in the total risk compensation.

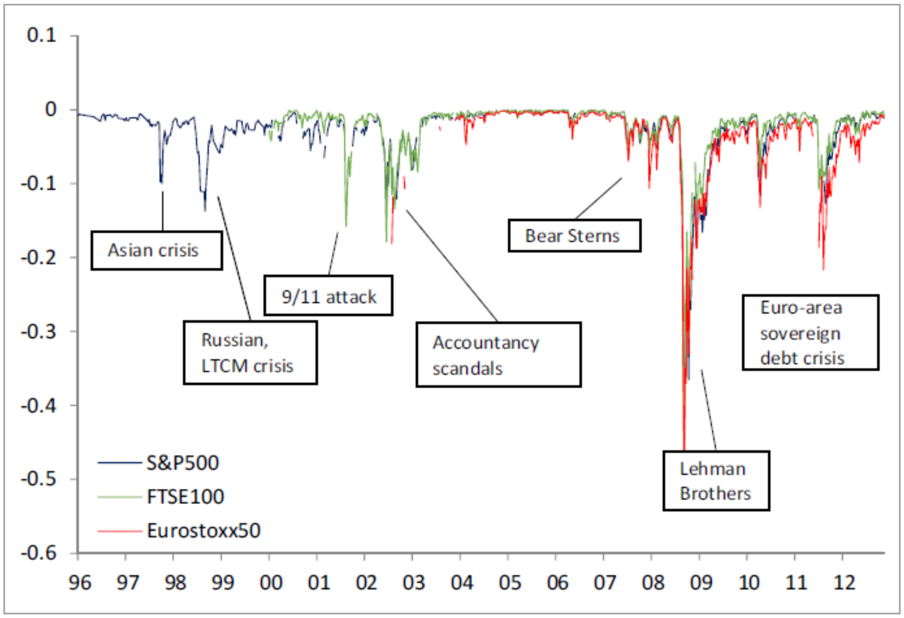
The result of high impact of crash risk on the market compensation for risk is in line with rare disasters literature. Rietz (1988) and subsequently Barro (2006) and Wachter (2013) highlight this phenomenon in theoretical macro-financial models. From that perspective my finding simply empirically reinforces their analysis.

**1.4.4. New evidence on contagion.** The main question of this paper is the existence of the volatility contagion. This question is answered by the comparison of cross-market correlations of crash risk premia VRP(0.9) against correlations of the non-crash risk premia VRP(tr).

Table 1.6 summarizes the key result of this paper - **crash risk premia co-move by more than the premia for non-crash risk across all three equity indices.** This indicates that large negative events (market crashes) have more global impact than other 'regular' events. This proves the existence of volatility contagion on equity markets. It should be once more underlined that, in contrast to the existing literature, this test for market contagion does not depend on a crash event, but is based on market pricing of crash risk.

Moreover, crash premia VRP(0.9) seem to be driven by a common factor. In fact, simple principal component analysis indicates that the first principal component of three crash

FIGURE 1.2. Crash-Risk Variance Risk Premia (0.9)




*Notes:* The figure shows evolution of crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with a -10% jump in the price of the underlying index futures) for S&P500, Eurostoxx50 and FTSE100. Labels depict major global market events. Missing data for S&P500 and Eurostoxx50 are due to gaps in the options datasets. The figure represents weekly averages of VRP(0.9).

TABLE 1.6. Pairwise correlations of the VRP(0.9) and VRP(tr)

| | Pearson's correlation | |
|---|---|---|
| | VRP(tr) | VRP(0.9) |
| S&P500 Eurostoxx50 | 0.5214 | 0.9559 |
| S&P500 FTSE100 | 0.3026 | 0.9631 |
| FTSE100 Eurostoxx50 | 0.2508 | 0.9624 |

*Notes:* The table reports pairwise correlations of three index pairs for two measures: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with a -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Pairwise correlation are calculated on a common sample of weekly data for all three indices from March 2002 to December 2012. The table clearly shows that correlations of crash-risk premia VRP(0.9) are substantially higher than correlations of the non-crash risk premia VRP(tr).

premia VRP(0.9) describes 97% of total data variability, whereas in case of the reminder of the volatility premia VRP(tr) it amounts to 87%.

Crash premia VRP(0.9) are quite volatile and susceptible to market adverse events (like the collapse of Lehman Brothers), hence one might suspect that the high correlation results are driven solely by outliers. In order to check whether presented results are robust to outliers, I also look at two non-parametric measures of dependence, namely Kendall's $\tau$ and Sperman's $\rho$. Table 7 shows that even under those measures of dependence, crash premia

Table 1.7. Non-parametric dependence measures of VRP(0.9) and VRP(tr)

| | Kendall‘s $\tau$ | | Spearman‘s $\rho$ | |
|---|---|---|---|---|
| | VRP(tr) | VRP(0.9) | VRP(tr) | VRP(0.9) |
| S&P500 Eurostoxx50 | 0.5648 | 0.7972 | 0.7154 | 0.9430 |
| S&P500 FTSE100 | 0.3919 | 0.7996 | 0.5334 | 0.9475 |
| FTSE100 Eurostoxx50 | 0.3088 | 0.7888 | 0.4120 | 0.9369 |

*Notes:* The table reports non-parametric pairwise dependence measures of three index pairs for two premia measures: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with a -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Two non-parametric measures are Kendall's $\tau$ and Sperman's $\rho$. These measures are used as they should be more robust to outliers than the simple correlation coefficient. Dependence measures are calculated on a common sample of weekly data for all three indices from March 2002 to December 2012. The table confirms findings documented by a simple correlation coefficient in Table 1.6.

VRP(0.9) are more closely co-moving than the non-crash premia VRP(tr). This reinforces the existence of volatility contagion.

Kendal's $\tau$ for crash premia are markedly lower than in the case of Pearson's correlations, but still higher than the correlations of non-crash premia, whereas Sperman's $\rho$ dependence measures are in line with Pearson's correlation numbers. If anything, this simple robustness exercise indicates that the outliers are rather decreasing the non-crash premia correlations, but still they are always lower than the correlation of crash premia.

Market correlations are perceived to be unstable over longer periods of time. In order to overcome this problem, as the last robustness check, I have extended my analysis to allow for the dynamic correlations. A simple rolling window analysis, presented in Appendix H, shows that correlations in question are indeed unstable. Moreover, Forbes and Rigobon (2002) pointed out that this type of simple analysis might be biased due to heterogeneity of individual shocks.

As mentioned earlier, I solve both issues by looking at the VAR(SIC)-DCC(1,1) model, as it allows for time varying: correlations and individual volatilities. The number of lags for each series is determined by the Schwartz information criteria. The models shows that the conditional correlation of dynamic innovations are indeed time-varying, for crash risk premia VRP(0.9) as well as for non-crash premia VRP(tr). In fact, the data rejected the Constant Conditional Correlation model of Bollerslev et al. (1988) that is embedded in the DCC specification.

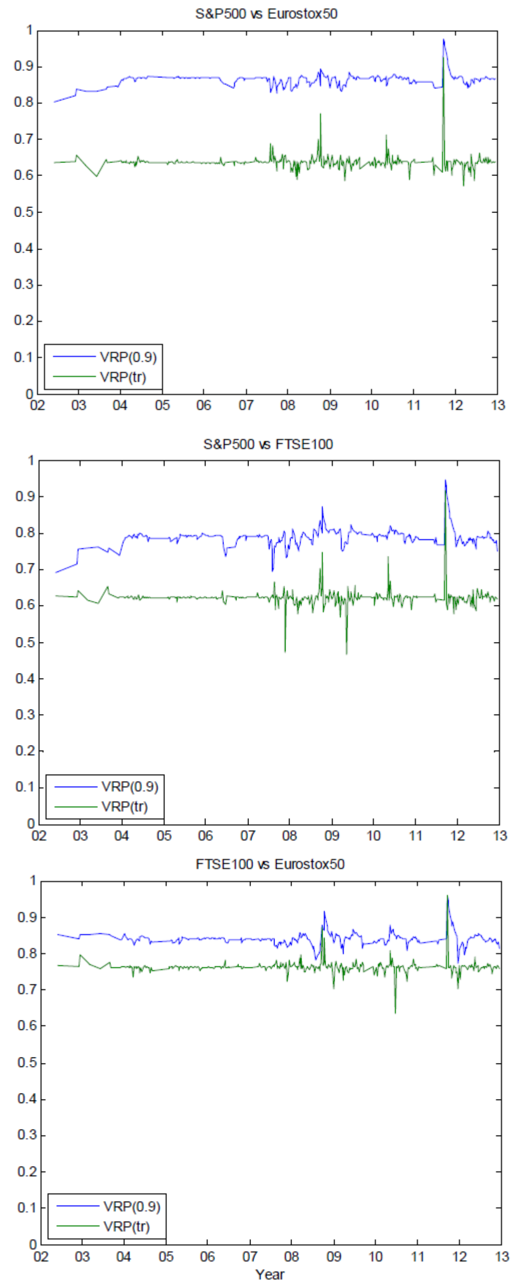
Even though correlations are time-varying the main result of the paper remains in place as correlations of crash premia VRP(0.9) are always higher than correlations of non-crash premia VRP(tr) (see Figure 1.3). Finally, the result of higher co-movement of crash premia also holds when the level equation follows different processes, namely AR(1), AR(SIC), VAR(1).[12] This highlights the robustness of the key result.

## 1.5. Conclusions

In this study I showed that the volatility premia investors require to compensate for crash risks are more closely co-moving across different equity markets than volatility premia required for non-crash risks. This result implies that investors perceive crash risks to have more global impact than other risks, hence pointing to market contagion. This study uses a novel approach to assess the volatility contagion. Unlike previous studies that compare market co-movement in crisis times with co-movement in 'tranquil' times, I compare co-movement of market variance premia for market crash risk with co-movement of non-crash risk variance premia. This allows me to circumvent many of the econometric issues that existing studies suffer from. More precisely, I do not have problems with dating crisis periods or having short crisis data samples. Finally, it should be underlined again that the main result of the paper is robust to different measures of premia co-movement as well as to possible time variation in correlations.

[12] Graphs of dynamic correlations under different level equations, can be found in Appendix

FIGURE 1.3. Time varying conditional correlations calculated on the basis of weekly data by VAR(SIC)-DCC(1,1) model




*Notes:* Figures show the dynamic correlations of three index pairs for two premia measures over time: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with a -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Dynamic correlations are calculated using the Dynamic Conditional Correlation model of Engle (2002). The model is based on a common sample of weekly data for all three indices from March 2002 to December 2012. The level equation of either VRP(0.9) or VRP(tr) for all three indices is modelled jointly as a VAR process, where the number of lags is selected using Bayesian information criterion.

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## Appendix A - Time-decay approximation

The dataset used in this study has one substantial drawback - the time to maturity of options is much longer than in the Bollerslev and Todorov (2011) study (see Table A.1), except for FTSE100. Consequently the estimator of the tail measure could be contaminated by the diffusion process. This in turn may bias my estimates of the Generalized Pareto Distribution leading to an inaccurate inference about tail-risk premia. In order to circumvent this problem I use all available maturities of options to estimate the time-decay patterns. This allows me to calculate the theoretical value of option that has 14 days to maturity. I choose this number of days to maturity to match exactly the median number of days to maturity in the Bollerslev and Todorov (2011) study.

Table A.1. Maturities of the closest to maturity options

| Index | Minimum | Maximum | Median |
|---|---|---|---|
| BT: S&P500 | 5 | x | 14 |
| S&P500 | 6 | 75 | 33 |
| FTSE100 | 5 | 29 | 15 |
| Eurostoxx50 | 5 | 74 | 36 |

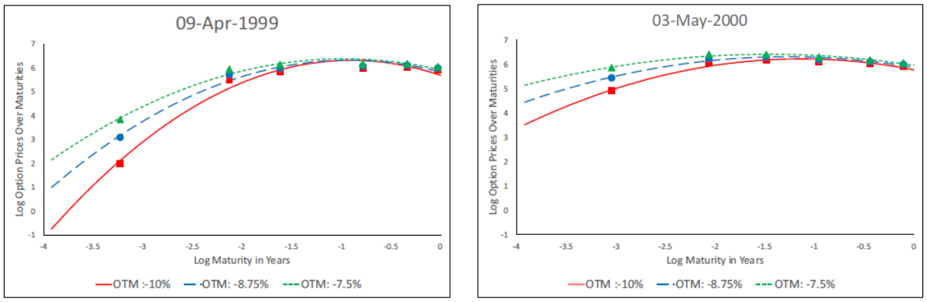
*Notes*: The table reports minimum, maximum and median days-to-maturity of the closest to maturity option used in my dataset as well as options used in the original Bollerslev and Todorov (2011) study. Only the median days to maturity for the FTSE100 roughly matches the one of Bollerslev and Todorov (2011).

Out-of-the-money options at the maturity have zero value. However, the order of convergence over time to that value depends largely on the process governing the underlying asset's price dynamics. Carr and Wu (2003) showed that the time decay (or the order of convergence) of out-of-the money options is dominated by the presence of jumps. They showed that if the price of the underlying asset follows a jump process or a jump-diffusion proces, then the value of the out-of-the-money option will converge more slowly to zero than in the case of a strict diffusion process. They also showed that the time decay of option prices can be closely approximated by the following polynomial:

$$ln\left(\frac{P}{T}\right) = a(lnT)^2 + b(lnT) + c \quad \text{(A.1)}$$

This approximation equation is valid regardless whether the underlying process exhibiting jumps or not. If the underlying equity process has no jumps the fitted line should have a greater slope close to the zero maturity (as the price of the option is falling faster than the time to maturity), whereas if it exhibits jumps the time-decay plot should be flatter (see Figure A.1). In this study I fit this polynomial for each day of the data - since the perception of the jump probability might change over time. The fitted line allows me to calculate the theoretical option value for the exact 14 days to maturity.

FIGURE A.1. Time-decay plots with fitted polynomial. Source: Carr and Wu (2003)




*Notes:* Figures depict time-decay plots for FTSE100 index options. Markers represent actual option prices while lines represent the fitted polynomial (see equation A.1). The left panel shows time-decay of FTSE100 index options on the 9th of April 1999, a very tranquil period when jumps were very unlikely. The right panel shows time-decay FTSE100 index options in 3rd of May 2000, a more volatile period when jumps were more likely. The dates are chosen to match the graph in Carr and Wu (2003) article so as to make a comparison. In each figure the three lines, from the bottom to the top, represent three moneyness levels of out-of-the-money option prices: -10% (red, solid line), -8.75% (blue, dashed line) and -7.5% (green, dotted line).

TABLE A.2. The proportion of maturity nodes in the data

| Number of options | S&P 500 | FTSE 100 | Eurostoxx 50 |
|---|---|---|---|
| 6 | 17% | 91% | 86% |
| 5 | 18% | 9% | 7% |
| 4 | 65% | 0% | 6% |

*Notes:* The table reports the structure of available options data used in this study. For S&P500 and FTSE100 options data ranges from January 1996 to December 2013 with many missing points for S&P500 in the earlier part of the sample. The Eurostoxx50 options data ranges from March 2002 to December 2013, and also exhibits missing data in the earlier part of the sample. Missing data points are due to the fact that for those dates only 3 options with different maturities were available. Those data points were discarded.

The number of options used in the approximation varies over time and is driven by the data availability. I use 3 to 6 option maturities to fit the polynomial - Table A.2 shows details for each index.[13] I should expect to get the best results for the FTSE100 index as its option data displays the highest quality - shortest maturities and most of the dataset is covered by 6 maturities. However given that the S&P500 index is the only one present in the original Bollerslev and Todorov (2011) study I will use this to start my robustness check.

First of all it might be noted that the dynamics of tail measures calculated on the bias of the approximation follows nearly the same pattern as the one of Bollerslev and Todorov (2011) (see Figure A.2). The two biggest differences are a jump in the tail measure in the early 1996 that is only present in my calculation and a more pronounced response of my tail measure to the 'dotcom bubble' burst in the late 2001. Unfortunately I do not have

[13] It should be noted that for certain periods I only had 3 options at my disposal. Those data-points were removed from the dataset, leading to signicant number of missing datapoints for S&P500 and Eurostoxx50 series especially visible before 2003.

Table A.3. GMM estimates of the Q-tail parameters

| | BT: S&P500 | | S&P500 | |
|---|---|---|---|---|
| | LT | RT | LT | RT |
| $\xi$ | 0.2581 | 0.0793 | 0.2570 | 0.0615 |
| | (0.0282) | (0.0147) | (0.0130) | (0.0161) |
| $\sigma$ | 0.0497 | 0.0238 | 0.0513 | 0.0242 |
| | (0.0021) | (0.0010) | (0.0009) | (0.0006) |
| $\alpha v$ | 0.9888 | 0.5551 | 1.1431 | 0.7266 |
| | (0.0525) | (0.0443) | (0.0142) | (0.0156) |

*Notes:* The table compares parameter estimates obtained by Bollerslev and Todorov (2011), the BT: S&P500 column, and estimates obtained in this article where I use approximated 14-day to maturity options. LT and RT denote estimates for the left tail and right tail, respectively. Estimates are based on the same sample ranging from January 1996 to June 2007. It should be noted that the sample used by me has some missing data points prior to January 2003, moreover in the main article the calculations are based on the more up-to-date sample. Standard errors are reported in parenthesis.

Table A.4. Annualized jump intensities implied by the Q-tail distributions

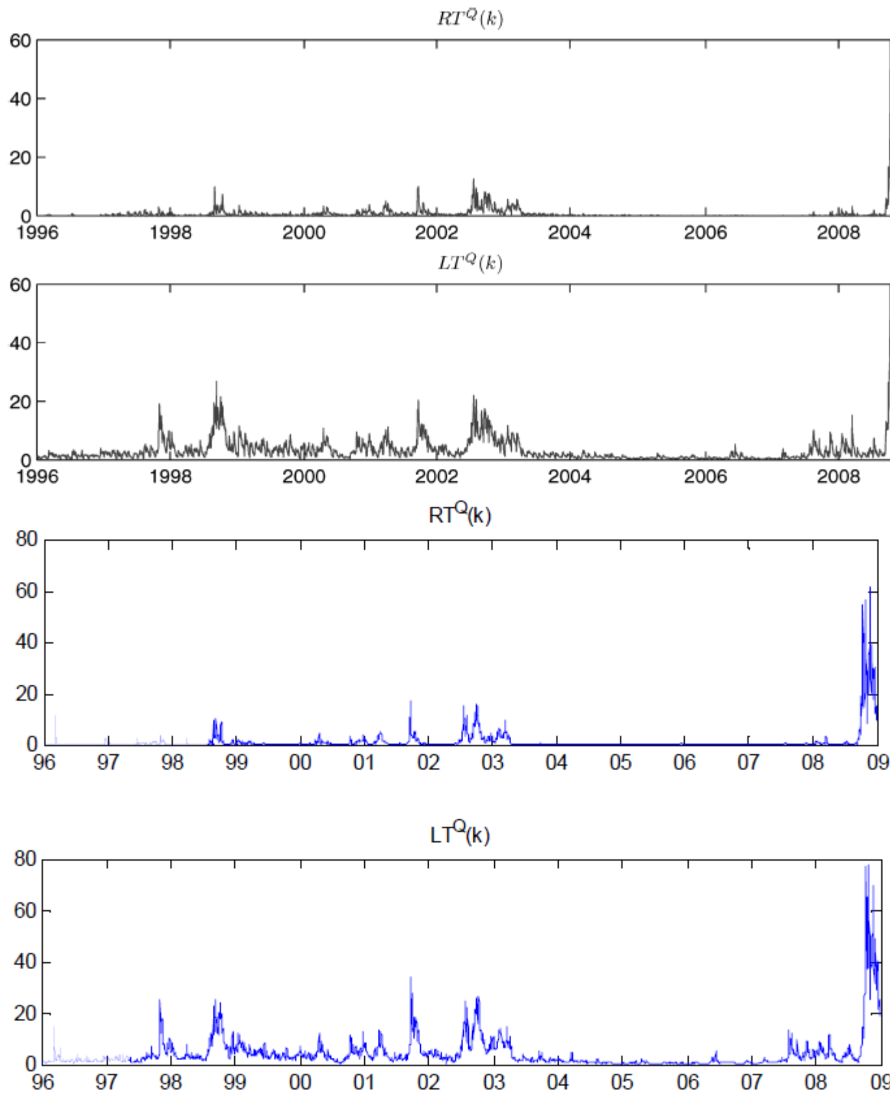
| Jump Size | BT:S&P500 | S&P500 |
|---|---|---|
| >7.5% | 0.5551 | 0.7266 |
| >10% | 0.2026 | 0.2666 |
| >20% | 0.0069 | 0.0082 |
| <-7.5% | 0.9888 | 1.1431 |
| <-10% | 0.5640 | 0.6627 |
| <-20% | 0.0862 | 0.1052 |

*Notes:* The table compares estimated average jump intensities over January 1996 to June 2007 sample obtained by Bollerslev and Todorov (2011), the BT: S&P500 column, and estimates obtained in this article where I use approximated 14-day to maturity options. Jump sizes are in terms of percentage changes in price levels. It should be noted that the sample used by me has some missing data points prior to January 2003, moreover in the main article the calculations are based on the more up-to-date sample.

the original time-series data of tail measures computed by Bollerslev and Todorov (2011), so I cannot calculate any goodness-of-fit measure. Yet, I can compare the GMM estimation results (see Table A.3). The estimates of the GPD are very close to each other especially for the left tail, as this tail is estimated with a higher accuracy. The only substantial difference is slightly higher estimates of the jump intensity parameters. However as one may note from the final results of the structure of the jump probabilities, the differences are not very large (see Table A.4). Judging by the sole comparison of my results to the ones of Bollerslev and Todorov (2011), it appears that the approximation does a very good job.

Yet, it is still important to see how well the approximation does with other indices. Here I cannot rely on others results, as to the best of my knowledge I am the first one to estimate those measures for other indices, namely Eurostoxx50 and FTSE100. Consequently I have looked at two fit measures and the volatility of the theoretical prices for different sets of

FIGURE A.2. Tail measures comparison between Bollerslev and Todorov (2011) study and this article.




*Notes:* Figures depict tail measures for the left $LT^Q(k)$ and the right tail $RT^Q(k)$ calculated from S&P500 options, where $k = 0.9$ and $k = 1.1$ for the left and the right tail, respectively. The first two panels are from the Bollerslev and Todorov (2011) article, where available closest-to-maturity options were used. Last two panels are based on my own calculations, where theoretical 14-day-to-maturity options are used to calculate tail measures.

maturity structures (see Table A.5 and Figure A.3). The simple goodness-of-fit measure ($R^2$) does not seem to be a good metric. It is exceptionally high for all indices as the dataset has only a small number of nodes. The MAPE of the fit evaluated only at the 14-days to maturity also seems to be very small, except for the FTSE100. In that case the MAPE value is ballooned by having a denominator very close to zero. It is very difficult to drive any

TABLE A.5. The fit of the time-decay polynomial

| | S&P 500 | FTSE 100 | Eurostoxx 50 |
|---|---|---|---|
| $R^2$ of the polynomial for 6 different moneyness levels | | | |
| Minimum | 98.95% | 86.04% | 99.53% |
| Average | 99.99% | 99.91% | 99.99% |
| Percentage error of predicted price for 14-days to maturity option | | | |
| MAPE | 0.25% | 2.86% | 0.36% |
| Maximum | 3.47% | 56.43% | 3.81% |

*Notes:* The table reports different measures of fit of 14-day to maturity option prices by the estimated polynomial. The top part of the table reports minimum and average determination coefficients $R^2$ of the daily regressions of the option time decay polynomial. The bottom part reports average and maximum observed percentage error of polynomial implied 14-day to maturity option price relative to the actual 14-day to maturity option price. Naturally, the bottom part uses only the observations where the actual 14-days-to-maturity options data were available.

conclusions from those simple fit metrics as they are based on an insufficient number of data points for each polynomial.

In order to overcome the problem of an insufficient number of data points I have looked at volatilities of theoretical 14-days to maturity option prices approximated using option prices with different maturities. In principle the volatility of the theoretical price should not depend on the set of nodes uses in the approximation (at least not to much). Of course if I extrapolate the 14-days price from a big distance the error of fit might generate a higher error than if I use actual maturities very close to the 14 days. Nonetheless it seems informative to investigate how much of the extra volatility is being caused by having distant maturities while performing the approximation. Figure A.3 presents inter-quartile ranges for theoretical 14-days prices.[14] The volatility of the theoretical price rises across minimum volatility pointing to certain losses caused by the approximation, but the increase does not seem to be excessive.

All in all it seems that the approximation is giving a good proxy for the original method especially as the estimates do not differ too much from the original study.

## Appendix B - GMM conditions to estimate GPD parameters in the Q-measure

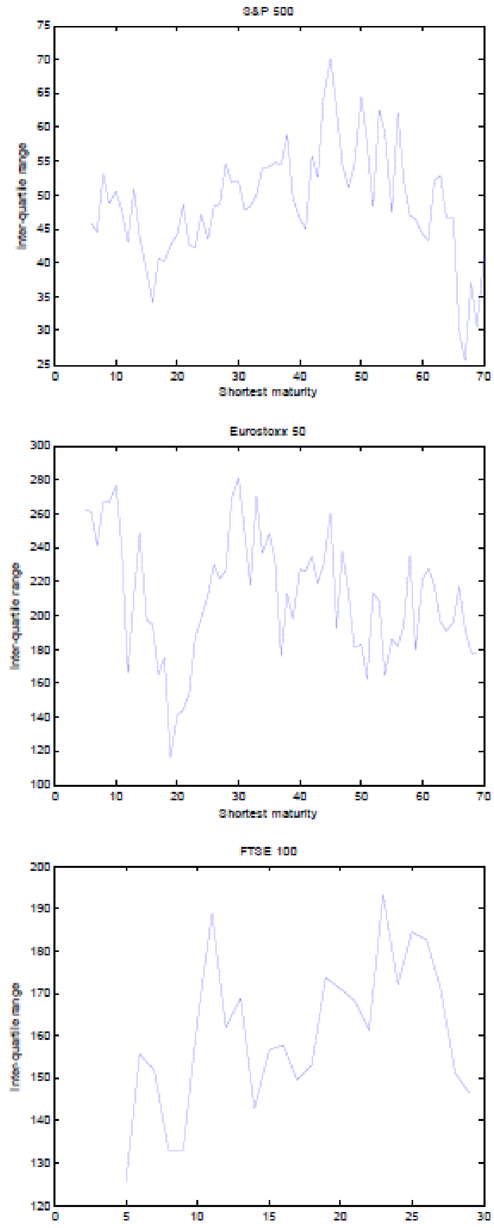
The aim of the GMM estimation for the Q-measure is to find the following vector of parameters for each tail:

$$\theta^Q = [\alpha_Q^{\pm}\bar{v}_{\psi}^{Q\pm}(tr^{\pm}); \xi_Q^{\pm}; \sigma_Q^{\pm}]$$

Those parameters are found by fulfilling the following three moment conditions:

[14]Inter-quartile range is being used instead of standard deviations to make the measure robust to outliers.

Figure A.3. Inter-quartile ranges of the theoretical 14-day to maturity option prices for different set of maturities used in the approximation




*Notes:* Figures depict inter-quartile ranges of the approximated (theoretical) 14-day to maturity option price. For all indices -10% out of the money options were approximated and used for the approximation. Horizontal axis denotes the shortest maturity used for the approximation. Inter-quartile ranges are used instead of standard deviations to circumvent the impact of outliers. For S&P500 and FTSE100 data ranged from January 1996 to December 2013 with many missing points for S&P500 in the earlier part of the sample. For Eurostoxx50 data ranged from March 2002 to December 2013, also exhibiting missing data in the earlier part of the sample.

$$E(LT_t^Q(k)) = \alpha_Q^- \bar{v}_\psi^{Q-}(tr^-)\frac{\xi_Q^-}{\xi_Q^-+1}\left(e^k\right)^{1+1/\xi_Q^-}\left(\frac{\xi_Q^-}{\sigma_Q^-}\right)^{-1/\xi_Q^-} *$$
$$*{}_2F_1\left(1+\frac{1}{\xi_Q^-};\frac{1}{\xi_Q^-};2+\frac{1}{\xi_Q^-};\frac{tr^-\frac{\xi_Q^-}{\sigma_Q^-}-1}{e^{-k}\frac{\xi_Q^-}{\sigma_Q^-}}\right)$$
$$E(RT_t^Q(k)) = \alpha_Q^+ \bar{v}_\psi^{Q+}(tr^+)\frac{\sigma_Q^+}{1-\xi_Q^+}\left(1+\frac{\xi_Q^+}{\sigma_Q^+}(e^k-1-tr^+)\right)^{1-1/\xi_Q^+}$$

where ${}_2F_1$ is a hypergeometric function and $E(LT_t^Q(k))$ and $E(RT_t^Q(k))$ are sample averages of the introduced tail-measures for right and left tails respectively. Standard errors of estimates are obtained using the delta method.

Parameter estimates for the right tail are presented in the Appendix F.

## Appendix C - Realized and continuous variation

In order to compute P-measure components of the VRP and the crash risk VRP(k) we need to compute realized variance (RV) and extract the continuous variation ($\sigma_t^2$) from it.

Daily RV is computed using 5-minute high frequency intra-day data on prices of index futures ($F_t$). More specically, RV is a sum of squared changes of log prices of index futures ($f_t$) scaled-up by the average overnight contribution $O$:

$$RV_t = \left[\sum_{i=1}^{n-1}\left(f_{t+i\Delta}-f_{t+(i-1)\Delta}\right)^2\right]*O$$

where $n$ is the number of daily prices available in the data, $\Delta$denotes the 5- minute time increment, and the overnight scaling factor $O$ is computed in the following way: $O = 1+\frac{\sum_{t=1}^{T}(f_t-f_{t-1+(n-1)\Delta})/T}{\sum_{t=1}^{T}\left(\sum_{i=1}^{n-1}\left(f_{t+i\Delta}-f_{t+(i-1)\Delta}\right)\right)/T}$.

In calculating continuous variation ($\sigma_t^2$) I follow directly the methodology suggested by Mancini (2001). Essentially the calculations resemble those for RV, with the exception that only the change in log prices that are smaller than the time-varying threshold $\alpha_t$ are added:

$$\sigma_t^2 = \left[\sum_{i=1}^{n-1}\left(f_{t+i\Delta}-f_{t+(i-1)\Delta}\right)^2\mathbb{I}_{\left\{|f_{t+i\Delta}-f_{t+(i-1)\Delta}|\le\alpha_t\right\}}\right]*O$$

where $\mathbb{I}$ is an indicator function amounting to one if the absolute change falls below the threshold $\alpha_t$ and zero otherwise.

The time-varying threshold $\alpha$ should take into account the intra-day volatility patterns as well as time varying volatility across days. In order to control for the first one I estimate daily volatility patterns for each futures index. First I set a general truncation level $\bar{\alpha}$ for the whole dataset, so that my calculations are not biased by outliers. The general truncation level $\bar{\alpha}$ is based on the average sample volatility for 5-minute log-price change, measured by the bi-power variation:

$$\bar{\alpha} = 3\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\sum_{i=2}^{n-1}|f_{t+i\Delta} - f_{t+(i-1)\Delta}||f_{t+(i-1)\Delta} - f_{t+(i-2)\Delta}|}\left(\frac{1}{n}\right)^{0.49}$$

In turn, this threshold is used to calculate the average log-price variation for every 5-minutes of the trading day (only for the data falling below the threshold):

$$Var_i = \frac{\sum_{t=1}^{T}\left(f_{t+i\Delta} - f_{t+(i-1)\Delta}\right)^2 \mathbb{I}_{\left\{|f_{t+i\Delta}-f_{t+(i-1)\Delta}|\leq\bar{\alpha}\right\}}}{\sum_{t=1}^{T}\mathbb{I}_{\left\{|f_{t+i\Delta}-f_{t+(i-1)\Delta}|\leq\bar{\alpha}\right\}}}$$

Finally, in order to obtain the time-of-day factor ($TOD_i$) I normalize each 5-minutes variation ($Var_i$) by the total sample truncated variation ($Var_{TOT}$):

$$Var_{TOT} = \frac{\sum_{t=1}^{T}\sum_{i=1}^{n-1}\left(f_{t+i\Delta} - f_{t+(i-1)\Delta}\right)^2 \mathbb{I}_{\left\{|f_{t+i\Delta}-f_{t+(i-1)\Delta}|\leq\bar{\alpha}\right\}}}{\sum_{t=1}^{T}\sum_{i=1}^{n-1}\mathbb{I}_{\left\{|f_{t+i\Delta}-f_{t+(i-1)\Delta}|\leq\bar{\alpha}\right\}}}$$

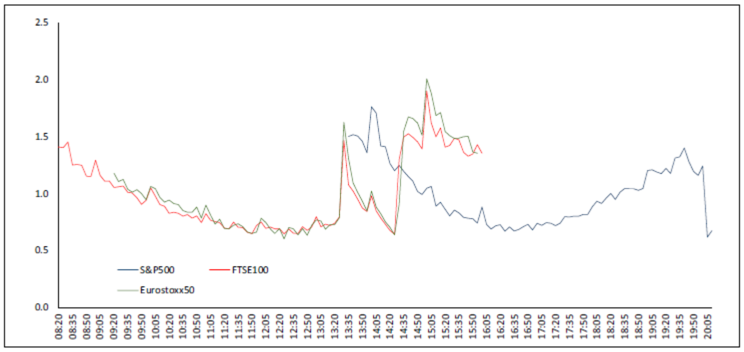
$$TOD_i = \frac{Var_i}{Var_{TOT}}$$

Figure C.1 plots TOD factors for all three analyzed indices on the standardized GMT scale. All time of day volatility patterns roughly exhibit a U shape, showing that most of the volatility comes at the beginning and closing of trading time. In addition European indices, Eurostoxx50 and FTSE100, also experience a large increase in volatility at the opening time of the New York Stock Exchange. Whereas the closure of the European trading has a rather minuscule impact on the S&P500 daily volatility pattern.

The daily dynamic pattern for the time-varying threshold is captured by linking threshold value $\alpha_t$ with lagged values of estimated continuous volatility per 5 minute log-price change $\sigma_{t-1}/\sum_{i=1}^{n-1}\mathbb{I}_{\left\{|f_{t-1+i\Delta}-f_{t-1+(i-1)\Delta}|\leq\alpha_{t-1}\right\}}$. Taking both time-of-day factor and lagged continuous volatility I obtain formula for the time-varying threshold:

$$\alpha_{t,i} = 3\frac{\sigma_{t-1}}{\left(\sum_{i=1}^{n-1}\mathbb{I}_{\left\{|f_{t-1+i\Delta}-f_{t-1+(i-1)\Delta}|\leq\alpha_{t-1}\right\}}\right)^{0.49}}TOD_i$$

Figure C.1. Time-of-day factor




*Notes:* The figure shows the estimated time-of-day factor for the S&P500, FTSE100 and Eurostoxx50, the x-axis is GMT. The estimates are based on 5-minute high frequency data on futures prices from January 1996 to December 2012 for S&P500 and FTSE100, and from January 1999 to December 2012 for Eurostoxx50.

## Appendix D - GMM conditions to estimate GPD and intensity parameters in the P measure

The aim of the GMM estimation of the P measure is to find the following vector of parameters for each tail:

$$\theta^P = [\alpha_0^{\pm}\bar{v}_{\psi}^{\pm}(tr^{\pm}); \alpha_1^{\pm}\bar{v}_{\psi}^{\pm}(tr^{\pm}); \xi^{\pm}; \sigma^{\pm}]$$

The four moments conditions are as follows:

$$\frac{1}{N}\sum_{t=1}^{N}\sum_{j=1}^{n-1}\phi_i^{\pm}\left(\psi^{\pm}(\Delta_j^{n,t}p) - tr^{\pm}\right)1_{\{\psi^{\pm}(\Delta_j^{n,t}p)>tr^{\pm}\}} = 0 \quad i = 1,2$$

$$\frac{1}{N}\sum_{t=1}^{N}\sum_{j=1}^{n-1}1_{\{\psi^{\pm}(\Delta_j^{n,t}p)>tr^{\pm}\}} - \alpha_0^{\pm}\bar{v}_{\psi}^{\pm}(tr^{\pm}) - \alpha_1^{\pm}\bar{v}_{\psi}^{\pm}(tr^{\pm})CV_t = 0$$

$$\frac{1}{N}\sum_{t=2}^{N}\left(\sum_{j=1}^{n-1}1_{\{\psi^{\pm}(\Delta_j^{n,t}p)>tr^{\pm}\}} - \alpha_0^{\pm}\bar{v}_{\psi}^{\pm}(tr^{\pm}) - \alpha_1^{\pm}\bar{v}_{\psi}^{\pm}(tr^{\pm})CV_t\right)CV_{t-1} = 0$$

where:

$$\phi_1^{\pm}(u) = -\frac{1}{\sigma^{\pm}} + \frac{\xi^{\pm}u}{(\sigma^{\pm})^2}\left(1 + \frac{1}{\xi^{\pm}}\right)\left(1 + \frac{\xi^{\pm}u}{\sigma^{\pm}}\right)^{-1}$$

$$\phi_2^{\pm}(u) = \frac{1}{(\xi^{\pm})^2}ln\left(1 + \frac{\xi^{\pm}u}{\sigma^{\pm}}\right) - \frac{u}{\sigma^{\pm}}\left(1 + \frac{1}{\xi^{\pm}}\right)\left(1 + \frac{\xi^{\pm}u}{\sigma^{\pm}}\right)^{-1}$$

## Appendix E - A short guide on how to get VRP(k) from the GMM estimates

This is a very short and basic instruction on how to derive VRP(k) for any given threshold k based on estimates. All of the following results are based on the derivations presented in the appendix of the Bollerslev and Todorov (2011) paper.

Let us have a look at the tail volatility measure first. The measure can be presented as a sum of two components:

$$\int_{x>k} x^2 v(x)dx = 2\bar{v}_{\psi}^{+}(tr^{+}) * K_1 + k^2\bar{v}_{\psi}^{+}(e^k - 1)$$

The first part of the sum is directly determined by my estimates. For the selected threshold of $tr^{+} = 0.075$ I have estimated the value directly:

$$\bar{v}_{\psi}^{+}(tr^{+}) = \alpha_{Q}^{+}\bar{v}_{\psi}^{Q+}(0.075)$$

The multiplier $K_1$ is also directly defined by the estimated parameters:

$$K_1 = e^{-k/\xi^{+}}\xi^{+}\left(\frac{\xi^{+}}{\sigma^{+}}\right)^{-1/\xi^{+}}[\xi^{+}{}_3F_2\left(\frac{1}{\xi^{+}},\frac{1}{\xi^{+}},\frac{1}{\xi^{+}};1+\frac{1}{\xi^{+}},1+\frac{1}{\xi^{+}};\frac{\frac{\xi^{+}}{\sigma^{+}}(tr^{+}+1)-1}{e^k\frac{\xi^{+}}{\sigma^{+}}}\right)$$
$$+k_2F_1\left(\frac{1}{\xi^{+}},\frac{1}{\xi^{+}};1+\frac{1}{\xi^{+}};\frac{\frac{\xi^{+}}{\sigma^{+}}(tr^{+}+1)-1}{e^k\frac{\xi^{+}}{\sigma^{+}}}\right)]$$

The second part of the sum can be obtained from the approximation to the GPD. Following Bollerslev and Todorov (2011) I assume that for a large threshold value the following approximation holds with equality:

$$1-\frac{\bar{v}_{\psi}^{+}(u+x)}{\bar{v}_{\psi}^{+}(x)} = G(u;\sigma^{+},\xi^{+})$$

where $G()$ denotes a GPD. Assuming that $x = tr^{+}$, $u = e^k - 1 - tr^{+}$ and $tr^{+} = 0.075$, it is quite straight forward that:

$$\bar{v}_{\psi}^{+}(e^k - 1) = \left[1 - G\left(e^k - 1 - tr+;\sigma^{+},\xi^{+}\right)\right]\bar{v}_{\psi}^{+}\left(tr^{+}\right)$$

## Appendix F - Q-measure estimates of the right tail

TABLE F.1. Q-measure: estimation results for the right tail

| | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| $\xi$ | 0.1530 | 0.1143 | 0.1015 |
| | (0.0115) | (0.0114) | (0.0122) |
| $\sigma$ | 0.0278 | 0.0329 | 0.0272 |
| | (0.0006) | (0.0006) | (0.0004) |
| $\alpha\upsilon$ | 0.8049 | 1.1443 | 0.7383 |
| | (0.0184) | (0.0258) | (0.0156) |

*Notes:* The table reports estimated parameters of the generalized Pareto distribution of the right-tail under the risk neutral Q-measure: $\xi$ is the estimate of the shape parameter and $\sigma$ is the estimate of the scale parameter. $\alpha\upsilon$ is the estimate of the average annualized jump intensity of $+7.5\%$ jump in the price level. The estimates are based on S&P500 and FTSE100 options data from January 1996 to December 2013 and Eurostoxx50 options data from March 2002 to December 2013. The log-moneyness of options used to estimate parameters were 1.1000, 1.0875 and 1.0750. Estimated standard errors are reported in parentheses.

TABLE F.2. Q-measure: annualized jump intensity estimates for the right tail

| Jump Size | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| $>$7.5% | 0.8049 | 1.1443 | 0.7383 |
| $>$10% | 0.3462 | 0.5523 | 0.3065 |
| $>$20% | 0.0262 | 0.0488 | 0.0170 |

*Notes:* The table reports annualized average jump intensities under the Q-measure i.e. implied by the option prices. Jump sizes are in terms of percentage changes in price levels. In the case of S&P500 and FTSE100, averages are calculated from January 1996 to December 2013, and for Eurostoxx50 averages are calculated from March 2002 to December 2013. All the reported figures are based on generalized Pareto distribution estimates reported in Table F.1.

## Appendix G - P-measure estimates for the right tail

Table G.1. P-measure: estimation results for the right tail

| | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| $\xi$ | 0.2088 | 0.1648 | 0.2218 |
| | (0.0671) | (0.0739) | (0.0415) |
| $100\sigma$ | 0.1834 | 0.1955 | 0.1714 |
| | (0.0161) | (0.0189) | (0.0092) |
| $\alpha_0$ | -0.0020 | -0.0013 | -0.0028 |
| | (0.0001) | (0.0001) | (0.0001) |
| $\alpha_1$ | 0.0396 | 0.0291 | 0.0402 |
| | (0.0006) | (0.0005) | (0.0006) |

*Notes:* The table reports estimated parameters of the generalized Pareto distribution of the right-tail under the physical P-measure: $\xi$ is the estimate of the shape parameter and $\sigma$ is the estimate of the scale parameter. $\alpha_0$ and $\alpha_1$ are estimates of parameters of equation 1.11 linking jump intensities to the time-varying continuous volatilities. The estimates are based on high frequency 5-minute futures prices from January 1996 to December 2012 for S&P500 and FTSE100, and from January 1999 to December 2012 for Eurostoxx50. Estimated standard errors are reported in parentheses.

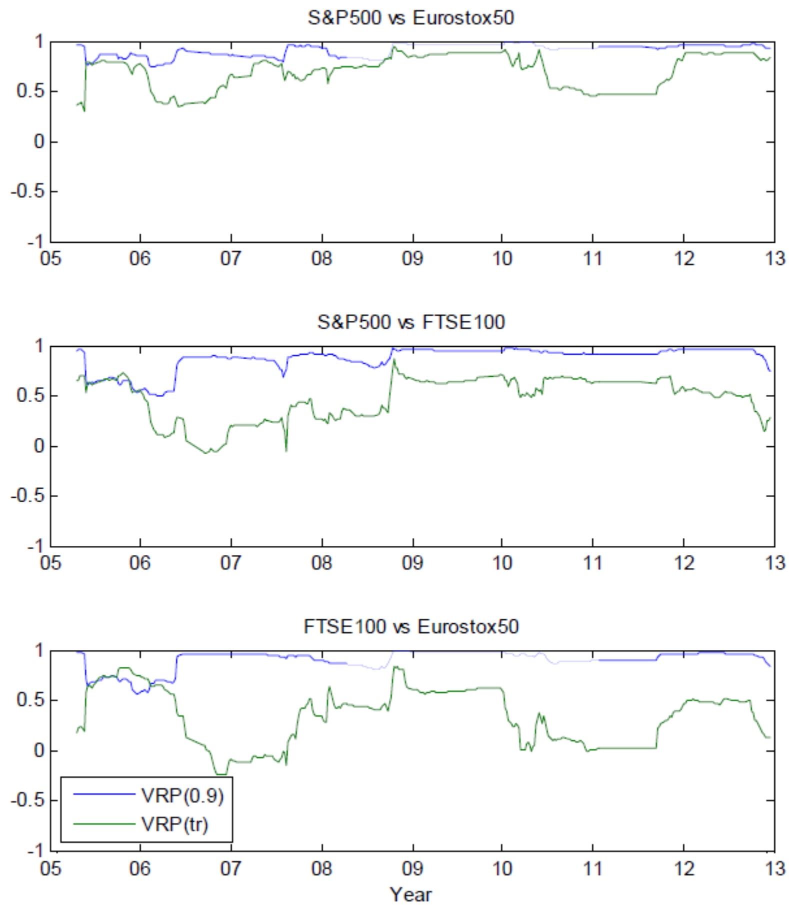
Table G.2. P-measure: annualized jump intensity estimates for the right tail

| Jump Size | S&P500 | Eurostoxx50 | FTSE100 |
|---|---|---|---|
| >7.5% | 0.0062 | 0.0016 | 0.0187 |
| >10% | 0.0016 | 0.0003 | 0.0052 |
| >20% | 0.0001 | 0.0000 | 0.0002 |

*Notes:* The table reports annualized average jump intensities under the P-measure i.e. based on the high frequency data estimation. Jump sizes are in terms of percentage changes in price levels. In case of S&P500 and FTSE100 averages are calculated from January 1996 to December 2012, and for Eurostoxx50 averages are calculated from January 1999 to December 2012. All the reported figures are based on generalized Pareto distribution estimates reported in Table G.1.

## Appendix H - Rolling Window correlations

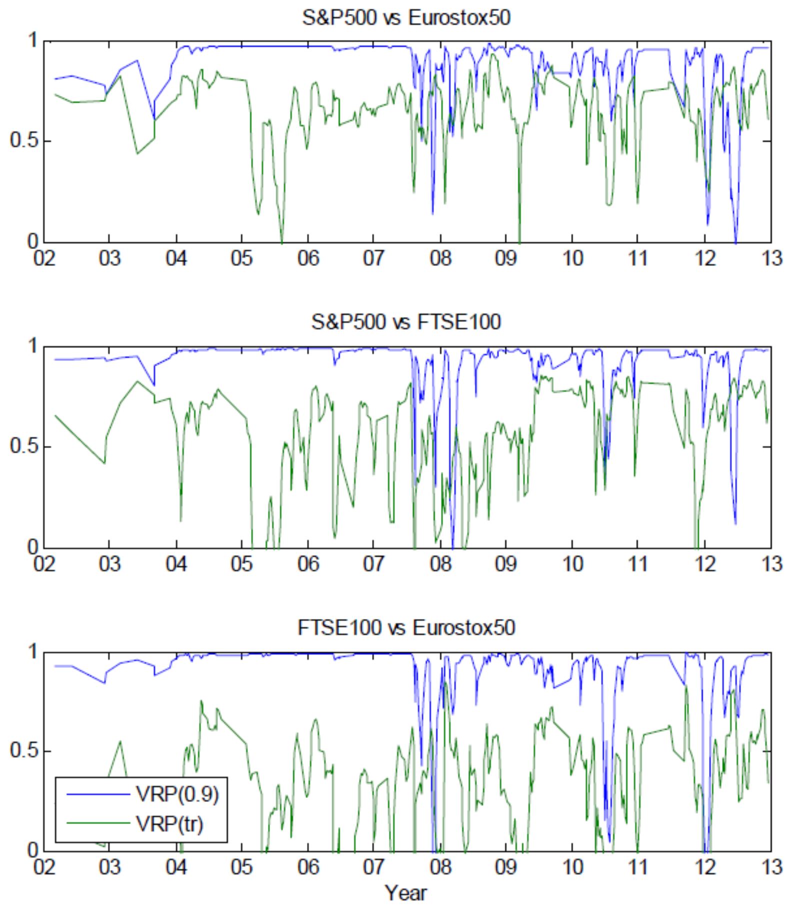
FIGURE H.1. Time varying correlations of VRP(tr) and VRP(0.9) for different index pairs




*Notes:* Figures show time patterns of the 50-week rolling window r-Pearson correlation coefficients between different indices for crash-risk premia VRP(0.9) and non-crash-risk premia VRP(tr). Correlation coefficients are calculated on a common sample of weekly data for all three indices from March 2002 to December 2012.

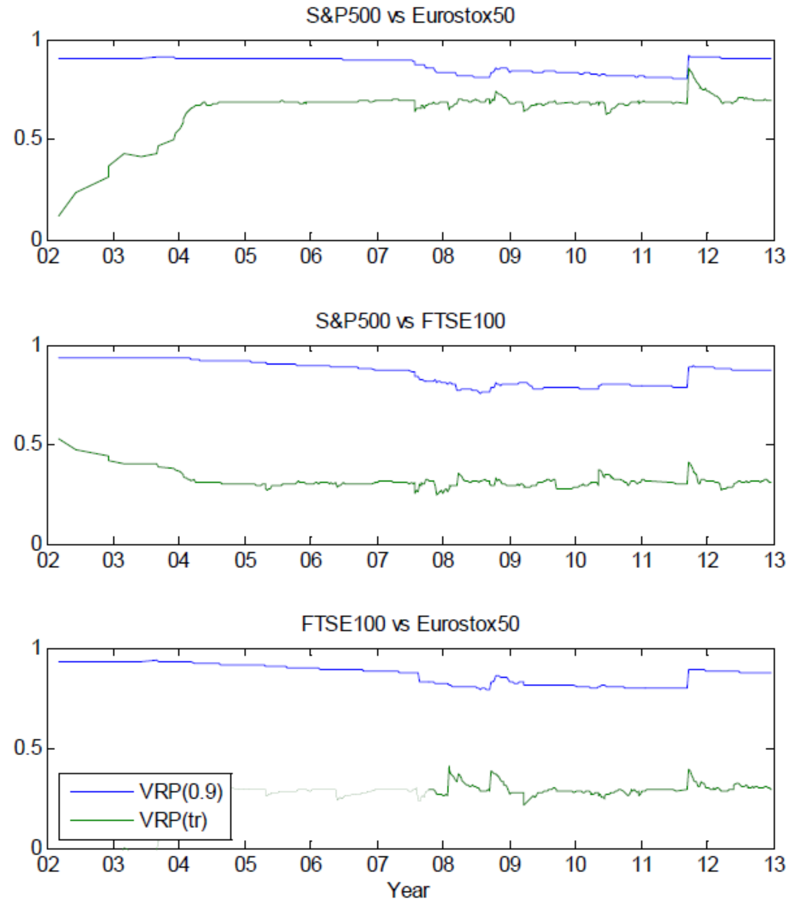
## Appendix I - Dynamic correlations with different level equations

FIGURE I.1. Dynamic conditional correlations on pure de-meaned data



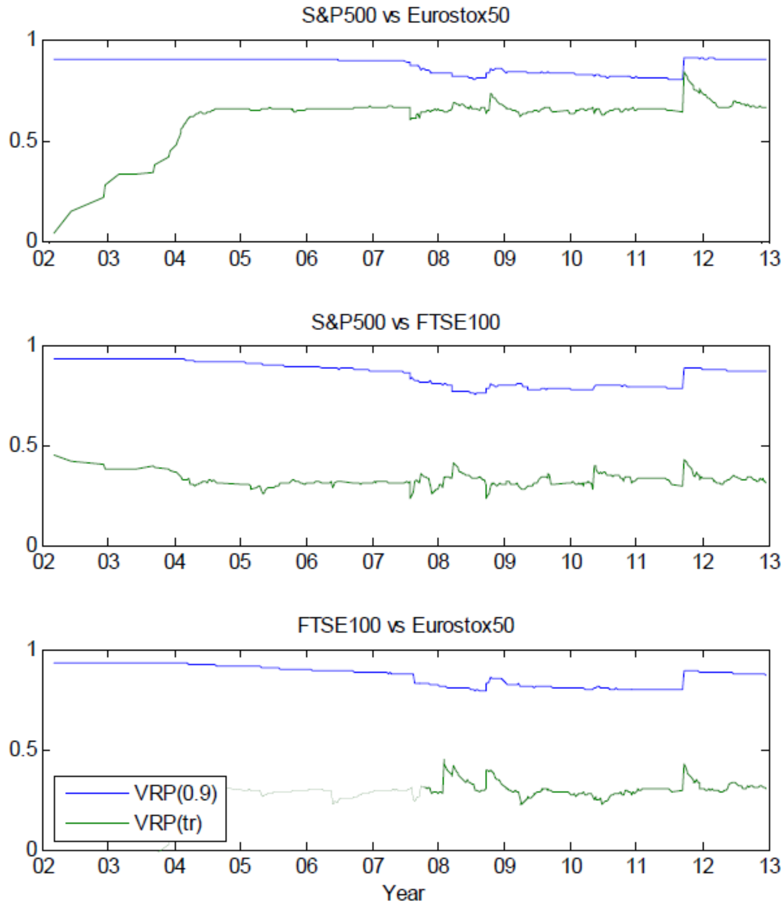

*Notes:* Figures show time patterns of the dynamic correlations of three index pairs for two premia measures: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Dynamic correlations are calculated using Dynamic Conditional Correlation model of Engle (2002). The model is based on a common sample of weekly data for all three indices from March 2002 to December 2012. The level equation of either VRP(0.9) or VRP(tr) for all three indices is modelled as a constant, i.e. each level equation only de-means the data and does not account for any individual index persistence.

FIGURE I.2. Dynamic conditional correlations, where the level equation is modelled as an AR(1) process



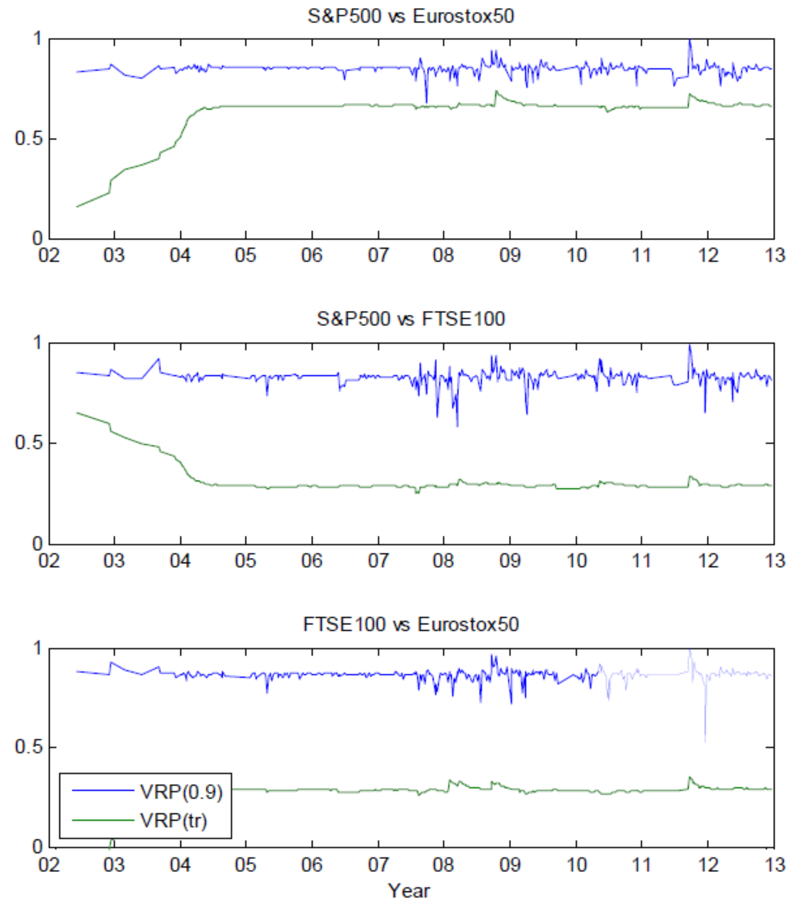

*Notes:* Figures show time patterns of the dynamic correlations of three index pairs for two premia measures: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Dynamic correlations are calculated using Dynamic Conditional Correlation model of Engle (2002). The model is based on a common sample of weekly data for all three indices from March 2002 to December 2012. The level equation of either VRP(0.9) or VRP(tr) for all three indices is modelled individually as an AR(1) process.

FIGURE I.3. Dynamic conditional correlations, where the level equation is modelled as an AR(SIC) process




*Notes:* Figures show time patterns of the dynamic correlations of three index pairs for two premia measures: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Dynamic correlations are calculated using Dynamic Conditional Correlation model of Engle (2002). The model is based on a common sample of weekly data for all three indices from March 2002 to December 2012. The level equation of either VRP(0.9) or VRP(tr) for all three indices is modelled individually as an AR(SIC) process, where the number of lags is selected using Bayesian information criterion.

FIGURE I.4. Dynamic conditional correlations, where the level equation is modelled as a VAR(1) process




*Notes:* Figures show time patterns of the dynamic correlations of three index pairs for two premia measures: crash-risk VRP(0.9) (i.e. the premium for holding volatility risk associated with -10% jump in the price of the underlying index futures) and non-crash-risk VRP(tr) (i.e. the premium for holding volatility risk not related to the market crash). Dynamic correlations are calculated using Dynamic Conditional Correlation model of Engle (2002). The model is based on a common sample of weekly data for all three indices from March 2002 to December 2012. The level equation of either VRP(0.9) or VRP(tr) for all three indices is modelled jointly as a VAR(1) process.

CHAPTER 2

# The Rietz-Barro crash risk: Does it bias the estimates of the risk aversion coefficient?


## Abstract

This paper assesses the impact of the Peso problem on the econometric estimates of the risk aversion coefficient. Rietz (1988) and subsequently Barro (2006) showed that the introduction of the crash risk allows the canonical general equilibrium framework to generate data consistent equity premia even under low risk aversion of the representative agents. They argue that the original data used to calibrate these models suffer from a Peso problem (i.e. does not encounter a crash state). To the best of my knowledge the impact of their Peso problem on the estimation of the risk aversion coefficient has not to date been evaluated. This paper seeks to remedy this. I find that crash states that are internalized by economic agents, but are not realized in the sample, generate only a small bias in the estimates of the risk aversion coefficient.

I also show that the introduction of the crash state has a strong bearing on the representative agents portfolio composition. In fact, under the internalized crash state scenario, agents exhibit positive bond holdings even in a frictionless environment.



**Keywords:** Equity Premium Puzzle, Risk Aversion, Crash Risk, GMM, Monte-Carlo, Peso Problem.

## 2.1. Introduction

The Equity Premium Puzzle (originally proposed by Mehra and Prescott (1985)) remains an important puzzle of modern finance literature. The key problem is the fact that only under implausibly high values of risk aversion parameter are general equilibrium models able to generate the equity premia observed in the data.

Rietz (1988) and, subsequently, Barro (2006) showed that introducing a crash state (a severe, albeit unlikely, drop in consumption) into a simple general equilibrium model can justify 'excessive' returns even under a reasonable risk aversion coefficient. Moreover, they point out that the crash itself is not observed in the standard samples analyzed pointing to the Peso problem.

As a consequence of the Peso problem we should expect that the estimates of a risk aversion coefficient will be biased upwards and potentially be high. However, studies focusing on the Euler equation estimation do not necessarily show this. For example, Hansen and Singleton (1982) get a low risk aversion coefficient estimate (c.a. 1), whereas Hall (1988) gets a much higher estimate (c.a. 10), but still lower than the value needed in a classical general equilibrium calibration (above 50, see Cochrane (2009)).

In this paper I assess the potential impact of the anticipated crash risk by consumers on the estimate of a risk aversion coefficient. The study is based on a battery of Monte-Carlo experiments assessing the size of the bias in the risk aversion estimate induced by a Rietz-Barro crash state. Two estimation techniques are applied in the study, the one of Hansen and Singleton (1982) as it should be resilient to different distribution assumptions, and the simple OLS as it is a natural benchmark.

I find that the values of the average bias generated by sizable crash risks (of the magnitude suggested by Rietz (1988)) is very small. Moreover, it turns out that even under classical assumptions i.e. without the Peso problem, the linear estimator does very poorly whereas the Hansen and Singleton (1982) estimator is efficient.

I also address another important puzzle linked to the Equity Premium Puzzle, the problem of portfolio composition. Heaton and Lucas (1997) claim that under canonical assumptions, economic agents never hold positive amounts of bonds, even though it clearly stands in contrast with the stylized facts. Their partial equilibrium model generates policy functions under which agents only use equity investments as buffer savings. Bonaparte et al. (2012) show that one may achieve positive bond holdings if equity investment is subject to portfolio adjustment costs. My model, in contrast to the existing literature, shows that the presence of tail dependent crash in endowment and equity income may generate positive bond holdings even in a frictionless environment.

The remainder of the paper is organized as follows. Section 2 discusses the literature on equity premia. Section 3 presents households portfolio decision problem with tail dependent crash . Section 4 elaborates on numerical solutions of the model under different parametrizations. Section 5 discusses the Monte Carlo experiment and its results. Finally, Section 6 concludes.

## 2.2. Literature

Mehra and Prescott (1985) have showed that the average return to equity investment over the years 1889-1978 amounted roughly 7.0%, whereas a return to holding short term bonds was only 0.8%. This observation led to an Equity Premium Puzzle (EPP) that states that the equity premium (difference between the average equity return and bond return) cannot be justified as the sole compensation for risk under plausible levels of a risk aversion parameter. More specifically Mehra and Prescott (1985) use a simple Lucas-tree type of economy (see Lucas Jr (1978)) with a power utility function to calculate the prices of equity and bonds in the general equilibrium set-up. They take the data on consumption for the US and assume that they represent the true stochastic consumption process of a representative household.[1] They find that the model is unable to generate the equity premium observed in the data under any values of parameters in the utility function from plausible domains i.e. risk aversion $\gamma\epsilon(0,10)$ and $\beta\epsilon(0,1)$. They find that the maximum size of equity premium that can be justified amounts to around 2p.p., which obviously stands in contrast to the data. Cochrane (2005) points out that a value of at least $\gamma > 50$ for the risk aversion coefficient would be needed to attain the observed premium, and an even higher value to explain low returns to bonds[2].

Rietz (1988) showed that the EPP disappears if we introduce an extra state ('crash state') of consumption to the original model. The crash state is characterized by a very low consumption value and by a low probability attached to that state. Rietz argued that the data used for a Mehra and Prescott (1985) study are biased and if we had been able to use a dataset containing events like revolutions or wars, or at least better reflecting Great Depression, we would have observed substantial but rare drops in consumption and consequently in equity prices.

The problem of an anticipated negative event, though not present in the data, is known as a Peso problem and was originally used to explain excessive returns to currency speculation (see Lewis (1994)). Rietz's solution, although very convincing and simple, was largely criticized by the literature (see Mehra and Prescott (1988)). First of all, other researches found it implausible and difficult to prove that the Peso problem really exists in a dataset covering 100 years of data. Secondly, addressing the idea of revolutions and wars in other countries Campbell (2000) points out that it is difficult to believe that in such events bonds will still pay

[1] They approximate the process by two and five discrete states of consumption.

[2] For a more detailed analysis of the EPP puzzle please refer to a splendid survey by Kocherlakota (1996).

the promised values[3]. It seems that these two issues have been successfully resolved. Ursúa (2011) presented a new global dataset on consumption and equity returns showing that crash type of events occurred in the last century. Second, Bollerslev and Todorov (2011) for equity and Jurek (2014) for currency speculation showed that deep out-of-the-money options that could only protect investment from crash states, are quite valuable. Hence implying that the excessive premia are mainly a reward for exposure to a crash risk. This empirical evidence also showed that market participants internalize a possibility of equity or currency market crash. At the same time one could notice that US CDS (i.e. instruments protecting from bonds default) do not discount any reasonable possibility of default in the US treasuries.

It seems straight forward that as a Peso problem has such a deep impact on a general equilibrium model it should have a parallel effect on the estimates of the risk aversion parameter. However, contrary to calibrated general equilibrium models, estimation of risk aversion parameter based on either linearized Euler equation or on Hansen and Singleton (1982) methodology leads to ambiguous results. For example, Hansen and Singleton (1982) on the base of aggregate data find the risk aversion parameter to be close to 1, whereas Hall (1988) using microdata finds this coefficient to be c.a. 10.[4] Hence still substantially below the general equilibrium prediction. This finding turns us to the key question of this paper, i.e. What is the size of the bias in the risk aversion coefficient estimate that can be attributed to a crash-Peso event?

Literature has also proposed solutions other than Rietz's. The EPP was addressed mainly along the lines of different utility functions, habit formation, borrowing constraints and taxes, and heterogeneous agents with incomplete markets. In case of utility function modifications, most notable is the usage of Epstin-Zin-Weil (EZW) preferences that allows for a split of the elasticity of intertemporal substitution and the risk aversion coefficient (see Epstein and Zin (1991)). Barro (2006) combines the crash risk with EZW preferences showing that his model closely tracks the actual data processes, but also claiming that EZW preferences cannot address the EPP by itself. Campbell and Cochrane (1999b) focus on habit formation and show that time varying risk aversion could potentially address the EPP. Their model predicts that risk aversion is time varying and highly counter cyclical, which might be a disputable finding. Another solution may lie within the relaxation of the market completeness assumption. Mankiw (1986) showed that idiosyncratic risk could make the representative consumer seem more risk averse than he actually is. The additional departure from the basic representative household might be due to borrowing constraints varying over the lifecycle, for example Constantinides et al. (2002) show that an age varying budget constraint makes

[3]The same point on the constant return to bonds in the crash state was also made by DeLong in reference to Barro (2006) article.

[4]It should be noted that Hall (1988) estimates the Elasticity of Intertemporal Substitution, which for the given utility function is just an inverse of the risk aversion.

equity and consumption paths change over the lifecycle. However, most of those solutions point to a failure of a representative household framework which is contrary to Rietz (1988), who shows that the baseline model still can be saved.

## 2.3. The Model

In this section I describe in detail the decision theoretic model of portfolio choice. This model is used to analyze the impact of rare events on portfolio composition under different model parametrizations. It is also used as a tool to obtain a Data Generating Process (DGP) for key variables used in the Monte-Carlo study of the risk aversion estimator.

**2.3.1. Household decision problem.** I build a simple portfolio choice model to obtain policy functions determining the demand for bonds and equities and households consumption patterns. Since I am interested in household's portfolio allocation under different stochastic processes for endowment and equity returns, I focus on a partial equilibrium framework. In fact, my model very closely resembles the one of Heaton and Lucas (1997). In that model stock and bond returns as well as endowment[5] are treated as exogenous stochastic variables that will be defined precisely in one of the following sections. But their model does not internalize a potential crash. Let us first look at the household optimization problem.

*Household maximization problem.* Households choose sequences of consumption ($C_t$), equity holdings ($A_t$) and bond holdings ($B_t$) that maximize their expected lifetime utility:

$$max_{C_t,A_t,B_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{2.2}$$

Future utility is discounted at the time-invariant rate $\beta$. The household's maximization problem is subject to a sequence of budget constraints:

$$C_t = Y_t + R_t^f B_{t-1} + R_t^e A_{t-1} - A_t - B_t \tag{2.3}$$

In each period expenditure on consumption ($C_t$) is financed by an endowment $Y_t$, the gross return on bonds $R^f B_{t-1}$ and equities $R_t^e A_{t-1}$, both bought in the previous period $(t-1)$, less current period $(t)$ investment in bonds ($B_t$) and equities ($A_t$). In this model I assume that the gross return on bonds $(R^f)$[6] is time-invariant, while endowment ($Y_t$) and gross return on equities $(R_t^e)$[7] are both stochastic. Hence I have two sources of uncertainty in the model.

In addition, it is also assumed that households cannot borrow nor take short positions on equities at any point in time:

[5]In Heaton and Lucas (1997) article endowment is referred to as labor income, but there is no labour supply choice in the model.

[6]$R^f = 1 + r^f$ where $r_t^f$ denotes the return on bonds.

[7]$R_t = 1 + r_t$ where $r_t$ is the stochastic return to investment made in period t-1.

$$B_t \geq 0 \tag{2.4}$$

$$A_t \geq 0 \tag{2.5}$$

Following a vast literature on portfolio choice I assume contemporaneous utility to be described by a constant relative risk aversion (CRRA) function:

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} \tag{2.6}$$

where $\gamma$ denotes the risk aversion parameter. The greater the $\gamma$, the more risk averse the household is. Please note that Heaton and Lucas (1997) also consider the case of habit persistence in the utility function, but as I will show later there is no need for habit persistance to generate positive bond holdings.

**2.3.2. Stochastic environment.** Since I opted for a partial equilibrium framework, endowment and equity returns are treated as exogenous stochastic variables. Consequently, I need to specify the joint stochastic process generating the endowment and gross returns on equities.

I model equity returns as a simple AR(1) process. Parameters of the process are set to the exact same values as in the study of Heaton and Lucas (1997):

$$R_t^e = \alpha + \lambda R_{t-1}^e + \epsilon \qquad \epsilon \sim N(0, \sigma_\epsilon) \tag{2.7}$$

where parameters are found to be: $\alpha = 1.0775$, $\lambda = 0.0$ and $\sigma_\epsilon = 0.157$. This means that in the base scenario I do not allow for any persistence in equity returns. Yet, I keep this formulation of the stochastic process as I will allow for persistent equity returns in one of the auxiliary scenarios.

The endowment process is also described by a simple AR(1). Yet again, I follow Heaton and Lucas (1997) and apply the parametrization estimated by them:

$$log(Y_t) = \alpha_Y * (1 - \lambda_Y) + \lambda_Y log(Y_{t-1}) + \epsilon_Y \qquad \epsilon_Y \sim N(0, \sigma_{\epsilon_Y}) \tag{2.8}$$

where parameters are estimated to be: $\alpha_Y = log(100)$, $\lambda_Y = 0.53$ and $\sigma_{\epsilon_Y} = 0.24$. The average income does not match any actual data, it is treated as the numeraire of the model.

Both processes are discretized using the Tauchen (1986) algorithm. Equity returns are described by $N_R$ states with corresponding Markov transition matrix $\Pi_R$. Endowment is represented by $N_Y$ states with $\Pi_Y$ Markov transition matrix[8]. Finally, as pointed out by Heaton and Lucas (1997) or more recently by Bonaparte and Cooper (2009), the endowment is not correlated with equity returns. Consequently, I assume that the two processes are independent in 'normal' times, hence the joint stochastic process is described by $N_R * N_Y$ states with Markov transition matrix being a simple Kronecker product of the two underling Markov matrices: $\Pi = \Pi_R \otimes \Pi_Y$.

The crash state is introduced as an additional possible state. In the baseline scenario this state occurs simultaneously for the endowment and the gross equity return. Following Rietz (1988) and Barro (2006), I introduce the crash to be equally probable from any 'normal' state. Moreover, in the period directly following the crash, any state of nature may happen with equal probability except for another crash. I assume that there cannot be two subsequent crashes. This is why the final stochastic process is described by $N_R * N_Y + 1$ states and the following Markov transition matrix:

$$\Pi^{Crash} = \begin{bmatrix} (1-p) * \Pi & p \\ \frac{1}{N_R * N_Y} & 0 \end{bmatrix} \tag{2.9}$$

The crash is fully defined by two parameters: its probability $p$ and the size of the crash state defined by a multiplier of standard deviation $m_{crash}$. More precisely, the return to equity investment or endowment in a crash state is defined as its mean value minus $m_{crash}$ times the standard deviation of the original process. Following Rietz (1988), in the baseline scenario the crash probability is set equal to $p = 3$ and the multiplier is set to $m_{crash} = 3$. This formulation means that the crash state is fully tail-dependent i.e. if a crash occurs in the endowment it also occurs in the equity return.[9]

**2.3.3. Solution method.** The household's optimization problem described by the set of equations from 2.2 to 2.6, along with it's stochastic environment described by equations from 2.7 to 2.9 can be solved using many different techniques. For example, one could use value function iterations, policy function iterations or policy function approximation using Chebyshev polynomials. Heaton and Lucas (1997) use both - methods based on policy function iterations and on policy function approximations. In contrast, I use the method of value function iterations due to its robustness. This method guarantees to find an optimal policy function even if this function is highly non-linear and/or the stochastic set-up is very non-Gaussian.[10] The latter is definitely the case in the presence of rare crashes. In order to

[8] In the baseline scenario I assume both processes to have 5 states. The exact numbers for 5 states under baseline scenario are available in the Appendix A

[9] An alternative formulation of the crash risk could model the crash in endowment and in the equity market separately and join their distribution with some form of dependency, for example using Gumbel copula.

[10] Figures B.1 and B.2 in Appendix B represent sample policy functions for the average level of endowment $Y = 100$ and average return on equity $R^e = 1.0775$.

use the value function iteration method, I need to re-formulate the maximization problem into it's recursive representation.

The household's problem in recursive form can be written as:

$$v(Y,R^e,A,B) = max_{A',B'}\{u(Y+R^fB+R^eA-A'-B')+ +\beta E_{Y',R'|Y,R}v(Y',R'^e,A',B')\} \tag{2.10}$$

Please note that I have combined the maximization problem from equation 2.2 with the budget constraint given by equation 2.3. The value function is a function of four state variables. Current endowment ($Y$) and return on equity ($R^e$) are the two exogenous state variables, while equity holdings ($A$) and bond holdings ($B$) are the two endogenous state variables. Households decide on the future equity ($A'$) and bond ($B'$) holdings, hence they are their choice variables. It should be also noted that no borrowing and no short-sale constraints are in place, hence $A' \geq 0$ and $B' \geq 0$. The value function depicted by equation 2.10 simply states that the current value function is equal to the maximum of the sum of current utility and the expected discounted value function of the next period. Expectations are conditional on the current realization of exogenous state variables, $Y$ and $R^e$. In fact in our baseline parametrization expectations are going to be solely conditional on endowment since equity returns are independent.

The recursive problem formulated by equation 2.10 along with the no borrowing and no short-selling conditions is a standard portfolio allocation problem. As shown in Adda and Cooper (2003) this problem suffices the Blackwell sufficient conditions for contraction mapping, hence it can be solved using value function iterations. The solutions to the model are two policy functions $A' = g_1(Y,A,B,R^e)$ and $B' = g_2(Y,A,B,R^e)$, which determine equity and bond investments, respectively.

There is however one caveat that has to be stated before we proceed. The problem is bounded as long as the model has a sufficiently low discount rate. As shown in Chamberlain and Wilson (2000) in the case of the stochastic endowment, $\beta * R^f < 1$ to guarantee that the model is bounded. Unfortunately, to the best of my knowledge, there is no simple rule on how to set $\beta$ in the case of both the stochastic endowment and the stochastic return on equity. That is why, in the baseline parametrization, I set $\beta = 0.88$. This number is lower than the one proposed by Heaton and Lucas (1997) - $\beta = 0.9$, but is in line with the value estimated by Bonaparte and Cooper (2009).

## 2.4. Numerical solutions

It is difficult to investigate policy functions as they are multidimensional objects - investment decisions depend on exogenous and endogenous state variables. Hence, in order to evaluate the impact of crash-risk as well as to understand model's sensitivity to different

TABLE 2.1. Baseline simulation results

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Median bond holdings | 0.00 | 0.15 | 0.21 | 18.61 | 47.27 |
| | (0.00) | (0.67) | (0.96) | (1.92) | (1.13) |
| Median equity holdings | 22.94 | 65.71 | 103.66 | 127.60 | 125.56 |
| | (2.57) | (6.16) | (9.87) | (17.27) | (16.45) |
| Std. dev. of consumption | 19.47 | 17.52 | 17.32 | 17.35 | 16.90 |
| | (0.66) | (0.89) | (1.18) | (1.35) | (1.46) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$ and tail-dependent crash risk. For each simulations 1100 exogenous observations were drawn, but first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

parametrizations, I use model simulations. More precisely I report summary statistics on equity and bond holdings and on average volatilities of consumption. I run 1000 separate simulations. For each simulation I draw 1100 realizations of stochastic exogenous state variables: endowment and equity returns. Using these stochastic realizations and calculated policy functions, I calculate bond and equity investment time-series as well as the corresponding consumption time-series. In order to minimize the effects of initial equity and bond holdings I drop the first 100 data points for each series. Since both bond and equity holdings are bounded by zero, which may lead to the skew in their distributions, I decided to report time-series medians rather than means. All tables report averages and standard deviations of medians for each time-series with the exception of consumption for which the average standard deviation is reported.[11]

Since the focus of this paper is on the effect of the Peso problem, all the stochastic realizations are generated by stochastic processes without crash states, i.e. in none of the simulations the crash actually occur. But the policy functions are calculated, assuming that agents factor into their decisions the potential crash risk, unless stated otherwise.

*Baseline parametrization.* The baseline model parametrization consists of: the discount rate set to a relatively low level $\beta = 0.88$, 'normal' times endowment and equity returns described by independent AR(1) processes with parameters set to match the values of Heaton and Lucas (1997), and the fully tail dependent crash risk characterized by the probability of crash $p = 3\%$ and the size of crash defined as the expected value less three standard deviations of the standard shock. Table 2.1 gives summary statistics for the baseline model under different levels of risk aversion. I am treating these results as the benchmark results for alternative parametrizations.

*Crash-risk.* First let us look at the impact of the crash risk on the average portfolio structure. Panel (a) in Table 2.2 reports average positive bond holdings for higher levels of

[11]In case the reader is interested in mean values rather than medians, they can refer to Appendix C, where mean values are reported.

risk-aversion. This stands in contrast with findings of Heaton and Lucas (1997) who were unable to generate positive bond holdings, however they did not consider a tail-dependent crash event. In fact panel (a) in Table 2.2 shows that if households do not internalize crash risk they will not hold any bonds in their portfolio, which is in line with the existing literature. The large impact of crash risk on agents portfolio composition might have two sources. The first source is related to the fact that crash risk increases equity volatility as well as it decreases the expected payoff (in our baseline calibration expected equity return falls from 7.75% to 6.38% ). The second source comes from the fact that crash risk is tail dependent i.e. both endowment and equity crash happens at the same time, hence equity savings are not the best tool to hedge against endowment crash risk. Panel (b) in Table 2.2 reports average bond holdings if households internalize crash risk, but they assume that crash risks are independent of each other. In this scenario households do not hold any bonds, hence the decrease in equity premium does not generate bond holdings. This shows that the tail-dependency effect is the most important for generating positive bond holdings in a simple framework.

It should be also noted that the existence of additional risk - in the form of crash - increases average wealth holdings as agents require a higher buffer for potential crises periods.

*Variation in risk-aversion coefficient $\gamma$.* Tables 2.1 and 2.2 show clearly that financial wealth is a positive function of the risk aversion coefficient, under models with and without internalized crash risk. For example, in the baseline scenario, Table 2.1 when $\gamma = 1$, household's financial wealth holdings amount only to 23% of the average endowment, while when $\gamma = 5$ households financial wealth holdings amount to 172% of the average endowment. An increase in the risk aversion also increases consumption smoothing as basic buffer-saving theory would predict. Moreover it is notable that one needs a risk-aversion coefficient of at least 4 to obtain significant bond holdings, keeping other parameters constant.

*Increase in discount rate $\beta$.* The model is sensitive to the choice of the discount rate. Table 2.3 reports summary statistics for two models with higher than the baseline discount rates $\beta = \{0.9, 0.92\}$. These results show that an increase in households patience leads to a large increase in average financial wealth holdings. For example, when households exhibit a medium level of risk aversion, $\gamma = 3$ under baseline discount rate ($\beta = 0.88$), their average financial wealth amounts to 104% of the average income, while under a high discount rate ($\beta = 0.92$) the average financial wealth increases to 178%. This is in line with the common sense as more patient households are more willing to invest. In addition, an increase in the discount rate changes portfolio shares as agents have higher stocks of equities and they tend to keep lower buffer bond savings. However for higher values of risk aversion they exhibit positive bond holdings.

TABLE 2.2. Simulation results under different stochastic process

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| (a) No crash risk | | | | | |
| Median bond holdings | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Median equity holdings | 9.87 | 33.72 | 57.55 | 81.55 | 105.30 |
| | (1.48) | (4.28) | (7.12) | (10.11) | (12.93) |
| Std. dev. of consumption | 20.89 | 18.05 | 17.00 | 16.69 | 16.76 |
| | (0.59) | (0.79) | (1.01) | (1.20) | (1.34) |
| (b) Crash risk without tail dependency | | | | | |
| Median bond holdings | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Median equity holdings | 18.66 | 51.08 | 84.04 | 118.52 | 152.69 |
| | (2.10) | (4.61) | (7.92) | (11.77) | (15.01) |
| Std. dev. of consumption | 19.98 | 17.56 | 16.90 | 16.99 | 17.47 |
| | (0.63) | (0.82) | (1.08) | (1.29) | (1.41) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$. For each simulations 1100 exogenous observations were drawn, but first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors. Panel (a) reports descriptive statistics where agents do not internalize crash neither in endowment nor in equity returns. Panel (b) reports descriptive statistics where agents internalize crash risk in endowment and equity returns, but assume that these crash states happen independently.

*Decrease in the probability or the size of crash.* At the beginning of this section I showed the importance of tail-dependency of crash for the size and composition of financial wealth. Since the parametrization of the crash risk itself is done in an ad-hoc way, it is also informative to see what happens to the baseline model when the probability or size of crash risk is lower. Panel (a) of Table 2.4 reports average equity and bond holdings if the crash probability is 2%, instead of 3% used in the baseline parametrization. In this case we get much lower average bond holdings. In fact only under high levels of risk aversion ($\gamma = 5$) do households exhibit positive bond holdings. Lower probability of crash risk, also makes equities more attractive as the expected equity return increases and returns volatility decreases. Consequently panel (a) reports higher equity holdings for large levels of risk aversion than in the baseline parametrization. Panel (b) of Table 2.4 reports average equity and bond holdings, when households internalize tail-dependent crash risk with the probability of 3%, but with the size of crash limited to -2.5 standard deviations of the average shock. This change has a very big impact on portfolio composition - under any of the considered levels of risk aversion households on average do not hold positive amounts of bonds. All in all, this exercise shows

TABLE 2.3. Simulation results under higher levels of discount rate

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| (a) Moderate discount rate: $\beta = 0.9$ | | | | | |
| Median bond holdings | 0.00 | 0.00 | 0.00 | 9.57 | 42.71 |
| | (0.00) | (0.00) | (0.00) | (2.23) | (1.95) |
| Median equity holdings | 38.79 | 89.87 | 135.31 | 174.92 | 167.54 |
| | (3.66) | (7.82) | (12.95) | (17.23) | (14.22) |
| Std. dev. of consumption | 18.16 | 17.09 | 17.46 | 18.04 | 18.00 |
| | (0.75) | (1.09) | (1.34) | (1.43) | (1.51) |
| (b) High discount rate: $\beta = 0.92$ | | | | | |
| Median bond holdings | 0.00 | 0.00 | 0.00 | 4.56 | 41.68 |
| | (0.00) | (0.00) | (0.00) | (3.02) | (4.69) |
| Median equity holdings | 71.86 | 132.65 | 178.21 | 211.61 | 196.72 |
| | (6.87) | (12.77) | (15.95) | (12.89) | (8.45) |
| Std. dev. of consumption | 16.92 | 17.24 | 18.21 | 19.04 | 19.39 |
| | (1.01) | (1.32) | (1.43) | (1.44) | (1.49) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on models with endowment and equity returns processes as in the baseline scenario and with tail-dependent crash. The table presents simulation results for higher discount rate than in the baseline scenario. For each simulation 1100 exogenous observations were drawn, but the first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

that not only the tail-dependency but also the size and severity of the crash is important to generate positive bond holdings.

*Different equity returns process.* In the baseline parametrization equity returns follow exactly the same process as in Heaton and Lucas (1997), where equity returns are independently distributed with average returns: $E(R^e) = 1.0775$ and standard deviations of shocks to returns amounts to $\sigma_\epsilon = 0.157$. In addition, under the baseline scenario the risk free rate is equal to $R^f = 1.02$. As an additional scenario I look at the impact of equity returns parametrized to match Mehra and Prescott (1985) data. This parametrization is based on US equity returns from 1889 to 1978. Under that parametrization equities exhibit lower average returns: $E(R^e) = 1.0698$, higher standard deviations of shocks to returns: $\sigma_\epsilon = 0.166$ and in particular some equity return persistence $\lambda_R = 0.11$. This would make equities less attractive, but in Mehra and Prescott (1985) calibration bond returns are also lower: $R^f = 1.008$, leading to the overall equity premium being higher than in the Heaton and Lucas (1997) parametrization. Table 2.5 reports summary statistics for simulations where equities follow the Mehra and Prescott (1985) calibration. It is apparent that lower returns to equities decreased substantially overall equity holdings. Moreover households decided to hold more bonds than in the baseline scenario even though returns to bonds are negligible.

TABLE 2.4. Simulation results under different crash probabilities and different crash sizes

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| (a) Probability of the crash decreased to 2% | | | | | |
| Median bond holdings | 0.00 | 0.00 | 0.00 | 0.00 | 13.42 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (2.39) |
| Median equity holdings | 18.88 | 60.74 | 96.96 | 134.17 | 164.47 |
| | (2.40) | (5.22) | (8.66) | (12.75) | (17.88) |
| Std. dev. of consumption | 19.78 | 17.44 | 17.18 | 17.47 | 17.89 |
| | (0.64) | (0.87) | (1.13) | (1.33) | (1.42) |
| (b) Size of the crash reduced to $-2.5 * \sigma$ | | | | | |
| Median bond holdings | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Median equity holdings | 15.81 | 51.11 | 84.58 | 120.09 | 155.79 |
| | (1.88) | (4.86) | (8.32) | (12.35) | (15.63) |
| Std. dev. of consumption | 20.11 | 17.46 | 16.89 | 17.07 | 17.54 |
| | (0.63) | (0.84) | (1.11) | (1.32) | (1.43) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$, endowment process as in the baseline scenario and tail-dependent crash. The equity process is parametrized to match 1889-1978 data for US as in Mehra and Prescott (1985): $R^f = 1.008$, $E(R^e) = 1.0698$, $\lambda_{R^e} = 0.11$ and $\sigma_\epsilon = 0.166$. For each simulation 1100 exogenous observations were drawn, but the first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

TABLE 2.5. Simulation results for Mehra and Prescott (1985) parametrization of equity returns

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Median bond holdings | 0.00 | 5.81 | 13.95 | 31.12 | 55.02 |
| | (0.00) | (2.47) | (2.65) | (1.57) | (2.53) |
| Median equity holdings | 16.10 | 49.89 | 78.94 | 91.75 | 96.56 |
| | (2.38) | (7.54) | (10.97) | (13.12) | (14.38) |
| Std. dev. of consumption | 19.85 | 17.67 | 17.27 | 17.00 | 16.86 |
| | (0.64) | (0.88) | (1.14) | (1.33) | (1.49) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$, an endowment process as in the baseline scenario and a tail-dependent crash. The equity process is parametrized to match 1889-1978 data for US as in Mehra and Prescott (1985): $R^f = 1.008$, $E(R^e) = 1.0698$, $\lambda_{R^e} = 0.11$ and $\sigma_\epsilon = 0.166$. For each simulation 1100 exogenous observations were drawn, but the first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

## 2.5. Monte-Carlo experiment

**2.5.1. Simulation.** The Monte-Carlo experiment aims to gauge the bias of the risk-aversion coefficient attributed to the Peso problem. This type of experiment is comprised of two parts: one that simulates an artificial environment and another that uses simulated numbers to estimate the parameters in question. Simulations of time series closely resemble experiments done in the previous section. The key difference is that for each parameter scenario I simulate a time series for the model where crash risk is and is not incorporated by household expectations.

More specifically, the simulation scenario is comprised of 10000 replications. For each replication I draw two series: one for returns to equity and one for endowments.[12]. Each series contains T observations. It is very important to note that both stochastic processes are drawn from processes not containing crash states, hence crash never happens in my sample. Once the exogenous stochastic process is set, I apply two types of policy functions: the one not internalizing the crash state and the one internalizing the crash state. This allows me to compare estimates where there are no internalized crashes with estimates where there is a Peso problem. Once I have generated artificial data I can use them to compute a set of risk aversion estimates. The next subsection describes in detail the estimators used.

**2.5.2. Estimation methods.** In my exercise I focus solely on the estimation of the risk aversion coefficient $\gamma$, assuming that the discount factor $\beta$ is known. I focus on two methods: the Hansen and Singleton (1982) non-linear generalized method of moments (GMM) estimator of the Euler equation and the simple ordinary least squares (OLS) regression of the log-linearized Euler equation. I use the GMM estimator as it uses the Euler equation directly and does not impose any binding assumptions on the stochastic part of the model. The OLS is used as the natural benchmark, but as noted by Carroll (2001), it performs very poorly.

Euler equations. From the optimization problem described by 2.2 and 2.3 we can easily derive the first order conditions with respect to equity (known as the Euler equation):

$$\beta E_0[\frac{u'(C_{t+1})}{u'(C_t)}R^e_t] = 1 \tag{2.11}$$

In addition, in the case of the CRRA utility function, the Euler equation simplifies to a relationship of the expected consumption growth and the expected equity return. Moreover the theory predicts that agents should use all the available information $\Omega_t$ while forming

[12]In addition to T observations, there are 100 extra draws treated as a burn in for the process, just to decrease the impact of initial conditions.

expectations, hence we can re-write (2.11) as:

$$E[\beta R_t^e \left[\frac{C_{t+1}}{C_t}\right]^{-\gamma} - 1|\Omega_t] = 0 \tag{2.12}$$

The Hansen and Singleton (1982) method exploits directly the above presented condition using it as a key moment in the GMM set up. Moreover, they also form additional moments using the orthogonality condition to the available information. In this paper I apply two versions of the Hansen and Singleton (1982) estimator. In the first one I only use the Euler equation and I do not use any additional orthogonalization conditions i.e. I assume that $\Omega$ consists only of a constant. Consequently I estimate one parameter using one moment condition. This means that my estimator is just identified. In the second GMM estimator I use lagged values of consumption, equity returns and endowment to enrich the structure of the information set $\Omega$. In this case I have four moment conditions and one parameter to be estimated hence the model is over-identified, therefore I weigh moments with an optimal weighting matrix obtained using the "true" parameters. It should be noted that in the baseline parametrization only endowment is a strong instrument, as it is persistent.

The second estimation method is just an OLS estimation of the following equation:

$$ln\left(\frac{C_{t+1}}{C_t}\right) = \alpha_0 + \alpha_1 ln(R_{t+1}) + \xi_{t+1}$$

where $\alpha_1$ is the estimate of the *Elasticity of Intertemporal Substitution* (EIS), which is just the inverse of the risk-aversion parameter. In addition, the constant is calibrated, and not estimated, as it exhibits information on the discount rate. As mentioned earlier this method was applied by Hall (1988), yielding a relatively high risk aversion coefficient. In my set-up the key issue with this method is that the assumption of log normal distribution of the underlying variables is definitely violated. Carroll (2001) has largely criticized the use of the log-linearized Euler equations to estimate the risk-aversion coefficient and my results confirm his finding.

**2.5.3. Results.** Table 2.6 reports average estimates of the risk aversion coefficient for the canonical model without crash risk for different parametrizations and different sample sizes. Surprisingly an upward bias of the estimated risk-aversion parameter is noticeable, in the case of GMM estimators - see panels (a) and (b). Adda and Cooper (2003) attribute this fact to the no-borrowing/no-short sale constraint, as at the constraint we have a corner solution in our optimization problem, hence the Euler equation might not hold. GMM estimators are also consistent, since the increase in the sample sizes leads to the decrease in estimates variance. Looking at variance of the estimators allows us to notice additional feature - over-identified GMM estimator is more efficient than the single condition GMM.

TABLE 2.6. Monte Carlo estimates of the risk aversion coefficient for model without crash risk

| (a) Just identified GMM | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\beta$ | | | 0.88 | | | 0.90 | |
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Mean($\hat{\gamma}$) | 1.92 | 3.50 | 4.45 | 1.68 | 2.96 | 3.80 |
| | Std dev.($\hat{\gamma}$) | 1.86 | 1.17 | 1.14 | 2.11 | 1.99 | 1.89 |
| T=400 | Mean($\hat{\gamma}$) | 2.07 | 3.43 | 4.26 | 1.35 | 3.25 | 3.87 |
| | Std dev.($\hat{\gamma}$) | 1.53 | 0.39 | 0.50 | 2.08 | 0.80 | 0.77 |
| T=1200 | Mean($\hat{\gamma}$) | 2.40 | 3.39 | 4.22 | 1.60 | 3.28 | 3.83 |
| | Std dev.($\hat{\gamma}$) | 0.87 | 0.23 | 0.28 | 1.84 | 0.28 | 0.30 |
| (b) Over-identified GMM | | | | | | | |
| $\beta$ | | | 0.88 | | | 0.90 | |
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Mean($\hat{\gamma}$) | 2.40 | 3.37 | 4.24 | 2.26 | 3.03 | 3.61 |
| | Std dev.($\hat{\gamma}$) | 0.65 | 0.90 | 1.07 | 1.17 | 1.47 | 1.69 |
| T=400 | Mean($\hat{\gamma}$) | 2.36 | 3.25 | 4.09 | 2.30 | 3.06 | 3.55 |
| | Std dev.($\hat{\gamma}$) | 0.28 | 0.40 | 0.51 | 0.45 | 0.70 | 0.92 |
| T=1200 | Mean($\hat{\gamma}$) | 2.33 | 3.22 | 4.05 | 2.29 | 3.08 | 3.54 |
| | Std dev.($\hat{\gamma}$) | 0.16 | 0.23 | 0.28 | 0.22 | 0.29 | 0.34 |
| (c) Ordinary Least Squares | | | | | | | |
| $\beta$ | | | 0.88 | | | 0.90 | |
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Mean($\hat{\gamma}$) | 4.62 | 5.36 | 6.07 | 4.40 | 5.29 | 4.98 |
| | Std dev.($\hat{\gamma}$) | 7.81 | 19.27 | 11.92 | 12.09 | 5.50 | 6.83 |
| T=400 | Mean($\hat{\gamma}$) | 4.00 | 4.72 | 5.06 | 4.23 | 4.78 | 4.85 |
| | Std dev.($\hat{\gamma}$) | 0.78 | 0.82 | 0.85 | 0.72 | 0.70 | 0.71 |
| T=1200 | Mean($\hat{\gamma}$) | 3.90 | 4.63 | 4.92 | 4.16 | 4.72 | 4.79 |
| | Std dev.($\hat{\gamma}$) | 0.38 | 0.43 | 0.40 | 0.39 | 0.41 | 0.39 |

*Notes:* The table reports average values, as well as their standard deviations, of the estimated risk-aversion coefficent for the model where households do not internalize crises. For each parameter values 10000 time series of length T are simulated. These series are used to estimate risk aversion parameter. Panel (a) reports Hansen and Singleton (1982) estimates, obtained using only one-moment condition. Panel (b) reports Hansen and Singleton (1982) estimates, obtained from over-identified model, where past values of consumption, equity returns and endowment where used as instruments. Panel (c) reports estimates obtained by ordinary least squares.

This estimator feature is especially pronounced for small samples (T=100) and fades away once the sample is big.

Table 2.6 also reports OLS estimates for the canonical model without any crash risk. Clearly estimates obtained from OLS are biased and the sign of the bias cannot be easily determined. In addition, even though the variance of the estimates decreases when the size of sample increases, it is much larger than the variance for corresponding GMM estimates.

Table 2.7 reports estimates of the risk-aversion coefficient from the data simulated by the model with the Peso problem (households anticipate crash risk, but it never happens in the sample). Panels (a) and (b) show that the GMM estimates of the risk aversion coefficient are biased upwards (as we expected). Moreover, even though the estimates of the risk aversion coefficients are getting tighter when the sample size increases, the bias does not diminish. This means that the bias induced by the Peso problem is not a small sample phenomenon. It is rather a manifestation of the fact that sample misses important information.

Panel (c) in Table 2.7 reports OLS estimates, surprisingly these estimates have smaller bias than the estimates of the model without the Peso problem. Still, the results confirm that the OLS estimator of the log-linearized Euler equation does not have good properties and should not be used to estimate risk aversion coefficient.

Figure 2.1 shows distributions of risk aversions coefficients estimates obtained from the over-identified GMM estimator for the model with and without Peso problem. Even though the distributions show the aparent bias in the average estimate of the risk aversion coefficient due to the Peso problem, their shapes do not differ substantially. Distributions of the risk aversion estimates obtained from small samples ($T = 100$) are wide and have a small positive skew. Yet, both of these estimator features diminish when the sample size increases.

Let us look again at the estimator bias generated by the Peso problem. Table 2.8 reports that the average bias ranges 0.65 to 1.71, depending on estimation method, size of the underlying risk aversion coefficient and the sample size. It is easy to note that on average the over-identified method produces a smaller bias. This might be related to an increased efficiency of the over-identified estimator. Table 2.8 also implies that the higher the true risk aversion coefficient the lower the bias of the estimate. In order to understand this phenomenon it is informative to look again at the Euler equation 2.12. If we could observe the true distribution of equity returns, the Euler equation would hold with the true gamma, but since we do not observe the crash state our average equity return is biased upwards. Consequently, for the Euler equation to hold, the $E\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$ has to decrease. Since we observe the positive growth rate of consumption, the estimate of $\gamma$ gets biased upwards. In case of higher financial wealth holdings, the growth rate of consumption is higher. This in turn means that the $\gamma$ has too be less biased to fulfill the Euler condition.

These observations raise another issue. In our set-up the crash risk pushes highly risk-averse agents to hold bonds rather than equities. This could potentially decrease the consumption growth rate and again increase the bias of the risk aversion coefficient. In order to assess this effect I compare the biases for the model with Peso problem where crash risk is tail dependent (Table 2.8) to the model where crash risks are independent (Panel (a) of Table 2.9). Note that the first model exhibited significant bond holdings for high risk aversion levels and the second model did not exhibit any bond holdings. In fact pairwise biases of

TABLE 2.7. Monte Carlo estimates of the risk aversion coefficient for model with anticipated crash risk

| (a) Just identified GMM | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\beta$ | | | 0.88 | | | 0.90 | |
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Mean($\hat{\gamma}$) | 3.62 | 4.69 | 5.38 | 3.10 | 3.89 | 3.92 |
| | Std dev.($\hat{\gamma}$) | 0.89 | 1.03 | 1.31 | 1.69 | 1.88 | 2.29 |
| T=400 | Mean($\hat{\gamma}$) | 3.63 | 4.58 | 5.17 | 3.43 | 4.09 | 4.01 |
| | Std dev.($\hat{\gamma}$) | 0.31 | 0.42 | 0.61 | 0.55 | 0.64 | 1.15 |
| T=1200 | Mean($\hat{\gamma}$) | 3.60 | 4.55 | 5.10 | 3.43 | 4.08 | 4.01 |
| | Std dev.($\hat{\gamma}$) | 0.18 | 0.25 | 0.36 | 0.22 | 0.32 | 0.43 |
| (b) Over-identified GMM | | | | | | | |
| $\beta$ | | | 0.88 | | | 0.90 | |
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Mean($\hat{\gamma}$) | 3.36 | 4.44 | 5.04 | 3.00 | 3.57 | 3.35 |
| | Std dev.($\hat{\gamma}$) | 0.75 | 0.99 | 1.42 | 1.36 | 1.83 | 2.49 |
| T=400 | Mean($\hat{\gamma}$) | 3.36 | 4.35 | 4.81 | 3.14 | 3.65 | 3.27 |
| | Std dev.($\hat{\gamma}$) | 0.32 | 0.45 | 0.71 | 0.56 | 0.97 | 1.61 |
| T=1200 | Mean($\hat{\gamma}$) | 3.33 | 4.32 | 4.71 | 3.15 | 3.69 | 3.36 |
| | Std dev.($\hat{\gamma}$) | 0.19 | 0.26 | 0.44 | 0.22 | 0.37 | 0.75 |
| (c) Ordinary Least Squares | | | | | | | |
| $\beta$ | | | 0.88 | | | 0.90 | |
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Mean($\hat{\gamma}$) | 3.71 | 4.23 | 4.43 | 3.75 | 4.08 | 3.90 |
| | Std dev.($\hat{\gamma}$) | 1.37 | 1.55 | 1.38 | 1.08 | 1.76 | 1.16 |
| T=400 | Mean($\hat{\gamma}$) | 3.49 | 3.97 | 4.20 | 3.57 | 3.83 | 3.74 |
| | Std dev.($\hat{\gamma}$) | 0.44 | 0.47 | 0.59 | 0.39 | 0.42 | 0.47 |
| T=1200 | Mean($\hat{\gamma}$) | 3.44 | 3.93 | 4.12 | 3.56 | 3.80 | 3.70 |
| | Std dev.($\hat{\gamma}$) | 0.23 | 0.27 | 0.29 | 0.22 | 0.25 | 0.27 |

*Notes:* The table reports average values, as well as their standard deviations, of the estimated risk-aversion coefficient for the model where households internalize crises, but it never happens in the analyzed sample - Peso problem. For each parameter values 10000 time series of length T are simulated. These series are used to estimate risk aversion parameter. Panel (a) reports Hansen and Singleton (1982) estimates, obtained using only one-moment condition. Panel (b) reports Hansen and Singleton (1982) estimates, obtained from over-identified model, where past values of consumption, equity returns and endowment where used as instruments. Panel (c) reports estimates obtained by ordinary least squares.

the model with non tail dependent crash exhibits smaller bias, but the differences in biases is very small and could be attributed to numerical error. It seems that, even though the tail dependency has a substantial effect on the household's portfolio composition it has hardly any effect on the estimate of the risk aversion coefficient.

Finally, let us look at the risk aversion coefficient bias reported for the Mehra and Prescott (1985) parametrization (see panel (b) of Table 2.9). It is very easy to notice that the average

TABLE 2.8. Pairwise bias of risk aversion coefficient estimates

| | | Just-identified GMM | | | Over-identified GMM | | |
|---|---|---|---|---|---|---|---|
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| T=100 | Average bias | 1.71 | 1.23 | 0.93 | 0.97 | 1.07 | 0.80 |
| | Std of the bias | 1.51 | 0.78 | 1.00 | 0.29 | 0.51 | 1.08 |
| T=400 | Average bias | 1.58 | 1.16 | 0.91 | 1.00 | 1.10 | 0.72 |
| | Std of the bias | 1.36 | 0.18 | 0.47 | 0.08 | 0.21 | 0.59 |
| T=1200 | Average bias | 1.18 | 1.17 | 0.88 | 1.00 | 1.11 | 0.65 |
| | Std of the bias | 0.73 | 0.09 | 0.28 | 0.04 | 0.12 | 0.37 |

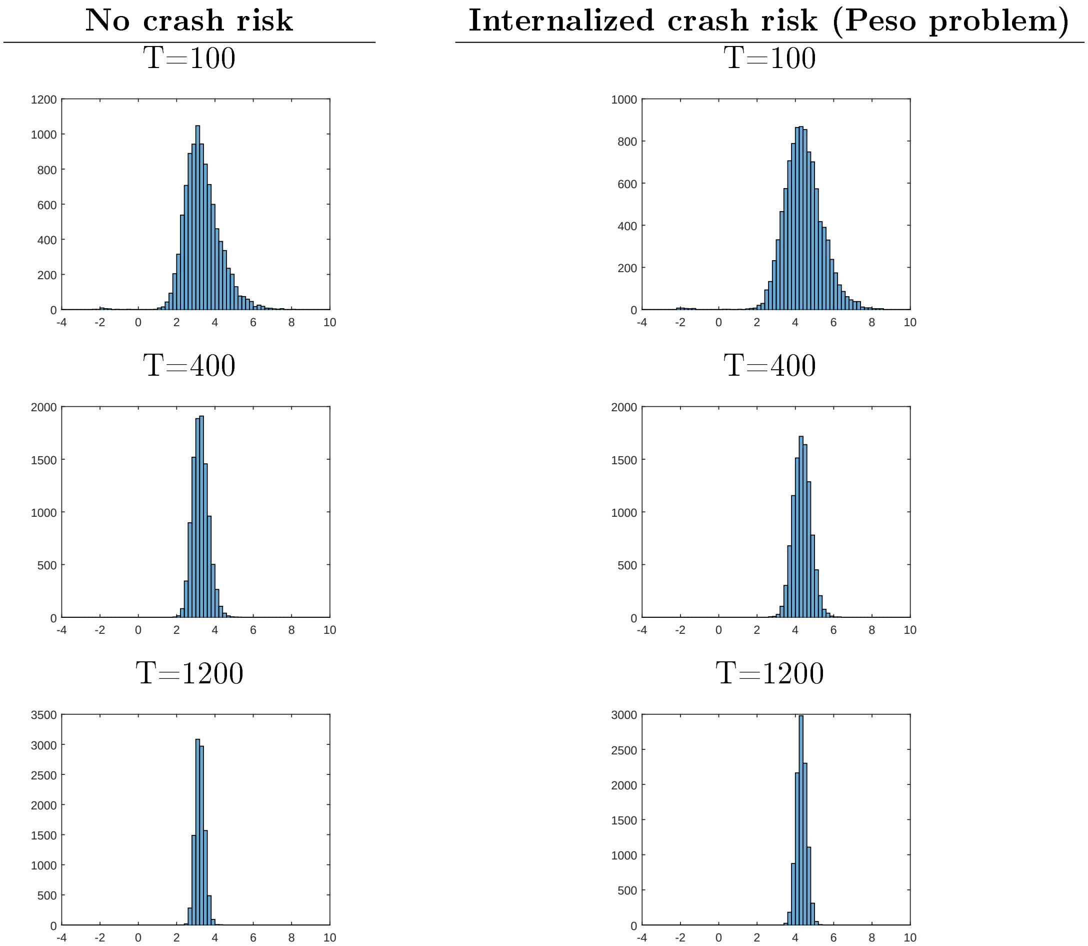
*Notes:* The table reports average pairwise bias of the risk-aversion coefficient, as well as its standard deviation, due to the Peso problem. All calculations are done for the model with low discount rate $\beta = 0.88$. Each parametrization is replicated 10000 times. For each parametrization two set of series are produced, one where households do not internalize crash and one in which they internalize tail dependent crash. On the basis of these simulated data two risk-aversion coefficients are estimated and the table reports the difference between these parameters. Two estimators are applied: just-identified GMM where we use Euler equation as the only moment condition, and over-identified GMM where we use past values of consumption, equity returns and endowment as additional orthogonalization conditions.

TABLE 2.9. Pairwise bias of risk aversion coefficient estimates

| | | Just-identified GMM | | | Over-identified GMM | | |
|---|---|---|---|---|---|---|---|
| $\gamma$ | | 2 | 3 | 4 | 2 | 3 | 4 |
| (a) Model with independent crash risks | | | | | | | |
| T=100 | Average bias | 1.36 | 0.98 | 0.80 | 0.66 | 0.82 | 0.70 |
| | Std of the bias | 1.53 | 0.77 | 0.74 | 0.30 | 0.36 | 0.71 |
| T=400 | Average bias | 1.28 | 0.93 | 0.84 | 0.69 | 0.86 | 0.70 |
| | Std of the bias | 1.38 | 0.14 | 0.31 | 0.07 | 0.14 | 0.39 |
| T=1200 | Average bias | 0.88 | 0.94 | 0.84 | 0.69 | 0.87 | 0.68 |
| | Std of the bias | 0.73 | 0.07 | 0.19 | 0.04 | 0.08 | 0.25 |
| (b) Model with Mehara-Prescott (1985) parametrization | | | | | | | |
| T=100 | Average bias | 1.42 | 0.98 | 0.87 | 0.80 | 0.85 | 0.80 |
| | Std of the bias | 1.41 | 0.68 | 0.63 | 0.29 | 0.40 | 0.67 |
| T=400 | Average bias | 1.25 | 0.93 | 0.87 | 0.83 | 0.88 | 0.77 |
| | Std of the bias | 1.17 | 0.14 | 0.29 | 0.08 | 0.15 | 0.37 |
| T=1200 | Average bias | 0.95 | 0.94 | 0.86 | 0.83 | 0.88 | 0.75 |
| | Std of the bias | 0.52 | 0.08 | 0.17 | 0.05 | 0.09 | 0.24 |

*Notes:* The table reports average pairwise bias of the risk-aversion coefficient, as well as its standard deviation, due to the Peso problem. All calculations are done for the model with low discount rate $\beta = 0.88$. Each parametrization is replicated 10000 times. For each parametrization two set of series are produced, one where households do not internalize crash and one in which they internalize tail dependent crash. On the basis of these simulated data two risk-aversion coefficients are estimated and the table reports the difference between these parameters. Two estimators are applied: just-identified GMM where we use Euler equation as the only moment condition, and over-identified GMM where we use past values of consumption, equity returns and endowment as additional orthogonalization conditions.

FIGURE 2.1. Distributions of risk aversion coefficient estimates




*Notes:* The figure depicts distributions of risk aversion coefficient estimates obtained using over-identified GMM. The underlying model used for simulations has a discount rate set to $\beta = 0.88$ and the risk aversion parameter set to $\gamma = 3$. For the model with incorporated crash risk, the probability of crash is $p = 3\%$ and the size of the crash is determined by a negative shock of three standard deviations, $m = 3$.

bias is small. This clearly stands in contrast with the change of the calibrated risk aversion coefficient for the general equilibrium model with and without crash state.

## 2.6. Conclusions

This paper investigates the impact of a tail-dependent crash event on portfolio composition and bias of the estimates of the risk-aversion coefficient. I show that tail-dependent crash risk, i.e. crash risk that hits at the same time the equity market and household's endowment, leads to positive bond holdings in the canonical households investment-saving choice model. I also find that the Peso problem (the fact that households internalize Barro-Rietz type of crash, but this crash is never observed in the actual empirical sample) biases the Hansen and

Singleton (1982) estimates of the risk aversion coefficient. However, the size of this bias is small relative to the change of the risk aversion coefficient needed to generate realistic equity premium in general equilibrium models. Therefore, estimates of the risk aversion coefficient suffer less from Peso problem. Moreover, the fact that empirical analyses find moderate estimates of risk aversion coefficient does not necessarily rule out the possibility that households internalize market crash. Finally, the analysis confirms that the OLS estimator is a very bad tool to estimate risk aversion coefficient as previously noted by Carroll (2001).

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## Appendix A: Baseline state variables and Markov matrices

In the baseline parametrization I discretize endowment into five following states:

$$Y = \begin{bmatrix} 57.55 \\ 78.77 \\ 100.00 \\ 121.23 \\ 142.45 \end{bmatrix}$$

The minimum state is established as 1.5 endowment standard deviation from mean endowment (100). Markov transition matrix associated with that state is:

$$\pi_Y = \begin{bmatrix} 0.35 & 0.34 & 0.23 & 0.07 & 0.01 \\ 0.20 & 0.32 & 0.30 & 0.14 & 0.04 \\ 0.09 & 0.24 & 0.34 & 0.24 & 0.09 \\ 0.04 & 0.14 & 0.30 & 0.32 & 0.20 \\ 0.01 & 0.07 & 0.23 & 0.34 & 0.35 \end{bmatrix}$$

.

The process for equity returns is discretized into five states as well:

$$R^e = \begin{bmatrix} 0.842 \\ 0.960 \\ 1.078 \\ 1.195 \\ 1.313 \end{bmatrix}$$

where, just like in the case of endowment, the lowest state is calculated as average equity return minus 1.5 standard deviation of average equity return. The associated Markov transition matrix is:

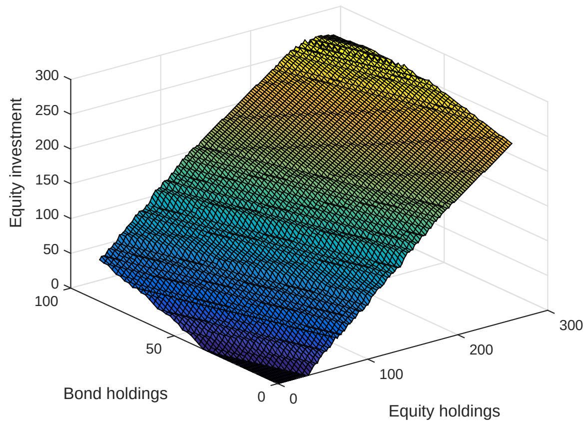
$$\pi_R = \begin{bmatrix} 0.13 & 0.22 & 0.30 & 0.22 & 0.13 \\ 0.13 & 0.22 & 0.30 & 0.22 & 0.13 \\ 0.13 & 0.22 & 0.30 & 0.22 & 0.13 \\ 0.13 & 0.22 & 0.30 & 0.22 & 0.13 \\ 0.13 & 0.22 & 0.30 & 0.22 & 0.13 \end{bmatrix}$$

As it is easy to notice, the transition matrix has all rows the same, as in the baseline calibration equity returns are assumed to be independent.

Finally the additional crash state is parametrized as additional endowment state $Y_{Crash} = [15.1]$ and $R^e_{Crash} = [0.607]$, both are defined as the average state minus three standard deviations of the underlying process.

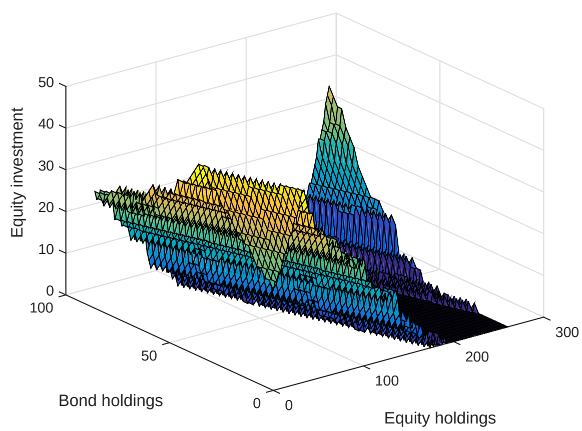
## Appendix B: Sample policy functions

FIGURE B.1. Equity investment policy function




*Notes:* The figure depicts equity investment policy function for the model with tail dependent crash risk, low discount factor $\beta = 0.88$ and moderate risk aversion $\gamma = 4$. Represented policy function is for endowment set to 100.

FIGURE B.2. Bond investment policy function




*Notes:* The figure depicts bond investment policy function for the model with tail dependent crash risk, low discount factor $\beta = 0.88$ and moderate risk aversion $\gamma = 4$. Represented policy function is for endowment set to 100.

## Appendix C: Numerical results - mean

TABLE C.1. Baseline simulation results

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Mean bond holdings | 0.28 | 6.32 | 9.59 | 21.69 | 50.66 |
| | (0.04) | (0.75) | (1.28) | (1.59) | (1.55) |
| Mean equity holdings | 27.22 | 66.83 | 105.69 | 129.77 | 123.25 |
| | (2.08) | (5.99) | (9.79) | (11.88) | (11.13) |
| Std. dev. of consumption | 19.47 | 17.52 | 17.32 | 17.35 | 16.90 |
| | (0.66) | (0.89) | (1.18) | (1.35) | (1.46) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$ and tail-dependent crash risk. For each simulations 1100 exogenous observations were drawn, but first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

TABLE C.2. Simulation results under different stochastic process

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| (a) No crash risk | | | | | |
| Mean bond holdings | 0.00 | 0.01 | 0.23 | 1.06 | 2.63 |
| | (0.00) | (0.04) | (0.27) | (0.71) | (1.25) |
| Mean equity holdings | 15.67 | 42.62 | 68.70 | 93.35 | 115.64 |
| | (1.49) | (4.26) | (7.06) | (9.31) | (10.86) |
| Std. dev. of consumption | 20.89 | 18.05 | 17.00 | 16.69 | 16.76 |
| | (0.59) | (0.79) | (1.01) | (1.20) | (1.34) |
| (b) Crash risk without tail dependency | | | | | |
| Mean bond holdings | 0.00 | 0.03 | 0.68 | 3.05 | 7.55 |
| | (0.00) | (0.07) | (0.52) | (1.33) | (2.29) |
| Mean equity holdings | 23.01 | 58.78 | 94.63 | 127.85 | 155.22 |
| | (1.72) | (4.59) | (7.73) | (9.81) | (10.48) |
| Std. dev. of consumption | 19.98 | 17.56 | 16.90 | 16.99 | 17.47 |
| | (0.63) | (0.82) | (1.08) | (1.29) | (1.41) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$. For each simulations 1100 exogenous observations were drawn, but first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors. Panel (a) reports descriptive statistics where agents do not internalize crash neither in endowment nor in equity returns. Panel (b) reports descriptive statistics where agents internalize crash risk in endowment and equity returns, but assume that these crash states happen independently.

Table C.3. Simulation results under higher levels of discount rate

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| (a) Moderate discount rate: $\beta = 0.9$ | | | | | |
| Mean bond holdings | 0.09 | 3.98 | 8.76 | 19.77 | 48.66 |
| | (0.02) | (0.68) | (1.50) | (2.34) | (2.20) |
| Mean equity holdings | 44.21 | 95.87 | 137.21 | 162.45 | 152.98 |
| | (3.36) | (8.13) | (11.01) | (11.59) | (10.88) |
| Std. dev. of consumption | 18.16 | 17.09 | 17.46 | 18.04 | 18.00 |
| | (0.75) | (1.09) | (1.34) | (1.43) | (1.51) |
| (b) High discount rate: $\beta = 0.92$ | | | | | |
| Mean bond holdings | 0.18 | 4.87 | 12.01 | 23.41 | 48.91 |
| | (0.19) | (1.32) | (2.31) | (3.22) | (2.90) |
| Mean equity holdings | 80.50 | 137.97 | 171.09 | 189.06 | 179.13 |
| | (6.73) | (10.32) | (10.90) | (9.96) | (9.42) |
| Std. dev. of consumption | 16.92 | 17.24 | 18.21 | 19.04 | 19.39 |
| | (1.01) | (1.32) | (1.43) | (1.44) | (1.49) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on models with endowment and equity returns processes as in the baseline scenario and with tail-dependent crash. The table presents simulation results for higher discount rate than in the baseline scenario. For each simulation 1100 exogenous observations were drawn, but the first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

TABLE C.4. Simulation results under different crash probabilities and different crash sizes

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| (a) Probability of the crash decreased to 2% | | | | | |
| Mean bond holdings | 0.00 | 2.71 | 5.90 | 9.24 | 20.38 |
| | (0.00) | (0.39) | (0.92) | (1.52) | (2.10) |
| Mean equity holdings | 24.09 | 64.95 | 101.93 | 135.85 | 155.42 |
| | (1.91) | (5.41) | (8.88) | (10.96) | (11.79) |
| Std. dev. of consumption | 19.78 | 17.44 | 17.18 | 17.47 | 17.89 |
| | (0.64) | (0.87) | (1.13) | (1.33) | (1.42) |
| (b) Size of the crash reduced to $-2.5 * \sigma$ | | | | | |
| Mean bond holdings | 0.00 | 0.23 | 2.26 | 5.18 | 10.35 |
| | (0.00) | (0.08) | (0.57) | (1.35) | (2.35) |
| Mean equity holdings | 21.16 | 58.24 | 93.43 | 126.88 | 154.51 |
| | (1.75) | (4.81) | (8.27) | (10.55) | (11.13) |
| Std. dev. of consumption | 20.11 | 17.46 | 16.89 | 17.07 | 17.54 |
| | (0.63) | (0.84) | (1.11) | (1.32) | (1.43) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$, endowment process as in the baseline scenario and tail-dependent crash. The equity process is parametrized to match 1889-1978 data for US as in Mehra and Prescott (1985): $R^f = 1.008$, $E(R^e) = 1.0698$, $\lambda_{R^e} = 0.11$ and $\sigma_\epsilon = 0.166$. For each simulation 1100 exogenous observations were drawn, but the first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

TABLE C.5. Simulation results for Mehra and Prescott (1985) parametrization of equity returns

| Gamma | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Mean bond holdings | 1.92 | 10.22 | 17.53 | 33.73 | 54.95 |
| | (0.16) | (0.77) | (1.08) | (1.27) | (1.43) |
| Mean equity holdings | 21.75 | 56.49 | 86.97 | 102.88 | 106.33 |
| | (1.85) | (5.55) | (8.76) | (10.16) | (10.62) |
| Std. dev. of consumption | 19.85 | 17.67 | 17.27 | 17.00 | 16.86 |
| | (0.64) | (0.88) | (1.14) | (1.33) | (1.49) |

*Notes:* The table reports averages across 1000 simulations of descriptive statistics for each time series. Simulations are based on a model with a discount rate $\beta = 0.88$, an endowment process as in the baseline scenario and a tail-dependent crash. The equity process is parametrized to match 1889-1978 data for US as in Mehra and Prescott (1985): $R^f = 1.008$, $E(R^e) = 1.0698$, $\lambda_{R^e} = 0.11$ and $\sigma_\epsilon = 0.166$. For each simulation 1100 exogenous observations were drawn, but the first 100 observations were discarded to minimize the impact of initial conditions. The numbers in brackets report standard errors.

CHAPTER 3

# Overseas unspanned factors and domestic bond returns

joint with Andrew Meldrum (Bank of England) and Peter Spencer (University of York)

**Abstract**

Using data on government bonds in Germany and the US, we show that 'overseas unspanned factors' - constructed from the components of overseas yields that are uncorrelated with domestic yields - have significant explanatory power for subsequent *domestic* bond returns. This result is remarkably robust, holding for different sample periods, as well as out of sample. By adding our overseas unspanned factors to simple dynamic term structure models, we show that shocks to those factors have large and persistent effects on domestic yield curves. Dynamic term structure models that omit information about foreign bond yields are therefore likely to be misspecified.

*Keywords:* return-forecasting regressions, dynamic term structure models.

JEL: E43, G12.

## 3.1. Introduction

Using data on government bond yields in Germany and the USA, this paper shows that a factor extracted from the part of overseas yields that is orthogonal to domestic yields can explain a substantial part of subsequent domestic bond returns. Moreover, this 'overseas unspanned factor' has significant *additional* predictive power for domestic bond returns relative to the information contained in the domestic yield curve. The result is remarkably robust, holding for different sample periods as well as out-of-sample.

A large number of studies have demonstrated that most of the variation in government bond yields over different maturities within a single country can be explained by the first three principal components of domestic yields (typically labelled as level, slope and curvature - e.g. Litterman and Scheinkman (1991)). Models of the term structure that specify bond yields as linear functions of three or more principal components are therefore likely to achieve a high in-sample fit to the cross section of yields. That does not, however, imply that three domestic principal components are sufficient for modelling the *time-series* behaviour of yields. Previous studies have shown that other variables, unspanned by level, slope and curvature, have significant explanatory power for USexcess returns. These include other factors extracted from domestic bond yields (Cochrane and Piazessi (2005) and Duffee (2011b)) and macroeconomic variables (Joslin et al. (2014)). This paper extends this emerging literature on unspanned factors in the term structure by demonstrating that an 'overseas unspanned factor' extracted from *overseas* yields but unspanned by domestic yields is an important predictor of future domestic yields.

We use a simple two-stage regression-based method to construct our overseas unspanned factors. We first regress bond yields from the 'foreign' country on a cross-section of yields from the 'domestic' country, thereby obtaining the components of foreign yields that are orthogonal to domestic yields. We then construct our overseas unspanned factor as a linear combination of these orthogonal components at different maturities, with the weights chosen to maximise fit to excess bond returns averaged across maturities.

To assess the information content of this factor, we include it in two sets of empirical exercises: (i) return-forecasting regressions; and (ii) dynamic factor models of bond yields. We highlight the following results from these empirical exercises. First, in return-forecasting regressions with a twelve-month holding period, the overseas unspanned factor has a statistically significant coefficient for all maturity returns; and excluding it results in substantially worse in-sample fit, particularly for German returns and at short maturities. Second, these results are remarkably robust and do not appear to be a result of in-sample over-fitting: they hold for alternative samples, out-of-sample and if we extend the analysis to consider returns on UK bonds. Third, in the dynamic factor model for German yields, a one standard deviation shock to our overseas unspanned factor is followed by a decline in yields of up to 70 basis points; in the model of US yields, the largest reaction is somewhat smaller but still reasonably substantial, at around 40 basis points. And fourth, shocks to the overseas unspanned factors also account for a substantial portion of the unexpected variation in long-term bond yields - for example, they account for around 40-50% of forecast error variance of German yields over a ten-year forecast horizon. This proportion is lower for the US but still non-negligible (around 15%).

Our approach to constructing our overseas unspanned factor is similar to that used by Cochrane and Piazessi (2005). They construct a 'return-forecasting factor' as a single linear combination of US forward rates and then show that this factor can explain a substantial part of US excess bond returns. Dahlquist and Hasseltoft (2013) find similar results to Cochrane and Piazessi (2005) for Germany, Switzerland and the UK (as well as for the US); and that a global factor constructed as a GDP-weighted average of the local return-forecasting factors raises the explanatory power of return-forecasting regressions relative to versions that only include the local return-forecasting factors - for countries other than the US.[1] There are, however, three important differences between Dahlquist and Hasseltoft (2013) and the present study. First, we show that there is information in foreign yields which is not reflected in *any* linear combination of domestic yields (not just the single linear combination they use as a domestic return-forecasting factor). Second, our overseas unspanned factor contains no information extracted from domestic yields, whereas the Dahlquist and Hasseltoft (2013) global factor is a weighted average of local factors from the different countries. So it is clear in our case that the return-forecasting ability of the overseas unspanned factor does not derive from its containing information about current domestic yields. And third, Dahlquist and Hasseltoft (2013) find that their global factor does not help to explain excess returns in the US, whereas we show that there is information in overseas yields that is relevant for explaining US returns. These three differences are particularly important when building dynamic term structure models, since our paper clearly demonstrate that we cannot capture all of the information relevant for modelling the time-series dynamics of yields simply by adding more factors extracted from domestic yield curves, even for the US.

Our dynamic factor models of yields - which we estimate separately for yields in each country - are broadly similar to the model of Diebold and Li (2006) in that they model the time-series dynamics of the factors driving bond yields using a Vector-Autoregression and have a simple cross-sectional mapping between factors and yields. The non-standard feature of our model is that we incorporate the respective overseas unspanned factors as state variables alongside principal components of local yields. We can motivate this by appealing to a no-arbitrage term structure model with unspanned factors, similar to Joslin et al. (2014) (we provide further detail on this point in Appendix A). While we do not impose

[1]Zhu (2015) shows that such a global return-forecasting factor can predict returns out of sample for Germany, Japan, the UK and the US.

no-arbitrage restrictions on the cross section of yields,[2] this is unlikely to imply a materially different mapping between the factors and bond yields, however, so such an exercise would add little to the contribution of this paper (Duffee (2011a) provides a discussion of the impact of no-arbitrage restrictions on yield forecasts from dynamic term structure models).

Our interest in overseas unspanned factors can be motivated by the fact that they allow us to achieve a partial identification of directional effects in interdependent global markets. While a number of studies have found that yields in multiple countries can be explained by a small number of factors extracted from the pooled data set, sometimes interpreted as 'global factors' (e.g. Diebold et al. (2008) and Kaminska et al. (2013) among others), it is hard to identify what structural shocks drive these factors. Such models beg the question of whether the international correlations and factors reflect common shocks or spillovers from one country to another. Reflecting this problem, recent research on global business cycle models has moved away from reliance upon global factors to developing multi-country models with explicit cross-country spillover effects (e.g. Diebold and Yilmaz (2015)). Our focus on unspanned factors allows us to identify similar cross-country spillovers. We should acknowledge, however, that our identification of spillover effects is only partial, since the *domestic* yield curve factors in our models inevitably reflect the impact of global factors that are spanned by domestic yields as well as genuinely domestic influences. Ciccarelli and Garcia (2015) use Stock and Watson (2005) techniques to decompose these factors into global and domestic components, but we do not attempt to make this distinction in this paper, simply identifying directional effects from the unspanned components.

Section 2 of this paper summarizes the US and German data sets we use and demonstrates the extent to which these is unspanned information in overseas yields. The return-forecasting regressions including several robustness checks are presented in Section 3 and the dynamic term structure model in Section 4. Section 5 concludes.

## 3.2. The unspanned component of overseas yields

**3.2.1. Data.** Our data set consists of estimates of German and US end-month zero-coupon yields from January 1990 until December 2014, with maturities of 6 months and 1, 2, 3, 5, 7 and 10 years. For the US, we use the estimates of Gürkaynak et al. (2007) using the Svensson (1994) parametric method, which are updated and published by the Federal Reserve Board.[3] For Germany, we use estimates published by the Bundesbank, also estimated using the Svensson method.[4] In Sections 4 and 5 we also report results of extensions to cover the

[2]For example, as the affine term structure models of Duffie and Kan (1996) and Duffee (2002). Dahlquist and Hasseltoft (2013) estimate no-arbitrage term structure models that include their global factor.
[3]Available at: http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html.
[4]Available at: http://www.bundesbank.de/Navigation/EN/Statistics/Money_and_capital_markets/Interest_rates_and_yields/Term_structure_of_interest_rates/term_structure_of_interest_rates.html.

UK; estimates of UK zero-coupon yields are published by the Bank of England and computed using the smoothed cubic spline method of Anderson and Sleath (2001).[5]

Table 3.1 reports summary statistics of the US and German yields at selected maturities. As is well known, the average term structures are upward sloping, the volatility of yields declines slowly with maturity and yields are highly persistent, with autocorrelation coefficients close to one for all maturities. For example, the average US six-month and ten-year yields are approximately 3.3% and 5.1% respectively; whereas the equivalent averages for Germany are 3.5% and 4.8%. The average German yield curve is therefore a little flatter than the average US yield curve (the average spread between the ten-year and six-month yield is 1.9 percentage points in the US and 1.4 percentage points in Germany). The standard deviation of the US six-month and ten-year yields are 2.3% and 1.8% respectively; with corresponding standard deviations of 2.6% and 2.0% in Germany.

TABLE 3.1. Summary statistics of nominal zero-coupon yields

| Maturity (months) | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|---|---|
| (a) United States | | | | | | | |
| Mean | 3.275 | 3.406 | 3.673 | 3.923 | 4.365 | 4.729 | 5.135 |
| Minimum | 0.089 | 0.099 | 0.188 | 0.306 | 0.627 | 1.007 | 1.552 |
| Maximum | 8.382 | 8.568 | 8.780 | 8.863 | 8.909 | 8.919 | 8.924 |
| Standard deviation | 2.338 | 2.373 | 2.343 | 2.264 | 2.087 | 1.939 | 1.781 |
| AR(1) coefficient | 0.992 | 0.992 | 0.990 | 0.989 | 0.988 | 0.987 | 0.987 |
| (b) Germany | | | | | | | |
| Mean | 3.472 | 3.528 | 3.695 | 3.883 | 4.238 | 4.525 | 4.841 |
| Minimum | -0.113 | -0.125 | -0.124 | -0.095 | 0.032 | 0.231 | 0.615 |
| Maximum | 9.630 | 9.470 | 9.130 | 9.089 | 9.240 | 9.286 | 9.229 |
| Standard deviation | 2.623 | 2.572 | 2.506 | 2.444 | 2.317 | 2.193 | 2.027 |
| AR(1) coefficient | 0.994 | 0.995 | 0.994 | 0.995 | 0.996 | 0.997 | 0.998 |

*Notes:* All numbers except for the AR(1) coefficients are in annualized percentage points. The AR(1) coefficient reports the first-order autocorrelation coefficient from an AR(1) model including an intercept, estimated using OLS. The sample ranges from January 1990 to December 2014.

Table 3.2 reports correlations of domestic yields across maturities for the two countries separately. As is well known, yields of nearby maturities within a single country are strongly correlated - for example, the seven- and ten-year yields have a correlation greater than 0.995 in both the US and Germany. The correlations between very short and very long maturity yields are somewhat weaker but are still positive - for example, the correlations between the six-month and ten-year yields are 0.85 in the US and 0.90 in Germany.

Table 3.3 reports correlations of yields *across countries*. Cross-country correlations are strongly positive for all pairs of yields and are generally higher for longer maturity yields.

[5]Available at: http://www.bankofengland.co.uk/statistics/pages/yieldcurve/default.aspx.

TABLE 3.2. Correlations of yields across maturities within a single country

| Maturity (months) | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|---|---|
| (a) United States | | | | | | | |
| 6 | 1.000 | 0.997 | 0.983 | 0.965 | 0.927 | 0.890 | 0.845 |
| 12 | 0.997 | 1.000 | 0.994 | 0.981 | 0.948 | 0.915 | 0.874 |
| 24 | 0.983 | 0.994 | 1.000 | 0.996 | 0.976 | 0.951 | 0.916 |
| 36 | 0.965 | 0.981 | 0.996 | 1.000 | 0.991 | 0.973 | 0.945 |
| 60 | 0.927 | 0.948 | 0.976 | 0.991 | 1.000 | 0.995 | 0.979 |
| 84 | 0.890 | 0.915 | 0.951 | 0.973 | 0.995 | 1.000 | 0.995 |
| 120 | 0.845 | 0.874 | 0.916 | 0.945 | 0.979 | 0.995 | 1.000 |
| (b) Germany | | | | | | | |
| | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
| 6 | 1.000 | 0.998 | 0.989 | 0.977 | 0.953 | 0.931 | 0.903 |
| 12 | 0.998 | 1.000 | 0.996 | 0.988 | 0.966 | 0.945 | 0.918 |
| 24 | 0.989 | 0.996 | 1.000 | 0.997 | 0.984 | 0.967 | 0.943 |
| 36 | 0.977 | 0.988 | 0.997 | 1.000 | 0.994 | 0.982 | 0.963 |
| 60 | 0.953 | 0.966 | 0.984 | 0.994 | 1.000 | 0.997 | 0.986 |
| 84 | 0.931 | 0.945 | 0.967 | 0.982 | 0.997 | 1.000 | 0.996 |
| 120 | 0.903 | 0.918 | 0.943 | 0.963 | 0.986 | 0.996 | 1.000 |

*Notes:* The table reports r-Pearson pairwise correlation coefficients computed for end-month values of the considered maturities for the period January 1990 to December 2014.

For some maturities, we note that the foreign yield with the highest correlation does not necessarily have the same maturity. In particular, German yields are generally more highly correlated with longer maturity US yields than with the US yield of the corresponding maturity. This suggests that when we are analyzing the extent to which foreign and domestic yield curves contain the same information we cannot just focus on bivariate correlations between yields of the same maturity; rather, we should consider whether a given yield is spanned by the full set of maturities in the other country. We return to this issue in the following sub-section.

**3.2.2. Unspanned overseas information.** The simple correlation analysis above demonstrates a high degree of co-movement of bond yields across the two countries. But the fact that the cross-country correlations are less than one shows that there is nevertheless *some* information in yields that is specific to individual countries. To isolate the information in the yields of country $j$ that is not (linearly) spanned by yields in country $i \neq j$, we regress yields in country $j$ on yields from country $i$:

$$y_{n,t}^{(j)} = \beta_0 + \beta_6 y_{6,t}^{(i)} + \beta_{12} y_{12,t}^{(i)} + ... + \beta_{120} y_{120,t}^{(i)} + u_{n,t}^{(j)}, \tag{3.1}$$

for $n = 6, 12, 24, 36, 60, 84, 120$ and where $y_{n,t}^{(i)}$ is the time-$t$, $n$-period yield for country $i$.

TABLE 3.3. Correlations of yields across countries

| Germany \ United States | | | | | | | |
|---|---|---|---|---|---|---|---|
| Maturity (months) | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
| 6 | 0.711 | 0.731 | 0.767 | 0.797 | 0.840 | 0.864 | 0.880 |
| 12 | 0.733 | 0.754 | 0.790 | 0.820 | 0.861 | 0.884 | 0.899 |
| 24 | 0.758 | 0.780 | 0.819 | 0.848 | 0.889 | 0.911 | 0.926 |
| 36 | 0.771 | 0.796 | 0.836 | 0.866 | 0.907 | 0.930 | 0.944 |
| 60 | 0.781 | 0.808 | 0.850 | 0.882 | 0.924 | 0.947 | 0.964 |
| 84 | 0.780 | 0.807 | 0.851 | 0.884 | 0.927 | 0.952 | 0.969 |
| 120 | 0.771 | 0.800 | 0.845 | 0.878 | 0.923 | 0.948 | 0.968 |

*Notes:* The table reports r-Pearson pairwise cross-country correlations of monthly yields for US and Germany computed for end-month values of the considered maturities for January 1990 to December 2014. German yields are in rows and US yields are in columns. For example, the number 0.758 from the third row and first column reports the correlation between 24-month German yield and the 6-month US yield.

Panel (a) of Table 3.4 reports the $R^2$ statistics for these regressions. These are consistent with the general pattern observed in the cross-country correlation analysis reported above. Yields in the foreign country can explain a large proportion of the variation in domestic long-term yields: the $R^2$s for the ten-year yields are both close to 0.95. At shorter maturities, the $R^2$s are lower: regressing the six-month US yield on German yields gives an $R^2$ of 0.66; and regressing the six-month German yield on US yields gives an $R^2$ of 0.81.

Panel (b) of Table 3.4 reports results from restricted versions of (3.1) in which the only regressors are a constant and the matched maturity yield in country $i$ (i.e. regressing $y_{n,t}^{(j)}$ on $y_{n,t}^{(i)}$). The $R^2$ statistics are substantially lower and F-tests of the implied zero restrictions suggest that they should be strongly rejected in all cases. Similar to the correlation analysis in the previous sub-section, this shows that when analyzing the common information in international term structures, we cannot necessarily just consider bivariate relationships between yields that have the same maturity.

## 3.3. Return regressions

**3.3.1. An unspanned overseas return-forecasting factor.** As discussed above, when specifying a dynamic term structure model, it may be important to include variables unspanned by the yield curve - and which therefore do not improve the cross-sectional fit of the model - but are nevertheless important for predicting future yields (Joslin et al. (2014)); and we can use simple reduced-form return-forecasting regressions to provide an indication of whether there are such unspanned factors in the yield curve (Appendix A provides future motivation for these regressions). In this section, we therefore turn to the question of whether the information in the foreign yield curve that is orthogonal to domestic yields is nevertheless useful for explaining domestic excess returns.

TABLE 3.4. Regressions of domestic yields on foreign yields

| Maturity (months) | | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|---|---|---|
| (a) Multivariate regressions | | | | | | | | |
| United States | $R^2$ | 0.66 | 0.70 | 0.76 | 0.81 | 0.88 | 0.92 | 0.95 |
| Germany | $R^2$ | 0.81 | 0.84 | 0.88 | 0.91 | 0.95 | 0.96 | 0.96 |
| (b) Univariate regressions | | | | | | | | |
| United States | $R^2$ | 0.50 | 0.57 | 0.67 | 0.75 | 0.85 | 0.90 | 0.94 |
| | F-test (p-values) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |
| Germany | $R^2$ | 0.50 | 0.57 | 0.67 | 0.75 | 0.85 | 0.90 | 0.94 |
| | F-test (p-values) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) | (0.00) |

*Notes:* Panel (a) of the table shows $R^2$ statistics for regressions of yields in the relevant country on a constant and yields with maturities of 6,12, 24, 36, 60, 84 and 120 months from the other country (equation (3.1)). Panel (b) shows the $R^2$ statistics for regressions of yields in the relevant country on a constant and the single yield from the other country with the same maturity. Figures in brackets in panel (b) show the p-values of F-tests of the restrictions that all omitted regressors included in the regressions reported panel (a) are equal to zero. The sample ranges from January 1990 to December 2014.

With seven different maturities for each country, the dimensions of the orthogonal information contained in the seven residuals $u_{n,t}^{(j)}$ for $n = 6, 12, 24, 36, 60, 84, 120$ from (3.1) is clearly large. But it turns out that the large majority of the information contained in those residuals that is relevant for forecasting country $i$ returns can be summarised in a single 'overseas unspanned factor' (OUF).

Note first that the excess return from holding a country $i$ $n$-month bond between times $t$ and $t+12$ is defined as

$$rx_{n,t,t+12}^{(i)} = \log\left(P_{n-12,t+12}^{(i)}\right) - \log\left(P_{n,t}^{(i)}\right) - y_{12,t}^{(i)}, \tag{3.2}$$

where $P_{n,t}^{(i)}$ is the time-$t$ price of an $n$-period bond. To construct a single linear combination of the information in the residuals from (3.1), we regress the average excess return on country $i$ bonds of different maturities between times $t$ and $t+12$ on the time-$t$ components of all foreign yields orthogonal to domestic yields (i.e. $u_{n,t}^{(j)}$):

$$\overline{rx}_{t,t+12}^{(i)} = \gamma_0 + \gamma' \mathbf{u}_t^{(j)} + \varepsilon_{t,t+12}^{(i)}, \tag{3.3}$$

Here, $\overline{rx}_{t,t+12}^{(i)}$ denotes the average 12-month excess return on 2-, 3-, 5-, 7- and 10-year bonds and $\mathbf{u}_t^{(j)} = \left[u_{6,t}^{(j)}, u_{12,t}^{(j)}, u_{24,t}^{(j)}, ..., u_{120,t}^{(j)}\right]'$. Our return-forecasting factor, which we denote $z_t^{(j)}$ below, is given by the fitted value from this regression ($z_t^{(j)} = \widehat{\gamma}\mathbf{u}_t^{(j)}$). This is similar to the procedure in Cochrane and Piazessi (2005), although their regressors are domestic forward rates.

We can evaluate how well this single-factor specification explains excess returns on bonds across different maturities in a second step, by running separate regressions of the form

$$rx^{(i)}_{n,t,t+12} = \alpha_0 + \alpha_n z^{(j)}_t + \widetilde{\varepsilon}^{(i)}_{n,t,t+12} \tag{3.4}$$

for $n = 24, 36, 60, 84, 120$. The $R^2$s from these regressions are in the region of 0.1-0.2 for the US and 0.2-0.4 for Germany (Table 3.5). In both cases, there is information in overseas yields, unspanned by domestic yields, which can explain a substantial part of the variation in domestic excess returns.

TABLE 3.5. $R^2$ of regression of excess bond returns on single and multiple unspanned factors

| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| (a) United States | | | | | |
| Single-factor specification | 0.107 | 0.138 | 0.166 | 0.171 | 0.161 |
| Unrestricted | 0.172 | 0.176 | 0.173 | 0.172 | 0.179 |
| (b) Germany | | | | | |
| Single-factor specification | 0.335 | 0.351 | 0.338 | 0.296 | 0.223 |
| Unrestricted | 0.357 | 0.363 | 0.340 | 0.296 | 0.228 |

*Notes:* The table reports $R^2$ statistics for two models. The 'single-factor specification' refers to regressions of excess bond returns on a constant and the overseas unspanned factor (3.4). The 'unrestricted' specification refers to regressions of excess bond returns on a constant and the components of all considered domestic yields orthogonal to overseas yields (3.5). The sample ranges from January 1990 to December 2014.

Fitting a model with a single-factor obtained from the two-step procedure of estimating (3.3) and then (3.4) does of course involve some loss of information. To evaluate how well our single factor captures the relevant information contained in all the residuals $u^{(j)}_{n,t}$, we can also estimate the unrestricted version of (3.4):[6]

$$rx^{(i)}_{n,t,t+12} = \gamma_{0,n} + \gamma'_n \mathbf{u}^{(j)}_t + \varepsilon^{(i)}_{n,t,t+12} \tag{3.5}$$

for $n = 24, 36, 60, 84, 120$. The $R^2$s from these regressions are also shown in Table 3.5 (the rows headed 'unrestricted'). In almost all cases, these are very similar to those obtained from the single-factor model (3.4), i.e. there is little information lost by using the single-factor specification.

**3.3.2. Does the overseas unspanned factor contain information for predicting returns relative to the domestic yield curve?** We next assess the extent of the *marginal* information in the unspanned portion of overseas yields - relative to the information contained in the domestic term structure - by estimating regressions of the form

$$rx^{(i)}_{n,t,t+12} = \kappa_0 + \kappa' \mathbf{y}^{(i)}_t + \alpha_n z^{(j)}_t + \eta^{(i)}_{n,t+12} \tag{3.6}$$

[6]This is similar to the approach taken by Cochrane and Piazessi (2005) when considering the return-forecasting information in domestic forward rates.

where $\mathbf{y}_t^{(i)} = \left[y_{6,t}^{(i)}, y_{12,t}^{(i)}, ..., y_{120,t}^{(i)}\right]'$ denotes a vector of all considered yields for country $i$. Return-forecasting regressions usually have fewer explanatory variables than this, so it is worth emphasizing that the point we are making here is not necessarily that a model with so many variables is desirable in absolute terms; rather, the point of the exercise is to show that *no* linear combination of the considered domestic yields can replicate the information contained in the overseas unspanned factor - hence why we include all seven as explanatory variables.

Table 3.6 reports results from estimating (3.6) and from a version with $\alpha_n$ restricted to zero. For both the US and Germany as the domestic country $i$, the increase in the explanatory power of the regression, measured by its $R^2$, is substantial - from about 0.35 to 0.5 for US returns and from about 0.2 to 0.5 for German returns. In both cases, the change in the $R^2$ is strongly significant based on the bootstrap procedure by Bauer and Hamilton (2015) (our implementation of this bootstrap is explained in detail in Appendix B). And the coefficients on the overseas unspanned factor $\alpha_n$ are also individually strongly statistically significant. In summary, therefore, there is clearly statistically and economically significant information in overseas yield curves, unspanned by domestic yields, which is nevertheless important for predicting future domestic bond returns.

**3.3.3. Interpreting the overseas unspanned factor.** Clearly, the way in which our overseas unspanned factor is constructed by regressing returns on many different orthogonal components of overseas yields (3.3) means that it is not straightforward to attach an interpretation. However, it turns out that they are reasonably highly correlated with spreads between observed yield curve factors. The US overseas unspanned factor (which we include in regressions explaining German excess returns) is highly correlated with the spread between the first principal components of yields (i.e. the 'level factors') in the two countries (Figure 3.1; we provide further details on these principal components in Section 4). And the German overseas unspanned factor (which we include in regressions explaining US excess returns) is highly correlated with the spread between the third principal components (i.e. the 'curvature factors') in the two countries (Figure 3.2).[7]

**3.3.4. Robustness tests.** Our paper is not the first to find a variable which appears to predict future bond returns. In general, however, a problem in this literature is a lack of robustness: results are particular to the considered sample period or disappear out-of-sample. This may be a particular concern in our case, given the high colinearity of the regressors

[7]We have considered whether any the OUFs co-move with several important financial market or macroeconomic variables: a measure of implied equity market volatility (the VIX); a measure of banking sector credit risk (TED spreads); and a real activity indictor (industrial production). In simple bivariate regressions, none of these potential explanatory variables have significant coefficients (at 5% significance level). Of course the above list of financial/macroeconomic indicators does not cover all the possibilities, but it seems that it is difficult to match any of our OUF with financial market or macroeconomic variables.

TABLE 3.6. Regression of excess bond returns on domestic yields and the unspanned overseas factor

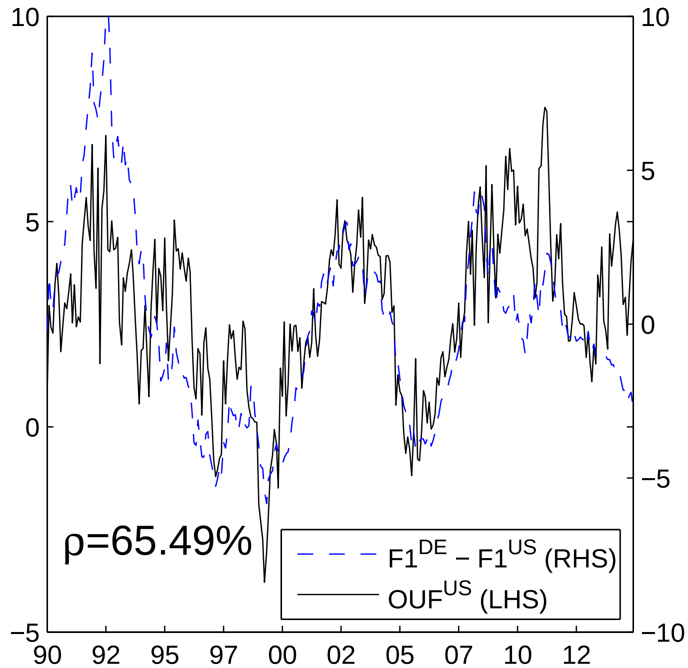
| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| (a) United States | | | | | |
| $\alpha_n$ | 0.24 | 0.52 | 1.01 | 1.39 | 1.81 |
| t-statistics | (7.5) | (8.6) | (9.8) | (10.0) | (9.4) |
| | [-4.3,4.1] | [-4.3,4.1] | [-4.3,4.1] | [-4.3,4.1] | [-4.2,4.1] |
| $R^2$ including OUF | 0.47 | 0.49 | 0.52 | 0.52 | 0.50 |
| $R^2$ restricted $\alpha_n = 0$ | 0.37 | 0.35 | 0.35 | 0.35 | 0.34 |
| $\Delta R^2$ | 0.10 | 0.14 | 0.16 | 0.17 | 0.16 |
| | [0.04] | [0.04] | [0.04] | [0.04] | [0.04] |
| (b) Germany | | | | | |
| $\alpha_n$ | 0.30 | 0.61 | 1.08 | 1.36 | 1.58 |
| t-statistics | (13.6) | (14.5) | (14.1) | (12.7) | (10.5) |
| | [-4.9,4.9] | [-4.9,4.9] | [-4.9,4.9] | [-4.9,4.8] | [-4.9,4.8] |
| $R^2$ including OUF | 0.50 | 0.54 | 0.54 | 0.50 | 0.46 |
| $R^2$ restricted $\alpha_n = 0$ | 0.17 | 0.20 | 0.21 | 0.21 | 0.24 |
| $\Delta R^2$ | 0.33 | 0.34 | 0.33 | 0.29 | 0.22 |
| | [0.06] | [0.06] | [0.06] | [0.06] | [0.06] |

*Notes:* The table reports estimated parameters from regressions of excess bond returns on a constant, seven domestic yields and the overseas unspanned factor ($\alpha_n$), i.e. equation (3.6). Numbers in parentheses report the values of t-statisitcs and numbers in brackets refer to the 95% confidence interval for these t-statistics obtained using the Bauer and Hamilton (2015) bootstrap procedure. The final two rows of each section (a) and (b) report the $R^2$ statistics from models with and without the overseas unspanned factor ('Including OUF' and 'Restricted' respectively). Numbers in brackets refer to the 95% critical value for the change in the $R^2$. The sample ranges from January 1990 to December 2014.

in the construction of the return-forecasting factor (3.3). Viewed in that light, however, our results appear to be remarkably robust. Most importantly, the overseas unspanned factor significantly improves forecasts of returns out-of-sample. Our results also hold across a number of different sub-samples and when we consider alternative domestic yield curve variables. While the results are weaker if we consider a six-month investment horizon, our overseas unspanned factors can still provide a statistically significant improvement in the predictability of domestic returns. Finally, we also show that very similar results apply if we extend our analysis to include the UK as a third country in our analysis.

*Different sample periods.* A potential concern about the results reported above is that the sample period we use contains two obvious potential structural breaks: the introduction of the euro in January 1999 and the fall in short-term nominal interest rates close to the zero lower bound during the recent financial crisis. Consequently we first consider three sub-sample periods: (i) the pre-euro period (January 1990-December 1998); (ii) the post-euro period (January 1999-December 2014); and (iii) the pre-lower bound period (January 1990-December 2007). Tables 3.8 and 3.8 report $R^2$s for models including and excluding the

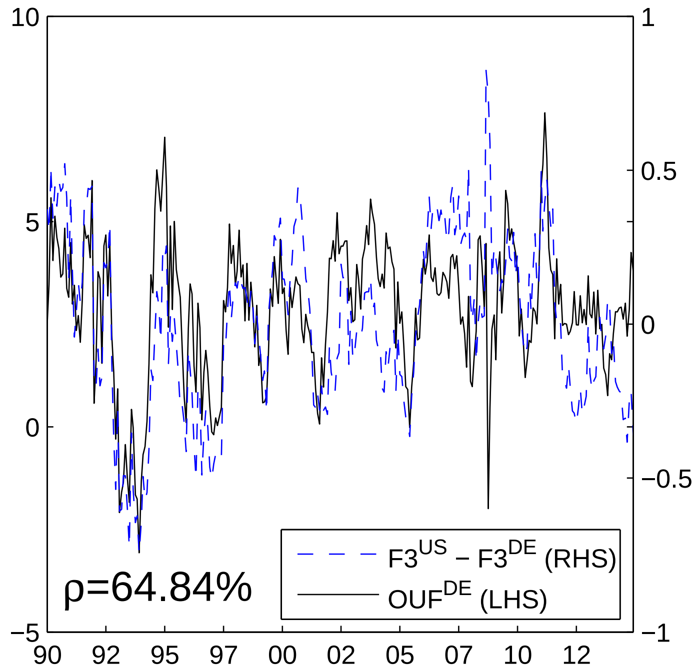
FIGURE 3.1. Overseas unspanned factor extracted from US yield curve.




*Notes:* The figure depicts the overseas unspanned factor (OUF$^{US}$) extracted from US yield curve, i.e. the component of the US yield curve that is unspanned by German yields. $F1^{DE} - F1^{US}$ depicts the difference between first principal components ('levels') of German and US yield curve, respectively.

overseas unspanned factor for the different sub-samples. The goodness of fit varies across samples, yet the overall $R^2$s remain high for models including the overseas unspanned factor, ranging from 47% to 82%. Most importantly, in all cases the fit of the regressions that exclude the overseas unspanned factor are worse, particularly for German short-maturity returns. The coefficients on the overseas unspanned factor are strongly statistically significant in all cases.

FIGURE 3.2. Overseas unspanned factor extracted from German yield curve.




*Notes:* The figure depicts the overseas unspanned factor ($\mathrm{OUF}^{DE}$) extracted from German yield curve, i.e. the component of the German yield curve that is unspanned by US yields. $F3^{US} - F3^{DE}$ depicts the difference between third principal components ('curvature') of US and German yield curve, respectively.

TABLE 3.7. Regressions of excess returns on domestic yields and the overseas unspanned factor for different sub-samples

| Maturity (months) | 24 | 36 | 60 | 84 | 120 | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|
| | (a) United States | | | | | (b) Germany | | | | |
| (i) Full sample: 1990-2014 | | | | | | | | | | |
| $\alpha_n$ | 0.24 | 0.52 | 1.01 | 1.39 | 1.81 | 0.30 | 0.61 | 1.08 | 1.36 | 1.58 |
| t-statistics | (7.5) | (8.6) | (9.8) | (10.0) | (9.4) | (13.6) | (14.5) | (14.1) | (12.7) | (10.5) |
| | [-4.3,4.1] | [-4.3,4.1] | [-4.3,4.1] | [-4.3,4.1] | [-4.2,4.1] | [-4.9,4.9] | [-4.9,4.9] | [-4.9,4.9] | [-4.9,4.8] | [-4.9,4.8] |
| $R^2$ including OUF | 0.47 | 0.49 | 0.52 | 0.52 | 0.50 | 0.50 | 0.54 | 0.54 | 0.50 | 0.46 |
| $R^2$ restricted $\alpha_n = 0$ | 0.37 | 0.35 | 0.35 | 0.35 | 0.34 | 0.17 | 0.20 | 0.21 | 0.21 | 0.24 |
| $\Delta R^2$ | 0.10 | 0.14 | 0.16 | 0.17 | 0.16 | 0.33 | 0.34 | 0.33 | 0.29 | 0.22 |
| | [0.04] | [0.04] | [0.04] | [0.04] | [0.04] | [0.06] | [0.06] | [0.06] | [0.06] | [0.06] |
| (ii) Pre-ZLB sample 1990-2007 | | | | | | | | | | |
| $\alpha_n$ | 0.22 | 0.49 | 0.98 | 1.41 | 1.93 | 0.31 | 0.63 | 1.09 | 1.33 | 1.50 |
| t-statistics | (7.4) | (8.8) | (10.4) | (11.0) | (11.2) | (10.9) | (11.4) | (10.8) | (9.5) | (7.5) |
| | [-4.3,4.5] | [-4.3,4.4] | [-4.3,4.4] | [-4.4,4.4] | [-4.4,4.4] | [-5.2,5.1] | [-5.2,5.2] | [-5.1,5.2] | [-5.1,5.1] | [-5.1,5.0] |
| $R^2$ including OUF | 0.67 | 0.68 | 0.69 | 0.67 | 0.64 | 0.58 | 0.62 | 0.62 | 0.59 | 0.54 |
| $R^2$ restricted $\alpha_n = 0$ | 0.58 | 0.56 | 0.51 | 0.47 | 0.42 | 0.32 | 0.37 | 0.40 | 0.41 | 0.41 |
| $\Delta R^2$ | 0.09 | 0.12 | 0.18 | 0.20 | 0.22 | 0.26 | 0.25 | 0.22 | 0.18 | 0.13 |
| | [0.05] | [0.05] | [0.05] | [0.05] | [0.05] | [0.08] | [0.08] | [0.08] | [0.08] | [0.08] |

*Notes:* The table reports results from regressions of excess bond returns on a constant, the seven considered domestic yields and the overseas unspanned factor - i.e. equation (3.6), estimated for the indicated sample periods. Numbers in parentheses report the values of t-statisitcs and numbers in brackets refer to the 95% confidence interval for these t-statistics obtained using the Bauer and Hamilton (2015) bootstrap procedure. The final two rows of each part of the table report the $R^2$ statistics from models with and without the overseas unspanned factor ('Including OUF' and 'Restricted' respectively). Numbers in brackets refer to the 95% critical value for the change in the $R^2$.

TABLE 3.8. Regressions of excess returns on domestic yields and the overseas unspanned factor for different sub-samples

| Maturity (months) | 24 | 36 | 60 | 84 | 120 | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|
| | (a) United States | | | | | (b) Germany | | | | |
| (iii) Pre-euro sample: 1990-1998 | | | | | | | | | | |
| $\alpha_n$ | 0.13 | 0.32 | 0.71 | 1.08 | 1.58 | 0.20 | 0.51 | 1.06 | 1.45 | 1.83 |
| t-statistics | (5.8) | (7.0) | (8.1) | (8.6) | (8.9) | (4.3) | (5.8) | (7.1) | (6.9) | (5.8) |
| | [-4.3,4.4] | [-4.3,4.3] | [-4.2,4.3] | [-4.2,4.3] | [-4.2,4.3] | [-4.7,4.8] | [-4.7,4.8] | [-4.7,4.9] | [-4.7,4.8] | [-4.7,4.9] |
| $R^2$ including OUF | 0.82 | 0.81 | 0.77 | 0.74 | 0.71 | 0.56 | 0.64 | 0.69 | 0.65 | 0.57 |
| $R^2$ restricted $\alpha_n = 0$ | 0.75 | 0.70 | 0.60 | 0.52 | 0.44 | 0.47 | 0.51 | 0.50 | 0.46 | 0.41 |
| $\Delta R^2$ | 0.07 | 0.11 | 0.17 | 0.22 | 0.27 | 0.09 | 0.13 | 0.19 | 0.19 | 0.16 |
| | (0.07) | (0.07) | (0.07) | (0.08) | (0.08) | (0.13) | (0.13) | (0.13) | (0.13) | (0.13) |
| (iv) Post-euro sample: 1999-2014 | | | | | | | | | | |
| $\alpha_n$ | 0.29 | 0.58 | 1.05 | 1.33 | 1.47 | 0.33 | 0.64 | 1.08 | 1.35 | 1.57 |
| t-statistics | (7.0) | (8.2) | (9.2) | (8.6) | (6.6) | (27.2) | (25.9) | (21.9) | (18.4) | (14.7) |
| | [-4.0,4.0] | [-4.0,3.9] | [-3.9,3.9] | [-3.9,3.9] | [-3.8,3.9] | [-4.9,4.8] | [-5.0,4.8] | [-4.9,4.8] | [-4.9,4.9] | [-4.9,4.9] |
| $R^2$ including OUF | 0.59 | 0.63 | 0.67 | 0.67 | 0.62 | 0.83 | 0.81 | 0.76 | 0.71 | 0.65 |
| $R^2$ restricted $\alpha_n = 0$ | 0.47 | 0.48 | 0.51 | 0.53 | 0.53 | 0.10 | 0.08 | 0.09 | 0.13 | 0.21 |
| $\Delta R^2$ | 0.12 | 0.15 | 0.16 | 0.14 | 0.09 | 0.73 | 0.73 | 0.67 | 0.58 | 0.44 |
| | (0.05) | (0.05) | (0.04) | (0.04) | (0.04) | (0.08) | (0.08) | (0.08) | (0.08) | (0.08) |

*Notes:* The table reports results from regressions of excess bond returns on a constant, the seven considered domestic yields and the overseas unspanned factor - i.e. equation (3.6), estimated for the indicated sample periods. Numbers in parentheses report the values of t-statisitcs and numbers in brackets refer to the 95% confidence interval for these t-statistics obtained using the Bauer and Hamilton (2015) bootstrap procedure. The final two rows of each part of the table report the $R^2$ statistics from models with and without the overseas unspanned factor ('Including OUF' and 'Restricted' respectively). Numbers in brackets refer to the 95% critical value for the change in the $R^2$.

*Out-of-sample performance.* We next evaluate whether the increase in explanatory power from including our overseas unspanned factors holds out of sample. In our forecasting exercise we estimate the models using rolling windows of 120 monthly observations to generate 168 forecasts. More precisely, we start by estimating the model using the ten-year period January 1990-December 1999 and construct a twelve-month ahead forecast of returns for the period ending December 2000. We then move the estimation period on by one month (i.e. February 1999 to January 2000) and repeat. Table 3.9 reports root mean squared forecast error (RMSFE) statistics from this forecasting exercise for different maturities, computed across all the resulting 168 forecasts.

The RMSFE for the model including the unspanned overseas factor is lower than for the restricted model for all maturity returns in both countries. Giacomini and White (2006) tests of the statistical significance of the improvements in forecasting performance show that models including the unspanned overseas factor perform *significantly* better at forecasting returns, with the single exception of German ten-year bonds. The model including the overseas unspanned factor even out-performs a random walk for US seven- and ten-year bonds and for all maturities for Germany. In summary, therefore, our results are remarkably robust out of sample, which should substantially alleviate concerns that they are an artefact of in-sample over-fitting.

TABLE 3.9. Root Mean Squared Forecast Error of out-of-sample excess return predictions

| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| (a) United States | | | | | |
| Random walk | 1.16 | 2.35 | 4.70 | 7.08 | 10.62 |
| Restricted $\alpha_n = 0$ | 2.15 | 4.18 | 7.26 | 9.63 | 12.54 |
| Including OUF | 1.78** | 3.22** | 5.24** | 6.77** | 8.79** |
| (b) Germany | | | | | |
| Random walk | 1.34 | 2.60 | 4.80 | 6.60 | 8.85 |
| Restricted $\alpha_n = 0$ | 1.41 | 2.73 | 4.79 | 6.18 | 7.55 |
| Including OUF | 0.78*** | 1.56*** | 3.12*** | 4.59** | 6.51 |

*Notes:* The table reports Root Mean Square Forecast Errors for excess bond returns for three different forecasting models: a random walk i.e. a simple naive forecast; and our benchmark model both including the overseas unspanned factor and excluding it ('Restricted' and 'Including OUF' respectively). All model parameters, as well as the OUFs are computed using 10-year rolling samples (i.e. 120 months). All numbers reported are in annualized percentage points. Asterisks indicate significance levels from Giacomini-White test (see Giacomini and White (2006)) assessing the difference of forecasting power between the models excluding and including the overseas unspanned factor: ***,**, * denote significance at $p = 0.01$, $p = 0.05$ and $p = 0.1$ respectively for the best performing model. The sample ranges from January 1990 to December 2014, implying a forecasting period of January 2000 to December 2014.

*Alternative domestic yield curve variables.* As explained above, the primary purpose of our return-forecasting regression (3.6) is to demonstrate that there is information contained in the overseas unspanned factor which is not reflected in any linear combination of the considered domestic yields - i.e. it is not necessarily to show that this is the 'best' forecasting model of yields. Indeed, it is plausible that a more parsimonious model would deliver superior out-of-sample forecasts of returns to those presented in Section 3.4.2. In this sub-section we show that our specification nevertheless performs favourably out-of-sample compared with three more parsimonious alternatives.

All of the alternative models we consider here can be written as

$$rx^{(i)}_{n,t,t+12} = \widetilde{\kappa}_0 + \widetilde{\kappa}'\mathbf{x}^{(i)}_t + \widetilde{\alpha}_n z^{(j)}_t + \widetilde{\eta}^{(i)}_{n,t+12}, \tag{3.7}$$

where $\mathbf{x}^{(i)}_t$ is a vector of variables constructed from domestic yields for country $i$. In all cases we also consider versions of the models that exclude the overseas unspanned factor ($z^{(j)}_t$).

The first alternative model uses the first three principal components of domestic yields, which is fairly standard number in the dynamic term structure literature. The second uses a purely domestic return-forecasting factor constructed a broadly similar way to Cochrane and Piazessi (2005) - i.e. regressing average excess returns on *ex ante* domestic forward rates. Specifically, we first regress average excess returns on bonds with 2, 3, 5, 7 and 10 years to maturity on a vector of domestic forward rates $\mathbf{f}_t^{(i)} = \left[f_{12,t}^{(i)}, f_{24,t}^{(i)}, f_{36,t}^{(i)}, f_{60,t}^{(i)}, f_{84,t}^{(i)}, f_{120,t}^{(i)}\right]'$:[8]

$$\overline{rx}_{t,t+12}^{(i)} = \theta_0 + \theta \mathbf{f}_t^{(i)} + \epsilon_{t,t+12}^{(i)}. \tag{3.8}$$

The domestic return-forecasting 'CP factor' is the fitted value from this regression. The third alternative model includes both the first three domestic principal components and the domestic CP factor. Table 3.10 reports the results of out-of-sample forecasting exercises for these more parsimonious alternative models, reporting the RMSFE for different maturity excess returns from the different models. We adopt two coding schemes to assist in reading the table. First, a bold number indicates the best performing model out of our benchmark specification and the three alternatives. A box round a number indicates which is the best performing model if we also include a random walk in the set of considered models.

We highlight the following results. First, in most cases, our benchmark specification is actually the best performing model; the only exceptions are for German long yields returns, although the differences compared with the benchmark model are small in these cases. Second, in almost all cases the versions of the models that include the overseas unspanned factor perform significantly better than the versions that exclude it, according to Giacomini and White (2006) tests of their comparative predictive ability. Here, the only exception is the model of US returns based on three domestic principal components, which performs slightly better if the overseas unspanned factor is excluded, although in this case the differences are *not* statistically significant. Third, our specification compares quite favourably with a random walk. For Germany, the benchmark model substantially out-performs a random walk at all maturities, whereas for the US it does so for the longer-maturity returns (seven and ten years).

*Different investment horizons.* In our analysis above, we have focused on twelve-month excess returns, in line with much of the literature on return predictability, including the related studies by Cochrane and Piazessi (2005) and Dahlquist and Hasseltoft (2013). In this section, we examine whether our results hold if we consider shorter holding periods. Specifically, we assess the information content of domestic and overseas unspanned factors for one- and six-month excess returns by estimating (3.6) with left-hand side variables changed to one- and six-month excess returns respectively.

Tables 3.11 and 3.12 report $R^2$ coefficients for models with different investment horizons. For the 6-month investment horizon, both domestic yields and unspanned overseas factors still contain substantial information about future excess returns, although the gain from including the unspanned overseas factor (in terms of the increase in $R^2$) is around half that for the 12-month horizon. At the one-month investment horizon return predictability is generally substantially lower and there is negligible gain from including the overseas unspanned factor.

[8]The data sources for forward rates are the same as those described in Section 3.

TABLE 3.10. Root mean squared forecast error of excess returns predictions from different models estimated over 10 years of data

| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| (a) United States | | | | | |
| Random walk | 1.163 | 2.353 | 4.698 | 7.076 | 10.619 |
| 7 Local factors | 2.153 | 4.175 | 7.263 | 9.628 | 12.540 |
| 7 Local factors and z | **1.782**** | **3.223**** | **5.238**** | **6.772**** | **8.786**** |
| 3 Local factors | 2.221 | 4.257 | 7.144 | 9.117 | 11.359 |
| 3 Local factors and z | 2.369 | 4.495 | 7.382 | 9.157 | 10.838 |
| CP factor | 2.202 | 4.351 | 7.649 | 10.009 | 12.536 |
| CP factor and z | 1.797** | 3.502** | 6.089*** | 7.924*** | 9.890** |
| 3 Local factors and CP factor | 2.405 | 4.614 | 7.746 | 9.887 | 12.315 |
| 3 Local factors and CP factor and z | 1.976** | 3.667** | 5.918** | 7.412** | 9.189** |
| (b) Germany | | | | | |
| Random walk | 1.336 | 2.597 | 4.800 | 6.597 | 8.853 |
| 7 Local factors | 1.407 | 2.732 | 4.788 | 6.185 | 7.552 |
| 7 Local factors and z | **0.784***** | **1.561***** | 3.125*** | 4.588** | 6.509 |
| 3 Local factors | 1.305 | 2.521 | 4.405 | 5.717 | 7.102 |
| 3 Local factors and z | 0.808*** | 1.616*** | 3.176*** | 4.603** | 6.486 |
| CP factor | 1.260 | 2.484 | 4.550 | 6.174 | 8.125 |
| CP factor and z | 0.856** | 1.642** | 3.144*** | 4.568** | 6.575** |
| 3 Local factors and CP factor | 1.392 | 2.709 | 4.763 | 6.152 | 7.491 |
| 3 Local factors and CP factor and z | 0.802** | 1.575*** | **3.068***** | **4.432**** | **6.227** |

*Notes:* The table reports Root Mean Square Forecast Errors for excess bond returns for five forecasting models: (i) a random walk; (ii) the benchmark model including seven domestic yields and the overseas unspanned factor (z); (iii) a model with three domestic principal components and the overseas unspanned factor; (iv) a model with our 'CP' factor and the overseas unspanned factor; and (v) a model that includes three domestic principal components, our CP factor and the overseas unspanned factor. All model parameters, as well as the domestic principal components and overseas unspanned factors are computed using 10-year rolling samples (i.e. 120 months). All numbers reported are in annualized percentage points. Asterisks indicate significance levels from Giacomini-White test (see Giacomini and White (2006)) assessing the difference of forecasting power between the considered model and the version without the overseas unspanned factor: ***,**, * denote significance at $p = 0.01$, $p = 0.05$ and $p = 0.1$ respectively for the best performing model. The sample ranges from January 1990 to December 2014, implying a forecasting period of January 2000 to December 2014.

This clearly indicates that the information content of unspanned overseas factors is more substantial for longer horizons, which is consistent with previous studies showing that bond return predictability increases with the holding period (e.g. Fama and Bliss (1987)).

TABLE 3.11. Regression of excess bond returns on domestic yields and the unspanned overseas factor for 1-month holding period

| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| (a) United States | | | | | |
| $\alpha_n$ | 0.20 | 0.31 | 0.36 | 0.20 | -0.15 |
| t-statistics | (1.0) | (0.9) | (0.6) | (0.3) | (-0.1) |
| | [-2.0,2.0] | [-2.0,2.0] | [-2.0,2.0] | [-2.0,2.1] | [-2.0,2.1] |
| $R^2$ including OUF | 0.10 | 0.09 | 0.08 | 0.08 | 0.07 |
| $R^2$ restricted $\alpha_n = 0$ | 0.09 | 0.09 | 0.08 | 0.08 | 0.07 |
| $\Delta R^2$ | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 |
| | [0.00] | [0.00] | [0.01] | [0.01] | [0.01] |
| (b) Germany | | | | | |
| $\alpha_n$ | 0.44 | 0.72 | 1.17 | 1.45 | 1.67 |
| t-statistics | (3.0) | (3.1) | (3.1) | (2.9) | (2.4) |
| | [-2.1,2.0] | [-2.1,2.0] | [-2.1,2.0] | [-2.1,2.0] | [-2.1,2.0] |
| $R^2$ including OUF | 0.12 | 0.09 | 0.07 | 0.06 | 0.04 |
| $R^2$ restricted $\alpha_n = 0$ | 0.09 | 0.05 | 0.03 | 0.03 | 0.02 |
| $\Delta R^2$ | 0.03 | 0.04 | 0.04 | 0.03 | 0.02 |
| | [0.00] | [0.01] | [0.01] | [0.01] | [0.01] |

*Notes:* The table reports results from regressions of one-month excess bond returns on an intercept, seven domestic yields and the overseas unspanned factor - i.e. equation (3.6). For each holding period the table reports the estimate of the coefficient on the overseas unspanned factor ($\alpha_n$). Numbers in parentheses report the values of t-statisitcs and numbers in brackets refer to the 95% confidence interval for these t-statistics obtained using the Bauer and Hamilton (2015) bootstrap procedure. The final two rows of each part of the table report the $R^2$ statistics from models with and without the overseas unspanned factor ('Including OUF' and 'Restricted' respectively). Numbers in brackets refer to the 95% critical value for the change in the $R^2$. The sample ranges from January 1990 to December 2014.

*Incorporating the UK into the analysis.* In this sub-section, we show that similar results hold if we extend the analysis to cover the excess returns on UK bonds. We first estimate two overseas unspanned factors using the procedure explained previously: one each from the components of US and German yields that are orthogonal to UK yields. More precisely, we first estimate (3.1) and then (3.3) with the US as country $j$ and the UK as country $i$ to obtain an overseas unspanned factor $z_t^{(US)}$. We then repeat the process with Germany as country $j$ to obtain an overseas unspanned factor $z_t^{(DE)}$. We then assess whether either of these factors contains information for predicting UK returns relative to the information contained in the UK term structure by estimating extended versions of (3.6):

$$rx_{n,t,t+12}^{(UK)} = \kappa_0 + \kappa' \mathbf{y}_t^{(UK)} + \alpha_{n,US} z_t^{(US)} + \alpha_{n,DE} z_t^{(DE)} + \eta_{n,t+12}^{(UK)} \tag{3.9}$$

Table 3.13 reports $R^2$ coefficients from versions of this regression with different combinations of the overseas unspanned factors. Including either of the overseas unspanned factors causes the $R^2$ to rise substantially, particularly at short maturities, although the difference

TABLE 3.12. Regression of excess bond returns on domestic yields and the unspanned overseas factor for 6-month holding period

| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| (a) United States | | | | | |
| $\alpha_n$ | 0.41 | 0.76 | 1.36 | 1.81 | 2.34 |
| t-statistics | (5.8) | (6.5) | (6.9) | (6.8) | (6.4) |
| | [-3.6,3.6] | [-3.6,3.6] | [-3.7,3.6] | [-3.6,3.7] | [-3.6,3.7] |
| $R^2$ including OUF | 0.32 | 0.32 | 0.33 | 0.32 | 0.30 |
| $R^2$ restricted $\alpha_n = 0$ | 0.24 | 0.22 | 0.22 | 0.21 | 0.20 |
| $\Delta R^2$ | 0.08 | 0.10 | 0.11 | 0.09 | 0.10 |
| | [0.03] | [0.04] | [0.04] | [0.04] | [0.04] |
| (b) Germany | | | | | |
| $\alpha_n$ | 0.48 | 0.83 | 1.37 | 1.71 | 2.00 |
| t-statistics | (9.3) | (9.6) | (9.4) | (8.7) | (7.6) |
| | [-3.7,3.7] | [-3.8,3.8] | [-3.9,3.8] | [-3.9,3.9] | [-4.0,4.0] |
| $R^2$ including OUF | 0.36 | 0.36 | 0.33 | 0.30 | 0.27 |
| $R^2$ restricted $\alpha_n = 0$ | 0.17 | 0.15 | 0.12 | 0.11 | 0.11 |
| $\Delta R^2$ | 0.19 | 0.21 | 0.21 | 0.19 | 0.16 |
| | [0.05] | [0.06] | [0.06] | [0.06] | [0.06] |

*Notes:* The table reports results from regressions of six-month excess bond returns on an intercept, seven domestic yields and the overseas unspanned factor - i.e. equation (3.6). For each holding period the table reports the estimate of the coefficient on the overseas unspanned factor ($\alpha_n$). Numbers in parentheses report the values of t-statisitcs and numbers in brackets refer to the 95% confidence interval for these t-statistics obtained using the Bauer and Hamilton (2015) bootstrap procedure. The final two rows of each part of the table report the $R^2$ statistics from models with and without the overseas unspanned factor ('Including OUF' and 'Restricted' respectively). Numbers in brackets refer to the 95% critical value for the change in the $R^2$. The sample ranges from January 1990 to December 2014.

is greater when the US factor is added. For example, the model with no overseas unspanned factors has an $R^2$ of 0.23 for the excess return on the two-year bond; this rises to 0.51 for the model including the US unspanned factor; or 0.37 for the model including the German factor. Including both overseas factors raises the $R^2$ a little further.

Table 3.14 reports results from an out-of-sample forecasting exercise for UK returns, analogous to those reported in Section 3.4.2. The best performing model for all maturities is the one that includes both the US and German overseas unspanned factors and the improvement relative to a model that only includes domestic yields is strongly statistically significant according to Giacomini and White (2006) tests. The model with both overseas unspanned factors even out-performs a random walk for maturities longer than five years.

## 3.4. A dynamic term structure model

**3.4.1. Model.** In this section, we use our preceding results to motivate a simple dynamic term structure model. Specifically, for each country we consider a first-order VAR of the form:

TABLE 3.13. United Kingdom excess bond returns regressions

| Maturity (months) | 24 | 36 | 60 | 84 | 120 |
|---|---|---|---|---|---|
| $\alpha_{n,US}$ | 0.274 | 0.529 | 0.860 | 1.049 | 1.188 |
| t-statistics | (10.4) | (11.3) | (10.5) | (9.0) | (7.1) |
| | [-4.5,4.6] | [-4.5,4.6] | [-4.5,4.6] | [-4.5,4.7] | [-4.6,4.7] |
| $\alpha_{n,DE}$ | 0.157 | 0.382 | 0.740 | 0.934 | 0.996 |
| t-statistics | (4.9) | (6.8) | (7.4) | (6.6) | (5.0) |
| | [-3.9,3.9] | [-3.9,3.9] | [-3.9,3.9] | [-3.9,3.9] | [-4.0,3.9] |
| (a) Including $z_t^{US}$ and $z_t^{DE}$ | 0.547 | 0.605 | 0.610 | 0.580 | 0.529 |
| (b) Restricted $\alpha_{n,DE} = 0$ | 0.508 | 0.540 | 0.533 | 0.513 | 0.487 |
| (c) Restricted $\alpha_{n,US} = 0$ | 0.372 | 0.422 | 0.457 | 0.457 | 0.442 |
| (d) Restricted $\alpha_{n,DE} = 0$ and $\alpha_{n,US} = 0$ | 0.231 | 0.231 | 0.255 | 0.288 | 0.332 |
| $\Delta R^2 = R^2_{(a)} - R^2_{(d)}$ | 0.316 | 0.374 | 0.355 | 0.292 | 0.197 |
| | [0.066] | [0.071] | [0.075] | [0.076] | [0.075] |

*Notes:* The table reports results from regressions of UK excess bond returns on a constant, seven domestic yields and two overseas factors for US and Germany - i.e. equation (3.9). The table reports the estimates of the coefficients on the overseas unspanned factors ($\alpha_{n,US}$ and $\alpha_{n,DE}$). Numbers in parentheses report the values of t-statisitcs and numbers in brackets refer to the 95% confidence interval for these t-statistics obtained using the Bauer and Hamilton (2015) bootstrap procedure. The final four rows report the $R^2$ statistics from models with different combinations of the two overseas unspanned factors. Numbers in brackets refer to the 95% critical value for the change in the $R^2$. The sample ranges from January 1990 to December 2014.

TABLE 3.14. Root mean squared forecast error of out-of-sample UK excess return predictions

| Maturity (months) | 24 | 36 | 60 | 82 | 120 |
|---|---|---|---|---|---|
| Random walk | 1.376 | 2.533 | 4.657 | 6.742 | 9.607 |
| Restricted $\alpha_{n,DE}$= 0 and $\alpha_{n,US}$= 0 | 2.273 | 4.055 | 6.330 | 7.689 | 8.854 |
| Including $z_t^{DE}$ | 1.890** | 3.355** | 5.269** | 6.489** | 7.732** |
| Including $z_t^{US}$ | 1.778*** | 3.122*** | 4.944*** | 6.203*** | 7.657** |
| Including $z_t^{US}$ and $z_t^{DE}$ | 1.589*** | 2.767*** | 4.399*** | 5.591*** | 7.108*** |

*Notes:* The table reports Root Mean Square Forecast Errors for UK excess bond returns for five forecasting models: a random walk and four restricted and unrestricted versions of equation (3.9). All model parameters, as well as the OUFs are computed using 10-year rolling samples (i.e. 120 months). All numbers reported are in annualized percentage points. Asterisks indicate significance levels from Giacomini-White test (see Giacomini and White (2006)) assessing the difference of forecasting power between the considered model and the version without either overseas unspanned factor: ***,**, * denote significance at $p = 0.01$, $p = 0.05$ and $p = 0.1$ respectively for the best performing model. The sample ranges from January 1990 to December 2014, implying a forecasting period of January 2000 to December 2014.

$$\begin{aligned} \mathbf{m}_t^{(i)} &= \mu + \mathbf{\Phi}\mathbf{m}_{t-12}^{(i)} + \mathbf{\Sigma}\mathbf{v}_t \\ \mathbf{v}_t &\sim i.i.d.\left(\mathbf{0}, \mathbf{I}\right). \end{aligned} \tag{3.10}$$

Here, the $4 \times 1$ vector $\mathbf{m}_t = \left[\mathbf{x}_t^{(i)\prime}, z_t^{(j)}\right]'$ collects the first three principal components of domestic yields ($\mathbf{x}_t^{(i)\prime}$) and the overseas unspanned factor ($z_t^{(j)}$); and $\boldsymbol{\Sigma}$ is a lower triangular matrix. We use a lag of twelve months in the VAR, rather than the more standard single month lag in the dynamic term structure literature. We justify this choice by appealing to the results in the previous section: return predictability is substantially stronger at lags of twelve months than one month. We estimate the model using our benchmark sample (i.e. January 1990-December 2014), which means that we have 288 overlapping sample points with which to estimate the model.

We can motivate the choice of three domestic principal components - which is standard in the term structure literature - by referring to a preliminary principal components analysis of domestic yields. In both countries the first three principal components collectively account for more than 99.9% of the variation in the considered bond yields (Table 3.15). As is standard, the loadings on the first ('level') principal component have the same sign and are relatively constant across maturities. For the second ('slope') principal component, the loadings are increasing with maturity, while for the third ('curvature'), the loadings are higher at very short and very long maturities.

TABLE 3.15. Principal component analysis of domesetic bond yields

| | Cum. prop. explained | PC loadings (maturities in months) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (Percentage) | 6 | 12 | 24 | 36 | 60 | 84 | 120 |
| (a) United States | | | | | | | | |
| PC1 | 96.444 | 0.403 | 0.414 | 0.414 | 0.401 | 0.366 | 0.334 | 0.298 |
| PC2 | 99.859 | -0.506 | -0.386 | -0.153 | 0.034 | 0.288 | 0.435 | 0.546 |
| PC3 | 99.985 | 0.551 | 0.069 | -0.403 | -0.480 | -0.215 | 0.130 | 0.485 |
| PC4 | 99.998 | 0.409 | -0.408 | -0.365 | 0.086 | 0.489 | 0.249 | -0.474 |
| PC5 | 100.000 | -0.313 | 0.604 | -0.168 | -0.418 | 0.168 | 0.447 | -0.328 |
| PC6 | 100.000 | 0.112 | -0.348 | 0.524 | -0.215 | -0.436 | 0.562 | -0.200 |
| PC7 | 100.000 | -0.035 | 0.150 | -0.456 | 0.616 | -0.528 | 0.324 | -0.072 |
| (b) Germany | | | | | | | | |
| PC1 | 97.604 | 0.412 | 0.407 | 0.400 | 0.391 | 0.368 | 0.345 | 0.313 |
| PC2 | 99.852 | -0.501 | -0.387 | -0.170 | 0.016 | 0.281 | 0.434 | 0.551 |
| PC3 | 99.994 | 0.589 | 0.013 | -0.421 | -0.463 | -0.208 | 0.101 | 0.456 |
| PC4 | 99.999 | -0.430 | 0.548 | 0.278 | -0.201 | -0.432 | -0.158 | 0.431 |
| PC5 | 100.000 | -0.202 | 0.537 | -0.323 | -0.368 | 0.294 | 0.439 | -0.390 |
| PC6 | 100.000 | 0.080 | -0.291 | 0.556 | -0.344 | -0.350 | 0.560 | -0.213 |
| PC7 | 100.000 | 0.022 | -0.104 | 0.378 | -0.583 | 0.589 | -0.389 | 0.087 |

*Notes:* Table reports the cumulative proportion of the variation in the considered yields explained by successive principal components (PCs) and loadings of PC on different maturity bond yields. The sample used ranges from January 1990 to December 2014.

The model (3.10) specifies the time-series dynamics of the factors that drive bond yields, analogous to (A.9) in a standard no-arbitrage term structure model. Given that the domestic yield curve factors are principal components of yields, our model also has an affine cross-sectional mapping between the factors and current yields:

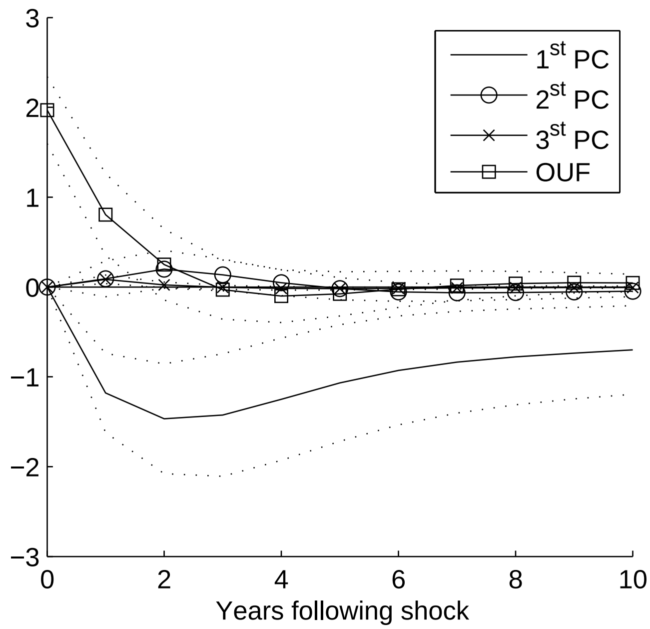
$$\mathbf{y}_t^{(i)} = \mathbf{A}^{(i)} + \mathbf{B}^{(i)}\mathbf{x}_t^{(i)} + \mathbf{w}_t^{(i)}. \tag{3.11}$$

Here, $\mathbf{y}_t^{(i)} = \left[y_{6,t}^{(i)}, y_{12,t}^{(i)}, ..., y_{120,t}^{(i)}\right]'$ is a $7 \times 1$ vector of bond yields observed at time $t$; the coefficients $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(i)}$ are determined by the relevant principal component loadings; and $\mathbf{w}_t^{(i)}$ is a measurement error.

We identify the impact of shocks to the overseas unspanned factor using a Cholesky factorization, ordering it last in the VAR (3.10). While Cholesky identification is sensitive to the ordering of variables in the VAR, ordering the overseas unspanned factor last makes intuitive sense in this case, given that it is orthogonal to domestic yields by construction: the assumption that $\mathbf{\Sigma}$ is lower triangular means that a shock to the final element of $\mathbf{v}_t$ is one that has an impact on the overseas unspanned factor but no contemporaneous impact on domestic yields.

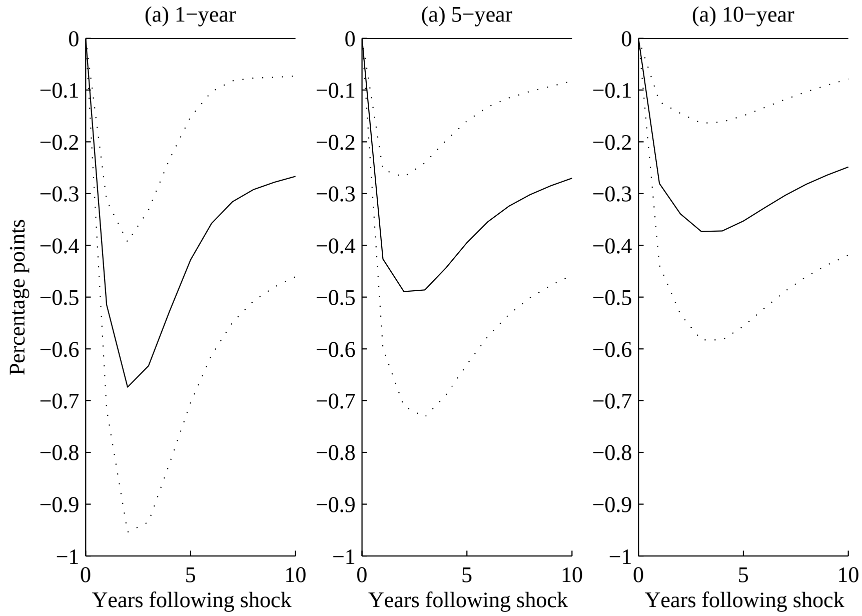
**3.4.2. Results.** Our results suggest that the impact of shocks to overseas unspanned factors on domestic bond yields can be substantial and persistent. Figure 3.3 shows impulse response functions for the local principal components from the model with Germany as the domestic country. Following a one standard deviation shock to the overseas unspanned factor the first principal component (i.e. the level of the yield curve) falls and the second and third principal components (i.e. the slope and curvature) rise, although the effect on the level is much larger and more persistent than on the other domestic principal components. Figure 3.4 translates this into the reaction of yields of different maturities. The shock is followed by a drop in domestic yields (as explained above, there is no contemporaneous reaction by construction). This reaction is largest for short maturity yields: six-month to three-year yields all fall by around 50 basis points twelve months after the shock, while the fall in the ten-year yield is only about 30 basis points. The peak impact on short maturity yields comes after two years but longer maturity yields continue to fall for four years after the shock. After about seven years, the remaining effect is roughly equal across the yield curve (as the impacts on the slope and curvature factors have largely died out).

FIGURE 3.3. German yield curve factors response to an innovation in the unspanned overseas factor.




*Notes:* The figure depicts impulse responses of the first three principal components of German yield curve ($1^{st}$ PC, $2^{nd}$ PC and $3^{rd}$ PC) and unspanned overseas factor (OUF) to a one standard deviation shock to the unspanned overseas factor.

FIGURE 3.4. German yields response to an innovation to the unspanned overseas factor.



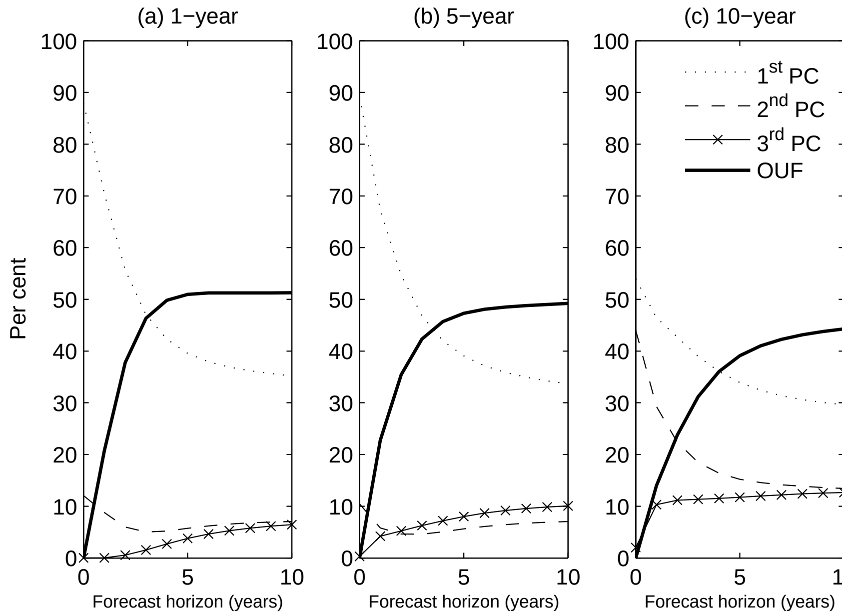

*Notes:* The figure depicts impulse response of different maturity German yields to a one standard deviation shock to the unspanned overseas factor (OUF).

Figure 3.5 decomposes the variance of forecast errors for selected maturity yields into the contributions from innovations to different factors, for different forecast horizons.[9] Panel (a) shows results for the one-year yield, panel (b) for the five-year yield and panel (c) for the ten-year yield. At short forecast horizons, the majority of the forecast errors are explained by the level factor, with a smaller contribution from the slope and a negligible contribution from the curvature. The contribution of the overseas unspanned factor grows with maturity; at the ten-year forecast horizon it accounts for more than 40% of the variance of forecast horizons, with the largest contribution at shorter maturities. At forecast horizons longer than three or four years (depending on the maturity) the overseas unspanned factor accounts for more of the forecast error variance than the level factor.

Figures 3.6 and 3.7 report the equivalent impulse response functions from the model with the US as the domestic country. Similar to the case of Germany, the US level factor falls following the shock to the overseas unspanned factor, although the impact is much less persistent. The peak response of all yields (around 30 to 40 basis points) comes twelve months following the shock. The forecast error variance decompositions are also somewhat different for the US (Figure 3.8): the proportion explained by the overseas unspanned factor is somewhat smaller than for Germany, although it still reaches about 15% for the ten-year

[9] Appendix A explains how these are computed.

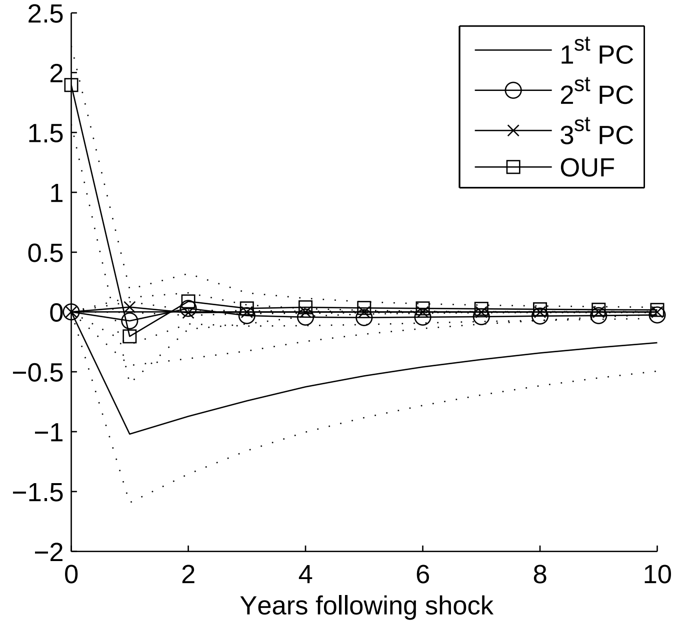
FIGURE 3.5. Forecast error variance decomposition of German yields.




*Notes:* The figure depicts forecast error variance decompositions for three different yields for forecast horizons of up to 10 years. Panels (a), (b) and (c) report decompositions for 1-year, 5-year and 10-year German yields, respectively. Each panel shows the proportion of the yield forecast error variance accounted by the first three principal components ($1^{st}$ PC, $2^{nd}$ PC and $3^{rd}$ PC) of the German yield curve and the unspanned overseas factor (OUF).

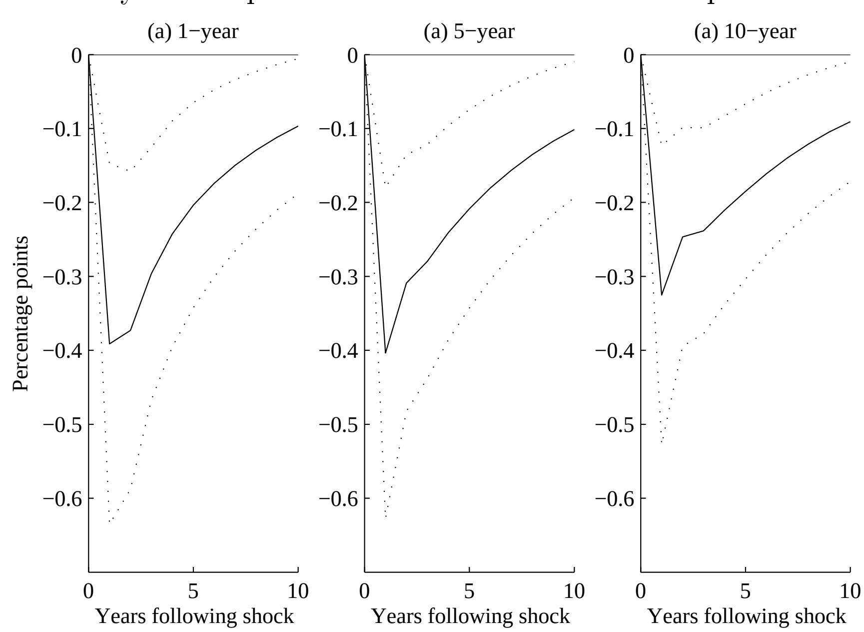
forecast horizon (with shocks to all of the domestic yield curve factors playing a relatively more important role in explaining forecast errors).

FIGURE 3.6. US yield curve factors response to an innovation in the unspanned overseas factor.



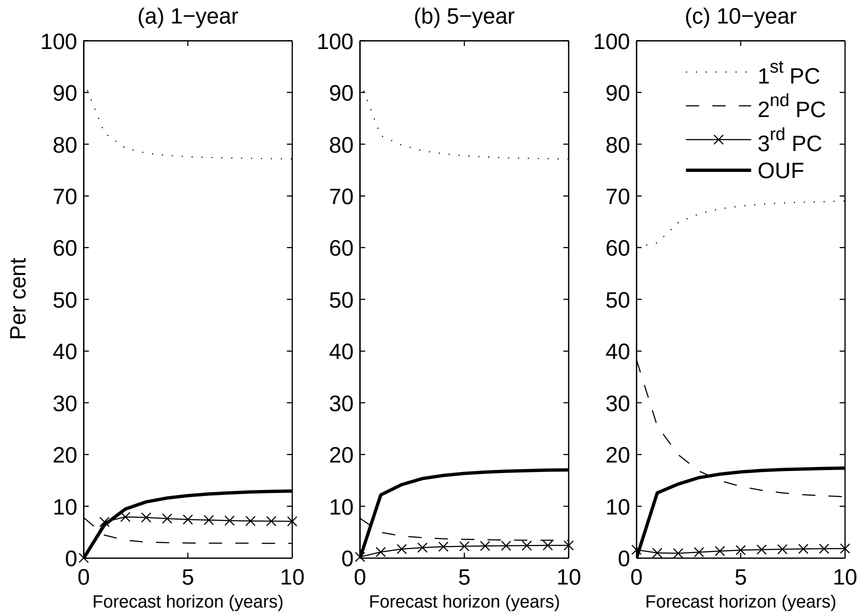

*Notes:* The figure depicts impulse responses of the first three principal components of US yield curve ($1^{st}$ PC, $2^{nd}$ PC and $3^{rd}$ PC) and the unspanned overseas factor (OUF) to a one standard deviation shock to the unspanned overseas factor.

FIGURE 3.7. US yields response to an innovation to the unspanned overseas factor.




*Notes:* The figure depicts impulse response of different maturity US yields to a one standard deviation shock to the unspanned overseas factor (OUF).

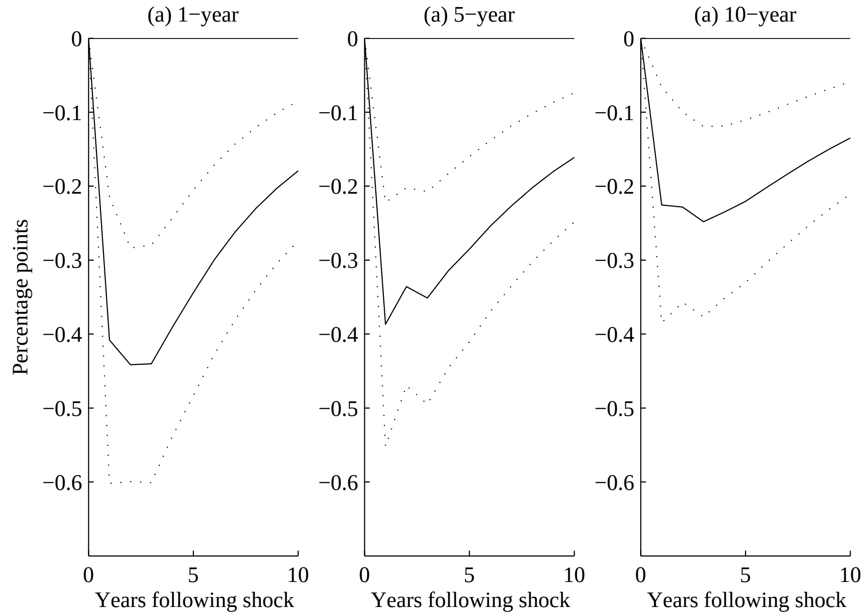
FIGURE 3.8. Variance decomposition of US yields.




*Notes:* The figure depicts forecast error variance decompositions for three different yields for forecast horizons of up to 10 years. Panels (a), (b) and (c) report decompositions for 1-year, 5-year and 10-year US yields, respectively. Each panel shows the proportion of the yield forecast error variance accounted by the first three principal components ($1^{st}$ PC, $2^{nd}$ PC and $3^{rd}$ PC) of the US yield curve and the unspanned overseas factor (OUF).

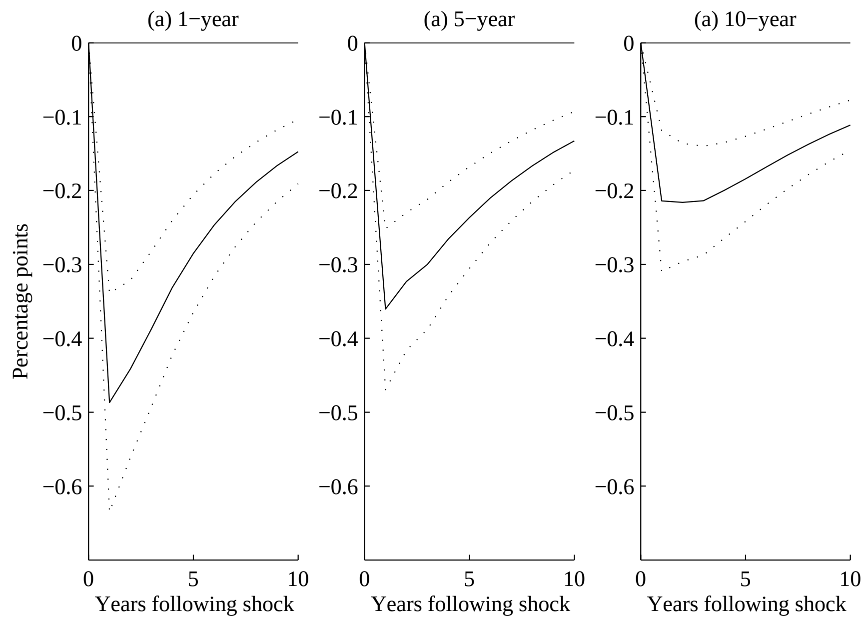
**3.4.3. Robustness: a dynamic term structure model for the UK.** In this subsection, we examine whether we obtain similar results if we estimate a dynamic term structure model of UK yields that includes both US and German unspanned factors, constructed as described in Section 3.4.4. The US factor is the penultimate variable in the time-series VAR and the German variable is the final variable; this implies that shocks to the US factor can have a contemporaneous impact on the German factor but not vice versa. First, Figures 3.9 and 3.10 show the responses of UK yields following a one standard deviation shock to the US and German unspanned factors respectively. Similar to the results for the two-country models reported above, yields fall following the shock - in this case, by up to about 50 basis points - and the effect is persistent. The overseas unspanned factors explain a substantial part of the forecast error variance of yields (Figure 3.11). For example, for the ten-year yield each accounts for 10-20% of the variance of ten-year ahead forecast errors.

FIGURE 3.9. UK yields response to an innovation to the German unspanned overseas factor.




*Notes:* The figure depicts impulse response of different maturity UK yields to a one standard deviation shock to the German unspanned overseas factor ($z_t^{DE}$).

FIGURE 3.10. UK yields response to an innovation to the US unspanned overseas factor.




*Notes:* The figure depicts impulse response of different maturity UK yields to a one standard deviation shock to the US unspanned overseas factor ($z_t^{US}$).

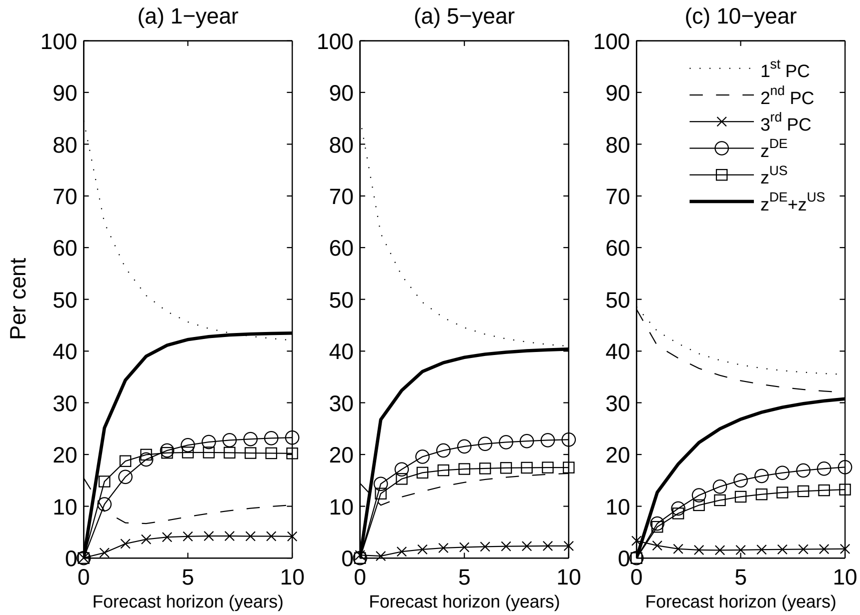
## 3.5. Conclusions

The recent literature on unspanned factors in the term structure of interest rates argues that there is a non-trivial portion of information that is not contained in the yield curve, but helps to predict yields' dynamics. This article argues that there is important information contained in foreign yields, which is not contained in (spanned by) domestic yields and that helps to predict future moves of domestic yields.

More specifically, we show that there is important information spanned by the German yield curve, but unspanned by the US yield curve, which helps forecasting future dynamics of US yields and vice versa. We use simple return-forecasting regressions to prove that the overseas unspanned factors matter, both in- and out-of-sample. We also show that this result is robust to different sample selections as well as to different specification of domestic yield curve factors. In addition, we find that it is not only a US-DE phenomenon. We also show that US and German factors unspanned by the UK yield curve have substantial predictive power for UK yields. An advantage of the modular structure of our approach for adding different countries mean that this analysis would be straightforward to extend to other countries.

Our results are especially important for dynamic factor models of bond yields. Current state of the art models focus only on domestic yields, hence, in the light of our findings,

FIGURE 3.11. Variance decomposition of UK yields.




*Notes:* The figure depicts forecast error variance decompositions for three different yields for forecast horizons of up to 10 years. Panels (a), (b) and (c) report decompositions for 1-year, 5-year and 10-year UK yields, respectively. Each panel shows the proportion of the yield forecast error variance accounted by the first three principal components ($1^{st}$ PC, $2^{nd}$ PC and $3^{rd}$ PC) of the UK yield curve and the two unspanned overseas factors, namely: the German unspanned overseas factor ($z_t^{DE}$) and the US unspanned overseas factor ($z_t^{US}$).

they lack important information and are potentially misspecified. In fact, when we enrich simple dynamic term structural model, consisting of the first 3 principal components, with overseas unspanned factor we find that shocks to this factor drive sizeable portions of future yields variation. This effect is especially pronounced for German and UK yields, but is also significant for US yields.

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## Appendix A - Motivation for the dynamic term structure model

Although we do not estimate no-arbitrage term structure models in this paper, we can nevertheless motivate our empirical exercises by appealing to the standard Gaussian dynamic no-arbitrage affine term structure models (ATSM) of Duffie and Kan (1996) and Duffee (2002). These models have four basic building blocks. First, the assumption of no arbitrage implies that the price at time $t$ of an $n$-period default-free zero-coupon bond ($P_t^{(n)}$) is given by

$$P_t^{(n)} = E_t^{\mathbb{Q}} \left[ \exp\left(-r_t\right) P_{t+1}^{(n-1)} \right], \tag{A.1}$$

where $r_t$ is the risk-free one-period interest rate and expectations are formed with respect to the risk-neutral probability measure, denoted $\mathbb{Q}$. Second, the short-term interest rate is an affine function of a $K \times 1$ vector of unobserved pricing factors ($\mathbf{x}_t$):

$$r_t = \delta_0 + \delta_1' \mathbf{x}_t. \tag{A.2}$$

Third, the pricing factors follow a Gaussian Vector Autoregression (VAR) under $\mathbb{Q}$:

$$\begin{aligned} \mathbf{x}_{t+1} &= \mu^{\mathbb{Q}} + \mathbf{\Phi}^{\mathbb{Q}} \mathbf{x}_t + \mathbf{\Sigma} \mathbf{v}_{t+1}^{\mathbb{Q}} \\ \mathbf{v}_{t+1}^{\mathbb{Q}} &\sim i.i.d. \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right). \end{aligned} \tag{A.3}$$

Under these assumptions, $n$-period bond yields turn out to be affine functions of the state variables:

$$y_t^{(n)} = -\frac{1}{n} \log P_t^{(n)} = -\frac{1}{n} \left( a_n + \mathbf{b}_n' \mathbf{x}_t \right), \tag{A.4}$$

where the coefficients $a_n$ and $\mathbf{b}_n$ follow the recursive equations

$$a_n = a_{n-1} + \mathbf{b}_{n-1}' \mu^{\mathbb{Q}} + \frac{1}{2} \mathbf{b}_{n-1}' \mathbf{\Sigma} \mathbf{\Sigma}' \mathbf{b}_{n-1} - \delta_0 \tag{A.5}$$

$$\mathbf{b}_n' = \mathbf{b}_{n-1}' \mathbf{\Phi}^{\mathbb{Q}} - \delta_1. \tag{A.6}$$

Finally, the Radon-Nikodym derivative which relates the time-series and risk-neutral dynamics takes the form

$$\left( \frac{d\mathbb{P}}{d\mathbb{Q}} \right)_{t+1} = \exp\left[ -\frac{1}{2} \lambda_t' \lambda_t + \lambda_t' \mathbf{v}_{t+1} \right] \tag{A.7}$$

where the prices of risk ($\lambda_t$) are affine in the pricing factors, as proposed by Duffee (2002):

$$\lambda_t = \mathbf{\Sigma}^{-1} \left( \lambda_0 + \mathbf{\Lambda}_1 \mathbf{x}_t \right). \tag{A.8}$$

This implies that the factors also follow a Gaussian VAR(1) under the time-series measure:

$$\begin{aligned} \mathbf{x}_{t+1} &= \mu + \mathbf{\Phi} \mathbf{x}_t + \mathbf{\Sigma} \mathbf{v}_{t+1} \\ \mathbf{v}_{t+1} &\sim i.i.d. \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right), \end{aligned} \tag{A.9}$$

where

$$\mu = \mu^{\mathbb{Q}} + \boldsymbol{\Sigma}\lambda_0 \tag{A.10}$$

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathbb{Q}} + \boldsymbol{\Sigma}\boldsymbol{\Lambda}_1. \tag{A.11}$$

In a model with unobserved factors, we must impose additional identification restrictions. Here we consider the normalization of Dai and Singleton (2000),[10] where $\delta_1 = \mathbf{1}$, $\mu^{\mathbb{Q}} = \mathbf{0}$, $\boldsymbol{\Phi}^{\mathbb{Q}}$ is a diagonal matrix and $\boldsymbol{\Sigma}$ is lower triangular.[11] All other parameters are unrestricted.

To capture the case where we have some factors that are unspanned by current yields, we can partition the vector of factors into $K_s < K$ factors that are spanned by the yield curve ($\mathbf{x}_t^s$) and $K - K_s$ unspanned factors ($\mathbf{x}_t^u$), i.e. $\mathbf{x}_t = \begin{bmatrix} \mathbf{x}_t^{s\prime} & \mathbf{x}_t^{u\prime} \end{bmatrix}'$, as in Joslin et al. (2014). Given the normalization that $\boldsymbol{\Phi}^{\mathbb{Q}}$ is diagonal, the assumption that $\mathbf{x}_t^u$ is unspanned implies zero restrictions on the elements of $\delta_1$ corresponding to the unspanned factors, i.e. $\delta_1 = \begin{bmatrix} \mathbf{1}'_{K_s\times 1} & \mathbf{0}'_{(K-K_s)\times 1} \end{bmatrix}'$ (where we have also imposed that the elements of $\delta_1$ corresponding to the spanned factors are normalized to one, as explained above). It is not possible to identify the prices of unspanned factors in such a model, so we can set the corresponding elements of the prices of risk to zero, i.e.:

$$\lambda_0 = \begin{bmatrix} \lambda_0^s \\ \mathbf{0}_{(K-K_s)\times 1} \end{bmatrix}$$

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} \boldsymbol{\Lambda}_1^{ss} & \boldsymbol{\Lambda}_1^{su} \\ \mathbf{0}_{(K-K_s)\times K_s} & \mathbf{0}_{(K-K_s)\times(K-K_s)} \end{bmatrix}.$$

The one-period excess return on an $n$-period bond is defined as

$$rx_{t+1}^{(n-1)} = \log P_{t+1}^{(n-1)} - \log P_t^{(n)} - r_t. \tag{A.12}$$

Using (A.2), (A.4)-(A.6) and (A.9)-(A.11) in (A.12) gives[12]

$$rx_{t+1}^{(n-1)} = -\frac{1}{2}\mathbf{b}'_{n-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{b}_{n-1} + \mathbf{b}'_{n-1}\boldsymbol{\Sigma}\lambda_0 + \mathbf{b}'_{n-1}\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\mathbf{x}_t + \mathbf{b}'_{n-1}\boldsymbol{\Sigma}\mathbf{v}_{t+1}. \tag{A.13}$$

The first two terms on the right-hand side of (A.13) are constant. The final term is the unexpected component of excess returns. The third term captures the time-variation in expected returns, which depends on the price of risk parameters ($\boldsymbol{\Lambda}_1$). Taking expectations of both sides of (A.13) gives

$$E_t\left[rx_{t+1}^{(n-1)}\right] = -\frac{1}{2}\mathbf{b}'_{n-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{b}_{n-1} + \mathbf{b}'_{n-1}\boldsymbol{\Sigma}\lambda_0 + \mathbf{b}'_{n-1}\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\mathbf{x}_t \tag{A.14}$$

[10]Other normalizations are feasible, e.g. the scheme proposed by Joslin et al. (2011).

[11]Hamilton and Wu (2012) show that identification also requires an additional restriction on the ordering of the elements of $\boldsymbol{\Phi}^{\mathbb{Q}}$.

[12]Abrahams et al. (2015) provide a fuller derivation of the following equation.

which is equivalent to equation (15) in Cochrane and Piazessi (2008). Our reduced-form regressions reported in Section 3 involve regressing excess returns of different maturities on a constant and various factors, some of which are extracted from domestic yields and some of which are unspanned by domestic yields by construction. We can motivate these regressions by appealing to (A.14): if an unspanned factor has a non-zero slope coefficient in these unrestricted regressions, this factor must affect the price of one or more of the spanned factors.[13] And if an unspanned factor enters the price of risk, it must also enter the time series dynamics of yields (A.9), which have an analogous specification to that of the dynamic factor model reported in Section 4 (as explained above, the only difference between an ATSM and our factor model is in the cross-sectional relationship between factors and yields, which is unlikely to make a material difference to our results).

## Appendix B - Bauer and Hamilton (2015) bootstrap procedure

In this appendix, we explain how we implement the procedure for computing confidence intervals for the return-forecasting regressions proposed by Bauer and Hamilton (2015). Our return-forecasting regressions take the general form

$$rx^{(i)}_{n,t,t+h} = \kappa_0 + \kappa' \mathbf{y}^{(i)}_t + \alpha_n z^{(j)}_t + \eta^{(i)}_{n,t+12} \tag{B.1}$$

Bauer and Hamilton propose a bootstrap procedure to simulate the distribution of the coefficient on the unspanned factor (i.e. $\alpha_n$) and the increase in the $R^2$ of (B.1) resulting from the inclusion of the unspanned factor $z^{(j)}_t$ under the null hypothesis that $\alpha_n = 0$. In our implementation of their procedure, we first estimate separate VAR(1) models for the spanned yield curve factors for country $i$ (in our case, $\mathbf{y}^{(i)}_t = \left[y^{(i)}_{6,t}, y^{(i)}_{12,t}, y^{(i)}_{24,t}, ..., y^{(i)}_{120,t}\right]'$) and the unspanned factor for country $j$ $z^{(j)}_t$:

$$\mathbf{y}^{(i)}_{t+1} = \mu_y + \mathbf{\Phi}_y \mathbf{y}^{(i)}_t + \mathbf{v}^{(i)}_{t+1} \tag{B.2}$$

$$z^{(j)}_{t+1} = \mu_z + \phi_z z^{(j)}_t + w^{(j)}_{t+1} \tag{B.3}$$

We assume that yields with maturities other than those included in $\mathbf{y}^{(i)}_t$ (such as the six-year yield) are given by affine functions of $\mathbf{y}^{(i)}_t$, i.e.

$$y^{(i)}_{n,t} = a_n + b'_n \mathbf{y}^{(i)}_t + e^{(i)}_{n,t} \tag{B.4}$$

Given these assumptions, we use a residual bootstrap to produce 10,000 draws of the domestic yields for country $i$ and the country-$j$ unspanned factor. In each bootstrapped sample,

[13] We can illustrate this easily using a $2 \times 2$ example in which the second factor is unspanned (i.e. the loading on this factor is $b_{n-1,2} = 0$ for all $n$). In this case, the third term simplifies to $\mathbf{b}'_{n-1} \mathbf{\Sigma} \mathbf{\Lambda}_1 \mathbf{x}_t = b_{n-1,1} \sigma_{11} (\lambda_{11} x_{1,t} + \lambda_{12} x_{2,t})$. A non-zero slope coefficient on $x_{2,t}$ in a regression of excess returns on a constant and the factors requires that $\lambda_{12} \neq 0$, i.e. that the unspanned factor affects the price of the spanned factor.

the country-$j$ unspanned factor has no predictive power for domestic returns by construction (consistent with the null hypothesis). For each bootstrapped sample, we compute 12-month returns on domestic yields with maturities of $1, 2, ..., 10$ years, obtaining the required yields not included in $\mathbf{y}_t^{(i)}$ using (B.4).[14] We then estimate (B.1) for each bootstrapped sample, as well as a restricted version with $\alpha_n = 0$. The critical values for $\alpha_n$ and the increase in $R^2$ reported in the text are the $97.5th$ percentile of the bootstrapped distributions.

## Appendix C - Forecast error variance decompositions

Forecast error variance decomposition is another useful tool to assess the impact of unspanned overseas factors. In order to compute variance decomposiotions of yields' forecast errors, we assume that there is no measurement error in (3.11). Using our factor specification we can then re-write (3.10) as:

$$\mathbf{y}_t^{(i)} = \mathbf{A}^{(i)} + \widetilde{\mathbf{B}}^{(i)}\mathbf{f}_t^{(i)}, \tag{C.1}$$

where $\widetilde{B} = \begin{bmatrix} B^i \\ 0 \end{bmatrix}$. This allows us to map forecast variance error decomposition of different factors into forecast variance decomposition of yields. As we look at annual forecasting horizons, for simplicity we drop monthly time notation and denote time in annual units, ex. t+1 means t plus 1 year. Taking into account (equation C.1), we can define h-year forecast Mean Squared Forecast Errors matrix as:

$$\Omega_y(h) = \sum_{i=0}^{h-1} \left( \widetilde{B}\Phi^i \Sigma \Sigma' (\Phi^i)' \widetilde{B}' \right). \tag{C.2}$$

Note that $\Phi^i \Sigma$ is simply the i-th parameters of the VMA representation of VAR(3.10). More importantly diagonal elements of $\Omega_y(h)$ are the h-year MSFE of the j-th yield - $\Omega_{y_j}(h)$. The contribution of innovations in factor k to the h-year MSFE of yield j is given by:

$$\sum_{i=0}^{h-1} \left( e_j' \widetilde{B} \Phi^i \Sigma e_k \right)^2, \tag{C.3}$$

where $e_j$ is the j-th colum of the identity matrix. Dividing the contribution (C.3) by total h-year MSFE of j-th yields we obtain the proportion of the h-year ahead forecast error variance of yield j accounted by an innovation to the k-th factor:

[14] Unlike Bauer and Hamilton (2015), we ignore the measurement error on these yields in the bootstrap. Given that we are effectively estimating a seven-factor yield curve model, these measurement errors will be tiny.

$$\omega_{j,k,h} = \frac{\sum_{i=0}^{h-1} \left( e_j' \widetilde{B} \Phi^i \Sigma e_k \right)^2}{\Omega_{y_j}(h)} \tag{C.4}$$