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A NOTE ON THE DEMAND THEORY OF THE WEAK AXIOMS

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ABSTRACT: This note characterizes the Weak Weak Axiom of Revealed Preference and Wald's Weak Axiom for not necessarily differentiable demand functions. A theorem which generalizes a previous result of Kihlstrom, Mas-Colell and Sonneschein (1976) and Hildenbrand and Jerison (1989) is presented. It is also shown that the theorem cannot be extended to the set of not necessarily homogeneous demand functions. The method of proof used in this note is very simple and lends itself to an immediate geometrical interpretation; it is also shown that it can be usefully employed to obtain simpler proofs of other results found in this literature.

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A NOTE ON THE DEMAND THEORY OF THE WEAK AXIOMS

Introduction

This note deals with essentially two properties of demand functions, the Weak Weak Axiom of Revealed Preference (WWA) and Wald's Weak Axiom (WALD). WWA is a milder version of Samuelson's Weak Axiom of Revealed Preference and was first introduced by Hicks (1956). Similarly, WALD is a milder version of a condition on aggregate excess demand functions proposed by A. Wald in the context of general equilibrium theory. The relevance of these two concepts is not confined to the theory of individual choice but also to equilibrium analysis. Although WWA and WALD are among the weakest conditions of consistency of consumer behaviour, when possessed by market demand they may turn out to be strong enough to ensure uniqueness of equilibrium.

This note characterizes WWA and WALD for demand functions which are not necessarily differentiable. In particular it provides a Theorem which extends a previous result of Kihlstrom, Mas-Colell and Sonneschein (1976) and Hildenbrand and Jerison (1989) which characterizes WWA in terms of price derivatives of demand functions and Slutsky compensated demand functions.[1] By means of a

[1] Specifically, we refer to Theorem 1 and 3 in Kihlstrom *et al.* (1976) and to Theorem 1 in Hildenbrand and Jerison (1989).

recent result of John (1991) and a numerical example it is also shown that the the Theorem cannot be extended to the larger set of not necessarily homogeneous demand functions.

At the same time, this note offers a somewhat more general approach to the analysis of the demand theory of the weak axioms which allows very simple proofs. The method of proof used in most of the results obtained in this note is very simple and lends itself to an immediate geometrical interpretation. It is also shown that this method can be usefully employed to obtain simpler proofs of others existing results found in this literature.

In the next section the basic notation and definitions are introduced and a preliminary result essentially taken from Hildenbrand and Kirman (1988) is presented. Section 2 contains the Theorem with a graphical illustration of the proof. Section 3 presents a result of John (1991) and a numerical example which show, among other things, that the Theorem cannot be extended. Finally, the last section gives another application of the method of proof proposed.

1. Notation and Definitions

A demand function is defined as a continuous function $\mathbf{f}: P \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{+}$, where P is the set of strictly positive prices $P \subset \mathbb{R}^{\ell}_{++}$, satisfying *budget equality*, i.e. $\mathbf{p} \cdot \mathbf{f}(\mathbf{p}, w) = w$. A demand function is *homogeneous* if $\mathbf{f}(t\mathbf{p}, tw) = \mathbf{f}(\mathbf{p}, w)$ for $t > 0$. The set of demand functions is

denoted by $\mathcal{F}$, the subset of homogeneous demand functions by $\mathcal{F}_h$. An analogous notation is introduced for continuously differentiable demand functions, i.e. respectively C^1 and C^1_h.

The Slutsky compensated demand function at prices $\mathbf{q}$ and relative to point $(\mathbf{p}, w) \in \mathbb{R}^{\ell+1}_{++}$ is defined by

$$\mathbf{s}(\mathbf{q}) = \mathbf{f}\big(\mathbf{q}, \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w)\big);$$

the vector $\mathbf{s}(\mathbf{q})$ is the consumption bundle that the consumer would demand if the prices changed from $\mathbf{p}$ to $\mathbf{q}$ and his nominal income were compensated so as to keep unchanged his 'real income'. The compensated income is $w' = \mathbf{q} \cdot \mathbf{s}(\mathbf{q}) = \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w)$ and recall that $\mathbf{s}(\mathbf{p}) = \mathbf{f}(\mathbf{p}, w)$.

Here is a list of properties of demand functions that are needed in the sequel.

DEFINITION 1. A demand function $\mathbf{f}(\mathbf{p}, w)$ in $\mathcal{F}$ is said to satisfy property:

LD ('Law of Demand') if $\mathbf{f}$ is monotone, i.e.

$$(\mathbf{q} - \mathbf{p}) \cdot \big[\mathbf{f}(\mathbf{q}, w) - \mathbf{f}(\mathbf{p}, w)\big] \leq 0,$$

for all $\mathbf{p}$, $\mathbf{q}$ and w.

GLD (Generalized Law of Demand) if the compensated demand function $\mathbf{s}(\mathbf{q})$ is monotone, i.e.[2]

$$\mathbf{p} \cdot \big[\mathbf{s}(\mathbf{q}) - \mathbf{f}(\mathbf{p}, w)\big] \geq 0,$$

[2] We have used the fact that, by construction, $\mathbf{q} \cdot [\mathbf{s}(\mathbf{q}) - \mathbf{f}(\mathbf{p}, w)] = 0$.

for all $\mathbf{q}$ and all $(\mathbf{p}, w)$.

RM (Restricted Monotonicity) if $\mathbf{f}$ is monotone on the set of prices $P_f = \{\mathbf{q} \in P \mid \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) = w\}$, i.e.

$$\mathbf{p} \cdot [\mathbf{f}(\mathbf{q}, w) - \mathbf{f}(\mathbf{p}, w)] \geq 0,$$

for all $\mathbf{p}$ and w and for all $\mathbf{q} \in P_f$.[3] Equivalently RM can be defined as

$$\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) = w \qquad \text{implies} \qquad \mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w) \geq w.$$

WWA if for all $(\mathbf{p}, w)$ and $(\mathbf{q}, \hat{w})$

$$\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, \hat{w}) \leq w \qquad \text{implies} \qquad \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) \geq \hat{w}.$$

WALD (Wald's Weak Axiom) if for all $\mathbf{p}$, $\mathbf{q}$ and w,

$$\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w) \leq w \qquad \text{implies} \qquad \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) \geq w.$$

It is evident from the definitions that when demand functions are homogeneous the properties WWA and WALD are exactly the same thing; so that in $\mathcal{F}_h$ we will not make any distinction between them. Let us introduce three additional properties for demand functions in C^1.

DEFINITION 2. A demand function $\mathbf{f}(\mathbf{p}, w)$ in C^1 is said to satisfy property:

[3] Clearly we have exploited the fact that $\mathbf{q} \cdot [\mathbf{f}(\mathbf{q}, w) - \mathbf{f}(\mathbf{p}, w)] = 0$ for $\mathbf{q} \in P_f$.

∂LD if the matrix of price derivatives of the demand function, $\partial_p \mathbf{f}(\mathbf{p}, w) = [\partial f_i(\mathbf{p}, w)/\partial p_j]_{i,j}$, is negative semi-definite, i.e.

$$\mathbf{v} \cdot \partial_p \mathbf{f}(\mathbf{p}, w) \cdot \mathbf{v} \leq 0$$

for all $\mathbf{v} \in \mathbb{R}^\ell$.

∂RM if the matrix of price derivatives of the demand function, $\partial_p \mathbf{f}(\mathbf{p}, w)$ is negative semi-definite, on the hyperplane $T_f = \{\mathbf{v} \in \mathbb{R}^\ell \mid \mathbf{v} \cdot \mathbf{f}(\mathbf{p}, w) = 0\}$, i.e.

$$\mathbf{v} \cdot \partial_p \mathbf{f}(\mathbf{p}, w) \cdot \mathbf{v} \leq 0$$

for all $\mathbf{v} \in T_f$.

NSD if the matrix of substitution terms, $S(\mathbf{p}, w)$, i.e. the Jacobian of the compensated demand evaluated at $\mathbf{q} = \mathbf{p}$, is negative semi-definite,

$$\mathbf{v} \cdot S(\mathbf{p}, w) \cdot \mathbf{v} \leq 0$$

for all $\mathbf{v} \in \mathbb{R}^\ell$.

It is quite intuitive that the properties LD, RM and GLD are respectively the finite counterparts of properties ∂LD, ∂RM and NSD. We shall make this statement precise.

PROPOSITION 1. *For demand functions in C^1 the following propositions hold:*
(a) LD $\iff$ ∂LD.

(b) $RM \iff \partial RM$.
(c) $GLD \iff NSD$.

The proof of Proposition 1 is in the Appendix and is essentially taken, with minor differences, from Hildenbrand and Kirman (1988). It is important to stress that homogeneity of demand functions is not required by Proposition 1.

In 1976 Kihlstrom, Mas-Colell and Sonnenschein characterized WWA in terms of price derivatives of demand functions; in particular they proved that, for demand functions in C_h^1, the properties WWA, ∂RM and NSD are equivalent.[4] By Proposition 1, it is clear that the above result can be rephrased by substituting RM and GLD respectively for ∂RM and NSD.

2. The Main Result

In this section we shall strengthen the result of Kihlstrom *et al.* (1976) by extending the equivalence between WWA, GLD and RM to the larger set $\mathcal{F}_h$ of not necessarily differentiable homogeneous demand functions.

[4] A complete and simpler proof of the result of Kihlstrom *et al.* (1976) can be found in Hildenbrand and Jerison (1989)

The main result is derived from two lemmas that we shall present separately since they are of interest of their own. Indeed, the lemmas have a greater generality since they do not require homogeneity of demand functions. Moreover, both of them can be seen as applications of the same method of proof. As will be seen in the last section, this method can also be used to obtain simpler proofs of other existing results. The following lemma characterizes the Weak Weak Axiom in terms of monotonicity of the Slutsky compensated demand.

LEMMA 1. *For demand functions in $\mathcal{F}$ the properties GLD and WWA are equivalent.*

Proof.
WWA $\Rightarrow$ GLD. Let $\mathbf{p}$ and $\mathbf{q}$ be in P, and take $\mathbf{f}(\mathbf{p}, w)$ and $\mathbf{f}(\mathbf{q}, w')$ where $w' = \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w)$, i.e. $\mathbf{f}(\mathbf{q}, w') = \mathbf{s}(\mathbf{q})$ the compensated demand. Since $\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) = w'$, WWA implies that $\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w') \geq w$, i.e. $\mathbf{p} \cdot [\mathbf{s}(\mathbf{q}) - \mathbf{f}(\mathbf{p}, w)] \geq 0$.

GLD $\Rightarrow$ WWA. We will show that when WWA is violated then GLD is not satisfied. Let us assume that WWA is violated, i.e. there exists $(\mathbf{p}, w)$ and $(\mathbf{q}, w')$ such that $\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w') \leq w$ and $\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) < w'$ and consider the function

$$g(t) = \mathbf{q} \cdot \mathbf{f}(t\mathbf{p}, w)$$

for $t \in (0, 1]$. From the above assumption one has $g(1) < w'$. We want to show that there exists $t^* \in (0, 1)$ such that $g(t^*) =$

$\mathbf{q} \cdot \mathbf{f}(t^*\mathbf{p}, w) = w'$. This is not difficult to establish. First, notice that from the continuity of the demand function $g(t)$ is continuous. Second, by budget equality,

$$\mathbf{p} \cdot \mathbf{f}(t\mathbf{p}, w) = \frac{w}{t}$$

then, since the above expression tends to infinity as t goes to zero and $\mathbf{p}$ is finite, at least one of the components of $\mathbf{f}(t\mathbf{p}, w)$ goes to infinity as t goes to zero, which means that $g(t) \to \infty$ as $t \to 0$, since $\mathbf{q} \gg \mathbf{0}$.

Then we have established that there exists $0 < t^* < 1$ such that $\mathbf{q} \cdot \mathbf{f}(t^*\mathbf{p}, w) = w'$. Let us consider the effect on demand of a price change from $t^*\mathbf{p}$ to $\mathbf{q}$ and notice that $\mathbf{f}(\mathbf{q}, w')$ is the compensated demand at point $(t^*\mathbf{p}, w)$, indeed $\mathbf{s}(\mathbf{q}) = \mathbf{f}(\mathbf{q}, \mathbf{q} \cdot \mathbf{f}(t^*\mathbf{p}, w)) = \mathbf{f}(\mathbf{q}, w')$. We will show that GLD is violated, i.e.

$$t^*\mathbf{p} \cdot \left[\mathbf{f}(\mathbf{q}, w') - \mathbf{f}(t^*\mathbf{p}, w)\right] < 0.$$

In fact the left-hand side is equal to $t^*\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w') - w$, where t^* is strictly less than 1 and, by assumption, $\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w') \leq w$. ■

The proof of the proposition GLD $\Rightarrow$ WWA can be given a very straightforward graphical illustration. Figure 1 shows a case where $\mathbf{f}(\mathbf{p}, w)$ and $\mathbf{f}(\mathbf{q}, w')$ violate WWA. The dashed line corresponds to the budget hyperplane at prices $t^*\mathbf{p}$ and income w. By construction the demand $\mathbf{f}(t^*\mathbf{p}, w)$ must lie at the intersection between the budget hyperplanes $T(t^*\mathbf{p}, w)$ and $T(\mathbf{q}, w')$. Therefore, the angle between the vector $t^*\mathbf{p}$ and the vector $\left[\mathbf{f}(\mathbf{q}, w') - \mathbf{f}(t^*\mathbf{p}, w)\right]$ must be greater than $\pi/2$.

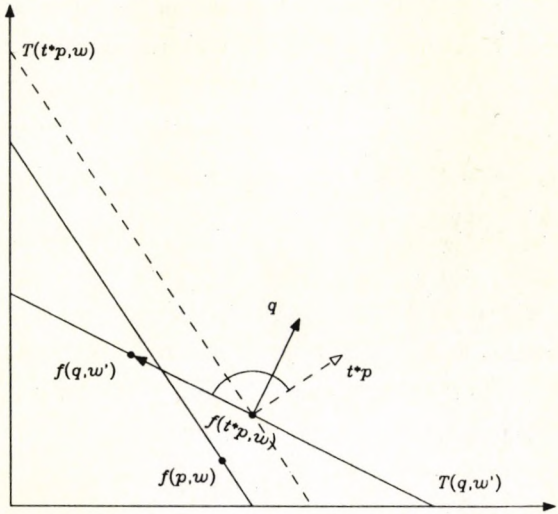

Figure 1.

The next lemma simply says that Wald's Weak Axiom amounts to requiring the monotonicity of the demand function on a particular subset of prices (precisely those prices which do not induce any income effect).

LEMMA 2. *For demand functions in $\mathcal{F}$ the properties RM and WALD are equivalent.*

Proof.
That WALD $\Rightarrow$ RM is immediate from the definitions. We shall prove the converse.
RM $\Rightarrow$ WALD. Let us assume that $\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w) \leq w$; we have to show that $\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) \geq w$. Consider the demand $\mathbf{f}(t\mathbf{q}, w)$ and define the scalar t^* from

$$\mathbf{p} \cdot \mathbf{f}(t^*\mathbf{q}, w) = w.$$

Clearly if $\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w) = w$, then $t^* = 1$; if $\mathbf{p} \cdot \mathbf{f}(\mathbf{q}, w) < w$, consider the following function

$$g(t) = \mathbf{p} \cdot \mathbf{f}(t\mathbf{q}, w)$$

which is continuous and $g(1) < w$. Using the same argument as that of Lemma 1, one can prove that $g(t) \to \infty$ as $t \to 0$ so that there exists $0 < t^* < 1$. From RM and $\mathbf{p} \cdot \mathbf{f}(t^*\mathbf{q}, w) = w$ one obtains $t^*\mathbf{q} \cdot [\mathbf{f}(\mathbf{p}, w) - \mathbf{f}(t^*\mathbf{q}, w)] \geq 0$ so that

$$t^*\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) \geq w.$$

Then since $0 < t^* \leq 1$ it must be $\mathbf{q} \cdot f(\mathbf{p}, w) \geq w$. ■

By virtue of Proposition 1, Lemma 1 and Lemma 2 allow to characterize WWA and WALD in terms of properties of the derivatives of demand functions.

REMARK. For demand functions in C^1, (i) WWA is equivalent to NSD; (ii) WALD is equivalent to ∂RM.

By noting that WWA and WALD are exactly the same thing when demand functions are homogeneous, Lemma 1 and Lemma 2 are sufficient to prove the following:

THEOREM. *For demand functions in $\mathcal{F}_h$ the properties WWA, GLD and RM are equivalent.*

For later reference as well as to test the validity of the Theorem we shall present a result of equivalence between the properties of Restricted Monotonicity and Generalized Law of Demand.

LEMMA 3. *For demand functions in $\mathcal{F}$ the property GLD implies RM. For demand functions in $\mathcal{F}_h$ the properties GLD and RM are equivalent.*

Proof.
GLD $\Rightarrow$ RM. Let us take $\mathbf{q} \in P_f$; one has to show that when GLD holds the following expression is non negative,

$$\mathbf{p} \cdot \left[\mathbf{f}(\mathbf{q}, w) - \mathbf{f}(\mathbf{p}, w)\right] \geq 0.$$

This is immediate, by noting that for $\mathbf{q} \in P_f$ demand is equal to compensated demand, i.e. $\mathbf{s}(\mathbf{q}) = \mathbf{f}(\mathbf{q}, \mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w)) = \mathbf{f}(\mathbf{q}, w)$.

RM $\Rightarrow$ GLD. Any vector $\mathbf{q}(t) \in P$ can be expressed as a linear combination of two vectors, i.e. $\mathbf{q}(t) = \mathbf{v} + t\mathbf{p}$, where $t > 0$ and $\mathbf{v} \in T_f = \{\mathbf{v} \in \mathbb{R}^\ell \mid \mathbf{v} \cdot \mathbf{f}(\mathbf{p}, w) = 0\}$, the subspace of vectors orthogonal to $\mathbf{f}(\mathbf{p}, w)$. Let us consider the following expression

$$\mathbf{p} \cdot \left[\mathbf{f}\big(\mathbf{q}(t), \mathbf{q}(t) \cdot \mathbf{f}(\mathbf{p}, w)\big) - f(\mathbf{p}, w)\right];$$

we have to show that if RM is satisfied the above expression is non-negative. By the definition of $\mathbf{q}(t)$ and orthogonality of $\mathbf{v}$, we have

that $\mathbf{q}(t) \cdot \mathbf{f}(\mathbf{p}, w) = tw$ and by applying homogeneity the above expression can be rewritten as:

$$\mathbf{p} \cdot \left[\mathbf{f}\left(\frac{\mathbf{q}(t)}{t}, w\right) - f(\mathbf{p}, w)\right];$$

furthermore, since $\mathbf{q}(t)/t \cdot \mathbf{f}(\mathbf{p}, w) = w$ one has that the price vector $\mathbf{q}(t)/t \in P_f$. Therefore, if RM holds the displayed expression must be non-negative. ∎

It is important to notice that in the the proof of implication RM⇒GLD we had to use homogeneity.

As the last remark notice that the Theorem and Proposition 1 establish the result of Kihlstrom *et al.* (1976). Indeed, since $C_h^1 \subset \mathcal{F}_h$, the Theorem implies that the equivalence also holds in C_h^1; part (b) and (c) of Proposition 1 do the rest.

3. Some Remarks on Homogeneity

As we have seen in the proof of Lemma 3, homogeneity plays its role in the implication RM ⇒ GLD; this is why we could not extend the equivalence established in the Theorem to the set of not necessarily homogeneous demand functions. Actually, as we shall see below, this equivalence cannot be extended to the set $\mathcal{F}$.

In a recent paper, John (1991) proved that WWA implies homogeneity. By following the same line of proof it is immediate that the same holds for GLD.

PROPOSITION 2. *For demand functions in $\mathcal{F}$ both WWA and GLD imply homogeneity.*

Proof.[5]
We have to prove that $\mathbf{f}(\mathbf{p}, w) = \mathbf{f}(\alpha\mathbf{p}, \alpha w)$, for $\alpha > 0$. It is sufficient to show that

$$\mathbf{q} \cdot \left[\mathbf{f}(\mathbf{p}, w) - \mathbf{f}(\alpha\mathbf{p}, \alpha w)\right] \leq 0,$$

for all $\mathbf{q} \in P$. Let us normalize $\mathbf{q}$ so that $\mathbf{q} \cdot \mathbf{f}(\alpha\mathbf{p}, \alpha w) = w$; therefore, to prove the proposition one has to show that $\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) \leq w$. Consider the price vectors $\mathbf{q}(t) = t\mathbf{q} + (1-t)\mathbf{p}$, with $t \in [0, 1]$. Clearly, $\mathbf{q}(t) \cdot \mathbf{f}(\alpha\mathbf{p}, \alpha w) = w$ so that both WWA and GLD imply

$$\mathbf{p} \cdot \mathbf{f}(\mathbf{q}(t), w) \geq w. \tag{1}$$

From budget identity and $\mathbf{q}(t) \cdot \mathbf{f}(\mathbf{q}(t), w) = w$ one gets

$$t\left[\mathbf{q} \cdot \mathbf{f}(\mathbf{q}(t), w) - w\right] + (1-t)\left[\mathbf{p} \cdot \mathbf{f}(\mathbf{q}(t), w) - w\right] = 0;$$

the above expression and inequality (1) imply that $\mathbf{q} \cdot \mathbf{f}(\mathbf{q}(t), w) \leq w$, then, letting t go to 0, by continuity $\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) \leq w$. ∎

Clearly if one can show that Wald's Weak Axiom or Restricted Monotonicity do not imply homogeneity of the demand function then

[5] The proof is taken from John (1991).

the equivalence stated in the Theorem cannot be extended any further. Here is the example.

Example. Let us consider the function

$$\mathbf{f}(\mathbf{p}, w) = \Big(\frac{\log(w+1)}{p_1} \qquad \frac{w - \log(w+1)}{p_2}\Big),$$

for all $\mathbf{p}$ and w. It is easily seen that it is continuously differentiable, satisfies budget equality, but it is not homogeneous. Take the matrix of price derivatives,

$$\partial_p \mathbf{f}(\mathbf{p}, w) = \begin{pmatrix} -\frac{\log(w+1)}{p_1^2} & 0 \\ 0 & -(\frac{w-\log(w+1)}{p_2^2}) \end{pmatrix}.$$

The matrix is clearly negative definite so that the demand function is strictly monotone and certainly satisfies RM and Wald's Weak Axiom.

This very simple example allows us to make some interesting remarks. In the first place, irrespective of whether demand functions are differentiable or not, even the 'Law of Demand' in its strongest version is not a sufficient condition for homogeneity. Therefore, if one is interested in modelling consumer behaviour but is not prepared to assume absence of money illusion one can still retain some degree of consistency of choice by adopting Wald's (Weak) Axiom or any other property related to the monotonicity of demand functions.

On the other hand, by Proposition 2, if one is interested in properties of demand functions related to the monotonicity of the Slutsky

compensated demand, such as WWA or Samuelson Weak Axiom of Revealed Preference, one is left less freedom in modelling consumer behaviour since he cannot help assuming implicitly homogeneity.

4. Concluding Remarks

We have not drawn all the conclusions from the results presented in this note. Lemma 1, 2, 3, and Proposition 1 and 2 can be combined in various way in order to provide other results available in the literature. For example we have already deduced the above mentioned results of Kihlstrom *et al.* (1976) and Hildenbrand and Jerison (1989). Another example is Theorem 3 in John (1991) where it is established the equivalence between NSD and ∂RM plus homogeneity. We do not need to prove this result. As one can easily verify it can be deduced by the Lemmas and the Propositions presented in this paper.

Let us conclude this work with a final remark which shows how the method of proof adopted in the Lemmas can be usefully applied to obtain simple proofs of other results available in the literature. The following definitions are needed.

Definition 3. A demand function $\mathbf{f}(\mathbf{p}, w)$ in $\mathcal{F}_h$ is said to satisfy property:

SGLD (Strong Generalized Law of Demand) if the compensated demand function $\mathbf{s}(\mathbf{q})$ is strictly monotone for all w and all linearly independent $\mathbf{p}$ and $\mathbf{q}$, i.e.

$$\mathbf{p} \cdot [\mathbf{s}(\mathbf{q}) - \mathbf{f}(\mathbf{p}, w)] > 0,$$

for all $(\mathbf{p}, w)$ and all $\mathbf{q} \neq \lambda \mathbf{p}$, $\lambda > 0$.

WARP Weak Axiom of Revealed Preference if for all $(\mathbf{p}, w)$ and $(\mathbf{q}, \hat{w})$

$$\mathbf{f}(\mathbf{p}, w) \neq \mathbf{f}(\mathbf{q}, \hat{w}) \quad \text{and} \quad \mathbf{p} \cdot \mathbf{f}(\mathbf{q}, \hat{w}) \leq w$$

imply

$$\mathbf{q} \cdot \mathbf{f}(\mathbf{p}, w) > \hat{w}.$$

One more definition for continuously differentiable demand functions:

DEFINITION 4. A demand function $\mathbf{f}(\mathbf{p}, w)$ in C_h^1 is said to satisfy property

ND if the matrix of substitution terms, $S(\mathbf{p}, w)$, i.e. the Jacobian of the compensated demand evaluated at $\mathbf{q} = \mathbf{p}$, is negative definite on the hyperplane $T_p = \{\mathbf{v} \in \mathbb{R}^\ell \mid \mathbf{v} \cdot \mathbf{p} = 0\}$, i.e.

$$\mathbf{v} \cdot S(\mathbf{p}, w) \cdot \mathbf{v} < 0$$

for all $\mathbf{v} \in T_p$.

REMARK. For demand functions in $\mathcal{F}_h$ the property SGLD implies WARP.

Proof.

Take $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{f}(\mathbf{p},1) \neq \mathbf{f}(\mathbf{q},1)$ and $\mathbf{p}\cdot\mathbf{f}(\mathbf{q},1) \leq 1$; one has to show that $\mathbf{q}\cdot\mathbf{f}(\mathbf{p},1) > 1$. By using the same argument as in Lemma 1, there exists $0 < t^* \leq 1$ such that $\mathbf{p}\cdot\mathbf{f}(t^*\mathbf{q},1) = 1$; therefore, $\mathbf{f}(\mathbf{p},1)$ is none other than the Slutsky compensated demand at point $(t^*\mathbf{q},1)$, indeed, $\mathbf{f}(\mathbf{p},\mathbf{p}\cdot\mathbf{f}(t^*\mathbf{q},1)) = \mathbf{f}(\mathbf{p},1)$. Then SGLD implies $t^*\mathbf{q}\cdot\mathbf{f}(\mathbf{p},1) > 1$, i.e. $\mathbf{q}\cdot\mathbf{f}(\mathbf{p},1) > 1$. ∎

The Remark is basically Lemma 2 of Kihlstrom *et al.* (1976). Moreover, by noting that for demand functions in C_h^1, ND implies SGLD it follows at once that ND is a sufficient condition for the Weak Axiom of Revealed Preference. This result is exactly Theorem 2 of Kihlstrom *et al.* (1976) or equivalently the Remark of Hildenbrand and Jerison (1989).

Appendix

Proof[6] of Proposition 1.

We shall give a formal proof of part (a); the proof of (b) and (c) is very similar and is left to the reader. Let us show, first, that if $\partial_p\mathbf{f}$ is negative semi-definite then LD holds. For prices $\mathbf{p}$ and $\mathbf{q} \in P$

[6] This proof was adapted from the proof of Lemma 6.1, in Hildenbrand and Kirman (1988), p. 220.

define the vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$ and consider the convex combination $\mathbf{q}(t) = t\mathbf{q} + (1-t)\mathbf{p}$, with $0 \leq t \leq 1$. Define the function

$$g(t) = \mathbf{v} \cdot \big[\mathbf{f}(\mathbf{q}(t), w) - \mathbf{f}(\mathbf{p}, w)\big].$$

Since $\mathbf{q}(0) = \mathbf{p}$ and $\mathbf{q}(1) = \mathbf{q}$ one has, $g(0) = 0$ and $g(1) = \mathbf{v} \cdot \big[\mathbf{f}(\mathbf{q}, w) - \mathbf{f}(\mathbf{p}, w)\big]$. Differentiating $g(t)$ yields

$$g'(t) = \mathbf{v} \cdot \partial_p \mathbf{f}(\mathbf{q}(t), w) \cdot \mathbf{v}.$$

Since the matrix $\partial_p \mathbf{f}$ is negative semi-definite, the function $g(t)$ is non increasing and taking into account that $g(0) = 0$, the function cannot be positive, so that

$$g(1) = \mathbf{v} \cdot \big[\mathbf{f}(\mathbf{q}, w) - \mathbf{f}(\mathbf{p}, w)\big] \leq 0.$$

To prove the converse notice that since $\mathbf{q}(t) - \mathbf{p} = t\mathbf{v}$, the function $g(t)$ can be rewritten as

$$g(t) = \frac{1}{t}(\mathbf{q}(t) - \mathbf{p}) \cdot \big[\mathbf{f}(\mathbf{q}(t), w) - \mathbf{f}(\mathbf{p}, w)\big].$$

One has to prove that the matrix of price derivatives is negative semi-definite. If LD holds the function $g(t)$ cannot be positive, $g(t) \leq 0$, for $t > 0$, and since $g(0) = 0$ the slope of the function in $t = 0$ must be non positive. ∎

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