European University Institute

DEPARTMENT OF ECONOMICS

Information and Credit Frictions in Financial Markets

Vincent Maurin

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

Florence, 15 September, 2016

European University Institute
Department of Economics

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Researcher declaration to accompany the submission of written work

I, Vincent Maurin, certify that I am the author of the work *Information and Credit Frictions in Financial Markets*. I have presented for examination for the PhD thesis at the European University Institute. I also certify that this is solely my own original work, other than where I have clearly indicated, in this declaration and in the thesis, that it is the work of others.

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I confirm that chapter 3 was jointly co-authored with Professor Piero Gottardi and Cyril Monnet and I contributed 40% of the work.

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Abstract

This thesis investigates theoretically how information and credit frictions affect the functioning of financial markets. I suggest that asymmetry of information about opaque assets may cause instability. The widespread use of these assets as collateral contributed to a severe credit contraction during the recent crisis. In this context, I propose a theory of collateral re-use, a technique designed to overcome shortages of high quality assets to secure credit.

The first chapter shows that information frictions generate liquidity fluctuations whereby asset prices move endogenously. In the model, buyers meet sellers in a decentralized market and do not know their asset quality. Prices and volume increase with the average quality of sellers since buyers are more willing to trade. However, high trading volume depletes the pool of future high quality sellers. Cyclical equilibria in price and volume are thus sustained endogenously. Temporary asset purchase programs can revive the market and smooth out fluctuations. Finally, I show that increasing market centralization may harm liquidity provision and reduce welfare.

The second chapter introduces collateral re-use in an economy where agents face limited commitment and must pledge a durable asset to borrow. Lenders may then re-sell a pledged asset or re-pledge it to secure further borrowing. Since lenders may now default and fail to return the collateral, net gains from collateral circulation are ambiguous. I show that benefits are larger in decentralized markets when agents trade trough intermediaries. The third chapter, joint with Piero Gottardi and Cyril Monnet, complements this analysis, focusing on repurchase agreements. In a repo, the borrower sells an asset to raise income and commits to a repurchase price to limit exposure to future market risk. If defaulting borrowers incur a cost over and above the loss of collateral, re-use is beneficial and increases leverage. We show that intermediation now arises endogenously: trustworthy agents - those with high cost of default - re-use collateral to borrow on behalf of riskier counterparties.

The beginning and end of a matter
are not always seen at once.

Herodotus, The Histories, VII, 51

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Finally, I would like to dedicate this thesis to my girlfriend Alice, a true princess.

Florence, July 14th 2016

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Chapter 1

Liquidity Fluctuations in Over the Counter Markets

1.1 Introduction

The recent 2007 crisis started with a widespread shortage of liquidity in the financial system after a prolonged boom. Difficulties to sell or finance securities on secondary markets triggered a collapse in the issuance of many assets, impacting credit and ultimately the real economy. The severity of the bust prompted a two stage response from policy-makers acting both as market participants and market designers. During the crisis, the US Fed provided credit lines and purchased assets such as Mortgage Backed Securities to prop up trading and liquidity. In Europe, fears of government losses or uncertainty regarding their optimal design sometimes delayed the implementation of these programs[1]. In a second ongoing phase, regulators have started to overhaul various segments of financial markets. For instance, the Market in Financial Instruments Directive[2] requires *"all standardized derivatives to be traded on organized and transparent venues"*. For many assets, transactions indeed take place Over The Counter (OTC) where trading frictions and opacity may cause illiquidity and instability[3]. Increased transparency and competition are generally seen as desirable features of centralized platforms.

In this paper, I propose a theory based on asymmetry of information to explain why liquid OTC markets can become illiquid. Endogenous variations in the supply of high quality assets generate price and volume swings. I use the model to study an asset purchase program designed to revive market liquidity in bad times. I show that

[1] "*Too little, too late*" was the financial press widespread reception to the ECB 60 billion-a-month bond buying plan announced in January 2015.

[2] MIFID II, Regulation 600/2014 of the European Parliament. In the US, Title VII of the Dodd Frank in the US contains similar provisions.

[3] On its website, the IMF referring to OTC markets explains that *"some types of market arrangements can very quickly become disorderly, dysfunctional, or otherwise unstable"*

large interventions might be too costly and that tight government budget constraints dampen the effect of the policy. Finally, I argue that in the presence of asymmetry of information, a decentralized market might perform better in providing liquidity. The introduction of a more transparent trading platform, in line with the target of current reforms, can thus decrease welfare.

In the model, agents have different valuations for a long-term asset and thus gain from trade. Preferences switch over time so that buyers may ultimately need to resell an asset previously acquired. In the OTC market, agents match bilaterally and the buyer makes a Take It Or Leave It offer. There are two qualities of the asset which either pays a high or a low dividend. The key friction is that only the seller knows the quality of the asset he holds as well as his own valuation. Hence, trade may fail to occur because of adverse selection as in Akerlof (1970). A pooling price offer to attract high quality sellers entails losses on low quality assets. When the share of high quality sellers is too low, a buyer thus reduces his bid to trade only with low quality sellers. We call market liquidity the ease to sell a good quality asset. Liquidity can fluctuate because both the value of a lemon to a buyer and the composition of the pool of sellers are endogenous. First, high future prices raise the resale value of a lemon. This increases buyers' willingness to pay today for an asset of unknown quality. Hence high future liquidity begets present liquidity since then, buyers are more likely to offer a price at which high quality assets trade. Dynamics can be more intricate, however, since the composition of the pool of sellers responds negatively to market liquidity. If the market was liquid in the past, high quality assets are in the hands of high valuation agents who do not wish to sell. Buyers thus mostly meet low quality sellers who want to flip their lemon, regardless of their private valuation. A low price offer becomes more profitable, so that liquidity can generate illiquidity. This *composition effect* also leaves room for recovery. If the market is illiquid, selling pressure from high quality asset owners accumulates over time as high valuation agents switch to low valuation. Buyers meet increasingly more good quality sellers and, after some time, may offer a pooling price.

When the discount factor and the probability of switching type are low, the *composition effect* is strong and equilibrium cycles exist in the absence of any aggregate shock. For $T \geq 2$, a T period cycle consists of $T - 1$ trough periods where only lemons trade and 1 peak period where both qualities trade. The market price for the low quality asset increases during the trough to reflect high offers at the next peak. Indeed, traders know that the accumulation of selling pressure for high quality assets will lead future buyers to offer a pooling price. To the best of my knowledge, this paper is the first to characterize cycles in this environment. Foucault et al. (2013) describes "'make-take" liquidity cycles in electronic markets which share many features with the dynamics in my model[4]. The model also sheds light on boom and bust episodes commonly associated to financial crises. During the trough of the cycle, investors exhibit *speculative behavior* in the words of Harrison and Kreps (1978) as they know they buy a lemon but pay an increasing premium that captures future resale gains. The low quality asset appears like a hot potato that agents pass to the next investor in line. Since the pool quality may take time to reach the peak whereas a pooling offer immediately clears the market, the model can rationalize slow build-ups followed by fast crashes.

Cycles emerge for an intermediate share of high quality assets in the economy. In that region, there also exists a steady state equilibrium in mixed strategy where buyers randomize between the pooling and the separating price. Intuitively, fundamentals are neither favorable enough for the market to be fully liquid nor so bad that good quality assets never trade. Partial illiquidity materializes either through buyers' randomization in a steady state or through cyclical dynamics. I show that fluctuations in a cycle typically entail a surplus loss with respect to the steady state. There is too much trade at the peak of the cycle and too little at the bottom. Surplus would improve by propping up (resp. taming) liquidity in the trough (resp. at the peak).

Illiquid and unstable markets call for policy interventions. I study an asset purchase program by a benevolent government who is bound to resell the assets purchased. The combination of an asset purchase program together with a resale constraint fits the description of many policies implemented during the financial

[4]The sharp fall and progressive build-up of liquidity after a period with high volume and high price also evokes Duffie (2010)'s account of price movements after good news. In my model, at the peak of a cycle, there is "good news" about the quality of assets for sale as the pool contains many high quality assets. Liquidity then falls because of the composition effect.

crisis[5]. In my model, the resale constraint proceeds from a natural assumption that the government values assets less than private agents[6]. In addition, a revertible intervention does not affect the fundamentals of the economy in the long run. I show that this asset purchase program can still increase surplus when the economy is in a cycle. Buying lemons jump-starts the OTC market as it improves the average asset quality private buyers face in the trough. Reselling lemons at the peak may then reduce prices and trading volume in line with steady state levels. When the objective is to maximize aggregate surplus, the government weighs the benefits from jump-starting and then stabilizing the market with the asset holdings costs. My numerical analysis shows that achieving the first objective sometimes requires buying too many assets and the intervention becomes undesirable.

I show that the program is not self-financing. Although he eventually resells assets, the government runs a loss. Indeed, he enjoys a lower utility for the dividends of the asset but also pays a premium to induce participation in the program. As we observed, buying lemons increases liquidity. However, better conditions in the OTC market raise the outside (market) option of lemon holders and in turn the price the government must pay. To finance the shortfall, I allow the government to tax transactions in the resale period. Then, I show numerically that Pareto improvements are possible but budget-neutral interventions need be smaller. Indeed, while flattening fluctuations raises surplus, riding the liquidity cycle relaxes the government budget constraint in two ways. First, maintaining high liquidity in the resale period reduces the capital loss of the government who can quote a high resale price. Second, it increases the tax base to make up for this loss.

Finally, Section 1.5 introduces a more transparent trading infrastructure called Exchange. Buyers now post and commit to terms of trade prior to meeting a counterparty. Sellers thus observe prices posted by all buyers. With bilateral matching, they may face a different level of rationing for each price. I show that the resulting

[5] After its first intervention in the Mortgage Backed Security Market in 2008, the Federal Reserve was eager to sell back the assets it had purchased. Since then, it has adopted a buy and hold strategy.

[6] Intuitively, a public entity does not value the potential services (borrowing, hedging..) attached to holding assets. Asset purchase programs would then generate some misallocation. To quote Singh (2013), *"some central banks purchases of good collateral have contributed to shrinkage in the pledged collateral market"*

increase in competition in the Exchange lowers liquidity and may decrease welfare. On the upside, price posting economizes on search costs. Indeed buyers and sellers can coordinate on a sub-market and price competition avoids spending on wasteful search cost. However, the availability of multiple offers induces sellers to try and signal their asset quality. High quality sellers should thus choose higher prices where they face a lower probability of trading. I show that in equilibrium, high quality assets do not trade at all. Both dimensions of private information - valuation and asset quality - matter for this extreme result. The key ingredient is the presence of high valuation lemon owners who have no gains from trade with buyers - a "no gap at the middle condition". These traders block transactions of high quality assets as they would otherwise mimic high quality sellers. Hence, competition generated by a transparent centralized market may harm liquidity provision. In the opaque OTC market, local monopsony power protects buyers from competition, allowing for a pooling outcome.

Aggregate surplus is higher in the OTC market than in the Exchange when the share of high quality assets is sufficiently large. In that region, the realized gains from trade on high quality assets overcome the inefficiencies attached to random search and bargaining. This comparison suggests that, to some extent, opaqueness of the asset traded and the trading structure are complement. In the presence of asymmetry of information about the asset value, a transparent and competitive exchange is not necessarily desirable.

Relation to the literature

A strand of literature has identified self-fulfilling expectations as a mechanism for cycles and chaotic dynamics in the absence of aggregate shocks. Boldrin and Woodford (1990) provides a survey of early endogenous business cycle models. More recently, a series of works including Gu et al. (2013) or Rocheteau and Wright (2013) highlighted the contribution of credit constraints in generating such dynamics. In my model, cycles rather hinge on variations in the composition of the pool of sellers - a backward-looking variable -, and equilibrium multiplicity is not as severe.

This paper relates more closely to the growing literature on dynamic markets with asymmetric information. My contribution is to show that this environment is prone to liquidity fluctuations. Recent works (e.g. Deneckere and Liang, 2006, Camargo and Lester, 2014, Moreno and Wooders, 2013) have emphasized that, as a screening device, trading delay is tantamount to rationing in a static environment. Bad assets trade first while high quality sellers are willing to wait. Frozen lemon markets thus eventually thaw over time endogenously or thanks to the arrival of news as in Daley and Green (2012). This force leading to separation of sellers is absent with re-trade. My paper is thus closer to Chiu and Koeppl (2014) since there also, the lemon problem does not vanish over time[7]. Monopsonist buyers also offer terms of trade after matching and may pool sellers if chances to obtain a good quality asset are high. However, their model does not disentangle the preference switching process from the trading process. This natural feature is an important element to identify cycles. In addition, as I discuss later, my revertible policy differs from their permanent asset purchase program. Finally, their paper does not discuss the role of the market structure or transparency.

I compare indeed the OTC equilibrium to that of an directed search environment (called Exchange) where buyers post and commit to terms of trade before meeting. There, as in Guerrieri and Shimer (2014) and Chang (2014), building on Guerrieri et al. (2010) and the pioneering work of Gale (1996), separation obtains through rationing at different prices[8]. As a difference with these works, I show that the possibility to resell assets with two dimensions of private information exacerbates market illiquidity. High valuation owners who try to flip their lemon form a middle type with whom buyers do not gain from trade. Adverse selection is thus more severe and high quality assets do not trade. As a consequence, liquidity is lower than in the OTC market when the equilibrium is pooling in that market. In a common value environment, Hörner and Vieille (2009) and Fuchs and Skrzypacz (2015) showed that pre-trade information may come at the expense of liquidity. If buyers can observe the offers a seller rejected, the latter can use this information as a signal. In my model of the Exchange, it is rather the possibility to observe current

[7] Hellwig and Zhang (2013) add endogenous information acquisition to this framework and show that equilibria with different degree of adverse selection and liquidity can coexist. In my model, liquidity varies over time in a given equilibrium

[8] Exclusivity for sellers is crucial to generate separation in this environment. Kurlat (2015) relaxes this assumption and obtains a pooling outcome. See Wilson (1980) for a useful discussion on this issue.

terms of trade and not inconclusive past offers that leads to separation.

Although the Exchange improves trading efficiency for low quality assets, market centralization and transparency can thus reduce aggregate surplus. In the seminal search model of Duffie et al. (2005), trading takes place under symmetric information and only the first effect is present. There, the centralized benchmark unambiguously dominates the market with frictions. My result complements the findings of Malamud and Rostek (2014) who highlight the surprising role of market power as a potential force against centralization. In their model, the market structure is also exogenous. Michelacci and Suarez (2006) or Bolton et al. (2014) can endogenize traders' choice by imposing symmetric information in the OTC market, while I maintain asymmetric information in both platforms.

Finally, the government purchase program for lemon markets I study, shares many features with the policy experiment of Philippon and Skreta (2012) Tirole (2012) or Chiu and Koeppl (2014). Unlike these papers and some mentioned above, I impose a realistic constraint on the government to revert the policy, that is to resell the assets purchased. Although the intervention may not change the fundamentals of the economy in the long run, it can prop up and then stabilize liquidity in a cycle. If implemented in steady state, this program would have at best no effect on aggregate surplus. As in Fuchs and Skrzypacz (2015) or Faria-e-Castro et al. (2015) with a different focus, I discuss the interaction between designing and funding the policy. To generate a high taxing profit in the resale period, the government somewhat rides the liquidity cycle and the intervention does not flatten fluctuations as much. The government then leans against the wind by buying low and selling high, like the market-maker of Weill (2007).

The rest of the paper is organized as follows. Section 1.2 introduces the model. In Section 1.3, I describe the main dynamic effects and solve for stationary equilibria including cycles. Section 1.4 discusses the welfare implications of liquidity fluctuations and studies an asset purchase program aimed at restoring liquidity. Section 1.5 analyzes a market structure change by allowing agents to post prices before meetings. Finally, Section 1.6 concludes. Proofs are in Appendix 1.7.2.

1.2 The Model

1.2.1 Environment

Time is discrete and runs forever $t = 0, 1, .., \infty$. The economy is populated by a large continuum of infinitely-lived agents with discount factor $\delta < 1$. They consume a non-storable numeraire good c and dividends d from assets. Agents can have either low ($i = 1$) or high valuation ($i = 2$) for the dividends with the following instantaneous preferences:

$$u^i(c, d) = c + \tau^i d$$

where $1 = \tau^1 < \tau^2 = \tau$. Agents with a higher private valuation like the (dividends of the) asset more. Valuation is persistent but may switch from one period to the next with probability $\gamma \in (0, 1/2)$. This Markov Process is identical and independently distributed across agents. The valuation of an agent is private information[9]. Agents are endowed with e units of the numeraire good every period. There is an infinitely lived asset in fixed supply S with two varieties denoted H (High) and L (Low) with share q and $1-q$ respectively. Variety L pays dividend $d_L > 0$ in every period while variety H pays dividend $d_H > d_L$. The variety or quality is private information to the current holder of the asset. The asset is indivisible and agents may hold either zero or one unit[10].

Asset owners enter date $t \geq 1$ carrying their holdings from date $t-1$. At the beginning of the period, valuations can switch and agents may wish to trade. Non asset owners must pay a cost $\kappa > 0$ to enter a decentralized market where they offer terms of trade to asset owners. Section 1.2.2 describes the market structure in detail. The key friction is the asymmetry of information between buyers and sellers. At the end of the period, dividends pay off and buyers discover the quality of the asset purchased if any. The economy then moves on to period $t+1$.

With two valuations and three possible asset holding status, there are effectively

[9] Private valuation may capture different services attached to holding an asset (hedging, collateral). Since their type switch agents with high valuations may need to resell an asset purchased as it is usual in secondary markets.

[10] I make this assumption for tractability. The indivisibility comes without loss of generality as buyers may offer contracts with probabilities of trade.

six types of agents in the economy. Let thus (τ^i, a) denote an agent with private valuation $\tau^i \in \{\tau^1, \tau^2\}$ and asset holding $a \in \{0, L, H\}$ where by convention $a = 0$ means no asset. Importantly, there are two dimensions of private information for an asset owner: valuation and asset quality. Let now $\mu_a^i(t)$ be the mass of of (τ^i, a) agents in period t after the type switch but before the market opens. With this notation, $\{\mu_a^i(0)\}_{a=0,L,H}^{i=1,2}$ is the initial distribution of asset holdings. These quantities verify the following balance equations:

$$\begin{cases} \mu_H^1(t) + \mu_H^2(t) = Sq \\ \mu_L^1(t) + \mu_L^2(t) = S(1-q) \end{cases} \tag{1.1}$$

The total supply of each variety of the asset must match the total holdings of this variety across the population in any period t. I now impose a series of assumptions on the parameters of the model. The main restriction disciplines the degree of adverse selection:

$$\tau d_L < d_H \tag{LC}$$

I will refer to (LC) as the lemon condition. In a static environment, the value of dividend d_L to a type τ^2 agent lies below the value of dividend d_H to a type τ^1 agent. This will imply that any price acceptable by a (τ^1, H) owner to sell his asset is also acceptable for a type (τ^2, L) owner. The following assumptions are technical:

$$\frac{\tau d_H}{1-\delta} \leq e \tag{A1}$$

$$\kappa \leq \bar{\kappa}(\gamma, \delta, \tau, d_L, d_H), \tag{A2}$$

Condition (A1) ensures that trade does not fail for lack of funds. Observe indeed that the left hand side is the present discounted value of asset H when held in every period by a high valuation agent. Condition (A2) guarantees that search costs are small enough to preserve gains from trade and non-owners find it optimal to search. The expression of $\bar{\kappa}(\gamma, \delta, \tau, d_L, d_H)$ is in Appendix 1.7.1. The important observation is that $\bar{\kappa}$ does not depend on the share of H assets q.

1.2.2 The OTC market

Until Section 1.5 with price posting, I consider a market structure with random search and ex-post offers, called Over The Counter or OTC. Trading is decentralized. A non-owner must pay the search cost $\kappa > 0$ to match with at most one asset owner. In a match, the non-owner makes a Take It Or Leave It (TIOLI) offer[11] to the asset owner. I first describe the matching and bargaining stage in a given period $t \geq 0$ and then turn to the model dynamics. Definition 2 then introduces the concept of the OTC equilibrium.

Matching

Asset owners with total mass S simply wait for a match. The mass of active non-owners or buyers in period t, $\mu^B(t)$ results from an entry decision detailed later. The matching function is of the Leontieff type. Precisely, the probability $\lambda^S(t)$ (resp. $\lambda^B(t)$) for a seller to meet a buyer (resp. for a buyer to meet a seller) in period t is:

$$\lambda^S(t) = \min\left\{\frac{\mu^B(t)}{S}, 1\right\}, \quad \lambda^B(t) = \min\left\{\frac{S}{\mu^B(t)}, 1\right\}$$

Search frictions are minimal because the short side of the market finds a counterparty for sure[12]. Search is random so that a matched buyer meets an owner with type (τ^i, a) with probability $\mu^i_a(t)/S$ where $i \in \{1, 2\}$ and $a \in \{L, H\}$. This is the fraction of that type in the population of asset owners. Private information and randomness generate inefficiencies because agents with no gains from trade may meet.

Stage Bargaining Game

In a match, the buyer does not know the quality of the asset held by the seller. He offers a price[13] to trade that the seller may accept or refuse. Formally, a strategy for

[11] We follow most of the literature on bargaining with common value in giving the bargaining power to the uninformed party. The search costs matters primarily to compare the OTC and the Exchange structure in a meaningful way. In equilibrium, buyers will indeed always make zero profit. Since they do not compete simultaneously in price in the OTC market, this would not be possible without the cost.

[12] I make this assumption for tractability. Most of the search models set in continuous time a la Duffie et al. (2005) use a purely random search technology but specific functional forms as well. In discrete time however, Guerrieri and Shimer (2014) use a similar matching function.

[13] We can extend the set of possible offers to contracts formed by a price and a probability to

buyer $(\tau^k, 0)$ where $k \in \{1, 2\}$ in period t is a distribution $\Pi^k(t, .)$ over the real line. We let $Supp(\Pi^k(t, .))$ denote the support of this distribution. A strategy for owner (τ^i, a) in period t is a probability $\alpha_a^i(t, p) \in [0, 1]$ to accept offer p. To introduce the primitives of the bargaining game, let $\{v_a^i(t)\}_{a=0,L,H}^{i=1,2}$ be the value functions in period t before the market opens, net of the value of the endowment stream. Then:

$$\bar{v}_a^i(t+1) := (1-\gamma)v_a^i(t+1) + \gamma v_a^j(t+1), \quad a \in \{0, L, H\}, i \in \{1, 2\}, j \neq i$$

is agent (τ^i, a) expected future utility from holding asset a, given that he may switch valuation. We can now write down payoffs in the bargaining game. When offered p, owner with type (τ^i, a) solves:

$$\max_{\alpha_a^i(t,p)} \alpha_a^i(t,p)\big[p + \delta\bar{v}_0^i(t+1)\big] + (1-\alpha_a^i(t,p))\big[\tau^i d_a + \delta\bar{v}_a^i(t+1)\big] \tag{1.2}$$

If he accepts, a seller obtains the price p and the future utility from being a non owner $\delta\bar{v}_0^i(t+1)$. If he refuses the offer, he enjoys the dividend $\tau^i d_a$ and obtains next period expected utility from the asset $\delta\bar{v}_a^i(t+1)$. For buyer $(\tau^k, 0)$, price p^* is optimal if:

$$p^* \in \arg\max_p \left\{ \sum_{\substack{i=1,2\\a=L,H}} \frac{\mu_a^i(t)}{S} \Big(\alpha_a^i(t,p)\left[\tau^k d_a + \delta\bar{v}_a^k(t+1) - p\right] + (1-\alpha_a^i(t,p))\delta\bar{v}_0^k(t+1)\Big) \right\} \tag{1.3}$$

Under asymmetry of information, the buyer forms expectations over the asset owner type he matched with. When offering p, he obtains the asset with probability $\alpha_a^i(t, p)$ if he meets seller (τ^i, a). He then enjoys the current dividend from the asset $\tau^k d_a$ and its future value $\delta\bar{v}_a^k(t+1)$ minus the price he pays p. Otherwise, the buyer goes on to the next period where his utility is $\bar{v}_0^k(t+1)$. We can now introduce the solution concept for the stage bargaining game.

trade. An offer would thus be a menu of such contracts. It can be shown that buyers do not use this extra dimension to screen sellers, that is all proposed contract have probability of trade equal to 1. This result by Samuelson (1984) obtains because preferences are linear.

Definition 1 (Bargaining Equilibrium). *For value functions* $\{v_a^i(t+1)\}_{a=0,L,H}^{i=1,2}$*, the bargaining equilibrium of period* t *is given by probabilities* $\{\alpha_a^i(t,p)\}_{a=L,H}^{i=1,2}$ *and distribution* $\{\Pi^k(t,.)\}^{k=1,2}$ *such that*

1. *Probability* $\alpha_a^i(t,p)$ *solves seller's problem* (1.2) *for any* $p \in \mathbb{R}$.

2. *A buyer offers* p*, that is* $p \in Supp(\Pi^k(t,.))$ *if* p *solves* (1.3).

Subgame perfection follows from the requirement that sellers reply optimally to any price, including out of equilibrium offers. Although the primitives of the game are ultimately endogenous, we may partially characterize the bargaining equilibrium of period t, using the reservation values for each type of asset owner. Define:

$$r_a^i(t) := \tau^i d_a + \delta(\bar{v}_a^i(t+1) - \bar{v}_0^i(t+1)), \quad i = 1,2, \quad a = L, H \tag{1.4}$$

In words, $r_a^i(t)$ is the net value attached to holding asset a for agent i in period t over not owning an asset. The label reservation value comes from the seller problem (1.2). Indeed, an asset owner (τ^i, a) would never accept an offer below $r_a^i(t)$. Reservation values are thus inversely related to the eagerness to sell the asset which is the relevant statistic for each type of asset owner. The following Lemma simplifies the description of the bargaining equilibrium, anticipating on Definition 2:

Lemma 1. *In any OTC equilibrium (Definition 2), the following statements hold:*

1. *Type ranking:*
$$r_L^1(t) < r_L^2(t) < r_H^1(t) < r_H^2(t) \tag{1.5}$$

2. *Only type* $(\tau^2, 0)$ *search and* $Supp(\Pi^2(t,.)) \in \{r_L^1(t), r_H^1(t)\}$.
   Let $\pi(t) := \Pi^2(t, r_H^1(t)) - \Pi^2(t, r_L^1(t))$ *be the probability of a pooling offer* $r_H^1(t)$.

To prove this Lemma, we anticipate on equilibrium definition 2 and use the free entry condition. Buyers make zero profit which simplifies the expression of (1.4) for the reservation values[14]. Agents with low valuation or low asset quality accept a

[14] In Chiu and Koeppl (2014), this is not an issue because the only way a type τ^1 asset owner switches back to type τ^2 is by selling his asset. Observe that the difficulty also vanishes in the case where types are iid across time, that is $\gamma = 1/2$ since then $\bar{v}_a^1(t) = \bar{v}_a^2(t)$ for $a \in \{0, L, H\}$

lower price to sell. In particular, Assumption (LC) implies that $r_L^2(t) < r_H^1(t)$. Even high valuation owners of lemons are more eager to sell than low valuation owners of H assets. From (1.5), it is clear that $(\tau^1, 0)$ non-owners do not gain from trade with any asset owner type. Hence they do not pay the cost κ to search. A buyer is thus an agent of type τ^2. Buyers only target low valuation asset owners with whom they have gains from trade. From (1.5) however, an offer addressed to (τ^1, H) owners attracts all L asset owners. In the following, we refer to the probability of a pooling offer $\pi(t)$ as market liquidity. Indeed it measures the probability that a (τ^1, H) asset owner sells his asset. Building on Lemma 1, the buyer's problem boils down to a binary choice between a separating price offer $r_L^1(t)$ and a pooling offer $r_H^1(t) > r_L^1(t)$. Let us write his profit $v^B(t, \pi)$ from randomization π in period t:

$$v^B(t,\pi) := \frac{1}{S}\left\{\pi\Big[\mu_H^1(t)(r_H^2(t) - r_H^1(t)) + S(1-q)(r_L^2(t) - r_H^1(t))\Big]\right. \\ \left. + (1-\pi)\mu_L^1(t)(r_L^2(t) - r_L^1(t))\right\} + \delta\bar{v}_0^2(t+1) \tag{1.6}$$

Equation (1.6) captures the standard rent-efficiency trade-off. The pooling offer $r_H^1(t)$ (weight π) attracts high quality assets but generates losses $r_H^1 - r_L^2(t)$ on low quality assets. With a separating offer $r_L^1(t)$ (weight $1-\pi$), buyers forgo gains from trade on the H asset to extract rents $r_L^2(t) - r_L^1(t)$ from (τ^1, L) sellers. Illiquidity materializes in this case. For simplicity, we let $v^B(t)$ denote the value of $v^B(t, \pi)$ at the optimum, which is the utility of a matched buyer.

Buyers entry

Non-owners decide whether to search for an asset, given the matching probability $\lambda^B(t)$ and the outcome of the bargaining game $v^B(t)$. Non-owner $(\tau^2, 0)$ obtains a net payoff equal to $-\kappa + \lambda^B(t)v^B(t) + (1-\lambda^B(t))\delta\bar{v}_0^2(t+1)$ from searching. Otherwise, he goes on to the next period with utility $\delta\bar{v}_0^2(t+1)$. The equilibrium mass of buyers derives from the optimal search choice of non-owners:

$$\mu^B(t) = \begin{cases} 0 & \text{if } -\kappa + \left[v^B(t) - \delta\bar{v}_0^2(t+1)\right] < 0 \\ \frac{S(v^B(t) - \delta\bar{v}_0^2(t+1))}{\kappa} & \text{otherwise} \end{cases} \tag{1.7}$$

Assumption (A2) ensures that the search cost κ is small enough so that the last case prevails in equilibrium, that is non-owners do enter as buyers.

Dynamics

An agent valuation for an asset depends both on its reservation value and the price he may obtain for the asset in the OTC market. Precisely, for $i \in \{1, 2\}$,

$$v_H^i(t) = \tau^i d_H + \delta \bar{v}_H^i(t+1) \tag{1.8}$$

$$v_L^i(t) = \tau^i d_L + \delta \bar{v}_L^i(t+1) + \lambda^S(t)\pi(t)[r_H^1(t) - r_L^i(t)] \tag{1.9}$$

Equation (1.8) shows that H asset owners are at most indifferent between trading today and waiting next period. Indeed, they never receive an offer above their reservation value. Low quality asset owners earn information rents when matched - the second term in (1.9). These are proportional to the probability $\pi(t)$ of a pooling offer and the difference between the pooling price $r_H^1(t)$ and the reservation value $r_L^i(t)$. For non asset owners, we obtain:

$$v_0^1(t) = \delta \bar{v}_0^1(t+1) \tag{1.10}$$

$$v_0^2(t) = \max\left\{0, -\kappa + \lambda^B(t)\left[v^B(t) - \delta \bar{v}_0^2(t+1)\right]\right\} + \delta \bar{v}_0^2(t+1) \tag{1.11}$$

Non-owners $(\tau^1, 0)$ might only become active buyers if they switch type. Non-owners $(\tau^2, 0)$ decide whether to search and become buyers.

Finally, we need to characterize the evolution of asset ownership across time. For a given asset, the owner type might change because the original owner sold the asset or switched valuation. Figure 1.1 describes these dynamics for the H asset. Consider the $\mu_H^1(t)$ assets initially held by (τ^1, H) agents at the beginning of period t. Some agents fail to find a match with probability $1 - \lambda^S(t)$ or do not trade in a match with probability $\lambda^S(t)(1 - \pi(t))$. Overall, a fraction $1 - \lambda^S(t)\pi(t)$ of these assets is not traded. Summing over the solid lines in Figure 1.1, we thus obtain:

$$\begin{aligned}\mu_H^1(t+1) &= \left[(1-\gamma)(1-\lambda^S(t)\pi(t)) + \gamma\lambda^S(t)\pi(t)\right]\mu_H^1(t) + \gamma\mu_H^2(t) \\ &= \gamma S q + (1-2\gamma)(1-\lambda^S(t)\pi(t))\mu_H^1(t) \end{aligned} \tag{1.12}$$

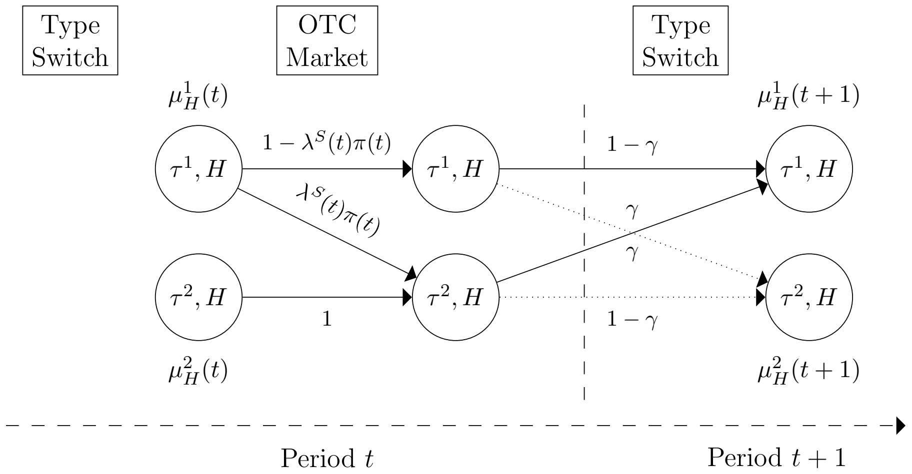


Figure 1.1: Law of motion for the H asset

where I used the balance condition $\mu_H^2(t) = \gamma S q - \mu_H^1(t)$ in the last line. We may similarly derive the law of motion for L assets with the difference that (τ^1, L) agents fail to trade only when they do not find a match. We obtain:

$$\mu_L^1(t+1) = \gamma S(1-q) + (1-2\gamma)(1-\lambda^S(t))\mu_L^1(t) \tag{1.13}$$

Equation (1.12) highlights the effect of past buyers' offer on the composition of the pool of sellers. The mass $\mu_H^1(t+1)$ of H asset owners looking to sell in period $t+1$ decreases with the probability $\pi(t)$ of a pooling price in period t. Intuitively, favorable offers in period t deplete the pool from high quality sellers in period $t+1$. For (τ^1, L) owners, $\pi(t)$ affects the price received but not the trading probability as they always sell their asset. Overall, market liquidity thus affects negatively the pool of sellers, and hence future market liquidity. We can now introduce the definition of a stationary OTC equilibrium.

Definition 2 (OTC Equilibrium)**.** *An OTC equilibrium is a collection of value functions* $\{v_a^i(t)\}_{a=0,L,H}^{i=1,2}$ *and reservation functions* $\{r_a^i(t)\}_{a=L,H}^{i=1,2}$*, a distribution of asset owners* $\{\mu_a^i(t)\}_{a=L,H}^{i=1,2}$*, a mass of buyers* $\mu^B(t)$ *and a probability* $\pi(t)$ *of a high price offer* $r_H^1(t)$ *for any* t *such that:*

1. *Buyers' offers verify* $\pi(t) \in \arg\max_\pi v^B(t,\pi)$ *and* $\mu^B(t)$ *verifies condition* (1.7)*.*
2. *Functions* $\{v_a^i(t)\}_{a=0,L,H}^{i=1,2}$ *and* $\{r_a^i(t)\}_{a=L,H}^{i=1,2}$ *verify equations* (1.9)*-*(1.11) *and* (1.4)*.*

3. *Distribution* $\{\mu_a^i(t)\}_{a=L,H}^{i=1,2}$ *verifies equations* (1.1) *and law of motion* (1.12)-(1.13)

4. ***Stationary property:*** $\exists\, T \in \mathbb{N}_+$ *such that for all endogenous variables* z

$$z(t+T) = z(T)$$

The lowest T *for which this property holds is the period of the equilibrium.*

By imposing the stationary property, I focus on the long-run dynamics of the model, hence the absence of reference to the initial distribution $\{\mu_a^i(0)\}_{a=L,H}^{i=1,2}$. The permanent component of liquidity fluctuations captures most of the model dynamics. Observe that an equilibrium with period $T = 1$ is a steady sate. The main result of the paper is to show that there can be other (cyclical) stationary equilibria with period $T \geq 2$.

1.3 Equilibrium Liquidity Dynamics

We first present a series of preliminary results. We then characterize equilibrium cycles in Section 1.3.1 and steady states in Section 1.3.2.

Lemma 2. *In any OTC equilibrium, the following statements hold*

i) Agents (τ^i, H) *obtain their autarky payoff:*

$$\forall\, t, i \in \{1,2\}, \quad v_H^i(t) = r_H^i := \frac{(1-\delta)\tau^i + \delta\gamma(\tau+1)}{(1-\delta)\big[1-\delta(1-2\gamma)\big]} d_H$$

ii) Buyers make zero profit: $\forall\, t$, $v_0^i(t) = 0$, *for* $i = 1,2$ *and equilibrium entry is*

$$\mu^B(t) = \frac{Sv^B(t)}{\kappa}$$

iii) Sellers find a match with probability 1*:* $\forall\, t$, $\lambda^S(t) = 1$.

As he receives offers through sequential matching, an H asset owners enjoys the same utility than in autarky[15]. Part *ii*) follows from free-entry since buyers

[15] The market would then shut down if asset owners were to pay the search cost κ, a result known as the Diamond paradox. Interestingly, this result also arises as an equilibrium outcome in Section 1.5 where buyers compete simultaneously for sellers.

enter as long as they earn a positive profit. For part *iii*), assumption (A2) further ensures that buyers' entry is enough to match all asset owners. This implies that trading does not fail for lack of buyers and that market illiquidity is fully driven by asymmetry of information and adverse selection[16].

We now identify the two main dynamic forces at play in the model. Using the expression for buyers' profit $v^B(t,\pi)$ from (1.6), let us derive the net profit from a pooling offer over a separating offer:

$$v^B(t,1) - v^B(t,0) = \frac{\mu_H^1(t)}{S}(r_H^2 - r_H^1) - (1-q)\big[r_H^1 - r_L^2(t) + \gamma(r_L^2(t) - r_L^1(t))\big] \quad (1.14)$$

where we replaced $\mu_L^1(t)$ and $r_H^i(t)$ by $\gamma S(1-q)$ and r_H^i respectively, using Lemma 2. Pooling becomes more advantageous as the share of high quality assets for sale $\mu_H^1(t)/S$ or the reservation values $\{r_L^i(t)\}^{i=1,2}$ for the L asset increase. The following Lemma shows how these quantities react to past and future prices to shape today buyers' trade-off (1.14).

Lemma 3. *In any OTC equilibrium, the following statements hold:*

i) Competition effect: $\pi(t)$ *increases with* $\{\pi(t+l)\}_{l=1,..,\infty}$

ii) Composition effect: $\pi(t)$ *decreases with* $\pi(t-1)$.

Part *i*) states that the probability of a pooling offer today increases with the probability of pooling offers in the future. The crucial insight is that higher future prices raise the value of a lemon both to the prospective buyer and the current owner. First, the ex-post loss on lemons $r_H^1 - r_L^2(t)$ from a pooling offer decreases as the buyer's valuation $r_L^2(t)$ goes up. When he knows he can resell the lemon at a good price tomorrow, the buyer is inclined to offer a (high) pooling price although the quality is uncertain. Second, future high prices also squeezes the buyer's margin $r_L^2(t) - r_L^1(t)$ on a low offer $r_L^1(t)$. The high offer $r_H^1(t)$ thus becomes more relatively more profitable. To see this, suppose that a lemon owner receives a pooling offer the

[16]The fact that sellers find a buyer with probability 1 is a byproduct of the efficient rationing property attached to the matching function. The analysis can accommodate a matching function of the form $\alpha \min\{\mu^B(t), \mu^S(t)\}$ with $\alpha < 1$. Under appropriate modifications to A2, we would obtain similar results.

next period, that is $\pi(t+1)=1$. Then this resale price determines the future value of the lemon both for the current and the new potential owner. Hence, the gains from a separating offer boil down to the difference in the current dividend valuation $(\tau-1)d_L$. Gains from trade would be larger if agents were to stay in autarky forever after (or $\pi(t+l)=0$ for $l\geq 1$). Both components of the *competition effect* work in the same direction so that $\pi(t)$ increases with $\{\pi(t+l)\}_{l=1,..,\infty}$, that is future liquidity begets present liquidity. The *competition effect* captures complementarity across time in decisions to pool sellers, which is a source of equilibrium multiplicity.

Part *ii*) establishes that the probability of a pooling offer today $\pi(t)$ depends negatively on the probability $\pi(t-1)$ in the last period. The *composition effect* hinges on the endogenous asset holdings dynamics captured by law of motion (1.12). Suppose indeed that liquidity was high in the last period, or $\pi(t-1)=1$. This pooling offer clears the market for all assets. In particular, the implicit supply of H assets, $\mu^1_H(t)=\gamma Sq$ reaches its lowest point in period t. As the pool of sellers now contains mostly L assets, buyers should find it more profitable to make a low separating offer. But if liquidity is indeed low today, that is $\pi(t)=0$, we obtain:

$$\mu^1_H(t+1)-\mu^1_H(t)=2\gamma(Sq/2-\mu^1_H(t))>0$$

The distribution tomorrow becomes more favorable to a pooling offer because the implicit supply of H assets for sale increases. Delaying trade thus improves the pool of sellers through the accumulation of selling pressure of H assets. The composition effect creates a negative relationship between present and future liquidity which is key to equilibrium fluctuations. The relative strength of the composition and the competition effect then determines the nature of equilibrium.

1.3.1 Liquidity Cycles

For $T\geq 2$, I solve for OTC equilibria involving pure strategies[17] for buyers, that is where $\pi(t)\in\{0,1\}$ for any t. Lemma 4 first shows that liquidity cannot be high in two consecutive dates.

[17] If anything, cycles with pure strategies generates starker price fluctuations and are thus harder to sustain. In the next section, I also characterize mixed-strategy equilibria steady state ($T=1$).

Lemma 4. *In an OTC equilibrium with period $T \geq 2$, if $\pi(t) = 1$, then $\pi(t+1) = 0$, that is a period with low liquidity always follows a period with high liquidity.*

The intuition for Lemma 4 follows from our previous discussion. After a pooling offer in period t, $\mu_H^1(t+1) = \gamma Sq$ using law of motion (1.12). The distribution of assets becomes least favorable to pooling[18] and buyers prefer to offer the low separating price. When they do so, the quality in the pool of sellers improves but it might take time before buyers find it optimal to pool sellers again depending on the speed of the type switching process. Hence, there can be several consecutive periods with low liquidity $\pi = 0$.

For a given period T, we label without ambiguity by 0 the peak dates $t, t+T, t+2T$ where $\pi(t) = 1$. In these periods, buyers offer a pooling price r_H^1. Labels $1, 2, ..., T-1$ are for the intermediate "trough" dates when buyers offer the separating price $r_{L,T}^1(t)$. The additional subscript T captures the dependence of endogenous variables on the cycle length when relevant. We may now state the main Proposition of the paper.

Proposition 1. *Let $T \geq 2$. There exists thresholds $(\underline{q}_T, \bar{q}_T)$ such that an OTC equilibrium of period T exists if and only if $q \in [\underline{q}_T, \bar{q}_T]$ and*

$$\frac{1-(1-2\gamma)^T}{1-(1-2\gamma)^{T-1}} \geq \frac{r_H^1 - (1-\gamma) r_{L,T}^2(0) - \gamma r_{L,T}^1(0)}{r_H^1 - (1-\gamma)\tau d_L - \gamma d_L - \delta r_H^1} \qquad (E_T)$$

where for $i \in \{1, 2\}$ and $t = 0, .., T$

$$r_{L,T}^i(t) = \left[\frac{1-\delta^{T-t}}{1-\delta}(\tau+1) + (-1)^i \frac{1-(\delta(1-2\gamma))^{T-t}}{1-\delta(1-2\gamma)}(\tau-1)\right]\frac{d_L}{2} + \delta^{T-t} r_H^1 \qquad (1.15)$$

For $T \geq 2$, $(E_{T+1}) \Rightarrow (E_T)$ and $\bar{q}_{T+1} < \underline{q}_T$.
In a cycle of period T, the mass of (τ^1, H) agents is

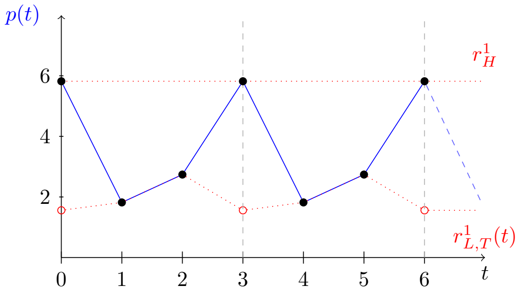
$$\mu_{H,T}^1(t) = \frac{1-(1-2\gamma)^t}{2} Sq, \quad t = 1, .., T \qquad (1.16)$$

Observe first that, in a cycle, agents' valuation for a lemon $r_{L,T}^i(t)$ weighs the

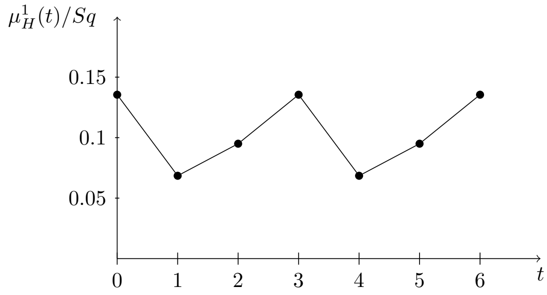
[18] The argument thus relies partially on the fact that search frictions are mild, or $\lambda^S(t) = 1$. Intuitively though, entry by buyers should precisely be larger and frictions less severe for sellers at the peak of the cycle.

holding value over the remaining trough periods and the pooling price they will receive at the peak, in $T-t$ periods. During the trough, equation (1.16) captures the accumulation of selling pressure of H assets that leads to the peak. Let us now interpret the existence condition (E_T) for a cycle of period T. The left hand side measures the relative improvement in the pool quality between the last date of the trough and the peak as can be seen from (1.16). It thus captures the benefits in date $T-1$ from waiting one more period to make a pooling offer. The right hand side represents the costs from waiting. Indeed, in period $T-1$, losses on lemons from a pooling offer would be low since a buyer can resell at the pooling price next period. In period 0 however, acquiring lemons is costly because a buyer can flip them at a high price only T periods after the purchase. A cycle thus exists if the improvement in the pool quality from the composition effect overcomes the loss increase from the competition effect. Adding one period to the cycle leaves more time for the pool to improve. Hence longer period cycles exist for lower values of the share of H assets q, that is $\underline{q}_{T+1} < \underline{q}_T$. Observe that as T increases though, the last period marginal improvement in the pool quality $\mu^1_{H,T}(T) - \mu^1_{H,T}(T-1)$ goes down. On the other hand, buyers' incentives to anticipate the gains from trade with H assets go up. Low periods equilibrium cycles are thus easier to sustain, that is $(E_{T+1}) \Rightarrow (E_T)$ for $T \geq 2$.

Figure 1.2 illustrates price and composition dynamics for a 3 period cycle. In Figure 1.2a, the solid blue line represents the transaction price over the cycle. It coincides with the reservation value $r^1_{L,T}(t)$ (lower dashed red line) of (τ^1, L) owners during the trough and equals the pooling offer r^1_H (upper dashed red line) at the peak. After the peak, Figure 1.2b illustrates the drop in the supply of H assets $\mu^1_{H,T}(t)$ measured as a fraction of the total quantity of H assets. During the trough, only L assets are traded but the price increases. Indeed, it must reflect (τ^1, L) sellers' outside option of waiting for a better offer in $t = 3$. In particular, the asset trades above the full information price although buyers know for sure they are buying a L asset. We can interpret this premium as a bubble component reflecting the future high value of the lemon at the peak of the cycle. These dynamics then evoke the "hot potato" story for a financial crisis. Agents know they purchase bad assets at inflated prices but ride the bubble to resell them at an even higher price



(a) Prices and Reservation Values



(b) Asset Supply of quality H

Figure 1.2: A 3 Period Cycle ($\delta = 0.3$, $d_L = 1$, $d_H = 4$, $\tau = 2$, $\gamma = 0.05$)

in the future. When they do (at the peak of the cycle), the price drops significantly and the bubble bursts. More generally, our analysis shows that liquidity fluctuations arise naturally in markets with asymmetry of information. Prices reflect the average quality of assets offered for sale but the supply responds endogenously to past prices.

1.3.2 Steady State Equilibria

This section describes steady state equilibria, that are OTC equilibria of period $T = 1$, this time both in mixed and in pure buyers' strategy. I shorten the presentation since the characterization mostly serves as a basis for welfare comparison with liquidity cycles. I drop the time arguments for endogenous variables.

Proposition 2. *There exists two thresholds in the share of H assets in the economy $(\underline{q}, \bar{q}) \in [0,1]^2$ such that the only steady state equilibria are the following:*

i) When the share q is low, $q \leq \underline{q}$, there is a separating equilibrium $\pi = 0$.

ii) When the share q is high, $q \geq \bar{q}$, there is a pooling equilibrium $\pi = 1$.

iii) For $q \in (\min\{\underline{q}, \bar{q}\}, \max\{\underline{q}, \bar{q}\})$, there exists an equilibrium in mixed strategy $\pi(q) \in (0,1)$.

When $\underline{q} \leq \bar{q}$, there is a unique equilibrium for any value of q. When $\underline{q} \geq \bar{q}$, equilibria i), ii) and iii) coexist on $[\bar{q}, \underline{q}]$.

Intuitively, a pooling equilibrium with high liquidity $\pi = 1$ may only exist if the share of H assets q is high enough. When q is too low, buyers thus cater only to

(τ^1, L) owners and the equilibrium is separating with $\pi = 0$. For intermediate values of q, a partial equilibrium exists with $\pi(q) \in (0,1)$. The mixed strategy equilibrium highlights the tension between the composition and the competition effect. As π goes up, lemons become more valuable and pooling offers more profitable. The competition effect thus favors pooling. However, the steady state share of H quality assets decreases with π. Indeed, from law of motion (1.12), we obtain:

$$\mu_H^1(\pi) = \frac{\gamma}{2\gamma + \pi(1-2\gamma)} Sq \tag{1.17}$$

Hence, through the composition effect, higher market liquidity makes it less profitable for buyers to offer a pooling price. These forces work against one another so that mixed strategy equilibria exist for an open interval of values of q.

When the upper bound of the pooling region $\underline{q}$ exceeds the lower bound $\bar{q}$ of the pooling region, a separating, a pooling and a mixed equilibrium coexist. The proof to Proposition 2 shows that the multiplicity condition $\underline{q} \geq \bar{q}$ writes:

$$\frac{\delta\gamma(1-\delta)(1-2\gamma)(\tau-1)d_L}{(1-\delta)(d_H - \tau d_L) + \gamma\delta(\tau+1)(d_H - d_L) + \gamma(1-\delta)(\tau-1)d_L} \geq 1 - \delta - 2\gamma \tag{1.18}$$

In the limit case where $d_L \to 0$, the condition writes[19] $\delta > 1 - 2\gamma$. From equation (1.17), $1 - 2\gamma$ measures the sensitivity of the H asset supply μ_H^1 to liquidity π and thus captures the strength of the *composition effect*. The discount factor δ determines the weight assigned to future payoff and thus the strength of the competition effect. Hence, in the parameter region with steady state multiplicity, the *competition effect* dominates the *composition effect*. We now gather the existence results from Propositions 1 and 2 to provide a complete picture of OTC equilibria. In particular, we are interested in the nature of the steady state equilibrium in the region where cycles exist.

Corollary 1. *An OTC equilibrium of period $T \geq 2$ exists when there is a unique steady state equilibrium, that is $(E_T) \Rightarrow \underline{q} \leq \bar{q}$. A cyclical equilibrium exists in the region where the steady state is in mixed strategy, that is $[\underline{q}_T, \bar{q}_T] \subset [\underline{q}, \bar{q}]$ with $\bar{q}_2 = \bar{q}$.*

[19] This is the case analyzed by Chiu and Koeppl (2014) in a continuous time environment. Although, strictly speaking, this case is not well-defined in my model because of Assumption (A2), it is useful to form intuition.

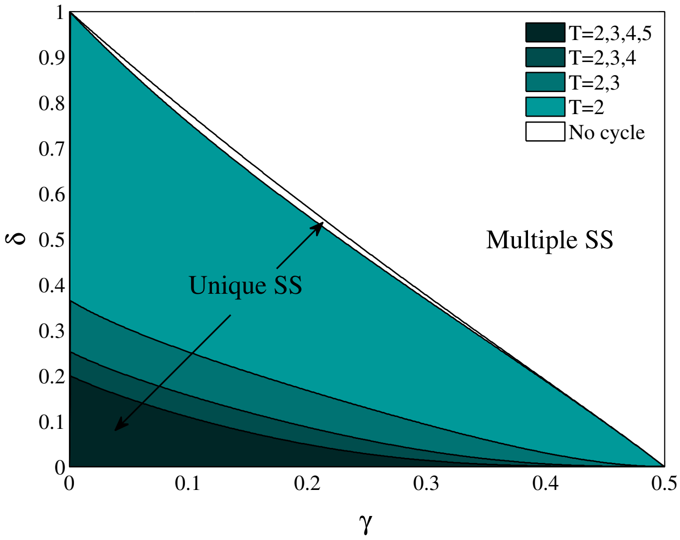


Figure 1.3: Equilibrium Existence ($d_L = 1$, $d_H = 4$, $\tau = 3$)

The proof of Proposition 1 establishes these results. First, cycles exist when the composition effect is strong whereas steady state multiplicity relies on the competition effect. Second, there is a natural relationship between mixed strategy steady state and cycles. When the share of H quality assets is intermediate, neither pooling nor separating can be sustained in every period. In a steady state, liquidity spreads out evenly across periods as sellers face a constant probability to receive a high price. In a cycle, liquidity fluctuates with the average quality in the pool of sellers and terms of trade change over time. Figure 1.3 describes the equilibrium regions in the (γ, δ) parameter space. The uppermost downward sloping line represents equation (1.18) and breaks the parameter space into two regions. In the bottom left part, there is a unique steady state for each value of q. Within this sub-region, equilibrium cycles may exist. Higher value of the period T corresponds to darker shades. The figure shows that cycles with period $T \geq 3$ require low discount factors[20]. Hence, in the rest of the analysis we focus on period 2 cycles (lighter shade on Figure 1.3), that is (E_T) holds for $T = 2$ only and $q \in [\underline{q}_2, \bar{q}]$

[20]This is in part driven by our focus on equilibrium cycles in pure buyers' strategies.

1.3.3 Welfare Comparison

Aggregate surplus $W(t)$ is defined recursively as follows:

$$W(t) = S(1-q)\tau d_L + \mu_H^1(t)(1-\pi(t))d_H + \left[\mu_H^1(t)\pi(t) + \mu_H^2(t)\right]\tau d_H - \mu^B(t)\kappa + \delta W(t+1) \tag{1.19}$$

The first three terms correspond to allocative efficiency. With symmetric information, type τ^2 agents would hold all the assets after trading. Market illiquidity, generates misallocation of a fraction $\mu_H^1(t)(1-\pi(t))$ of the H assets. The fourth term measures trade costs which are proportional to equilibrium entry. Since buyers get to make TIOLI offers, entry is typically inefficient[21]. The last term is self-explanatory.

In this section, the subscript ss (resp. cy) refers to endogenous variables in the steady state (resp. 2 period cycle). We thus denote W_{ss} welfare in steady state and $(W_{cy}(0), W_{cy}(1))$ welfare in the high and low date of cycle. Since liquidity fluctuates between $\pi_{cy}(0) = 1 > \pi_{ss}$ and $\pi_{cy}(1) = 0 < \pi_{ss}$ in a cycle, one could guess that welfare in a cycle also fluctuates around the steady state level W_{ss}. We show however that fluctuations entail a dynamic welfare loss with respect to the steady state.

Proposition 3. *There exists $\hat{q} \in [\underline{q}_2, \bar{q}]$ such that for all $q \in (\hat{q}, \bar{q}]$,*

$$W_{ss} > W_{cy}(0) > W_{cy}(1)$$

that is the surplus in a steady state is greater than in a cycle in every date.

Proposition 3 shows that when $q \in [\hat{q}, \bar{q}]$ the steady state equilibrium dominates the cyclical equilibrium, irrespectively of the "starting date" for the cycle. The striking part of the result is that surplus may be lower at the peak of the cycle than in steady state, that is $W_{ss} \geq W_{cy}(0)$, although trading volume and liquidity are maximal at a peak date. However, high liquidity at the peak lowers future market liquidity through the composition effect and generates dynamic misallocation of the H asset - the fourth term in (1.19). When q is close to $\bar{q}$, $\pi_{ss}(q) \to 1$, that is the steady state equilibrium features full liquidity in the limit. In a cycle however, every two periods some H assets are not traded. With the composition effect, there

[21] In general search frictions could also affect allocative efficiency if sellers can fail to meet buyers, that is $\lambda^S(t) < 1$, which is not the case in equilibrium here.

is too much trade at the peak and too little in the trough. When $q \in [\underline{q}, \hat{q}]$, the first inequality is reversed and surplus evaluated at the peak is higher than in steady sate. The comparison then depends on the reference date for the cycle although the average value of W_{cy} lies below W_{ss}. The unambiguous welfare ranking when $q \in [\hat{q}, \bar{q}]$ suggests that policy interventions can be desirable when the economy is in a cycle. Intuitively, a benevolent policy maker would seek to prop up (resp. tame) trading in the trough (resp. peak) of the cycle. In the following Section, we show how an asset purchase program can achieve this objective.

1.4 Liquidity and Policy Intervention

In this section, I study an asset purchase program by a benevolent government who seeks to jump-start the market but must resell the assets he purchased. Unlike previous works including Tirole (2012) and Chiu and Koeppl (2014), I can thus capture a realistic policy constraint because the government must eventually close his position. With respect to a permanent policy, a revertible program does not change the fundamentals of the economy in the long-run. In the following, I characterize feasible policies analytically and provide numerical results for the surplus maximizing policy. Although its effects are smaller, I also show that a budget-neutral policy is feasible and can increase welfare in the Pareto sense. Formally, the benevolent government is a large agent with the same discount factor δ as private agents and preferences:

$$u^G(c, d) = c + \tau^G d$$

over the consumption good and the dividends where $\tau^G \in [0, \tau]$. The government derives a lower utility from the dividends than type τ^2 agents, as for instance, he would not value potential services (borrowing, hedging) from holding assets. The government has deep pockets and can hold many assets.

1.4.1 Timing and Policy Design

Prior to the intervention, the economy is in an equilibrium cycle of period $T = 2$. Let t_{int} be the intervention date which corresponds to a low date of the cycle. At the beginning of period t_{int}, the distribution of agents across assets is thus $\{\mu^i_{a,cy}(1)\}^{i=1,2}_{a=L,H}$. We divide periods t_{int} and $t_{int}+1$ into phases $i)$ and $ii)$ as follows:

1. Date t_{int}: Purchase.

   i) The government announces that he will buy up to S^G assets at unit price P^G. Asset owners may apply and sell their asset to the government.

   ii) OTC market with asset owners who have not participated in the program.

2. Period $t_{int}+1$: Resale.

   i) The government quotes a resale price R^G at which he resells all assets purchased in t_{int}. Non-asset owners may apply to purchase these assets.

   ii) OTC market with asset owners including buyers of step $i)$.

The division of period t_{int} into steps $i)$ and $ii)$ is important. By removing assets before the OTC market opens, the government can affect the distribution of sellers faced by buyers[22] and hence the probability of a pooling offer. For simplicity, we abstract from issues related to the timing of the exit strategy since the government must resell assets one period after. A policy is a triplet $(S^G, P^G, R^G) \in \mathbb{R}^3_+$ where S^G is the program size, P^G the purchase price and R^G the resale price. Besides the obligation to resell assets, I impose a medium-run stabilization constraint on the intervention.

In order to explicit these constraints, I take as given the sequence $(\pi(t_{int}), \pi(t_{int}+1), ...)$ of buyers' offers. In equilibrium, this sequence will be consistent with the policy. The stabilization constraint sets the mass of (τ^1, H) agents in period $t_{int}+2$ to:

$$\mu^1_H(t_{int}+2) = \mu^1_{H,ss} \tag{SC}$$

When (SC) holds, we know from Proposition 2 that the steady state is a continuation equilibrium from period $t_{int}+2$ onward. We can thus avoid dealing with equilibrium transitions and focus on the short-term policy trade-off. The resale constraint puts

[22]Buying assets in the private OTC market would not have an impact if sellers are captive. If they are not, they could use the government program as a credible threat to induce more competitive offers from buyers which generates higher liquidity. We are interested in a policy that is actually implemented in equilibrium rather than an equilibrium selection device.

an upper bound on the price R^G the government can ask to sell his assets. This bound depends on the selection of assets acquired in period t_{int} and thus on agents' participation decisions. Agent (τ^i, a) for $i \in \{1,2\}$ and $a \in \{L, H\}$ opts in the program if and only if $P^G \geq v_a^i(t_{int})$ that is when[23] the price offered exceeds the utility he expects from trading in the OTC market at t_{int}. Observe that the outside option $v_a^i(t_{int})$ is endogenous since it depends on the equilibrium induced by the intervention. The crucial insight from Lemma 1 is that L asset owners have a lower market utility and would thus be the first to opt in. This leaves room for the policy to improve the distribution of assets in the OTC market. Since the government has no more information than private buyers, he obtains a random selection (S_L^G, S_H^G) of assets held by those agents who opt in where for $a \in \{L, H\}$:

$$S_a^G = \min\left\{1, \frac{S^G}{\sum_{j,a'} \mu_{a',cy}^j(1)\mathbb{1}_{\{P^G \geq v_{a'}^j(t_{int})\}}}\right\} \sum_{i=1,2} \mathbb{1}_{\{P^G \geq v_a^i(t_{int})\}}\mu_{a,cy}^i(1) \tag{1.20}$$

where the term between curly brackets captures rationing of sellers when the program is over-subscribed. We assume that S^G can adjust so that the program is never under-subscribed. The government cannot quote a resale price R^G higher than the buyer's valuation for the average asset from the government pool:

$$R^G \leq v_L^2(t_{int}+1)S_L^G/S + v_H^2(t_{int}+1)S_H^G/S \tag{RC}$$

Although conditions (SC) and (RC) bear on endogenous objects, they ultimately constrain the policy that induces this equilibrium. Let us now define the government payoff v^G as:

$$v^G = S^G(\delta R^G - P^G) + \left[S_H^G d_H + S_L^G d_L\right]\tau^G \tag{GP}$$

The first term is the capital gain. The second term measures the government valuation for the dividends and depends on the selection of assets acquired by the government. In the numerical analysis of Section 1.4.2, we first allow the government to run a loss $v^G < 0$ and then impose budget-neutrality. We may now define

[23]If $P^G = v_a^i(t_{int})$, the agent is indifferent between opting in or out of the program and could randomize. We can dismiss this concern because the government could induce participation strictly by raising the price by an infinitesimal amount

formally a feasible policy

Definition 3. *A policy* (S^G, P^G, R^G) *is feasible if there exists* $(\pi(t_{int}), \pi(t_{int}+1)) \in [0,1]^2$ *such that:*

1. *Given the sequence of buyers' strategies* $(\pi(t_{int}), \pi(t_{int}+1), \pi_{ss}, \pi_{ss}, ...)$, *the resale constraint* (RC) *and the stabilization constraint* (SC) *hold.*
2. *Strategies* $(\pi(t_{int}), \pi(t_{int}+1))$ *are optimal for buyers, that is for* $l \in \{0, 1\}$

$$\pi(t_{int}+l) \in \arg\max_{\pi} v^B(t_{int}+l, \pi)$$

given that $\pi(t+l) = \pi_{ss}$ *for* $l \geq 2$.

We now take as granted that the equilibrium with a feasible intervention is the steady state π_{ss} from period $t_{int}+2$ onward. In words, a feasible policy satisfies the resale (RC) and the stabilization constraints (SC) in an equilibrium induced by this policy[24]. The first objective of the policy is to maximize the net surplus gain ΔW_{int} from the intervention:

$$\begin{aligned}\Delta W(t_{int}) = &\underbrace{\left[\gamma Sq - (1-\pi(t_{int}))\mu_H^1(t_{int}) - \delta(1-\pi(t_{int}+1))\mu_H^1(t_{int}+1)\right](\tau-1)d_H}_{\text{Short-Term Trading Gains}} \\ &- \underbrace{(S_H^G d_H + S_L^G d_L)(\tau - \tau^G)}_{\text{Holding Costs}} + \underbrace{\left[\mu_C^B(1) - \mu^B(t_{int}) + \delta(\mu_C^B(0) - \mu^B(t_{int}+1))\right]}_{\text{Trading Costs Difference}} \kappa \\ &+ \underbrace{\delta^2(W_{ss} - W_{cy}(1))}_{\text{Long-Run Gains}} \end{aligned} \tag{1.21}$$

The short term trading gains account for the increase in liquidity from 0 to $\pi(t_{int})$ in period t_{int}, but also its potential decrease from 1 to $\pi(t_{int}+1)$ in period $t_{int}+1$ when the government resells assets. The holding costs are negative because the government must hold asset he values less than private agents. The long-run gains are positive because surplus is higher in steady state (reached after 2 periods) than in the low date of the cycle. The purchase and resale prices P^G and R^G do not enter expression (1.21) since transfers are neutral with linear utility. Still, the level of

[24] A feasible policy and an induced equilibrium thus solve a fixed-point problem. Although this is a relevant concern, we cannot claim the policy uniquely implements this equilibrium.

the intervention price P^G matters as it induces a particular selection of applicants through the participation constraint.

1.4.2 Welfare Improving Policy: Numerical Analysis

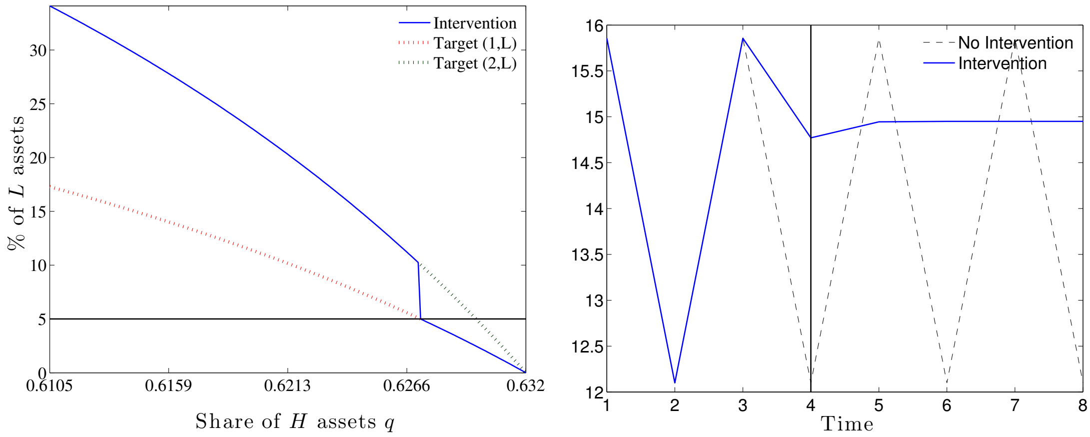
In this section, I use the numerical values reported in Table 1.1. Deep parameters are in bold characters. Although the 2 period cycle exists for less than 2.5% of the possible range of values of q, the features of the intervention may change significantly over this range as we show below. Appendix 1.7.3 describes the step to construct

$\boldsymbol{\delta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\tau}$	$\boldsymbol{d_L}$	$\boldsymbol{d_H}$	$\boldsymbol{\tau^G}$	$\underline{q}_2$	$\bar{q}_2$
0.7	0.05	3	1	4	0	0.610	0.632

Table 1.1: Benchmark parameter values

and rank feasible policies. Essentially, with discrete types of asset owner, the policy selection boils down to a discrete choice over these four types. The selection then reduces to a binary choice as a welfare maximizing policy should not target H asset owners. Intuitively, removing H from the market does not improve liquidity in the intervention period t_{int}.

Figure 1.4 illustrates the results. On the left panel, the red dotted line shows the intervention size targeting (τ^1, L) agents whereas the green dotted line is for (τ^2, L) agents. The blue line is the selection of the surplus maximizing intervention. The most efficient intervention targets (τ^1, L) agents. However, these agents only hold $\gamma = 5\%$ of the L assets - the horizontal line on the graph. For a larger intervention, the government needs to target (τ^2, L) owners. The pecking order from low to high valuation owners of lemons arises because buying L assets from (τ^1, L) is more efficient to jump-start the market. Indeed, it decreases both the cost from a pooling offer and the benefits from a separating offer for buyers. The latter effect is not present for (τ^2, L) agents. We see that the size of the intervention decreases with the share of H quality assets q. To understand this result, observe that the gap between the target supply of H quality assets $\mu^1_{H,ss}(q)$ imposed by the stabilization



(a) Size of the Intervention (% L assets) **(b)** OTC Market Price path ($q = 0.6213$)

Figure 1.4: Revertible Asset Purchase Program

constraint (SC) and that in the trough of a cycle $\mu^1_{cy,H}(q)$ is

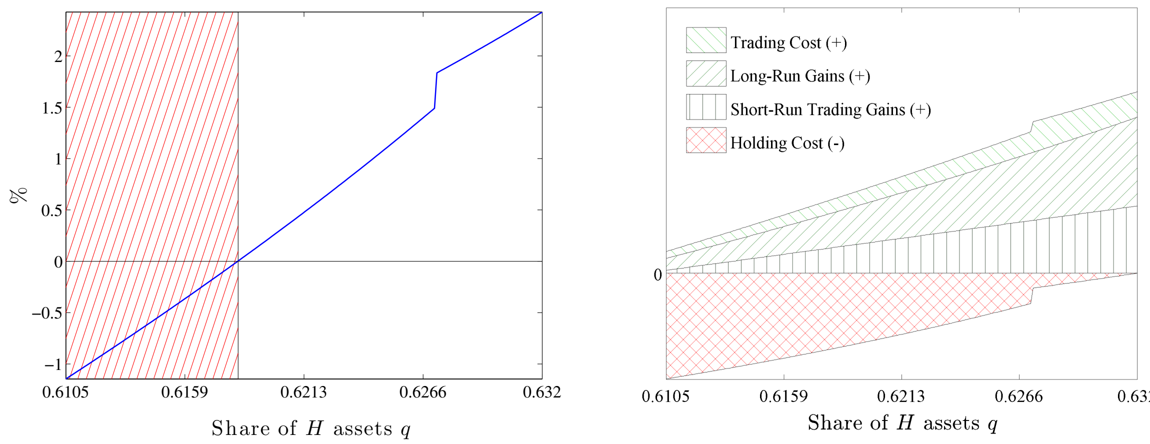
$$\mu^1_{H,ss}(q) - \mu^1_{cy,H}(q) = (1-2\gamma)\frac{\gamma S q(1-\pi_{ss}(q))}{2\gamma + \pi_{ss}(q)(1-2\gamma)} \xrightarrow[q \to \bar{q}]{} 0$$

An intervention targeting exclusively (τ^1, L) agents suffices to bridge this gap if q is high.

Figure 1.4b represents the average price path in the OTC market with the intervention (solid line) for the median value $q = 0.6213$. Before $t_{int} = 4$, it coincides with the no-intervention price[25] (dashed line). The asset purchase program smoothes out fluctuations very quickly. Indeed, the price in the intervention period nearly reaches the equilibrium steady state price of period $t_{int} + 2$. The picture looks similar for other values of q. The next Section shows how these results change with a budget-balanced intervention.

On figure 1.5, the left panel plots the net surplus gain ΔW_{int} whereas the right panel highlights the contribution of each component according to the decomposition of equation (1.21). Surplus gains increase in q together with the inverse of the program size. A large intervention might even be undesirable (the hatched area in figure 1.5a). Figure 1.5b shows indeed that for low values of q, government holding costs

[25] Announcing the policy in $t_{int} - 1 = 3$ would have no effect here. If anything, a pooling offer in that period then becomes even more profitable but the equilibrium would not change.



(a) Welfare Gains (%) **(b)** Contributions of Welfare Components

Figure 1.5: Efficiency properties

proportional to $(\tau - \tau^G)S^G$ dominate. As the intervention size decreases for higher values of q, the positive components take over. In particular, welfare improves with the long-term stabilization gains and the short-term trading gains.

1.4.3 Budget-Neutral and Pareto Improving Intervention

We find that the government earns a negative net return of -30% across values of q. Holding costs that are proportional to the difference in valuation $\tau - \tau^G = 3$ contribute significantly to this loss. However, even when $\tau^G = \tau = 3$, the average net return is still around -7%. Indeed, this number also reflects a premium the government pays to induce participation from asset owners. Suppose indeed that q is low (below the kink on Figure 1.4a). The government must attract (τ^2, L) owners and offer at least their market value of $P^G = v_L^2(t_{int})$. This price also compensates asset owners for the information rents earned in the OTC market while the government obtains L assets for sure. In comparison, the maximum price a buyer would pay in the OTC market for a L asset is $r_L^2(t_{int})$. Hence, the government pays a premium:

$$v_L^2(t_{int}) - r_L^2(t_{int}) = \pi(t_{int})(r_H^1 - r_L^2(t_{int})) > 0$$

Interestingly, the higher liquidity $\pi(t_{int})$ in the market following the purchase, the bigger this premium. The objective to increase liquidity thus raises the government losses. It is then natural to ask whether a budget-neutral policy is feasible. Indeed, a loss-making intervention does not constitute a Pareto improvement as private agents who would contribute those funds are worse-off despite surplus gains.

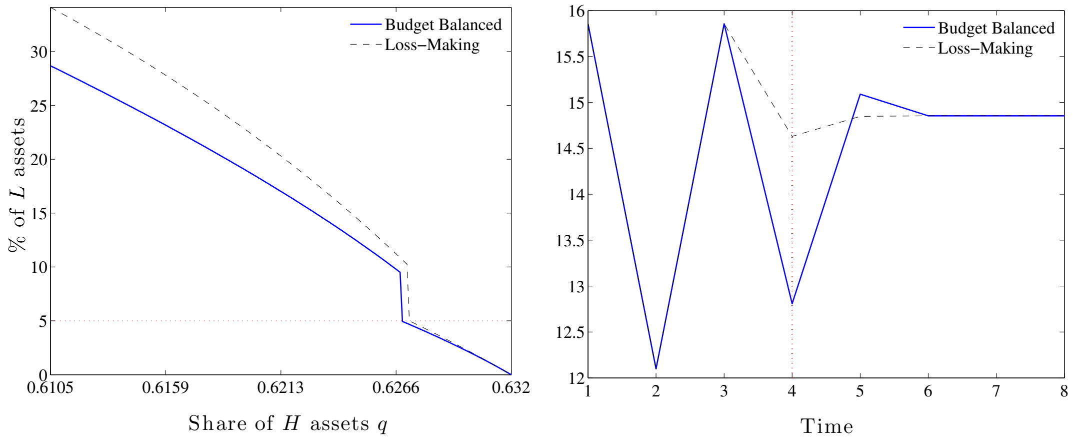
To finance the intervention, the government may now tax transactions in the OTC market of period $t_{int} + 1$. Every buyer who purchases an asset must pay $\sigma^G > 0$ units of the reference good c. The main point of this part of the analysis is to stress the interaction between designing and financing the policy. Indeed, a transaction tax distorts buyers' trade-off between a pooling and a separating offer. To see this, let us write the expression for the net gains from a pooling offer with tax σ^G adapting equation (1.14):

$$\begin{aligned} v^B(t,1) - v^B(t,0) = & \frac{\mu_H^1(t)}{S}(r_H^2 - r_H^1) - (1-q)(r_H^1 - r_L^2(t)) + \frac{\mu_L^1(t)}{S}(r_L^2(t) - r_L^1(t)) \\ & - \sigma^G \left[\frac{\mu_H^1(t) + \mu_L^2(t)}{S} \right] \end{aligned} \tag{1.22}$$

The first line is similar to equation (1.14) while the second line shows the effect of the tax. When offering a pooling price r_H^1, a buyer increases his trading probability and thus its tax payment proportionally to the mass of traders who only accept that offer, that is agents (τ^2, L) and (τ^1, H). Ceteris paribus, the tax thus lowers the benefits from a pooling offer. Hence, if the government were to set the tax naively so as to offset the loss, he would actually destroy liquidity in period $t_{int} + 1$. Hence, all the parameters $(S^G, P^G, R^G, \sigma^G)$ of the policy must now be determined jointly. The government payoff with tax verifies:

$$v^G := S^G(\delta R^G - P^G) + \sigma^G \big[\mu_L^1(t_{in}+1) + \pi(t_{int}+1)(\mu_H^1(t_{int}+1) + \mu_L^2(t_{int}+1))\big]$$

where in expression (GP), we used $\tau^G = 0$. The term between brackets is the volume of trade in period $t_{int} + 1$. A feasible budget-neutral policy is a policy feasible according to Definition 3 with the additional budget-neutrality constraint $v^G = 0$.



(a) Size of the Intervention **(b)** OTC Market Price path ($q = 0.62$)

Figure 1.6: Budget Balanced Program

Numerical Results

Across values of $q \in [0.6195, 0.6302]$ where the intervention increases aggregate welfare, an average 20% of surplus is lost because of the budget-balance constraint[26]. Figure 1.6 provides some intuition by comparing intervention size and price paths with the constraint (solid line) and without (dashed line). The intervention becomes smaller in order to reduce the holding costs. Interestingly, the asset purchase does not flatten fluctuations as much. Indeed, maintaining fluctuations between t_{int} and $t_{int}+1$ relax the budget-balance constraint in two ways. First, it allows for a high resale price R^G for the government assets. Second, increasing liquidity in period t_{int} with a large intervention would decrease the mass of (τ^1, H) asset owners in period $t_{int}+1$ through the composition effect. These asset owners precisely belong to the implicit tax base in period $t_{int}+1$ as show by equation (1.22). The budget balance condition thus creates a trade-off between jump-starting the market to increase surplus and riding the cycle to finance the intervention.

Asset Purchase vs. Subsidy

We finally compare[27] the asset purchase program to another feasible budget-neutral

[26]The range of values of q where the policy increases surplus is $q \geq 0.6195$ instead of $q \geq 0.6183$ previously,which is 6% smaller. Hence the effect appears larger on the intensive margin.

[27]In the US, during the recent financial crisis, the Public Private Investment Program for legacy Mortgage Backed Securities was a form of subsidy while the SBA 7(a) Securities Purchase Program

intervention: a subsidy χ^G in period t_{int}, also financed by a tax σ^G in period $t_{int}+1$. As we discussed, purchasing lemons modifies the composition of the pool of sellers since the relative probability to find a H asset in the OTC market is:

$$\frac{\mu_H^1(t_{int})}{\mu_H^1(t_{int}) + \mu_L^2(t_{int}) + \mu_L^1(t_{int})} = \frac{\gamma S q}{\gamma S q + S(1-q) - S^G}$$

The effect of the purchase size S^G increases in q. Instead, the subsidy increases the net gain from a pooling offer by:

$$\frac{\Delta v^B(t_{int})}{\partial \chi^G} = \frac{\mu_H^1(t_{int}) + \mu_L^2(t_{int})}{S} = \gamma q + (1-\gamma)(1-q)$$

where we used equation (1.22) because a subsidy is merely a negative tax. This time, the effect is larger when q is small. In addition, the government needs not hold asset in this case. Figure 1.7 plots the surplus gains for the subsidy and for the asset purchase program under various values of τ^G. Besides our benchmark $\tau^G = 0$, we also consider the case where the government has the same valuation for the asset as low valuation ($\tau^G = 1$) and high valuations ($\tau^G = \tau$) agents. The results show that for a given value of τ^G, the asset purchase becomes relatively more attractive as q increases, in line with our informal analysis. Second, the asset purchase program performs better for high values of τ^G since the costs from holding assets go down.

We have shown that a revertible asset purchase program can jump-start the market in the short-run and stabilize it in the long-run. However, large interventions entail important misallocation costs for the government and budget neutral policies have more limited effects. In the next section, we study the impact of a structural change to the OTC market on equilibrium liquidity.

1.5 Liquidity and Market Structure

In the wake of the financial crisis, regulators pointed at the very structure of OTC markets as a source of illiquidity and instability[28]. In my model, random search with

for small business loans was a direct asset purchase.

[28]On its website, the IMF referring to OTC markets explains that *"some types of market arrangements can very quickly become disorderly, dysfunctional, or otherwise unstable"*. Ongoing Dodd-Frank and EMIR reforms notably mandate central clearing of OTC instruments such as

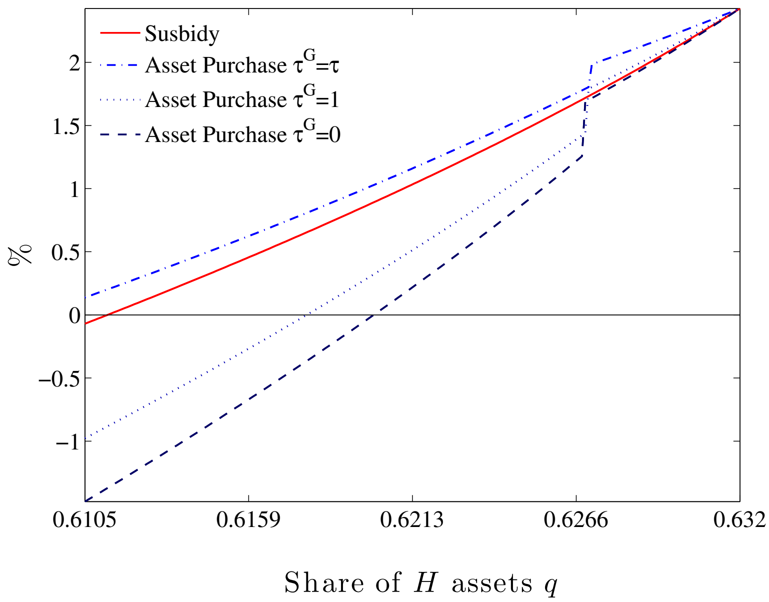


Figure 1.7: Welfare gains (%): Subsidy vs. Asset Purchase

ex-post bargaining generates several inefficiencies. First, buyers enter as long as they can make profit while fewer buyers could match all sellers. Resources are thus wasted on search costs. Second, with captive sellers, buyers reduce demand to extract rent, lowering trading volume and liquidity. In a centralized market place, a trader implicitly observes demand and supply from the rest of the market simultaneously and may signal his interest in trading. To capture these differences, I modify the model to allow buyers to post and commit to prices before meeting a counterparty while keeping the matching technology unchanged. Pre-trade transparency improves as sellers may pick their trading price from those posted by buyers. I show however that while the Exchange brings traders together more efficiently, liquidity shuts down for high quality assets.

1.5.1 Exchange

Formally, an exchange in period t is a continuum of markets $p \in \mathbb{R}^+$ where p is a price for the asset[29]. Each market p in period t is characterized by the ratio of

derivatives and swaps with a stated objective to increase transparency and competition.

[29]There is no loss in generality in assuming that buyers post prices and not contracts since rationing plays the same role as probabilities of trade. Implicitly, the OTC market has only one such sub-market where all asset owners and buyers must go.

buyers to sellers $\theta(t,p)$ and a belief vector $\{\gamma_a^i(t,p)\}_{a=L,H}^{i=1,2}$ about the share of each type of asset owner in market p. Each agent takes these quantities as given.

Matching

The bilateral matching technology is identical. Buyers and sellers choose the market they want to trade in, taking $\theta(t,p)$ as given. The probability for a seller (resp. a buyer) to meet a buyer (resp. a seller) in market p, in period t is:

$$\lambda^S(t,p) = \min\{\theta(t,p), 1\}, \quad \lambda^B(t,p) = \min\left\{\theta(t,p)^{-1}, 1\right\}$$

Hence, for a seller, $\theta(t,p)$ measures the extent of rationing in market p. Importantly, owners can not sell their asset in two different markets p and $\hat{p} \neq p$ in the same period. Hence, an attempt to sell at a price p is a commitment not to try and sell at a different price $p' < p$. This can act as a signal of quality if sellers expect rationing at high prices[30]. As in the OTC market though, exclusivity only restricts intra-period trades.

Sellers Problem

Asset owner (τ^i, a) chooses the market which maximizes his utility:

$$\begin{aligned} v_a^i(t) &= \max_{p\in\mathbb{R}^+} v_a^i(t,p) \\ \text{where} \quad v_a^i(t,p) &= \lambda^S(t,p)(p - r_a^i(t)) + \tau^i d_a + \delta\bar{v}_a^i(t+1) \end{aligned} \tag{1.23}$$

For an asset owner, $v_a^i(t,p)$ is the utility from trading in market p. Asset owners may always choose a very high price where $\theta(t,p) = 0$ if they do not want to trade. One can interpret a decision to sell at a high price with rationing as a limit order while a decision to sell at a low price without rationing would be a market order.

[30]Models of competitive adverse selection such as Gale (1996), (Guerrieri and Shimer, 2014), Chang (2014) also impose this exclusivity assumption. Kurlat (2015) allows for non-exclusivity in a static model and obtains pooling. For an analysis of non-exclusivity with a strategic equilibrium concept, see Biais et al. (2000) and Attar et al. (2011).

Buyers Problem

Let $\gamma_a^i(t,p) \in [0,1]$ denote buyers' belief about the share of type (τ^i, a) in market p in period t. The buyer's payoff from market p for non asset owner τ^2 writes:

$$v^B(t,p) = \lambda^B(t,p)\Big[\big(\gamma_H^2(t,p)+\gamma_H^1(t,p)\big)r_H^2(t)+\big(\gamma_L^2(t,p)+\gamma_L^1(t,p)\big)r_L^2(t)-p\Big]+\delta\bar{v}_0^2(t+1,p) \tag{1.24}$$

A buyer cares about the quality of the asset a not the type τ^i of the seller. We let $\mu^B(t,.)$ be the measure of buyers over markets $p \in \mathbb{R}_+$ with support $\mathcal{P}(t)$ and define:

$$\mu^B(t) = \int_{\mathcal{P}(t)} \mu^B(t,p)\mathrm{d}p$$

as the total mass of buyers.

Law of Motion

The law of motion for types (τ^1, a) writes:

$$\mu_a^1(t+1) = \Big[1-\gamma-(1-2\gamma)\int \gamma_a^1(t,p)\lambda^B(t,p)\mu^B(t,p)\mathrm{d}p\Big] + \gamma\mu_a^2(t) \tag{1.25}$$

The expression is similar to the one derived for OTC markets except that agents might visit different markets p with different trading probabilities[31].

Beliefs

On markets where trade takes place, beliefs $\{\gamma_a^i(t,p)\}_{a=L,H}^{i=1,2}$ shall reflect the distribution of sellers choosing this market. A complete description of the exchange requires buyers to form expectations about inactive markets $p \notin \mathcal{P}(t)$ as well. Many pessimistic equilibria can be sustained if buyers believe sellers would supply the L asset in inactive markets. We thus impose a refinement similar to Gale (1996) and Guerrieri et al. (2010). On inactive markets, buyers expect to see asset owners who find it most profitable to deviate to that market. This belief refinement formalized

[31] The formula above seems convoluted but economizes on notation as we do not need to introduce measure of sellers $\mu_a^i(t,p)$ in the market. We would have

$$\lambda^S(t,p)\mu_a^i(t,p) = \lambda^B(t,p)\mu^B(t,p)\gamma_a^i(t,p)$$

in Point 2 of Definition 4 essentially adapts Cho and Kreps (1987) to a competitive environment. We refer to the papers mentioned above for a more extensive discussion.

Definition 4 (Exchange Equilibrium). *An Equilibrium of the Exchange is given by value functions* $\{v_a^i(t)\}_{a=L,H}^{i=1,2}$*, distributions* $\{\mu_a^i(t)\}_{a=L,H}^{i=1,2}$ *a measure* $\mu^B(p,t)$ *with support* $\mathcal{P}(t)$ *and total mass* $\mu^B(t)$*, a rationing function* $\theta(t,p) : \mathbb{R}^+ \mapsto \mathbb{R}^+ \cup \{\infty\}$ *and belief function* $\gamma(t,p) : \mathbb{R}_+ \mapsto \Delta^4$ *for any* t *such that*

1. *Buyers optimality and free entry. For all* $p \in \mathbb{R}^+$, $\mathcal{P}(t) = \arg\max_p v^B(t,p) - \kappa$ *and* $\mu^B(t)$ *is determined by* (1.7)*.*
2. *Sellers optimality. For all* $p \in \mathbb{R}^+$*,* $i = 1,2$ *and* $a \in \{L,H\}$*,* $v_a^i(t) \geq v_a^i(t,p)$ *with equality if* $\theta(t,p) < \infty$ *and* $\gamma_a^i(t,p) > 0$*.*
3. *Market Clearing. For* $i = 1,2$ *and* $a = L,H$

$$\int_{\mathcal{P}(t)} \frac{\gamma_a^i(t,p)}{\theta(t,p)} \mu^B(t,p) \mathrm{d}p \leq \mu_a^i(t)$$

   with equality if $v_a^i(t) > \tau^i d_a + \delta \bar{v}_a^i(t+1)$

4. *Law of motion :* $\{\mu_a^i(t)\}_{a=L,H}^{i=1,2}$ *verify* (1.25) *and balance conditions* (1.1)*.*
5. ***Stationary Property*** *:* $\exists T \in \mathbb{N}_+$ *such that for all endogenous variables* z

$$z(t+T) = z(t)$$

The definition follows closely Guerrieri et al. (2010) and Guerrieri and Shimer (2014). Active markets $\mathcal{P}(t)$ are those buyers choose to visit. Point 2 formalizes the requirement that sellers choose the market(s) which maximizes their utility. In addition, on markets where $\theta(t,p) < \infty$, buyers should expect to see sellers who are indifferent between that market and their optimal choice. This is formally the refinement we discussed above. Point 3 ensures that supply on active markets is consistent with buyers beliefs. When one asset owners might find it optimal not to trade, that is $v_a^i(t) = \tau^i d_a + \delta \bar{v}_a^i(t+1)$, they can supply their asset on inactive markets where $\theta(t,p) = 0$. Finally, point 4 says that the mass of owners depends on past trading decisions since asset owners face different levels of rationing on each market. Point 5 is the stationary property already present in Definition 2 for an OTC equilibrium.

1.5.2 Equilibrium

As in the construction of the OTC equilibrium, an important statistic is the ordering of types of asset owners.

Lemma 5. *In any exchange equilibrium, for all t,*

$$r_L^1(t) \leq r_L^2(t) < r_H^1(t) \leq r_H^2(t) \tag{1.26}$$

I omit the proof which can be readily adapted from that of Lemma 1 and does not rely on the price formation process. The important information for our analysis is the monotone relationship between types: agents with a lower quality assets are more eager to sell, independently of their private valuation for the asset[32]. In an Exchange, it means that sellers with a higher type in the sense of Lemma 5 would accept (more) rationing to trade at higher prices and signal their quality. When type τ^2 agents hold L assets, which they do to realize gains from trade, Proposition 4 shows that this logic leading to separation has an extreme consequence as no market opens for H assets.

Proposition 4. *The unique exchange equilibrium is a steady state where*

i) Buyers make zero profit $\forall t$, $v_0^2 = 0$.

ii) Owners (τ^1, L) trade in the only open market $\mathcal{P} = \{p_L\}$ where

$$p_L := \frac{\tau d_L - \delta\gamma\kappa}{1-\delta} - \kappa$$

[32] Guerrieri and Shimer (2014) only consider private information about the asset dividend. In Guerrieri and Shimer (2015), the two dimensions actually collapse to one with a monotonicity assumption similar to my lemon condition (LC). Chang (2014) relaxes this assumption to obtain bunching in equilibrium. In my model, while the monotonicity condition follows closely from (LC), it is endogenous.

Other asset owners do not trade. Reservation values for L asset owners are

$$r_L^1 = d_L + \delta p_L + \delta\gamma\kappa$$
$$r_L^2 = p_L + \kappa$$

iii) The rationing function θ *satisfies* $\theta(p) = \frac{p_L - r_L^1}{p - r_L^1}$ *if* $p \in [p_L, p_L + \kappa]$ *and* $\theta(p) = 0$ *otherwise. The belief function is*

$$(\gamma_L^1, \gamma_H^1 \gamma_L^2, \gamma_H^2)(p) = \begin{cases} (1,0,0,0) & \text{if} \quad p \in [p_L, p_L + \kappa] \\ (0,0,1,0) & \text{if} \quad p > r_L^2 \end{cases}$$

iv) The masses of traders are

$$\begin{cases} \mu_L^1 = \gamma S(1-q) \\ \mu_L^2 = (1-\gamma)S(1-q) \end{cases} \qquad \begin{cases} \mu_H^1 = Sq/2 \\ \mu_H^2 = Sq/2 \end{cases}$$

while equilibrium entry $\mu^B(.,.)$ *is an atom of mass* μ_L^1 *at* p_L.

The last part of the result shows that the mass of buyers is equal to the mass of asset owners who sell in equilibrium, that is (τ^1, L) agents. Since asset owners can signal their willingness to trade, search from buyers is not random as in the OTC market. Competition between buyers drives equilibrium entry and search costs to the minimal level to support trade[33]. However, the existence of different prices with different level of rationing allows agents to signal the quality of their asset. The first consequence is that a pooling price p cannot be sustained as otherwise, H asset owners would want to deviate to a higher price $p' > p$. Higher rationing at that price, $\theta(., p') > \theta(., p)$ makes this signal credible for buyers who would then propose price p', a logic similar to the cream-skimming deviation in strategic models. The equilibrium is thus separating. In this environment, equilibrium rationing of (τ^1, H) owners is extreme since they do not trade: liquidity π is 0. Indeed, the monotonicity result $r_L^2 < r_H^1$ in Lemma 5 shows that (τ^2, L) agents would then like to trade in any

[33]The result that $\mu^B = \mu_L^1$ exactly comes from the matching function and the positive search costs $\kappa > 0$. If $\kappa = 0$, there could additional entry at no aggregate cost.

market chosen by type (τ^1, H) where $p > r_H^1$. These L asset owners with no gains from trade thus block trading for (τ^1, H) agents. As a result, misallocation is severe as half of the H assets are held by low valuation τ^1 agents. Selling pressure is large but does not lead to trade.

Both components of private information are important for the result. When buyers only ignore the quality of the asset proposed, a similar logic leads to separation between (τ^1, L) and (τ^1, H) agents but the latter trade with positive probability as long as $d_L > 0$. Otherwise, the so-called no gap at the bottom condition of the Akerlof (1970) model shut downs all trade. Asymmetry of information about valuation creates a middle type (τ^2, L) which prevents trade of the H asset. The lemon condition $\tau d_L < d_H$ is much weaker than $d_L = 0$ but also generates trading freeze of the H asset while the low quality market functions smoothly. In the OTC market as well, bi-dimensional private information reduces liquidity since buyers face disproportionately more L assets. However, ex-post offers protect buyers from cream-skimming deviations who may thus propose a pooling price if the share of H assets is high enough. Interestingly, the inter-temporal *competition effect* reinforces buyers' incentives to pool sellers in the OTC market and increases liquidity but intra-temporal competition reduces liquidity.

1.5.3 Welfare

We formalize the discussion above by comparing aggregate surplus across market structures. Let $W_E(q)$ be our measure of welfare in the unique stationary equilibrium of the exchange. In the OTC market, we focus on the steady state equilibrium and denote welfare by $W_{OTC}(q)$. This is the unique stationary equilibrium for $q \in [0, \underline{q}_2) \cup (\bar{q}, 1]$. In $q \in [\underline{q}_2, \bar{q}]$, our results would change quantitatively but not qualitatively when considering the cycle. The aggregate surplus gains from trading in an exchange rather than in the OTC market as a function of q are:

$$W_E(q) - W_{OTC}(q) = \frac{\mu^B(\pi(q)) - \gamma S(1-q)}{1-\delta}\kappa - \frac{Sq - 2\mu_H^1(\pi(q))(1-\pi(q))}{2(1-\delta)}(\tau - 1)d_H \tag{1.27}$$

The first term is positive and captures the gains from improving the meeting process with price posting. These gains have two sources. First, with random search, it

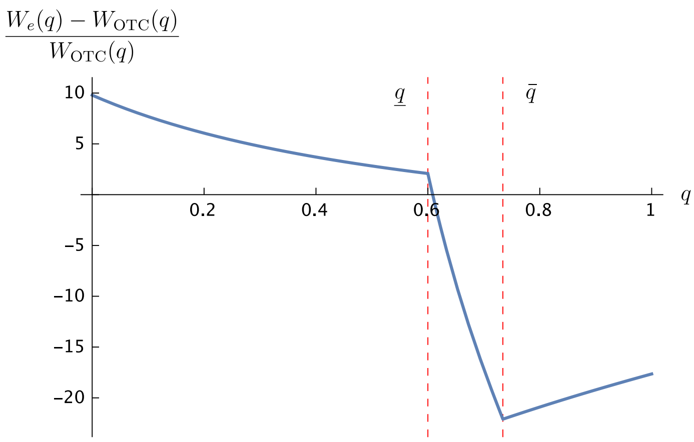


Figure 1.8: Welfare: Exchange vs. OTC (%)
Parameter Values: $\delta = 0.8$, $\gamma = 0.05$, $d_L = 2$, $d_H = 6$, $\tau = 2$

must be that at least S buyers enter to match all sellers. Second, OTC equilibrium entry $\mu^B(\pi(q))$ might even be higher because buyers do not compete in price. In the Exchange, the price adjusts so that entry matches exactly the mass of asset owners that are selling. The second term is negative and measures the difference in misallocation of the H asset. While half of the H assets are not properly allocated in the Exchange, this fraction falls to $\mu^1_H(\pi(q))(1 - \pi(q))/S$ in the OTC market since buyers may offer a pooling price ($\pi(q) > 0$). The following proposition derives the sign of expression (1.27) as a function of the share of H assets q.

Proposition 5. *There exists $q_W \in [\underline{q}, \bar{q}]$ such that $W_E(q) - W_{OTC}(q) \geq 0$ for $q \leq q_W$ and $W_E(q) - W_{OTC}(q) \leq 0$ otherwise.*

Proposition 5 states that when the share of H assets is above the threshold q_W aggregate surplus is higher in the OTC market. When $q \leq \underline{q}$, the OTC equilibrium is also separating ($\pi(q) = 0$) and only lemons are traded. Hence only the component related to trading costs plays a role and favors the Exchange. Pooling arises in the equilibrium of the OTC market when $q \geq \underline{q}$ and reduces the misallocation of H assets - the second term in (1.27). The threshold q_W is such that the realized gains from trade on the H assets overcome the difference in trading costs. Figure 1.8 plots the welfare difference of equation (1.27) in percentage as a function of q for specific numeric values of the other parameters. The dashed red lines delimit the region where the OTC steady state equilibrium is in mixed strategy, that is $q \in [\underline{q}, \bar{q}]$. In that region, the welfare difference becomes negative and decreases steeply because H assets start trading and the difference in trading costs per unit traded decrease with intertemporal competition in the OTC market. Observe that in the region where the OTC equilibrium is pooling in pure strategy, that is $[\bar{q}, 1]$, the welfare difference might increase. Although trade in the OTC market improves the allocation of the increasing mass of H quality assets, the lack of intertemporal competition for that asset also raises trading costs significantly. Still, the first effect dominates and in the limit $W_E(1) - W_{OTC}(1) < 0$.

Our analysis thus emphasizes the ambiguous role of pre-trade information for market efficiency in the presence of asymmetry of information. The centralization of the trading platform saves on trading costs as buyers do not have to search for potential sellers. Sellers in turn may observe all buyers' offer simultaneously rather than sequentially through search which reduces the inefficiencies due to monopoly pricing in the OTC market. However, competition lowers liquidity as sellers are now able to signal their type which hinders the correct allocation of high quality assets[34].

1.6 Conclusion

This paper presents a theory of endogenous liquidity fluctuations based on asymmetry of information and re-trade in secondary markets. I show that Over the Counter Market are prone to fluctuations where prices and trading volume vary in the absence of aggregate shocks. Equilibrium cycles are inefficient because of the dynamic externality attached to the composition effect. Hence although market conditions will eventually improve, it is desirable to bring liquidity forward in the short-run and stabilize the market in the long-run. I show that a revertible asset purchase program can achieve these objectives. However, our analysis highlights several limitations. First, the government asset purchase program interferes with the efficient allocation of assets in the economy. Indeed, private agents also value assets for their conve-

[34]This comparison leaves open the question of the endogenous choice of platforms by traders. Observe however that in our environment, (τ^1, H) sellers do not gain from trade. In the OTC market, the trading price is at most equal to their reservation value while they fail to trade in the Exchange. Hence pure OTC and pure Exchange market structures can be interpreted as a particular selection of equilibrium in the larger game where traders would choose their platform.

nience to realize transactions while the government does not. In addition, absent taxation, the government runs a loss because he must pay a premium to convince asset owners to part with their assets. The need to finance the intervention interacts with the design of the policy and mitigates its effect. The last part of the paper draws mixed conclusions about the current set of structural reforms of the OTC market. When buyers may post prices before meetings, matching is more efficient to bring buyers and sellers together but high quality assets become illiquid. Hence, the lack of competition and information in OTC markets can be desirable to foster liquidity.

This analysis still leaves many interesting questions open for policy design in asset markets with adverse selection. I imposed an immediate resale constraint to capture realistic constraint for policy makers in a tractable way. In practice, the timing and the pace of these exit strategies seems important. In addition, the supply of asset is fixed and constant in my model while it could react endogenously to liquidity in secondary markets. On the issue of competition and information, the comparison between OTC and Exchange highlights a negative role of transparency on liquidity. More work is needed to understand the coexistence of platforms with different degree of competition and opacity and its implication for efficiency.

1.7 Appendix

1.7.1 Assumption: Upper bound on search costs

I give the expression for the upper bound $\bar{\kappa}(\gamma, \delta, \tau, d_L, d_H)$ of the search cost κ in Assumption (A2). The condition ensure that buyers find it profitable to enter if they match for sure $\lambda^B = 1$ and face the least favorable prospects. Building on expression (1.14), we can provide a lower bound for the net gains from entering the market. Precisely

$$v^B \geq M(q) := \max \left\{ \gamma(1-q)(\tau-1)d_L, \gamma q \frac{\tau - 1}{1 - \delta(1-2\gamma)} d_H \right.$$
$$\left. -(1-q)\frac{(1-\delta)(d_H - \tau d_L) + \delta\gamma(\tau+1)(d_H - d_L}{(1-\delta)\big[1 - \delta(1-2\gamma)\big]} \right\}$$

Intuitively, the first (resp. second) argument is the minimum possible payoff from a separating (resp. pooling) offer. Hence we define $\bar{\kappa}(\gamma, \delta, \tau, d_L, d_H) := \min_q M(q)$. It is easy to check that this expression is strictly positive if $d_L > 0$ and $\gamma > 0$.

1.7.2 Proofs

Proof of Lemma 1

Part 1

We want to show that for any (τ^i, τ^j, a, a') such that $\tau^i d_a \geq \tau^j d_{a'}$, we have $r^i_a(t) \geq r^j_a(t)$. Anticipating equilibrium free entry condition, we know that non-owners make zero profit so that $r^i_a(t) = \tau^i d_a + \delta \bar{v}^i_a(t)$. Hence, we establish the following sufficient condition for the result: $\bar{v}^i_a(t) \geq \bar{v}^j_a(t)$. For this proof, let $\pi(t,p)$ be the probability that an owner receive offer p in period t. Since the matching technology is symmetric and buyers ignore seller's type, $\{\pi(t,p)\}_p$ is the same across asset owners. By optimality, agent (i,a) obtains a higher utility than if he behaves from t on-wards like agent (j,a') for $j \in \{i, -i\}$. Hence,

$$v^i_a(t) \geq r^i_a(t) + \lambda^S(t) \sum_{p \in \Gamma(t)} \pi(t,p) \alpha^j_{a'}(t,p)[p - r^i_a(t)]$$

where $\alpha^j_{a'}(t,p)$ is the acceptance probability of type (j,a'). Hence

$$v^i_a(t) - v^j_{a'}(t) \geq r^i_a(t) - r^j_{a'}(t) - \lambda^S(t) \sum_{p \in \Gamma(t)} \pi(t,p) \alpha^j_{a'}(t,p)[r^i_a(t) - r^j_{a'}(t)]$$

Using the expression for $r_a^i(t)$ and denoting $f_{a'}^j(t) = \lambda^S(t) \sum_{p\in\Gamma(t)} \pi(t,p)\alpha_{a'}^j(t,p) \leq 1$ we obtain

$$\begin{aligned} v_a^i(t) - v_{a'}^j(t) \geq &(1 - f_{a'}^j(t))(\tau^i d_a - \tau^j d_{a'}) + \delta(1 - f_{a'}^j(t))(\bar{v}_a^i(t) - \bar{v}_{a'}^j(t)) \\ =&(1 - f_{a'}^j(t))(\tau^i d_a - \tau^j d_{a'}) + \delta(1-\gamma)(1 - f_{a'}^j(t))(v_a^i(t+1) - v_{a'}^j(t+1)) \\ &+ \delta\gamma(1 - f_{a'}^j(t))(v_a^{-i}(t+1) - v_{a'}^{-j}(t+1)) \end{aligned}$$

Hence we can rewrite the following expression as

$$\begin{bmatrix} v_L^2(t) - v_L^1(t) \\ v_H^1(t) - v_L^2(t) \\ v_H^2(t) - v_H^1(t) \end{bmatrix} \geq \begin{bmatrix} (\tau - 1)d_L \\ d_H - \tau d_L \\ (\tau-1)d_H \end{bmatrix} + \delta M(t) \begin{bmatrix} v_L^2(t+1) - v_L^1(t+1) \\ v_H^1(t+1) - v_L^2(t+1) \\ v_H^2(t+1) - v_H^1(t+1) \end{bmatrix} \tag{1.28}$$

where

$$M(t) = \begin{bmatrix} (1 - f_L^1(t))(1-2\gamma) & 0 & 0 \\ \gamma(1 - f_L^2(t)) & 1 - f_L^2(t) & \gamma(1 - f_L^2(t)) \\ 0 & 0 & (1 - f_H^1(t))(1-2\gamma) \end{bmatrix}$$

Iterating on the inequality above and using the transversality condition $\lim_{t\to\infty} \delta^t v_a^i(t) = 0$, it follows that the left hand side of (1.28) is positive. It is then straightforward to show that the result extends to reservation values.

Part 2

From subgame perfection of the bargaining game, seller (τ^i, a) strategy is simply $\alpha_a^i(t,p) = \mathbf{1}_{p \geq r_a^i(t)}$ for $i = 1,2$ and $a = L, H$. A seller accepts any offer weakly above his reservation value. It follows immediately that a buyer may only offer one of these reservation values. To characterize the buyers' offer, let us rewrite the buyer's problem (1.3) as

$$Supp\left(\Pi^k(t,.)\right) = \arg\max_p \left\{ \sum_{\substack{i=1,2 \\ a=L,H}} \frac{\mu_a^i(t)}{S} \alpha_a^i(t,p)\big(r_a^k(t,p) - p\big) \right\} + \delta \bar{v}_0^k(t+1)$$

Consider first an agent $(\tau^1, 0)$. Since $r_a^2(t) - r_a^1(t) \geq 0$ for $a = L, H$, this agent weakly prefers not to make an offer. This means that not participating in the market is a strictly dominant strategy since searching costs $\kappa > 0$. Let us turn now to type $(\tau^2, 0)$ agents. Using law of motion (1.13)-(1.12), observe first that $\mu_a^1(t) > 0$ for $a = L, H$. In any period, there is a strictly positive mass of each type of asset owners due to the type switching process. We argue now that offer $r_a^2(t)$ is strictly dominated by offer $r_a^1(t)$. Indeed, by lowering his offer, the buyer makes a strictly greater profit on all types (τ^i, a') for which $r_{a'}^i(t) \leq r_a^1(t)$. In addition, while the lower offer fails to attract (τ^2, a) agents anymore, the buyer was breaking even on this group with the higher offer. Hence the only possible offers are $r_L^1(t)$ and $r_H^1(t)$.

Proof of Lemma 2

Point i). Since agent (τ^2, H) never trades and agent (τ^1, H) only receives offer (weakly) below his reservation value. By stationarity, we can drop the time argument in dynamic equation (1.8) to obtain the following system

$$r_H^1 = d_H + \delta\big[(1-\gamma)r_H^1 + \gamma r_H^2(t+1)\big]$$
$$r_H^2 = d_H + \delta\big[(1-\gamma)r_H^2 + \gamma r_H^1(t+1)\big]$$

Straightforward manipulations give:

$$r_H^i = \frac{(1-\delta)\tau^i + \delta\gamma(\tau+1)}{(1-\delta)\big[1-\delta(1-2\gamma)\big]} d_H, \quad i = 1, 2$$

Point ii). From the free entry condition (1.7) and dynamic equations (1.10)-(1.11) we obtain for $i = 1, 2$

$$v_0^i(t) = \delta \bar{v}_0^i(t+1)$$

By stationary property 5 of Definition 2 , it is immediate that for all t, and $i \in \{1, 2\}$, $v_0^i(t) = 0$.

Point iii). We want to show that in equilibrium $\mu^B(t) \geq S$. Suppose that the opposite inequality holds. Then the net profit from searching for a type τ^2 agent is $-\kappa + v^B(t)$ since the matching probability is 1 for a buyer. If this expression is strictly positive, buyers would enter and since there is no rivalry in the matching technology as long as $\mu^B(t) \leq S$, entry would be $\mu^B(t) \geq S$, proving the conjecture wrong. Using expression (1.6), a lower bound on $v^B(t)$ is

$$\begin{aligned} v^B(t) \geq \max \Bigg\{ &\gamma(1-q)(\tau-1)d_L, \gamma q \frac{\tau-1}{1-\delta(1-2\gamma)} d_H \\ &-(1-q)\frac{(1-\delta)(d_H - \tau d_L) + \delta\gamma(\tau+1)(d_H - d_L}{(1-\delta)\big[1-\delta(1-2\gamma)\big]} \Bigg\} := M(q) \end{aligned}$$

Our assumption that $\kappa \geq \min_q M(q) = \bar{\kappa}(\gamma, \delta, \tau, d_L, d_H)$ ensures that $-\kappa + v^B(t) > 0$ so that in equilibrium $\mu^B(t) \geq S$, that is $\lambda^S(t) = 1$.

Proof of Lemma 3

Observe first that $\pi(t)$ weakly increases when $v^B(t, 1) - v^B(t, 0)$ increases.

Competition effect

From equation (1.14), $v^B(t, 1) - v^B(t, 0)$ increases with $\{r_L^i(t)\}^{i=1,2}$. For $i = 1, 2$, $r_L^i(t) = \tau^i d_L + \delta \bar{v}_L^i(t+1)$ where $\bar{v}_L^i(t+1) = (1-\gamma)v_L^i(t+1) + \gamma v_L^j(t+1)$ for $j \neq i$. From dynamic equations (1.9), we have

$$v_L^i(t+1) = \pi(t+1)r_H^1 + (1 - \pi(t+1))\big[\tau^i d_L + \bar{v}^i(t+2)\big]$$

Since $r^1_H \geq r^i_L(t+1) := \tau^i d_L + \bar{v}^i(t+2)$ for $i = 1, 2$ by Lemma 1, the result follows.

Composition effect

From equation (1.14), $v^B(t,1) - v^B(t,0)$ increases with $\mu^1_H(t)$. From law of motion (1.12), we have

$$\mu^1_H(t) = \gamma Sq + (1-2\gamma)(1-\pi(t-1))\mu^1_H(t-1)$$

Since $\gamma < 1/2$, $1 - 2\gamma > 0$ and thus $\mu^1_H(t)$ decreases with $\pi(t-1)$.

Proof of Lemma 4

$\triangleright$ The proof is by contradiction. Let t_0 be such that $\pi(t_0) = 1$. Suppose then that $\pi(t_0+1) = 1$. We want to establish that $\pi(t_0+2) = 1$. Then by induction, it means that for all $t \geq t_0$, $\pi(t) = 1$, a contradiction with having a cycle of period $T \geq 2$.

Using law of motion (1.12), we have that $\mu^1_H(t_0+1) = \gamma Sq$, that is the distribution of assets is least favorable to a pooling offer in period t_0+1. Since $\pi(t_0+1) = 1$, we also have $\mu^1_H(t_0+2) = \gamma Sq$. Suppose now $\pi(t_0+2) = 0$. Then , the sequence of offers $\{\pi(t)\}_{t>t_0+1}$ is weakly less favorable to a pooling offer than the sequence $\{\pi(t)\}_{t>t_0+2}$ since it starts with 0. Hence, using Lemma 3, since $\pi(t)$ is weakly increasing in $\{\pi(t+l\}_{l=1,..,\infty}$, if $\pi(t_0+1) = 1$, it must be that $\pi(t_0+2) = 1$, a contradiction. $\triangleleft$

Proof of Proposition 1

In the following, we characterize a T period cycle and then derive conditions for it to be an equilibrium. Let us first write down the endogenous variables in the conjectured equilibrium. Using the results in the main text, we have

$$\mu^1_{H,T}(t) = \frac{1-(1-2\gamma)^t}{2} Sq, \qquad t = 1, .., T$$

Reservation values $\{r^i_{L,T}(.)\}^{i=1,2}$ verify the following equations:

$$r^i_{L,T}(t) = \begin{cases} \tau^i d_L + \delta\big[(1-\gamma) r^i_{L,T}(t+1) + \gamma r^j_{L,T}(t+1)\big] & \text{if} \quad t = 0, ..., T-2 \\ \tau^i d_L + \delta r^1_H & \text{if} \quad t = T-1 \end{cases}$$

where $j \neq i$. We obtain for $t = 0, .., T-1$

$$r^1_{L,T}(t) + r^2_{L,T}(t) = \frac{1-\delta^{T-t}}{1-\delta}(\tau+1)d_L + 2\delta^{T-t} r^1_H$$
$$r^2_{L,T}(t) - r^1_{L,T}(t) = \frac{1-\big[\delta(1-2\gamma)\big]^{T-t}}{1-\delta(1-2\gamma)}(\tau-1)d_L$$

from which we get for $t = 0, .., T-1$

$$r^1_{L,T}(t) = \left[\frac{1-\delta^{T-t}}{1-\delta}(\tau+1) - \frac{1-(\delta(1-2\gamma))^{T-t}}{1-\delta(1-2\gamma)}(\tau-1)\right]\frac{d_L}{2} + \delta^{T-t}r^1_H$$
$$r^2_{L,T}(t) = \left[\frac{1-\delta^{T-t}}{1-\delta}(\tau+1) + \frac{1-(\delta(1-2\gamma))^{T-t}}{1-(\delta(1-2\gamma))}(\tau-1)\right]\frac{d_L}{2} + \delta^{T-t}r^1_H$$

Since for all t and $i = 1, 2$, $r^i_{L,T}(t) \leq r^1_H$, we obtain by backward induction that

$$r^i_{L,T}(t) \leq r^i_{L,T}(t+1), \qquad t = 0, .., T-2$$

The net gain from a pooling offer writes

$$v^B_T(t,1) - v^B_T(t,0) = \mu^1_H(t)(r^2_H - r^1_H) - S(1-q)\big[r^1_H - (1-\gamma)r^2_{L,T}(t) - \gamma r^1_{L,T}(t)\big]$$

The conjecture is an equilibrium if this expression is strictly negative in periods $t = 1, .., T-1$ and strictly positive in period 0. From the analysis above, this expression is increasing over $[|1, T-1|]$. Hence, we need only to verify that $v^B_T(0,1) - v^B_T(0,0) > 0$ and $v^B_T(T-1,1) - v^B_T(T-1,0) < 0$. These conditions are respectively equivalent to

$$q \geq \underline{q}_T := \frac{2(r^1_H - r^2_{L,T}(0)) + 2\gamma\big(r^2_{L,T}(0) - r^1_{L,T}(0)\big)}{\big[1-(1-2\gamma)^T\big](r^2_H - r^1_H) + 2(r^1_H - r^2_{L,T}(0)) + 2\gamma\big(r^2_{L,T}(0) - r^1_{L,T}(0)\big)}$$
$$q \leq \bar{q}_T := \frac{2(r^1_H - r^2_{L,T}(T-1)) + 2\gamma\big(r^2_{L,T}(T-1) - r^1_{L,T}(T-1)\big)}{\big[1-(1-2\gamma)^{T-1}\big](r^2_H - r^1_H) + 2(r^1_H - r^2_L(T-1)) + 2\gamma\big(r^2_{L,T}(T-1) - r^1_{L,T}(T-1)\big)}$$

Hence the T periods cycle exists if and only if $\underline{q}_T \leq \bar{q}_T$ that is

$$\frac{1-(1-2\gamma)^T}{1-(1-2\gamma)^{T-1}} \geq \frac{r^1_H - r^2_{L,T}(0) + \gamma\big(r^2_{L,T}(0) - r^1_{L,T}(0)\big)}{(1-\delta)r^1_H - \tau d_L + \gamma(\tau-1)d_L} \qquad (E_T)$$

The LHS decreases with T. On the RHS, the denominator does not depend on T while the numerator is equal to $r^1_H - (1-\gamma)r^2_{L,T}(0) - r^1_{L,T}(0)$ and increases in T. This proves that for $T' \geq T$ $(E_{T'}) \Rightarrow (E_T)$

It is clear that $\bar{q}_T$ only depends on T through the first term of the denominator $1-(1-2\gamma)^{T-1}$. It is then immediate that the sequence $\{\bar{q}_T\}$ is decreasing in T. Cumbersome but straightforward computations show that $\bar{q}_2 = \bar{q}$ where $\bar{q}$ is defined in Proposition 2. Finally, we are left to show that $\bar{q}_{T+1} \leq \underline{q}_T$ for $T \geq 2$. From, the expression above, this is true if $r^i_{L,T+1}(T) \geq r^i_{L,T}(0)$ for $i = 1, 2$. By definition, we have $r^i_{L,T+1}(T) = r^i_{L,T}(T-1)$ and the result follows from the monotonicity of $r^i_{L,T}$.

Proof of Proposition 2

To prove Proposition 1, we proceed as follows. First, we write all endogenous variables as a function of π. Then we solve for a fixed point equation in π.

Step 1

Using law of motion (1.12), we obtain

$$\begin{aligned}\mu_H^1(\pi) &= \gamma S q + (1-2\gamma)(1-\pi)\mu_H^1(\pi)\\ &= \frac{(1-\gamma)\pi + (1-\pi)\gamma}{\pi + 2\gamma(1-\pi)} S q\end{aligned}$$

We determine the reservation values for L asset owners $(r_L^1(\pi), r_L^2(\pi))$ which solve

$$\begin{aligned}r_L^1(\pi) &= d_L + \delta\Big(\pi r_H^1 + (1-\pi)\big[(1-\gamma)r_L^1(\pi) + \gamma r_L^2(\pi)\big]\Big)\\ r_L^2(\pi) &= \tau d_L + \delta\Big(\pi r_H^1 + (1-\pi)\big[(1-\gamma)r_L^2(\pi) + \gamma r_L^1(\pi)\big]\Big)\end{aligned}$$

Hence

$$\begin{aligned}r_L^2(\pi) + r_L^1(\pi) &= (\tau+1)d_L + 2\delta\pi r_H^1 + \delta(1-\pi)\big[r_L^2(\pi) + r_L^1(\pi)\big]\\ r_L^2(\pi) - r_L^1(\pi) &= (\tau-1)d_L + \delta(1-\pi)(1-2\gamma)(r_L^2(\pi) - r_L^1(\pi))\end{aligned}$$

From which we obtain for $i = 1, 2$

$$r_L^i(\pi) = \tau^i d_L + \frac{1}{1-(1-\pi)\delta}\left[\pi\delta r_H^1 + (1-\pi)\delta\left(\tau^i d_L + \gamma\frac{\tau^j - \tau^i}{1-\delta(1-\pi)(1-2\gamma)} d_L\right)\right]$$

Step 2

Using the buyer's problem (1.14), with a slight abuse of notation, let us write $v^B(\pi, \hat{\pi})$ where the first argument is the strategy played by other buyers and the second argument is the strategy of an individual buyer. An equilibrium π must verify

$$\pi = \begin{cases} 0 & \text{if } v^B(0,0) \geq v^B(0,1)\\ \in (0,1) & \text{if } v^B(\pi,0) = v^B(\pi,1)\\ 1 & \text{if } v^B(1,1) \geq v^B(1,0)\end{cases} \tag{1.29}$$

We can first characterize too cutoffs $(\underline{q}, \bar{q})$ for the existence of the pure strategy equilibria $\pi^* = 0$ and $\pi^* = 1$. Plugging the expressions obtained above, we have

$$\begin{aligned}v^B(0,1) - v^B(0,0) = &\frac{q}{2}\frac{\tau-1}{1-\delta(1-2\gamma)} d_H - (1-q)\frac{(1-\delta)(d_H - v d_L) + \delta\gamma(v+1)(d_H - d_L)}{(1-\delta)(1-\delta(1-2\gamma))}\\ &-\gamma(1-q)\frac{\tau-1}{1-\delta(1-2\gamma)} d_L\end{aligned}$$

and thus $v^B(0,1) - v^B(0,0) \leq 0$ if and only if

$$\begin{aligned}&q(1-\delta)(\tau-1)d_H - 2(1-q)(1-\delta)(d_H - \tau d_L) - 2(1-q)\delta\gamma(\tau+1)(d_H - d_L)\\&- 2\gamma(1-q)(1-\delta)(\tau-1)d_L \leq 0\end{aligned}$$

We thus obtain the threshold $\underline{q}$ introduced in the main text:

$$\begin{aligned}\underline{q} &= \frac{2a}{(1-\delta)(\tau-1)d_H + 2a}\\ a &= (1-\delta)(d_H - \tau d_L) + \gamma\delta(\tau+1)(d_H - d_L) + \gamma(1-\delta)(\tau-1)d_L\end{aligned}$$

Similarly, we have

$$v^B(1,1) - v^B(1,0) = \gamma q \frac{\tau-1}{1-\delta(1-2\gamma)} d_H - (1-q)\left[\frac{1-\delta+\delta\gamma(\tau+1)}{1-\delta(1-2\gamma)} d_H - \tau d_L\right] - \gamma(1-q)(\tau-1)d_L$$

and thus $v^B(1,1) \geq v^B(1,0)$ if and only if

$$\begin{aligned}&\gamma q(\tau-1)d_H - (1-q)(1-\delta)d_H - (1-q)\delta\gamma(\tau+1)d_H + (1-q)(1-\delta(1-2\gamma))\tau d_L\\&- \gamma(1-q)(1-\delta(1-2\gamma))(\tau-1)d_L \geq 0\end{aligned}$$

We obtain

$$\bar{q} = \frac{a-b}{\gamma(\tau-1)d_H + a - b}, \quad \text{where} \quad b = \delta\gamma(1-2\gamma)(\tau-1)d_L$$

To derive mixed strategy equilibria, let us focus on the case $\underline{q} < \bar{q}$ (a similar argument applies when the multiplicity condition holds). Then for $q \in (\underline{q}, \bar{q})$, we have $v^B(0,1) - v^B(0,0) > 0$ and $v^B(1,1) - v^B(1,0) < 0$. Hence by continuity of $v^B(.,1) - v^B(.,0)$ in its first argument, there exists $\pi(q) \in (0,1)$ such that $v^B(\pi(q),0) = v^B(\pi(q),1)$. We are left to prove uniqueness of this mixed strategy equilibrium. We have

$$\begin{aligned}v^B(\pi,1) - v^B(\pi,0) =& \frac{\gamma S q}{2\gamma + \pi(1-2\gamma)}(r_H^2 - r_H^1) - S(1-q)\frac{(1-\delta)r_H^1 - (\gamma + (1-\gamma)\tau)d_L}{1-(1-\pi)\delta}\\&- S(1-q)\frac{(1-\pi)(1-2\gamma)(\tau-1)\delta\gamma d_L}{\left[1-(1-\pi)\delta\right]\left[1-\delta(1-\pi)(1-2\gamma)\right]}\end{aligned}$$

The expression above shows that the zeros of $v^B(.,1) - v^B(.,0)$ are solutions to a second order equation in π which may have at most 2 real roots. Hence, the second root cannot belong to $(0,1)$ as otherwise the expression would need to change sign twice on $(0,1)$ and have a third root. This concludes the proof.

Proof of Proposition 3

Using equation (1.19), we obtain the following expression for steady state welfare:

$$W_{ss}(q) = \frac{S(1-q)\tau d_L + Sq\tau d_H}{1-\delta} - \frac{(1-\pi(q))\mu_H^1(\pi(q))(\tau-1)d_H + \mu_{ss}^B\kappa}{1-\delta}$$

For the high date of the cycle we obtain:

$$W_{cy}(0,q) = \frac{S(1-q)\tau d_L + Sq\tau d_H}{1-\delta} - \frac{\delta\gamma Sq(\tau-1)d_H + \mu_{cy}^B(0)\kappa + \delta\mu_{cy}^B(1)\kappa}{1-\delta^2}$$

Since buyers make zero profit in equilibrium, trade costs cover the gains from trade that is $\mu^B(t)\kappa = Sv^B(t)$ using equation (1.7). in equilibrium. In steady state, the trade costs are thus equal to $\mu_{ss}^B\kappa = \gamma S(1-q)(r_L^2(\pi(q)) - r_L^1(\pi(q))$. In a cycle, trade costs are respectively $\mu_{cy}^B(1)\kappa = \gamma S(1-q)(\tau-1)d_L$ in the trough and $\mu_{cy}^B(0)\kappa = \mu_{H,C}^1(0)(r_H^2 - r_H^1) + S(1-q)(r_{L,cy}^2(0) - r_H^1) > \mu_{cy}^B(1)\kappa$ at the peak. Hence we obtain

$$W_{ss,q} - W_{cy}(0,q) > \frac{1}{1-\delta}\left[\left(\frac{\delta}{1+\delta} - \frac{1-\pi(q)}{2\gamma + \pi(q)(1-2\gamma)}\right)\gamma Sq(\tau-1)d_H + (\mu_{cy}^B(1) - \mu_{ss}^B)\kappa\right]$$

As $q \to \bar{q}$, $\pi(q) \to 1$. Hence $\mu_{ss}^B \to \mu_{cy}^B(1)$. The second term in the brackets thus converges to 0. The first term however is bounded away from 0 as q converges towards $\bar{q}$. This proves the result.

Proof of Proposition 4

The proof is in two steps. First, I show that there cannot be a market where $\theta(t,p) > 0$ and $\gamma_H^i(t,p) > 0$. Finally, I show that the allocation in Proposition 4 is the only equilibrium possible.

Step 1

Observe first that in a stationary equilibrium, it must be that $\mu_L^2(t) > 0$ since agents have a positive probability to switch type The argument is by contradiction. Observe first that $\max \mathcal{P}(t) < r_H^2(t)$. Indeed, the maximum price buyers will pay for an asset is $r_H^2(t) - \kappa$, that is the value of a H asset minus the search cost. Hence, (τ^2, H) will never sell their asset. Practically, they choose a market $p > r_H^2(t)$ where $\theta(t,p) = 0$. Define now

$$\mathcal{P}_H(t) = \{p \in \mathcal{P}(t) \mid \gamma_H^1(t,p) > 0\}$$

The set $\mathcal{P}_H(t)$ is the set of active markets where H assets are for sale. We want to show that $\mathcal{P}_H(t) = \emptyset$. By (τ^1, H) sellers optimality condition, we have $\min \mathcal{P}_H(t) \geq r_H^1(t)$. Let $\bar{p}_H(t) = \max \mathcal{P}_H(t) = \max \mathcal{P}(t)$.

Suppose first that $\gamma_H^1(t,p) = 1$. It must be that (τ^2, L) sellers are at most indifferent about trading at that price. Let thus be

$$\bar{p}_L^2(t) = \max\{p \in \mathcal{P}(t) \mid \gamma^2(t,p) > 0\} < \bar{p}_H(t)$$

Market $\bar{p}_L^2(t)$ is the maximum price at which agents (τ^2, L) trade. For $\bar{p}_L^2(t)$ to be optimal, it must be that:

$$\lambda^S(t, \bar{p}_L^2(t))(\bar{p}_L^2(t) - r_L^2(t)) \geq \lambda^S(t, \bar{p}_H(t))(\bar{p}_H(t) - r_L^2(t))$$

In particular, we must have $\bar{p}_L^2(t) > r_L^2(t)$. Since $r_L^2(t) - \kappa$ is the maximum price a L asset can command, it must be that $\gamma_H^1(t,p') > 0$. By seller's optimality, agents (τ^1, H) must weakly prefer market $p_L^2(t)$ to any market $p' > \bar{p}_L^2(t)$, that is

$$\lambda^S(t, \bar{p}_L^2(t))(\bar{p}_L^2(t) - r_H^1(t)) \geq \lambda^S(t,p')(p' - r_H^1(t))$$

This is only possible if $\lambda^S(t,p') < \lambda^S(t, \bar{p}_L^2(t)) \leq 1$. Then since $r_L^2(t) < r_H^1(t)$, agents (τ^2, L) strictly prefer market $\bar{p}_L^2(t)$ over p'. Hence, using Part 2 of the Equilibrium definition, $\gamma_H^1(t,p') = 1$. Let us now write buyers profit in market $\bar{p}_L^2(t)$ and $p' > \bar{p}_L^2(t)$

$$\begin{aligned} v_0^2(t, \bar{p}_L^2(t)) &= -\kappa + \lambda^B(t, \bar{p}_L^2(t))\left[(\gamma_L^1(t) + \gamma_L^2(t,p))r_L^2(t) + \gamma_H^1(t,p)r_H^2(t) - \bar{p}_L^2(t)\right] \\ v_0^2(t,p') &= -\kappa + r_H^2(t) - p' \end{aligned}$$

Hence,

$$\lim_{p' \to \bar{p}_L^2(t)} v_0^2(t,p') > v_0^2(t, \bar{p}_L^2(t))$$

which is incompatible with buyers' optimality condition.

But if we suppose now that $\gamma_H^1(t, \bar{p}_H) < 1$, the same argument applies. Hence, we have shown that $\mathcal{P}_H(t) = \emptyset$.

Step 2

We have shown that only (τ^1, L) asset owners might trade. To conclude the proof we must derive the equilibrium price $p_L(t)$ for trade as well as equilibrium entry from non-owners. With free entry, buyers make zero profit so that $v_0^2(t, p_L) = 0$. This implies that

$$p_L(t) = \tau d_L + \delta \bar{v}_L^2(t+1) - \kappa$$

Asset owners (τ^1, L) find a match for sure while (τ^2, L) asset owners do not trade, so that

$$\begin{aligned} v_L^1(t) &= p_L(t) \\ v_L^2(t) &= \tau d_L + \delta v_L^2(t+1) \end{aligned}$$

Using these equations together with the stationary condition of Definition 4, we obtain that (p_L, v_L^2)

are constant over time and equal to

$$v_L^2 = \frac{\tau - \delta\gamma\kappa}{1-\delta} d_L$$
$$p_L = v_L^2 - \kappa$$

Finally, non-owners make zero profit upon entering if and only if $\lambda^B(t) = 1$. Hence $\mu^B(t) = \mu_L^1(t) = \gamma S(1-q)$.

Proof of Proposition 5

Consider first the case where $q \in [0, \underline{q}]$. In this region, expression (1.27) become:

$$W_E(q) - W_{OTC}(q) = \frac{\gamma S(1-q)}{1-\delta}\left[\frac{\tau - 1}{1-\delta(1-2\gamma)} - \kappa\right]$$

which is positive and decreasing in q. When $q \in [\underline{q}, \bar{q}]$, we obtain

$$\begin{aligned} W_E(q) - W_{OTC}(q) =& \frac{\gamma S(1-q)}{1-\delta}\left[\frac{\tau - 1}{1-\delta(1-\pi(q))(1-2\gamma)} - \kappa\right] \\ & - \frac{Sq\pi(q)}{(1-\delta)\big[2\gamma + \pi(q)(1-2\gamma)\big]}(\tau-1)d_H \end{aligned}$$

We have that $\pi(.)$ is strictly increasing in q over $q \in [\underline{q}, \bar{q}]$. Hence both terms of the expression above are increasing in q. We establish now that that this expression is negative when evaluated in $\bar{q}$

$$W_E(\bar{q}) - W_{OTC}(\bar{q}) = \frac{\gamma S(1-\bar{q})}{1-\delta}\big[(\tau-1)d_L - \kappa\big] - \frac{S\bar{q}}{2(1-\delta)}(\tau-1)d_H$$

It is sufficient to establish that $2\gamma(1-\bar{q})d_L - \bar{q}d_H \leq 0$. Using the expression derived in the proof of Proposition 2, straightforward computations show that this is the case. Finally, on the interval $[\bar{q}, 1]$, we have

$$\begin{aligned} W_E(q) - W_{OTC}(q) =& -\frac{Sq}{2}(\tau-1)\frac{(1-\delta)(1-2\gamma)}{(1-\delta)\big[1-\delta(1-2\gamma)\big]}d_H \\ & - S(1-q)\left[\frac{\kappa - \tau d_L}{1-\delta} + \frac{1-\delta+\delta\gamma(\tau+1)}{(1-\delta)\big[1-\delta(1-2\gamma)\big]}d_H\right] \end{aligned}$$

It is immediate to see that this expression is negative in $q = 1$. Since it is also linear in q, it is negative over $[\bar{q}, 1]$.

1.7.3 Numerical Exercise

We describe the method used to solve for equilibrium induced by feasible policies in Section 1.4.2. We show that each policy ultimately depends the identity of a targeted type and on variables

$(S^G, \pi(t_{int}), \pi(t_{int}+1))$ which solve three equations imposed by Definition 3. Let us first express the reservation values for $i \in \{1,2\}$. We have

$$\begin{aligned} r_L^i(t_{int}+1) &= r_{L,ss}^i \\ r_L^i(t_{int}) &= \tau^i d_L + \delta\Big(\pi(t_{int}+1) r_H^1 + (1-\pi(t_{int}+1))\big[(1-\gamma) r_L^i(t_{int}+1) + \gamma r_L^j(t_{int}+1)\big]\Big) \end{aligned}$$

The first equality follows from the fact that the economy is in a steady state equilibrium from period $t_{int}+2$ onward. We now turn to agents participation constraint. Reservation values in period t_{int} are determine exactly as before. From Lemma 1, we have that $v_a^i(t_{int}) \geq v_a^j(t_{int})$ if $\tau^i d_a \geq \tau^j d_{a'}$ for $(i,j) \in \{1,2\}^2$ and $(a,a') \in \{L,H\}^2$. We thus call targeted type the highest type willing to participate in the program. Given a targeted type, (τ^i, a), the purchase price P^G must verify

$$v_a^i(t_{int}) \leq P^G < v_{a'}^j(t_{int}), \quad \forall j \in \{1,2\},\ a' \in \{L,H\}, \text{ such that } \tau^j d_{a'} > \tau^i d_a$$

For a given size S^G, the value within the range does not change the effect of the policy and we thus set $P^G = v_a^i(t_{int})$. Similarly, we set R^G to the upper bound defined by the resale constraint (RC). For a given targeted type, the size of the intervention also pins down the selection of assets (S_H^G, S_L^G) through (1.20). Let us then express the masses of each type of trader in period $t_{int}+1$:

$$\begin{aligned} \mu_L^1(t_{int}+1) &= \gamma(S(1-q) - S_L^G), \\ \mu_H^1(t_{int}+1) &= \gamma(Sq - S_H^G) + (1-2\gamma)(1-\pi(t_{int}))\mu_H^1(t_{int}) \end{aligned}$$

In phase $i)$ of period $t_{int}+1$, the government sold his assets to τ^2 buyers. The masses of type τ^1 traders in phase $ii)$ of period $t_{int}+1$ thus obtain from law of motions (1.13)-(1.12) substituting total supply by non-government supply. It is now clear that for a given targeted type, an equilibrium with intervention is pinned down by a triplet $(S^G, \pi(t_{int}), \pi(t_{int}+1))$ verifying the three conditions in Definition 3 the stabilization constraint SC and the optimality of offer $(\pi(t_{int}), \pi(t_{int}+1))$ for buyers. We thus adopt the following numerical procedure:

1. Select the highest type (τ^i, a) to attract where $i \in \{1,2\}$, $a \in \{L,H\}$.
2. Derive the lower bound on the purchase price P^G and the upper bound on the resale price R^G as a function of $(\pi(t_{int}), \pi(t_{int}+1))$ thanks to (RC).
3. Solve for a fixed point in $(S^G, \pi(t_{int}), \pi(t_{int}+1))$ following Definition 3.
4. Rank the policies derived in Point 3. across types using surplus criterion (1.21).

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Chapter 2

Re-using the Collateral of Others

2.1 Introduction

In credit markets, lenders frequently require borrowers to post collateral as a protection against default. Households pledge their house as collateral in a mortgage while banks use financial assets such as government securities. Households with mortgages own and use their house. In financial markets however, the lender may become the owner of the asset. More precisely, counterparties can arrange for the legal transfer of the collateral from the borrower to the lender. This provision allows the lender to enjoy rehypothecation rights over the collateral received: he may sell or re-use the collateral for his own funding[1]. While rehypothecation may seem bizarre, the ISDA[2] reports that borrowers grant such rehypothecation rights in 73.7% of trades surveyed for swaps and derivatives. With rehypothecation, collateral circulates through credit chains. These chains also generate risk as borrowers may be wary of losing access to their re-used asset. On account of those risks, Canadian law prohibits rehypothecation and several pending reforms in the US and the EU[3] seek to limit re-use. The current debates relies mostly on qualitative assessments and the literature does not yet offer a formal analysis of rehypothecation. The mere size of collateralized financial markets where rehypothecation is common practice (derivatives alone represent $700 trillions of gross notional) thus motivates this analysis.

[1]Strictly speaking, rehypothecation refers to the onward use of collateral pledged in a financial relationship. In swaps or derivatives markets, this re-use right can be granted by the collateral pledgor in an annex to the contract. Rehypothecation also applies to the US repo market where the asset is pledged together with a contractual right to re-use. In the non-US repo market, counterparties transfer the title of the asset during the sale. Hence, (re)-use is a fundamental right of the collateral receiver rather than one granted by the pledgor and rehypothecation is not appropriate. We refer to the ICMA website for further details.

[2]International Swaps and Derivatives Association, 2013 Margin Survey

[3]Section VII of the Dodd-Frank Act on central clearing of swaps implies that collateral cannot be rehypothecated. Similarly, dealer banks now have to obligation to notify their clients that their collateral can be segregated. The EU's own EMIR regulation introduces similar requirements for cleared swaps.

This paper provides a model to account for the trade-off between circulation of collateral and re-use risk in a general equilibrium framework. My theory of rehypothecation builds on Geanakoplos (1996) where a durable asset is used as collateral to secure short positions in financial securities[4]. In this environment, borrowers cannot commit to repay a loan and thus need to post collateral. I add rehypothecation whereby lenders can re-use a fraction of the collateral received to secure their own short positions. Lenders however also face limited commitment when required to return the collateral they received. Indeed, with re-use rights, the (cash) borrower ultimately extends an asset loan to the lender against the initial cash loan. Hence, fundamentally, collateral cannot circulate without generating re-use risk.

The technical contribution of this paper is to frame a model of rehypothecation as a simple extension of Geanakoplos (1996). Its first building block is the distinction between segregated and unsegregated collateral. Only the latter may be re-used by traders. I also adapt the settlement mechanism to accommodate double sided limited commitment, resulting from ex-post default decisions. The main result of the paper establishes that rehypothecation proves redundant if agents may already trade *collateral efficient* securities without re-use. A precise combination of securities with optimal collateral requirements can deliver the same asset velocity as the original transaction with re-use. Hence, re-use is not a free lunch in this environment because (money) lenders become asset borrowers and face limited commitment. My analysis thus suggests that the implicit relationship between re-use and collateral efficiency in many recent studies should be revised[5].

The rest of the paper examines departures from the complete market benchmark to understand when rehypothecation can strictly increase efficiency. With incomplete markets, rehypothecation helps freeing up collateral and relaxes collateral constraints. Re-use then essentially completes the markets by allowing trades that were not permitted under an incomplete financial structure. I show that these

[4]Some important models of default with limited commitment use a different technology than collateral to enforce promises. In Kehoe and Levine (1993), defaulting agents are excluded from financial markets. Dubey et al. (2005) uses non-pecuniary penalties for default. However, my work focuses precisely on the utilization of collateral, hence this modeling approach

[5]Several papers focusing on collateral demand induced by central clearing including Galbiati and Soramaki (2013) and Duffie et al. (2014) factor rehypothecation as an unambiguous gain for dealer banks

gains from re-use are larger with decentralized trading. Consider indeed a situation where three agents A, B, C trade in a chain. A and C may only trade with B who effectively acts as an intermediary. If A wishes to borrow from C, B would first extend a collateralized loan to A and then fund this loan by borrowing collateralized from C. Hence, intermediation mechanically doubles the need for pledgeable asset. Rehypothecation is beneficial as it allows agent B to re-use the collateral he receives from A in his transaction with C. If A could trade directly with C, re-use would not be needed.

Finally, I also show how re-use may generate fragility in the settlement process when many agents must return a scarce asset to their counterparty. Indeed, rehypothecation creates additional collateral out of a fixed quantity of pledgeable asset. The system fragility to external shocks depends on the strength of this multiplier effect.

Rehypothecation received much attention from policy-making circles as the 2007 financial crisis exposed risky collateral management practices at some dealer banks[6]. Monnet (2011), Singh (2011) or Kirk et al. (2014) describe the trade-off between collateral circulation and collateral risk and provide rough measures of circulation at the aggregate or bank level. Bottazzi et al. (2012), account for re-use of securities through repurchase agreements but sidestep the commitment issue attached to returning securities. In Andolfatto et al. (2014), the value of a long-term relationship between traders mitigate the commitment issue and effectively substitutes for collateral. Their different focus on monetary policy complements the analysis in this paper. Monnet and Nellen (2012) analyze different collateral segregation regimes with pure limited commitment. This paper combines the features from the contributions outlined above.

The need to re-use collateral suggests that pledgeable assets might be too scarce to sustain an efficient level of borrowing in the economy[7]. Concerns about collateral scarcity gained prominence as Central Banks massive purchases of Treasuries reduce the supply of collateral-eligible assets, a point developed in Araújo et al. (2013). Re-use of collateral resembles other financial innovations such as tranching

[6] The 2011 failure of MF Global, a broker dealer who took bets on the European debt market re-using the collateral of its clients brought the practice to light.

[7] See CGFS (2013) for a recent take on the empirical debate around asset scarcity.

or pyramiding studied in Fostel and Geanakoplos (2012) or Gottardi and Kubler (2014) to economize on the pledgeable asset. Somewhat surprisingly, I show that in the absence of frictions, re-use fails to deliver the same efficiency gains. When markets are incomplete however, rehypothecation proves useful if pyramiding or tranching are not possible for legal or technical reasons. In practice, the widespread use of plain vanilla assets[8] as collateral justifies rehypothecation over pyramiding.

A the aggregate level, rehypothecation contributes to the creation of credit chains resembling that of Kiyotaki and Moore (1997). As in their model, an unanticipated shock propagates through the system when agents hold large gross positions (cf Section 2.6). This effect is reminiscent of contagion in interbank networks as in Allen and Gale (2000) or clearing mechanisms a la Eisenberg and Noe (2001). Rehypothecation may also generate contagion through the need to return the asset used as collateral. Indeed, when the same piece of collateral secured many loans, settlement and collateral delivery might not proceed smoothly. I show that a contract feature for settlement based on actual market practice mitigates these concerns.

Finally, rehypothecation stresses the analogy between collateral and money. As means of exchange, both determine the economy's ability to trade efficiently in an environment with limited commitment. The concept of money velocity extends to collateral with rehypothecation as the same piece of asset can back several transactions. However, money ultimately replaces credit transactions with spot trades for exchanging goods. On the contrary, collateral fundamentally backs credit transactions for future state contingent commodities. Hence limited commitment is an issue for re-using collateral and limits the gains from rehypothecation.

The paper is organized as follows. In Section 2.2, I present a simple example to illustrate the main mechanisms at play with rehypothecation. Section 2.3 then introduces the general model and discusses its novel features. The main irrelevance result on rehypothecation appears on Section 2.4 while Section 2.5 shows how rehypothecation can restore efficiency in an incomplete market environment. Section 2.6 features a discussion on settlement and fragility. Finally, Section 2.7 concludes.

[8] The ISDA 2013 Margin Survey reports that cash and government securities make up for more than 90% of collateral pledged by a sample of financial institutions

2.2 Re-using collateral: an example

This example illustrates how rehypothecation facilitates trade and increases welfare when collateral is scarce and markets are (collateral)-incomplete. I strip down the presentation to avoid repetitions with the exposition of the model in Section 2.3.

Example 1

Consider a two period ($t = 0, 1$) competitive one good economy with 3 types of agents, $i = A, B, C$. There are two states of the world in $t = 1$ denoted $s = 1, 2$ with equal probability $\pi(s) = 1/2$. Agents are risk-averse, have identical VNM preferences and do not discount period 1 payoffs. Agent i endowment is denoted by e_0^i in period 0 and $e_1^i(s)$ in period 1 and state s. Let $e \in (1, \infty)$. We have

$$\begin{cases} e_0^A = e + 3/4 \\ e_1^A(1) = e - 1/2 \\ e_1^A(2) = e - 1 \end{cases} \qquad \begin{cases} e_0^B = e + 1/4 \\ e_1^B(1) = e - 1/2 \\ e_1^B(2) = e \end{cases} \qquad \begin{cases} e_0^C = e - 1 \\ e_1^C(1) = e \\ e_1^C(2) = e - 1 \end{cases}$$

In addition, agent C is endowed with one unit of the a tree that delivers $x(s) = s$ units of the consumption good in state s of period 1. With this endowment specification, there is no aggregate uncertainty in this economy since

$$x(s) + \sum_{i=A,B,C} e_1^i(s) = 3e, \quad s = 1, 2$$

In period 0, agents may trade the tree and a riskless bond in zero net supply with face value 1. Let agent i's portfolio be given by (θ^i, ϕ^i) where θ^i (resp. ϕ^i) denotes agent i holdings of the tree (resp. the bond).

Symmetric Efficient Allocation

In this example, total endowment is constant in very date/state and equal to $3e$. Because agents are risk-averse, consumption across states must be constant in a Pareto Efficient allocation. Hence, the symmetric efficient allocation has any agent consuming $c^* = e$ in every date/state.

Decentralization

The symmetric efficient allocation is also the competitive equilibrium of this economy. Agents can trade two assets (the tree and the bond) with payoff matrix

$$R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

With two states and two non-colinear assets, a unique portfolio profile finances c^*

$$(\theta^A, \phi^A) = (1/2, 0) \qquad (\theta^B, \phi^B) = (-1/2, 1) \qquad (\theta^C, \phi^C) = (1, -1)$$

Collateral constraints without rehypothecation

We suppose now that agents can default at no cost on promises to deliver goods. However, they may use the tree as collateral to secure borrowing. Hence, a bond seller (i.e. a borrower) must post one unit of the tree as collateral for every bond unit sold. Observe that this is the minimum quantity to ensure repayment by a borrower in state $s = 1$. This friction leads to the following constraint on portfolios.

$$\theta^i + \min\{0, \phi^i\} \geq 0 \tag{1a}$$

One can read (1a) as the collateral constraint of agent i. Observe that (1a) nests a no short-sale constraint $\theta^i \geq 0$ on the tree. Shorting the tree indeed amounts to a non-credible promise to pay $x(s)$ in state s. Agent B's portfolio violates (1a) since he would need to sell the tree short. We show below that this becomes possible with rehypothecation.

Rehypothecation

With the trades outlined above, agent B would receive 1 unit of the tree as collateral since he has a long position $\phi^B = 1$. Assume that he can re-use half of this collateral for his own trades. Formally, constraint (1a) becomes:

$$\theta^i + 1/2 \max\{0, \phi^i\} + \min\{0, \phi^i\} \geq 0 \tag{1b}$$

Now B may sell half of the tree he receives as collateral as if it were his own. It is easy to verify that the portfolios described above verify collateral constraints (1b) and thus that agents reach allocation c^*.

With rehypothecation, however, agent B issued an implicit promise to return 1/2 units of the tree. We are thus left to check that agent B wants to make good on his promise by buying the tree back from A. We argue that he does since

$$1 - 1/2x(s) \geq 0, \quad s = 1, 2$$

The left hand side is agent B's payoff when he delivers. It costs $1/2x(s) = 1/2s$ to buy back the re-used tree and he obtains 1 from the bond payment. The right hand side is the payoff from failing to deliver. In this case, agent C obligation towards agent B is canceled and the later gets 0. It is thus crucial that agent B's failure to deliver the collateral cancels the obligation to repay of the debtor C.

Collateral backs promises by borrowers to pay in consumption goods and make them credible. Credible promises by borrowers then back lenders' promises to return the asset used as collateral. Two elements were important for this analysis though:

(i) Agent B may only re-use half of the collateral he receives but he is liable up to the full value of posted collateral upon default. As we show in the main model, this wedge between posted collateral and re-usable collateral is crucial.

(ii) Agent B is able to buy back the tree in period 1. In general, many re-users might try to buy back the same piece of collateral simultaneously, generating a bottleneck effect. We will see that a well-defined settlement process or a right to return equivalent collateral allow to sidestep this difficulty.

In this example, agent B re-uses the collateral to sell it to agent A. In general, the collateral receiver could also re-pledge, that is using the pledged collateral to secure additional borrowing in the same or different debt instruments. This will depend on which collateralized trades are the most valuable. However, the example suggests that agents who re-use collateral must be given proper incentives to return it to their counterparties. This friction limits the economic value of rehypothecation. The rest of the paper is concerned mainly with identifying environments where re-use may bring about strict welfare gains, as in this example.

2.3 The Model

2.3.1 Physical Environment

Consider a pure exchange economy with two periods $t = 0, 1$ and several states of the world $s = 1, ...S$ in period 1. The economy consists of a set of agents $\mathcal{I} = \{1, 2, ..., I\}$ and a perishable consumption good. Each type i represents a continuum of agents of mass 1. Agent i endowment of the consumption good is $e^i = (e^i_0, e^i_1(1), ..., e^i_1(S)) \in \mathbb{R}^{S+1}$. Preferences are Von Neumann-Morgenstern over consumption streams. Function u^i denotes instantaneous utility so that agent i's preferences over consumption bundle $(c_0, \boldsymbol{c}_1)$ are given by

$$U^i(c_0, \boldsymbol{c}_1) = u^i(c_0) + \sum_{s=1}^{S} \pi(s) u^i(c_1(s))$$

where $\pi(s)$ denotes the probability that state s realizes in period 1. Utility functions $u^i : \mathbb{R}_{++} \to \mathbb{R}$ are strictly monotone, $\mathcal{C}^2$, strictly concave and verify the Inada condition : $\lim_{c \to 0} u^i_c(c) = \infty$.

The economy is endowed with θ_0 units of a tradable Lucas tree: a durable asset[9] which delivers a quantity $x(s)$ of consumption good in state $s \in \{1, ...S\}$ of period 1 but no dividend in period 0. Agent i initially holds $\theta^i_0 > 0$ units of the tree. Each agent has two different accounts called a **segregated** and a **non-segregated** account to handle his holdings of the tree. This distinction becomes effective when I describe securities in the financial environment. Agent i total endowment in state s is then given by $\omega^i_1(s) = e^i_1(s) + \theta^i_0 x(s)$. The main friction of the model is that the first component of endowment, $\boldsymbol{e}^i_1$ cannot be pledged. This means that that no agent can be liable beyond his holdings of the durable asset[10].

[9]There are actually two goods in the economy: the perishable consumption good and the tree. Here we assume that the intrinsic valuation for the tree depends only on the quantity of consumption goods (dividends) it yields in period 1. The only difference is that in multiple good economies with incomplete markets, pecuniary externalities can arise independently of collateral re-use.

[10]The lack of punishment besides the loss of collateral plays an important role for rehypothecation as illustrated in Gottardi et al. (2016). The threat of bankruptcy or reputation cost could help discipline agents but would also reduce the usefulness of collateral which is the object of study of this paper.

2.3.2 Financial Environment

Securities

With limited pledgeability, agents must use the tree as collateral to borrow. Lenders can re-use the collateral they receive to secure their own borrowing. A financial security (or simply a security) is thus defined as follows :

Definition 1: *A security j is a triplet $(\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j) \in \mathbb{R}^S_+ \times \mathbb{R}_+ \times [0,1]$ where $\bar{R}_j(s)$ is the promised amount to be paid in state s, $\bar{k}_j$ is the quantity of tree to be posted as collateral and α_j is the fraction of this collateral to be held in the buyer's segregated account.*

In the following, I call seller or borrower (resp. buyer or lender) the agent who sells (resp. buys) a security. My definition with rehypothecation collapses to the standard concept of a collateralized security of Geanakoplos (1996) and others when $\alpha_j = 1$. When $\alpha_j < 1$, the security grants *rehypothecation rights* to the buyer. The set of all tradable securities is $\mathcal{J}$. The transfers involved by the sale of a security j are illustrated on Figure 2.1. The seller must then transfer $\bar{k}_j$ units of the tree as collateral, a fraction α_j of which is segregated[11]. The buyer may use the $1 - \alpha_j$ fraction of non-segregated collateral to carry his own transactions. With

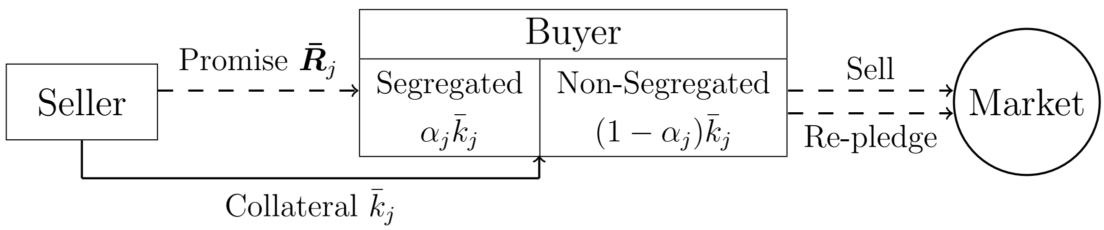

Figure 2.1: Sale of security j

limited commitment, borrowers or lenders may opportunistically renege on their promises. To make a distinction, I call **Failure** the action of a lender not turning

[11]Practically, the segregated collateral could be stored in a third party's dedicated account. In the tri-party repo market, BNY Mellon and JP Morgan provide this collateral storage facility.

back collateral[12] as in Johnson (1997). The terminology **Default** is saved for the decision of the borrower not to repay. Ultimately, rehypothecation introduces a double sided limited commitment (henceforth 2SLC) problem. I now analyze default and failure pattern for securities in $\mathcal{J}$.

Default and Securities Payoff

I now characterize securities' payoff provided that borrowers may default on repayments and lenders may fail to return collateral. An important and realistic feature of the bankruptcy process is that an agent is freed of his obligations if the counterparty does not comply. We guess and verify that an agent takes as given that the counterparty would comply. Consider first the decision of an agent short in security j. He will default on the promised amount $\bar{R}_j(s)$ if it exceeds the value of the collateral he is entitled to, that is:

$$\bar{R}_j(s) \geq \bar{k}_j x(s) \qquad \text{(Borrower Default)}$$

Let us now turn to an agent long in security j. He will return the non-segregated amount of the tree if the payment from the security lies below the value of the tree he may hold on to, that is:

$$R_j(s) \leq (1-\alpha_j)\bar{k}_j x(s) \qquad \text{(Lender Fail)}$$

The lender effectively has a short position in the spot market for the tree equal to $-(1-\alpha_j)\bar{k}_j$ covered by the payment from security j he has a long position in. Observed indeed that borrowers and lenders have countervailing incentives so that decisions to default do not overlap. In other contingencies, payment and delivery follow contractual obligations. We can thus write the actual payoff of a security j

[12] Strictly speaking, Fleming and Garbade (2005) explain that failure (to settle) is not a default event but a delay in delivering the tree. Because such delays can matter when re-used collateral must be returned, I analyze this issue in Section 2.6

as

$$R_j(s) = \begin{cases} (1-\alpha_j)\bar{k}_j x(s) & \text{if} \quad \bar{R}_j(s) < (1-\alpha_j)\bar{k}_j x(s) \\ \bar{R}_j(s) & \text{if} \quad \bar{R}_j(s) \in [(1-\alpha_j)\bar{k}_j x(s), \bar{k}_j x(s)] \\ \bar{k}_j x(s) & \text{if} \quad \bar{R}_j(s) > \bar{k}_j x(s) \end{cases} \tag{2.2}$$

Observe that all three components of a security matter to determine the actual payoff but traders' anonymity is preserved. Partial segregation (that is $\alpha_j < 1$) creates a wedge between the default threshold for a borrower and the failure point for a lender. With complete segregation ($\alpha_j = 1$), equation (2.2) simplifies to $R_j(s) = \min\{\bar{R}_j(s), \bar{k}_j d(s)\}$ as only borrowers face a strategic choice. Alternatively, without segregation ($\alpha_j = 0$), the payoff is given by $R_j(s) = \bar{k}_j x(s)$. The transaction cannot be distinguished from a sale of the tree "pledged" as collateral. When α_j is interior, security j is made of two components. A sale of $(1-\alpha_j)\bar{k}_j$ units of the tree and security j' with payoff $R_{j'}(s) = R_j(s) - (1-\alpha_j)\bar{k}_j x(s)$, with $\bar{k}_{j'} = \alpha_j \bar{k}_j$ and $\alpha_{j'} = 1$. This simple observation is the basis for the spanning result in Lemma 2.

I denote $\mathcal{E}(\mathcal{J})$ the economy where the set of agents $\mathcal{I}$, preferences $\mathcal{U} := \{u^i\}^{i\in\mathcal{I}}$ and endowments $\boldsymbol{\mathcal{W}} = \{\boldsymbol{e}^i, \theta_0^i\}^{i\in\mathcal{I}}$ are fixed. Securities in $\mathcal{J}$ are traded competitively. The matrix $\boldsymbol{R} \in \mathcal{M}_{(S)\times J+1}$ collects the payoffs of securities $\mathcal{J}$.

2.3.3 Agent's Optimization Problem and Equilibrium

Consumer Problem

Denote θ^i agent i's position in the market for the tree. Short security positions need to be backed up by collateral while long positions may grant re-use rights. We then distinguish purchases $\phi_j^{i+} \geq 0$ and sales $\phi_j^{i-} \geq 0$ of security $j \in \mathcal{J}$. Prices of collateral and securities are respectively $p \in \mathbb{R}_+$ and $\boldsymbol{q} \in \mathbb{R}_+^J$. Given asset prices and security payoffs as given by (2.2) - agent i solves:

$$\max_{\{\boldsymbol{c},\theta,\boldsymbol{\phi}\}} \quad u^i(c_0^i) + \sum_{s\in\mathcal{S}} \pi(s) u^i(c_1^i(s)) \tag{2.3}$$

$$\text{s. to} \quad c_0^i + \sum_{j\in\mathcal{J}} q_j \phi_j^{i+} + p\theta^i \le e_0^i + p\theta_0^i + \sum_{j\in\mathcal{J}} q_j \phi_j^{i-} \tag{2.4}$$

$$c_1^i(s) + \sum_{j\in\mathcal{J}} \phi_j^{i-} R_j(s) \le e_1^i(s) + \theta^i x(s) + \sum_{j\in\mathcal{J}} \phi_j^{i+} R_j(s) \qquad \forall s \in \mathcal{S} \tag{2.5}$$

$$\theta^i + \underbrace{\sum_{j\in\mathcal{J}} (1-\alpha_j)\bar{k}_j \phi_j^{i+}}_{\text{Re-use}} - \sum_{j\in\mathcal{J}} \bar{k}_j \phi_j^{i-} \ge 0 \tag{2.6}$$

Equations (2.4)-(2.5) are budget constraints in period 0 and 1 respectively. Equation (2.6) is the collateral constraint. An agent must have enough asset to cover all his short positions $\sum_{j\in\mathcal{J}} \bar{k}_j \phi_j^{i-}$. As usual, this asset can be bought in the spot market (the first term θ^i). With rehypothecation however, an agent might also re-use collateral he receives as a lender. Indeed, when he acquires a long position ϕ_j^{i+} in security j, agent i may re-use a fraction $1-\alpha_j$ from the $\bar{k}_j$ units of collateral he receives. This re-usable collateral can be sold (θ^i goes down) or re-pledged for a short position $\phi_{j'}^{i-}$. In particular, agents may effectively hold a negative position in the tree market $\theta^i < 0$ if they receive collateral through long positions. This feature is reminiscent of Bottazzi et al. (2012) where the Box Constraint implies that agents cannot issue securities they have not borrowed first.

Equilibrium Definition

A competitive equilibrium of economy $\mathcal{E}(\mathcal{J})$ is a feasible allocation $(\boldsymbol{c}_0, \boldsymbol{c}_1) \in \mathbb{R}^{(S+1)\times I}$, a price vector $(p, \boldsymbol{q}) \in \mathbb{R}_+^{J+1}$ such that $\forall i \in \mathcal{I}$, $(c_0^i, \boldsymbol{c}_1^i)$ solves agent i's optimization problem (2.3)-(2.6) and securities market clear, that is:

$$\text{(Feasibility)} \qquad \sum_{i=1}^{I} c_0^i \le \sum_{i=1}^{I} e_0^i, \qquad \sum_{i=1}^{I} c_1^i(s) \le \sum_{i=1}^{I} e_1^i(s) + \theta_0 x(s) \quad \forall s = 1..S \tag{2.7}$$

$$\text{(Market Clearing)} \qquad \sum_{i=1}^{I} \theta^i = \theta_0, \qquad \sum_{i=1}^{I} (\phi_j^{i+} - \phi_j^{i-}) = 0, \quad \forall j = 1..J \tag{2.8}$$

Proposition 1 : *Economy $\mathcal{E}(\mathcal{J})$ admits a competitive equilibrium.*

Proof : $\triangleright$ Given the regularity conditions imposed on utility functions, the existence of equilibrium follows as in the standard 1SLC environment. This result can be seen as a particular case of Theorem 1 in Geanakoplos and Zame (2013) .$\triangleleft$

The existence result would hold in most 2 period financial market models with any linear constraint (this is the case of collateral constraint 2.6). Hence, the contribution here consists in setting a general equilibrium model of rehypothecation in a familiar environment to exploit standard results. Existence of equilibrium does not require any new technique, hence the outside reference for the proof.

2.3.4 Discussion of the assumptions

Rehypothecation Rights

In the model, the re-usable fraction of collateral may be specified between 0 and 1. In practice rehypothecation rights follow from a binary decision by counterparties: either complete or no segregation. In the repo market for example, as we explained the full right of re-use is implicit to the contract. Still, current or upcoming regulations aim to put a cap on the re-usable amount of collateral[13]. My framework would thus accommodate these possible developments with a lower bound $\bar{\alpha}$ on α_j. More generally, partial segregation or limited re-use may arise because of market illiquidity or other technical impediments in the settlement process. At the macro level, Singh provides an estimate of collateral velocity around 3, hinting at a value of $\alpha = 1/3$ in my model.

Collateral Pledges and Settlement

In the model, it is assumed that collateral transfers take place simultaneously at the trading and settlement stage. Consider for instance agent B in our introductory example. He can only sell the asset to agent A after he receives it from a pledge

[13]In the US, brokers/dealers may re-use an amount up to 140% of the debit balance. See SEC Rule 15c3-3. The debit balance is an equilibrium object in our model hence the different formulation as a fraction of total collateral posted of their client under Regulation T. In the EU, future reforms contemplate the possibility to impose partial segregation.

by agent C. I assume here that these steps take place simultaneously under a single price for the tree. Appendix A accounts precisely for collateral circulation by introducing trading and settlement rounds. There, agent B would receive collateral during round 1 and re-sell it during round 2 of the trading process. Essentially, he cannot use the proceeds of the re-usable part of collateral to finance the loan he extends to agent C anymore. In this sequential process, each piece of asset can only be pledged (resp. returned) once in each trading (resp. settlement) round. We show in Appendix A the equivalence between the two representations.

Observe that the model also abstracts from trading or search frictions whereby lenders might have difficulty to locate a piece of asset they re-used. In practice, some Master Agreements contain provisions to smooth settlement. This is the case of the English Credit Support Annex to the ISDA Master Agreement which only bind receivers to return "equivalent" property. Counterparties may thus agree on the delivery of a different security than that pledged initially, thereby recognizing the potential difficulty to settle trades when a certain type of collateral is scarce[14]. In this environment, this would imply that a lender can either return k units of asset or pay $kx(s)$ in consumption goods to the borrower, that is the consumption value of collateral.

The next section presents the main irrelevance result of the paper which states that rehypothecation is redundant when markets are complete in a sense made precise below.

2.4 Collateral Scarcity

In this model, agents trade in order to share risk and smooth consumption over time. The ability to conduct these trades might be limited by market incompleteness as in the standard GEI literature. More importantly, when agents face collateral constraints, the availability of the tree proves crucial for efficiency. In the extreme case where the economy has no tree, $\bar{\theta}_0 = 0$, agents cannot commit to repay any debt and

[14]See Monnet (2011) for a comparison of the different Credit Support Annexes. Unfortunately, the ISDA does not provide detailed numbers on the use of each of this Annex. In practice, other impediments to locate a security like search frictions help rationalize this provision

stay in autarky. Singh (2011)'s emphasis on the link between rehypothecation and collateral velocity suggests that re-using collateral allows to save on a potentially scarce resource and ultimately improves market efficiency. The rest of this paper analyzes this claim and characterizes financial environments where it holds.

Proposition 2, the main result of this paper, plays down the importance of rehypothecation. Indeed, I show that without restrictions on security design, agents are equally well-off trading securities without rehypothecation rights. In other words, welfare gains from collateral re-use may only materialize in a situation of market incompleteness.

2.4.1 Replicating Financial Structure

The first step towards Proposition 2 is to show that some securities in a financial structure $\mathcal{J}$ may be redundant. The following definition formalizes this concept in our environment with collateral constraints.

Definition 3 : A financial structure $\mathcal{J}_1$ replicates another structure $\mathcal{J}$ iff $\forall j \in \mathcal{J}$, $\exists \mathcal{J}_1(j) \subset \mathcal{J}_1$, $(\theta, \boldsymbol{\phi}) \in \mathbb{R}_+ \times \mathbb{R}_+^{|\mathcal{J}_1(j)|}$ such that

(i) $\forall s \in \mathcal{S}, \quad \theta x(s) + \sum_{j_1 \in \mathcal{J}_1(j)} \phi_{j_1} \boldsymbol{R}_{j_1}(s) = \boldsymbol{R}_j(s)$

(ii) $\theta + \sum_{j_1 \in \mathcal{J}_1(j)} \phi_{j_1} \bar{k}_{j_1} \leq \bar{k}_j$

(iii) $\sum_{j_1 \in \mathcal{J}_1(j)} \alpha_{j_1} \phi_{j_1} \bar{k}_{j_1} \leq \alpha_j \bar{k}_j$

$(\theta, \boldsymbol{\phi})$ is called a replicating portfolio and we denote $\mathcal{J}_1 \in Sp(\mathcal{J})$

In words, for every security of $\mathcal{J}$, there must be a portfolio made of the tree and securities in the set $\mathcal{J}_1$ that delivers the same payoff. This is the standard spanning feature of a replicating portfolio. In addition, selling this portfolio does not require more collateral than the original security. Finally, the replicating portfolio leaves the same amount of asset free to use for the agent long in the portfolio. Criteria (ii) and (iii) acknowledge the role of the tree as collateral and recognizes the importance of tree availability or unencumbered collateral to carry trades. We can now introduce the following lemma:

Lemma 1: Let $\mathcal{J}$ and $\mathcal{J}_1$ be such that $\mathcal{J}_1 \in Sp(\mathcal{J})$. Then, any equilibrium allocation of $\mathcal{E}(\mathcal{J} \cup \mathcal{J}_1)$ is an equilibrium allocation of $\mathcal{E}(\mathcal{J}_1)$.

The proof is in Appendix 2.8.2. Lemma 1 can be seen as an extension of Proposition 2 in Araújo et al. (2012) for rehypothecation. There is no loss of generality in restricting trades to a replicating financial structure. Intuitively, we can substitute every security j of $\mathcal{J}\backslash\mathcal{J}_1$ traded in equilibrium by its replicating portfolio in $\mathcal{J}_1$. This result is of limited interest unless the former entails some genuine restrictions. Precisely, the next Lemma will show that securities with rehypothecation rights are redundant.

Consider indeed the unrestricted set of securities $\mathcal{J}_0$ together with a subset of $\mathcal{J}_* \subset \mathcal{J}_0$ of securities without rehypothecation rights.

$$\mathcal{J}_0 = \left\{(\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j) \in \mathbb{R}^S_+ \times \mathbb{R}_{++} \times [0,1]\right\}$$
$$\mathcal{J}_* = \left\{(\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j) \in \times_{s=1..S}\{0, x(s)\} \times \{1\} \times \{1\}\right\}$$

Observe that $\mathcal{J}_*$ contains but is not equal to the set of (collateralized) Arrow Securities. This arises because of security-specific collateral constraints. Suppose indeed that $S = 4$ and $x(s) = 1$ for all s and that an agent wants to sell the payoff $(0,0,1,1)$. With Arrow securities only, he needs to short $(0,0,1,0)$ and $(0,0,0,1)$ hence requiring 2 units of collateral. Only 1 is necessary if the security is available.[15]. The next Lemma show that the no-rehypothecation structure $\mathcal{J}_*$ replicates the unrestricted financial structure $\mathcal{J}_0$

Lemma 2 : $\mathcal{J}_* \in Sp(\mathcal{J}_0)$.

[15]This argument also makes clear that with ex-post (or equivalently portfolio wide) collateral constraints as in Chien et al. (2011), Arrow Securities do suffice. For Credit Default Swaps (CDS) transactions, initial margins are a mix of both. Still, I consider that security-specific collateral constraints better approximate market practice in general.

The proof adapted from Kilenthong (2011) is in Appendix 2.8.2. Lemma 1 thus shows that for spanning purposes and collateral utilization, a smaller set of securities without rehypothecation suffices[16]. Intuitively, a transaction with rehypothecation can be broken down into two components : a collateral sale and a residual full segregation security. To see this, let us consider a generic security $j = (\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j)$. The actual payoff $\boldsymbol{R}_j$ verifies

$$R_j(s) = (1-\alpha_j)\bar{k}_j x(s) + \underbrace{\left[R_j(s) - (1-\alpha_j)\bar{k}_j x(s)\right]}_{R_{j_2}(s)} \tag{2.9}$$

The first term on the RHS amounts to a sale of the unsegregated $(1-\alpha_j)\bar{k}_j$ units of the tree. The second term is the residual payoff $R_{j_2}(s) \in [0, \alpha_j \bar{k}_j x(s)]$. Observe that this is equal to the payoff a security j_2 with payoff $R_{j_2}(s)$ and collateral requirement $\alpha_j \bar{k}_j$. It is clear that the replication uses the same amount of tree $\bar{k}_j$ and also segregates $(1-\alpha_j)\bar{k}_j$ units of the tree. If such operation is feasible for any security j, rehypothecation becomes redundant. The proof of Lemma 2 shows how to realize this substitution for any generic security thanks to the no re-use securities in $\mathcal{J}^*$

2.4.2 Neutrality under Complete Markets

We can now state the main result of this paper. A complete financial structure without rehypothecation rights $\mathcal{J}_*$ can deliver the same outcomes as the unconstrained structure $\mathcal{J}_0$.

Proposition 2 *Any equilibrium allocation of economy $\mathcal{E}(\mathcal{J}_0)$ is an equilibrium allocation of $\mathcal{E}(\mathcal{J}_*)$. Under collateral-complete markets, rehypothecation is redundant.*

The proof follows directly from Lemma 1 and 2. Proposition 2 reads as a negative statement. Rehypothecation does not effectively improve collateral use and allocations when the standard securities in $\mathcal{J}_*$ are available. This seems to con-

[16]It can be shown that subsets of $\mathcal{J}_*$ containing only $2^{S-1}-1$ securities also span $\mathcal{J}_0$ but this is of limited interest to our analysis

tradict our initial intuition that re-use relaxes collateral constraints. However, the collateral receiver might default on the asset loan that he receives. This affects the security payoff ex-post and limits the benefits from re-usability of collateral.

Proposition 2 thus states that rehypothecation is no more efficient than standard security specific margins without re-use. This is an interesting, albeit surprising result, in comparison with other existing techniques for collateral re-use. Asset specific margins indeed entail two types of restriction, given the fundamental pledgeability friction of our environment. First, a given piece of collateral cannot back several securities (no tranching). Second, only the tree itself, not financial securities, can serve as collateral (no pyramiding). Gottardi and Kubler (2014) precisely show that pyramiding or tranching realize the efficient use of the tree as collateral[17]. The relatively higher performance of pyramiding against rehypothecation is striking since both techniques have a similar flavor of collateral re-use. The key difference is that pyramiding allows to use financial wealth as collateral in a more subtle way since any long position can be posted as collateral. With rehypothecation, only the tree may be posted. In addition, pyramiding abstracts from the limited commitment problem of the lender since he may never re-use more than the value of a debt contract he is long in. Although proposition 2 suggests that traders should prefer pyramiding to rehypothecation, in practice, the use of pyramiding is restricted to some asset-backed securities markets while rehypothecation appears more prevalent. Although we abstract from these issues here, observe that collateral is good only to the extent that it is easily recognizable and that the corresponding asset market is liquid. This might be the case for Treasuries that are re-used extensively but not for complex financial products resulting from pyramiding.[18].

[17] We refer the reader to this paper for a formal definition of these concepts as well as that of a feasible allocation with limited pledgeability. In our two periods environment, the additional feasibility requirement is that an allocation must verify

$$\forall i \in \mathcal{I},\, \forall s \in \mathcal{S}, \quad c_1^i(s) \geq e_1^i(s)$$

which is simply stating that financial wealth cannot be negative

[18] Abstracting from these issues, in Gottardi et al. (2016), we consider an environment with limited instead of no-commitment and show hat rehypothecation can be beneficial. Muley (2016) also considers a simpler model with heterogeneity in commitment power and shows that rehypothecation can then dominate pyramiding.

2.5 Incomplete Collateral Markets

The literature has emphasized a number of reasons why markets may be incomplete[19] and lack the securities needed for Proposition 2. This section considers exogenously incomplete markets in the sense of Proposition 2 and shows that rehypothecation may be strictly desirable. To do so, I revisit first Example 1 to identify precisely the role of rehypothecation. Then Example 2 shows that the gains from re-using collateral in incomplete markets are magnified with decentralized trade. This last result usefully rationalizes rehypothecation patterns in repos, swaps or derivatives markets where trade takes place Over the Counter.

2.5.1 Example 1: Market incompleteness and short sale constraint

Let us remind the setting of Example 1 introduced in Section 2.2. There are 3 types of agents, $i = A, B, C.$ and two states of the world in $t = 1$ denoted $s = 1, 2$ with equal probability $\pi(s) = 1/2$. Agents have identical VNM preferences and do not discount period 1 payoffs. Let $e \in (1, \infty)$ be given. Endowments are as follows.

$$\begin{cases} e_0^A = e + 3/4 \\ e_1^A(1) = e - 1/2 \\ e_1^A(2) = e - 1 \end{cases} \qquad \begin{cases} e_0^B = e + 1/4 \\ e_1^B(1) = e - 1/2 \\ e_1^B(2) = e \end{cases} \qquad \begin{cases} e_0^C = e - 1 \\ e_1^C(1) = e \\ e_1^C(2) = e - 1 \end{cases}$$

In addition, agent C is endowed with one unit of the a tree that delivers $x(s) = s$ units of the consumption good in state s of period 1. With our notation $\theta_0^C = 1$ while $\theta_0^A = \theta_0^B = 0$. As we highlighted before, the symmetric Pareto efficient allocation $\boldsymbol{c}^*$ features constant consumption across agents, time and states equal to e.

[19] Market incompleteness is exogenous in my model since I do not model the supply side for financial securities. Private information about aggregate or idiosyncratic outcomes sometimes justify why some securities are not traded. In the first case, DeMarzo and Duffie (1999) and more recently Dang et al. (2012) show that debt-like securities are desirable because of their low information sensitivity. For the second case, there exists a long tradition of Bewley models where idiosyncratic shocks are private information and hence uninsurable. Even without such frictions, Carvajal et al. (2012) prove that security designers might find it optimal not to complete the market.

Complete Markets

Suppose two Arrow securities j_1, j_2 are available to trade. Security j_s pays 1 in state s and zero in the other state and requires $1/s$ units of collateral. Define $\mathbf{R_C}$ (where C stands for complete) the matrix of payoff :

$$\mathbf{R_C} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

where the first column contains the tree payoff. Period 1 transfers implied by allocation $\mathbf{c}^*$ may be decentralized with portfolios $\{(\theta^i, \phi^i_{j_1}, \phi^i_{j_2})\}^{i=A,B,C}$ if

$$\begin{aligned}
& & \sum_i \theta^i & = 1 \\
&\text{For } s = 1,2 & \sum_i \phi^i_{j_s} & = 0 \\
&\forall i \in \{A,B,C\}, & \begin{bmatrix} e \\ e \end{bmatrix} & = \begin{bmatrix} e^i_1(1) \\ e^i_1(2) \end{bmatrix} + \mathbf{R_C}.(\theta^i, \phi^i_{j_1}, \phi^i_{j_2}) \\
& & 0 & \leq \theta^i + \min\{\phi^i_{j_1}, 0\} + \frac{1}{2}\min\{\phi^i_{j_1}, 0\}
\end{aligned}$$

One can easily check that the following portfolios are a solution to the problem above:

$$(\theta^A, \phi^A_{j_1}, \phi^A_{j_2}) = (1/2, 0, 0) \qquad (\theta^B, \phi^B_{j_1}, \phi^B_{j_2}) = (0, 1/2, 0) \qquad (\theta^C, \phi^C_{j_1}, \phi^C_{j_2}) = (1/2, -1/2, 0)$$

Finally, prices for the tree and the securities are equal to their state weighted payoff since the allocation exhibits perfect risk sharing and constant consumption. Thus we have $p = 3/2$ and $q_{j_1} = q_{j_2} = 1$. Arrow securities are collateral efficient because agents can diversify idiosyncratic risk with a minimal amount of the tree.

Incomplete Markets : Bond Economy

We now turn back to the incomplete (i.e. collateral inefficient) financial structure where agents can only trade the tree and a bond with face value equal to 1. The payoff matrix $\mathbf{R}_I$ (where I stands for incomplete) thus writes

$$\mathbf{R_I} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

As shown in Section 2.2, granting rehypothecation rights on half of the collateral pledged for a bond allows to finance c^*. The collateral constrain thus writes

$$\forall i \in \{A, B, C\}, \quad \theta^i + \frac{1}{2}\max\{\phi^i, 0\} + \min\{\phi^i, 0\} \geq 0$$

To finance c^*, agent C borrows one unit from agent B who then sells half of the tree obtained as collateral to agent A. Agents thus hold the following portfolios

$$(\theta^A, \phi^A) = (1/2, 0) \qquad (\theta^B, \phi^B) = (-1/2, 1) \qquad (\theta^C, \phi^C) = (1, -1)$$

Fundamentally, agent B wishes to increase consumption by $(1/2, 0)$ in state 1 and 2 respectively. Under complete markets, he needs 1/2 units of Arrow Security, mobilizing 1/2 units of collateral: the tree agent C has to post. In the bond economy however, agent B must take a long position of 1 unit in the bond and sell 1/2 units of tree: this locks 1 unit of the tree as collateral. With rehypothecation, 1/2 units of this collateral can be re-used, leaving as before only 1/2 units segregated. Ultimately, with incomplete markets, rehypothecation frees up tree pledged as collateral for which segregation is not fundamentally required for risk-sharing. In fact, rehypothecation makes use of the slack in the lender's limited commitment constraint which for a no re-use security $j = (\bar{\boldsymbol{R}}_j, \bar{k}_j, 1)$ writes as

$$1 - \bar{\alpha}_j = \min_{s \in \mathcal{S}} \frac{R_j(s)}{\bar{k}_j(s)x(s)}$$

One can check that this number indeed equals 1/2 for the bond collateralized by one unit of the tree.

As we mentioned before, the irrelevance result illustrated with this example may not provide a positive theory of rehypothecation. One should not observe collateral re-use in the first place if some collateral efficient securities could be traded instead.

Again, this analysis remains agnostic as to why markets would be incomplete in the first place. The purpose of this example was rather to show the positive effect of re-use given that such a friction exists.

2.5.2 Centralized vs. Bilateral Trading

This section emphasizes rehypothecation gains with decentralized trading. Again, given the equivalence results of Section 2.4, this analysis is meaningful only with an incomplete market environment. I show that higher collateral needs with bilateral trade can justify rehypothecation, when it is not warranted in a centralized market.

Example 2

To account for decentralized trading, at least 3 agents $i = A, B, C$ are needed. There are again 2 states of the world. The asset pays off $x(s) = s$ for $s = 1, 2$. Agents have the same utility function and do not discount period 1 payoffs. Let $e \in (1, \infty)$ be given and consider the following endowments:

$$\begin{cases} e_0^A = e + 1/2 \\ e_1^A(s) = e - 1/2 \\ \theta_0^A = 0 \end{cases} \qquad \begin{cases} e_0^B = e + 1/2 \\ e_1^B(s) = e - 1/2 \\ \theta_0^B = 0 \end{cases} \qquad \begin{cases} e_0^C = e - 1 \\ e_1^C(s) = e + 1 - s \\ \theta_0^C = 1 \end{cases}$$

Given convex preferences and the absence of discounting, it is easy to see that the symmetric Pareto optimal allocation is $c_t^i = c^* = e$ for all $i = A, B, C$ and all $t = 0, 1$. Agent C wishes to borrow while agents 1 and 2 desire to lend. To put it otherwise, agents want to trade endowment across time and not across states as in 2.5.1. Hence, a non-contingent bond with face value $R(s) = 1$ appears as the perfect instrument. For borrowers not to default, it needs to be collateralized by one unit of asset so $\bar{k} = 1$. Finally, for lenders not to fail, rehypothecation should be limited by $\alpha \geq \frac{1}{2}$, exactly as in the previous example.

Centralized Trading

With centralized trading, the bond decentralizes the efficient allocation and agents need not re-use collateral. For this, agent C borrows 1 while agents A and B both lend 1/2. This trades verifies agent C's collateral constraint since $\phi^{C,-} = 1 \leq \theta_0^C = 1$. The allocation c^* is supported by price $q = 1$ for the bond given perfect risk sharing and absence of discounting. The inequalities above are also equivalent to the fundamental constraints on the limited pledgeability of income and no other trade pattern can improve upon this one.

Decentralized Trading

Suppose now that trading is bilateral as shown in figure 2.2 so that agent B is essential to trading[20]. Precisely, to implement c^* agent B needs to lend 1 to agent C and borrow 1/2 from agent A. Hence, he is still a net lender of 1/2 but he holds both long and short positions in the bond. The continuous (resp. dahsed) curved arrows illustrate the transfer of cash (resp. collateral) in period 0.

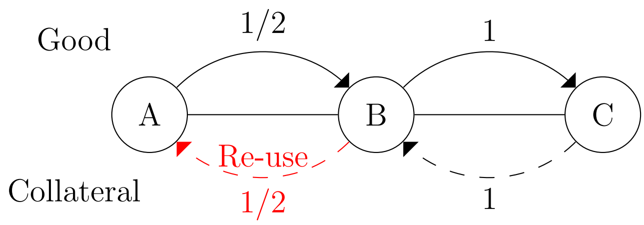

Figure 2.2: Bilateral Trades

Since 3/2 units are lent on aggregate, 3/2 units of collateral must be posted while there is only 1 unit of tree. With rehypothecation, agent B can re-use a fraction 1/2 of the collateral posted by agent C to borrow in turn from agent A. While 3/2 units of collateral are posted, total physical holdings of the tree still sum up to 1. After trading took place in period 0, we can decompose holdings between segregated and unsegregated account for every agent as follows :

$$(\theta_S^A, \theta_U^A) = (1/4, 1/4) \qquad (\theta_S^B, \theta_U^B) = (1/2, 0) \qquad (\theta_S^C, \theta_U^C) = (0, 0)$$

When borrowing from agent A, agent B recycles half of the collateral he receives

[20] The only difference with our benchmark model is that decentralization impose sub-markets clearing constraints for the tree and securities, that is in the AB and BC sub-markets. I do not develop further as the pattern of trades described below clearly highlights the role of these new constraints.

from agent C. We are left to verify that agents who receive collateral do not want to fail on the tree loan they receive. As in the previous example, the no fail constraint writes

$$\bar{R}_j(s) \geq \frac{1}{2}x(s) \quad \Leftrightarrow \quad 1 \geq \frac{1}{2}s$$

which holds for $s = 1, 2$. In period 1, A thus delivers the 1/2 units he holds against the payment of 1/2 from B who can then return 1 unit of collateral to A against a payment of 1. Hence, the final holder of the tree is agent C as required by the optimal portfolio.

Bilateral trading implies double collateral posting for the intermediary (agent B) to finance the trade. This cost disappears in a centralized market where agents A and C can trade directly as explained before. On OTC markets, intermediaries typically hold both short and long positions in the same security to match trading needs of dispersed customers[21]. When collateral must be posted, the gross trade mechanically requires more collateral than the net transaction. Rehypothecation generates more collateral out of a given amount of tree to meet this increasing demand.

2.6 Fragility

By definition, rehypothecation allows a lender to re-use a piece of collateral pledged by a borrower. If he exerts his right, the lender may not have the asset on his balance sheet upon settlement of their transaction. Assumption **A**, inspired by the English Credit Support Annex, neutralizes this concern as a cash payment may substitute for the physical asset delivery. However, some borrowers may not wish to see their collateral transformed and can opt for a different scheme, the New York Credit Support Annex.

In this case, the lender would purchase the asset in the market to make the contractual delivery. In my framework, Appendix 2.8.1 indeed shows that a market for the tree in period 1 allows agents to make up for their short positions before

[21] This is the essence of a matched-book repo trade when broker-dealers engage simultaneously in a repo with one customer and a reverse repo with another customer. Atkeson et al. (2013) also report that top dealer banks have large gross positions but small net positions in the CDS market

turning back the collateral. The analysis also stresses the importance of a well-behaved sequential process to ensure smooth settlement. Sequential settlement is crucial when agents may not purchase back the tree simultaneously. This arises when the total collateral pledged exceeds the quantity of tree available in the economy, i.e:

$$\theta_0 < \sum_i \sum_j \bar{k}_j \phi_j^{i,-} \tag{2.10}$$

Inequality (2.10) involves endogenous variables rather than deep parameters. However, this condition typically holds in equilibrium since the purpose of rehypothecation is precisely to create more collateral out of a given quantity of pledgeable asset. Then, the need for a sequential settlement process described in Appendix 2.8.1 proceeds from a resource constraint on the physical availability of the tree.

The rest of the paper presents modifications to the original set-up which generate fragility specific to rehypothecation in the settlement process. Section 2.6.1 introduces discounting during settlement to create a trade-off between early and late delivery. I show that the trade-off works in favor of borrowers because they can postpone payment until they receive their asset back, a point emphasized in Fleming and Garbade (2005) for settlement fails. In Section 2.6.2, I discuss how exogenous default might propagate along credit chains when (2.10) holds. For simplicity, In introduce these shocks as zero probability events. Hence, the interest of this analysis rests on the amplification mechanism specific to rehypothecation.

2.6.1 Delivery Race for Collateral

In this section, we consider Example 2 and look at the impact of an unanticipated preference shock in period 1 which affects discounting between settlement sub-periods. Hence, the equilibrium positions derived in Section 2.5.2 are still valid. Remember that agent C borrows one unit of consumption good from agent B, pledging 1 unit of tree with payoff $x(s) = s$. In turn, agent B borrows 1/2 units from agent A, re-using half of the collateral he received from agent C. Hence, condition (2.10) holds with the left and right hand side respectively equal to 1 and 3/2. Then, 2 rounds of settlement $\tau = 1, 2$ are necessary to clear all equilibrium positions since agent B must receive the collateral from A before he can return it to C.

When the shock materializes, agents discount $\tau = 2$ payoffs at rate $\delta < 1$. Hence, with obvious notations, consumption in period 1 is given by

$$c_1^i(s) = c_{1,1}^i(s) + \delta c_{1,2}^i(s)$$

Agents may save the consumption good between $\tau = 1$ and $\tau = 2$ with net interest rate $r = (1-\delta)/\delta$. The tree pays its dividend in round $\tau = 2$ which is normalized to $x(s)/\delta$. This ensures that the contribution of the tree to $c_i^1(s)$ is still $x(s)$ per unit whether it is sold in round $\tau = 1$ or consumed in period $\tau = 2$. Indeed, the natural price of the tree in round 1 would thus be $p_{1,1}^*(s) = \delta \times 1/\delta x(s) = x(s)$. A possible interpretation of this preference shock is an unexpected increases of overnight rates on reserves from 0 to $r > 0$. Settlement sub-periods then correspond to two different business days. With discounting, the physical necessity to deliver over 2 rounds has real consequences unlike in Appendix 2.8.1 where $\delta = 1$. [22].

We now introduce the definition of a settlement equilibrium given positions traded in period 0. Appendix 2.8.1 provides a more formal treatment.

Definition: *A settlement equilibrium in state s is a pair of prices $\{p_{1,\tau}(s)\}_{\tau=1,2}$, trades, default $\{d_{1,\tau}(s)\}_{\tau=1,2}$ and fail $\{f_{1,\tau}(s)\}_{\tau=1,2}$ decisions by borrowers and lenders respectively such that :*

i) $\{d_{1,\tau}(s), f_{1,\tau}(s)\}_{,\tau=1,2}$ *are optimal given* $\{p_{1,\tau}(s)\}_{\tau=1,2}$

ii) Tree market clears in every sub-period $\tau = 1, 2$

We now construct heuristically the settlement equilibrium when $\delta < 1$. Observe first that borrowers actually benefit from late delivery since the present value of collateral is $x(s)$ while the present value of the payment is $\delta < 1$. Hence, since borrowers did not default when $\delta = 1$, they do not when $\delta < 1$. Lenders, on the other side of the trade, lose with a late delivery which drives up the price of the asset in sub-period $\tau = 1$. I will now determine the price $p_{1,1}(s)$ of the tree in the first settlement round and whether lenders decide to fail or not. For this, one must

[22] Although relying on an unanticipated shock does not provide a fully satisfying answer, we observe that such a shock would have no consequence absent rehypothecation

equalize the value to deliver early and late for a lender. This writes

$$1 - 1/2p_{1,1}(s) = \delta - 1/2x(s)$$

To deliver in the first period, an agent must buy 1/2 unit of tree at price $p_{1,1}(s)$. For an agent who holds the collateral, this is the opportunity cost of not selling it. Delivering in period 2 costs only $p_{1,2}(s) = x(s)$ but the security payment is discounted by δ. In state $s = 1$, the clearing price is $p_{1,1}(s) = 3 - 2\delta > p^*_{1,1}(s)$. In state $s = 2$, the RHS is negative and lenders fail. One can then work out consumption in period 1 for all three agents

$$\begin{cases} c_1^A(1) = e \\ c_1^A(2) = e \end{cases} \qquad \begin{cases} c_1^B(1) = e - 1 + \delta \\ c_1^B(2) = e \end{cases} \qquad \begin{cases} c_1^C(1) = e + 1 - \delta \\ c_1^C(2) = e \end{cases}$$

Hence there is an ex-post transfer from agent B to agent C for re-using his collateral. This arises because collateral is scarce and all lenders cannot deliver in round 1. This does not affect agent A who holds the collateral he needs to deliver before settlement starts. If δ is sufficiently close to 1, the benefits from collateral circulation should still overweight the settlement costs. Importantly, sequential delivery is necessary because condition (2.10) holds. Otherwise, all agents could settle positions in the first round of trading by buying the tree. Hence, the settlement costs associated to rehypothecation materialize only when the pledgeable asset is scarce[23].

2.6.2 Exogenous Default and Propagation

In this model, default and fails are purely opportunistic and apply to securities independently from one another. A default on a security does not affect the remaining business of a trader or his ability to access spot markets. Precisely, an agents fails when the spot market price of the (unsegregated) tree exceeds the payment from the security he holds. The availability and liquidity of the spot market trade also

[23]This issue is often raised in the debate on rehypothecation. Terence Duffy, President of the Chicago Mercantile Exchange (CME) put it in a simple but compelling way during his MF Global House hearing: *"if [20 of us] are all looking to get the same security back at the same time [..], 19 of us are gonna have a problem"*

facilitates rehypothecation. Indeed, it also allows to settle positions smoothly along a collateral re-use chain even when some agent defaults. The tree previously held by this agent will be available in the spot market to settle remaining trades. In financial markets, default or fails may have more severe consequences or arise for exogenous motives. Banning defaulting agents from spot market would have little impact in this model. Indeed, Lemma 2 shows that we can consider default and fail-free securities without loss of generality. However, rehypothecation may amplify the impact of non-opportunistic default.

Suppose indeed that an agent i may go out of business between period 0 and period 1 with some exogenous probability η. In this case, any unsegregated collateral held by agent i is lost (because of the length of bankruptcy proceedings for example). Creditors may keep the collateral they received from this agent while debtors can only collect the segregated fraction of the asset they posted. Without rehypothecation, all the collateral is segregated. Hence both creditors and debtors are better-off ex-post after agent i's default. Importantly, they can still pay or deliver the asset for all transactions not implying agent i. With rehypothecation, the tree available in the economy to settle positions is reduced by $\sum_j (1-\alpha_j)\bar{k}_j\phi_j^{i,+}$, the amount of unsegregated collateral held by agent i. Hence, there might not be enough collateral to settle all remaining positions in the economy if (2.10) holds. Indeed, in this case, some of the unsegregated collateral held by agent i might be crucial to settle other positions. To fix ideas, consider against Example 2 of Section 2.5.2. If agent A disappears, there is only 3/4 units of collateral while the total demand from B for settlement with C amounts to 1 unit. Hence, a fraction 1/4 of type B agents must default. In addition, the price of the tree shoots up because of scarcity, affecting payoffs of type B agents who actually deliver. Hence, with rehypothecation, an exogenous default of type A agents would trigger endogenous default from some type B agents.

Both these examples show the importance of collateral scarcity in generating fragility during settlement. Indeed, when (2.10) holds, many credit transactions are ultimately backed by a limited amount of pledgeable asset. High aggregate leverage paves the way for coordination problems in settlement or contagion through credit chains. We hope to develop these interesting considerations in future research.

2.7 Conclusion

This paper introduces rehypothecation in a competitive economy where agents post collateral to short securities. Re-use may facilitate the circulation of collateral but limited commitment also affects the implicit asset lending transaction. I show that rehypothecation can be replaced by an efficient financial structure without rehypothecation which delivers the same velocity of collateral. Hence, my results mitigate the claim that a ban on rehypothecation would severely affect secured financial markets. In the presence of market incompleteness however, the irrelevance result breaks down as rehypothecation allows to free (inefficiently) encumbered collateral and increase welfare. Decentralized trading magnifies these gains as intermediaries typically need to take long and short positions simultaneously, which are costly in terms of collateral use. Finally, I showed that when aggregate collateral is scarce, settlement of positions along collateral chains is fragile.

This theory of rehypothecation fundamentally stresses the relationship between market incompleteness and collateral velocity. In general, other frictions[24] not modeled in this paper could explain why dealer banks find it valuable to re-use collateral pledged by customers. The analysis of Section 2.6 also calls for a better understanding of collateral circulation and settlement of positions with re-used collateral.

Since rehypothecation does not improve upon our complete market benchmark, this analysis further stresses the importance of asset availability for collateralized financial markets. A clear consensus about the empirical relevance of such asset scarcity has yet to emerge (cf CGFS (2013) for a recent overview). Still, the increasing demand for high quality assets to be used as collateral appears as a major trend in financial markets. In particular, the regulatory effort towards more central clearing of OTC trades may have significant effects on collateral demand. The implications of these trends for asset prices, public and private asset supply as well as unconventional monetary policy still rank high on research agendas.

[24]Singh (2011) reports that collateral desk at dealer banks should be ‘self-funding” to avoid dipping in their own balance sheet for collateral. On a similar note, Kirk et al. (2014) believe that dealer banks might find it ‘uneconomical” to use their own collateral.

2.8 Appendix

2.8.1 Sequential trading and settlement

In this section, I present a variation of the model with the same physical environment but where trading and settlement takes place over several subperiods. This departure allows to get a better understanding of collateral circulation and accommodates the need for sequential settlement when collateral is scarce and Assumption **A** does not hold.

I show that there exists a way to settle trades in an orderly fashion whereby securities traded first are settled last and conversely. In this configuration, by definition, collateral needs to deliver may not exceed the amount pledged during the corresponding trading round. In the absence of time discounting, agents agree to wait in line for settlement. In this context, in spite of its apparent importance, Assumption **A** is immaterial to our results in the following sense. Any equilibrium allocation of the model in the main text obtains as an equilibrium of the model introduced below.

Period 0 (resp. period 1) now contains T trading (resp. settlement) rounds. In each trading round $\tau \in \{1, .., T\}$, agent i may buy a quantity $\theta^{i,B}_{0,\tau}$ of the tree and take positions $(\phi^{i,+}_{\tau,j}, \phi^{i,-}_{\tau,j}) \in \mathbb{R}^2_+$ for securities $j \in \mathcal{J}$. In round τ, the tree trades competitively at price $p_{0,\tau}$ and security j trades competitively at price $q_{\tau,j}$. We assume that an agent trades with a representative pool of agents taking the opposite position so we need not index a security by the identity of the (seller,buyer) pair[25]. In period 1, uncertainty is resolved and the settlement stage takes place in a backward fashion. Securities traded in trading round $\tau \in \{1, .., T\}$ are settled in settlement round $T - \tau + 1$. The tree, which pays dividends after the last settlement round is traded at price $p_{1,\tau}(s)$ in round τ and state s. Trading the tree after uncertainty is resolved allows agents to get back the collateral they rehypothecated for delivery. Consumption in period 0 and 1 take place at the last round of the trading and settlement stages respectively. For every variable, subscript 0 (resp. 1) denotes the trading (resp. settlement) stage as before. The second subscript τ denotes the round. Figure 2.3 presents the time-line of the sequential trading model.

Period 0 : Trading stage

Agent i enters trading round $1 < \tau \leq T$ with (i) a quantity $\omega^i_{0,\tau-1}$ of consumption good (also called cash here), (ii) a quantity of tree $\theta^i_{0,\tau-1}$ and (iii) a record of securities traded during the $\tau - 1$ previous rounds $\mathbf{\Phi}^i_{\tau-1} = (\boldsymbol{\phi}^i_1, .., \boldsymbol{\phi}^i_{\tau-1}) \in \mathbb{R}^{2J\times(\tau-1)}$. In line with the main model, the initial values of the variables introduced above are $\omega^i_{0,0} := \omega^i_0$ and $\theta^i_{0,0} := \theta^i_0$. During round τ, given respective prices $(p_{0,\tau}, \boldsymbol{q}^{\cdot}_{\tau})$, he chooses new purchases of collateral $\theta^{i,B}_{0,\tau}$ and securities $\boldsymbol{\phi}^i_\tau \in \mathbb{R}^{2J}$.

[25]This is not an issue in the simultaneous model of the main text since decisions to default or fail are the same for all agents by construction

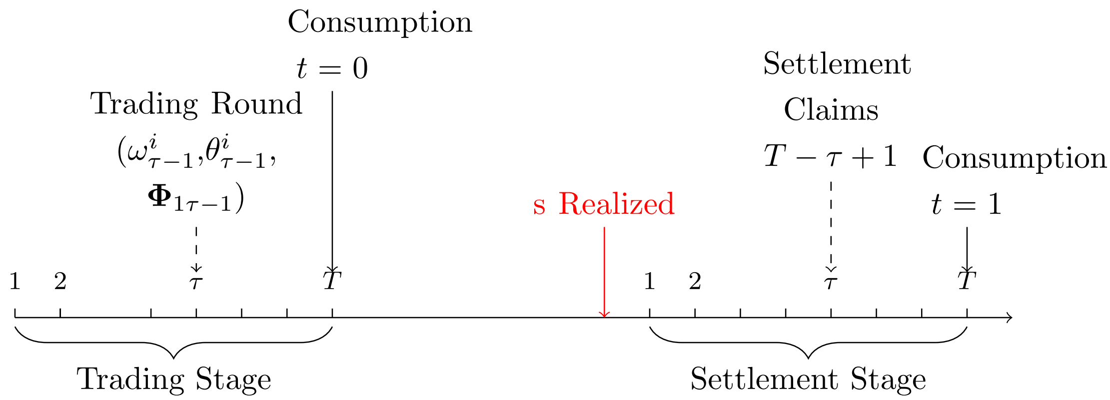

Figure 2.3: Timeline of the sequential trading model

The following equations then describe the evolution of the cash and tree accounts from one trading round[26] to the next:

$$\omega^i_{0,\tau} = \omega^i_{0,\tau-1} - p_{0,\tau}\theta^{i,B}_{0,\tau} - \boldsymbol{q}^i_\tau.\boldsymbol{\phi}^{i,+}_\tau + \boldsymbol{q}^i_\tau.\boldsymbol{\phi}^{i,-}_\tau \tag{2.11}$$

$$\theta^i_{0,\tau} = \theta^i_{0,\tau-1} + \theta^{i,B}_{0,\tau} - \sum_j \bar{k}_j \phi^{i,-}_{\tau,j} + \sum_j (1-\alpha_j)\bar{k}_j \phi^{i,+}_{\tau,j} \tag{2.12}$$

Furthermore, cash and collateral account balances must verify the following constraints in any period τ

$$\omega^i_{0,\tau} \geq 0 \tag{2.13}$$

$$\theta^i_{0,\tau-1} + \theta^{i,B}_{0,\tau} \geq \sum_j \bar{k}_j \phi^{i,-}_{\tau,j} \tag{2.14}$$

Constraint (2.13) requires agents to hold positive cash holdings at the end of every trading round. Inequality (2.14) is the the collateral constraint for trading round τ. Its formulation makes the circulation of collateral explicit. When an agent receives collateral, the re-usable part can only be sold or re-pledged in the next trading round. Hence, in a given trading round, no piece of tree is used twice as collateral. The simultaneous model abstracts from such cash and tree-in-advance constraints as it only requires that $\omega^i_{0,T} \geq 0$ and $\theta^i_{0,T} \geq 0$. Naturally, the agent consumes in period 0 whichever amount of consumption good he has left after he conducted his trades, that is

$$c^i_0 := \omega^i_{0,T} \tag{2.15}$$

[26]When he takes long positions in securities, agent i also adds up $\sum_j \alpha_j \bar{k}_j \phi^{i,+}_{\tau,j}$ to his segregated account. For expository convenience, I abstract from the mechanical accounting of tree inflows in the segregated account

Period 1 : Settlement stage

In settlement round $1 \leq \tau \leq T$, only security traded during trading round $T-\tau+1$ are settled if any. A trader i with a short position in security j $\phi^{i,-}_{j,T-\tau+1} > 0$ either repays the face value $\bar{R}_j(s)$ or defaults. A trader h with a long position $\phi^{h,+}_{j,T-\tau+1} > 0$ decides either to turn back the unsegregated collateral $(1-\alpha_j)\bar{k}_j$ or fails. Since uncertainty is resolved before settlement takes place, I abstract from indexing variables by the realized state s in the following.

As in the trading stage, I introduce the variables summarizing an agent's position upon entering settlement round τ, i.e. (i) consumption good (sometimes referred to as cash) in quantity $\omega^i_{1,\tau-1}$, (ii) a quantity of tree $\theta^i_{1,\tau-1}$ and (iii) securities yet to be settled $\mathbf{\Phi}^i_{T-\tau+1} = (\boldsymbol{\phi}^i_1, .., \boldsymbol{\phi}^i_{T-\tau-1}) \in \mathbb{R}^{2J\times(T-\tau-1)}$. At every settlement stage τ, an agent must decide whether to default on securities he shorted at trading stage $T-\tau+1$ and whether to deliver collateral on his long positions. To deliver collateral, he may either use his tree holdings or buy tree (quantity $\theta^{i,B}_{1,\tau}$) in the collateral market of round τ.

The initial values of the variables introduced above are $\omega^i_{1,0} := \omega^i_1$ and $\theta^i_{1,0} := \theta^i_{0,T}$. In each settlement round τ, given the tree price sequence $(p_{1,\tau}, ..., p_{1,T})$ agent i chooses optimally the tree purchase $\theta^{i,B}_{1,\tau}$, the default and fail decisions to maximize his period 1 consumption

$$c^i_1 = \omega^i_{1,T} + \theta^i_{1,T} x \tag{2.16}$$

In every settlement round τ, let us denote $d^i_{j,\tau}$ the decision of agent i to default on security $(j, T-\tau+1)$ and $f^i_{,j,\tau}$ the decision to fail on security $(j, T-\tau+1)$. Finally, *actual* payments and deliveries are ultimately conditional on counterparties' decisions which are taken as given[27]. During the settlement stage, the cash and collateral accounts of agent i evolve as follows:

$$\omega^i_{1,\tau} = \omega^i_{1,\tau-1} - p_{1,\tau}\theta^{i,B}_{1,\tau} + \sum_j (1-f^i_{j,\tau})(1-d_{j,\tau})\bar{R}_j \phi^{i,+}_{T-\tau+1,j} - \sum_j (1-d^i_{j,\tau})(1-f_{j,\tau})\bar{R}_j \phi^{i,-}_{T-\tau+1,j} \tag{2.17}$$

$$\begin{aligned} \theta^i_{1,\tau} =& \theta^i_{1,\tau-1} + \theta^{i,B}_{1,\tau} - \sum_j (1-f^i_{\tau,j})\bar{k}_j \left[(1-d_{\tau,j})(1-\alpha_j) - d_{\tau,j}\alpha_j \bar{k}_j\right] \phi^{i,+}_{T-\tau+1,j} \\ &+ \sum_j (1-d^i_{\tau,j})\bar{k}_j \left[(1-f_{\tau,j}) + \alpha_j f_{\tau,j}\right] \phi^{i,-}_{T-1+\tau,j} \end{aligned} \tag{2.18}$$

[27] Remember, that an agent trades with a representative sample of those agents who take an opposite position, so that there is pooling.

Finally, let us write down the liquidity constraints imposed by sequential settlement.

$$\omega^i_{1,\tau} \geq 0 \tag{2.19}$$

$$\theta^i_{1,\tau-1} + \theta^{i,B}_{1,\tau} \geq \sum_j (1 - f^i_{\tau,j})(1-\alpha_j)\bar{k}_j(1-d_{\tau,j})\phi^{i,+}_{T-\tau+1,j} \tag{2.20}$$

Equation (2.19) rules out inter-round credit because of limited commitment. Equation (2.20) states that in order to return collateral on long positions during round τ, an agent should either dip in its tree account or purchase the tree in the market. Essentially, he cannot use collateral that is returned to him from a short position in the same round. Symmetrically to the trading round where no piece of asset can be pledged twice in the same round, it cannot be returned twice in the same settlement round. These cash and tree-in-advance trading constraints may generate frictions along the sequential delivery process. Indeed, borrowers cannot use the proceeds of the sale of returned collateral to pay their loan in a given round[28].

Let us call $\boldsymbol{d}^{-i}$ and $\boldsymbol{f}^{-i}$ the $T \times (J-1) \times (I-1)$ vectors collecting decisions to fail and default by agents other than i. We label $c^*_1(\omega^i_1, \theta^i_{0,T}, \boldsymbol{\Phi}^i_{1T}, \boldsymbol{p}_1, \boldsymbol{d}^{-i}, \boldsymbol{f}^{-i})$ the optimal choice of a consumer with initial balances $(\omega^i_1, \theta^i_{0,T}, \boldsymbol{\Phi}^i_{1T})$, taking as given the sequence of collateral price $\boldsymbol{p}_1$ and other traders' decisions $(\boldsymbol{d}^{-i}, \boldsymbol{f}^{-i})$ where trader i optimizes with respect to $(\theta^{i,B}_{1,\tau}, \boldsymbol{d}^i_\tau, \boldsymbol{f}^i_\tau)_{\tau=1..T}$ given constraint (2.19)-(2.20).

Definition 5: Given initial positions $(\omega^i_1, \theta^i_{0,T}, \boldsymbol{\Phi}^i_{1T})^{i\in\mathcal{I}}$, and a state $s \in \mathcal{S}$, a **settlement equilibrium** is a price sequence $\boldsymbol{p}_1 = (p_{1,1}, .., p_{1,T})$, payments and delivery decisions $\{\boldsymbol{d}^i_\tau, \boldsymbol{f}^i_\tau\}_{\tau=1..T}$ for every agent $i \in \mathcal{I}$ and an allocation $\{c^i_1\}^{i\in\mathcal{I}}$ such that

1. $p_{1,\tau}$ clear the collateral market in round τ
2. $\forall i \in \mathcal{I}$, $c^i_1 = c^*_1(\omega^i_1, \theta^i_{0,T}, \boldsymbol{\Phi}^i_{1T}, \boldsymbol{p}_1, \boldsymbol{d}^{-i}, \boldsymbol{f}^{-i})$.
3. $\{\boldsymbol{d}^i_\tau, \boldsymbol{f}^i_\tau\}_{\tau=1..T}$ is consistent with individual decisions to default and fail.

In a settlement equilibrium of the sequential model, default and fail decisions might differ from that of the simultaneous model if liquidity constraints (2.13)-(2.13) bind or if the price sequence for the tree $\{p_{1,\tau}\}_{\tau=1..T}$ is non trivial. Equipped with the settlement equilibrium concept, we can naturally define an equilibrium of the sequential model.

Definition 6: Equilibrium An equilibrium of the sequential model is a trading price vector $(p_{0,\tau}, \boldsymbol{q}_\tau) \in \mathbb{R}^{(J+1)\times T}$, a settlement price vector $(\boldsymbol{p}_1(1), ..., \boldsymbol{p}_1(S)) \in \mathbb{R}^{S\times T}$, a default and fail map $(\boldsymbol{d}, \boldsymbol{f}) \in \mathbb{R}^{J\times I\times S\times T}$ and an allocation $(\boldsymbol{c}_0, \boldsymbol{c}_1) \in \mathbb{R}^{I\times(S+1)}$ such that

1. $\forall s \in \mathcal{S}$, $(\boldsymbol{c}_1(s), \boldsymbol{p}_1(s), \boldsymbol{d}(s), \boldsymbol{f}(s))$ form a settlement equilibrium given $(\omega^i_1(s), \theta^i_{0,T}, \boldsymbol{\Phi}^i_{1T})^{i\in\mathcal{I}}$

[28] Observe that the situation is asymmetric here since lenders can use the cash received from the borrower to purchase the tree he must return to this borrower. We could restore symmetry with an additional constraint on trades but this is not central to our argument.

2. $\forall i \in \mathcal{I}$,

$$(\theta^{i,U}_{0,T}, \boldsymbol{\Phi}^i_{1T}) \in \arg\max \quad u^i(\omega^i_{0,T}) + \sum_{s=1}^{S} \pi(s) u^i(c^1_i(s))$$
$$\text{where} \quad \omega^i_{0,T} \quad \text{is defined by} \quad (2.11) - (2.13)$$

3. Security and tree markets clear in each round where they are traded.

4. Agents form correct expectations about $\boldsymbol{p}_1$, $\boldsymbol{d}$ and $\boldsymbol{f}$.

Essentially, the sequential equilibrium concept involves more constraints than the equilibrium of the main text. First, agents must satisfy cash and tree-in-advance constraints (2.13)-(2.14) and (2.19) -(2.20) at every round of trading. In addition markets must clear in each round. In the simultaneous model, these round constraints all collapse into a single constraint.

The two models are equivalent if any equilibrium allocation of the simultaneous trading model of the main text is an equilibrium allocation of the sequential model developed above. In particular, we are interested for a fundamental settlement equilibrium where the tree price equals the dividend x in every round and default and fail decisions are similar to that of the main text. In the trading round, the price of the tree and securities should not depend on the round either. We now prove this equivalence.

Equivalence between the models

We show that an agent can finance any equilibrium consumption plan of the simultaneous model (model *Sim*) when facing the budget constraint of the sequential model (model *Seq*) where the securities and the tree trade in each round at their original price.

Proposition 3
Every equilibrium allocation of model *Sim* is budget feasible under model *Seq*.

Proof
$\triangleright$ Let $(\boldsymbol{c}, p, \boldsymbol{q})$ be an equilibrium of the *Sim* model. By definition, for every trader i, there exists a portfolio $(\theta^i, \boldsymbol{\phi}^i) \in \mathbb{R} \times \mathbb{R}^{2J}_+$ which finances c^i under $(p, \boldsymbol{q})$. We want to show that $\boldsymbol{c}$ can be financed despite the liquidity constraints of the *Seq* model. For this, we consider a fundamental settlement equilibrium of the *Seq* model with $p_{1,\tau} = x$ for all τ. In addition, the relevant trading prices are $p_{0,\tau} = p$ and $\boldsymbol{q}_\tau = \boldsymbol{q}$ for any τ, i.e. constant collateral and security prices. Denote $Sim\mathcal{BC}(p, \boldsymbol{q}, \omega_0, \theta_0)$ (resp. $Seq\mathcal{BC}_T(p, \boldsymbol{q}, \omega_0, \theta_0)$) the budget constraint in the *Sim* model (resp. the *Seq* model with T trading rounds). We need to show that . $c \in Seq\mathcal{BC}_T(p, \boldsymbol{q}, \omega_0, \theta_0)$ for some T.

Let us consider the following trades over two rounds :

1. Buy $\theta^B_{0,1} = \theta - \theta_0 + \frac{1}{p}\boldsymbol{q}.\boldsymbol{\phi}^+$ units of tree and short $\boldsymbol{\phi}^-_1 = \boldsymbol{\phi}^-$

2. Buy $\theta^B_{0,2} = -\boldsymbol{q}.\boldsymbol{\phi}^+$ units of tree and buy $\boldsymbol{\phi}^+$

It is easy to see that as a result of the two operations, the agent effectively holds total portfolio $(\theta, \boldsymbol{\phi})$ which finances $\boldsymbol{c}$ under prices $(p, \boldsymbol{q})$. In addition, the second round operation is budget neutral. We are left to verify that liquidity constraints in trading round 1 and settlement rounds 2 are verified.

For the first point, use the fact that $(p, \boldsymbol{q})$ are equilibrium prices of the simultaneous model. By absence of arbitrage, an agent cannot make profit by extending loans and selling the re-usable collateral. This writes:

$$\left(\frac{1}{p}\boldsymbol{q} - (1-\alpha)\bar{\boldsymbol{k}}\right).\boldsymbol{\phi}^+ \geq 0 \tag{2.21}$$

Thus, for the collateral constraint observe that

$$\theta^U_{0,0} + \theta^B_{0,1} = \theta_0 + \theta - \theta_0 + \frac{1}{p}\boldsymbol{q}.\boldsymbol{\phi}^+ \geq \theta + (1-\alpha)\bar{\boldsymbol{k}}.\boldsymbol{\phi}^+ \geq \bar{\boldsymbol{k}}.\boldsymbol{\phi}^- = \bar{\boldsymbol{k}}.\boldsymbol{\phi}^-_1$$

where the first inequality uses (2.21), the second is the collateral constraint (2.6) of the simultaneous model. For the liquidity constraint (2.13), note that

$$\omega_{0,1} = \omega_0 - p\theta^B_{0,1} + \boldsymbol{q}.\boldsymbol{\phi}^- = \omega_0 - p(\theta - \theta_0) - \boldsymbol{q}.\boldsymbol{\phi}^+ + \boldsymbol{q}.\boldsymbol{\phi}^- = c_0 \geq 0$$

where the inequality follows from $\boldsymbol{c} \in Sim\mathcal{BC}(p, \boldsymbol{q}, \omega_0, \theta_0)$

Consider now settlement round 2. Since this is the last settlement round, the agent may cash in the tree dividend to pay for short positions traded in round 1. Hence the cash in advance constraint for settlement does not bind because all the short positions where traded in trading round 1.

Finally, since budget feasible plans are the same and $\boldsymbol{c}$ is optimal in the simultaneous model under prices $(p, \boldsymbol{q})$, this is also the consumer's choice in the sequential model[29] under $(p, \boldsymbol{q})$. $\triangleleft$

2.8.2 Proofs

Proof of Lemma 1

$\triangleright$ Let $E := (\boldsymbol{c}, p, \boldsymbol{q})$ be an equilibrium in economy $\mathcal{E}(\mathcal{J} \cup \mathcal{J}_1)$ where $\mathcal{J}_1 \in Sp(\mathcal{J})$. By definition, for every agent i, there exists a portfolio $(\theta^i, \boldsymbol{\phi}^{i+}, \boldsymbol{\phi}^{i-}) \in \mathbb{R} \times \mathbb{R}_+^{|\mathcal{J} \cup \mathcal{J}_1|} \times \mathbb{R}_+^{|\mathcal{J} \cup \mathcal{J}_1|}$ of collateral and securities of $\mathcal{J} \cup \mathcal{J}_1$ which finances the allocation $\boldsymbol{c}^i$ under prices $(p_0, \boldsymbol{q_0})$ according to budget

[29]The proof for Proposition 3 relies on a budget set argument. One should not infer from the analysis above that we characterized the actual equilibrium trades in the sequential model leading to $\boldsymbol{c}$ under $(p, \boldsymbol{q})$, let alone that two trading rounds of the sequential model would suffice to reach the allocation in equilibrium. Indeed, while they are budget feasible on an individual basis, the trades described in the proof are not mutually consistent with the equilibrium requirement of market clearing.

constraint (2.4)-(2.6). In what follows, I show that $\boldsymbol{c}$ is an equilibrium allocation of economy $\mathcal{E}(\mathcal{J}_1)$ where only securities in $\mathcal{J}_1$ are available for trade. Let $E_1 = (\boldsymbol{c}, p_1, \boldsymbol{q}_1)$ this equilibrium to be constructed.

Every security is priced in E, even if it is not traded. Let us set prices for securities $(p_1, \boldsymbol{q}_1)$ of $\mathcal{J}_1$ in E_1 to their value in E. Since $\mathcal{J}_1 \subset \mathcal{J} \cup \mathcal{J}_1$ budget feasible allocations in E_1 are feasible in E.

Suppose now a security $j = (\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j) \in \mathcal{J} \backslash \mathcal{J}_1$ is traded in equilibrium E. By definition of $\mathcal{J}_1$, there exists a replicating portfolio $\psi(j) = (\theta(j), \boldsymbol{\phi}(j))$ verifying criteria $(i) - (iii)$ of Definition 3. For any agent i, replace every long position ϕ_j^{i+} (resp. short position ϕ_j^{i+}) in security j by ϕ_j^{i+} (resp. $-\phi_j^{i+}$) units of portfolio $\psi(j)$. The following points prove that the substitution achieves our goal

a) Using market clearing for security j in E, the securities in $\psi(j)$ verify market clearing in E_1. To put it otherwise, the substitution is resource neutral.

b) Second, any agent's payoff in period 1 stays identical by definition of $\psi(j)$. Furthermore, the replicating portfolio's price in E_0 equals that of the security. If the former were strictly lower, by monotonicity of preferences long agents would have bought $\psi(j)$ instead of j. If it were strictly higher, short agents would have sold $\psi(j)$ instead of j. Hence, the substitution is also cost neutral.

c) Finally, substituting $\psi(j)$ for j does not violate the collateral constraint. By construction the substitution does not require more collateral or more segregation. To see this, consider a long agent first. The net variation in the collateral constraint from this substitution is

$$\Delta\theta^+ = -(1-\alpha_j)\bar{k}_j + \theta(j) + \sum_{j_1 \in \psi(j)} (1-\alpha_{j_1})\phi_{j_1}\bar{k}_{j_1} \geq 0$$

The term to enter negatively is the quantity of collateral that can be re-pledged out of 1 unit of security j. The positive terms are the quantity of tree in the replicating portfolio and the re-usable collateral in the securities of $\psi(j)$.
For an agent short in j, the substitution yields

$$\Delta\theta^- = +k_j - \theta(j) - \sum_{j_1 \in psi(j)} \phi_{j_1}\bar{k}_{j_1} \geq 0$$

The term to enter positively is the collateral requirement for j. The first negative term accounts for the sale of $\theta(j)$ units of tree while the second one represents the collateral requirement to short the securities in $\psi(j)$.

Hence, we have shown that $\boldsymbol{c}$ is budget feasible with securities in $\mathcal{J}_1$. Since this is the optimal choice of agents under a larger budget set, $\boldsymbol{c}$ is the optimal choice of agents in $\mathcal{E}(\mathcal{J}_1)$ and thus constitutes an equilibrium allocation. $\lhd$

Proof of Lemma 2

$\triangleright$ Let $j = (\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j) \in \mathcal{J}_0$ and consider security $j' = (\boldsymbol{R}_j/\bar{k}_j, 1, \alpha_j)$ the face value of which is proportional to the actual payoff of j. Since $\bar{R}_{j'}(s) \in [(1-\alpha_j)x(s), x(s)]$, we have $R_{j'}(s) = \bar{R}_{j'}(s) = (1/\bar{k}_j)R_j$. Hence, security j can be replicated by $\bar{k}_j$ units of security j'. It is thus enough to find a replicating portfolio for j'. We can then restrict our attention to the following set :

$$\mathcal{J}_1 = \left\{ (\bar{\boldsymbol{R}}_j, \bar{k}_j, \alpha_j) \mid \bar{R}_j(s) \in [(1-\alpha_j)x(s), x(s)],\ \bar{k}_j = 1,\ \alpha_j \in [0,1] \right\} \subset \mathcal{J}_0$$

which is exactly the set of no-default/no-fail securities collateralized by one unit of the tree. Let now $j = (\bar{\boldsymbol{R}}, 1, \alpha) \in \mathcal{J}_1$ and re-order the states $s = 1, .., S$ so that

$$1 \geq \frac{\bar{R}(1)}{x(1)} \geq \frac{\bar{R}(2)}{x(2)} \geq \cdots \geq \frac{\bar{R}(S)}{x(S)} \geq (1-\alpha) \tag{2.22}$$

Whenever $\bar{R}(s) = \bar{R}(s')$, let the initial ordering prevail. Next consider the $S-1$ securities $\{j_1, j_2, .., j_{S-1}\} := \mathcal{J}_*(j) \subset \mathcal{J}_*$ which verify

$$\begin{aligned} R_{j_l}(s) &= x(s), \quad \text{if } 1 \leq s \leq S-l \\ R_{j_l}(s) &= 0, \qquad \text{otherwise} \end{aligned}$$

Security j_l has the same payoff as one unit of collateral in the first $S-l$ state. I now derive the portfolio $\psi(j) = (\theta, \phi_{j_1}, .., \phi_{j_{S-1}}) \in \mathbb{R}_+^S$ of tree and securities of $\mathcal{J}_*(j)$ to replicate j. To this effect, set

$$\begin{aligned} \forall l \geq 1, \quad \phi_{j_l} &= \frac{R(S-l)}{x(S-l)} - \frac{R(S-l+1)}{x(S-l+1)} \\ \theta &= \frac{R(S)}{x(S)} \end{aligned}$$

By construction, this portfolio replicates security j's payoff so that requirement (i) of Definition 3 holds. The collateral needed to sell portfolio $\psi(j)$ is

$$k(\psi(j)) = \theta + \sum_{l=1}^{S-1} \phi_{j_l} = \frac{R(1)}{x(1)} \leq 1 = \bar{k}_j$$

Finally, the collateral segregated when selling $\psi(j)$ is:

$$k(\psi(j)) = \sum_{l=1}^{S-1} \alpha_{j_l}\phi_{j_l} = \sum_{l=1}^{S-1} \phi_{j_l} \frac{R(1)}{x(1)} - \frac{R(S)}{x(S)} \leq \alpha_j$$

Hence conditions (ii) and (iii) also hold. We thus proved that any contract in $\mathcal{J}_0$ can be replicated by contracts in $\mathcal{J}_*$ according to Definition 3. $\triangleleft$

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Chapter 3

Repurchase Agreements

3.1 Introduction

According to Gorton and Metrick (2012), the financial panic of 2007-08 started with a run on the market for repurchase agreements (repos). Their paper was very influential in shaping our understanding of the crisis. It was quickly followed by many attempts to understand repo markets more deeply, both empirically and theoretically as well as calls to regulate these markets[1].

A repo is the sale of an asset combined with a forward contract that requires the original seller to repurchase the asset at a given price. Repos are different from simple collateralized loans in (at least) one important way. A repo lender obtains the legal title to the pledged collateral and can thus use the collateral during the length of the forward contract. This practice is known as re-use or re-hypothecation. With standard collateralized loans, borrowers must agree to grant the lender similar rights[2]. This special feature of repos has attracted a lot of attention from economists and regulators alike.

Repos are extensively used by market makers and dealer banks as well as other

[1]See Acharya (2010) "A Case for Reforming the Repo Market" and (FRBNY 2010)

[2]Aghion and Bolton (1992) argue that securities are characterize by cash-flow rights but also control rights. Collateralized loans grant neither cash-flow rights nor control rights over the collateral to the lender unless the counterparties sign an agreement for this purpose. As a sale of the asset, a repo automatically gives the lender full control rights over the security as well as over its cash-flows. Re-use rights follow directly from ownership rights. As Comotto (2014) explains, there is a subtle difference between US and EU law however. Under EU law, a repo is a transfer of the security's title to the lender. However, a repo in the US falls under New York law which is the predominant jurisdiction in the US. *"Under the law of New York, the transfer of title to collateral is not legally robust. In the event of a repo seller becoming insolvent, there is a material risk that the rights of the buyer to liquidate collateral could be successfully challenged in court. Consequently, the transfer of collateral in the US takes the form of the seller giving the buyer (1) a pledge, in which the collateral is transferred into the control of the buyer or his agent, and (2) the right to re-use the collateral at any time during the term of the repo, in other words, a right of re-hypothecation. The right of re-use of the pledged collateral (...) gives US repo the same legal effect as a transfer of title of collateral."* To conclude, although there are legal differences between re-use and rehypothecation, they are economically equivalent (see e.g. Singh, 2011) and we treat them as such in our analysis.

financial institutions as a source of funding, to acquire securities that are on specials, or simply to obtain a safe return on idle cash. As such, they are important to explain leverage in financial institutions land to determine the liquidity of certain assets. The Federal Reserve Bank, the central bank in the United States, and other central banks use repos to steer the short term nominal interest rate. The Fed's newly introduced reverse repos are considered an effective tool to increase the money market rate when there are large excess reserves. Repos thus became essential to the conduct of monetary policy. Finally, firms also rent capital and use collateralized borrowing and some forms of repos to finance their activities or hedge exposures (notably interest rate risk, see BIS, 1999). Hence, the use of repos also affects the real economy and macroeconomic activity.

Most existing research papers study specific aspects of the repo markets, e.g. exemption from automatic stay, fire sales, etc., taking the repo contract and most of its idiosyncrasies as given. These theories leave many fundamental questions unanswered, such as why are repos different from collateralized loans? What is the nature of the economic problem solved by the repo contract? To answer these questions, to understand the repo market and the effect of regulations, one cannot presume the existence or the design of repo contracts. In this paper we characterize a simple economic environment where repos emerge as the funding instrument of choice. More precisely, we borrow techniques from security design to derive the equilibrium collateralized contract. The interpretation as a repo contract is natural since the borrower ultimately sells an asset spot combined with a promise to repurchase at an agreed price.

The model has three periods and two types of agents, a natural borrower and a natural lender, both risk-averse. The borrower is endowed with an asset that yields an uncertain payoff in the last period. The payoff realization becomes known in the second period and is reflected in the second period price of the asset. To increase his consumption in the first period, the borrower could sell the asset to the lender in the spot market. However this trade will expose both parties to price risk in the second period. Instead, the borrower can obtain resources from the lender by selling the asset combined with a forward contract promising to repurchase the asset in period 2. Unlike with an outright sale, a constant repurchase price in a repo hedges market price risk. Under limited commitment however, the borrower might not honor his promise. Indeed, he may find it optimal to default if the value of collateral falls below the promised repayment[3]. We assume that in addition to the loss of the collateral, a defaulting borrower incurs a cost commensurate with the size of default. To avoid this wasteful default, the repurchase price of the repo contract cannot lie above a multiplier of the asset price proportional to this default cost parameter. In high states of the world or when the asset is abundant, this constraint does not bind and the repurchase price is constant. In low states of the world however, the asset pays very little and the borrower exhausts his borrowing capacity : the repurchase price increases with the spot market price.

Using this equilibrium contract we derive comparative statics for haircuts and liquidity premia. Haircuts increase with counterparty risk as a riskier agent can promise less income per unit of asset pledged. More risky collateral commands a larger haircut and a lower liquidity premium. Compared to a safe asset, a risky security pays less in bad times and more in good times. Since agents are constrained in bad times, this is precisely when collateral is valuable. Hence the liquidity premium is higher for the safe asset. In good times, agents do not exploit the higher value of the riskier collateral since the repurchase price becomes constant. Hence, compared to the safe asset, less of the risky asset's payoff is pledged and the haircut is larger.

In Section 3.4, we introduce collateral re-use. In a repo, the lender indeed acquires ownership of the asset used as collateral in the repo transaction. In our model, a lender might re-use a fraction of the asset he receives as collateral. We show that agents strictly prefer to re-use as it increases the borrowing capacity of the repo seller. To fix ideas, suppose the collateral is perfectly safe and pays \$100 in the second period. The net interest rate is 0 so that \$100 is also the price of the asset in period 1. With the extra cost for default, a borrower can promise to repay more than \$100 per unit of the asset in period 2, say \$110. The lender can then re-use some of the collateral by selling it back to the borrower. The latter can now pledge another \$110 per unit. With one round of re-use, the borrower netted an extra \$10 per unit.

[3] In practice, even in the absence of outright default, traders opportunistically delay the settlement of transactions, as documented by Fleming and Garbade (2005). In our model, the repurchase price can be made state-contingent to prevent default in equilibrium. The state-contingency somehow plays a similar role to margin adjustment or repricing in actual transactions.

These trades can be repeated as long as collateral can circulate. Overall, re-use has a multiplier effect since a borrower can pledge more income with the same quantity of the asset in those states where he is constrained[4]. Without the non-pecuniary penalty, this extra borrowing capacity disappears and and re-use does not affect the equilibrium allocation, a result in line with Maurin (2015). Overall, the model implies that collateral re-use should be more prevalent for assets that command low haircuts and when the lender's trading partners have low counterparty risk.

Finally, Section 3.5 discusses the implications of collateral re-use for the repo market structure. We argue that some participants naturally emerge as intermediaries when they can re-use collateral. In practice, dealer banks indeed make for a significant share of this market by intermediating between natural borrowers (say hedge funds) and lenders (say money market funds or MMF). This might seem puzzling if direct trading platforms are available for both parties to bypass the dealer bank[5]. Our model rationalizes intermediation with difference in trustworthiness and ability to re-deploy the collateral. In our example, the hedge fund delegates borrowing to the dealer bank if the later is more trustworthy. Although there are larger gains from trade with the MMF, the hedge fund prefers borrowing from the dealer bank if he is more efficient at re-using collateral. Indeed, through re-use, one unit pledged to the dealer bank can then support more borrowing in the chain of transactions. Our model thus provides an theory for repo intermediation based on the endogenous choice of trading relationships.

Relation to the literature

Gorton and Metrick (2012) argue that the recent crisis started with a run on repo whereby funding dropped dramatically for many financial institutions. Subsequent studies by Krishnamurty et al. (2014) and Copeland et al. (2014) have qualified this finding by showing that the run was specific to the bilateral segment of the repo market. Recent theoretical works indeed highlighted some features of repo contracts as sources of funding fragility. As a short-term debt instrument to finance long-term

[4]Our stripped down example suggested that re-use only works when haircuts are negative. This is an artefact of the assumption that the asset is perfectly safe. When it is risky, borrowers only want to pledge more income in low payoff states where they are constrained. In good states, they might still want to pledge less income than the future value of the asset. The haircut averages over states and might thus be positive.

[5]In the US, Direct Repo^{TM} provides this service

assets, Zhang (2014) and Martin et al. (2014) show that repos are subject to roll-over risk. Antinolfi et al. (2015) emphasize the trade-off attached to the exemption from automatic stay for repo collateral. Lenders easy access to the borrower's collateral may be privately optimal but collectively harmful in the presence of fire sales, a point also made by Infante (2013) and Kuong (2015).

These papers usually take repurchase agreements as given while we want to understand their emergence as a funding instrument. One natural question is to ask why borrowers do not simply sell the collateral to lenders? Several works including our highlight the role of the commitment to the repurchase price. In Narajabad and Monnet (2012), Tomura (2013) and Parlatore (2015), it allows lenders to avoid search frictions in the spot market when reselling the asset. In contrast, our model is fully competitive but assets' payoff are risky. As a result, repos are essential because the repurchase price provides hedging against price risk. Bigio (2015) and Madison (2016) emphasize asymmetry of information about the quality of the asset. There, the commitment to repurchase insulates uninformed buyers from the information-sensitive part of the asset cash flow. The repo contract thus resembles a leasing agreement as in Hendel and Lizzeri (2002) or the optimal debt financing arrangement of DeMarzo and Duffie (1999), both of which mitigate adverse selection. Our model has symmetric information but assets payoff are random. With uncertainty, agents may also want to pledge less than the future value of the cash flow when it is expected to be high (the hedging component). Besides the different economic motivation, these works essentially identify repos with standard collateralized loans. We account for the sale of collateral in a repo by considering re-use. In addition, our theory rationalizes haircuts since borrowers choose repos when they could obtain more income in the spot market.

To derive the repo contract, we follow Geanakoplos (1996) , Araújo et al. (2000) and Geanakoplos and Zame (2014) where collateralized promises traded by agents are selected in equilibrium. Our model differs from theirs as we allow for an extra non-pecuniary penalty for default in the spirit of Dubey et al. (2005). While our results on the design of repo contracts carry through without this penalty, it is crucial for the results in Section 3.4 and 3.5 related to collateral re-use.

In the second part of the paper, we indeed account for the transfer of the legal title to the collateral to the lender, opening the possibility for re-use. Singh and Aitken (2010) and Singh (2011) argue that collateral re-use or rehypothecation lubricates transactions in the financial system[6]. However rehypothecation may entail risks for collateral pledgers as explained by Monnet (2011). While Bottazzi et al. (2012) or Andolfatto et al. (2014) abstract from the limited commitment problem of the collateral receiver, Maurin (2015) shows that re-use risk seriously mitigates the benefits from circulation. In our model indeed, re-use relaxes collateral constraints only thanks to the extra penalty for default besides the collateral loss. Asset re-use then plays a role similar to pyramiding (see Gottardi and Kubler, 2015). One difference is that lenders re-use the collateral backing the debt rather than the debt itself as collateral. We stress the role of collateral re-use in explaining repo market intermediation as in Infante (2015) and Muley (2015). Unlike these papers, intermediation arises endogenously in our model as trustworthy agents re-use the collateral from risky counterparties to borrow on their behalf. In an empirical paper, Issa and Jarnecic (2016) indeed suggested that the fee based view of repo intermediation whereby dealers gain from differences in haircuts does not stand in the data.

The structure of the paper is as follows. We present the model and the complete market benchmark in Section 3.2. We analyze the optimal repo contracts, including properties for haircuts, liquidity premiums, and repo rates in Section 3.3. In Section 3.4, we allow for collateral re-use and study intermediation in Section 3.5. Finally, Section 3.6 concludes.

3.2 The Model

3.2.1 Setting

The economy lasts three dates, $t = 1, 2, 3$. There are two types of agents $i = 1, 2$ and only one good each period. Both agents have endowment ω in all but the last period. Agent 1 is also endowed with a units of an asset while agent 2 has none. This asset pays dividend s in date 3. The dividend is distributed according to a cumulative distribution function $F(s)$ with support $\mathcal{S} = [\underline{s}, \bar{s}]$ and mean $E[s] = 1$.

[6] Fuhrer et al. (2015) estimate an average 5% re-use rate in the Swiss repo market over 2006-2013.

In date 2, the realization of s in date 3 is known to all agents. This is an easy way to model price risk at date 2.

Preferences from consumption profile (c_1, c_2, c_3) for agent 1 and 2 are

$$\begin{aligned} U^1(c_1, c_2, c_3) &= c_1 + v(c_2) + c_3 \\ U^2(c_1, c_2, c_3) &= c_1 + u(c_2) + \beta c_3 \end{aligned}$$

where $\beta < 1$, $u(.)$ and $v(.)$ are respectively strictly concave and concave functions. We assume $u'(\omega) > v'(\omega)$ and $u'(2\omega) < v'(0)$, so that there are gains from transferring resources from agent 1 to agent 2 in date 2 and the optimal allocation is interior. These preferences contain two important elements. First, as $\beta < 1$, agent 2 values less consumption in date 3 so that agent 1 is the natural holder of the asset in that period. Second, agents with concave utility function dislike consumption variability in period 2 .

While they may want to engage in borrowing and lending, agents are not able to fully commit to future promised payments. When agent i defaults on a promised repayment r, he suffers a loss $\theta_i r$, where $\theta_i \in [0, 1]$, measured in consumption units. The punishment is a deadweight cost, not transferable to the lender[7]. This non-transferability feature rules out unsecured credit, except in the limit case $\theta = 1$. Indeed, from the borrower point of view, the cost of defaulting θr strictly exceeds the cost of repaying r so that he would always default. The lender does not gain anything from a default since he does not appropriate the amount θr. This gives a natural role for the asset to be used as collateral[8]. We specify feasible collateralized contracts in the next subsections. Besides the punishment and the loss of collateral, there are no other stigma attached to default such as market exclusion. In the paper,

[7] A natural interpretation of our assumptions is that a defaulting borrower incurs non-pecuniary costs (bankruptcy procedure, hassle, loss of reputation). This costs cannot be monetized and transferred to the lender. An important feature is that the cost increases in the size of the default. The linearity is just for tractability. For simplicity, we measure this cost in terms of consumption units. Observe that u and v should be defined over negative real numbers since in theory, the punishment can be very large.

[8] If the punishment were transferable, agents could actually borrow as much as they want by defaulting and there would be no role for collateral. Our specification ultimately nests two familiar cases. When $\theta = 1$, agents always deliver on their promises so that limited commitment has no bite and collateral is useless. When $\theta = 0$, agents suffer only the loss of the asset used as collateral like in standard models by Geanakoplos (1996) and others.

we sometimes interpret the severity of the punishment θ_i as a proxy for the quality or trustworthiness of agent i.

When trading collateralized contracts or repos, lenders may re-use the asset pledged by the borrower. Specifically, lender i is able to re-use at most ν_i of the collateral he receives where $\nu_i \in [0, 1]$. We interpret ν_i as a measure of the operational efficiency of a trader in re-deploying collateral for his own trades[9].

The environment is a simple set-up where a repo contract arises naturally. Limited commitment implies that the value of the borrowers' debt must be in line with the (uncertain) market value of their collateral. But agents dislike risk and will try to hedge using a forward contract - the repurchase leg of the repo. Our model speaks to repos and not only collateralized loans because the lender may re-use the asset pledged as collateral. All markets are competitive with price-taking agents.

3.2.2 Perfect commitment

When $\theta_1 = \theta_2 = 1$, agents can perfectly commit to future promises. Unsecured credit is possible so that markets are complete. The resulting equilibrium allocation is efficient. Hence, marginal rates of substitution should be equalized whenever possible. We guess that in equilibrium, this is the case between the first and the second period[10]. Let c_t^i denote agent i consumption in period t. We obtain the following equilibrium conditions:

$$\begin{cases} u'(c_{2,*}^2) = v'(2\omega - c_{2,*}^2) \\ c_{3,*}^2 = 0 \end{cases} \tag{3.1}$$

where we used the resource constraint of period 2 to substitute for $c_{2,*}^1 = 2\omega - c_{2,*}^2$. The implicit prices for period 2 and 3 consumption are respectively $u'(c_{2,*}^2)$ and 1. Intuitively, since $\beta < 1$, agent 2 does not consume in period 3 because he has a lower marginal utility than agent 1. To pin down the equilibrium allocation completely,

[9]Singh (2011) discusses the role played by collateral desks at large dealer banks in channeling these assets across different business lines. These desks might not be available for less sophisticated repo market participants such as money market mutual funds or pension funds

[10]Conjecturing that marginal rates of substitution are equalized between the second and the third period, we find a contradiction since the resulting allocation is not budget feasible at the implied market prices.

we use the budget constraint of agent 2 and obtain $c^2_{1,*} = \omega - u'(c^2_{2,*})(c^2_{2,*} - \omega)$. This expression is positive if :

$$\omega \geq u'(c^2_{2,*})(c^2_{2,*} - \omega) \tag{3.2}$$

which we assume in the remainder of the text. In equilibrium, agent 1 borrows $c^2_{2,*} - \omega$ at a net interest rate $r^* = 1/u'(c^2_{2,*}) - 1$. Observe that agents consumption $(c^1_{2,*}, c^2_{2,*})$ in period 2 is deterministic although the asset payoff s is already known. Indeed, risk averse agents prefer a smooth consumption profile.

3.2.3 Incomplete Markets with Limited Commitment

In this section, we let $\theta_i \in [0, 1)$. Markets are incomplete because agents cannot write unsecured debt contracts and must use the asset to support trades. As we will see, the quantity of asset in the economy will now determine the equilibrium allocation. To build our intuition, we show first that agents cannot achieve the complete market allocation by using only spot trades, as it exposes them to price risk.

Spot Transactions and Efficiency

Agents can only trade the asset in a spot market. The price in period 1 (resp. period 2 and state s) is denoted p_1 (resp. $p_2(s)$). The price in period 2 reflects the future known payoff s of the asset. Let us denote a^i_1 (resp. $a^i_2(s)$) the asset holdings of agent i after trading in period 1 (resp. period 2 and date s). The budget constraints of agent 2 in period 1 and 2 write

$$c^2_1 = \omega + p_1 a^2_1$$
$$c^2_2(s) = \omega + p_2(s)(a^2_1 - a^2_2(s))$$

Using spot trades, agent 2 can implicitly lend to agent 1 if he buys the asset in period 1, that is $a^2_1 > 0$ and re-sells it in period 2, that is $a^2_2(s) < a^2_1$. We give a formal characterization of the equilibrium in the Appendix. Here, we stress our main point: a combination of spot trades can never finance the first-best allocation (3.1). Since agent 2 does not want to consume in period 3 ($\beta < 1$), he would resell any asset bought in period 1 so that $a_2^2(s) = 0$. This implies $c_2^2(s) = \omega + p_2(s)a_1^2$. Agent 2 consumption varies with s because of price risk. However, the first best consumption level $c_{2,*}^2$ is deterministic. This inefficiency arises because spot trades are too crude an instrument to transfer wealth across time. In particular, asset price risk generates undesirable consumption variability in period 2. As we will see, the repo allows agents to commit to a repurchase price to hedge against the asset payoff variability.

Trading in Spot and Repo Markets

In this section, we specify the agents' problem when they can also trade repo contracts. A repo is the combination of a sale of an asset with a forward contract to buy it back. The forward leg may provide insurance against future price risk. However, under limited commitment, agents may not always make good on the promises implicit to that forward contract.

Repos

A repo contract at date 1 is a schedule $f = \{f(s)\}_{s \in \mathcal{S}}$ where $f(s)$ is the price at which the period 1 seller agrees to repurchase the asset in state s of period 2. The seller i transfers one unit of the asset to the buyer j per unit of repo traded. In exchange, he receives $q_{ij}(f)$ per unit which is the price of the repo. We explain below how this price may depend on traders' type. A repo f is similar to a collateralized loan where the seller/borrower obtains $q_{ij}(f)$ per unit of asset pledged against a promised repayment schedule $\{f(s)\}_{s \in \mathcal{S}}$. In this model, repos differ from collateralized loans because lender j can re-use a fraction ν_j of the asset pledged.

Borrower and Lender Default

When entering a repo contract, the borrower promises to repurchase the asset at a pre-agreed price while the lender promises to return the re-usable collateral at that price. Hence, a dual limited commitment problem arises. As we stated above, upon default, an agent loses his entitlement (the collateral or the cash) and incurs a penalty proportional to the size of the default and his trustworthiness θ. Consider a trade of one unit of repo contract f between borrower i and lender j. This comes without loss of generality because of the linearity of the punishment. If the borrower defaults in state s, he loses the unit of the asset pledged and incurs a penalty with pecuniary cost $\theta_i f(s)$. Borrower i would thus repay if and only if

$$p_2(s) + \theta_i f(s) \geq f(s)$$

The left hand side is the cost of defaulting, that is the sum of the market value of the asset and the penalty. The right hand side is the cost of repaying the loan. Therefore, the borrower makes good on his promise to pay $f(s)$ in state s whenever,

$$f(s) \leq \frac{p_2(s)}{1-\theta_i} \tag{3.3}$$

The right hand side is the maximum agent i can promise to repay per unit of asset he holds. When $\theta_i > 0$, he may pledge more income in state s than the value $p_2(s)$ of the asset.

The lender acquires and promises to return the collateral. Observe however that he can only re-use a fraction ν_j of this collateral. We assume that he can deposit the non re-usable fraction $1-\nu_j$ with a collateral custodian[11]. We call this collateral segregated. If he chooses this option, he may only abscond with the re-usable fraction ν_j of the collateral. It is easy to understand why this is optimal for him ex-ante. First, he is less likely to default ex-post. Second, by definition, he would not derive ownership benefits from keeping the non re-usable collateral on his balance sheet. Therefore, lender j does not default when:

$$f(s) + \theta_j \nu_j p_2(s) \geq \nu_j p_2(s)$$

The left hand side is the cost of defaulting. It includes the loss of the promised payment $f(s)$ and the penalty $\theta_j \nu_j p_2(s)$, proportional to the value of the collateral he defaults upon. The right hand side is the cost of returning the re-usable units of collateral at market value[12]. Finally, the lender no-default constraint writes:

[11]In the tri-party repo market, BNY Mellon and JP Morgan provide these services.

[12]A lender might re-use collateral and not have in on his balance sheet when he must return it to the borrower. However, observe that he can always purchase the relevant quantity of the asset in the spot market to satisfy his obligation. The lender thus effectively covers a short position $-\nu_j$ in the asset.

$$f(s) \geq \nu_j(1-\theta_j)p_2(s) \tag{3.4}$$

With state contingent repurchase prices, there is no loss in generality in focusing on repo contracts satisfying no-default constraints (3.3) and (3.4). Indeed, for every contract with equilibrium default, there exists another default-free contract that agents weakly prefer to trade. We then define the set of no-default repo contracts $\mathcal{F}_{ij}$ between two agents i and j as a function of the period 2 spot market price $\mathbf{p}_2 = \{p_2(s)\}_{s \in \mathcal{S}}$

$$\mathcal{F}_{ij}(\mathbf{p}_2) = \left\{ f \mid \forall\, s \in [\underline{s}, \bar{s}]\,,\ \nu_j(1-\theta_j)p_2(s) \leq f(s) \leq \frac{p_2(s)}{1-\theta_i} \right\} \tag{3.5}$$

Observe that the set above is stable by linear combination with positive coefficients. In addition, all contracts require pledging one unit of the asset. Hence for any combination of multiple contracts sold by i, there exists an equivalent trade of a single repo contract. In the following, we thus call without ambiguity f_{12} and f_{21} the equilibrium contracts.

Agents optimization problem.

We can now write the agent's optimization problem. We let b^{ij} (resp l^{ij}) denote the amount agent i borrows (resp. lends) with j using equilibrium contract f_{ij} (resp f_{ji}). We call q_{ij} denotes the price of the equilibrium contract f_{ij} traded by agents i and j. When indexing a contract, the subscript ij reflects the equilibrium choice of repos by agents i and j. The subscript ij also indexes the price because some repo contracts have different prices when traded by different pairs of agents because of heterogeneous incentives to default. For simplicity, we write $q_{ij} := q_{ij}(f_{ij})$.

$$\max_{a_t^i, b^{ij}, l^{ij}} E\left[U^i(c_1^i, c_2^i(s), c_3^i(s))\right] \tag{3.6}$$

$$\text{subject to} \quad c_1^i = \omega + p_1(a_0^i - a_1^i) + q_{ij}b^{ij} - q_{ji}l^{ij} \tag{3.7}$$

$$c_2^i(s) = \omega + p_2(s)(a_1^i - a_2^i(s)) - f_{ij}(s)b^{ij} + f_{ji}l^{ij} \tag{3.8}$$

$$c_3^i(s) = a_2^i(s)s \tag{3.9}$$

$$a_1^i + \nu_j l^{ij} \geq b^{ij} \tag{3.10}$$

$$b^{ij} \geq 0 \tag{3.11}$$

$$l^{ij} \geq 0 \tag{3.12}$$

At date 1, agent i has resources $\omega + p_1 a_0^i$ and chooses asset holding a_1^i, lending ℓ^{ij} and borrowing b^{ij}. Given these decisions, his resources at date 2 is the endowment ω and the value of his asset holdings $p_2(s)a_1^i$ as well the net value of the repo positions $f_{ij}(s)\ell^{ij} - f(s)b^{ij}$. Equation (3.10) is the collateral constraint of agent i. A borrower (an agent for which $b > 0$) must hold one asset per unit of repo contract sold. He can buy these assets either in the spot market ($a_1 > 0$) or in the repo market ($l > 0$). In the latter case, however, only a fraction ν_j of the asset purchased can be re-used. The collateral constraint also shows that a lender can take a short position on the spot market. Let indeed $b = 0$ and $l > 0$. Then, it can be that $a_1 < 0$ if $\nu > 0$. With re-use, a lender acquires ownership of the asset pledged by the lender and can then sell it. The only difference with a regular sale is the commitment to return the asset to the agent who initially sold it.

Definition 1. *Repo equilibrium*

An equilibrium is a system of spot prices p_1 and $\mathbf{p_2} = \{p_2(s)\}_{s\in\mathcal{S}}$, a pair of repo contracts $(f_{12}, f_{21}) \in \mathcal{F}_{12}(\mathbf{p_2}) \times \mathcal{F}_{21}(\mathbf{p_2})$ and their prices q_{12} and q_{21}, and allocations $\{c_t^i(s), a_t^i, \ell^{ij}, b^{ij}\}_{t=1..3, s\in\mathcal{S}}^{i=1,2.j\neq i}$ such that

1. *$\{c_t^i(s), a_t^i, \ell^{ij}, b^{ij}\}_{t=1..3, s\in\mathcal{S}}^{j\neq i}$ solves agent $i = 1, 2$ problem* (3.6)-(3.12).

2. *Markets clear, that is $a_2^1 + a_1^2 = a$ and $b^{ij} = l^{ji}$ for $i = 1, 2$ and $j \neq i$*

3. *For any contract $\tilde{f} \notin \{f_{12}, f_{21}\}$, there exists a price $q(\tilde{f})$ such that agents do not trade this contract.*

Points 1 and 2 are self-explanatory. Point 3 is a natural requirement to characterize the repo contracts traded in equilibrium. A repo contract is chosen by the agents if they do not wish to trade an alternative contract $\tilde{f}$. For example, if $\tilde{f} \in \mathcal{F}_{12}(\mathbf{p_2})$, the implicit equilibrium price $q(\tilde{f})$ must be too low (resp too high) for agent 1 (resp. agent 2) to wish to sell (resp. to buy) this contract. Hence, with our equilibrium definition, all contracts are available to trade and agents select their preferred contracts taking prices as given.

3.3 Equilibrium contract with no re-use

In this section, we solve for equilibrium when agents cannot re-use collateral. Since $\nu = 0$, repo contracts are indistinguishable from collateralized loans. We show below that in equilibrium agents need not trade spot and agent 1 only sells a repo $f = \{f(s)\}_{s \in \mathcal{S}}$ to agent 2. This is intuitive since agent 1 is the natural borrower.

To gain intuition, remember that agent 1 wants to borrow in period 1 to repay $c^2_{2,*} - \omega$ in period 2. Consider the following trade pattern. Agent 1 sells all his asset in a repo, that is $b^{12} = a$ and does not trade spot. The maximum per-unit payoff of the repo is $p_2(s)/(1-\theta_1)$. Hence, in period 2 and state s, using his budget constraint, agent 2 's consumption can be at most $c^2_2(s) = \omega + ap_2(s)/(1-\theta)$. In low states s, this amount may fall short of $c^2_{2,*}$. Since agents are not satiated then, the repurchase price should indeed be $f(s) = p_2(s)/(1-\theta_1)$. In high states however, this would raise agent 2 consumption too much. There, $f(s)$ should be constant. Define indeed s^* as the solution to

$$c^2_{2,*} = \omega + \frac{ap_2(s^*)}{(1-\theta)} = \omega + \frac{as^*}{v'(c^1_{2,*})(1-\theta)}. \tag{3.13}$$

This is the minimal state where the first-best allocation can be financed. The second equality follows from the observation that $p_2(s) = s/v'(c^1_2(s))$ since agent 1 is the natural holder of the asset into period 3. Observe that s^* is decreasing with a and θ. Therefore, it is easier to achieve the first best level of consumption the larger the stock of asset and the more agent 1 is able to commit. We have the following result.

Proposition 1. *Define $p_2(s)$ as the unique solution - increasing in s- to*

$$\begin{cases} p_2(s)v'\left(\omega - a\dfrac{p_2(s)}{1-\theta_1}\right) - s = 0 & \textit{if} \quad s < s^* \\ p_2(s) = s/v'(c^1_{2,*}) & \textit{if} \quad s \geq s^* \end{cases} \tag{3.14}$$

There is a unique equilibrium allocation with repo contract f where:

1. *If $s^* \geq \bar{s}$ (a is low), $f(s) = p_2(s)/(1-\theta_1)$ for all $s \in \mathcal{S}$*
2. *If $s^* \in [\underline{s}, \bar{s}]$ (a is intermediate),*

$$f(s) = \begin{cases} \dfrac{p_2(s)}{1-\theta_1} & \textit{for } s \leq s^* \\ \dfrac{p_2(s^*)}{(1-\theta_1)} & \textit{for } s \geq s^* \end{cases} \tag{3.15}$$

3. *If $s^* \leq \underline{s}$ (a is high), $f(s) = f^*$ for all $s \in \mathcal{S}$ where $f^* \in [\frac{p_2(s^*)}{(1-\theta_1)}, \frac{p_2(\bar{s})}{(1-\theta_1)}]$.*

In equilibrium, agents strictly prefer to trade repo over any combination of repo and spot trades in cases 1 and 2. They are indifferent to using a combination of both in case 3.

The equilibrium contract reflects the optimal use of the collateral value. As we explained, agent 1 can indeed pledge at most $p_2(s)/(1-\theta_1)$ per unit of asset in state s. This amount increases in s together with the collateral value $p_2(s)$. When the collateral value is low, for $s \leq s^*$, the borrowing constraint of agent 1 is binding and the repurchase price $f(s)$ is equal to this maximal amount. However, when the collateral value is high, agent 1 does not want to borrow above the first best amount. Hence, the repurchase price becomes flat for $s \geq s^*$. We call this the hedging motive. The proof of Proposition 1 in the Appendix formalizes this argument ensuring that agents do not want to trade another contract $\tilde{f}$.

It is interesting to emphasize why agents prefer trading repo rather than spot. Suppose indeed that agent 1 sells the asset spot in period 1 and buys it back at the spot market price $p_2(s)$ in period 2. This is formally equivalent to a repo contract $\tilde{f}$ with $\tilde{f}(s) = p_2(s)$. This alternative trade is dominated for two reasons. When the collateral value is low, agent 1 can increase the amount he pledges from $p_2(s)$

to $p_2(s)/(1-\theta_1)$ with a repo. More importantly, when the collateral value is high, the equilibrium repo limits the repayment to agent 2 to the first best level. In the case where $\theta_1 = 0$ (no punishment), our results thus carry through but only the last motive to trade a repo is present[13].

The equilibrium repo in (3.15) has a state contingent repurchase price. When $\theta_1 = 0$, it is equivalent to a repo contract with constant repurchase price $\hat{f}$ and default. Let us indeed set $\hat{f} = p_2(s^*)$ for all states s. When $s \geq s^*$, agent 1 repays since the collateral is worth more than the payment he has to make, that is $p_2(s) > p_2(s^*)$. In low states however, he finds it optimal to default. The lender's payment is then equal to the collateral value $p_2(s)$. As a result, the effective repurchase price schedule is exactly as in Proposition 1. When $\theta_1 > 0$, there is a deadweight cost of default. Hence, contracts with constant repurchase prices come with a loss of generality. Indeed, a contract with a high (constant) value of f generates costly default in low states. A contract with a low (constant) value of f reduces default but also the amount pledged in any state. When restricted to constant repurchase schedules, the equilibrium contract is non-trivial and depends on the relative strength of the borrowing and the hedging motive.

3.3.1 Haircuts, liquidity premium, and repo rates

In this section, we derive the equilibrium properties of the liquidity premium and repo haircut. We compare the haircuts and liquidity premia of two assets with different risk profile. We also investigate the role of counterparty risk, as measured by θ. We define the liquidity premium $\mathcal{L}$ as the difference between the spot price of the asset in period 1 and its holding value. We thus have

$$\mathcal{L} = p_1 - E[s]$$

The holding value $E[s]$ follows naturally from the preferences of agent 1. The liquidity premium is also the shadow price of the collateral constraint. Hence, whenever the asset is scarce and agents are constrained, the asset bears a positive liquidity premium. Using the equilibrium characterization, we can relate the liquidity premium

[13] As we explain in the next Section however, $\theta_1 > 0$ is necessary to explain re-use.

to the repo contract and the allocation:

$$\mathcal{L} = E[f(s)(u'(c_2^2(s)) - v'(c_2^1(s))]$$

The liquidity premium is positive if there exist (low) states where agents are constrained because they cannot increase borrowing. In those states, $u'(c_2^2(s)) > v'(c_2^1(s))$ which implies $\mathcal{L} > 0$.

The repo haircut is the difference between the spot market price and the repo price. Indeed, it costs p_1 to obtain 1 unit of the asset, which can be pledged as collateral to borrow q. So to purchase 1 unit of the asset, an agent needs $p_1 - q$ which is the downpayment or haircut[14].

$$\mathcal{H} \equiv p_1 - q = E[(p_2(s) - f(s))v'(c_2^1(s))] \tag{3.16}$$

where the second equality follows from the first order condition of agent 1 with respect to spot and repo trades. Finally, the repo rate is

$$1 + r = \frac{\mathbb{E}[f(s)]}{q} = \frac{\mathbb{E}[f(s)]}{\mathbb{E}[f(s)u'(c_2^2(s))]}. \tag{3.17}$$

When agents are constrained (case i) and ii) of Proposition 1), we have $u'(c_2^2(s)) > u'(c_{2,*}^2)$ for $s \in [\underline{s}, s^*]$ so that $1 + r < 1 + r^*$. Agent 2 would like to lend at the frictionless interest rate $1+r^*$. However, agent 1 cannot increase borrowing since he runs out of collateral. The interest rate must then fall for agent 2 to be indifferent. Interestingly, $r < r^*$ when the liquidity premium $\mathcal{L}$ is strictly positive. Remember that a positive liquidity premium precisely indicates collateral scarcity. Net repo rates r can thus be negative for assets with large liquidity premium. This is consistent[15] with market data as reported in ICMA (2013). We now derive the haircut and liquidity premium for the equilibrium repo f.

[14]An alternative but equivalent definition is $(p_1 - q)/q$.

[15]The ICMA (2013) reports that "The demand for some assets can become so strong that the repo rate on that particular asset falls to zero or even goes negative. The repo market is the only financial market in which a negative rate of return is not an anomaly." (p.12) and in footnote 6 "negative repo rates have been a frequent occurrence and can be deeply negative." Also, see Duffie (1996), or Vayanos and Weill (2008).

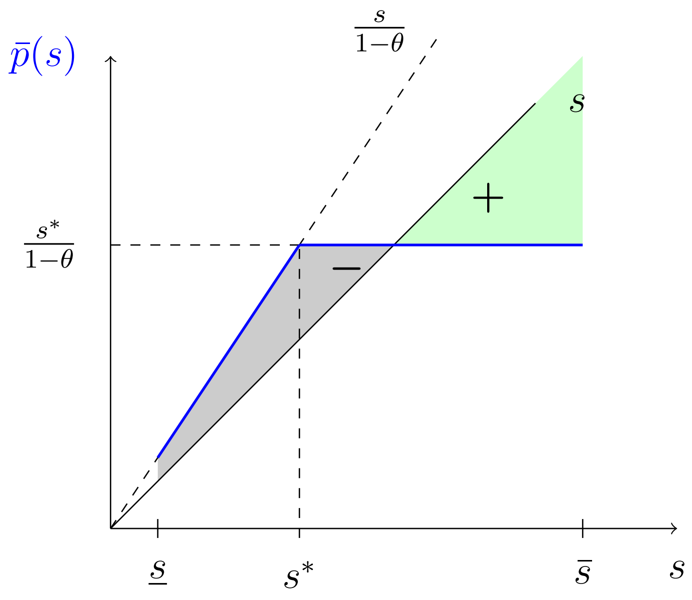

Figure 3.1: Repo haircuts

Corollary 1. *The haircut and liquidity premium are:*

$$\mathcal{L} = \int_{\bar{s}}^{s^*} \frac{s}{1-\theta_1} \left[\frac{u'\left(\omega + a\frac{p_2(s)}{1-\theta}\right)}{v'\left(\omega - a\frac{p_2(s)}{1-\theta}\right)} - 1 \right] \mathrm{d}F(s)$$

$$\mathcal{H} = -\frac{\theta}{1-\theta}\int_{\underline{s}}^{s^*} s\mathrm{d}F(s) + \int_{s^*}^{\bar{s}} \left(s - \frac{s^*}{1-\theta}\right)\mathrm{d}F(s)$$

when $s^* \geq \underline{s}$, *where* $p_2(s)$ *is the period 2 spot market price defined in* (3.14). *When agents can reach the first-best allocation in all states, that is* $s^* \leq \underline{s}$, *the liquidity premium is* $\mathcal{L} = 0$ *and the haircut lies in the following range:*

$$\mathcal{H} \in \left[E[s] - \frac{\underline{s}}{1-\theta}, E[s] - \frac{s^*}{1-\theta}\right]$$

As Figure 3.1 shows, the borrowing and hedging motives have opposite effects on the size of the haircut. In the states $s < s^*$ where agents are constrained, the borrower uses the maximum pledgeable capacity $p_2(s)/(1-\theta_1)$ per unit while the asset price trades at $p_2(s)$. From expression (3.16), this contributes negatively to the haircut. However, in states $s \geq s^*$, agent 1 does not wish to borrow more than $c^2_{2,*} - \omega$. Hence, he does not use the full collateral value of the asset. In particular, the repayment $f(s)$ is flat while the asset value $p_2(s)$ increases with s. This contributes positively to the haircut. The overall sign of the haircut depends on the weights on both regions in the distribution of s. This simple discussion suggests that when $\theta_1 = 0$, haircuts are always positive. Finally, observe that the haircut is not pinned down when $s^* \leq \underline{s}$ since several (constant) repurchase prices f are possible in equilibrium.

As we discussed, the liquidity premium captures the value of the asset as an instrument to borrow over and above its holding value. This premium is zero when agents are not constrained in any state, that is $s^* \leq \underline{s}$ as shown by the expression above. When $s^* > \underline{s}$, the liquidity premium is an average of the pledging capacity of the asset $s/(1-\theta_1)$ multiplied by the wedge in marginal utilities

Counterparty risk

We now perform a comparative static exercise varying θ, a proxy for counterparty quality. Indeed, a higher θ implies a higher punishment from defaulting and thus a superior ability to honor debt. Although there is no default in equilibrium, the equilibrium contract reflects default risk. Using the expression derived in Corollary 1, we obtain that haircuts increase with counterparty risk, or:

$$\frac{\partial \mathcal{H}}{\partial \theta_1} = -\frac{1}{(1-\theta_1)^2} \int_{\underline{s}}^{s^*} s dF(s) \leq 0$$

Indeed, as Figure 3.2 shows, a higher θ_1 increases the amount a borrower can raise per unit of the asset pledged. This naturally leads to a decrease in the haircut, by increasing the size of the region where $f(s) > p_2(s)$ while leaving the other region unchanged.

When it comes to the liquidity premium $\mathcal{L}$, counterparty quality θ_1 has an ambiguous effect. First, remember that agent 1 can pledge at most $ap_2(s)/(1-\theta_1)$ in state s. Hence, an increase in θ raises the pledgeable amount[16]. Agent 1 can

[16]This argument abstracts from the negative equilibrium impact of θ on the spot market price $p_2(s)$ which is pinned down by the relationship $p_2(s)v'(\omega - ap_2(s)/(1-\theta)) - s = 0$ for $s \leq s^*$. However, one can easily show that the net effect is positive, that is $\partial[p_2(s)/(1-\theta)]\partial\theta > 0$.

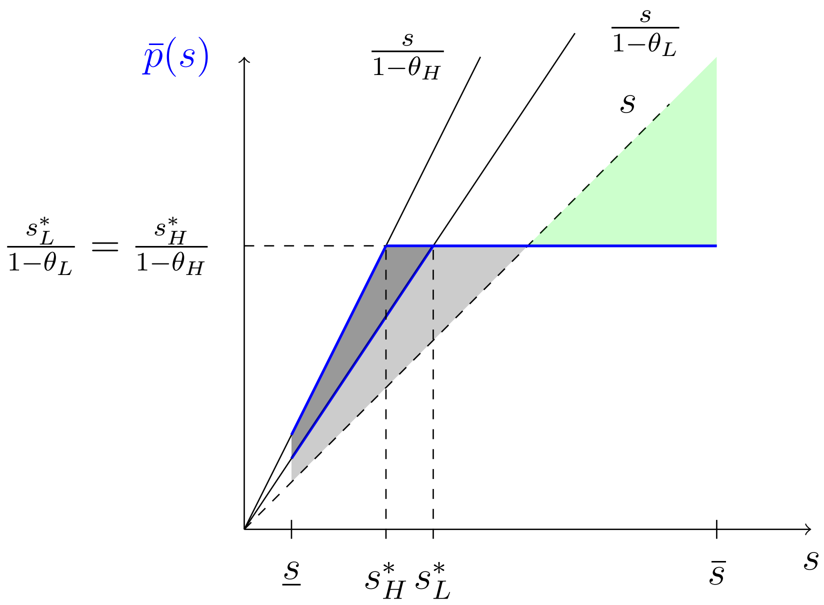

Figure 3.2: Influence of θ, with $\theta_H > \theta_L$

thus borrow more in states $s < s^*$, which reduces the wedge $u'(c_2^2(s))/v'(c_2^1(s)) - 1$ between marginal utilities. This effect, similar to an increase in the asset available a, tends to reduce the liquidity premium. However, θ_1 also increases the slope of the repurchase price $1/(1-\theta_1)$ on those states where the agents are constrained. As more income can be pledged when this is most valuable, the asset becomes a better borrowing instrument, which raises its price. Observe that this second effect does not arise when we vary the asset supply a. Thus, counterparty quality θ_1 can have a non-monotonic impact on the liquidity premium $\mathcal{L}$.

Asset risk

We now want to compare haircuts and liquidity premium as a function of asset riskiness. For this purpose, we introduce two assets with different risk profiles but perfectly correlated payoffs[17]. We compute the liquidity premium of the safer asset relative to the riskier, the haircuts that both assets carry, and the repo rates. As

[17] We can prove similar results, in the one asset case, by considering a mean preserving spread. However, we would then compare quantities across equilibrium rather than within an equilibrium as we do here.

before, $s \sim F[\underline{s}, \bar{s}]$ but there are now two assets $i = A, B$ with payoffs $\rho_i(s)$:

$$\rho_i(s) = s + \alpha_i(s - \mathbb{E}[s]),$$

where $\alpha_B > \alpha_A = 0$. With $\alpha_A = 0$, asset A is our benchmark asset. Since $\alpha_B > 0$, asset B has the same mean but a higher variance than asset A. Indeed $Var[\rho_\alpha] = (1 + \alpha)Var[s]$. We choose to consider two assets with perfectly correlated payoff to ignore the effect of risk sharing on the structure of the repo contract. Agent 1 is endowed with a units of asset A and b units of asset B, while agent 2 does not hold any of the assets. It is relatively straightforward to extend the equilibrium analysis of the previous section to this new economy with two assets. The set of available contracts consists of feasible repos using assets A and B. For each asset $i = A, B$, the repo contract f_i uses the maximum pledgeable capacity up to the state where the first best level of consumption can be reached. We then prove the following result.

Proposition 2. *The safer asset A always has a higher liquidity premium and a lower haircut than the riskier asset B.*

The key intuition behind the result is the misallocation of collateral value induced by a mean preserving spread. Asset A and B have the same expected payoff. However, since $\rho_B(s) - \rho_A(s) = \alpha_B(s - E[s])$, the risky asset pays relatively more in high states (upside risk) and less in low states (downside risk). Given that agents are constrained for low values of s, this is precisely when collateral is valuable. Since the safe asset A pays more in these states, it carries a larger liquidity premium. We now turn to the haircut. In high states, the riskier asset B has a higher payoff which means that more income can be pledged compared to asset A. However, agent 1 does not wish to borrow over the first best level. Hence agents do not exploit the the higher collateral value of the risky asset in high states, implying a larger haircut. Observe that without this hedging motive, asset risk would have no impact on the haircut.

So far, repos are indistinguishable from collateralized loans. Indeed, with $\nu = 0$, the asset is immobile once pledged in a repo. The next two sections show that allowing for re-use delivers new predictions. First, re-use increases the borrowing capacity of agent 1. Second, the possibility to re-use collateral may lead to endogenous intermediation in equilibrium.

3.4 The multiplier effect of re-use

In this section, we analyze the impact of collateral re-use on equilibrium contracts and allocations. This is a natural feature of a repo trade where the collateral is sold to the lender who acquires ownership rights. Re-use has been very much under scrutiny following the crisis (see Singh and Aitken, 2010) since a default on re-used collateral may affect several agents along a credit chain. While we do not model the consequence of such default cascades, we provide the foundations for this analysis by highlighting the benefits of re-use. The lender, agent 2 is now able to re-use collateral, that is $\nu_2 > 0$. We set $\nu_1 = 0$ for agent 1, the borrower to simplify the exposition and discuss the role of this assumption after Proposition 3.

To understand the potential benefits, consider the equilibrium without re-use. Agent 2 (the lender) holds collateral pledged by agent 1. Re-use frees up a fraction ν_2 of this collateral. Suppose agent 2 then sells ϵ units where ϵ is small to agent 1 at the equilibrium price p_1. The marginal gain for agent 1 is null since buying the asset is feasible without re-use. The marginal gain to agent 2 is

$$\frac{\partial U^2}{\partial \epsilon} = p_1 - E[p_2(s)u'(c_2^2(s))] = \eta_1^2$$

where η_1^2 is the shadow price of the asset for agent 2. Using the equilibrium characterization, we obtain:

$$\eta_1^2 = \frac{\theta_1}{1-\theta_1} \int_{\underline{s}}^{s^*} \left(u'(c_2^2(s)) - v'(c_2^1(s))\right) p_2(s) dF(s)$$

Hence, this marginal gain is strictly positive when $s^* > \underline{s}$ (agents are constrained) and $\theta_1 > 0$. To understand this last condition, observe that agent 1 may now sell in a repo the re-used asset he bought spot from agent 2. He can thus pledge $p_2(s)/(1-\theta_1)$

per unit. The net transfer however is

$$-p_2(s) + \frac{p_2(s)}{1-\theta_1} = \frac{\theta_1}{1-\theta_1} p_2(s)$$

since he first bought the asset spot from agent 2. This transfer is positive and increases agent 1's borrowing only if $\theta_1 > 0$.

These steps can be repeated over multiple rounds. Agent 1 initially owns a unit of the asset. In the first round, he can pledge $ap_2(s)/(1-\theta_1)$, just as in the no re-use case. After this trade, agent 2 has $\nu_2 a$ units of re-usable asset. Given our argument above, agent 1 can then pledge an additional $\frac{\theta_1}{1-\theta_1}\nu_2 a p_2(s)$ in state s. After this operation, agent 2 has $(\nu_2)^2 a$ units of re-usable asset. Iterating over these rounds infinitely, the total pledgeable amount per unit of asset in state s obtains:

$$\begin{aligned} M_{12} p_2(s) :&= \frac{p_2(s)}{1-\theta_1} + \sum_{i=1}^{\infty} (\nu_2)^i \frac{\theta_1}{1-\theta_1} p_2(s) \\ &= \frac{1}{1-\nu_2}\left[\frac{1}{1-\theta_1} - \nu_2\right] p_2(s) \end{aligned} \tag{3.18}$$

where we call M_{12} the borrowing multiplier between agents 1 and 2. This expression is strictly increasing in ν_2 as long as $\theta_1 > 0$. Again, the role of trustworthiness θ for re-use appears clearly.

In this analysis, we guessed that agent 2, the lender, returns the collateral. Indeed, our construction implicitly relied on a repo contract f_{12} with $f_{12}(s) = p_2(s)/(1-\theta)$. This satisfies the no-default constraint (3.4) of the lender, agent 2.

Let us now define $s^*(\nu_2)$ as the minimal state above which agent 1 can pledge enough income to finance the first best allocation, that is:

$$\omega + aM_{12}p_2(s^*(\nu_2)) = c^2_{2,*}.$$

We can then introduce the candidate equilibrium repo contract $f(\nu_2)$ where:

$$f(s,\nu_2) = \begin{cases} \dfrac{p_2(s)}{1-\theta_1} & \text{if } s < s^*(\nu_2) \\ \dfrac{s^*(\nu_2)}{(1-\theta_1)v'(c^1_{2,*})} + \dfrac{\nu_2(s-s^*(\nu_2))}{v'(c^1_{2,*})} & \text{if } s \geq s^*(\nu_2) \end{cases} \tag{3.19}$$

The following Proposition establishes that agents trade this contract in an equilibrium with re-use:

Proposition 3. ***Collateral Re-use.***

Let $\nu_2 \in (0,1)$ be the fraction of collateral agent 2 can re-use. Collateral re-use is strictly preferred whenever $\theta_1 > 0$ and the first best allocation is achieved for any $\nu_2 \geq \nu^*$ defined as

$$\nu^* = \frac{s^*(0) - \underline{s}}{s^*(0) - (1-\theta)\underline{s}}.$$

The (essentially) unique equilibrium repo contract is $f(\nu)$ defined in (3.19).

As we discussed before, when $\theta_1 > 0$, re-use strictly increases the amount agent 1 can pledge to agent 2. This is valuable when agents are constrained and want to expand borrowing in low states. From the expression of M_{12} in (3.18), it is clear that for ν_2 high enough, the first-best allocation can even be financed in the lowest state $\underline{s}$. One can obtain the expression for ν^* by setting $s^*(\nu_2) = \underline{s}$. The equilibrium repo contract has $f(s,\nu_2) = p_2(s)/(1-\theta_1)$ in those states $s \leq s^*(\nu_2)$, where agents are still constrained. There, agents use the maximum pledgeable amount. On states $s \geq s^*(\nu_2)$, the repo schedule is not flat anymore when $\nu_2 > 0$. The second component $\nu_2(s - s^*(\nu_2))$ corrects for the short position taken by agent 2 in the spot market. Indeed, in every round of re-use, agent 2 sells a fraction ν_2 of the collateral he receives as a pledge from agent 1. He must then return this collateral in period 2 or equivalently cover his short position in the asset. Since he re-sells a fraction ν_2 of every unit, it is not surprising that the second term is proportional to ν_2.

It may seem that agent 1 would only be willing to engage in re-use if the haircut is negative. Indeed, buying 1 unit of asset from agent 2 to pledge it back yields a net gain of $-p_1 + p_F = -\mathcal{H}$ in period 1. This intuition proves correct when there is no uncertainty, that is $s = E[s] = 1$ for all s. Then, since $f = p_2/(1-\theta_1)$, expression (3.16) shows that $\mathcal{H} \leq 0$ indeed. Intuitively, the borrower would benefit only if he increases consumption in period 1. When there is uncertainty, the above logic is incomplete since agent 1 may also gain by transferring consumption across states. If the haircut is positive, agent 1 benefits by decreasing his consumption in period 1 and in the low states of period 2 to increase it in the high states of period 2.

The liquidity premium $\mathcal{L}$ can exhibit non-monotonicity in the re-use factor ν_2. Indeed, while re-use relaxes the collateral constraint, it also increases the amount pledgeable in states where agents are constrained. This last effect makes the asset more valuable and can increase the liquidity premium. These two effects are reminiscent of the comparative statics with respect to the commitment power θ. Finally, our model predicts that re-use is helpful when collateral is most scarce (that is $s^*(0) > \underline{s}$) and there is evidence that this is indeed the case (see Fuhrer et al., 2015).

Remark 1. The role of ν_1

Since agent 2 *is the natural lender, it seems that the re-use capacity of the borrower ν_1 should play no role. The proof in the appendix shows indeed that Proposition 3 holds not only when $\nu_1 = 0$ but that ν_1 should not be too large either. We provide an informal discussion for the role of ν_1 here. Observe first that agent* 2 *is free to re-use the asset as he wishes. He may either re-sell it spot or re-pledge it in a repo to agent* 1*. In the discussion leading to Proposition 3, we implicitly guessed that he prefers the first option. We now argue that this is indeed the case if:*

$$-p_2(s) + M_{12}p_2(s) \geq -\nu_1(1-\theta_1)p_2(s) + \nu_1 M_{12}p_2(s)$$

The left hand side are the gains from re-selling the asset spot. The right hand side are the gains from re-selling the asset in a repo where $f_{21}(s) = \nu_1(1-\theta_1)p_2(s)$. Observe from (3.4) *that this is the repo with the minimum repurchase price that avoids agent* 1 *default as a lender. On each side of the inequality, the first term is the period* 2 *transfer from agent* 2 *to agent* 1*. It enters with a minus sign because agent* 1 *actually wants to borrow, which implies decreasing his consumption in period* 2*. The second term is the gain derived from agent* 1 *holding some asset. This is equal to the quantity of asset he acquires from agent* 2 *times the borrowing multiplier $M_{12}p_2(s)$. We may now understand why agent* 2 *may want to use repo f_{21} rather* *than a spot sale. The benefit is the smaller transfer from agent* 2 *to agent* 1 *in period* 2 *while the cost is the segregation of* $(1-\nu_1)$ *units by agent* 1*. Intuitively, when* ν_1 *is large enough the benefit dominates. Elementary transformations of this inequality yields the following condition.*

$$\frac{\nu_1(1-\nu_2)}{1-\nu_2\nu_1}(2-\theta_1) < 1 \tag{3.20}$$

The left hand side is monotonic in ν_1 *, the fraction of pledged collateral agent* 1 *can re-use. The equilibrium characterization in Proposition 3 is thus valid not only when* $\nu_1 = 0$ *but whenever* (3.20) *holds. When it does not hold, agent* 2 *re-sells in a repo to agent* 1*. In equilibrium, this will also affect the contract sold by agent* 1 *to agent* 2*. Although, the equilibrium contracts may change, the core intuition remains. Collateral re-use allows agent* 2 *to sell the asset back to agent* 1*, whether spot or repo, for him to increase the amount he borrows.*

3.5 Collateral Re-use and Intermediation

In their guide to the repo market, Baklanova et al. (2015) state that "*dealers operate as intermediaries between those who lend cash collateralized by securities, and those who seek funding*". To fix ideas, let us consider the following chain of trades. First, a hedge fund who needs cash borrows from a dealer bank through a repo. The dealer bank then taps in a money market fund (MMF) cash pool through another repo to finance the transaction. Figure 3.3 illustrates this pattern of repo intermediation. Since direct trading platforms such as Direct Repo™ in the US are available, it may seem puzzling that a significant share of the repo market is intermediated. In this section, we explain these chain of trades based on heterogeneity in trustworthiness between the hedge fund and the dealer bank. A remarkable feature of this equilibrium is that intermediation arises endogenously although in our example, the hedge fund would be free to trade with the MMF[18].

[18] Our analysis thus extends Infante (2015) and Muley (2015) which assume intermediation exogenously. We do not account for the possible institutional differences between the two trades involved. Indeed, repos between hedge funds and dealer banks are typically bilateral whereas dealer banks and hedge funds often trade via a tri-party agent (in the US, these are JP Morgan

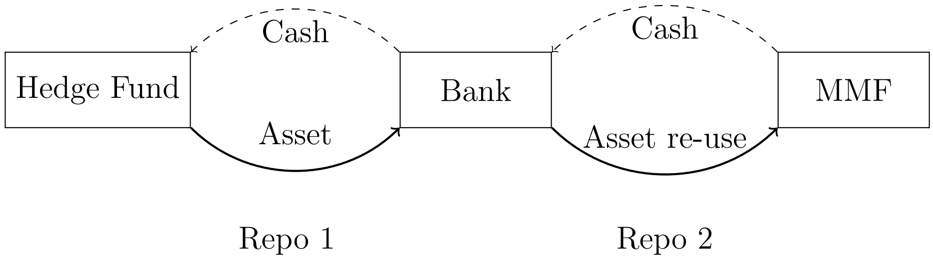

Figure 3.3: Intermediation with Repo

We change the economy slightly for this purpose. For simplicity, we assume that agent 1 has linear preferences, that is :

$$U^1(c_1, c_{2,}c_3) = c_1 + \delta c_2 + c_3$$

This is a particular case of our general framework with $v(x) = \delta x$. We also introduce a third type of agent named B, for Banker. Agent B has no asset initially. He is endowed with ω in period 1 and 2 and has the following preferences:

$$U^B(c_1, c_2, c_3) = c_1 + \delta_B c_2 + c_3$$

where $\delta \leq \delta_B < u'(\omega)$. Under this assumption, agent B also wants to borrow from agent 2 but he has lower gains from trade than agent 1. We set $\theta_B > \theta$ so that the Bank has a higher trustworthiness than agent 1. The corresponding greater borrowing capacity will explain why agent B can play a role as an intermediary. We will say that there is intermediation when agent 1 chooses not to trade directly with 2 in a repo contract. The last section already discussed the role of the re-use factor of the borrower ν_1. For simplicity, we thus set $\nu_1 = 0$ here.

3.5.1 Intermediation via spot trades

We assume first that agents 1 and B have the preferences, $\delta = \delta_B$ and only differ in their trustworthiness with $\theta_1 < \theta_B$. Although agents 1 and B have no gains from

and BNY Mellon) which acts as a collateral custodian. See Federal Reserve Bank of New York (2010) for a discussion of Tri-Party repo. Still, our analysis provides a fundamental explanation for this market segmentation.

trade, we show that the latter plays an active role as an intermediary.

Proposition 4. *Let $\delta = \delta_B$ and $\theta_1 < \theta_B$ and suppose the asset is too scarce to reach the first-best allocation. Then, in equilibrium, agent 1 sells his asset spot. Agent B buys the asset and iterates on repo trades with agent 2.*

The striking feature in Proposition 4 is that agent 1, does not trade a repo with agent 2, the natural lender while he holds the asset. It means that there exists no repo contract $\tilde{f}_{12}$ that these agents wish to trade. Instead, in equilibrium, 1 sells the asset to agent B for the latter to use as collateral with agent 2. Observe that once he buys the asset from agent 1, agent B substitutes for agent 1 as a borrower with agent 2. In particular, the equilibrium repo contract f_{B2} is the same[19] as (3.19), replacing θ_1 with θ_B.

Without agent B, agent 1 would borrow in a repo from agent 2 as before. However, agent B may pledge more income to agent 2 due to its higher trustworthiness $\theta_B > \theta$. In a competitive equilibrium, agent B makes no profit as an intermediary. Hence, his higher borrowing capacity with agent 2 is fully reflected in the spot price he pays for the asset to agent 1. As a result, agent 1 now prefers to sell his asset and delegates borrowing to a more trustworthy agent. When $\delta = \delta_B$, intermediation takes place via a spot trade between agents 1 and B and not via a repo. Observe indeed that there are no direct gains from trade between 1 and B. As a result, agents 1 and B do not value the extra borrowing capacity from a repo when $\theta_1 > 0$. To the contrary, trading repo is costly because a fraction $1 - \nu_B$ of the asset could not be used by agent B to borrow from 2. This trade-off is no longer trivial when $\delta < \delta_B$ and a chain of repos may emerge as we show in the next subsection.

When the asset is not scarce, the first best allocation whereby $u'(c^2_{2,*}) = \delta$ is attainable. In this case, other equilibrium trades are possible. In particular, agent B could be inactive if agent 1 has enough asset to compensate for his low trustworthiness θ_1. An interesting implication of our result is thus that intermediation should

[19] As before, agents B and 2 do multiple rounds of repo thanks to re-use by agent 2. Whether agent 2 re-sells spot rather than repo the collateral pledged by agent B depends on the following condition:

$$\frac{\nu_B(1-\nu_2)}{1-\nu_2\nu_B}(2-\theta_B) < 1$$

This expression is similar to (3.20) replacing ν_1 by ν_B and θ_1 by θ_B.

be observed precisely when collateral is scarce. When $\delta_B = \delta$, the Bank essentially acts as a proxy by "selling" his higher trustworthiness to the risky agent.

3.5.2 Chain of repos

We now let $\delta < \delta_B$ and show that an equilibrium with a chain of repos exists. When $\delta < \delta_{\mathrm{B}}$, agents 1 and B have direct gains from trade. A repo sale can now be valuable because it increases the transfer with respect to a spot sale when $\theta_1 > 0$. Since a fraction $(1 - \nu_B)$ of the asset pledged is segregated, this may dominate a spot sale only if $\delta_B - \delta$ is large enough.

If agents trade in a chain of repo, agent B acts both as a lender with agent 1 and as a borrower vis a vis agent 2. This creates a competing use for the asset. Indeed, when he holds one unit, agent B may either sell the asset back to agent 1 for him to increase borrowing or use it to borrow from agent 2. A key observation is that in equilibrium, he will be marginally indifferent between these two options. Suppose for instance that he strictly prefers to re-use to borrow from agent 2. This means that some asset collateralizing trade between agent 1 and B is misallocated and should rather support trade between B and 2. As a result, agents 1 and B would rather trade spot as before. Intuitively, indifference is possible if the gains from trade between agents 1 and B (proportional to $\delta_B - \delta$) are not too different from those between agents B and 2 (proportional to $u'(\omega) - \delta_B$).

Finally, agent 1 must prefer trading in a repo with agent B rather than with agent 2 while gains from trade are larger with 2. However, with heterogeneity in re-use factors ν, one unit of pledged collateral can be redeployed at different rates by each counterparty. Indeed, we have shown that the multiplier between borrower i and lender j is:

$$M_{ij} = \frac{1}{1-\nu_j}\left[\frac{1}{1-\theta_i} - \nu_j\right] \quad i = B, 2 \tag{3.21}$$

We conjecture that agent 1 will prefer to trade with B if the larger borrowing multiplier compensates for the lower gains from trade. We may now state the exact conditions under which a chain of repo can arise in equilibrium.

Proposition 5. ***Intermediation equilibrium.***

An intermediation equilibrium with a chain of repos f_{1B} and f_{B2} exists iff

$$\frac{1}{1-\theta_B} - \frac{1}{1-\theta_1} \geq (1-\nu_B)M_{B2} \tag{3.22}$$

$$\overline{\delta} \geq \delta_B \geq \underline{\delta} \tag{3.23}$$

where the thresholds verify $\delta < \underline{\delta} < \overline{\delta} < u'(\omega)$ and depend on all the parameter values. Agents 1 and B trade using repo f_{1B} given by

$$f_{1B}(s) = \frac{s}{1-\theta_1} \qquad \forall s \in [\underline{s}, \bar{s}] \tag{3.24}$$

Observe first that the repo contract f_{1B} between agents 1 and B does not reflect any hedging motive since both agents are risk neutral. Uncertainty does not play any crucial role for the rest of the equilibrium characterization. To simplify the interpretation of Proposition 5, we thus assume that $s = 1$ for the discussion. When involved in a repo chain, agent B acquires re-usable collateral from agent 1. He may either re-resell this collateral to agent 1 to increase borrowing or re-pledge the collateral to agent 2. As we discussed, in equilibrium, he must be marginally indifferent between both usages. The indifference condition can be written as follows:

$$M_{B2}[u'(c_2^2) - \delta_B] = (M_{1B} - 1)(\delta_B - \delta) \tag{3.25}$$

The left hand side is the gain from pledging collateral to agent 2. It is equal to the gains from trade times the borrowing multiplier M_{B2} between the two agents. The left hand side is the gain from re-selling the asset to agent 1. The term in factor of the gains from trade is $M_{1B} - 1$ since the asset must first be sold for agent 1 to borrow, if it is initially held by agent B. Condition (3.23) should now be clearer. If δ_B is too close to δ, the right hand side is necessarily smaller than the left hand side. In this case, agent 1 and B would rather trade spot as in Proposition 4.

We must still understand why agent 1 should trade with agent B rather than with agent 2 . He can indeed sell one unit of asset in repo f_{1B} to either agent. We argue that the first option dominates the second if:

$$\frac{\delta_B - \delta}{1-\theta_1} + \nu_H M_{B2}(u'(c_2^2) - \delta_B) \geq \frac{u'(c_2^2) - \delta}{1-\theta_1} + \nu_2(M_{B2} - 1)(u'(c_2^2) - \delta_B)$$

The left hand side (resp. right hand side) measures the gains from selling the asset in a repo to agent B (resp. 2). The first component is the direct gain from trade. This is obviously larger with agent 2 since $u'(\omega) > \delta_B$. However, with re-use, there are also indirect gains (the second term) since the collateral can be redeployed. If agent 1 sells to agent B, ν_H units can support borrowing with agent 2. If sold to agent 2, ν_2 units can be re-used. These re-use gains may be larger with B if $\nu_B > \nu_2$. The possibility to re-use collateral thus explains why seemingly dominated trades (here between 1 and B) can take place[20]. Equilibrium condition (3.22) obtains from straightforward manipulation of the inequality above. When we set $\nu_2 = 0$, condition (3.22) nicely reads as a cost-benefit analysis of intermediation. Indeed, it collapses to:

$$1 - \nu_B \leq \frac{\theta_B - \theta_1}{1-\theta_1}$$

The left hand side is the fraction of collateral immobilized when going through agent B to trade. The right hand side is the (normalized) extra borrowing capacity $\theta_B - \theta_1$ of agent B.

To summarize, an agent may become a dealer if he is more trustworthy than the natural borrower and more efficient at re-deploying collateral than the natural lender. Our analysis thus shows that repo intermediation arises endogenously out of fundamental heterogeneity between traders. Existing models of repo intermediation typically take the chain of possible trades as exogenous. Our endogenous approach to intermediation is helpful to rationalize several features of the repo market. First, we can explain why intermediating repo is still popular despite the emergence of direct trading platforms. Second, in exogenous intermediation models, dealers typically

[20] One can show that Condition (3.22) is equivalent to

$$M_{1B}(\delta_B - \delta) \geq M_{12}(u'(c_2^2) - \delta)$$

From the point of view of agent 1, borrowing from agent B dominates if the multiplier M_{1B} is larger than M_{12} although gains from trade are smaller ($\delta_B - \delta \leq u'(c_2^2) - \delta$). Again this is possible only if $\nu_B > \nu_2$.

gain and collect fees by charging higher haircuts to borrowers. In our model, the haircut paid by the borrower to the bank may very well be smaller than the one paid by the bank to the lender. Using data from the Australian repo market, Issa and Jarnecic (2016) show that this is indeed the case in most transactions.

3.6 Conclusion

We analyzed a simple model of repurchase agreement with limited commitment and price risk. Unlike a combination of sale and repurchase in the spot market, a repo contract provides insurance against the asset price risk. We introduce counterparty risk as heterogeneous cost from defaulting on the promised repurchase price. We showed that the repo haircut is an increasing function of counterparty risk and a decreasing function of the asset inherent risk. Safe assets naturally command a higher liquidity premium than risky ones. Our model targets repos since we allow agents to re-use collateral. We showed that re-use increases borrowing through a multiplier effect. In addition, it can explain intermediation whereby trustworthy agents borrow on behalf of riskier counterparties.

Our simple model delivers rich implications about the repo market but leaves many venues for future research. We argued that counterparty risk is a fundamental determinant for the terms of trade in repo contracts. It would be interesting to analyze the impact of clearing on repo market activity since clearing often implies novation by a central counterparty. Novation bears some similarities with intermediation although terms of trades cannot be adjusted and risk may be concentrated on a single agent. When it comes to re-use, besides the limit on the amount of collateral that can be re-deployed, we assumed a frictionless process. Traders establish and settle positions smoothly although many rounds of re-use may be involved. This may not be the case anymore in the presence of frictions in the spot market for instance. Recent theoretical papers have shown that secured lending markets can be fragile. Although we did not investigate this aspect in the present work, we believe collateral re-use may add to this fragility.

3.7 Appendix

3.7.1 Equilibrium analysis of spot trade only

We prove the following Proposition that characterize spot trade equilibria.

Proposition 6. *When agents can only trade spot, there exists a threshold $\bar{a}_{spot}$ such that*

1. *Low asset quantity: if $a < \bar{a}_{spot}$, then agent 1 sells his entire asset holdings at date 1. The liquidity premium $\mathcal{L}$ is strictly positive.*

2. *High asset quantity:if $a \geq \bar{a}_{spot}$, then agent 1 sells less than a at date 1. The liquidity premium is $\mathcal{L} = 0$.*

Deriving the first order conditions, the following system of equations characterize the equilibrium.

$$\begin{aligned}
c_3^1(s) &= \omega + as, \\
c_2^1(s) &= \omega - p_2(s)(a_1^2 - a_2^2(s)), \\
-p_2(s)v'(c_2^1(s)) &= s + \xi_2^1(s), && (3.26)\\
-p_2(s)u'(c_2^2(s)) &= \delta s + \xi_2^2(s), && (3.27)\\
-p_1 + E\left[p_2(s)v'(c_2^1(s))\right] + \xi_1^1 &= 0, \\
-p_1 + E\left[p_2(s)u'(c_2^2(s))\right] + \xi_1^2 &= 0, \\
\xi_1^1 \xi_1^2 &= 0
\end{aligned}$$

where ξ_t^i is the Lagrange multiplier on the no-short sale constraint of agent i in period t. Given that $u'(\omega) > v'(\omega)$, one can easily check that $\xi_2^1(s) = 0$ for all s. This is natural since agent 1 who does not discount period 3 payoffs is the natural holder of the asset. By the same logic, we have that $\xi_1^2 = 0$. Agent 2 must buy a positive quantity of the asset since otherwise gains from trade are left on the table. From equation 3.26, it is easy to realize that $p_2(s)$ is increasing in s. Moreover there exists $\hat{s}(a_1^2)$ such that $a_2^2(s)$ is equal to 0 for $s \leq \hat{s}(a_1^2)$ and solves $p_2(s)u'(\omega + p_2(s)(a_1^2 - a_2^2(s)) = \delta s$ otherwise. Agent 2 carries positive holdings of the asset into period 3 in those high states $s > \hat{s}(a_1^2)$ where re-selling everything would increase too much his period 2 consumption. Focusing now on period 1, we are left to pin down a_1^2, the quantity agent 2 initially buys from agent 1.

$$\begin{aligned}
-p_1 + E\left[p_2(s)v'(c_2^1(s))\right] + \xi_1^1 &= 0,\\
-p_1 + E\left[p_2(s)u'(c_2^2(s))\right] &= 0,\\
c_2^1(s) &= \omega - p_2(s)a_1^2\\
c_3^1(s) &= 2\omega + as
\end{aligned}$$

so

$$\xi_1^1 = E\left\{p_2(s)\left[u'(c_2^2(s)) - v'(c_2^1(s))\right]\right\} \tag{3.28}$$

To solve for the equilibrium price $p_2(s)$ and the quantity sold a_1^2, let us introduce the following system:.

$$\begin{aligned}
p_2(s)v'(\omega - p_2(s)a_1^2) &= s\\
G(a_1^2) &= E\left\{p_2(s)\left[u'(\omega + p_2(s)a_1^2) - v'(\omega - p_2(s)a_1^2)\right]\right\}
\end{aligned}$$

The first equation implicitly defines $p_2(s)$ as a function of s and a_1^2, using equation 3.26. The Implicit Function Theorem shows that $p_2(s)$ depends negatively on a_1^2. The total derivative of G with respect to a_1^2 is equal to

$$G'(a_1^2) = \int_{\underline{s}}^{\hat{s}(a_1^2)} \left[\frac{\partial p_2(s)}{\partial a_1^2}\left\{u'(\omega + p_2(s)a_1^2) + a_1^2 p_2(s)u''(\omega + p_2(s)a_1^2)\right\} + p_2(s)^2 u''(\omega + p_2(s)a_1^2)\right] \mathrm{d}F(s)$$

This expression is strictly negative if the coefficient of relative risk aversion of u is less than 1. Define then $\bar{a}_{spot}$ as the unique solution to $G(a_2^2) = 0$. Two cases are then possible: *i)* $a \geq \bar{a}_{spot}$ and $\xi_1^1 = 0$ and $a_1^2 = \bar{a}_{spot}$ or *ii)* $a < \bar{a}_{spot}$ and $\xi_1^1 > 0$ that is $a_1^2 = a$.

3.7.2 Proof of Proposition 1

In the absence of re-use ($\nu = 0$), the set of no-default repo contracts for agent $i \in \{1, 2\}$ at a given spot market price schedule $\boldsymbol{p_2} = \{p_2(s)\}_{s \in \mathcal{S}}$ is:

$$\mathcal{F}_i(\mathbf{p}_2) = \left\{ f \in \mathcal{C}^0[\underline{s}, \bar{s}] \mid 0 \leq f(s) \leq \frac{p_2(s)}{1 - \theta_i} \right\}$$

We proceed in two steps. First we characterize the equilibrium repurchase contract $f \in \mathcal{F}_1(\mathbf{p}_2)$ for a given spot price schedule $\mathbf{p}_2$, using the fact that agents must not be willing to trade any other feasible contract. Then we characterize spot market prices $\mathbf{p}_2$ compatible with the equilibrium.

First observe that $\mathcal{F}_i$ is stable under linear combinations with positive coefficient (convex cone).

Hence a combination of contracts shorted by i can be replicated by a single contract. Let f_{ij} be the unique repo contract in which agent i borrows $b^{ij} \geq 0$ from agent j. We observe also that in equilibrium, agent 2 needs not borrow, that is $b^{21} = 0$ and that spot trading is redundant, that is $a_1^1 = a_2^1(s) = a$ for all s wlog. Indeed, agent 1 is the natural borrower since $u'(\omega) > v'(\omega)$ and the payment schedule from a spot transaction $\mathbf{p}_2$ is included in the set of feasible repos $\mathcal{F}(\mathbf{p}_2)$. Hence, we will only consider a repo contract where agent 1 is the borrower that we call f for simplicity.

The equilibrium conditions when agents trade repo f are:

$$\begin{aligned}
-p_1 + E[p_2(s)v'(c_2^1(s)] + \gamma_1^1 &= 0, \\
-p_F + E\left[f(s)v'(c_2^1(s))\right] + \gamma_1^1 &= 0. \\
-p_1 + E\left[f(s)u'(c_2^2(s))\right] + \gamma_1^2 &= 0, \\
-p_F + E\left[f(s)u'(c_2^2(s))\right] &= 0. \\
-p_2(s)v'(c_2^1(s)) - s &= 0, \\
\xi_1^1\xi_1^2 &= 0, \\
c_2^1(s) &= \omega - f(s)b^{12}, \\
c_2^2(s) &= \omega + f(s)b^{12}
\end{aligned}$$

We used the fact that agent 1 will be the marginal holder of the asset into period 3. We can also derive the marginal willingness to pay for any contract $\tilde{f} \in \mathcal{F}_{12}(\mathbf{p}_2)$ for both agents. In other words, we derive the minimum (resp. maximum) price $\tilde{q}_{12}^1(\tilde{f})$ and $\tilde{q}_{12}^2(\tilde{f})$ at which agent 1 (resp. agent 2) is ready to sell (resp. to buy) an infinitesimal amount of contract $\tilde{F}$.

$$\begin{aligned}
\tilde{q}_{12}^1(\tilde{f}) &= E\left[\tilde{f}(s)v'(c_2^1(s))\right] + \gamma_1^1 \\
\tilde{q}_{12}^2(\tilde{f}) &= E\left[\tilde{f}(s)u'(c_2^2(s))\right]
\end{aligned}$$

For agents not to trade contract $\tilde{f}$ in equilibrium, the following inequality must hold:

$$\tilde{q}_{12}^2(\tilde{f}) \leq \tilde{q}_{12}^1(\tilde{f}) \tag{3.29}$$

Indeed, if this inequality holds, there is an equilibrium price $\tilde{q}_{12}(\tilde{f}) \in [\tilde{q}_{12}^1(\tilde{f}), \tilde{q}_{12}^2(\tilde{f})]$ such that agents' optimal trade in $\tilde{f}$ is 0. We will use this inequality to show that the equilibrium f is the contract characterized in Proposition 1. There are two cases.

i) $\gamma_1^1 = 0$: agent 1 is unconstrained.

Then agents 1 and 2's (marginal) valuation for any contract $\tilde{f} \in \mathcal{F}_1(\mathbf{p}_2)$ must coincide, that is:

$$E\left[\tilde{f}(s)u'(c_2^2(s))\right] = E\left[\tilde{f}(s)v'(c_2^1(s))\right] \tag{3.30}$$

where $c_2^2(s) = \omega + f(s)b^{12}$. Suppose there is an open interval $(s_1, s_2) \in \mathcal{S}$ such that for all $s \in (s_1, s_2)$, $u'(c_2^2(s)) - v'(c_2^1(s)) = 0$ and has a constant sign. Let us then consider the piece-wise linear schedule $\tilde{f}$ such that $\tilde{f}(\underline{s}) = \tilde{f}(s_1) = \tilde{f}(s_2) = \tilde{f}(\bar{s}) = 0$ and $\tilde{f}(s_1/2+s_2/2) = s_1$. The schedule $\tilde{f} \in \mathcal{F}_1$ would violate equality (3.30). It means that there cannot be an open interval on which $u'(c_2^2(s)) - v'(c_2^1(s)) \neq 0$. Hence, by continuity, we must have for all $s \in \mathcal{S}$, $u'(c_2^2(s)) = v'(c_2^1(s))$, that is $c_2^2(s) = c_{2,*}^2$. This means that f is constant and in particular that agent 2 can finance $c_{2,*}^2$ in the lowest state $\underline{s}$:

$$c_{2,*}^2 = \omega + f(\underline{s})b^{12} \leq \omega + a\frac{p_2(\underline{s})}{1-\theta_1} = \omega + a\frac{\underline{s}}{v'(c_{2,*}^1)(1-\theta_1)}$$

where we can replace $p_2(s) = s/v'(c_{2,*}^1)$, using the spot market equilibrium condition in period 2 and the fact that $c_2^1(s) = c_{2,*}^1$. This may hold, only if $s^* \leq \underline{s}$. In that case, although the equilibrium allocation is unique, the contracts traded are not. The expression of $c_2^2(s)$ only pins down[21] the product $b^{12}f$, and the repurchase price f may lie anywhere in the interval $[\frac{s^*}{(1-\theta_1)v'(c_{2,*}^1)}, \frac{\underline{s}}{(1-\theta_1)v'(c_{2,*}^1)}]$.

ii) $\gamma_1^1 > 0$: agent 1 is constrained.

This means that $b^{12} = a$. Rewriting 3.29 using equilibrium conditions, we obtain:

$$E\left[\left(f(s) - \tilde{f}(s)\right)\left(u'(c_2^2(s)) - v'(c_2^1(s))\right)\right] \geq 0 \tag{3.31}$$

Let us now define a partition of $\mathcal{S}$ as follows

$$\mathcal{S}^+(\mathbf{p}_2) = \left\{s \in \mathcal{S} \mid \omega + a\frac{p_2(s)}{1-\theta_1} \geq c_{2,*}^2\right\}, \quad \mathcal{S}^-(\mathbf{p}_2) = \left\{s \in \mathcal{S} \mid \omega + a\frac{p_2(s)}{1-\theta_1} < c_{2,*}^2\right\}$$

Intuitively, $\mathcal{S}^+(\mathbf{p}_2)$ is the union of intervals (by continuity) where the first-best allocation is attainable given $\mathbf{p}_2$. We have $\mathcal{S}^+(\mathbf{p}_2) \cup \mathcal{S}^+(\mathbf{p}_2) = \mathcal{S}$ also by continuity. We argue first that $f(s) = s^*/(1-\theta_1)$ for $s \in \mathcal{S}^+(\mathbf{p}_2)$. If f lies below this constant, be definition of s^* , we have $u'(c_2^2(.)) - v'(c_2^1(.)) > 0$. Any $\tilde{f}$ lying slightly above $\bar{p}$ would violate (3.31). A similar argument can be applied to show that f cannot lie above $s^*/(1-\theta_1)$ for $s \in \mathcal{S}^+(\mathbf{p}_2)$. Now, we argue that $f(s) = p_2(s)/(1-\theta_1)$ for $s \in \mathcal{S}^-(\mathbf{p}_2)$. Indeed, by definition, for all $s \in \mathcal{S}^-(\mathbf{p}_2)$, $u'(c_2^2(.)) - v'(c_2^1(.)) > 0$ so that any schedule $\tilde{p}$ above $\bar{p}$ is feasible and would again violate (3.31). Hence, we have fully defined the equilibrium f as a function of $\mathbf{p}_2$.

We now characterize the fixed point defining equilibrium $\mathbf{p}_2$. Given equilibrium trades and the

[21] In addition, agent 2 could also buy the asset spot to sell it in a repo F_2. In any case, having agent 1 sell a units of contract $\bar{p} = s^*/(1-\theta)$ is an equilibrium since agents do not (strictly) want to trade another contract.

equilibrium contract traded, we have:

$$\begin{cases} p_2(s)v'\left(\omega - a\frac{p_2(s)}{1-\theta_1}\right) = s & s \in \mathcal{S}^-(\mathbf{p}_2) \\ p_2(s)v'(c^1_{2,*}) = s & s \in \mathcal{S}^+(\mathbf{p}_2) \end{cases}$$

We have that $c^1_2(s) < c^1_{2,*}$ for $s \in \mathcal{S}^-(\mathbf{p}_2)$. Suppose there exists $s^+ \in \mathcal{S}^+(p_2)$. Since $p_2(s) > p_2(s^+)$ for $s > s^+$, we have that $[s^+, \bar{s}] \in \mathcal{S}^+(\mathbf{p_2})$. In this case, $\mathcal{S}^+(\mathbf{p}_2)$ is an interval containing the larger elements of $\mathcal{S}$. We are left to show that its minimal element is s^* defined in (3.13). Clearly, $s^* \in \mathcal{S}^+(\mathbf{p_2})$. Consider now $\hat{s} < s^*$. By definition of s^*, we have that

$$\omega + a\frac{\hat{s}}{v'(c^1_{2,*})(1-\theta_1)} < c^2_{2,*}$$

In words, the first best allocation cannot be reached if the spot market price is equal to its "fundamental value" that is $p_2(\hat{s}) = \hat{s}/v'(c^1_{2,*})$. This means that $\hat{s} \in \mathcal{S}^-(\mathbf{p}_2)$ as otherwise, we would have $f(\hat{s}) = s^*/(1-\theta_1)$ and $p_2(\hat{s}) = \hat{s}/v'(c^1_{2,*})$.

To conclude, the equilibrium contract f and spot market price $\mathbf{p}_2$ verify the following equations

$$\text{If} \quad s < s^*, \quad \begin{cases} p_2(s)\left(v'\omega - a\frac{p_2(s)}{1-\theta_1}\right) - s & = 0 \\ f(s) & = \frac{p_2(s)}{1-\theta_1} \end{cases}$$

$$\text{If} \quad s \geq s^*, \quad \begin{cases} p_2(s) & = s \\ f(s) & = \frac{s^*}{1-\theta_1} \end{cases}$$

3.7.3 Proof of Proposition 2

Proof. Building on the case with one asset, we can characterize the equilibrium as follows. Define s^{**} as the minimal state where the first best allocation can be reached.

$$\omega + \frac{a\rho_A(s^{**}) + b\rho_B(s^{**})}{(1-\theta_1)v'(c^1_{2,*})} = c^2_{2,*}.$$

Then the repayment schedule for asset i is

$$f_i(s) = \begin{cases} \dfrac{p_{2,i}(s)}{1-\theta} & \text{for } s \leq s^{**}, \\ \dfrac{\rho_i(s^{**})}{(1-\theta)v'(c^1_{2,*})} & \text{for } s \geq s^{**}. \end{cases}$$

where $(p_{2,A}(s), p_{2,B}(s))$ are the spot market prices of asset A and B respectively in period 2, state s. They are defined as follows for $i = A, B$:

$$\begin{cases} p_{2,i}(s)v'\left(\omega + \frac{ap_{2,A}(s)+bp_{2,B}(s)}{(1-\theta)}\right) - \rho_i(s) = 0 & s \leq s^{**} \\ p_{2,i}(s)v'(c^1_{2,*}) = \rho_i(s) & s > s^{**} \end{cases}$$

for $s < s^{**}$.

The liquidity premium for asset $i = A, B$ is

$$\mathcal{L}_i = \int_{\underline{s}}^{s^{**}} \frac{\rho_i(s)}{1-\theta}\left[\frac{u'\left(c^2_2(s)\right)}{v'(c^1_2(s))} - 1\right] dF(s)$$

Hence,

$$\begin{aligned} \cdot\mathcal{L}_{a,b} &= \mathcal{L}_A - \mathcal{L}_B \\ &= \int_{\underline{s}}^{s^{**}} \frac{s-\rho_\alpha(s)}{1-\theta}\left[\frac{u'\left(c^2_2(s)\right)}{v'(c^1_2(s))} - 1\right] dF(s) \\ &= -\frac{\alpha}{1-\theta}\int_{\underline{s}}^{s^{**}} (s-\mathbb{E}[s])\left[\frac{u'\left(c^2_2(s)\right)}{v'(c^1_2(s))} - 1\right] dF(s) \\ &> 0 \end{aligned}$$

where the inequality follows from the fact that the integral is negative over the integration range.

The haircut as a function of α is:

$$\begin{aligned} \mathcal{H}_i(\alpha) &= p_{1,i} - q_i \\ &= \mathbb{E}\left[(p_{2,i}(s) - f_i(s))v'(c^1_2(s))\right] \\ &= \mathbb{E}[p_{2,i}(s)v'(c^1_2(s))] - \int_{\underline{s}}^{s^*} \frac{p_{2,i}(s)}{1-\theta_1}v'(c^1_{2,s})\mathrm{d}F(s) - \int_{\underline{s}}^{s^*} \frac{p_{2,i}(s^*)}{1-\theta_1}v'(c^1_{2,*})\mathrm{d}F(s) \\ &= \mathbb{E}[s] - \int_{\underline{s}}^{s^{**}} \frac{\rho_i(s)}{1-\theta}dF(s) - \int_{s^{**}}^{\bar{s}} \frac{\rho_i(s^{**})}{1-\theta}dF(s) \\ &= \mathbb{E}[s] - \int_{\underline{s}}^{s^{**}} \frac{(1+\alpha_i)s - \alpha_i\mu}{1-\theta}dF(s) - \int_{s^{**}}^{\bar{s}} \frac{(1+\alpha_i)s^{**} - \alpha_i\mu}{1-\theta}dF(s) \\ &= \mathbb{E}[s] + \frac{\alpha_i\mu}{1-\theta} - \frac{(1+\alpha_i)}{1-\theta}\left[\int_{\underline{s}}^{s^{**}} sdF(s) + \int_{s^{**}}^{\bar{s}} s^{**}dF(s)\right] \end{aligned}$$

The term in brackets is less than $\mathbb{E}[s]$ therefore, for all assets A and B such that $\alpha_A < \alpha_B$ we obtain

$$\mathcal{H}_A < \mathcal{H}_B$$

i.e. the safe asset always commands a lower haircut than the risky asset.

□

3.7.4 Proof of Proposition 3

Proof. The same arguments apply to establish that agent 2 does not borrow in a repo so that we need to consider only one repo contract $f(\nu_2) \in \mathcal{F}_{12}(\mathbf{p}_2)$. However, spot trades are non trivial because agent 2 can now re-sell collateral pledged by agent 1. The equilibrium conditions write:

$$\begin{aligned}
-p_1 + E[sv'(c_2^1(s))] + \gamma_1^1 &= 0, \\
-q_{12} + E\left[f(s,\nu)v'(c_2^1(s))\right] + \gamma_1^1 &= 0. \\
-p_1 + E\left[su'(c_2^2(s))\right] + \gamma_1^2 &= 0, \\
-q_{12} + E\left[f(s,\nu)u'(c_2^2(s))\right] + \nu_2\gamma_1^2 &= 0. \\
c_2^1(s) &= \omega - f(s,\nu)b^{12} + p_2(s)(a_1^1 - a) \\
c_2^2(s) &= \omega + f(s,\nu)b^{12} + p_2(s)a_1^2
\end{aligned}$$

We only look at the case where the collateral constraint binds as otherwise agents may reach the first-best allocation and the analysis is straightforward. In this case:

$$a_1^1 = b^{12} \tag{3.32}$$

$$a_1^2 = -\nu_2\ell^{21} \tag{3.33}$$

Using clearing in the spot market, we have $a_1^1 + a_1^2 = a$. Market clearing for for repo requires $b^{12} = \ell^{21}$. Summing (3.32) and (3.33) we obtain

$$\begin{aligned}
a_1^1 &= b^{12} = \frac{a}{1-\nu_2} \\
a_1^2 &= -\nu b^{12} = -\frac{\nu_2}{1-\nu_2}a
\end{aligned}$$

We can write agent 2 consumption as

$$c_2^2(s) = \omega + \frac{a}{1-\nu_2}(f(s,\nu_2) - \nu_2 p_2(s))$$

We can then adapt the proof of the no re-use case, observing that $f(s,\nu_2) \leq p_2(s)/(1-\theta_1)$ as before and replacing $f(s)$ by $(f(s,\nu_2)-\nu_2 p_2(s))/(1-\nu_2)$. As a consequence, the repurchase schedule finances the first-best consumption profile $(c_{2,*}^1, c_{2,*}^2)$ whenever possible and hits the borrowing limit otherwise. We thus have:

$$\bar{p}(s,\nu) = \begin{cases} \frac{p_2(s)}{1-\theta} & \text{if } s < s^*(\nu) \\ \frac{s^*(\nu_2)}{(1-\theta)v'(c_{2,*}^1)} + \frac{\nu(s-s^*(\nu_2))}{v'(c_{2,*}^1)} & \text{if } s \geq s^*(\nu) \end{cases}$$

where $s^*(\nu_2)$ is implicitly defined as follows

$$\omega + \frac{as^*(\nu_2)}{(1-\nu_2)v'(c^1_{2,*})}\left[\frac{1}{1-\theta} - \nu_2\right] = c^2_{2,*}.$$

Since $v \to \frac{1-(1-\theta)v}{1-v}$ is increasing in v, $s^*(\nu_2)$ is decreasing in v_2 and $\lim_{\nu_2 \to 1} s^*(\nu_2) < 0$.

Assuming now that $\nu_1 > 0$, we provide a formal argument for the claim in Remark 1. Agent 2 does not want to sell in a repo if for all $\tilde{f}_{21} \in \mathcal{F}_{21}(\mathbf{p}_2)$, we have:

$$E[\tilde{f}_{21}(s)u'(c^2_2(s))] + \gamma^2_1 \geq E[\tilde{f}_{21}(s)v'(c^1_2(s))] + \nu_1\gamma^1_1$$

Using the equilibrium characterization, we obtain the following inequality:

$$E\left[\tilde{f}_{21}(s)\left(u'(c^2_2(s)) - v'(c^1_2(s))\right)\right] \geq \frac{1}{1-\nu_2} E\left[(\nu_1(f(s,\nu_2) - \nu_2 p_2(s)) - f(s,\nu_2) + p_2(s))\left(u'(c^2_2(s)) - v'(c^1_2(s))\right)\right]$$

Using the expression for $f(s,\nu_2)$ we derived and $\tilde{f}_{21} = \nu_1(1-\theta_1)p_2(s)$ (the contract for which the inequality above is the most difficult to satisfy), we obtain:

$$\begin{aligned}
& (1-\theta_1)\nu_1 && \geq \frac{(1-\nu_1\nu_2)(1-\theta_1) - (1-\nu_1)}{(1-\theta_1)(1-\nu_2)} \\
\Leftrightarrow \quad & \nu_1(1-\nu_2)(1-\theta_1)^2 && \geq \nu_1(1-\nu_2) - \theta_1(1-\nu_1\nu_2) \\
\Leftrightarrow \quad & \theta_1(1-\nu_2\nu_1) && \geq \nu_1(1-\nu_2)\theta_1(2-\theta_1)
\end{aligned}$$

This is equivalent to condition (3.20).

□

3.7.5 Proof of Proposition 4 and 5

In this section, we prove the following result, that nests Proposition 7 and 8. Define first $s^*(b,\theta_i,\nu)$ for $i = 1, B$, as the solution in s^* to:

$$u'\left(\omega + \frac{bs^*}{1-\nu}\left[\frac{1}{1-\theta_i} - \nu\right]\right) = \delta_i,$$

and for $i = 1, B$, define the repo contract $f_{i2}(b)$ implicitly as a function of the amount borrowed b:

$$f_{i2}(b,\nu_2,s) = \begin{cases} \frac{p_2(s)}{1-\theta_i} & \text{if } s < s^*(b,\theta_i,\nu_2) \\ \frac{s^*(b,\theta_i,\nu)}{\delta(1-\theta_i)} + \frac{\nu_2(s - s^*(b,\theta_i,\nu_2))}{\delta} & \text{if } s \geq s^*(b,\theta_i,\nu_2) \end{cases} \tag{3.34}$$

As before, $s^*(b,\theta_i,\nu_2)$ is the threshold in s above which marginal rates of substitution between a type i and 2 agents can be equalized, given, in particular, the amount of asset b available. Since agent 1 is the marginal holder of the asset into period 3, we know that $p_2(s) = s/\delta$.

Proposition. *Let f_{1B} be the repo contract given by :*

$$f_{1B}(s) = \frac{p_2(s)}{1-\theta_1} \qquad \forall s \in [\underline{s}, \bar{s}] \tag{3.35}$$

and $\hat{b}$ the solution to:

$$\int_{\underline{s}}^{s^*(\hat{b},\theta_B,\nu_2)} \left[u'\left(\omega + \frac{\hat{b}}{1-\nu_2}\left[\frac{1}{1-\theta_B} - \nu_2\right]\right) - \delta_B \right] \mathrm{p_2(s)d}F(s)$$
$$= \frac{(\delta_B - \delta)\theta_1}{(1-\nu_B)(1-\theta_1)} \frac{(1-\theta_B)(1-\nu_2)}{1-(1-\theta_B)\nu_2} E[p_2(s)] \tag{3.36}$$

The equilibrium features intermediation iff $\hat{b} > 0$. Three cases are then possible:

1. *$\hat{b} > a$. Agent 1 sells the asset spot to B who borrows $b_*^{B2} = a/(1-\nu_2)$ from agent 2 using repo $f_{B2}(a, \nu_2)$.* [Proposition 4]

2. *$\hat{b} \in [\nu_H a, a]$. Agent 1 uses a combination of a spot and repo sale with f_{1B} with B. Agent B borrows $b^{B2} = \hat{b}/(1-\nu_2)$ from agent 2 using repo $f_{B2}(b^{B2}, \nu_2)$*

3. *$\hat{b} \in [0, \nu_H a]$. Agent 1 borrows from B using repo F_{1B} and B borrows $b^{B2} = \hat{b}/(1-\nu_2)$ from agent 2 using repo $f_{B2}(b^{B2}, \nu_2)$.* [Proposition 5]

In case 2 and 3, the following condition is necessary :

$$\frac{1}{1-\theta_B} - \frac{1}{1-\theta_1} \geq (1-\nu_B)\frac{1-\nu_2(1-\theta_B)}{(1-\nu_2)(1-\theta_B)}$$

In all three cases the amount b_*^{1B} borrowed by agent 1 from B is given by:

$$b_*^{LH} = \frac{a - (1-\nu_2)b_*^{B2}}{1-\nu_B} \tag{3.37}$$

Proof. Under our conjecture, agents 1 and B may trade in a repo f_{1B} and agents B and 2 can trade in a repo f_{L2}. As usual, agents may also trade in the spot market. We then derive the conditions and characterize the repo contracts f_{LH} and f_{H2} for this conjecture to be an equilibrium.

Step 1: Agents problem and first order conditions

Observing that agent 1 will be the final holder of the asset, as before, we can write his optimization problem as follows:

$$\max_{a_1^1, b^{1B}} \omega + p_1(a - a_1^1) + q_{1B}b^{1B} + E\left[\delta\left(\omega + p_2(s)a_1^1 - f_{1B}(s)b^{1B}\right) + a.s\right]$$
$$\begin{aligned} \text{s.to} \qquad a_1^1 &\geq b^{1B} \quad (\gamma_1^1) \\ b^{1B} &\geq 0 \quad (\xi_{1B}) \end{aligned}$$

While the problem of agent H is:

$$\begin{aligned}
&\max_{a_1^B, l^{B1}, b^{B1}} \omega - p_1 a_1^B - q_{1B}\ell^{B1} + q_{B2} b^{B2} \\
&+\delta_B E\left[\omega + p_2(s) a_1^B + f_{1B}(s)\ell^{B1} - f_{B2}(s) b^{B2}\right] \\
&\qquad \text{s.to} \quad a_1^B + \nu_B \ell^{B1} \geq b^{B2} \quad (\gamma_1^B) \\
&\qquad\qquad \ell^{B1} \geq 0 \quad (\xi_{B1}) \\
&\qquad\qquad b^{B2} \geq 0 \quad (\xi_{B2})
\end{aligned}$$

Recall that a_1^B is the spot market trade of agent B. The variable ℓ^{B1} is the amount agent B lends to 1. Every unit of loans yields agent B a fraction ν_B of re-usable asset. These units can be re-sold spot, which decreases a_1^B, or re-pledged to agent 2, which increases b^{B2}. Finally, agent 2 solves

$$\begin{aligned}
&\max_{a_1^2, l^{2B}} \omega - p_1 a_1^2 - q_{B2}\ell^{2B} + E\left[u\left(\omega + s a_1^2 + f_{B2}(s)\ell^{2B}\right)\right] \\
&\qquad \text{s.t. } a_1^2 + \nu_2 \ell^{2B} \geq 0 \quad (\gamma_1^2) \\
&\qquad \ell^{2B} \geq 0 \quad (\xi_{2B})
\end{aligned}$$

Let us now write down the first order conditions for our 3 agents:

$$-p_1 + \delta E[p_2(s)] + \gamma_1^1 = 0 \tag{3.38}$$

$$q_{1B} - \delta E[f_{1B} s)] - \gamma_1^1 + \xi_{1B} = 0 \tag{3.39}$$

$$-p_1 + \delta_B E[p_2(s)] + \gamma_1^B = 0 \tag{3.40}$$

$$-q_{1B} + \delta_B E[f_{1B}(s)] + \nu_B \gamma_1^B + \xi_{B1} = 0 \tag{3.41}$$

$$+q_{B2} - \delta_B E\left[f_{B2}(s)\right] - \gamma_1^B = 0 \tag{3.42}$$

$$-p_1 + E\left[p_2(s) u'(c_2^2(s))\right] + \gamma_1^2 = 0 \tag{3.43}$$

$$-q_{B2} + E\left[f_{B2}(s) u'(c_2^2(s))\right] + \nu_2 \gamma_1^2 = 0 \tag{3.44}$$

Market clearing implies that $b^{ij} = \ell^{ji}$ for each pair of agents (i, j). Hence, we only use the notation b in the following. Observe that we introduced the positivity constraint on the amount borrowed by 1 to B as these agents may not use a repo transaction but a spot trade exclusively. A quick examination shows that all three collateral constraints bind, that is $\gamma_1^1 > 0$, $\gamma_1^B > 0$ and $\gamma_1^2 > 0$. This implies that:

$$\begin{aligned}
a_1^1 &= b^{1B} \\
a_1^B + \nu_B b^{1B} &= b^{B2} \\
a_1^2 + \nu_2 b^{B2} &= 0
\end{aligned}$$

while market clearing for the asset yields:

$$a_1^1 + a_1^B + a_1^2 = a$$

Using this last equations together with the collateral constraints above, we obtain equation (3.37), that is

$$a = (1 - \nu_B) b^{1B} + (1 - \nu_2) b^{B2}$$

Quick manipulations of equations (3.38) to (3.44) give the following expressions for the Lagrange multipliers associated to the collateral constraints:

$$\gamma_1^2 = \frac{1}{(1-\nu_2)} E\left[(f_{B2}(s) - p_2(s))\left(u'(c_2^2(s)) - \delta_B\right)\right]$$
$$\gamma_1^B = \frac{1}{(1-\nu_2)} E\left[(f_{B2}(s) - \nu_2 p_2(s))\left(u'(c_2^2(s)) - \delta_B\right)\right]$$
$$\gamma_1^1 = \gamma_1^B + (\delta_B - \delta) E[p_2(s)]$$

Let b^* be the amount of asset available for the repo between $1H$ and 2 so that $b_*^{B2} = b^*/(1-\nu_2)$

$$c_2^2(s) = \omega + b_*^{B2}(f_{B2}(s, b^*, \nu_2) - \nu_2 p_2(s))$$

Step 2 : Equilibrium Repo contracts

i) Equilibrium repo contract f_{1B} between 1 and B

With usual equilibrium selection argument, agents 1 and B are not willing to trade contract $\tilde{f}_{1B}$ if and only if

$$\delta E[\tilde{f}_{1B}(s)] + \gamma_1^1 \geq \delta_B E[\tilde{f}_{1B}(s)] + \nu_B \gamma_1^B$$

If $b_{1B} > 0$, from equations (3.39) and (3.41), we obtain

$$(\delta_B - \delta) E\left[f_{1B}(s) - \tilde{f}_{1B}(s)\right] \geq 0$$

which, if $\delta_B > \delta_1$, may only hold if

$$f_{1B}(s) = \frac{p_2(s)}{1 - \theta_L}, \quad \forall s.$$

If $b_{1B} = 0$ (i.e. agents 1 and B trade only spot) then we must have :

$$\frac{(\delta_B - \delta)\,\theta_1}{1 - \theta_1} E[p_2(s)] \leq (1 - \nu_B)\gamma_B^1$$

ii) Equilibrium repo contract f_{B2} between B and 2.

For a given amount b^* of asset available, agents B and 2 trade as in the previous section replacing a by b^*. Using our previous results, the equilibrium repo contract is $f_{B2}(b^*, \nu_2)$ defined in (3.34).

Step 3 : Determination of b^*

We now determine the endogenous amount b^* available for the repo trade between $1H$ and 2 in order to describe the equilibrium completely. This determines in particular whether agents 1 and B trade spot.

i) $b^{1B} = 0$.

Since agent 1 collateral constraint binds, it must be that $a_1^1 = 0$ and $a_1^B = \frac{a}{1-\nu_2}$. This implies that $b^* = a$ and $b^{B2} = \frac{a}{1-\nu_2}$. Agent 2 consumption is

$$c_2^2(s) = \begin{cases} \omega + \frac{ap_2(s)}{1-\nu_2}\left[\frac{1}{1-\theta_B} - \nu_2\right] & \text{if } s < s^*(a, \nu_2, \theta_B) \\ c_{2,*}^2 & \text{if } s \geq s^*(a, \nu_2, \theta_B) \end{cases}$$

This is an equilibrium if $\xi_{1B} + \xi_{B1} \geq 0$. Using equations (3.39) and (3.41), the condition writes

$$\gamma_1^1 \geq \nu_B \gamma_1^B + \frac{\delta_B - \delta}{1-\theta_1} E[p_2(s)]$$

which, using our derivations above, can be rewritten:

$$\frac{1-(1-\theta_B)\nu_2}{(1-\theta_B)(1-\nu_2)} \int_{\underline{s}}^{s^*(a,\theta_B,\nu_2)} \left[u'\left(c_2^2(s)\right) - \delta_B\right] p_2(\mathrm{s}) \mathrm{d}F(s) \geq \frac{(\delta_B - \delta)\theta_1}{(1-\nu_B)(1-\theta_1)} E[p_2(s)]$$

Since the mapping

$$a \to \int_{\underline{s}}^{s^*(a,\theta_B,\nu_2)} \left[u'\left(c_2^2(a,s)\right) - \delta_B\right] \mathrm{p}_2(\mathrm{s}) \mathrm{d}F(s)$$

is decreasing in its argument, the condition above is equivalent to condition $\hat{b} > a$ of case 1.

ii) $b^{1B} > 0$

In this case, we obtain two expressions for γ_1^B which impose the following equality:

$$\frac{1-(1-\theta_B)\nu_2}{(1-\theta_B)(1-\nu_2)} \int_{\underline{s}}^{s^*(a,\theta_B,\nu_2)} \left[u'\left(c_2^2(s)\right) - \delta_B\right] p_2(\mathrm{s}) \mathrm{d}F(s) = \frac{(\delta_B - \delta)\theta_1}{(1-\nu_B)(1-\theta_1)} E[p_2(s)]$$

and pins down the amount $b^* \in [0, a]$ available for B to use in repo f_{B2}. Suppose first that

$b^* \in [\nu_B a, a]$. Then it must be that agent $1H$ buys a fraction of the asset from $1L$. If they only trade in a repo, the maximum amount of free collateral available to $1H$ is $\nu_B a$. Suppose now that $b^* \in [\nu_B a, a]$, then using equation (3.37) and agent 1 collateral constraint, we obtain $a_1^1 = b^{1B} > a$. Since agent $1L$ initially owns $a_0^1 = a$, it means that he is a net buyer of the asset (through$B$ re-selling in the repo). Hence, agent 1 and B only use repo f_{1B}.

Step 4 : No profitable contract between 1 **and** 2

We need to check that intermediation is optimal, that is agents 1 and 2 do not want to trade directly a contract $\tilde{f}_{12}$. This requires

$$\delta E[\tilde{f}_{L2}(s)] + \gamma_1^1 \geq E[\tilde{f}_{12}(s)u'(c_2^2(s))] + \nu_2 \gamma_1^2$$

We can rewrite the condition as

$$(1-\nu_2)\gamma_1^2 \geq E\left[\left(\tilde{f}_{12}(s) - p_2(s)\right)\left(u'(c_2^2(s)) - \delta\right)\right]$$

Using the expression of γ_1^2, we obtain:

$$E\left[\left(f_{B2}(s) - p_2(s)\right)\left(u'(c_2^2(s)) - \delta_B\right)\right] \geq E\left[\left(\tilde{f}_{12}(s) - p_2(s)\right)\left(u'(c_2^2(s)) - \delta\right)\right]$$

or,

$$E\left[\left(f_{B2}(s) - \tilde{f}_{12}(s)\right)\left(u'(c_2^2(s)) - \delta_B\right)\right] \geq \frac{\theta_1(\delta_B - \delta)}{1-\theta_1} E[p_2(s)]$$

In the LHS, we use $\tilde{f}_{12} = p_2(s)/(1-\theta_1)$ to find the tightest bound:

$$\left(\frac{1}{1-\theta_H} - \frac{1}{1-\theta_L}\right) \int_{\underline{s}}^{s^*(b^*,,\theta,\nu_2)} p_2(s)\left(u'\left(c_2^2(s)\right) - \delta_B\right) \geq \frac{\theta_L(\delta_H - \delta_L)}{1-\theta_L} E[p_2(s)]$$
$$\frac{(1-\nu_2)(1-\theta_B)}{1-(1-\theta_B)\nu_2}\left[\frac{1}{1-\theta_B} - \frac{1}{1-\theta_1}\right]\gamma_1^B \geq \frac{\theta_L(\delta_H - \delta_L)}{1-\theta_L} E[p_2(s)]$$

\

where the last line follows from the expression for γ_1^B derived above. We know from (3.38) -(3.41) that the RHS lies below $(1-\nu_B)\gamma_1^B$ and that it is equal when $b_{1B} > 0$. In this latter case, we can rewrite the necessary condition above as

$$\left(1 - \frac{1-\theta_B}{1-\theta_1}\right)\frac{1}{1-\nu_2(1-\theta_B)} \geq \frac{1-\nu_B}{1-\nu_2}$$

which is sufficient condition (5). The condition will also be necessary whenever $b^{1B} > 0$.

□

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