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Polynomially Cointegrated Systems and their Representations: A Synthesis

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Abstract

This paper presents a discussion of polynomial cointegration and a synthesis of various ways generalised cointegrated systems for a multivariate time series process may be represented. Using the Smith-McMillan canonical form of a rational polynomial matrix we describe the null-space structure of higher order - and in particular I(2) - cointegrated systems and we show how different representations such as the error correction model, the common stochastic trends model and various triangular array decompositions, can be derived within this unifying framework. Hence we extend the results of Hylleberg and Mizon (1989) to more general systems. The different representations provide different insights into distinct features of multivariate systems that may simultaneously contain several types of equilibrium behaviour. One obvious case arises when a model contains both higher order integrated and possibly seasonally integrated time series and can be represented in a lower dimensional space implying that the cointegrating equilibria may be expressed as polynomials in the lag operator. The implied long run equilibria may not then seem to be contemporanous in the underlying economic variables. Such non-contemporaneous equilibrium relationships may often have little appeal in terms of economic intuition and we briefly discuss how the specification of appropriately defined "state" variables may provide a more straightforward representation of economic equilibria for cointegrated systems.

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1. Introduction.

It is widely recognized that cointegration as defined by Engle and Granger (1987) at the zero frequency, gives rise to a reduced rank condition on the autoregressive and moving average (rational) polynomial matrices characterising a multivariable time series process. For instance, assume that x_t is a $p\times1$ vector time series integrated of order 1 with the Wold representation

$$\Delta x_t = C(B)\varepsilon_t \tag{1.1}$$

where $C(B)$ is a rational polynomial matrix of full normal rank. Cointegration occurs when this matrix is of reduced rank $r<p$ at a particular frequency, say the zero frequency. In this situation a $p\times r$ vector α exists with the property that

$$\alpha' C(1) = 0 \tag{1.2}$$

and any vector lying in the left nullspace of $C(1)$ spanned by α is considered to be a cointegrating vector. The nullity provides the cointegrating rank r which indicates that r stationary independent linear combinations exist amongst the levels of the series, that is, since $C(B)=C(1)+(1-B)C^{*}(B)$ we have that

$$\alpha' x_t = \alpha' C^{*}(B)\varepsilon_t \tag{1.3}$$

is a stationary process. The concept of cointegration, as initially conceived, is limited in the sense that only the zero frequency of I(1) cointegrated time series is considered although, in general, a much richer class of cointegrated systems may exist with integration of higher orders occuring and also potentially over a range of frequencies. For example Hylleberg et al. (1990) and Engle et al. (1993) analyze seasonal frequencies and the idea of seasonally cointegrated time series and they show how reduced rank conditions similar to (1.3) can be obtained when a seasonally integrated vector time series has a common seasonal pattern. In these more general cointegrated systems the left nullspace may often most easily be described in terms of a *polynomial* space such that the cointegrating vectors which span the space are polynomials in the lag operator rather than vectors. Many other types of common non-stationary and possibly stationary features within a vector time series are plausible, see e.g. Vahid and Engle (1991). For instance series may share a common business cycle so that frequencies other than the zero and seasonal frequencies may be of interest in defining an economic equilibrium that relates economic behaviour at several frequencies. Similarly, Yoo (1986), Salmon (1988), Granger and Lee (1988,89), Engle and Yoo (1991), Gregoir and Laroque (1991) and Johansen (1988a, 1992a) amongst others have shown how higher order cointegration can imply polynomial cointegrating vectors when both the levels and first difference of a variable may be needed to induce full cointegration[1] among the set of variables.

A practical and theoretical difficulty arises in the empirical analysis of such polynomially cointegrated systems in the separation of the short run dynamics from the apparently non contemporaneous interaction within the "equilibrium" relationship. Starting an empirical analysis with a given set of economic variables may lead to the conclusion that an apparently "dynamic" equilibrium relation exists between these variables and calls into question the nature of the notion of equilibrium being considered. Assume for instance that two time series x_t and y_t are cointegrated of order CI(2,1), i.e. such that the individual series are I(2) but a particular linear combination is I(1). It is then a possibility that when considering these two series jointly with say Δx_t, that the three series will cointegrate to a stationary I(0) relation so that the variables $(x_t, y_t, \Delta x_t)$ exhibit full cointegration. The problem is that arbitrary lags of the x_t and y_t series together with other dynamic transformations or differences of the variables are also potentially able to enter (polynomially) cointegrating relationships. In this situation the transient dynamics in disequilibrium will also be different from case to case. One approach to resolving this problem was offered by Salmon (1988) who introduced the notion of an internal model for a multivariate system which imposes further conditions, through the identification of a *minimal*[2] polynomial basis for the cointegrating space and a stability condition on the short run dynamics that may serve to separate the equilibrium and disequilibrium dynamics[3]. This issue does not arise for I(1) systems where minimality is invariably obtained naturally with contemporaneous relations among the given variables but for I(2) systems, say, defining the variables and equilibrium relations to achieve minimality delivers a cointegration space of minimal polynomial order (potentially non polynomial) and the separation of

[1] By full cointegration we mean that the vector time series is reduced to I(0). (See Davidson (1991)).

[2] By minimality here we are assuming that the polynomial degrees are of lowest order and potentially zero yielding a vector space.

[3] Notice that in the I(1) case the internal model just delivers the standard characterisation of the cointegrating basis of the system. It is this characterisation which becomes potentially ill defined in higher order cases and for models that are nonstationary at different frequencies.

the short run dynamics from the equilibrium follows naturally. Consistent with this approach we may interpret polynomial cointegration vectors as indicating the appropriate transformations that should be made on the 'natural' economic variables to define a set of state variables that would occur in the contemporaneous relations of the economic equilibrium defined in these state variables. A similar train of thought can also be found in the work of Johansen (1988a, 1992a) and Davidson (1991) for instance where the concepts of 'balancing' and 'full cointegration' have been developed.

The major purpose of the present paper is to illustrate in a unifying way how polynomially cointegrated systems can be characterised and represented. Our analysis exploits knowledge of the nullspace structure of polynomial matrices and we make extensive use of the Smith-McMillan form of a rational polynomial matrix since it explicitly indicates the I(2), I(1) and I(0) spaces of the system. The Smith-McMillan decomposition and the linkage to Johansen's (1988) notion of 'balanced systems' is discussed in the next section of the paper which also serves to clarify and describe the tools for the subsequent discussion in section 3 where various representations of I(2) systems are reported. In particular we consider the vector autoregressive representation, the error correction representation, various parametric and non-parametric triangular array decompositions and the common stochastic trends representation. These different representations provide different insights into the implicit structure of multivariate systems that simultaneously contain several types of equilibrium behaviour. We thus generalize the synthesis of representations for the I(1) case reported in Hylleberg and Mizon (1988) to more general systems. Although the paper synthesizes the main results that already exist in the literature, it should also be noted that the analysis does not include some aspects covered in Gregoir and Laroque (1991, 1993), and d'Autumn (1992) who adopt different approaches which in part are based on other canonical forms for multivariate systems.

2. The nullspace structure of cointegrated time series.

Fundamentally, the cointegration properties of a vector process are a statement about the singularity or rank deficiency of the *C(B)* matrix of the Wold representation as defined in (1.1) at particular frequencies. The initial benchmark description we gave in the introduction considered the simplest case where x_t was I(1) and the single cointegrating vectors did not take polynomial arguments. In order to generalize this approach we follow Engle and Yoo (1991) and assume that the x_t series has non-stationary features, but when a scalar polynomial filter $\delta(B)$ is applied the resulting series is stationary with Wold representation

$$\delta(B)x_t = C(B)\varepsilon_t \tag{2.1}$$

where $\delta(z^{-1})$ is a polynomial of order q with roots at $z=\theta_1, \theta_2, \ldots \theta_q$ which may be complex; hence we may write $\delta(z^{-1}) = \Pi_{i=1}^{q}(1-\theta_i z^{-1})$. We assume that all the roots of the filter $\delta(z^{-1})$ lie on or outside the unit disc. For instance, if $\delta(z^{-1})=(1-z^{-1})^2$ then two roots at $z=1$ exist and x_t is I(2); if $\delta(z^{-1})=(1-z^{-4})$, x_t is a quarterly seasonally integrated series with roots at $\pm 1, \pm i$ (where $i^2=-1$) and so forth. We shall call the vector space

$$F(z^{-1}) = \{ \; \alpha_i(z^{-1}) \mid \alpha_i(z^{-1})' C(z^{-1}) = 0 \; \} \tag{2.2}$$

the polynomial left null space of $C(z^{-1})$ where each element $\alpha_i(z^{-1})$ of $F(z^{-1})$ is of dimension $p \times 1$ and $F(z^{-1})$ is assumed to have r linearly independent columns. For most economic models $C(z^{-1})$ will invariably be of full normal rank[4] and hence the relevant nullspaces will be those defined at a particular frequency, where say $z=\theta_i$. However, in order to provide a complete characterisation of these spaces we need to consider the possibililty that the vectors spanning the null spaces are polynomials in the lag operator.

In order to describe this null-space structure we will use the Smith-McMillan form of a rational polynomial matrix. This canonical representation displays the system poles and zeros in a form that can be easily seen and hence facilitates our subsequent analysis.

LEMMA 1. THE SMITH-MCMILLAN FORM, (See e.g. Kailath (1980)). *Let $C(z^{-1})$ be a rational polynomial matrix of full normal rank p, which is finite for all z on or within the unit circle, then the Smith-McMillan form of $C(z^{-1})$ is given by*

$$C(z^{-1}) = U^{-1}(z^{-1}) M(z^{-1}) V^{-1}(z^{-1}) \tag{2.3}$$

where $M(z^{-1})$ is of the form

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[4] By 'full normal rank' we mean that det$C(z^{-1})$ is different from zero when expressed in terms of z. However, when evaluated at a particular value of z the determinant may turn out to be zero whereby the C matrix is singular at this particular frequency. Note that in essence the lack of full normal rank of the $C(z^{-1})$ matrix will imply that identities amongst the economic variables have been included in the specification of the multivariable system.

$$M(z^{-1})=\mathrm{diag}\left\{\frac{\varepsilon_i(z^{-1})}{\psi_i(z^{-1})}\right\}, \qquad i=1,2,.....,p$$

and $\{\varepsilon_i(z^{-1}), \psi_j(z^{-1})\}$ *are relatively prime polynomials for* $i=j$ *but not necessarily for* $i\neq j$, *in other words,* $\{\varepsilon_i(z^{-1}), \psi_i(z^{-1})\}$ *have no common factors. Furthermore by construction* $\psi_{i+1}(z^{-1})$ *divides* $\psi_i(z^{-1})$ *for* $i=1,...,r-1$, *whilst* $\varepsilon_{i+1}(z^{-1})$ *is divisible by* $\varepsilon_i(z^{-1})$. $U(z^{-1})$ *and* $V(z^{-1})$ *are unimodular matrices which are therefore invertible at all frequencies but nonunique.*

Note that although $U(z^{-1})$ and $V(z^{-1})$ are not uniquely determined, $M(z^{-1})$ unambigously contains the zeros and poles of the C-matrix as the roots of $\varepsilon_i(z^{-1})$ and $\psi_i(z^{-1})$ respectively. From the Smith-McMillan form of $C(z^{-1})$ we can see that

$$U(z^{-1})C(z^{-1}) = M(z^{-1})\, V^{-1}(z^{-1}) \tag{2.4}$$

and so in accordance with the above discussion it follows that any row of $U(z^{-1})$ which leads to a zero row on the RHS of (2.4) will belong to the nullspace of $C(z^{-1})$ and thus to the set $F(z^{-1})$. However, the $C(z^{-1})$ and $M(z^{-1})$ matrices will invariably be matrices of full normal rank; if $C(z^{-1})$ was not of full normal rank it would appear that the underlying variables would be cointegrated across *all* frequencies, as discussed by Salmon (1988), with different cointegrating relationships at each frequency. It is difficult to visualize an economically meaningful argument for this case and for most practical situations we need therefore to consider the left null space for $C(z^{-1})$ evaluated at specific frequencies where $z=\theta_i$, $i=1,..q$, for instance the zeros that give rise to the non-stationarity of the system. The Smith-McMillan form is an especially effective tool for identifying these frequencies. Note simply, that the coprimeness of $\varepsilon_i(z^{-1})$ and $\psi_j(z^{-1})$ allows us to write $M(z^{-1})$ as a matrix fraction description given by

$$M(z^{-1})=\varepsilon(z^{-1})\Psi^{-1}(z^{-1}) \tag{2.5}$$

where $\varepsilon(z^{-1})=\mathrm{diag}\{\varepsilon_i(z^{-1})\}$ and $\Psi(z^{-1})=\mathrm{diag}\{\psi_i(z^{-1})\}$. When evaluating the left nullspace of $C(z^{-1})$ at a particular frequency we therefore only need to focus on the zeros of the elements in $\varepsilon(z^{-1})$.

We shall concentrate in what follows on multiple unit roots at the zero frequency and hence the associated left null space of $C(z^{-1})$, however, if non-stationary roots at other frequencies are present then it will be necessary to expand the C-matrix around these roots. The following lemma originally proved by Lagrange and reported in Hylleberg et al. (1991) in a slightly different form achieves this expansion.

LEMMA 2. POLYNOMIAL EXPANSION. (See e.g. Hylleberg et al. (1991)). *Any rational polynomial matrix $C(z^{-1})$ which is finite valued at the distinct points $z=\theta_1, \theta_2, \ldots \theta_q$ can be expanded around these points to give*

$$C(z^{-1})=\sum_{i=1}^{q} C(\theta_i)\Pi_{j\neq i}\frac{(1-\theta_j z^{-1})}{(1-\theta_j/\theta_i)}+\Pi_{i=1}^{q}(1-\theta_i z^{-1})C^*(z^{-1}).$$

The remainder $C^*(z^{-1})$ is a unimodular rational polynomial matrix with no zeros at θ_i, $i=1,\ldots,q$.

The expansion of Lemma 2 is only valid when the roots θ_i are distinct. In the case that there exist multiple non-stationary roots, for most practical situations at the zero frequency, then a similar expansion can be conducted around these values for $C^*(z^{-1})$. Although we will concentrate on the zero frequency case the analysis covers unit roots at other frequencies and the question of the interaction of cointegrating spaces across frequencies is taken up in Haldrup and Salmon (1993).

Johansen's (1988a) idea of a *balancing* in which the system can be represented as a set of I(0) transformed variables with no redundant unit roots[5] becomes particularly important in the presence of multiple unit roots at a particular frequency. Johansen focused on zero frequency I(d) processes but as I(2) systems are most likely to arise in practice we will later limit our discussion to this case and examine how balancing can be characterized through the Smith-McMillan canonical form. For a general I(d) system $C(z^{-1})$ can be decomposed as follows

$$C(z^{-1})=\sum_{j=0}^{d-1}(1-z^{-1})^j C_j+(1-z^{-1})^d C_d(z^{-1}) \tag{2.6}$$

where $C_d(z^{-1})$ is a full rank matrix.

Johansen assumes that $C(z^{-1})$ is a holomorphic function which implies that it is sufficiently smooth to ensure all derivatives exist for all values of j in this expansion.

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[5] A similar analysis using a different approach can be found in Davidson (1991). See Clements (1990) for a discussion of the relationship between Johansen's notion of 'balancing' and Davidson's notion of 'full cointegration'.

In (2.6) the expansion is made around z=1, but we also need that $C(z^{-1})$ is non-singular for z≠1 which we have already assumed by definition. The left null spaces of the matrices C_j can now be defined as

$$F_j=\{\ \alpha_{ij}\in R^p \mid \alpha_{ij}'C_j=0\} \qquad \text{for } j=0,1,... \quad (2.7)$$

We also form the sets

$$M_j=F_0\cap F_1\cap...\cap F_j$$

where M_j is then the set of all vectors that span the null spaces for all of C_0 , C_1,...,C_j. Johansen proceeds by defining the index $n=\Sigma_{j=0}^{d}m_j$ where m_j is the dimension of M_j and n is thus the number of independent cointegration vectors reducing the order of integration by at least one degree. Notice, that m_d=0, so that in general M_j is empty for $j\geq d$. The idea behind this is that if α_{ij} is a vector belonging to M_j then

$$\alpha_{ij}'C(z^{-1})=(1-z^{-1})^{j+1}\alpha_{ij}'C_{j+1}(z^{-1})$$

such that j unit roots in $\Delta^d x_t=C(B)\varepsilon_t$ will cancel when weighted by the cointegrating vector $\alpha_{ij}\in M_j$. The condition for having a balanced system is that n=s, where s is the number of unit roots at z=1 in det $C(z^{-1})$=0. In general it can be shown that $s\geq n$ and when the equality does not hold there are thus too many unit roots in the system compared to the number of cointegrating relations; in this case we say we are in the unbalanced situation. As has been noted by Davidson (1991), this does not necessarily mean that the system cannot be put into a fully cointegrating form through further operations on the system. Balancing is therefore a sufficient but not a necessary condition for having full cointegration. One obvious way of formulating a balanced system is to appropriately redefine variables in such a way that we either increase n while holding s fixed *or*, as we shall see, to reduce s for a fixed value of n by removing the redundant roots following the data transformations. This is a possibility since, in defining n for say I(2) systems, the potential cointegration amongst cointegrated I(1) relations and differenced variables, has not been taken into account.

We illustate in the example below that Smith-McMillan decompositions can be found sequentially to redefine variables and thus to achieve balance as the transformations that are needed in each step are given by the vectors defining the relevant left null space. However, since these vectors are not uniquely determined we *also* need to strive for a minimal polynomial basis for the cointegration space which helps separating (and thus identifying) the equilibrium relations from the short run dynamics. Hence we need to impose further structure on the Smith-McMillan form for cointegrated systems to ensure the lefthand unimodular matrix is minimal.

EXAMPLE. Consider the model structure used in Granger and Lee (1988, 1989) with Wold representation

$$\Delta^2\begin{pmatrix}x_{1t}\\ x_{2t}\end{pmatrix}=\begin{pmatrix}\Delta & \Delta-1\\ \Delta & -1\end{pmatrix}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix},$$

implying that x_{1t} and x_{2t} are I(2) variables. It can be easily checked that $\det C(B)=-\Delta^2$, so $s=2$. We also have that

$$C_0=\begin{pmatrix}0 & -1\\ 0 & -1\end{pmatrix},\quad C_1=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

so $n=m_0+m_1=1+0=1<s$; hence we are in the unbalanced case indicating there are more unit roots in the system than independent cointegration vectors given x_{1t}, x_{2t}. Using the Smith-McMillan decomposition we can write the model as follows

$$\Delta^2\begin{pmatrix}x_{1t}\\ x_{2t}\end{pmatrix}=\begin{pmatrix}\Delta & (\Delta^{-1}-\Delta^{-2})\\ \Delta & -\Delta^{-2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \Delta^2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}.$$

The first matrix on the RHS - the "$U(B)^{-1}$"-matrix - is unimodular and the last row determines the left null space of $C(z^{-1})$ at $z=1$. By multiplying through by the inverse matrix

$$U(z^{-1})=\begin{pmatrix}\Delta^{-2} & (\Delta^{-1}-\Delta^{-2})\\ \Delta & -\Delta\end{pmatrix}$$

we can define new variables $x_t^*=U(z^{-1})x_t$ where

$$x_{1t}^*=\Delta^{-2}((x_{1t}-x_{2t})+\Delta x_{2t})\quad\text{and}$$
$$x_{2t}^*=\Delta(x_{1t}-x_{2t}).$$

Note that the polynomial cointegrating vector is given by $(\Delta, -\Delta)$ which is non-minimal. Hence we see that the new model

$$\Delta^2\begin{pmatrix} x_{1t}^* \\ x_{2t}^* \end{pmatrix} = \begin{pmatrix} (x_{1t}-x_{2t})+\Delta x_{2t} \\ \Delta^3(x_{1t}-x_{2t}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta^2 \end{pmatrix}\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

is balanced in the redefined variables since $n=s=2$. This model however contains redundant double unit roots in the last equation and we should also recognize though that the valid Wold representation of the system should have no such non-invertibility in the univariate representation and hence any redundant roots should be removed[6]. These redundant unit roots will not necessarily be apparent in the multivariate system. It is easily seen from the decomposition that the model then simplifies to the case of no unit roots in the transformed variables with $n=s=0$; so this model is of course in balance and we have identified the I(0) or equilibrium processes.

The route leading to the balanced model as specified above has the implication that a non-minimal and non-unique polynomial basis is used to transform the variables[7]; i.e. the $U(B)^{-1}$ matrix chosen to represent the left null-space gives rise to the polynomial cointegration vector $(\Delta, -\Delta)$, but since the U matrix in the Smith-McMillan canonical form is generally non-unique there is sufficient flexibility available to impose *further* restrictions on the choice of this unimodular matrix by requiring minimality. For the model in question, which has no short run dynamics, an alternative representation is

$$\Delta^2\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \Delta^2 \end{pmatrix}\begin{pmatrix} 0 & \Delta \\ \Delta^{-1} & \Delta^{-2} \end{pmatrix}\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}.$$

As can be seen the $U(B)$ matrix - the transformation matrix - is given in this case by

[6] In a general I(d) system the presence of redundant unit roots violates the stability condition of $C_d(z^{-1})$ as it is specified in (2.6).

[7] For instance a $(\Delta^2,-\Delta^2)$ polynomial cointegrating vector could have been used with a different Smith-McMillan decomposition.

$$U(z^{-1})=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

so in the transformed variables the model becomes

$$\Delta^2\begin{pmatrix}x_{1t}^{**}\\ x_{2t}^{**}\end{pmatrix}=\Delta^2\begin{pmatrix}(x_{1t}-x_{2t})\\ x_{2t}\end{pmatrix}=\begin{pmatrix}0 & \Delta\\ \Delta & -1\end{pmatrix}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}$$

We have in effect defined the appropriate first stage state variables $x_{1t}^{**}=(x_{1t}-x_{2t})$ and $x_{2t}^{**}=x_{2t}$ through the minimality condition. The advantage of so doing is that by choosing a minimal basis for the cointegration space we may be able to a set up a new system in contemporaneously related variables that may potentially have a more meaningful and intuitive economic interpretation. If in the example x_{1t} was production and x_{2t} sales such that $x_{1t}^{**}=(x_{1t}-x_{2t})$ defines the change of inventory, this would coincide with the definitions employed by Granger and Lee based on obvious economic criteria. In terms of the Wold representation where redundant unit roots have been removed, we can use the Smith-McMillan decomposition to write this as

$$\begin{pmatrix}\Delta x_{1t}^{**}\\ \Delta^2 x_{2t}^{**}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}$$

It can be easily checked that this new system in the appropriately chosen state variables is balanced as $n=s=1$. Note that essentially we have obtained balance by reducing s while keeping n fixed compared to the initial system by defining appropriate state variables. This contrasts the procedure of Johansen where s is kept fixed and n is increased. It is also revealed that for the new system the left null space is spanned by the vector (1, -1) which again satisfies the minimality condition. Hence, by choosing the state variables in this manner we obtain a 'minimal' polynomial basis for the cointegrating space together with an ability to examine the economic meaning of the state variables at each stage, exactly as suggested by Salmon (1988). Moreover, athough there are no short run dynamics in this example there is no interference in the process identifying the equilibrium with polynomial cointegration as may be obtained with the non-minimal reduction given by the $(\Delta,-\Delta)$ cointegrating vector. We may anticipate that the statistical properties of the two different approaches may differ when short run dynamics exist.

Finally the last step can be taken in the process of sequentially removing unit roots from the system by multiplying by the inverted LHS unimodular matrix. The resulting system is thus balanced with no redundant zeros, $n=s=0$, and with the equilibrium relations given directly by

$$x_{1t}^{**}+\Delta x_{2t}^{**}=x_{1t}-x_{2t}+\Delta x_{2t}=\varepsilon_{1t}$$
$$\Delta x_{1t}^{**}=\Delta(x_{1t}-x_{2t})=\varepsilon_{2t}.$$

The first decomposition in the example moved directly to I(0) or full cointegration with polynomial cointegration in the natural variables whereas the second moved to I(0) with non-polynomial cointegration in appropriately defined state variables[8].

Engle and Yoo (1991) follow the first of the above routes to give a precise characterization of when polynomial cointegration vectors might arise in multivariate systems. They use Smith-McMillan forms as well, but from the outset they assume the system is balanced or at least that the appropriate transformations have been performed to ensure balancing. To see this, consider the system

$$\Delta^2 x_t=C(B)\varepsilon_t=U^{-1}(B)M(B)V^{-1}(B)\varepsilon_t. \tag{2.8}$$

For a balanced I(2) system $M(B)$ is of the form

$$M(B)=\begin{bmatrix} 1 & & \\ & \Delta & \\ & & \Delta^2 \end{bmatrix}\begin{array}{l} p-r_1-r_2 \\ r_1 \\ r_2 \end{array} \tag{2.9}$$

and will contain all the system zeros of the $C(B)$-matrix[9]. We do not exclude the

[8] In the companion paper, Haldrup and Salmon (1993), we discuss the potential dangers of taking the direct cointegration route in which economically spurious relations may appear empirically.

[9] It should be emphasized here, that as the $M(B)$ matrix is specified it is assumed that all variables are I(2). This need not necessarily be the case in practical situations where a combination of variables of different orders of integration is rather likely. When this is the case we should recognize that the valid Wold representation does not contain redundant unit roots; for the very special case with no cointegration this would mean for instance that the M-matrix would be the identity matrix for the vector process $(x_{0t}, \Delta x_{1t}, \Delta^2 x_{2t})$ of I(0), I(1) and I(2) variables, respectively.

possibility that either r_1 or r_2 equals zero. The case $r_2 \neq 0$ is the most novel however as the analysis of I(1) systems has been well documented elsewhere. From (2.6) it follows that

$$C(z^{-1})=C(1)+C^{*}(1)(1-z^{-1})+C^{**}(z^{-1})(1-z^{-1})^2=C_0+C_1(1-z^{-1})+C_2(z^{-1})(1-z^{-1})^2 \quad (2.10)$$

and by the way $M(z^{-1})$ is defined it is easily seen that the left null spaces of C_0 and C_1 are of dimension r_1+r_2 and r_2, respectively, whilst $C_2(z^{-1})$ is of full rank. The null spaces will intersect by construction in such a way that $n=m_0+m_1=r_1+2r_2$; but this is exactly equal to the number of unit roots in the $C(z^{-1})$ matrix since $\det M(z^{-1})=(1-z^{-1})^s$ where $s=r_1+2r_2$ which ensures balancing.

Using the Smith-McMillan decomposition we can now give a precise description of the possible polynomial transformations that are needed to achieve full cointegration. Note simply that

$$U(B)\Delta^2 x_t=M(B)V^{-1}(B)\varepsilon_t, \quad (2.11)$$

and since $M(B)$ is diagonal, and thus commutes freely, we can write

$$U(B)\begin{bmatrix}\Delta^2 & & \\ & \Delta & \\ & & 1\end{bmatrix} x_t = V^{-1}(B)\,\varepsilon_t.$$

Hence the first $p\text{-}r_1\text{-}r_2$ relations are readily made stationary using the filter Δ^2. The next r_1 relations become stationary after the Δ-filter is used and the resulting variables are weighted by $\alpha_1(B)$ where $\alpha_1(B)$ follows from a conformable partitioning of $U(B)$ as

$$U'(B)=\left[U_1(B),\alpha_1(B),\alpha_2(B)\right]. \quad (2.12)$$

Engle and Yoo argue that $\alpha_1=\alpha_1(1)$ is a valid cointegrating vector. To see this, expand $\alpha_1(z^{-1})$ around $z=1$. Now the relevant relations read

$$\alpha_1'(B)\Delta x_t = \alpha_1'\Delta x_t + \alpha_1^{*\prime}(B)\Delta^2 x_t = \mathit{stationary}.$$

But $\Delta^2 x_t$, the second term on the RHS, is already stationary, so α_1 must be a valid non-polynomial cointegrating vector. This factorisation will thus ensure a minimal basis for the cointegration space in first differences, and thus uniquely separates the equilibrium relation and the disequilibrium dynamics. When proceeding to $\alpha_2(B)$ this argument will fail and cointegrating vectors will, in general, be polynomials in the lag operator. Consider the expansion

$$\alpha_2'(B)x_t = [\alpha_2' + \alpha_2^{*\prime}\Delta + \alpha_2^{**\prime}(B)\Delta^2]x_t \; - \; \textit{stationary}.$$

Clearly, the last term, $\alpha_2^{**\prime}(B)\Delta^2 x_t$ is stationary, but nothing guarantees that $\alpha_2' x_t$ is itself stationary given the presence of α_2^*. However,

$$\alpha_2' x_t + \alpha_2^{*\prime}\Delta x_t \; - \; \textit{stationary}$$

by definition. Again this choice of a polynomial cointegrating space is chosen to minimize the polynomial degree and is therefore unique in this respect. In another respect though we should note that although the polynomial order can be uniquely determined, the individual cointegrating (polynomial) vectors are non-unique since only the *space* spanned by these vectors is determined. This follows also from the Smith-McMillan form since $U(z^{-1})$ is non-unique.

Of course the arguments given above will extend to systems integrated of higher order than 2 and across frequencies. Engle and Yoo (1991) claim that when more than one unit root is eliminated by a polynomial cointegrating vector, then we may not be able to find a non-polynomial cointegrating vector and this only remains true provided we do not seek to redefine the set of state variables as previously discussed.

3. Representations of multivariate I(2)-processes.

In the previous section we focused on the MA (or Wold) representation of a multivariate time series process and used this representation to discuss various ways equilibrium relations could be described given their cointegration properties and the matching of unit roots in relation to rank reductions. In what follows we retain the notion of polynomial cointegration although we should keep in mind how polynomial cointegration may be removed by suitably defined transformations on the natural variables. When moving to other representations the notion of balancing is equally important and the question of whether excessive unit roots exist in the MA representation, will have a similar impact for say the vector autoregressive representation (VAR). Johansen (1988) also considers the VAR representation rather than the VMA representation and finds the relevant conditions in this case to ensure balancing. In what follows we assume balancing is already obtained and consider different ways of parametrizing I(2) systems and in so doing extend the synthesis of Hylleberg and Mizon (1989) to more general multivariate systems.

The benchmark for the various representations to be derived is the Wold representation (2.8) and the associated Smith-McMillan decomposition for a balanced system. In addition to the conformable partitioning of the *U(B)*-matrix in (2.12), we also define

$$V(B)=\left[V_1(B),\gamma_1(B),\gamma_2(B)\right]. \tag{3.1}$$

I. THE VECTOR AUTOREGRESSIVE REPRESENTATION.

The $p\times 1$ vector I(2)-process x_t with the Wold-representation given by (2.8) has the following autoregressive representation

$$A(B)x_t=\varepsilon_t$$

where

$$A(B)=A_0(1)+A_1(1)\Delta+A_2(B)\Delta^2.$$

The $A(z^{-1})$ matrix has the following properties:

a) *rank $A_0(1)=r_2$ and rank $A_1(1)=r_1+r_2$ provided $r_2\neq 0$. If $r_2=0$ a new VAR system for I(1) series may be defined and the standard I(1) analysis will apply. The matrix $A_2(z^{-1})$ is of full rank for all z.*

b) *The number of unit roots in $\det A(B)=0$ equals $r^*=2p-2r_2-r_1$.*

c) *A_0 and A_1 may further be written as*

   (i) $A_0=\gamma_2\alpha_2'$ *where γ_2,α_2 are of dimension $p\times r_2$, and,*

   (ii) $\gamma_2^{\perp\prime}A_1\alpha_2^{\perp}=\varphi\eta'$ *where φ and η are $(p-r_2)\times r_1$. The orthogonal matrices $\gamma_2^{\perp}$, $\alpha_2^{\perp}$ are $p\times(p-r_2)$ of full rank and satisfy $\gamma_2'\,\gamma_2^{\perp}=0$, and $\alpha_2'\,\alpha_2^{\perp}=0$.*

PROOF. In the stationary case the dual role of the MA and AR representations can be easily calculated by taking the inverse of the relevant polynomial matrices. For non-stationary processes this operation is invalid since the inverse of the polynomial matrix $C(z^{-1})$ not is summable when evaluated at the zero frequency where z=1, in other words the $C(1)$ matrix is singular. The Smith-McMillan form is useful in this respect since the unit roots causing the singularity will naturally cancel. Straightforward application of the Smith-McMillan-decomposition yields

$$A(B)x_t = V(B)\bar{M}(B)U(B)x_t = \varepsilon_t \tag{3.2}$$

where

$$\bar{M}(z^{-1}) = \begin{bmatrix} (1-z^{-1})^2 I_{p-r_1-r_2} & 0 & 0 \\ 0 & (1-z^{-1}) I_{r_1} & 0 \\ 0 & 0 & I_{r_2} \end{bmatrix}. \tag{3.3}$$

Since $V(z^{-1})$ and $U(z^{-1})$ are unimodular matrices it follows trivially that (*a*) and (*b*) are satisfied. Hence in general the number of unit roots for the entire multivariate system will be smaller than twice the dimension of the VAR-model as long as r_1, $r_2 \neq 0$, so that some cointegrating combinations of the series exist. In other words the number of unit roots in the $A(z^{-1})$ matrix is simply the rank deficiency at z=1 plus the number of I(2) components. It also follows from the Smith-McMillan decomposition and the $\bar{M}$-matrix that if r_2=0 a common (1-B) factor will occur in the AR and MA representation such that the standard I(1) analysis can be carried out on the differenced series. In this case the $A_1(1)$ matrix associated with the differenced system is of course of full rank.

The restrictions displayed in (*c*) are interesting because they exactly refer to the parametrization used by Johansen (1992a,b) in his I(2) representation of cointegrated VAR-systems, which also forms the basis for the estimation and inferential procedures suggested for I(2) analysis in the Gaussian VAR model, see also Paruolo (1993). This analysis is a further generalisation of the procedures initially reported in Johansen (1988b,1991) for the I(1) case. The first result follows directly by use of (2.11), (3.1), and (3.2), i.e.

$$A_0(1) = V(1)\bar{M}(1)U(1) = V(1)\begin{bmatrix} 0 & & \\ & 0 & \\ & & I_{r_2} \end{bmatrix}U(1) = \gamma_2 \alpha_2'. \tag{3.4}$$

To prove the second result for $A_1(1)$, we have that by defining $\alpha_2(B)=\alpha_2+\alpha_2^{*}\Delta+\alpha_2^{**}(B)\Delta^2$ and $\gamma_2(B)=\gamma_2+\gamma_2^{*}\Delta+\gamma_2^{**}(B)\Delta^2$, which provide a minimal polynomial basis amongst the I(2) variabels, then

$$A_1(1)=V(1)\left[\begin{pmatrix} 0 & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix}U(1)+\begin{pmatrix} 0 & & \\ & 0 & \\ & & I_{r_2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \alpha_2^{*\prime} \end{pmatrix}\right]+[0,0,\gamma_2^{*}]\begin{pmatrix} 0 & & \\ & 0 & \\ & & I_{r_2} \end{pmatrix}U(1) \tag{3.5}$$

$$= \gamma_1\alpha_1' + \gamma_2\alpha_2^{*\prime} + \gamma_2^{*}\alpha_2'.$$

Now, by specifying appropriate orthogonal matrices in the above manner, it follows that

$$\gamma_2^{\perp\prime}A_1\alpha_2^{\perp} = [\gamma_2^{\perp\prime}\gamma_1]\ [\alpha_1'\alpha_2^{\perp}] = \varphi\eta'$$

with the last two matrices straightforwardly defined. As discussed by Johansen (1992a,b), the reduced rank condition of A_1 depends on the reduced rank of A_0 and this is what makes his estimation and testing problems for the I(2) case non-trivial. The simultaneous estimation of model parameters is discussed in Johansen (1990) where a particular algorithm is suggested. A different method is reported in Johansen (1992b) which only relies on regression and reduced rank regression. The problem is solved by essentially first considering the reduced rank problem for A_0 and then treating as fixed the cointegration rank and the implied cointegration parameters spanning the cointegration space. Next a second reduced rank problem is solved with respect to A_1 after appropriate conditioning with respect to the short run parameters and the results found from the analysis in the first step. A distinct attraction of this approach is that after r_1 and r_2 have both been fixed, hypothesis testing reduces to standard Gaussian inference.

II. THE ERROR CORRECTION REPRESENTATION.

The $p\times 1$ vector I(2) process x_t has the following error correction representation:

$$D(B)\Delta^2 x_t = -\gamma_1(B)\, z_{1,t-1} - \gamma_2(B)\, z_{2,t-1} - \gamma_2(B)\, z_{3,t-1} + \varepsilon_t \tag{3.6}$$

where

$$z_{1,t}=\alpha_1'\Delta x_t\,,\quad z_{2,t}=\alpha_2'\Delta x_t\,,\quad \textit{and}\quad z_{3,t}=[\alpha_2'+\alpha_2^{*\prime}\Delta]x_t.$$

The matrices γ_1 *and* α_1 *are of orders* $p\times r_1$ *whilst* γ_2, α_2, α_2^* *are* $p\times r_2$. $D(z^{-1})$ *is a* $p\times p$ *polynomial matrix of full rank with all roots inside the unit circle.*

PROOF. The proof of the ECM representation is similar to that of Engle and Yoo (1991). As a starting point we partition $U(B)$ and $V(B)$ as in (2.12) and (3.1) and notice that $\bar{M}(B)$ can be factored as

$$\bar{M}(B) = (1-B)^2 I_p + B(1-B)\begin{bmatrix}0&0&0\\0&I_{r_1}&0\\0&0&I_{r_2}\end{bmatrix} + B\begin{bmatrix}0&0&0\\0&0&0\\0&0&I_{r_2}\end{bmatrix}. \tag{3.7}$$

From the autoregressive representation as it is written in (3.2) we get

$$\Big[V(B)U(B)(1-B)^2+\gamma_1(B)\alpha_1'(B)(1-B)B+\gamma_2(B)\alpha_2'(B)[(1-B)+1]B\Big]x_t=\varepsilon_t$$

so upon rearranging terms

$$V(B)U(B)\Delta^2 x_t = -\gamma_1(B)\alpha_1'(B)\Delta x_{t-1}-\gamma_2(B)\alpha_2'(B)\Delta x_{t-1}-\gamma_2(B)\alpha_2'(B)x_{t-1}.$$

This can be further rewritten as

$$\begin{aligned}&[V(B)U(B)+\gamma_1(B)\alpha_1^{*\prime}(B)B+\gamma_2(B)(\alpha_2^{*\prime}(B)+\alpha_2^{**\prime}(B))B]\Delta^2 x_t =\\ &D(B)\Delta^2 x_t = -\gamma_1(B)\alpha_1'\Delta x_{t-1}-\gamma_2(B)\alpha_2'\Delta x_{t-1}-\gamma_2(B)[\alpha_2'+\Delta\alpha_2^{*\prime}]x_{t-1}+\varepsilon_t\end{aligned}$$

with $D(B)$ directly defined. This matrix is of full rank by construction.

Notice the different forms of cointegration that will appear as error correction terms in the model. There are those combinations $\alpha_1'x_t$ which are I(1) but when differenced will reduce to I(0) relations; $z_{1t}=\alpha_1{}'\Delta x_t$. Hence these relations enter as cointegrating relations of the differences. The combinations $\alpha_2'\Delta x_t$ have the same property, but they also have the special feature that when being integrated up to levels the resulting I(1) variables cointegrate fully with the differences through the $\alpha_2^{*\prime}$ matrix. Essentially this property is what Granger and Lee (1988, 1989) refer to as *multicointegration*[10]. If some rows of $\alpha_2^{*\prime}$ are zero vectors we obtain CI(2,2) series where no differencing is needed to achieve stationarity. The above representation differs from the one reported in Engle and Yoo (1991) who only focus on a bivariate system in which the first error correction term in (3.6) will be absent and hence there will be no cointegrating relations amongst the differenced variables.

The error correction model as it is specified in (3.6) allows for some elements of x_t to be I(1) although this is not assumed from the outset. Reparametrizing x_t as $x_t=(x_{1t}', x_{2t}')'$ where x_{1t} and x_{2t} are assumed to be I(1) and I(2), respectively, (3.6) becomes

$$D(B)\begin{pmatrix}\Delta x_{1t}\\ \Delta^2 x_{2t}\end{pmatrix} = -\gamma_1(B)\alpha_1'\begin{pmatrix}x_{1t-1}\\ \Delta x_{2t-1}\end{pmatrix} - \gamma_2(B)\alpha_2'\begin{pmatrix}x_{1t-1}\\ \Delta x_{2t-1}\end{pmatrix}$$
$$-\gamma_2(B)\left[\alpha_2'\begin{pmatrix}\Delta^{-1} & 0\\ 0 & 1\end{pmatrix} + \alpha_2^{*\prime}\begin{pmatrix}1 & 0\\ 0 & \Delta\end{pmatrix}\right]\begin{pmatrix}x_{1t-1}\\ x_{2t-1}\end{pmatrix} + \begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}.$$

As can be seen the two first error correction terms are quite similar to the terms arising in an error correction model for I(1) variables The third term shows that the accumulation of the I(1) variable x_{1t}, which then is I(2) may be important to obtain full cointegration in the I(2) model, c.f. the Granger and Lee (1988,1989). However, this term may not be required since we permit some elements of α_2' associated with $\Delta^{-1}x_{t-1}$ to be zero.

In a similar way we may accomodate stationary - or I(0) series - in the system. In this case the notion of cointegration must be slightly redefined when using Smith-McMillan forms since the left null space of the MA form or the right null space of the AR form may consist of unit vectors as "cointegrating vectors". This possibility is explicitly considered in the VAR-approach to estimating cointegrating vectors by Johansen (1988b, 1991, 1992a) applied to both I(1) and I(2) systems. See also Clements (1990).

[10] Naturally this is a property characterising higher order cointegrated systems in general since an I(2) variable for instance can be defined to be an integrated I(1) series.

III. THE PARAMETRIC (BEWLEY) TRIANGULAR ARRAY DECOMPOSITION.

The x_t vector can be given the following parametric decomposition

$$\begin{aligned} 0 &= Q_1(B)\Delta^2 x_t + u_{1t} && (p-r_1-r_2)\times 1 \\ \alpha_1'\Delta x_t &= Q_2(B)\Delta^2 x_t + u_{2t} && (r_1\times 1) \\ \alpha_2' x_t + \alpha_2^{*\prime}\Delta x_t &= Q_3(B)\Delta^2 x_t + u_{3t} && (r_2\times 1) \end{aligned}$$

where $Q_i(z^{-1})$, $i=1,2,3$, are polynomial matrices with all roots strictly inside the unit circle.

PROOF. To see how the Bewley Representation, see Bewley (1979), may be derived, write the autoregressive representation as

$$[A_0(1)+A_1(1)\Delta]x_t = A_2(B)\Delta^2 x_t + \varepsilon_t,$$

where A_0 and A_1 are as previously defined in (3.4) and (3.5). Hence it follows that

$$A_0(1)+A_1(1)\Delta = \gamma_2(\alpha_2' + \alpha_2^{*\prime}\Delta) + \gamma_1\alpha_1'\Delta + \gamma_2^*\alpha_2'\Delta .$$

This can also be written more compactly in such a way that

$$(V_1(1), \gamma_1, \gamma_2+\gamma_2^*\Delta)\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1'\Delta & 0 \\ 0 & 0 & \alpha_2'+\alpha_2^{*\prime}\Delta \end{pmatrix} x_t = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_2^*\alpha_2^{*\prime} \end{pmatrix} + A_2(B)\right]\Delta^2 x_t + \varepsilon_t.$$

If we define $V^* = (V_1(1), \gamma_1, \gamma_2+\Delta\gamma_2^*)$ and note that this matrix is unimodular, then it is seen that

$$\begin{pmatrix} 0 \\ \alpha_1'\Delta x_t \\ \alpha_2' x_t + \alpha_2^{*\prime}\Delta x_t \end{pmatrix} = V^{*-1}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_2^*\alpha_2^{*\prime} \end{pmatrix} + A_2(B)\right]\Delta^2 x_t + V^{*-1}\varepsilon_t .$$

Now the representation above follows by appropriate definition of $Q_i(B)$ and u_{it}, i=1,2,3, as matrices of matching dimensions, i.e.

$$\begin{pmatrix} Q_1(B) \\ Q_2(B) \\ Q_3(B) \end{pmatrix} = V^{*-1}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_2^* \alpha_2^{*\prime} \end{pmatrix} + A_2(B)\right] \quad \text{and} \quad \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} = V^{*-1}\varepsilon_t.$$

The Bewley representation for cointegrated systems, which has previously been discussed by Hylleberg and Mizon (1989) for the I(1) case, shows that the long run and the short-run dynamic components of the x_t process can be parametrized separately. In the representation above this is done in the *minimal polynomial* sense. The first $(p-r_1-r_2)$ relations describe the interaction of the I(2) trends of the system that do not cointegrate. The next r_1 relations consist of linear combinations that cointegrate in first differences whilst the remaining r_2 equations are the fully cointegrating relations consisting of levels and first differences of the series. Note however, that the representation does not provide a mapping of equilibrium errors onto the remaining variables. The loadings of the error correction terms as they appear in the error correction model specification is an important feature of the dynamics of cointegrated systems which is totally neglected in the Bewley representation. The representation only focuses on the equilibrium relations. The neglect of the dynamic adjustment in response to disequilibrium shocks is made even more explicit in the parametrization of the triangular decomposition used by Phillips (1988) and generalized by Stock and Watson (1993).

IV. The Non-Parametric (Phillips-Stock-Watson) Triangular Array Decomposition.

Partition the x_t vector as $x_t=(x'_{1t}, x'_{2t}, x'_{3t})'$ where the components are of orders $p-r_1-r_2$, r_1 and r_2, respectively. The series are then related in the following way

$$\begin{aligned} \Delta^2 x_{1t} &= v_{1t} \\ \Delta x_{2t} &= \theta_{11}\Delta x_{1t}+v_{2t} \\ x_{3t} &= \theta_{21}x_{1t}+\theta_{22}x_{2t}+\theta_{23}\Delta x_{1t}+v_{3t} \end{aligned} \tag{3.8}$$

where $v_t=(v_{1t}',v_{2t}',v_{3t}')'$ *is a zero mean stationary process.*

PROOF. Many different parametrizations of triangular systems may be developed and Stock and Watson (1993) have provided a generalization of the I(1) decomposition of Phillips (1991). The non-parametric triangular representation can be derived along the following lines. Suppose x_t can be divided into the three separate components x_{1t}, x_{2t} and x_{3t}. The separation is made conformably with M(B) such that the dimensions of the components are $(p\text{-}r_1\text{-}r_2)\times 1$, $r_1\times 1$ and $r_2\times 1$, respectively. The representation is obtained by repeatedly premultiplying U(B) by elementary matrices of the type

$$\begin{bmatrix} I_{p-r_1-r_2} & 0 & 0 \\ 0 & I_{r_1} & 0 \\ 0 & E^*(B) & I_{r_2} \end{bmatrix}.$$

These operations are conducted until the U(B) matrix is triangular in the particular form specified in (3.9) below. In fact this triangularization procedure is the one adopted in the derivation of several different alternative canonical forms of polynomial matrices such as the Smith and Hermite forms, see e.g. Kailath (1980). We also define a matrix which creates unit matrices along the main diagonal after the original matrix has been block diagonalized. The product, $E(B)$, of the elementary matrices (making up the row operations and the diagonalising matrix) when applied to $U(B)$ delivers

$$E(B)U(B) = \begin{bmatrix} I_{p-r_1-r_2} & 0 & 0 \\ a_{11}(B) & I_{r_1} & 0 \\ a_{21}(B) & a_{22}(B) & I_{r_2} \end{bmatrix}. \tag{3.9}$$

Naturally the a(B)-matrices can be expressed in terms of the α(B) polynomial matrices defined above, including the cointegration parameters, but by defining the new parameters we are able to simplify the notation considerably. Next writing out the polynomial matrices comprising E(B)U(B) in accordance with $\bar{M}(B)E(B)U(B)x_t$ = $E(B)V(B)^{-1}\varepsilon_t$ we obtain

$$\begin{bmatrix} \Delta^2 x_{1t} \\ \Delta x_{2t} + a_{11}(1)\Delta x_{1t} + a_{11}^{*}(B)\Delta^2 x_{1t} \\ x_{3t} + a_{21}(1)x_{1t} + a_{22}(1)x_{2t} + a_{21}^{*}(1)\Delta x_{1t} + a_{22}^{*}(1)\Delta x_{2t} + a_{21}^{**}(B)\Delta^2 x_{1t} + a_{22}^{**}(B)\Delta^2 x_{2t} \end{bmatrix} = E(B)V(B)^{-1}\varepsilon_t.$$

Premultiplying by

$$F = \begin{bmatrix} I_{p-r_1-r_2} & 0 & 0 \\ 0 & I_{r_1} & 0 \\ 0 & -a_{22}^{*}(1) & I_{r_2} \end{bmatrix},$$

letting $\bar{E}(B)=FE(B)$ and reorganizing terms gives the expression

$$\begin{bmatrix} \Delta^2 x_{1t} \\ \Delta x_{2t} + a_{11}(1)\Delta x_{1t} \\ x_{3t} + a_{21}(1)x_{1t} + a_{22}(1)x_{2t} + (a_{21}^{*}(1) - a_{22}^{*}(1)a_{11}(1))\Delta x_{1t} \end{bmatrix} = \tag{3.10}$$

$$= \begin{bmatrix} I & 0 & 0 \\ -a_{11}^{*}(B) & I & 0 \\ -(a_{21}^{**}(B) - a_{22}^{*}(1)a_{11}^{*}(B)) + a_{22}^{**}(B)a_{11}(B) & -a_{22}^{**}(B)(1-B) & I \end{bmatrix} \bar{E}(B)V(B)^{-1}\varepsilon_t.$$

If we define the parameters $\theta_{11}=-a_{11}(1)$, $\theta_{21}=-a_{21}(1)$, $\theta_{22}=-a_{22}(1)$ and $\theta_{23}=-[a_{21}^{*}(1)-a_{22}^{*}(1)a_{11}(1)]$ and let the RHS of (3.10) be written more simply as the non-parametrically specified error process $v_t=(v_{1t}',v_{2t}',v_{3t}')'$, the triangular representation as defined above appears. Notice that v_t by construction is a zero mean stationary process since we have assumed that x_t is I(2) from the outset and no unit roots have been imposed through the elementary operations. Note however that $(v_{1t}',v_{2t}',v_{3t}')$ are correlated in a complicated way.

As can be seen the triangular array representation decomposes the x_t process into three different components of stochastic trends of different orders of integration. The first p-r_1-r_2 elements correspond to stochastic trends of order 2, the next r_1 components correspond to I(1) trends and finally the remaining r_2 terms are stationary I(0) components. One should note that as opposed to the (parametric) Bewley representation the x_t series is separated into *distinct* components where any relation to the remaining variables takes place through the error terms. The errors and their transient dynamics typically depend on the parameters of the cointegrating relations. The parametrization used above is interesting since it has been suggested by Phillips (1991) for I(1) systems, and recently by Stock and Watson (1993) for higher order integrated systems, as a convenient starting point for obtaining efficient estimates of the cointegrating parameters and for doing standard Gaussian inference.

Stock and Watson (1993) show that in order to obtain estimates and do inference the error process v_t in (3.8) can be orthogonalized in a particular way. First they define $v_t=H(B)\eta_t$ such that $E(\eta_t\eta_t')=I_p$. Next v_t is multiplied by a triangular polynomial matrix, $J(B)$, which generally will be two sided given that such a matrix can always be shown to exist. Now the resulting orthogonalized error process, e_t, can be written as

$$e_t = J(B)v_t = \begin{pmatrix} I_{p-r_1-r_2} & 0 & 0 \\ -d_{21}(B) & I_{r_1} & 0 \\ -d_{31}(B) & -d_{32}(B) & I_{r_2} \end{pmatrix} v_t = J(B)\,H(B)\eta_t = G(B)\eta_t \,.$$

Estimation relies on appropriate conditioning on variables determined from the top of the triangular representation (3.8) and since $J(B)$ is two-sided it is necessary to condition on both lags and leads of the stationary variables determined recursively in the model. This implies that if η_t is assumed Gaussian, the resulting likelihood function will have a non-standard factorisation. The block diagonal structure of the errors will imply however that cointegrating parameters of the model can be estimated efficiently by OLS or GLS equation by equation and inference will follow within the usual Gaussian framework. Notwithstanding, the estimators Stock and Watson propose can be shown to be asymptotically equivalent to Johansen's ML-estimator based on a full parametric specification of the vector error correction model.

It is a prerequisite for this analysis however that the cointegrating ranks r_1 and r_2 are known a priori which essentially means that the I(1) trends and the I(2) trends - i.e. the unit roots of the system - are known in advance. In addition knowledge of the error correction loadings and the short run dynamics, which may be of independent interest, are treated as a nuisance and absorbed in the residuals and hence cannot be estimated directly.

V. THE COMMON STOCHASTIC TRENDS REPRESENTATION.

Each component of the x_t series can be written as linear combinations of I(2) and I(1) trends plus a stationary component in the following way:

$$x_t = F\bar{\tau}_t + F^{*}\tau_t + w_t.$$

F and F^* are $p\times(p-r_1-r_2)$ and $p\times r_1$, respectively and the I(2) trends, $\bar{\tau}$, are defined as linear combinations of $\bar{S}_t=\Sigma_{s=1}^{t}\Sigma_{j=1}^{s}\varepsilon_j$. The I(1) trends τ_t are defined as linear combinations of $S_t=\Sigma_{j=1}^{t}\varepsilon_j$. Finally $w_t=C^{**}(B)\varepsilon_t$ is a stationary error.

PROOF. The proof follows the procedure used by Hylleberg and Mizon for the I(1) case. Define the following full rank $p\times p$ matrix $\mathbf{H}=[H^0/H^1/H^2]$ where H^i, i=0,1,2, are individually full column rank matrices of orders $p-r_1-r_2$, r_2 and r_1, respectively. H^i are mutually orthogonal and span R^p such that the following orderings of $\mathbf{H}$ can be made:

$\mathrm{H}=[H_\perp | H]$ where $H_\perp = H^0$ and $H=[H^1 | H^2]$ are full rank matrices and

$\mathrm{H}=[H^* | H_\perp^*]$ with $H^*=[H^0 | H^1]$ and $H_\perp^* = H^2$ also being of full rank.

H and H* further have the property that from the expansion of the Wold representation

$$C(B) = C(1) + \Delta C^*(1) + \Delta^2 C^{**}(B)$$

$C(1)H=0$ where H according to the above partitioning is $p\times(r_1+r_2)$ and $C^*(1)H^*=0$ where H^* is $p\times(p-r_1)$.

Hence we have that

$$\Delta^2 x_t = \{C(1)[H_\perp | H]\mathrm{H}^{-1} + \Delta C^*(1)[H^* | H_\perp^*]\mathrm{H}^{-1} + \Delta^2 C^{**}(B)]\}\varepsilon_t$$

or with the notation above

$$x_t = [C(1)H_\perp | 0]\mathrm{H}^{-1}\bar{S}_t + [0 | C^*(1)H_\perp^*]\mathrm{H}^{-1}S_t + C^{**}(B)\varepsilon_t = F\bar{\tau}_t + F^*\tau_t + w_t$$

where $F=C(1)H_\perp$ and $F^*=C^*(1)H_\perp^*$ with $\bar{\tau}_t$ being the first $p-r_1-r_2$ elements of $\mathrm{H}^{-1}\bar{S}_t$ and τ_t being the last r_1 elements of $\mathrm{H}^{-1}S_t$.

The common stochastic trends model for the I(1) case is originally due to Stock and Watson (1988) who developed the representation as a convenient framework in which to test the cointegration rank of a multivariate time series. The generalisation to the I(2) case shows that each of the series in x_t can be factored into separate components integrated of different orders. Although there are p elements in x_t, each series will consist of p-r_1-r_2 common I(2) trends, r_1 common I(1) trends and a stationary part. The representation given above suggests a natural starting point in order to extend Stock and Watson's method of testing for the cointegrating rank (at different orders of integration) although such a generalisation to our knowledge has not yet been made. However, procedures for conducting such an analysis are readily available using Johansen's (1992b) and Paruolo's (1993) reduced rank regression approach applied to I(2) systems.

4. Conclusions.

In this paper we have tried to provide a synthesis of the various ways generalised -and especially I(2) - cointegrated systems can be represented. The analysis has relied on the Smith-McMillan decomposition of a rational polynomial matrix which offers an elegant tool for considering the complicated interactions amongst variables of different orders of integration and possibly at different frequencies in a multivariate process. A general discussion of the nullspace structure of such systems was provided and Johansen's notion of 'balance' and the use of the Smith-McMillan form were compared.

The different representations we have considered in the paper provide different insights which help in describing distinct features of multivariate systems which may simultaneously contain several types of equilibrium relations. One possibility that arises in the class of models that we analyze is that the cointegrating equilibria may appear as polynomials in the lag operator and hence the implied long run equilibria may not seem to involve contemporanous relationships between the underlying economic variables. On grounds of economic intuition this may be difficult to justify, particularly when the suggested relationships are derived empirically from statistical rather than economic criteria. We have stressed the duality that arises between expressing a system as one with polynomial cointegrating relationships or alternatively defining transformations on the natural economic variables to deliver cointegrating *vectors* and we have illustrated how a sequential application of the Smith-McMillan form can be used to identify these transformations. The implications of these two alternative routes to analysing higher order polynomial cointegrated systems are considered further in Haldrup and Salmon (1993). Adopting a systematic state space approach to the analysis of higher order cointegrated systems may possibly aid the estimation and inference problems that arise in higher order integrated models and models with other frequency specific non-stationary behaviour. Aoki (1988, 1990) has developed this approach but there is much more that seems to be possible in analysing cointegrated systems from this point of view. For instance adopting the minimal polynomial basis route described above delivers the minimal state space representation of the cointegrated system.

5. Literature.

Aoki, M. (1988), "Cointegration, Error Correction and Aggregation in Dynamic models:A Comment", Oxford Bulletin of Economics and Statistics, 50.

Aoki, M. (1990), State Space Modelling of Time Series, Springer Verlag.

Bewley, R.A. (1979), "The Direct Estimation of the Equilibrium Response in a Linear Model", Economics Letters, Vol. 3.

Clements, M.P. (1990), "The Mathematical Structure of Models that exhibit Cointegration: A Survey of Recent Approaches", Applied Economics Discussion Papers, University of Oxford.

d'Autume A. (1992), "Deterministic Dynamics and Cointegration of Higher Orders", Universite de Paris, Sorbonne, working paper 92.43.

Davidson, J. (1991), "The Cointegration Properties of Vector Autoregressive Models", Journal of Time Series Analysis, 12, 41-62.

Engle, R.F. and C.W.J. Granger (1987), "Co-Integration and Error Correction : Representation, Estimation and Testing", Econometrica, 55, 251-276.

Engle, R.F., C.W.J. Granger, S. Hylleberg and H.S. Lee (1983), "Seasonal Cointegration - The Japanese Consumption Function", Journal of Econometrics, 55, 275-298.

Engle, R.F. and B.S. Yoo (1991), "Cointegrated Economic Time Series: An Overview with New Results", in R.F. Engle and C.W.J. Granger (eds.), Long Run Economic Relationships, 237-266, Oxford University Press.

Granger, C.W.J. and T.H. Lee (1988), "Multicointegration", in T.B. Fomby and G.F. Rhodes (eds), Advances in Econometrics; Cointegration, Spurious Regression and Unit Roots, 8, 71-84.

Granger, C.W.J. and T.H. Lee (1989), "Investigation of Production, Sales and Inventory Relationships using Multicointegration and Non-symmetric Error Correction Models", Journal of Applied Econometrics, 4, 145-159.

Gregoir, S. and G. Laroque (1991), "Multivariate Time Series : A General Error Correction Representation Theorem", INSEE working paper 9126.

Gregoir, S. and G. Laroque (1993), "Polynomial Cointegration : Estimation and Test", Forthcoming, Journal of Econometrics.

Haldrup, N. and M. Salmon (1993), "The State Space Approach to the Analysis of Cointegrated Systems", European University Institute, work in progress.

Hendry, D.F. and T. Von Ungern-Sternberg (1981), "Liquidity and Inflation Effects on

Consumer's Expenditure", in A.S. Deaton (ed.) Essays in the Theory and Measurement of Consumers Behaviour, Cambridge University Press.

Hylleberg, S., R.F. Engle, C.W.J. Granger, and B.S. Yoo (1990), "Seasonal Integration and Cointegration", Journal of Econometrics, 44, 215-238.

Hylleberg, S. and G.E. Mizon (1989), "Cointegration and Error Correction Mechanisms", The Economic Journal, 99, 113-125.

Johansen, S. (1988a), "The Mathematical Structure of Error Correction Models", Contemporary Mathematics, 80, 359-386.

Johansen, S. (1988b), "Statistical Analysis of Cointegration Vectors", Journal of Economic Dynamics and Control, 12, 231-254.

Johansen, S. (1991), "Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models." Econometrica 59, 1551-1580.

Johansen, S. (1992a), "A Representation of Vector Autoregressive Processes Integrated of Order 2", Econometric Theory, 8, 188-202.

Johansen, S. (1992b), "A Statistical Analysis of Cointegration for I(2) Variables", forthcoming, Econometric Theory.

Kailath, T. (1980), "Linear Systems", Prentice-Hall, Englewood Cliffs, NJ.

Paruolo, P. (1993), "On the Determination of Integration Indices in I(2) Systems", Mimeo, University of Bologna, Department of Statistics.

Phillips, P.C.B. (1991), "Optimal Inference in Cointegrated Systems", Econometrica, 59, 283-306.

Salmon, M. (1988), "Error Correction Models, Cointegration and the Internal Model Principle", Journal of Economic Dynamics and Control, 12, 523-549.

Stock, J.H. and M.W. Watson (1988), "Testing for Common Trends", Journal of the American Statistical Association, 83, 1097-1107.

Stock, J.H. and M.W. Watson (1993), "A Simple Estimator of Cointegrating Vectors in Higher Order Integrated Systems", Econometrica, 61, 783-820.

Vahid, F. and R.E. Engle (1991), "Common Trends and Common Cycles", working paper UCSD.

Yoo, B.S. (1986), "Multicointegrated Time Series and a Generalised Error Correction Model", Unpublished paper (Department of Economics, University of California, San Diego, CA).

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