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# Currency Option Pricing with Stochastic Interest Rates and Transaction Costs: A Theoretical Model

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**Abstract**

In this paper, we develop a currency option pricing model with stochastic interest rates and transactions costs when interest parity holds, and it is assumed that domestic and foreign bond prices have local variances that depend only on time. These additional parameters enter in a very simple way, through adjustment of the volatility in the Grabbe (1983) currency option pricing model. The "pure" Garman and Kohlhagen strategy holds only in the limiting case of constant interest rates and zero transactions costs.

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# Introduction

The arbitrage argument used by Black and Scholes (1973) to price options can no longer be used in the presence of nonzero transaction costs: since replicating the option by a dynamic strategy would be infinitely costly, no effective option price bounds are implied. Another assumption made by Black and Scholes in deriving their option pricing model, the constancy of interest rates, has become a matter of major concern to both academic and investment communities. Many studies, like Adams and Wyatt (1987), Choi and Hauser (1990), report pricing biases in European and American call options when interest rate uncertainty is not acknowledged in the model. For currency options, the problem is more complicated because (a) there are not one but two - domestic and foreign - interest rates to worry about, and (b) the international interest rate differential may dictate the rationality and timing of exercising options.

The option pricing model developed in this study will try to address both problems. We derive a currency option pricing model under stochastic interest rates and transactions costs. These additional parameters enter in a very simple way, through adjustment of the volatility in the Grabbe's (1983) formula, which itself represents the Black and Scholes model modified to currency contracts. As for the part related to stochastic interest rates, we use a modified Hilliard, Madura and Tucker (1991) approach which applies Vasicek's (1977) bond pricing model to the Grabbe's formula. It is assumed that the interest rate parity holds, and that domestic and foreign bond prices have local variances that depend only on time. As far as the transaction costs are concerned, we provide an extension of the Leland option replicating strategy to currency options with stochastic interest rates. Leland (1985) developed an approach to the problem of transaction costs in the case of stock options, in which the hedging strategy itself depends on the percent transaction costs and the revision interval.

The paper is organized as follows. In the first section, we present European currency call option models with stochastic interest rates. The second section is a short survey of some approaches to the problem of transaction costs in option pricing. Section 3 derives our currency option model inclusive of stochastic interest rates and transaction costs. Conclusions are drawn in the last section.

# 1 Pricing of European foreign currency call options with stochastic interest rates

The currency option pricing model under stochastic interest rates developed in this section is closely related to Garman and Kohlhagen (1983), Biger and Hull (1983), Giddy (1983), Grabbe (1983) currency option models except that the variance component, $\sigma^2$, is replaced by another one which depends on the form of the bond pricing model assumed. Models of this nature also have been investigated by Hsieh (1988), Rabinovitch (1989) and Hilliard, Madura, and Tucker (1991). We use Hilliard, Madura, and Tucker model with Vasicek's (1977) bond pricing model for both foreign and domestic bonds. We assume that the interest rate parity holds and bond prices have variances that depend only on time.

First paragraph defines the notation, assumptions and relations used in our study. Second paragraph presents the models of Garman and Kohlhagen, and Grabbe. Finally, in the last paragraph of this section we derive the model with stochastic interest rates.

## 1.1 Notation and assumptions

The following notation is employed in our paper:

| | | |
|---|---|---|
| $S(t)$ | $\equiv$ | the spot exchange rate (dollars per unit of foreign currency) at time $t$; |
| $T$ | $\equiv$ | the time until expiration; |
| $F(t,T)$ | $\equiv$ | the forward exchange rate at time $t$ for settlement at time $t+T$; |
| $B(t,T)$ | $\equiv$ | the domestic currency (US dollar) price of a pure discount bond which pays one unit of domestic currency (US dollar) at time $t+T$; |
| $B^*(t,T)$ | $\equiv$ | the foreign currency price of a pure discount bond which pays one unit of foreign currency at $t+T$; |
| $c(t)$ | $\equiv$ | the domestic currency (US dollar) price at time $t$ of an European call option written on one unit of foreign exchange, with exercise price $X$; |
| $r(t)$ | $\equiv$ | the instantaneous domestic (US) short rate of interest; |
| $r^*(t)$ | $\equiv$ | the instantaneous foreign short rate of interest; |
| $N(d)$ | $=$ | $\int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \exp(\frac{-x^2}{2}) dx$ |

The following assumptions and relations are used:

$$
\begin{aligned}
F(t,T) &= S(t)B^*(t,T)/B(t,T) \quad (\textit{interest rate parity}) && (1)\\
dS/S &= \mu_s(t)dt + \sigma_s dZ_s && (2)\\
dB/B &= \mu_b dt + \sigma_b(t,T)dZ_b && (3)\\
dB^*/B^* &= \mu_{b^*}dt + \sigma_{b^*}(t,T)dZ_{b^*} && (4)\\
dF/F &= \mu_f(t)dt + \sigma_f dZ_f && (5)\\
dr &= \alpha(r,t)dt + \sigma_r dZ_r && (6)\\
dr^* &= \mu(r^*,t)dt + \sigma_{r^*}dZ_{r^*} && (7)
\end{aligned}
$$

where $dZ_s, dZ_b, dZ_{b^*}, dZ_f, dZ_r, dZ_{r^*}$ are standard Wiener processes, $\mu_s$, $\mu_b$, $\mu_{b^*}$, $\mu_f$, $\alpha$, $\mu$ are instantaneous expected values of $S, B, B^*, F, r$ and $r^*$ respectively, $\sigma$-terms represent instantaneous standard deviations (volatilities). The $\mu_b$ and $\mu_{b^*}$ may depend on time and other stochastic variables, and the instantaneous volatility terms $\sigma_b$ and $\sigma_{b^*}$ depend only on time. The distribution of the instantaneous rates $r$ and $r^*$ depend on the assumptions made about $\alpha$ and $\mu$. Furthermore, the interest parity theorem is assumed to hold (equation 1). We also assume that forward and futures prices are the same. The symbol $F$ will be used to represent both the futures price and the forward price. Cox, Ingersoll, and Ross (1981) provide an arbitrage argument to show that when the risk-free rate is constant and the same for all maturities, forward prices and futures prices are the same. The argument can be extended to cover situations where the interest rate is a known function of time. When interest rates vary unpredictably, forward and futures prices are in theory no longer the same. However, the theoretical and empirical[1] differences between forward and futures prices are in most circumstances sufficiently small to be ignored. Some empirical research on this subject has been carried out. Cornell and Reinganum (1981) have for example, examined forward and futures prices on foreign currencies and found no significant differences.

Finally, we use another important result. As noted by Brenner, Courtadon and Subrahmanyam (1985), European options on the spot and on futures have identical prices when the futures contract has the same maturity as the options. This result is obvious when one considers that the spot and futures prices are identical at maturity of the

[1] In practice there are a number of factors not reflected in theoretical models which may cause forward and futures prices to be different. These include taxes, transaction costs, and the treatment of margins.

futures contract and that early exercise is prohibited when the option is European. For American options this is not always the case. However, when the domestic interest rate is higher than the foreign one in the period until the maturity of the option, the probability of early exercise of the option is almost equal to zero. In this case, American options on the spot exchange rate and on futures should have identical prices when the option and futures contract expire on the same date.

## 1.2 European currency option pricing models

Building on the classic model of Black and Scholes regarding European options on stock, any model of foreign currency options must incorporate foreign as well as domestic interest rates. This issue arises from the fact that default risk-free foreign bonds, as well as domestic bonds, represent a risk-free alternative to a hedge portfolio of spots and options on foreign exchange.

In 1983 Garman and Kohlhagen, Biger and Hull, Giddy, and Grabbe independently derived four models of foreign currency options. The first three models assume constant interest rates, while the latter assumes stochastic interest rates as reflected in the stochastic prices of discount bonds. The Garman and Kohlhagen, Biger and Hull, and Giddy model with constant interest rates is given by:

$$c = e^{-r^*T} S N(d_1) - e^{-rT} X N(d_2) \tag{8}$$

where

$d_1 = \frac{\ln(\frac{S}{X}) + (r - r^* + \frac{1}{2}\sigma_s^2)T}{\sigma_s \sqrt{T}}$,

$d_2 = d_1 - \sigma_s \sqrt{T}$.

This model is based on equation (2) which determines a dynamic diffusion process for $S$ and on the partial differential equation (which results from imposing riskfree arbitrage):

$$\frac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc + (r - r^*) S \frac{\partial c}{\partial S} + \frac{\partial c}{\partial T} = 0 \tag{9}$$

subject to the boundary condition: $c(S, T) = \max[0, S - X]$.

In contrast with this model, Grabbe considers the case of stochastic prices of pure discount bonds. He uses the approach applied by Merton (1973) to European stock options. In addition to equation

(2), he specifies the diffusion processes for the price of domestic $B$ and foreign pure discount bonds $B^*$ given by equations (3) and (4). Price changes of foreign bonds in domestic currency unit (G) can be stated as:

$$\begin{aligned} \frac{dG}{G} \equiv \frac{dSB^*}{SB^*} &= (\mu_s + \mu_{b^*} + \rho_{sb^*}\sigma_s\sigma_{b^*})dt + \sigma_s dZ_s + \sigma_{b^*} dZ_{b^*} \\ &\equiv \mu_G dt + \sigma_G dZ_G. \end{aligned} \tag{10}$$

Using arbitrage principle and applying Ito's lemma to the function $c(SB^*, B, X, T) = c(G, B, X, T)$, Grabbe obtains the following partial differential equation:

$$\frac{1}{2}\sigma_G^2 G^2 \frac{\partial^2 c}{\partial G^2} + \frac{\partial^2 c}{\partial G \partial B} GB\rho_{Gb}\sigma_G\sigma_b + \frac{1}{2}\sigma_b^2 B^2 \frac{\partial^2 c}{\partial B^2} - \frac{\partial c}{\partial T} = 0 \tag{11}$$

subject to the boundary conditions:

$$c(S(t+T), 1, X, 0) = \max[0, S(t+T) - X] \tag{12}$$

$$c(0, B(t,T), X, T) = 0. \tag{13}$$

The first boundary condition is the terminal value of the call option, which has to be greater than zero or the exercise value. The second boundary condition says that when the value of spot exchange is zero, the option to buy it has a zero value. An analytic solution to the European currency call derived by Grabbe is:

$$c = SB^* N(d_1) \text{ - XBN}(d_2) \tag{14}$$

where
$d_1 = \frac{\ln(\frac{SB^*}{XB} + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$,
$d_2 = d_1 - \sigma\sqrt{T}$.
The risk-free zero-coupon bond prices are related to interest rates via

$$\begin{aligned} B &= e^{-R(T-t)} \\ B^* &= e^{-R^*(T-t)} \end{aligned} \tag{15}$$

where $R$ and $R^*$ are the domestic and foreign rates of interest on riskless bonds that mature at time $T$. Therefore, the two models - Garman and

Kohlhagen, and Grabbe - are identical in form, but they differ in two aspects:

1. The instantaneous interest rates, $r$ and $r^*$ in Garman and Kohlhagen model, are replaced by the rates of interest, $R$ and $R^*$, on riskless bonds maturing at the same time as the option in the Grabbe model;

2. Given the uncertainties in bond interest rates, the variance of the Grabbe model reflects the covariances of spot exchange rates and prices of domestic and foreign bonds:

$$\sigma^2 = \frac{1}{T}\int_t^{t+T}(\sigma_b^2+\sigma_G^2-2\sigma_{Gb})du = \frac{1}{T}\int_t^{t+T}(\sigma_b^2+\sigma_G^2-2\sigma_G\sigma_b\rho_{Gb})du. \quad (16)$$

In Garman and Kohlhagen, in contrast, interest rates are deterministic, and the model's volatility reduces to:

$$\sigma^2 = \frac{1}{T}\int_t^{t+T}\sigma_s^2 du. \quad (17)$$

## 1.3 Derivation of the pricing formula with stochastic interest rates

In the previous paragraph, it can be seen that the Grabbe model assumed that the prices of domestic and foreign default-free discount bonds are a function of stochastic interest rates. Deriving an explicit valuation formula however, requires the specification of the stochastic process that governs interest rate changes. In this paragraph we present a derivation of a closed form valuation formula for a call option on foreign currency under the assumption that the interest rate follows a mean-reverting Ornstein-Uhlenbeck process. This derivation was previously made by Hilliard, Madura and Tucker (1991).
Using the interest parity relation (1), the European call option formula obtained by Grabbe may be rewritten as

$$c(F,t,T) = B(t,T)[F(t,T)N(d_1) - XN(d_2)] \quad (18)$$

where
$d_1 = \frac{\ln(\frac{F}{X})+\frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}}$
$d_2 = d_1 - \hat{\sigma}\sqrt{T}$,

$$\hat{\sigma}^2 = \frac{1}{T}\int_t^{t+T}\sigma_f^2 du \quad (19)$$

and $\sigma_f^2$ is the instantaneous variance of $dF/F$.

Note that $\hat{\sigma}^2$ here is exactly the same variable as in equation (16), as can be seen when applying Ito's lemma to the interest parity relation (1). What is remarkable about equation (16) is the disappearance of the price of spot exchange, which is the underlying asset on which the option is written. The reason for this result is that, given the current price of domestic currency discount bonds $B$, all of the relevant information concerning both the spot exchange rate as well as the foreign currency discount bond price that is necessary for option pricing, is already reflected in the forward rate. The fact that forward, spot rates and the price of foreign bonds are not independent follows, of course, from the Interest Parity Theorem. The diffusion of $dF/F$, from equation (5), can be rewritten using interest rate parity and Ito's lemma as

$$\begin{aligned}\frac{dF}{F} &= (\mu_s - \mu_b + \mu_{b^*} - \rho_{sb}\sigma_s\sigma_b + \rho_{sb^*}\sigma_s\sigma_{b^*} - \rho_{bb^*}\sigma_b\sigma_{b^*} + \sigma_b^2)dt \\ &\quad +\sigma_s dZ_s - \sigma_b dZ_b + \sigma_{b^*} dZ_{b^*} \\ &\equiv \mu_f dt + \sigma_f dZ_f. \end{aligned} \tag{20}$$

For a lognormal diffusion, $\text{Var}[\ln(\frac{F_{t+T}}{F_t})|F_t] = \int_t^{t+T} \text{Var}[\frac{dF}{F}]$,[2] so that the conditional variance of $\ln(\frac{F_{t+T}}{F_t})$ is

$$\begin{aligned}\sigma_f^2 &\equiv \text{Var}[\ln(\frac{F_{t+T}}{F_t})|F_t] \qquad (21)\\ &= \int_t^{t+T} (\sigma_s\sigma_{b^*} - \sigma_b) Cov(d\mathbf{Z}, d\mathbf{Z}')(\sigma_s\sigma_{b^*} - \sigma_b)' du \\ &= \int_t^{t+T} (\sigma_s\sigma_{b^*} - \sigma_b) Corr(d\mathbf{Z}, d\mathbf{Z}')(\sigma_s\sigma_{b^*} - \sigma_b)' du\end{aligned}$$

where

$$\text{Cov}(d\mathbf{Z}, d\mathbf{Z}') = \text{Corr}(d\mathbf{Z}, d\mathbf{Z}')dt = \begin{bmatrix} 1 & \rho_{sb^*} & \rho_{sb} \\ \rho_{sb^*} & 1 & \rho_{bb^*} \\ \rho_{sb} & \rho_{bb^*} & 1 \end{bmatrix} dt \tag{22}$$

and

$$d\mathbf{Z} = \begin{pmatrix} dZ_s & dZ_b & dZ_{b^*} \end{pmatrix}.$$

[2]See Hilliard, Madura and Tucker (1991).

Then

$$\sigma_f^2 = \int_t^{t+T} [\sigma_s^2 + \sigma_b^2 + \sigma_{b^*}^2 + 2(\rho_{sb^*}\sigma_s\sigma_{b^*} - \rho_{sb}\sigma_s\sigma_b - \rho_{bb^*}\sigma_b\sigma_{b^*})]du, \quad (23)$$

and from (19)

$$\hat{\sigma}^2 = \frac{1}{T}\int_t^{t+T} [\sigma_s^2 + \sigma_b^2 + \sigma_{b^*}^2 + 2(\rho_{sb^*}\sigma_s\sigma_{b^*} - \rho_{sb}\sigma_s\sigma_b - \rho_{bb^*}\sigma_b\sigma_{b^*})]du. \quad (24)$$

Formula (24) includes the variances of both domestic and foreign bond processes which can not be directly estimated from market data.[3] To obtain the functional form of the integrated variance $\hat{\sigma}^2$, we apply the version of Vasicek (1977) bond pricing model used by Hilliard, Madura, and Tucker (1991). Rabinovitch (1989) also uses this model to derive a stochastic interest rate model for pricing options on stocks and bonds.

Suppose as in Vasicek (1977), that markets are frictionless, all uncertainty in the term structure of interest rates is captured by the movements of the instantaneously riskless rate $r(t)$, and that its dynamics are given by an Ornstein-Uhlenbeck diffusion process:

$$dr(t) = \alpha(\mu - r)dt + \sigma_r dZ_r. \quad (25)$$

The Ornstein-Uhlenbeck process with $\alpha > 0$ is sometimes called the elastic random walk. It is a Markov process with normally distributed increments. In contrast to the random walk (the Wiener process), which is an integrated process and after a long time will diverge to infinite values, the Ornstein-Uhlenbeck process possesses a stationary distribution. According to (25), the instantaneously riskless rate is expected to drift towards the long-run mean level $\mu$, with a speed of adjustment $\alpha$. The stochastic element, which has a constant instantaneous variance $\sigma_r^2$, causes the process to fluctuate around the level $\mu$ in an erratic, but continuous fashion. Using an arbitrage argument, Vasicek derives a bond pricing model:

$$\begin{aligned} B(t,s,r) &= exp[\frac{1}{\alpha}(1 - e^{-\alpha(s-t)})(R(\infty) - r) - (s-t)R(\infty) \\ &\quad -\frac{\sigma_r^2}{4\alpha^3}(1 - e^{-\alpha(s-t)})^2] \end{aligned} \quad (26)$$

[3] These variances can be estimated from series of "pure" discount bond values computed using the relationship $B = e^{-r(t)T}$.

for $t \leq s$, where

$$R(\infty) = \mu + \frac{\sigma_r q}{\alpha} - \frac{1}{2}\frac{\sigma_r^2}{\alpha^2}, \tag{27}$$

and $q$ can be called the market price of risk, as it specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk; $q = (\mu_b - r)/\sigma_b$. The $q$ is assumed to be constant and independent of the level of the spot rate. The mean $\mu_b$ and standard deviation $\sigma_b$ of the instantaneous rate of return of a bond maturing at time $t+T$ are,

$$\mu_b(t,T) = r(t) + \frac{\sigma_r q}{\alpha}(1 - e^{-\alpha T}) = r(t) + \sigma_r q W(t,T) \tag{28}$$

$$\sigma_b(t,T) = \frac{\sigma_r}{\alpha}(1 - e^{-\alpha T}) = \sigma_r W(t,T) \tag{29}$$

where

$$W(t,T) = \frac{(1 - e^{-\alpha T})}{\alpha}. \tag{30}$$

The similar result holds for foreign interest rates:

$$\mu_{b^*} = r^*(t) + \sigma_{r^*} q^* W(t,T) \text{ and } \sigma_{b^*} = \sigma_{r^*} W(t,T).$$

The form of the parameter $W(t,T)$ determines the value of the conditional variance, $\hat{\sigma}^2$. If we take Vasicek's $W(t,T)$, given by equation (30), to compute the conditional variance, the resulting equations are lengthy and require estimates of parameters like $\alpha$ and $\beta$(this parameter corresponds to $\alpha$ in the case of foreign interest rates). For this reason we use an approximation of $W(t,T)$ given by Hilliard, Madura and Tucker (1991):

$$W(t,T) \cong T \tag{31}$$

which is exact as $\alpha$ and $\beta \to 0$(use De l'Hopital's Rule).[4]
In this case

$$\sigma_b = \sigma_r T, \tag{32}$$

$$\sigma_{b^*} = \sigma_{r^*} T. \tag{33}$$

Using (32) and (33) and integrating equation (24) gives

[4] From De l'Hopital's Rule: $\lim_{\alpha \to 0} \frac{f(\alpha)}{g(\alpha)} = \lim_{\alpha \to 0} \frac{f'(\alpha)}{g'(\alpha)}$ follows $\lim_{\alpha \to 0} \frac{(1-e^{-\alpha T})}{\alpha} = \lim_{\alpha \to 0} \frac{(Te^{-\alpha T})}{1} = T$.

$$
\begin{aligned}
\hat{\sigma}^2 &= \sigma_s^2 + \frac{T^2}{3}(\sigma_r^2 + \sigma_{r^*}^2 - 2\rho_{rr^*}\sigma_r\sigma_{r^*}) + T(\rho_{sr^*}\sigma_s\sigma_{r^*} - \rho_{sr}\sigma_s\sigma_r) \\
&= \sigma_s^2 + \frac{T^2}{3}(\sigma_r^2 + \sigma_{r^*}^2 - 2\sigma_{rr^*}) + T(\sigma_{sr^*} - \sigma_{sr}) \qquad (34)
\end{aligned}
$$

where $\sigma_{rr^*} = \rho_{rr^*}\sigma_r\sigma_{r^*}, \sigma_{sr^*} = \rho_{sr^*}\sigma_s\sigma_{r^*}$ and $\sigma_{sr} = \rho_{sr}\sigma_s\sigma_r$ are covariances, and $T$ is time to maturity. Under a constant interest rate scenario, equation (34) reduces to

$$\hat{\sigma}^2 = \sigma_s^2. \qquad (35)$$

This choice of parameter gives the currency option model of Garman and Kohlhagen (1983), Biger and Hull (1983), and Giddy (1983). In the Vasicek's model the term structure of interest rates takes the form [Vasicek(1977 p.186)]:

$$R(t,T) = R(\infty) + (r(t) - R(\infty))\frac{W(t,T)}{T} + \frac{\sigma_r^2}{4\alpha T}W(t,T)^2. \qquad (36)$$

The yield curves given by equation (36) starting at the current level r(t) of the spot rate for $T = 0$ approach a common asymptote $R(\infty)$ as $T \to \infty$. For values of r(t) smaller or equal to $R(\infty) - \frac{\sigma_r^2}{4\alpha^2}$ the yield curve is monotonically increasing. For values of r(t) larger than that but below $R(\infty) + \frac{\sigma_r^2}{2\alpha^2}$, it is humped curve. When r(t) is equal to or exceeds this last value, the yield curves are monotonically decreasing.[5]

However, in the case of our approximation given by equation (31), the term structure takes the form:

$$R(t,T) = r(t) + \frac{\sigma_r^2 T}{4\alpha} \qquad (37)$$

and for all values of r(t) the yield curve is increasing.[6] For $T = 0$ the yield is equal to the spot rate r(t), and approaches infinity as $T \to \infty$. When $\alpha \to 0$, the yield curves are explosively increasing and the bond price $B(t,T) \to 0$ even for small $T$.

Another problem of the Ornstein-Uhlenbeck process is the long-run possibility of negative interest rates. However, most options traded

[5]Since r(t) is normally distributed by virtue of the properties of the Ornstein-Uhlenbeck process, and R(t,T) is a linear function of r(t), it follows that R(t,T) is also normally distributed.

[6] Vasicek assumes that $\alpha > 0$.

on the organized exchanges expire in less than nine months. Rabinovitch (1989) shows that, given an initial positive interest rate and reasonable parameter values, the expected first-passage time of the process through the origin is longer than nine months. Thus, he suggests that the Ornstein-Uhlenbeck process may approximate the true but unknown interest rate process in the short run, and may be utilized in pricing short-lived options.

The Ornstein-Uhlenbeck process used by Vasicek has been used extensively by others in valuing stock options, bond options, futures, futures options and other types of contingent claims. Vasicek model is one of a broad class of interest rate models currently used by academic researchers and practitioners. A partial listing of these interest rate models includes those by Merton (1973), Cox, Ingersoll, and Ross (1980, 1985), Dothan (1978), Brennan and Schwartz (1980), Cox (1975), Cox and Ross (1976), Longstaff (1989, 1992), Hull and White (1990).[7] These models can be obtained from equation (38) simply by placing the appropriate restrictions on the four parameters $\alpha, \beta, \sigma$ and $\gamma$:[8]

$$dr(t) = (\alpha + \beta r)dt + \sigma r^{\gamma} dZ. \tag{38}$$

Among the interest rate models, Cox, Ingersoll, and Ross (1985) (henceforth, CIR) model presents some interesting features. Cox, Ingersoll, and Ross derive a general equilibrium model of the term structure in which the short-term interest rate is the single factor. CIR consider the alternative $\gamma = 0.5$.[9] In this case, $r$ can, in some circumstances, become zero but it can never become negative. Moreover, although CIR consider only one equilibrium, their framework allows for other interesting equilibria. These alternative equilibria are obtained by imposing boundary conditions on bond prices when the short-term interest rate reaches zero. Each equilibrium corresponds to a different assumption about the behavior of the short-term interest rate process at zero. This feature of the CIR framework is important since it provides an additional degree of freedom in developing general equilibrium term structure models that capture the actual properties of the term

[7]See Chan, Karolyi, Longstaff, and Sanders (1992) for an interesting comparison of eight continuous-time models of the short-term interest rate.

[8]Vasicek's model assumes $\gamma = 0$.

[9]Thus the model is $dr(t) = (\alpha + \beta r)dt + \sigma\sqrt{r}dZ$.

structure.[10]

Although innovative, the CIR model does not fully capture the observed properties of the term structure. It implies for example, that term premiums are monotone increasing functions of maturity.[11] Recent evidence in Fama (1984) and McCulloch (1987) however, suggests that the actual term premiums have a humped pattern. Moreover, the CIR model allows only two types of yield curves (monotone or humped); observed yield curves frequently display more complicated patterns. In addition, CIR show that their model can be used to price call and put European options. Their formulas are nevertheless rather more complicated, involving the noncentral chi-square distribution.

In this section we have presented European currency option pricing model with constant and stochastic interest rates. The deterministic interest rate version of this model was developed in 1983 by Garman and Kohlhagen, Biger and Hull, and Giddy. It is the Black and Scholes model modified to include the foreign interest rate. Both the domestic and foreign interest rates are constant. The European currency option model with stochastic discount bonds was derived in 1983 by Grabbe. It is based on Merton's model with proportional dividend. However, neither the Merton or Grabbe models explicitly assume stochastic processes for domestic and foreign interest rates. To do this, use must be made of a model of bond prices. We have presented a derivation of the currency call option model with stochastic interest rates developed by Hilliard, Madura and Tucker (HMT) in 1991. This model assumes that the interest rate parity holds and make use of Vasicek's bond pricing model for both foreign and domestic bonds. The stochastic interest rate component enter in a simple way, through the adjustment of the volatility in Grabbe's model. However, the HMT assumption, that interest rates and interest rates differentials follow a (nearly) random-walk should be verified empirically.

## 2 Transaction costs in option pricing

Costs of buying or selling a given financial asset are attributable to two separate costs:

- the broker's commission which can be either fixed or

[10]See Longstaff (1992) for multiple equilibria in CIR term structure framework.

[11]The term premium or liquidity premium it is a difference between the forward rates and expected spot rates. Vasicek model allows the term premium to be monotonically increasing or decreasing function of $T$, or to have a humped shape.

proportional to the value of the transaction and

- bid-ask spread.

These transaction costs invalidate the Black and Scholes arbitrage argument for option pricing, since continuous revision implies infinite trading. Because diffusion processes have infinite variation, continuous trading would be ruinously expensive, no matter how small the transaction costs might be as a percentage of the turnover.

The natural defense of the Black and Scholes approach is to assume that trading takes place only at discrete intervals. This will bound the transaction costs of the replicating strategy and, if trading takes place with reasonable frequency, hedging errors may be relatively small. Black and Scholes (1973), and Boyle and Emanuel (1980) argue that these errors will be uncorrelated with the market return, and can therefore be ignored if revision is reasonably frequent.

Problems however, arise with this argument in the presence of transaction costs. First, hedging errors exclusive of transaction costs will not be small unless portfolio revision is frequent. But transaction costs will rise (without limit) as the revision interval becomes shorter: it may be very costly to assure a given degree of accuracy in the replicating strategy before transaction costs. Paradoxically, we could find that the total cost of the replicating strategy exceeds that of the stock itself, even though the stock returns dominate the option return. Secondly, transaction costs themselves are random, and will add significantly to the error of the Black and Scholes replicating strategy. The cost of a replicating strategy must clearly include transaction costs. If we wish to continue to use an arbitrage argument to bound option prices, we are forced to consider the maximum transactions cost rather than simply the average. But transaction costs associated with replicating strategies are path-dependent: they depend not only on the initial and final stock prices, but also on the entire sequence of intermediate stock prices. Because of the path dependency and unboundedness of transaction costs, the uncertainty of transaction costs will not become small as the period of revision becomes shorter. One cannot hope for an arbitrarily good replication (no matter how expensive) by shortening the revision period. While replication errors exclusive of transaction costs will fall, they will not fall when transaction costs are included.

One possible approach is to look at expected transactions costs by following the Black and Scholes replicating portfolio in discrete time. Using a discrete time approximation within the continuous derivation, Gilster and Lee (1984) estimate expected transaction costs for Black and Scholes strategies.[12] They modify the Black and Scholes model by adding a rebalancing transactions cost term, $\alpha E(g)$, - which depends on the revision interval $\Delta t$ and transaction cost rate $\alpha$ - on the right hand side of the original Black and Scholes differential equation:

$$\frac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc + rS\frac{\partial c}{\partial S} + \frac{\partial c}{\partial T} = \alpha E(g) \tag{39}$$

where $g$ is the change in capital required for rebalancing, $\alpha$ is the transaction cost rate applied to $g$, and $E(g)$ is the expected value of $g$.

The change in capital required for rebalancing, $g$, consists of the change in the number of options $(\Delta n)$ at the changed price $-c(S + \Delta S, t + \Delta t)$ i.e.,[13]

$$\begin{aligned} g &= -c(S + \Delta S, t + \Delta t)\Delta n \\ &= \Delta S \frac{c\frac{\partial^2 c}{\partial S^2}}{\frac{\partial c}{\partial S}\Delta t} + \frac{c\frac{\partial^2 c}{\partial S \partial t}}{\frac{\partial c}{\partial S}} + \sigma_s^2 S^2 [-\frac{c\frac{\partial^2 c}{\partial S^2}}{(\frac{\partial c}{\partial S})^2} + \frac{1}{2}\frac{c\frac{\partial^3 c}{\partial S^3}}{\frac{\partial c}{\partial S}} + \frac{\partial^2 c}{\partial S^2}]. \end{aligned} \tag{40}$$

Equation (40) makes it clear that, as $\Delta t \to 0$, $g$ approaches infinity, yielding the disturbing result that, with continuous rebalancing, transaction costs will be infinite for any positive transactions cost rate. Gilster and Lee evaluate $E(g)$, and solve equations (39) and (40). The transaction cost adjustment, AC, to the Black and Scholes price will be the present value of the expected value of future transaction costs:

[12] There are a variety of possible applications of the Black and Scholes option hedge concept: any two of three elements of the hedge can be used to duplicate the third, thus providing the investor with a choice of mechanisms for achieving the same results. Gilster and Lee make use of the fact that a long position in stock and short position in call options (rebalanced) can be used to duplicate the behavior of a Treasury bill (risk-free investment). They call this strategy an "investment hedge". On the other hand, if the hedge consists of a long position in call options and a short position in stock, the hedge supplies funds which will cost the borrowing rate (a "borrowing hedge").

[13] In conducting the Black and Scholes hedging operations, we have two possibilities: to keep the number of calls constant, and make adjustments by buying or selling stock and bonds, or to make adjustments by keeping the number of shares of stock constant and buying or selling calls and bonds. If we adjust through the stock, there is no problem. If we insist on adjusting through the calls, the hedge can no longer be riskless. To remain hedged, the number of calls we would need to buy back depends on their value, not their price. Therefore, since we are uncertain about their price, we then become uncertain about the return from the hedge. Gilster and Lee assume that all rebalancing is conducted by adjusting the option portion of the hedge. It generally involves smaller dollar amounts but the hedging error can be higher than in the case of rebalancing by the stock.

$$AC = \alpha \int_0^\infty \int_t^{t^*} \exp[-r(t^* - u)]E(g)L'(x)dudx \qquad (41)$$

where $L'(S)$ is the lognormal density function of stock price. Equation (41) can be evaluated by numerical integration. Then, Gilster and Lee modify the Black and Scholes formula to include the effects of transaction costs and different borrowing and lending rates. They state that these market imperfections tend to offset each other yielding a bounded range of prices for each option.

However, their analysis is flawed by the fact that the equation (39) is not satisfied by the Black and Scholes strategy as they themselves admit. Equation (40) also presents a problem because, in principle, the transactions cost adjusted option price should be used to calculate the transaction cost adjustment and therefore, the use of unadjusted Black and Scholes call price to estimate $g$ is not entirely exact. Moreover, when the revision interval is much smaller than one day, AC can approach unacceptable results.

Leland (1985) proposes an other approach to overcome the problems related to with transaction costs mentioned earlier. He shows that there is an alternative replicating strategy to the Black and Scholes model for stocks, in which transaction costs remain bounded even as the revision period becomes shorter.[14] Leland's strategy replicates the option return inclusive of transaction cost, with an error which is uncorrelated with the market and approaches zero as the revision period becomes smaller.[15] Moreover, the transaction costs put bounds on option prices. Leland's alternative strategy depends upon the level of transaction costs and upon the revision interval (exogenously given). These additional parameters are introduced quite simply, through an adjustment of the volatility in the Black and Scholes formula.

The modified variance is bounded by $\hat{\sigma}^2_{\max}$ and $\hat{\sigma}^2_{\min}$. These two limits are given by:

$$\hat{\sigma}^2_{\max}(\sigma^2_s, k, \Delta t) \;=\; \sigma^2_s[1 + \frac{kE(\frac{\Delta S}{S})}{\sigma^2_s \Delta t}]$$

[14] In conducting the discrete-time hedging operations Leland keeps the number of calls constant, and makes adjustments by buying or selling stock and bonds.

[15] Discrete rebalancing interval applications of the continuous time option pricing model seem to pose no problem. Black and Scholes (1973) argue that the risk associated with short interval rebalancing is uncorrelated with the market, an argument later supported by Boyle and Emanuel (1980).

$$= \sigma_s^2[1 + \frac{k\sqrt{2/\pi}}{(\sigma_s\sqrt{\Delta t})}] \quad (42)$$

and

$$\hat{\sigma}^2_{\min}(\sigma_s^2, k, \Delta t) = \sigma_s^2[1 - \frac{kE(\frac{\Delta S}{S})}{\sigma_s^2 \Delta t}]$$

$$= \sigma_s^2[1 - \frac{k\sqrt{2/\pi}}{(\sigma_s\sqrt{\Delta t})}] \quad (43)$$

where $k$ represents the percent transaction cost measured as a fraction of the volume of transactions, $\Delta t$ is the revision period, $E(\frac{\Delta S}{S})$ is the expected value of $\frac{\Delta S}{S}$ in absolute value terms and $\sigma_s^2$ is an instantaneous variance of stock's rate of return.[16] The maximum modified variance, $\hat{\sigma}^2_{\max}$, used in the replicating strategy which duplicates *a long call option*[17] insures that as the readjustment interval becomes smaller, this strategy yields the option result almost surely, inclusive of transaction costs.

On the other hand, the minimum modified variance, $\hat{\sigma}^2_{\min}$, is applied to the hedging strategy which duplicates *a short call option.* [18] This strategy will also produce a hedging error after transaction costs which will be uncorrelated with the market, and will almost surely approach zero as $\Delta t$ becomes smaller.

These two limiting values of variance determine upper and lower bounds of option price. Therefore, $\hat{c}(\hat{\sigma}^2_{\max}, S, X, r, t)$ is the Black and Scholes option price based on the modified *maximum* variance including transaction costs and $\hat{c}(\hat{\sigma}^2_{\min}, S, X, r, t)$ is the Black and Scholes option price based on the modified *minimum* variance including transaction costs. If the price of an option exceeds $\hat{c}_{\max}$, we could make profits higher than the risk-free rate, by selling the option and buying the duplicating portfolio containing $\Delta$ long stocks and borrowing.[19] On the other hand, if the price of a call option is less than $\hat{c}_{\min}$, an investor could

[16] The expected value of $\Delta S/S$ taken in absolute value is equal to $E|\frac{\Delta S}{S}| = \sqrt{2/\pi}(\sigma_s\sqrt{\Delta t})$, which can be derived using the assumption that $\Delta S/S$ is normally distributed with mean zero.

[17] The hedge consists of a long position in stock and borrowing (selling bonds).

[18] The arbitrage portfolio consists of a short position in stock and lending (buying bonds).

[19] $\Delta$ is equal to first partial derivative of $\hat{c}_{\max}$ with respect to $S$, $\partial\hat{c}_{\max}/\partial S$. The amount of written bonds (borrowing) is equal to $\Delta S - \hat{c}_{\max}$. Later we have to follow the replicating strategy by adjusting our portfolio as described by the formula until the expiration date of the option. Then we close out the position which involves buying back

buy this "underpriced" option, "undo" it by following the offsetting replicating strategy, and make a return after transaction costs which exceeded the risk-free rate.[20]

The "total" transaction costs associated with the Leland's strategy duplicating a long call option are given by the difference between two Black and Scholes initial option values with the adjusted and no-adjusted volatility:

$$Z_1 = \hat{c}_{max0} - c_0.$$

As the revision interval $\Delta t \to 0$, $\hat{\sigma}_{\max} \to \infty$, and $\hat{c}_{max0} \to S_0$. Thus, $Z_1$ is bounded above by $S_0 - c_0$, implying that transaction costs are bounded as $\Delta t \to 0$. The transaction costs associated with the Leland's strategy duplicating a short call option are given by:[21]

$$Z_2 = c_0 - \hat{c}_{min0}.$$

As the revision interval $\Delta t \to 0$, $\hat{\sigma}_{\min} \to 0$, and $\hat{c}_{min0} \to 0$. Thus, $Z_2$ is bounded below by $c_0$, implying that transaction costs are bounded as $\Delta t \to 0$.[22] The fact that the transaction costs are bounded as the revision period becomes short is an important advantage in Leland's alternative strategy with respect to Black and Scholes replicating strategy. The "pure" Black and Scholes strategy holds in the limiting case of zero transaction costs when $k = 0$.

Merton (1992) examines the effects of transactions costs on derivative security pricing by using the two-period version of the Cox-Ross-Rubinstein binomial option pricing model when there are proportional

the option at its current market price, selling the stock and repaying the borrowing. It is true that closing out the position before expiration date might produce a loss that would more than offset our profit, but this loss could always be avoided by waiting until the expiration date.

[20] Initially, the investor buys the call option, sells shares of stock and buys bonds (investing or lending). Then he follows the replicating strategy by maintaining the neutral position ratio (the number of shares held for each call) until the maturity date of the option. Finally, he closes out his position by selling the option, buying the stock and selling the bonds (borrowing).

[21] Leland does not treat this case explicitly and implicitly assumes the symmetry of the two transaction costs, $Z_1$ and $Z_2$. However, as the revision interval becomes small, they are not equal and therefore the equation on the page 1300 of his paper is no longer valid. It should be as follows: $\hat{c}_{\max} - \hat{c}_{\min} \approx S_0$.

[22] $Z_1$ and $Z_2$ can be thought of as the cost of an insurance policy guaranteeing coverage of transaction costs, whatever those may actually be.

transaction costs on the underlying asset. There are no costs for transacting in the riskless security. In a discrete-time framework, he constructs a portfolio of the risky asset and riskless bonds that precisely replicates the option value at maturity of the option with transaction costs. He explores the spread in call option prices induced by transaction costs in the market of underlying asset. The symmetry of the bid and ask prices of the underlying asset around its zero-transaction-cost price does not imply a corresponding symmetry for the bid and ask prices of the call option. The average of the bid and ask prices of the option is a biased-high estimate of its zero-transaction-cost price.

Furthermore, Merton shows that the percentage spreads in the production costs of derivative securities can be many times larger than the spreads in their underlying securities. Hence, even with modest transaction costs for investors in traded securities, there is an economic function for financial intermediaries that specialize in creation of derivative securities and take advantage of economies of scale to produce them at a greatly reduced cost.

Boyle and Vorst (1992) also use the Cox-Ross-Rubinstein binomial option pricing model and extend Merton's analysis to several periods. They employ a discrete-time framework and construct the portfolio to replicate a long and short European call inclusive of proportional transaction costs. They start by obtaining the long call price in a one-period model. Then they extend this model to several periods and the initial long call price can be obtained by constructing the replicating portfolio backward from the maturity date. The procedure for obtaining the short call price is similar but not identical. As the zero transaction costs approach zero, the short call price and the long call price converge to the Cox-Ross-Rubinstein option price.

Furthermore, they derive a closed-form expression for the long call price which can be expressed as a discounted expectation under a new Markow process. This leads to an approximation for the long call price in terms of the ordinary Black and Scholes formula with a modified variance. The modified variance is given by:

$$\hat{\sigma}^2_{\max}(\sigma^2, k, \Delta t) = \sigma^2[1 + \frac{k}{(\sigma\sqrt{\Delta t})}]. \tag{44}$$

The variance adjustment obtained by Boyle and Vorst is similar to, but larger than that derived by Leland. Indeed, the two expressions for the variance are very similar, but where Leland has a factor of

$\sqrt{(2/\pi)}$, Boyle and Vorst have unity. Since $\sqrt{(2/\pi)} \approx 0.8$ the Boyle and Vorst's model leads to higher option values than Leland's. Boyle and Vorst also derive an analogous approximation for the short call price and note some interesting asymmetries between the properties of the long and short call prices.[23]

In this section we have presented several solutions to the problem of transaction costs in option pricing. The results of these approaches are quite different. Gilster and Lee add a rebalancing transaction cost term to the right-hand side of the original Black and Scholes differential equation and compute the transaction cost adjustment to the Black and Scholes call price. However, their modified differential equation is not satisfied by the Black and Scholes strategy and the hedging errors of *implicit* alternative replicating strategy do not approach zero when the revision period becomes small. Moreover, in their strategy, with continuous rebalancing, transaction costs will be infinite for any positive transaction cost.

These problems are solved by Leland who develops an alternative hedging strategy with transaction costs. The size of transaction costs and the frequency of revision enter through adjustment of the volatility in the Black and Scholes formula. Hedging errors of this strategy are uncorrelated with the market and approach zero as the revision interval becomes short. The transaction costs of option replication are bounded and provide upper and lower bounds on option prices.

Merton sets up the problem of proportional transaction costs on the stock in a discrete-time framework using the Cox-Ross-Rubinstein binomial option pricing model. He develops a replicating strategy which precisely replicates the option value at expiration inclusive of transaction costs. However, his approach is limited to the two-period case.

Boyle and Vorst extend the Merton's analysis to several periods. Their method proceeds by constructing the appropriate replicating portfolio with transaction costs at each trading interval. They also derive a simple Black and Scholes type approximation for the option prices with transaction costs. Transaction costs enter in the formula through the adjustment of the variance. This approach is similar to Leland's. However, the variance adjustment in the Boyle and Vorst

[23] When number of periods, $n$, is large (and $\Delta t$ is equal to $\frac{T}{n}$) the value of replicating a short call can be approximated by a Black and Scholes formula with a modified variance given by $\hat{\sigma}^2_{\min}(\sigma^2, k, \Delta t) = \sigma^2[1 - \frac{k}{(\sigma\sqrt{\Delta t})}]$. However, for some values of $k$ and $\Delta t$ the approximation might well lead to a negative modified variance.

model is larger than that derived by Leland.

The replicating strategies for option pricing proposed by Leland (1985), Merton (1992), and Boyle and Vorst (1992) imply finite transaction costs and still generate, with probability one, a payoff equal to that of the option. However, the strategies considered in these models are not chosen to satisfy some optimality criteria that investors may wish to meet. In the Leland-style transaction costs approach the frequency of portfolio revisions is exogenously given instead of being optimally chosen with respect to transaction costs or another state variable. Merton formulates a two-period replicating strategy but does not extend it to an arbitrary number of periods and consequently, is not in a position to determine the limiting value of the option price when time is allowed to become continuous. The exact replication approach used by Boyle and Vorst is not generally the most efficient method of replicating an option. Following the purchase or sale of an option, an intermediary would not typically choose to fully offset that transaction by means of sales and purchases of the underlying asset. A strategy of full offset would be unnecessarily costly and not to be pursued. As a consequence, their formulation does not match the optimal investment strategy with proportional transaction costs.

The optimality criterion for investors can be defined in at least two different ways. One definition is in terms of expected utility. In this approach, the chosen strategy should maximize the expected value of a constant relative risk-averse utility function, for a *given level of wealth.* Constantinides (1986) proposes an approximate solution to the portfolio choice problem in the presence of proportional transaction costs. The investor maximizes the expected value of his infinite-horizon utility function. Portfolio strategies (a proportion of the risky and risk-free assets in the portfolio) are computed numerically under the assumption that the investor in each period consumes a fixed proportion of his wealth. Dumas and Luciano (1991) assume that the investor does not consume along the way, but consumes everything at the terminal point in time. His objective is to maximize the expected utility derived from that terminal consumption. In contrast to Constantinides, their formulation of the portfolio strategy under proportional transaction costs leads to an exact solution. The exact solution is in the form of two control barriers. These set - for given level of transaction costs, investor's risk aversion, excess return on the risky asset, and variance of return on risky asset - the upper and lower limits (a proportion of risky and riskless assets) of imbalance in the portfolio, which will be tolerated before any action is taken. This implies that the frequency of portfolio revision is generally stochastic.

Another criterion is to minimize the initial cost of obtaining a *given terminal payoff* that is at least as large as that from the option being hedged. The advantage of the minimum cost criteria is that the optimal strategies are independent of an investor's preferences. Bensaid, Lesne, Pages and Scheinkman (1992) construct a dynamic programming algorithm to obtain the cost-minimizing trading strategy. However, in their algorithm, they introduce the entire path of the stock price process as a state variable. Thus, when the number of trading dates is large, the implementation of their algorithm for a general payoff is likely to be difficult. Edirisinghe, Naik and Uppal (1993) developed a two-stage dynamic programming model to account for fixed and variable trading costs, lot size constraints, and position limits on trading. Their least-cost replication strategy for hedging the payoff (convex or nonconvex) of contingent claims introduces the current stock and bond position of the investor as state variables. They show that in the presence of trading frictions, it is no longer optimal to revise one's portfolio in each period. Moreover, it is optimal to establish a larger position initially, and to reduce the amount of trading in later periods.

In spite of a strong conviction that options should be priced in an optimal portfolio-investment framework, these models give no straightforward and analytical solution to the option pricing problem. The Leland-style transaction costs approach, which assumes the fixed interval between portfolio rebalancing (in general non optimal), has an advantage in providing an analytical solution. It is for this reason that we have adopted Leland's approach in the next section applying it to the currency option pricing model with stochastic interest rates developed earlier in the first section.

# 3 European currency option pricing model with stochastic interest rates and transaction costs

In this section we derive a European currency option pricing model with both domestic and foreign stochastic interest rates and including transaction costs. We apply Leland's approach to transactions costs to Grabbe model of European currency options modified to stochastic interest rates (18) with conditional variance (34).

Let us apply our previous assumption that forward rates follow a stationary logarithmic diffusion process equation (5).[24] It can be rewritten as:

$$\frac{dF}{F} = \mu_f dt + \sigma_f \epsilon \sqrt{dt} \text{ since } dZ = \epsilon \sqrt{dt} \tag{45}$$

where $\epsilon$ is a normally distributed random variable with $E(\epsilon) = 0$ and $E(\epsilon^2) = 1$. The discrete-time version of this model of forward pricing sometimes known as Geometric Brownian Motion, is

$$\frac{\Delta F}{F} = \mu_f \Delta t + \sigma_f \epsilon \sqrt{\Delta t}. \tag{46}$$

Equation (46) shows that $\Delta F/F$ is normally distributed with mean $\mu_f \Delta t$ and standard deviation $\sigma_f \sqrt{\Delta t}$. In the absence of transaction costs but with possible continuous trading, Grabbe model modified to include stochastic interest rates is as follows:

$$c(F, t, T) = e^{-r(t)T}[F(t,T)N(d_1) - XN(d_2)] \tag{47}$$

where
$d_1 = \frac{\ln(\frac{F}{X}) + \frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}}$,
$d_2 = d_1 - \hat{\sigma}\sqrt{t}$,
$\hat{\sigma}^2 = \sigma_s^2 + \frac{T^2}{3}(\sigma_r^2 + \sigma_{r^*}^2 - 2\sigma_{rr^*}) + T(\sigma_{sr^*} - \sigma_{sr})$.

It can be verified that $c(F, t, T)$ satisfies the following partial differential equation,

$$\frac{1}{2}\hat{\sigma}^2 F^2 \frac{\partial^2 c}{\partial F^2} - rc + \frac{\partial c}{\partial t} = 0 \tag{48}$$

and the boundary condition:

$$c[F, X, t, r, \hat{\sigma}^2] = \max[F - X, 0]. \tag{49}$$

Consider now holding a fixed portfolio $\Pi$ of $\alpha$ futures (forward) contracts and $\beta$ dollars of the risk-free security (bonds) over the interval, $\Delta$t. This revision interval is expressed in a fraction of a year. Since it costs nothing to enter into a futures contract

$$\Pi = \beta. \tag{50}$$

[24] $dF/F$ follows normal distribution but $F$ follows lognormal distribution.

Over the interval $\Delta t$ the return to this portfolio will be

$$\Delta\Pi = \alpha F(\frac{\Delta F}{F}) + r\beta\Delta t. \tag{51}$$

Using a Taylor series expansion for a call option $c(F,T)$ we can calculate the change in its value over the same interval $\Delta t$,

$$\begin{aligned}\Delta c &= c(F+\Delta F, t+\Delta t) - c(F,t) \\ &= \frac{\partial c}{\partial F}F(\frac{\Delta F}{F}) + \frac{\partial c}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 c}{\partial F^2}F^2(\frac{\Delta F}{F})^2 + O(\Delta t^{3/2}). \end{aligned} \tag{52}$$

The difference, $\Delta ER$, between the change in value of the portfolio and the call option, which is the error of replicating portfolio strategy is given by

$$\begin{aligned}\Delta ER &= \Delta\Pi - \Delta c \qquad (53)\\ &= (\alpha F - \frac{\partial c}{\partial F}F)(\frac{\Delta F}{F}) + (r\beta - \frac{\partial c}{\partial t})\Delta t - \frac{1}{2}\frac{\partial^2 c}{\partial F^2}F^2(\frac{\Delta F}{F})^2 \\ &\quad + O(\Delta t^{3/2}).\end{aligned}$$

We can define a replicating portfolio as one for which, at the beginning of each interval $0, \Delta t, 2\Delta t, \ldots, t+T$,

$$\alpha = \frac{\partial c}{\partial F} \tag{54}$$

and

$$\beta = c. \tag{55}$$

Then substituting (54) and (55) into (53) and using (48) yields

$$\Delta ER = \frac{1}{2}\frac{\partial^2 c}{\partial F^2}F^2\left[\ \hat{\sigma}^2\Delta t - (\tfrac{\Delta F}{F})^2\ \right] + O(\Delta t^{3/2}). \tag{56}$$

Taking expectations, using (46) and the fact that forwards have zero expected drift since they have zero initial cost, gives

$$E[\Delta ER] = \frac{1}{2}\frac{\partial^2 c}{\partial F^2}F^2 E\left[\ \hat{\sigma}^2\Delta t - (\tfrac{\Delta F}{F})^2\ \right] + O(\Delta t^{3/2}) = O(\Delta t^{3/2}) \rightarrow 0. \tag{57}$$

In this replicating strategy, at each time period, $\Pi = \beta = c$ and $\Delta\Pi = \Delta c$. Thus, the portfolio $\Pi$ yields almost surely the option return $\max[F - X, 0]$ at $t + T$ as $\Delta t \to 0$.

We have ignored until now the impact of transaction costs on the performance of the replicating portfolio. The transaction costs are introduced in a very simple way, through an adjustment of the volatility in the Grabbe modified formula. The strategy depends upon the level of transaction costs and the time period between portfolio revision, in addition to the other standard variables of option pricing.

Let $k$ represent the round trip transaction cost, measured as a fraction of the volume of transaction in underlying asset and $\Delta t$ the revision period (frequency of revision). Define a new volatility which includes $k$ and $\Delta t$,[25]

$$\begin{aligned} \lambda^2_{\max}(\hat{\sigma}^2, k, \Delta t) &= \hat{\sigma}^2[1 + k + \frac{kE\mid \frac{\Delta F}{F} \mid}{\hat{\sigma}^2 \Delta t}] \\ &= \hat{\sigma}^2[1 + k + \frac{k\sqrt{2/\pi}}{(\hat{\sigma}\sqrt{\Delta t})}]. \end{aligned} \tag{58}$$

The expected value of $\frac{\Delta F}{F}$ taken in absolute value $E \mid \frac{\Delta F}{F} \mid$ is equal to

$$E \mid \tfrac{\Delta F}{F} \mid = \sqrt{2/\pi}(\hat{\sigma}\sqrt{\Delta t}) \tag{59}$$

which can be derived using the assumption that $\frac{\Delta F}{F}$ is normally distributed with mean zero.[26] This new volatility can be rewritten as a function of the initial volatility of exchange spot rate $\sigma_s^2$:

$$\begin{aligned} \lambda^2_{\max} &= \sigma_s^2\{[1 + \frac{T^2}{3\sigma_s^2}(\sigma_r^2 + \sigma_{r^*}^2 - 2\sigma_{rr^*}) + \frac{T}{\sigma_s^2}(\sigma_{sr^*} - \sigma_{sr})] \\ &\quad [1 + k + \frac{k\sqrt{2/\pi}}{\sqrt{\Delta t \sigma_s^2[1 + \frac{T^2}{3\sigma_s^2}(\sigma_r^2 + \sigma_{r^*}^2 - 2\sigma_{rr^*}) + \frac{T}{\sigma_s^2}(\sigma_{sr^*} - \sigma_{sr})]}}]\}. \end{aligned} \tag{60}$$

[25] Our formula for the modified variance, $\lambda^2_{\max}$, is different from the formula applied by Leland not only by the fact that it includes the stochastic interest component, $\hat{\sigma}^2$, but also by the fact that it contains the term $k$ which was omitted in Leland's formula. Simply, in the computation of transaction costs $TC$, equation (22) p.1290 of his paper, Leland neglected the term $(\Delta S)^2$ which could invalidate the results of his paper. With this term which cannot be omitted, the hedging error of the Leland's alternative strategy does not approach zero as $\Delta t$ becomes small. The solution consists of including a supplementary term, $k$, into the modified variance, equation (13) p.1289.

[26] Forwards have zero expected drift since they have zero initial cost.

Now, let us include this modified variance into the model of option pricing with stochastic interest rates (47)

$$\hat{c}(F, X, \lambda^2_{\max}, r, T) = e^{-r(t)T}[FN(\hat{d}_1) - XN(\hat{d}_2)] \tag{61}$$

where

$\hat{d}_1 = \frac{\ln(\frac{F}{X})+(\frac{1}{2}\lambda^2_{\max})T}{\lambda\sqrt{T}}$,

$\hat{d}_2 = \hat{d}_1 - \lambda\sqrt{T}$,

$\lambda^2_{\max} = \hat{\sigma}^2[1 + k + \frac{k\sqrt{2/\pi}}{(\hat{\sigma}\sqrt{\Delta t})}]$,

$\hat{\sigma}^2 = \sigma^2_s + \frac{T^2}{3}(\sigma^2_r + \sigma^2_{r^*} - 2\sigma_{rr^*}) + T(\sigma_{sr^*} - \sigma_{sr})$.

The call option $\hat{c}(F, X, \lambda^2_{\max}, r, T)$ satisfies the following partial differential equation,

$$\frac{1}{2}\lambda^2_{\max}F^2\frac{\partial^2\hat{c}_{\max}}{\partial F^2} - r\hat{c}_{\max} + \frac{\partial\hat{c}_{\max}}{\partial t} = 0. \tag{62}$$

Consider the replicating strategy $\alpha = \frac{\partial\hat{c}_{\max}}{\partial F}$ and $\beta = \hat{c}_{\max}$,[27] where $\hat{c}_{\max}$ is the modified Grabbe call price inclusive of stochastic interest rates and transaction costs. The after-transactions cost hedging error, $\Delta ER$, of the replicating portfolio over the interval, $\Delta t$ is given by

$$\Delta ER = \Delta\Pi - \Delta\hat{c}_{\max} - TC \tag{63}$$

where

$$\Delta\Pi = \alpha F(\frac{\Delta F}{F}) + r\beta\Delta t. \tag{64}$$

Using $\alpha = \frac{\partial\hat{c}_{\max}}{\partial F}$ and $\beta = \hat{c}_{\max}$ gives

$$\Delta\Pi = \frac{\partial\hat{c}_{\max}}{\partial F}F(\frac{\Delta F}{F}) + \hat{c}_{\max}r\Delta t \tag{65}$$

and

[27] As $\alpha$ and $\beta$ can be either positive or negative, we have to make an important distinction. The hedge strategy which consists of a long position in futures contracts ($\alpha > 0$) and borrowing ($\beta < 0$) duplicates a long call option. We use this strategy to duplicate the behavior of a transaction cost adjusted call option, $\hat{c}_{\max}$. We call this hedge *a long call hedge*. On the other hand, the hedge which consists of a short position in futures contracts ($\alpha < 0$) and buying bonds ($\beta > 0$) duplicates a short call option. We use this strategy to duplicate the behavior of $\hat{c}_{\min}$. We call this hedge *a short call hedge*. See Appendix A for more details on this replicating strategy.

$$
\begin{aligned}
\Delta\hat{c}_{\max} &= \hat{c}_{\max}(F+\Delta F, t+\Delta t) - \hat{c}_{\max}(F,t) \\
&= \frac{\partial \hat{c}_{\max}}{\partial F} F(\frac{\Delta F}{F}) + \frac{\partial \hat{c}_{\max}}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 (\frac{\Delta F}{F})^2 + O(\Delta t^{3/2}).
\end{aligned}
\tag{66}
$$

Transaction costs are given by

$$
\begin{aligned}
TC &= k \mid \Delta\alpha(F+\Delta F) \mid \\
&= k \mid \Delta \tfrac{\partial \hat{c}_{\max}}{\partial F}(F+\Delta F) \mid .
\end{aligned}
\tag{67}
$$

Applying a first-order Taylor series approximation for $\Delta \frac{\partial \hat{c}_{\max}}{\partial F}$, we obtain

$$
\begin{aligned}
TC &= k \mid [\tfrac{\partial \hat{c}_{\max}(F+\Delta F, t+\Delta t)}{\partial F} - \tfrac{\partial \hat{c}_{\max}(F,t)}{\partial F}](F+\Delta F) \mid \\
&= k \left| \tfrac{\partial^2 \hat{c}_{\max}(F,t)}{\partial F^2} \Delta F (F+\Delta F) \right| + O(\Delta t^{3/2}) \\
&= k \frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 \left[ (\tfrac{\Delta F}{F})^2 + \mid \tfrac{\Delta F}{F} \mid \right] + O(\Delta t^{3/2}).
\end{aligned}
\tag{68}
$$

The last line of (68) relies on the fact that $\frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 > 0$. Substituting equations (65), (66) and (68) in (63) gives:

$$
\begin{aligned}
\Delta ER &= (r\hat{c}_{\max} - \frac{\partial \hat{c}_{\max}}{\partial t})\Delta t - \frac{1}{2}\frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 (\frac{\Delta F}{F})^2 \\
&\quad - k \frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 \left[ (\tfrac{\Delta F}{F})^2 + \mid \tfrac{\Delta F}{F} \mid \right] + O(\Delta t^{3/2}).
\end{aligned}
\tag{69}
$$

Since $\hat{c}_{\max}$ satisfies the following partial differential equation (62), we may substitute for the first right-hand term in (69) to obtain

$$
\Delta ER = \frac{1}{2}\frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 \left[ \lambda^2_{\max}\Delta t - (1+k)(\tfrac{\Delta F}{F})^2 - k \mid \tfrac{\Delta F}{F} \mid \right] + O(\Delta t^{3/2}).
\tag{70}
$$

Substituting equation (58) into (70) yields

$$
\begin{aligned}
\Delta ER &= \frac{1}{2}\frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 [(1+k)\hat{\sigma}^2 \Delta t + kE \mid \tfrac{\Delta F}{F} \mid - (1+k)(\frac{\Delta F}{F})^2 \\
&\quad - k \mid \tfrac{\Delta F}{F} \mid ] + O(\Delta t^{3/2}).
\end{aligned}
\tag{71}
$$

Taking expectations yields

$$E[\Delta ER] = \frac{1}{2}\frac{\partial^2 \hat{c}_{\max}}{\partial F^2} F^2 E[(1+k)\hat{\sigma}^2 \Delta t + kE \mid \tfrac{\Delta F}{F} \mid - (1+k)(\frac{\Delta F}{F})^2 \\ -k \mid \tfrac{\Delta F}{F} \mid] + O(\Delta t^{3/2}) = O(\Delta t^{3/2}) \to 0. \quad (72)$$

The term into brackets is equal to zero because of the fact that $E\left[\, kE|\frac{\Delta F}{F}| - k|\frac{\Delta F}{F}|\,\right] = 0$, and $E\left[\,(1+k)\hat{\sigma}^2\Delta t - (1+k)(\frac{\Delta F}{F})^2\,\right] = 0.$[28] Thus, in the limit, as the readjustment interval becomes small, the modified hedging strategy yields the option result almost surely, inclusive of transaction costs. As one would expect, the "pure" Grabbe strategy holds only in the case when transaction costs become arbitrarily small.

Our model of currency option pricing with stochastic interest rates and transaction costs puts upper and lower bonds on the price of an option. On one hand the equation (61) sets an upper bound, $\hat{c}_{\max}$, since if the price exceed that amount the option could be constructed by the replicating strategy. On the other hand, the option price can be never less than $\hat{c}_{\min}$, where $\hat{c}_{\min}$ is given by the same equation (61) but with the different volatility $\lambda_{\min}$:[29]

$$\hat{c}(F, X, \lambda^2_{\min}, r, T) = e^{-r(t)T}[FN(\hat{d}_1) - XN(\hat{d}_2)] \quad (73)$$

where
$\hat{d}_1 = \frac{\ln(\frac{F}{X}) + (\frac{1}{2}\lambda^2_{\min})T}{\lambda_{\min}\sqrt{T}}$,
$\hat{d}_2 = \hat{d}_1 - \lambda_{\min}\sqrt{T}$,
$\lambda^2_{\min} = \hat{\sigma}^2[1 - k - \frac{k\sqrt{2/\pi}}{(\hat{\sigma}\sqrt{\Delta t})}]$ when $\hat{\sigma} > \frac{k}{(1-k)}\sqrt{\frac{2}{\pi\Delta T}}$ and
$\lambda^2_{\min} = 0$ when $\hat{\sigma} < \frac{k}{(1-k)}\sqrt{\frac{2}{\pi\Delta T}}$,
$\hat{\sigma}^2 = \sigma_s^2 + \frac{T^2}{3}(\sigma_r^2 + \sigma_{r^*}^2 - 2\sigma_{rr^*}) + T(\sigma_{sr^*} - \sigma_{sr})$.

If the price of an option exceeds $\hat{c}_{\max}$, we could make profits higher than the risk-free rate, by selling the option and buying the duplicating portfolio containing $\frac{\partial \hat{c}_{\max}}{\partial F}$ long futures contracts and selling $\frac{\partial \hat{c}_{\max}}{\partial F}F - \hat{c}_{\max}$ bonds (borrowing).[30] If the price of an call option is less than $\hat{c}_{\min}$, an investor could buy this "underpriced" option, "undo" it by following

[28] Forwards have zero expected drift since they have zero initial cost ($\mu\Delta t = 0$). Then, $\frac{\Delta F}{F} = \mu\Delta t + \hat{\sigma}\epsilon\sqrt{\Delta t} = \hat{\sigma}\epsilon\sqrt{\Delta t}$ and $(\frac{\Delta F}{F})^2 = \hat{\sigma}^2\epsilon^2\Delta t$. The expected value of the expression E$[\hat{\sigma}^2\Delta t - (\frac{\Delta F}{F})^2]$ is equal to zero because $E(\epsilon) = 0$ and $E(\epsilon^2) = 1$.

[29] See Appendix A for more details on the replicating strategy which duplicates *a short call* and puts lower bond on the option price.

[30] Later we have to follow the replicating strategy by adjusting our portfolio as described by the formula until the expiration date of the option. Then we close out the

the offsetting replicating strategy, and make a return after transaction costs which exceeded the risk-free rate.[31] Between these two transaction cost adjusted option prices, $\hat{c}_{\min}$ and $\hat{c}_{\max}$, will exist a no man's land in which option prices are too low for an investment hedge[32] to compete with a risk-free interest rate and too high for a borrowing hedge to compete with other forms of borrowing. In other terms, for option prices within this range, neither hedges duplicating *a long call option* nor hedges duplicating *a short call option* are particularly attractive.

In this section we have derived a European currency option pricing model with both stochastic interest rates and transaction costs. We applied Leland's methodology to the Hilliard, Madura, Tucker model with stochastic interest rates. These two additional components are introduced in formula through the adjustment of the volatility. We developed two hedging strategies, which can be used to replicate option returns inclusive of stochastic interest rates and transaction costs, with accuracy that increases as the revision interval becomes small. Our model enabled us to put upper and lower bounds, $\hat{c}_{\max}$ and $\hat{c}_{\min}$, on the price of an option.

---

position which involves buying back the option at its current market price, selling the stock and repaying the borrowing.

[31] Initially, the investor buys the call option, sells $\frac{\partial \hat{c}_{\min}}{\partial F}$ shares of stock and buys $\frac{\partial \hat{c}_{\min}}{\partial F} F - \hat{c}_{\min}$ bonds (investing or lending). Then he follows the replicating strategy by maintaining the neutral position ratio (the number of shares held for each call) until the maturity date of the option. Finally, he closes out his position by selling the option, buying the stock and selling the bonds (borrowing).

[32] *An investment hedge* consists of a long position in futures contract and a short position in call options. This hedge will require a positive net investment which will earn the risk-free rate. If the hedge consists of a short position in futures and a long position in call options, the hedge supplies funds which will cost the risk-free rate and it is called *a borrowing hedge*.

# Conclusion

In this paper, we develop a currency option pricing model with stochastic interest rates and transaction costs when interest parity holds, and it is assumed that domestic and foreign bond prices have local variances that depend only on time. We apply the Leland's technique for replicating option returns in the presence of transaction costs to the Grabbe formula modified to include the assumptions of the Vasicek bond pricing model. The stochastic interest rates and transaction costs are introduced in a simple way, through adjustment of the volatility in the Grabbe currency option pricing model. Hedging errors of the modified replicating strategies inclusive of stochastic interest rates and transaction costs are uncorrelated with the market and approach zero with more frequent revision. Our currency option pricing model therefore, puts upper and lower bounds on option prices. The symmetry of the bid and ask prices of the currency around its zero-transaction-cost price does not imply a corresponding symmetry for the bid and ask prices of the call option. The "pure" Black and Scholes strategy - which in the case of currency options is equivalent to the Garman and Kohlhagen formula - holds in the limiting case of constant interest rates and zero transactions costs. Whilst our analysis only dealt with European call options it can be extended to cover European put options. The put values can be derived from put-call parity. However, it is worth noting that in the replicating strategy for option pricing proposed by Leland and used in this paper, the frequency of portfolio revisions is exogenously given. As consequence, the Leland-style approach does not match the optimal investment strategy with proportional transaction costs. Constantinides (1986), Dumas and Luciano (1991) and Edirisinghe, Naik, and Uppal (1993) began to explore this important area.

# Appendix

## A Lower bound of an option price in the currency option model with stochastic interest rates and transaction costs

Consider a portfolio $\Pi$ of $\alpha$ sold futures (forward) contracts and $\beta$ dollars borrowed at the risk-free rate over the interval, $\Delta$t. Since it costs nothing to enter into a futures contract

$$\Pi = -\beta. \tag{A.1}$$

Over the interval, $\Delta t$ the return to this portfolio will be

$$\Delta\Pi = -\alpha F(\frac{\Delta F}{F}) - r\beta\Delta t. \tag{A.2}$$

At the same time, we write a call option $\hat{c}_{\min}$ where $\hat{c}_{\min}$ is the minimum price that the option can have. It is a modified Grabbe call price in which the stochastic interest rates and transaction costs enter through adjustment of the volatility $\lambda^2_{\min}$:

$$\hat{c}(F, X, \lambda^2_{\min}, r, T) = e^{-r(t)T}[FN(\hat{d}_1) - XN(\hat{d}_2)], \tag{A.3}$$

where
$\hat{d}_1 = \frac{\ln(\frac{F}{X})+(\frac{1}{2}\lambda^2_{\min})T}{\lambda_{\min}\sqrt{T}}$,
$\hat{d}_2 = \hat{d}_1 - \lambda_{\min}\sqrt{T}$,
$\lambda^2_{\min} = \hat{\sigma}^2[1 - k - \frac{k\sqrt{2/\pi}}{(\hat{\sigma}\sqrt{\Delta t})}]$ when $\hat{\sigma} > \frac{k}{(1-k)}\sqrt{\frac{2}{\pi\Delta T}}$ and
$\lambda^2_{\min} = 0$ when $\hat{\sigma} < \frac{k}{(1-k)}\sqrt{\frac{2}{\pi\Delta T}}$,
$\hat{\sigma}^2 = \sigma^2_s + \frac{T^2}{3}(\sigma^2_r + \sigma^2_{r^*} - 2\sigma_{rr^*}) + T(\sigma_{sr^*} - \sigma_{sr})$.

Now, consider the replicating strategy $\alpha = \frac{\partial\hat{c}_{\min}}{\partial F}$ and $\beta = \hat{c}_{\min}$. The after-transactions cost hedging error, $\Delta ER$, of the replicating portfolio over the interval, $\Delta t$ is given by:

$$\Delta ER = \Delta\Pi + \Delta\hat{c}_{\min} - TC. \tag{A.4}$$

Using $\alpha = \frac{\partial\hat{c}_{\min}}{\partial F}$ and $\beta = \hat{c}_{\min}$ equation (A.2) gives

$$\Delta\Pi = -\frac{\partial\hat{c}_{\min}}{\partial F}F(\frac{\Delta F}{F}) - \hat{c}_{\min}r\Delta t. \tag{A.5}$$

The $\Delta\hat{c}_{\min}$ is

$$\begin{aligned}\Delta\hat{c}_{\min} &= \hat{c}_{\min}(F+\Delta F, t+\Delta t) - \hat{c}_{\min}(F,t)\\ &= \frac{\partial\hat{c}_{\min}}{\partial F}F(\frac{\Delta F}{F}) + \frac{\partial\hat{c}_{\min}}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\hat{c}_{\min}}{\partial F^2}F^2(\frac{\Delta F}{F})^2 + O(\Delta t^{3/2}).\end{aligned} \tag{A.6}$$

Transaction costs are given by:

$$\begin{aligned}TC &= k \left| \Delta\alpha(F+\Delta F) \right|\\ &= k \left| \Delta\tfrac{\partial\hat{c}_{\min}}{\partial F}(F+\Delta F) \right|.\end{aligned} \tag{A.7}$$

Applying a first-order Taylor series approximation for $\Delta\frac{\partial\hat{c}_{\min}}{\partial F}$, we obtain

$$\begin{aligned}TC &= k \left| \left[\tfrac{\partial\hat{c}_{\min}(F+\Delta F,t+\Delta t)}{\partial F} - \tfrac{\partial\hat{c}_{\min}(F,t)}{\partial F}\right](F+\Delta F) \right|\\ &= k \left| \tfrac{\partial^2\hat{c}_{\min}(F,t)}{\partial F^2}\Delta F(F+\Delta F) \right| + O(\Delta t^{3/2})\\ &= k\frac{\partial^2\hat{c}_{\min}}{\partial F^2}F^2\left[\ (\tfrac{\Delta F}{F})^2 + \left| \tfrac{\Delta F}{F} \right|\ \right] + O(\Delta t^{3/2}).\end{aligned} \tag{A.8}$$

Substituting equations $(A.5), (A.6)$ and $(A.8)$ in $(A.4)$ gives

$$\begin{aligned}\Delta ER &= -(r\hat{c}_{\min} - \frac{\partial\hat{c}_{\min}}{\partial t})\Delta t + \frac{1}{2}\frac{\partial^2\hat{c}_{\min}}{\partial F^2}F^2(\frac{\Delta F}{F})^2 - k\frac{\partial^2\hat{c}_{\min}}{\partial F^2}F^2[(\frac{\Delta F}{F})^2\\ &\quad + \left| \tfrac{\Delta F}{F} \right|] + O(\Delta t^{3/2}).\end{aligned} \tag{A.9}$$

Since $\hat{c}_{\min}$ satisfies the following partial differential equation:

$\frac{1}{2}\lambda^2_{\max}F^2\frac{\partial^2\hat{c}_{\min}}{\partial F^2} - r\hat{c}_{\min} + \frac{\partial\hat{c}_{\min}}{\partial t} = 0,$

we may substitute for the first right-hand term in $(A.9)$ to give

$$\begin{aligned}\Delta ER &= \frac{1}{2}\frac{\partial^2\hat{c}_{\min}}{\partial F^2}F^2\left[\ -\lambda^2_{\min}\Delta t - (k-1)(\tfrac{\Delta F}{F})^2 - k\left| \tfrac{\Delta F}{F} \right|\ \right]\\ &\quad + O(\Delta t^{3/2}).\end{aligned} \tag{A.10}$$

Using equation:

$\lambda^2_{\min}(\hat{\sigma}^2, k, \Delta t) = \hat{\sigma}^2[1 - k - \frac{kE|\frac{\Delta F}{F}|}{\hat{\sigma}^2\Delta t}],$

and substituting into $(A.10)$ yields

$$
\begin{aligned}
\Delta ER &= \frac{1}{2}\frac{\partial^2 \hat{c}_{\min}}{\partial F^2} F^2[(k-1)\hat{\sigma}^2 \Delta t + kE \mid \tfrac{\Delta F}{F} \mid - (k-1)(\frac{\Delta F}{F})^2 \\
&\quad -k \mid \tfrac{\Delta F}{F} \mid] + O(\Delta t^{3/2}). \qquad \text{(A.11)}
\end{aligned}
$$

Taking expectations yields

$$
\begin{aligned}
E[\Delta ER] &= \frac{1}{2}\frac{\partial^2 \hat{c}_{\min}}{\partial F^2} F^2 E[(k-1)\hat{\sigma}^2 \Delta t + kE \mid \tfrac{\Delta F}{F} \mid - (k-1)(\frac{\Delta F}{F})^2 \\
&\quad -k \mid \tfrac{\Delta F}{F} \mid] + O(\Delta t^{3/2}) = O(\Delta t^{3/2}) \to 0. \qquad \text{(A.12)}
\end{aligned}
$$

The term into brackets is equal to zero because of the fact that $E\left[\ kE|\frac{\Delta F}{F}| - k|\frac{\Delta F}{F}|\ \right] = 0$, and $E\left[\ (k-1)\hat{\sigma}^2 \Delta t -^2 (k-1)(\frac{\Delta F}{F})\ \right] = 0.$[33] In this replicating strategy, at each time period, $\Pi = -\beta = -\hat{c}_{\min}$ and $\Delta\Pi = \Delta(-\hat{c}_{\min})$. Thus, the portfolio $\Pi$ yields almost surely the option return $\max[F - X, 0]$ at $t + T$ as $\Delta t \to 0$.

[33] Forwards have zero expected drift since they have zero initial cost ($\mu\Delta t = 0$). Then, $\frac{\Delta F}{F} = \mu\Delta t + \hat{\sigma}\epsilon\sqrt{\Delta t} = \hat{\sigma}\epsilon\sqrt{\Delta t}$ and $(\frac{\Delta F}{F})^2 = \hat{\sigma}^2\epsilon^2\Delta t$. The expected value of the expression $\mathrm{E}[\hat{\sigma}^2\Delta t - (\frac{\Delta F}{F})^2]$ is equal to zero because $E(\epsilon) = 0$ and $E(\epsilon^2) = 1$.

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