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Abstract

This paper analyses the impact of some assumptions of households' heterogeneity on the uniqueness and the stability property of the temporary equilibrium of a pure exchange economy. The set of households' characteristics includes expectations about future prices. A high dispersion of households on this space ensures the global stability of the short-run equilibrium, and in addition the convergence of the sequence of temporary equilibria. The heterogeneity of expectations stabilizes the dynamics with learning near any stationary price when the perfect foresight dynamics are locally well determinate. *Journal of Economics Literature* Classification Numbers: D11, D41, D50, D84, E1.

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The purpose of this paper is to study the properties of market demand of a very large and heterogeneous population, when short-run household demand is the outcome of a multi-period maximization program. In an economy where incomes are exogeneously given, the set of households' characteristics is the product of the set of income, the set of demand functions (for fixed household expectations) and the set of expectations. For exogeneous price expectations it has been shown that some assumption of heterogeneity ensures the monotonicity of market demand, see Grandmont (1992), Hildenbrand (1993) and Kneip (1993). When expectations are endogeneous, the variation of the short-run demand induced by a price change includes an intertemporal substitution effect. After a price change, the household reviews its expectations and decides whether to substitute present to future consumption. The question is whether broad classes of distributions on the set of households' characteristics including expectations yield again the monotonicity of market demand.

Heterogeneity of demand behaviour is defined by a high "dispersion" of the household behaviour with respect to one or the other of its endogeneous arguments. This definition is derived from the one introduced by Kneip (1993) which can be seen as a compromise between the theoretical analysis of Grandmont and the more empirical analysis of Hildenbrand. In this paper we introduce expectations about future prices in the households' decision rule. This leads to introduce two types of heterogeneity. First, the expected price vector for the future is now an argument of the demand function. Two households might adjust their demand in a different way after a small change of the expected prices *ceteris paribus*. That is, household demand might be dispersed along this argument. Furthermore, the expectations function itself, which formalizes the household's way of processing information, might as well be dispersed along any of its arguments.

Formally, we consider a parametric model of demand, that is each demand function is assumed to be parameterized by $\beta \in \mathcal{B}$[1]. Among

[1]The group of households whose demands are parameterized by $\beta \in \mathcal{B}$ might be the given population itself or some suitably stratified subpopulation. It is only required

the parameters β describing the demand function one can distinguish artificially the ones which describe the household's expectations, denote them ϵ and denote γ the remaining vector such that $(\gamma, \epsilon) = \beta$. The heterogeneity of demand can, then, be formalized by a "flat" density of the distribution on the parameter set $\mathcal{B}$, as proposed by Grandmont (1992). In order for a flat density of β to have any relation with heterogeneous demand behaviours, the structure of the parameter distribution should reflect the structure of the distribution of demand functions and expectations functions. Hence, one has to impose some qualitative restrictions on the parameterization of demand behaviour analogous to the ones introduced by Kneip (1993). Theses restrictions allow for the interpretation of a flat density of β as demand behaviours belonging to a large subset of the space of all admissible demand behaviours and being approximately "equally likely". This heterogeneity requirement includes, thereby, some heterogeneity of the expectations function. In this paper, we distinguish the two types of heterogeneity; the dispersion of demand along its arguments, including the vector of expected prices, and the dispersion of the expectations function along its arguments. The results obtained by Kneip (1993) for exogeneous expectations and exogeneous incomes are easily translated to this framework. We prove that when household demand is highly dispersed along the current price vector and the vector of expected prices, market demand is monotone. Furthermore, the required degree of heterogeneity of demand decreases as the heterogeneity of the expectations function increases. It is, as well, worth while noting that the negative definiteness of the matrix of substitution effects can be the outcome of a standard rationality requirement combined with an assumption of heterogeneity. This result is obtained for household matrices of substitution which are not necessarily negative definite. Note that, in the static framework, this assumption is usually deduced from the utility maximization requirement. However, in the multiperiod framework, the intertemporal substitution effect might go in any direction depending on the household's expectations, hence the a priori assumption of a negative definite matrix of substitution effect has no legitimacy under the

that demand of the selected group can be expected to be very heterogeneous and posses considerable income variations.

standard assumption on individual behaviour.

This analysis is, then, applied to the study of the short-run stability and uniqueness of the equilibrium of a pure exchange economy, when the concept of equilibrium is the temporary equilibrium.

> 'A temporary equilibrium or short-run Walrasian equilibrium is defined as a set of current prices that equate aggregate demand and supply on every market *at the date under consideration.* *Grandmont (1983)*

The current price is only required to ensure equilibrium in the current period. There exists a desequilibrium phenomena in the sense that at a given date the households' plans for the future are not coordinate and may be incompatible. The framework of temporary competitive equilibrium theory is seen as a step closer to reality because market prices are assumed to exist only for current goods. The purpose of the General Equilibrium Theory can be understood, as suggested by Malinvaud (1991), as to analyze the economic evolution as the outcome of a succession of temporary equilibria, the static model being a theoretical tool. The existence of temporary equilibria has already been proved in monetary economies (see for example Grandmont and Laroque (1973) Fuchs and Laroque (1976)). Therefore, we focus on the questions of uniqueness and stability of these equilibria.

Heterogeneity is again a solution to the short-run stability problem. Our assumption of heterogeneity and aggregation over endowments guarantees that market short-run demand is monotone in a restrictive sense (for price changes which preserve the mean income level). Such a property ensures that the equilibrium in each period is unique and stable for specific price adjustment processes and/or for specific price normalizations.

This result does not yield the stability of the one-period dynamical system under any tâtonnement processus described by

$$\partial_t p_i = G_i(Z_i(p))$$

where G_i is a sign preserving function with $G_i(0) = 0$ and differentiable with $G'_i > 0$. This processus is the most general formalization of the intuition of how markets should operate: the price rises when a good is in excess demand and decreases otherwise. The previous restricted monotonicity property ensured by behavioural heterogeneity and aggregation over endowments stabilizes the dynamical system only if it is induced by specific examples of the above processus or by specific price normalizations, namely the ones which preserve the mean income. When these restrictions are not required, even in very specific cases, for example, when all households are assumed to have homothetic preferences, and thereby market demand would be monotone if income were to be exogeneous, the income effects induced by changes in the relative income distribution are shown to be bad enough to destroy any nice structure of market demand. Nevertheless, it is clear that there exist cases where the one-period equilibrium is stable, like for example, when all households have Cobb-Douglas demand functions. Symmetrically, the requirement of a specific price normalization can be substituted by the requirement that all endowments are colinear, and stability is obtained if market demand is monotone for fixed incomes. These two assumptions have no a priori justification and are not very realistic. They are typically the kind of assumptions we shall try to avoid, since they restrict a priori the theoretical analysis to specific set of households' characteristics.

Note, however, that the main factor of instability is the direct dependence of income on prices due to the unrealistic definition of income as the value of the household's initial endowments evaluated at the current prices. Thus, a solution to short-run stability (under any standard tâtonnement process) can be to evaluate income at past prices. Such a time lag in the determination of income was introduced by Hens and Hildenbrand (1993) and is a quite natural assumption in a private ownership economy.

In a second part, we want to go beyond the one-period stability and analyze the stability problem as the evolution of short-run equilibria. We concentrate on the stability issue since one hope underlying part of the research is that the stability criteria in plausible dynamics with learning could be useful to reduce the embarassing multiplicity of long-run equilibria. We consider the framework of a monetary economy, more precisely, the model we retain is a generalisation to a temporary equilibrium context of Samuelson's pure consumption loan model introduced by Grandmont and Laroque (1973), and which dynamics has been studied by Fuchs and Laroque (1976) among others. In this overlapping generations model, we study the local stability of self-fulfilling expectations equilibria. Note that the convergence to self-fulfilling expectations can be as well analyzed in stochastic linear, or non linear macroeconomic models (see de Canio 1979, Marcet and Sargent 1988, 1989, Calvo 1988, Kurz 1989). The issue has also been treated in game theory, where convergence to particular equilibria is investigated, when players revise adaptively their expectations over time.

Two important characteristics of the model affecting dynamics are that households have a memory which lasts T periods and that the model includes now two generations of households. The dynamics with learning depend in a complex manner on the interaction between the dynamics of perfect foresight and the dynamics of the learning process. The relationship that may exist between the stability of the dynamics with learning, the prefect foresight stability or the local determinateness of a long run equilibrium was implemented for the one-dimensional case in Grandmont and Laroque (1986), for related results see also Tillman (1983). These relationships and the the mathematical sufficient conditions for stability of a stationary equilibrium when the state variable is multidimensional were studied respectivelly in Grandmont and Laroque (1990) and Fuchs and Laroque (1976).

We show that the heterogeneity of the demand function in the population along each of its arguments (including the expected price vector) ensures the local asymptotic stability of the dynamics with learning at any stationary temporary equilibrium. Again, the required degree of heterogeneity decreases with the increasing heterogeneity of the expectations function. Incidentally, stability is not the outcome of rational expectations, higher is the degree of heterogeneity of expectations higher is the probability that the stationary temporary equilibrium is stable.

It is worthy of note that the heterogeneity of expectations has to be restricted in order to be compatible with the natural assumption of stationarity of expectations, which requires that at the stationary price households' expectations are identical. However, for any deviation from the stationary price very different expectations are allowed. The household learning process is even allowed to be unstable. Nevertheless, the heterogeneity of individual instability leads to a stability property at the aggregate.

Despite expectations functions play an important role in the dynamics with learning their heterogeneity does not eliminate any feature of instability. The dynamics depend for an important part on the fundamental charateristics which affect the demand functions independently of the expectations function, and influence the perfect foresight dynamics.

First, it is important while noting that the stability of the perfect foresight dynamics does not require the same kind of heterogeneity than the stability of the dynamics with learning. While a high degree of heterogeneity of the demand function ensures the stability of the dynamics with learning it might lead to the instability of the perfect foresight dynamics. Stability of the perfect foresight dynamics requires, indeed, to restrict the heterogeneity of demand along the expected price vector. As an example, we show that in the simple case of only one non storable commodity, if household demand is highly dispersed along each of its arguments including the vector of expected prices, the stationary temporary equilibrium is a saddle point of the dynamics of perfect foresight, while the dynamics with learning is stable near stationary states.

This example illustrates as well the role of the heterogeneity of expectations, and underlines that there exist some cases where the heterogeneity of expectations eliminate any feature of instability from the dynamics. It is proved that if a stationary state is a saddle point of the perfect foresight dynamics, that is if the perfect foresight dynamics are locally determinate, then a high dispersion of the expectations function ensures that this stationary price is locally asymptotically stable in the dynamics with learning. This result is proved to remain valid in the multidimensional case. We consider that all households have identical preferences and identical endowments but differ in their way of processing information. If the fundamental characteristics ensure the local determinateness of the perfect foresight dynamics induced by some type of incentive to save at the aggregate the heterogeneity of expectations give enough structure to the dynamical system with learning to ensure its local asymptotic stability.

To conclude the one-dimensional model illustrates that the heterogeneity of expectations tends to stabilize the dynamics with learning around any stationary price. However, as long as the perfect foresight dynamics are not locally determinate one can find examples where despite a high dispersion of expectations the dynamics are unstable. This is the case when the stationary state is locally asymptotically stable in the dynamics of perfect foresight. The indeterminateness of the equilibrium trajectory in the perfect foresight dynamics implies that once expectations are introduced their heterogeneity cannot ensure enough structure to the dynamics with learning. The way expectations affect demand, which enters in the fundamental characteristics, plays a determinant role.

Any stability property previously obtained is structural following the theorem of Fuchs and Laroque, which states that, for the model under consideration, the qualitative behaviour of trajectories of the economy near stationary equilibria and cycles is preserved under small perturbations.

In a first section we describe the model. We prove the two main results, regarding short-run stability and the stability of the sequence of short-run equilibria, in Section 2 and 3 (respectively). As an illustration the example of an economy with only one non storable commodity is considered in Section 4. A last section is devoted to the study of dynamics once a time lag in the determination of income has been introduced. For this income determination which affects the dynamics, the requirement that household demand is highly dispersed with respect to all its arguments still ensures the local asymptotic stability of stationary temporary equilibria.

1 The Set-Up

1.1 The Model

There are l non storable commodities ($k = 2, \ldots, l+1$) and one storable commodity which is used to transfer wealth over time and which we shall call money ($k = 1$).

Households live two periods so that at each time t there are two living households of each type, one "young" and one "old". At the beginning of each period of their lives, households receive an endowment of the l non storable commodities which are known in advance. They trade on the spot market of the corresponding period. Young households can save by buying from the old households the money these had themselves bought in the previous period; the total monetary stock remains constant.

We shall describe the demand behaviour directly by the demand function and the expectations function rather than by the utility function.

We assume as in Grandmont and Laroque (1973) that the expectations about future prices of the young household in the current period depend on the current prices and on the T past price systems. This means that uncertainty concerns only the future prices and that households have a T-period memory. This property is determinant for the dynamics and might as well affect the properties of the set of stationary solutions. In a model with rational expectations Fourgeaud et al. (1985) show that the undesirable properties of the set of solutions of the model with unlimited memory disappear if the memory is bounded. In particular, there exists an infinite number of stationary solutions of the model with unlimited memory and every solution is characterized by an initial condition, while the model with bounded memory has a unique stationary solution and solutions are independent of initial conditions. Truncating the memory affects the dynamics in a way which, usually, is not a priori clear. Nevertheless, the present discussion about the asymptotic stability of stationary states obtained through heterogeneity requirement is not affected by the length of the memory, see Remarks 5 and 7. Hence, for the present analysis truncating households' memory is a reasonable solution to the difficulties involved with the unlimited memory. One may think of the memory lenght as very large with observations in the far distant past having a negligible influence on current forecasts. Then the bounded memory is a good approximation. Such a shape of the learning process is obtained, for example, when past observations are ponderated by some discount factor. This assumption has the convenient consequence that the dynamical system is time-independent.

Denote by s_t the sequence of price vectors $(p_t, p_{t-1}, \ldots, p_{t-T})$.

The set of prices we shall consider for money and non storable commodities is $P = \Re_{++}^{l+1}$, and the price normalization $p \in \Re_+ \times S$ where $S = \{p \in \Re^{l+1} \mid \sum_k p_k = 1, p_k > 0 \quad \forall k = 2, \ldots, l\}$. We do not allow the price of any commodity to be zero to ensure the differentiability of demand functions on P.

The household is now described by two demand behaviours corresponding to the two periods of his life. The action of the young household depends on the current price system and on its expectations about future prices p_{t+1}^e, while the action of the old household (born at time $(t-1)$) depends on his past action, that is on past prices p_{t-1}, on its past expectations about current prices p_t^e and on the current price system p_t.

Definition 1 *The demand function of the young household is a continuously differentiable function $f^y(p_t, p_{t+1}^e, x_t)$ of current prices p_t, of the price expectations formalized at the current date for the future period p_{t+1}^e and of the current income x_t.*

The demand function of the old household is a continuously differentiable function $f^o(p_t, p_{t-1}, p_t^e, x_t)$ of current prices p_t, of last period prices p_{t-1}, of the price expectations formalized at the previous period for the current date p_t^e and of the current income x_t. Both demands satisfy the budget identity for the period under consideration, that is to say

$$p_t^T f^y(p_t, p_{t+1}^e, x_t) = x_t = p_t^T \omega$$
$$p_t^T f^o(p_t, p_{t-1}, p_t^e, x_t) = x_t = p_t^T \omega$$

The first components $f^y_{t;1}(p_t, p^e_{t+1}, x_t)$ *and* $f^o_{t;1}(p_t, p_{t-1}, p^e_t, x_t)$ *describe demands for money, and the other components* $f^y_{t;k}(p_t, p^e_t, x_t)$ *and* $f^o_{t;k}(p_t, p_{t-1}, p^e_t, x_t)$ $\forall k \neq 1$ *represent demands for non storable commodities.*

In the sequel, we forget the subscript t of the current period when there is no ambiguity. We require the following assumptions.

Assumption 1 *An old household spends all the money it saved when it was young, that is*

$$-f^y_1(p_{-1}, p^e_{t+1}, x) \mid_{x=p^T\omega} = f^o_1(p, p_{-1}, p^e_t, x) \mid_{x=p^T\omega} \qquad \forall (p, s_{-1}) \in (\Re_+ \times S)^{T+1}$$

Assumption 2 *For constant consecutive price, the demand of money is inversely proportional to its price*

$$f^y_1(p, p, x) \mid_{x=p^T\omega} = (p_1)^{-1} f^y_1(\tilde{p}, \tilde{p}, \tilde{x}) \mid_{\tilde{x}=\tilde{p}^T\omega}$$

The demand of non storable commodities only depends on their relative prices

$$\begin{aligned} f^y_k(p, p, x) \mid_{x=p^T\omega} &= f^y_k(\tilde{p}, \tilde{p}, \tilde{x}) \mid_{\tilde{x}=\tilde{p}^T\omega} \\ f^o_k(p, p, p, x) \mid_{x=p^T\omega} &= f^o_k(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{x}) \mid_{\tilde{x}=\tilde{p}^T\omega} \end{aligned}$$

$\forall k = 2, \ldots, l$ *and* $\forall p \in (\Re_+ \times S)$, *where* $\tilde{p} = (1, p_2, \ldots, p_l)$.

We assume that the household information at the date t is s_t and summarize the households' way of processing that information by time-independent expectations functions[2]. The price expectation functions ψ^y and ψ^o are assumed to satisfy the following assumptions.

Assumption 3 ψ^y *and* ψ^o *are continuous and differentiable on* P^{T+1} *and takes values in* $\Re^l_+$. *Furthermore their derivative are continuous functions.* $\psi^y \in C^1(P^{T+1}, \Re^l)$ *and* $\psi^o \in C^1(P^{T+1}, \Re^l)$.

[2]Note that these functions are dependent on the length of the memory T.

Denote $\partial_{p_{-j}}\psi^y = C_j^\epsilon$ and $\partial_{p_{-j}}\psi^o = D_j^\epsilon$.

Remark 1 To simplify the presentation we assume that ψ^y and ψ^o are functions from P^{T+1} to $\Re^l$ rather than distributions of probability, that is mappings from P^{T+1} onto $\mathcal{M}(\Re^l)$. In such a framework temporary equilibria have been proved to exist (see Remark 3). However in terms of dynamics our results would not have been affected assuming that expectations of future prices are distributions of probability. One would just have had to require that these mapping were regular enough to get the continuity and differentiability of household demand with respect to current and past prices (see Christiansen, 1972, for the case where demands are deduced from the intertemporal maximization of utility functions).

Assumption 4 $\psi^y(P^{T+1}) \subset P$ *and* $\psi^o(P^{T+1}) \subset P$.

We require the following condition of consistency for the expectations function ψ^y and ψ^o:

Assumption 5

$$\psi(p, \ldots, p) = p \qquad \forall p \in P$$

In particular, this assumption implies that if the economy is at a stationary temporary equilibrium households do not make forecasting errors. This is the minimum rationality requirement needed for consistency with the standard notion of equilibrium.

Consider the definitions: Denote $\mathcal{A}$ the set of households' characteristics and μ the probability distribution of households on this set.

Definition 2 *A point p_t in P is a temporary equilibrium at time t if there exist s_{t-1} in P^T such that*

$$F(p_t, s_{t-1}) = \int_{\mathcal{A}} \omega d\mu$$

For each s_{t-1} the set of temporary equilibria will be labeled by $V(s_{t-1})$.

Definition 3 *A trajectory is an infinite sequence $\{p_t\}$ with $p_t \in P$ and t a whole number such that,*

$$p_t \in V(s_{t-1})$$

Remark 2 The total monetary stock

$$M(s) = \int_{\mathcal{A}} f_1^y(p, p_{t+1}^e, x) \mid_{\substack{x = p^T\omega \\ p_{t+1}^e = \psi^e(s)}} d\mu$$

is constant along a trajectory. This is the outcome of assumption 1.

Definition 4 *If a trajectory $\{p_t\}$ is the repetition of some $\bar{p}$, it is represented in P by a fixed point, which we shall call a stationary temporary equilibrium (STE).*

Denote W the set of STE of our economy.

As observed by Fuchs and Laroque (1976) the quantity theory of money holds in the model under consideration. Effectively, for any STE we observe that the aggregate demand for money F_1 is null while demands for non storable commodities do not depend on the price of money p_1. The property of dichotomy between the market of money and the commodity markets holds.

Remark 3 Consider the following additional assumptions:

Assumption 6 *Consumption sets are bounded below, f_k^y and f_k^o are bounded below whatever $k = 1, \ldots, l$.*

Denote $\bar{S}$ the interior of the set S and $\bar{S} \backslash S$ the complementary set of $\bar{S}$ in S. In addition, denote $\mathcal{A}$ the set of households' characteristics and μ the probability distribution of hoiuseholds on this set.

Assumption 7 *Desirability assumption: If $p \in (\Re_+ \times S)$ converges towards $p^0 \in \Re_+ \times \bar{S} \backslash S$, then $\int_{\mathcal{A}} \sum_{k=2}^{l} f_k^o(p, p_{-1}, p_t^e, x) \mid_{\substack{x = p^T\omega \\ p_{t+1}^e = \psi^o(s_{-1})}} d\mu$ converges towards $+\infty$ whatever $s_{-1} \in P^T$.*

Assumption 8 *The demand of money increases if its value is expected to increase. If* $p \in (\Re_+ \times S)$ *converges towards* $p^0 \in \{0\} \times S$ *and* $\psi_1^y(s) \geq c > 0$, *then* $\int_{\mathcal{A}} f_1^y(p, p_{t+1}^e, x) \mid_{\substack{x = p^T\omega \\ p_{t+1}^e = \psi^y(s)}} d\mu$ *converges towards* $+\infty$ *whatever* $\psi^y(s) \in P$.

We require as well that the expectations do not depend too much on the current price system.

Assumption 9 *If* $p \in P$ *converges towards* $p^0 \in \{0\} \times S$ *then whatever* $(p_{-1}, \ldots, p_{-T}) \in P^T$ *there exists* $c > 0$ *such that* $\psi_1^y(s) > c$.

Note that our heterogeneity requirement does not contradict this assumption, expectations about the future price of money belong to a bounded set and we require that they are highly dispersed on this set. Furthermore, it is worthy of note that this assumption is indeed quite artificial. Effectively, when demand behaviour is described directly by a demand function includinf the expectations function, one could have formalized Assumption 8 independently of the expectations functions and require, for a demand which is a function of income, current prices and past prices, that when the price of money converges towards zero the aggregate demand for money becomes infinite. The purpose of this assumption is only to recall the property of the expectations function required to prove the existence of temporary equilibria, when demand is the outcome of the maximization of a utility function under budget constraints. In addition our intuition is that Assumption 9 can be derived from behavioral heterogeneity. The requirement that the expectation function is quite dispersed along each of its arguments implies, as proved in Subsection 1.3, that the aggregate expectation function is less sensitive to a price change. This insensitivity property is formalized by small derivatives of the mean expectation function. Our conjecture is that this property (or some similar condition) combined with the rationality requirement of utility maximization will lead to Assumption 8.

Under the additional assumptions 8 and 9 Fuchs and Laroque (1976, p. 254) show that the set of temporary equilibria of our economy is non empty and compact valued.

Denote $\bar{f}$ the function describing the demand behaviour, that is the demand function where the expected prices have been substituted by their expressions in terms of current and past prices. One can add the two assumptions:

Assumption 10 *Second desirability assumption: If $q \in S$ converges towards $q^0 \in \Re_+ \times \bar{S} \backslash S$, then $\int_{\mathcal{A}} \sum_{k=2}^{l} \bar{f}_k^o(q, \ldots, q, x) \mid_{x=q^T\omega} d\mu$ converges to* $+\infty$

This assumption requires the existence of some aggregate preference for the present. As a result of Assumption 5 having observed during T period the price q which states that at least one non storable commodity price converges towards zero households expect the same price for the future. At the aggregate the current demand becomes infinite despite the price of one commodity is expected to be zero in the future.

Assumption 11 *In the long run there always exist potential lenders, this means that*

$$\int_{\mathcal{A}} \bar{f}_1^y(p, \ldots, p, x) \mid_{x=p^T\omega} d\mu > 0 \qquad \forall p \in P$$

Under these two additional assumptions the set of *stationary* temporary equilibria of our economy W is non empty and compact valued (see Theorem 2 of Fuchs and Laroque).

We restrict the set of admissible economies such that assumptions 1 through 11 hold, and focus on the question of stability of stationary temporary equilibria.

1.2 The Definition of Heterogeneity

The heterogeneity of the distribution of households' characteristics is defined by a measure of heterogeneity of the distribution of households on the set $\mathcal{A}$. It might be useful to decompose the population into suitable subpopulations, since some assumptions might be too restrictive when formalized for the whole population. We are only concerned with additive properties of market demand. Hence, we can focus on the mean demand of a given subpopulation and the result will follow for market demand by aggregation. For a given subgroup denote ν_y (resp. ν_o) the distribution of young households (resp. of old households) on the set of characteristics $\mathcal{A}_y$ (resp. $\mathcal{A}_o$). We are concerned by the properties of the aggregate short-run demand of the subgroup given by the formula:

$$\bar{F}(p, s_{-1}) = \int_{\mathcal{A}_y} \bar{f}(s, x) d\nu_y + \int_{\mathcal{A}_o} \bar{f}(p, s_{-1}, x) d\nu_o$$

In the sequel, we assume that market demand exists and is continuously differentiable in $(p, p_{-1}, \ldots, p_{-T}, p_{-T-1})$. Hence, $\bar{F} \in C^1(P^{T+2}, \Re^l_+)$ (this is, indeed, the outcome of the budget identity which yields a finite market demand under the requirement that the mean income is finite, and of the differentiability of the demand function and of the expectations function at the household level).[3].

The heterogeneity of initial endowments is trivially formalized by a continuous density function on a large set.

Assumption 12 *The distributions ν_y and ν_o are defined on complete separable metric spaces of households' characteristics $\mathcal{A}_y$ and $\mathcal{A}_o$, i.e. ν_y and ν_o are probability measures on the σ-field of Borelian subsets of $\mathcal{A}_y$ and $\mathcal{A}_o$. For a given $\omega \in \Omega$ the conditional distributions ν_y/ω and ν_o/ω exist. Furthermore the marginal distributions of initial endowment ω are*

[3]In the static model which does not include households' expectations, when demand is deduced from the maximization progamm of a utility function, differentiability of household demand is not the outcome of "standard assumptions on preferences". Nevertheless Dierker, Dierker and Trockel (1980a, 1980b and 1984) prove that heterogeneity has a smoothing impact through aggregation such that the differentiability of market demand can be deduced from an assumption of heterogeneity alone. More precisely, market demand is differentiable with respect to the price system if household demand is highly dispersed along the price system. Our conjecture is that such a result is still valid when household demand is deduced from a multiperiod maximization programm. Clearly, in this case, some assumption of a high dispersion of the expectations function along current and past price systems will be required to get the differentiability of ψ with respect to $(p_t, p_{t-1}, \ldots, p_{t-T})$.

defined on a compact set Ω *and possess continuous density functions* ρ_y *and* ρ_o *with finite means* $\bar{\omega} = \int_\Omega \omega\rho(\omega)d\omega$ *for* $\rho = \rho_y, \rho_o$.

The heterogeneity of the demand behaviour has not such an immediate formalization, since the set of demand functions and the set of expectations function are not complete separable metric spaces. One solution is to use the theoretical tool of the parameterization. One considers a homeomorphism between a restricted set of demand functions $\mathcal{F}$ and a complete separable metric space Γ. Symmetrically, one can considers a homeomorphism between a restricted set of expectations functions E and a complete separable metric space $\mathcal{E}$. Heterogeneity of demand behaviour is then defined by a measure of heterogeneity of the corresponding distribution on the parameter space $\mathcal{B} = \Gamma \times \mathcal{E}$. This means that, in a given subgroup, each demand behaviour is parameterized by a vector of parameters $\beta \in \mathcal{B}$[4]. The vector of parameters β is decomposed into two subvectors γ and ϵ, where γ describes the household's tastes and risk aversion, that is the household demand function, and ϵ parameterizes the household expectations function. First, we assume that there exists a probability density η on $\mathcal{B}$ such that

$$\int_{\mathcal{A}} f(s_t, x_t) d\nu/x = \int_{\mathcal{E}} f^{\beta}(s_t, x_t)\eta_x(\beta)d\beta$$

Denote $\beta_y \in \mathcal{B}_y$ the vector of parameters which describes the demand behaviour of the young household and $\beta_o \in \mathcal{B}_o$ the vector of parameters which describe the behaviour function of the old household. In the sequel, the restrictions imposed on the distribution of demand behaviour are required for both types of household, that is for $\beta = \beta_y$ (respectively $\beta = \beta_o$), $\nu/x = \nu_y/x$ (resp. $\nu/x = \nu_0/x$ and $\mathcal{B} = \mathcal{B}_y$ (resp. $\mathcal{B} = \mathcal{B}_o$). We shall make explicit the formalization for young households, and let the reader deduce the symmetrical formalization for old households.

[4] It might seem too restrictive to assume that the whole population follows a specific parametric model. The population is then decomposed into subgroups which can be decribed by distinct parametric models of demand. In each subgroup the demand function of a household is described by $\beta \in \mathcal{B}$.

Secondly, we assume that there exists a probability density ϕ on $\mathcal{E}$ such that

$$\int_{\mathcal{A}} f(s_t, x_t) d\mu/(x,\gamma) = \int_{\mathcal{E}} f^{\gamma}(p_t, \psi^{\epsilon}(s_t), x_t)\phi_{x,\gamma}(\epsilon)d\epsilon$$

Denote (x, γ)-households the subgroup of households with the current income x and a demand function described by γ. The density function of the distribution of β can be decomposed

$$\eta_x(\beta) = \int_{\Gamma} \phi_{x,\gamma}(\epsilon)\varphi_x(\gamma)d\gamma$$

Assumption 13 *1. Γ and $\mathcal{E}$ are compact subsets of complete, separable metric spaces.*[5]

2. The density φ and ϕ are continuously differentiable.

The question is now to quantify heterogeneity. For this purpose we shall adopt Grandmont's formalization of strong heterogeneity by a "flat" density of the parameter distribution and introduce the following measure of heterogeneity:

$$h(\eta, \rho) = \max_{i=1,\ldots,l} \int_{\mathcal{B}\times\Omega} \mid \partial_{\beta_i}\eta(\beta) \mid \rho(\omega)d\beta d\omega$$

Heterogeneity of the group of households increases as $h(\eta, \rho)$ converges towards. For a compact set $\mathcal{B}$ when $h(\eta, \rho) = 0$ η is the uniform distribution. Note that this measure is the aggregation over initial endowments of the one introduced by Grandmont (1992). This means that we only require that the distribution of the parameter β is flat in *tendency*, that is in average over the distribution of endowments. The idea is to exploit as well the aggregation over endowments which has by itself a structurizing impact (see for example Hildenbrand (1983)).

The problem arising with this formalization is that heterogeneity of the parameter distribution is a formal mathematical property which has no a priori relation to heterogeneity of demand behaviour, which refers to

[5] Γ and $\mathcal{E}$ might be, for example, the closures of open subspaces of $\Re^m$, where m is a positive whole number.

the distribution of the *function* $\bar{f}$. As an illustration, consider an extreme case. Suppose that $\mathcal{B} = \Re$, and that the parameterization of demand behaviour functions is such that $\bar{f}^{\beta} \neq \bar{f}^{\beta'}$ for all $\beta, \beta' \in [0,1]$ with $\beta \neq \beta'$, while $\bar{f}^{\beta} = \bar{f}^{0}$ if $\beta \notin [0,1]$. In this case a flat density of the parameter distribution means that a vast majority of the subpopulation possesses the same function $\bar{f}^{0}$. Hence, the set of parameterization under consideration has to be restricted such that a flat distribution on the parameter space induces very heterogenous demand vectors on the commodity space for given price systems $p_t, p_{t-1}, \ldots, p_{t-T}$, which are themselves induced by heterogeneous functions $\bar{f}$. Then, the parameterization will be interpretable in the sense that, first the degree of heterogeneity of the demand behaviour is evaluated by a measure of heterogeneity of the distribution of the *function* $\bar{f}$, secondly a high degree of heterogeneity induces a high dispersion of the demand vectors on the commodity space which might be induced by heterogeneous demand behaviour functions. Intuitively we speak of extremely heterogeneous groups of households if the two following postulates hold.

> **Postulate 1:** The class of admissible functions is large.
>
> **Postulate 2:** Demand behaviour functions of the group of households can be considered as chosen quite uniformly in a sense to be made precise from a large subset of the space of admissible functions.

Given the previous measure of heterogeneity, one has to restrict the class of parameterizations under consideration such that a flat distribution on the parameter space does mean a high heterogeneity of demand behaviour, that is such that it is compatible with the two above postulates. For this purpose we shall introduce some restrictions on the parameterization of the demand behaviour, as proposed by Kneip (1993). The main idea is to ensure that two significantly different parameters induce two significantly different demand behaviours at given price systems and endowments. That is, the *structure of the parameter space* has to reflect the *structure of the set of demand behaviour functions.* The idea of the restrictions imposed on the parameterization of the demand behaviour function is to compare the distribution of household demands with the distribution of the demands of these households when one argument has been affected. That is heterogeneity is defined in terms of dispersion along the arguments of the function under consideration. This means that we compare, for example, the distribution of the vector $\bar{f}^{\beta}(p_t, p_{t-1}, \ldots, p_{t-T}, x) = f^{\gamma}(p_t, x_t, \psi^{\epsilon}(p_t, p_{t-1}, \ldots, p_{t-T}))$ on the commodity space (denote ϑ its density function) with the one of the vector $\bar{f}^{\beta}(I_{\triangle}p_t, p_{t-1}, \ldots, p_{t-T}, x) = f^{\gamma}(I_{\triangle}p_t, x_t, \psi^{\epsilon}(I_{\triangle}p_t, p_{t-1}, \ldots, p_{t-T}))$. on the commodity space (denote $\vartheta_{\triangle}$ its density function), where $I_{\triangle}$ is a diagonal matrix with the k-th diagonal element $1 + \triangle_k$. The idea is that an extreme degree of heterogeneity should imply that both distributions are very flat. Hence, ϑ and $\vartheta_{\triangle}$ are of comparable degree of heterogeneity and incidently $\vartheta_{\triangle}$ should not be highly affected by a variation of $\triangle$. Then, we shall say that demand behaviour is highly dispersed along the price system.

Clearly, heterogeneity of demand behaviour is induced by two kinds of heterogeneity; heterogeneity due to the high dispersion of preferences and risk aversion which induces a high dispersion of the demand function along each of its arguments (including the expected price vector p_t^e) and the heterogeneity due to the dispersion of the expectations function itself.

In this paper, we formalize explicitely the two types of heterogeneity. We shall, thereby, consider the two measures of heterogeneity:

$$\begin{aligned} h(\varphi, \phi \times \rho) &= \max_{i=1,\ldots,l} \int_{\mathcal{B}\times\Omega} \mid \partial_{\gamma_i}\varphi(\gamma) \mid d\gamma\phi(\epsilon)d\epsilon\rho(\omega)d\omega \\ h(\phi, \varphi \times \rho) &= \max_{i=1,\ldots,l} \int_{\mathcal{B}\times\Omega} \mid \partial_{\epsilon_i}\phi(\epsilon) \mid d\epsilon\varphi(\gamma)d\gamma\rho(\omega)d\omega \end{aligned}$$

We shall restrict the set of parameterizations of the demand function under consideration such that a flat distribution on the parameter space Γ is in accordance with the two previous postulates when the function under consideration is the demand function. The parameterization has to be interpretable in the sense that, the degree of heterogeneity of the distribution of γ reflects some degree of heterogeneity of the distribution on the set of demand *functions* $\{f^{\gamma} \mid \gamma \in \Gamma\}$. More precisely, we impose that a high degree of heterogeneity of the parameter distribution induces a high dispersion of the demand vectors on the commodity space which might be induced by heterogeneous demand functions. Symmetrically, we shall restrict the set of parameterizations of the expectations function under consideration such that a flat distribution on the parameter space $\mathcal{E}$ is in accordance with the two previous postulates when the function under consideration is the expectations function. The parameterization has to be interpretable in the sense that, the degree of heterogeneity of the distribution of ϵ reflects some degree of heterogeneity of the distribution on the set of expectations *functions* $\{\psi^\epsilon \mid \epsilon \in \mathcal{E}\}$. More precisely, we impose that a high degree of heterogeneity of the parameter distribution induces a high dispersion of the expected price on the price set which might be induced by heterogeneous demand functions.

As far as heterogeneity of the demand function is concerned we consider Kneip's definition of heterogeneity, where heterogeneity of the demand function is defined in terms of dispersion of the demand function along one or the other of its arguments. A strong heterogeneity is formalized by a flat density of the parameter distribution on Γ for parameterizations of demand such that "flatness" implies effectively some heterogeneity of the demand function with respect to the argument considered. Note that, by opposite to Kneip's model, the vector of expected prices is now an argument of the demand function. We introduce a new type of heterogeneity, that is we shall assume as well that the demand function might be heterogeneous along the expected price vector. This means that the way expectations affect demand might as well be highly heterogeneous among households.

Heterogeneity of the demand function is formalized by a small value of $h(\varphi, \phi \times \rho)$. We first require that

Assumption 14 *There exists a finite parameter* $m_1 > 0$ *such that*

$$h(\varphi, \phi \times \rho) < m_1$$

As a result m_1 is a measure of heterogeneity of the demand function, as m_1 converges towards zero the demand function is more heterogeneous.

In order for a small parameter m_1 to mean effectively a high heterogeneity of the demand function (since this restriction affects only the parameter distribution) we require the parameterization of demand to satisfy assumptions below.

Typically, the distribution of demands at the actual price systems and income level and the distribution of demands for an hypothetical change of one argument do not have the same support. When for example the demand function satisfies the budget identity the support of the distribution of $f^{\beta}(p, p^e_{t+1}, x)$ in the commodity space is the hyperplane $\{y \in \Re^l_+ \mid p^T y = x\}$ while the support of the distribution of $f^{\beta}(p, p^e_{t+1}, (1+\triangle)x)$ in the commodity space is the hyperplane $\{y \in \Re^l_+ \mid p^T y = x(1+\triangle)\}$ which is parallel to the previous one with a higher norm. Hence, we shall work with the budget share expenditure functions $e^{\beta}(p, p^e_{t+1}, x) = p^T f^{\beta}(p, p^e_{t+1}, x)/x$ in order to allow for the comparison of density functions.

Consider for example the requirement that a flat density of the parameter distribution is interpretable as a strong heterogeneity of the demand function along the current price system.

The first condition on the parameterization allowing the above interpretation is in accordance with Postulate 1, and in particular, requires that there is a very large number of type of reaction of the demand function to a current price change. For a given (x, ϵ)-subpopulation, one can always associate to a given household another one who would have spent, at the same income level and for the same expected price vector, the same budget share on each commodity at a distinct current price system. To be more specific we require that:

Whatever $p, p^e_{t+1} \in \Re^l_+$, $x \in \Re_+$, $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such for any $(l+1)$-dimensional vector $\triangle = (\triangle_1, \ldots, \triangle_{l+1})^T$ close to the nul vector,

$$e^{\gamma'}(p, p^e_{t+1}, x) = e^{\gamma_y}(I_{\triangle} p, p^e_{t+1}, x) \tag{1}$$

This condition imposes a minor restriction in the sense that, if the class of demand functions $\{e^{\gamma} \mid \gamma \in \Gamma\}$ is large, then γ' will not be determined uniquely. Furthermore, this assumption is specified more in terms of vectors on the commodity space than in terms of functions. It requires that the class of demand vectors $\{e^{\gamma}(p, p^e_{t+1}, x) \in \Re^l_+ \mid \gamma \in \Gamma\}$ is large given (p, p^e_{t+1}, x). A high dispersion on the commodity space does not necessarily mean a high dispersion on the set of the budget share expenditure *functions*. Hence, we additionally require that for $\gamma, \gamma^* \in \Gamma$ close to each other the parameters $\gamma', \gamma^{*'} \in \Gamma$ with $e^{\gamma'}(p, p^e_{t+1}, x) = e^{\gamma}(I_\triangle p, p^e_{t+1}, x)$ and $e^{\gamma^{*'}}(p, p^e_{t+1}, x) = e^{\gamma^*}(I_\triangle p, p^e_{t+1}, x)$ can be chosen in such a way that

$$\| e^{\gamma'}(p, p^e_{t+1}, x) - e^{\gamma^{*'}}(p, p^e_{t+1}, x) \| = \| e^{\gamma}(I_\triangle p, p^e_{t+1}, x) - e^{\gamma^*}(I_\triangle p, p^e_{t+1}, x) \|$$

carries over

$$\| e^{\gamma'} - e^{\gamma^{*'}} \|_2 = \| e^{\gamma} - e^{\gamma^*} \|_2 \tag{2}$$

where $\| e^{\gamma} \|_2 = \sup_{q, q^e \in P} \sup_{y \in \Re_+} \| e^{\gamma}(q, q^e, y) \|$. Note that $\| \,.\, \|_2$ is a norm of the space of budget share *functions* $\{e^{\gamma} \mid \gamma \in \Gamma\}$. Hence, this second restriction affects the set of admissible functions of the subgroup. Clearly, Eq.1 and Eq.2 are fulfilled if the parameter space as well as the class of budget share expenditure functions $\{e^{\gamma} \in \mid \gamma \in \Gamma\}$ are large enough. It requires that for any current price system at a given expected price vector and a given income there exists in a neighborhood of the associated expenditure vector in the space of the budget share expenditure functions a continuum of heterogeneous budget share functions describing demands of other agents. Furthermore, demand functions are distinct in a continuous way, where continuity is defined with respect to the current price system. It follows from Eq.1 and Eq.2 that given p, p^e_{t+1}, x one can define mappings $\tau_\triangle$ from Γ onto Γ such that

$$e^{\tau_\triangle(\gamma)}(p, p^e_{t+1}, x) = e^{\gamma}(I_\triangle p, p^e_{t+1}, x)$$

with $\tau_0(\gamma) = \gamma$. In addition to $\triangle$, $\tau_\triangle$ might depend on p, p^e_{t+1} and x despite the notation does not state it, but one should always keep in mind this possible dependence. It seems to be a minor restrictions to require that the $\tau_\triangle$ be homeomorphisms from Γ onto Γ.

Note that γ' corresponds to γ if e^{γ} is invariant through the transformations under consideration, more precisely if the budget share expenditure functions are independent of the current price system. This feature characterized Cobb-Douglas demand functions. Therefore, the above assumption holds when the set of demand functions is any small subset of the class of Cobb-Douglas demand functions. Hence, we add the restriction that whatever $\gamma, \gamma^* \in \Gamma$ with $\gamma \neq \gamma*$ then $e^\gamma \neq e^{\gamma^*}$. To conclude we require that the class of admissible demand functions induced by the parameterization is large enough to be compatible with the following regularity conditions:

Assumption 15 *1. For a given $\triangle \leq 0$ the transformation $\tau_\triangle$ is a homeomorphism from Γ onto Γ. It holds $\tau_0(\gamma) = \gamma$ for all $\gamma \in \Gamma$. Furthermore, $\tau_\triangle$ and its inverse function $\tau_\triangle^{-1}$ are twice continuously differentiable functions of $(\triangle, \gamma)$.*

2. There exists a constant $C > 0$ such that $\| e^{\tau_\triangle(\beta)} - e^\gamma \|_2 \leq C\triangle$ forall $\gamma \in \Gamma$.

3. There is a $\delta > 0$ such that $\| e^{\tau_\triangle(\gamma)} - e^{\tau_\triangle(\gamma^*)} \|_2 = \| e^\gamma - e^{\gamma^*} \|_2$ holds for all $\gamma, \gamma^* \in \Gamma$ with $\| \gamma - \gamma^* \| < \delta$.

The density function φ of the distribution of the parameters γ induces a density function $\varphi_\triangle$ of the distribution of the transformed parameters $\gamma' = \tau_\triangle(\gamma)$, where:

$$\varphi_\triangle(\gamma) = \det(\partial_\gamma \tau_\triangle^{-1}(\gamma)).\varphi(\tau_\triangle^{-1}(\gamma))$$

For flatness of φ to possess any connection with the idea of heterogeneity of the demand function in accordance with Postulate 2, we require another restriction on the parameterization of demand.

A "strong heterogeneity" defined by a flat distribution on the parameter space Γ means highly heterogeneous demand functions with respect to the current price system, if the demand function is highly heterogeneous in terms of its reaction to a current price change. Intuitivelly, this should imply that for any small perturbations of the current price system the density function $\varphi_\triangle$ is also flat. That is, φ and $\varphi_\triangle$ are of a comparable degree of heterogeneity. We require that an interpretable parameterization reflects this fact. More precisely, there exists a constant

$c < \infty$ such that for every sufficiently small $\triangle$

$$ch(\varphi) \geq h(\varphi_\triangle)$$

Under this condition small values of $h(\varphi)$ imply small values of $h(\varphi_\triangle)$. It is easy to show (see Appendix A.1) that this property is implied by the assumption

Assumption 16 *The functions $\partial_{\gamma_r} \tau_\triangle^{-1}(\gamma)$ $\forall r$ are bounded, furthermore,*

$$\partial_{\gamma_r} det(\partial_\gamma \tau_\triangle^{-1}(\gamma)) \mid_{\triangle=0} = 0$$

To prove our main result we need indeed a slightly different assumption:

Assumption 17 *The functions $\partial_{\triangle_r} \tau_\triangle^{-1}(\gamma)$ $\forall r$ are bounded, furthermore,*

$$\partial_{\triangle_r} det(\partial_\gamma \tau_\triangle^{-1}(\gamma)) \mid_{\triangle=0} = 0$$

This assumption implies the very interesting property that

Property 1 *There exists $q > 0$ such that*

$$qh(\varphi) \geq \int_\Gamma \mid \partial_{\triangle_r} \varphi_\triangle(\gamma) \mid d\gamma = h^*(\varphi)$$

This property formalizes the intuition when the density function of the parameter distribution is flat, that is $h(\varphi)$ is small, then the density $\varphi_\triangle$ is not highly affected by a small variation of $\triangle$. If $h(\varphi, \phi \times \rho)$ is close to zero then $h^*(\varphi, \phi \times \rho)$ is as well close to zero. Furthermore, Kneip shows that Assumption 17 follows from previous assumptions if the model is interpretable in the sense that the heterogeneity of the parameter distribution represents heterogeneity of the distribution of demand functions. More precisely, the structure of the prameter space Γ reflects the structure of the set of budget share functions $\{e^\gamma \mid \gamma \in \Gamma\}$, $\varphi(\gamma)$ describes the probability structure of small neighborhoods of e^γ in the set of demand functions $\{e^\gamma \mid \gamma \in \Gamma\}$. Formally, there exists a function $\lambda : \Re_+ \mapsto \Re_+$ with $\lambda(\delta) \mapsto 0$ when $\delta \mapsto 0$ such that

$$\lim_{\delta \mapsto 0} \frac{P[e^c \in U_\delta(e^\gamma)]}{\lambda(\delta)} = \varphi(\gamma)$$

for any γ belonging to the interior of Γ, where γ are considered as the realizations of a random variable c which possesses the density φ and $U_\delta(e^\gamma) = \{e \in \{e^\gamma \mid \gamma \in \Gamma\} \mid\| e^\gamma - e \|_2 \leq \delta\}$.

If φ is very flat, then this condition implies that for very different functions e and e^* functions from a neighborhood $U_\delta(e)$ are approximately equally likely as functions from a neighborhood $U_\delta(e^*)$.

For a flat distribution on the parameter space Γ to imply a high dispersion of the demand function along the expected price vector or along income the restrictions required on the parameterization of demand are symmetrical. Note that when the dispersion along income is concerned the vector of perturbations $\triangle$ is reduced to one parameter, since income is a one-dimensional variable. If the parameterization satisfies the restrictions 15 and 17 for all arguments of the demand function then a flat density function of the parameter space is interpretable as a strong heterogeneity of the demand function along any of its arguments. Hence all "dimensions" of the demand function are considered and, intuitivelly, the resulting heterogeneity gets closer to the one stipulated in Postulates 1 and 2. Nevertheless, to get the equivalence with the two postulates stronger restrictions are clearly required. To make this slightly more precise, one can observe that the previous restrictions are formalized in terms of the reaction of the demand function to a variation of one of its arguments that is in terms of partial derivatives which are not enough to specify a function. To define heterogeneity in terms of the *functions* themselves global derivatives and derivatives of higher levels should be also considered.

What is at stake is not to define a theoretical concept of heterogeneity for art sake but to generate propositions capable of empirical refutations. Incidently, it is relevant to analyse heterogeneity of the demand function in terms of the resulting observable dispersion of the demand vector on the commodity space. Furthermore, it is very important to distinguish heterogeneity of the function along one or the other of its arguments, since the heterogeneity along one direction might have more realistic implications than when apprehended along another direction.

As an example, Kneip (1993) shows that a high enough degree of heterogeneity of the demand function along income, in tendency, ensures the positive semidefiniteness of the agggregate matrix of income effect. This property, which is equivalent to the assumption of Hildenbrand (1994) of increasing spread of household demand, is well supported by empirical data (see e.g. Hildenbrand (1994) and Kneip(1993)). By opposite, a strong heterogeneity of the demand function along the current price system, in tendency, (when demand is defined as a function of the current prices and income) has the unrealistic implication that the Jacobian matrix of market demand is almost diagonal with negative entries, this means that markets are perceived as separate at the aggregate.

To make explicit the distinct "dimensions" of the heterogeneity of the demand function we introduce a further notation. When the parameterization is restricted to allow for the interpretation of a flat density φ as a strong heterogeneity of the demand function along *income*, we denote as $m_{1,x} > 0$ the upper bound of

$$h^*(\varphi, \phi \times \rho) = \int_{\mathcal{B}\times\Omega} \mid \partial_{\triangle_r}\varphi_{\triangle}(\gamma) \mid \phi(\epsilon)\rho(\omega)d\beta d\omega < m_{1,x}$$

where $\triangle$ is any small perturbation of income. Under the assumptions 15 and 17, $m_{1,x}$ can be interpreted as a degree of heterogeneity of the demand function along income. As $m_{1,x}$ converges towards zero the demand function is more heterogeneous in terms of its reaction to a variation of income. We denote symmetrically $m_{1,p}$, $m_{1,p_{-1}}$ and m_{1,p^e} as the degrees of heterogeneity of the demand function along, respectively, the current price system, last period prices and the vector of expected prices. As established in Property 1 requiring a small m_1 is a stronger assumption than requiring a strong heterogeneity along one specific direction by stipulating a small parameter $m_{1,j}$ for $j \in \{p, p_{-1}, p^e, x\}$.

Remark 4 Note that, under the assumption of homogeneity of the demand function with respect to (p_t, p^e_{t+1}, x) (respectively with respect to (p_{t-1}, p_t, p^e_t, x) for old households), if the demand function is heterogeneous along the expected price vector and the current price system (resp.

along p_{t-1}, p_t and p_t^e) it is in addition heterogeneous along the income level, for the proof see Appendix A.2.

To define heterogeneity of the expectations function along its arguments, we transpose the previous definition of heterogeneity, in terms of the demand function introduced by Kneip (1993), to the case of the expectations function. Again, a strong heterogeneity is formalized by a flat distribution of the parameter ϵ in tendency. We require enough regularity of the density function ϕ to allow for the interpretation of a flat density of the parameter distribution as a strong heterogeneity of the expectations function along one or the other of its arguments.

Assumption 18 *There exists a finite parameter $m_2 > 0$ such that*

$$h(\phi, \varphi \times \rho) = \max_{i=1,\ldots,l} \int_{\mathcal{B}\times\Omega} \mid \partial_{\epsilon_i}\phi(\epsilon) \mid \varphi(\gamma)\rho(\omega) d\epsilon d\gamma d\omega < m_2$$

m_2 is a measure of heterogeneity of the expectations function in tendency as m_2 converges towards zero the expectations function is more heterogeneous. Again, to ensure that the structure of the parameter space reflects in some sense the structure of the set of expectations functions $\{\psi^\epsilon \mid \epsilon \in \mathcal{E}\}$ we impose some restrictions on the parameterization.

Before going any further note that, for conceptual consistency, we should restrict the degree of heterogeneity of the expectations function. In the sequel, we shall be concerned with the evolution of temporary equilibria. More precisely, we shall focus on the local asymptotic stability of stationary temporary equilibria. The notion of such an equilibrium yields the natural assumption of stationarity of expectations (assumption 5). This means that if the household has always observed the price p in past periods it should expect p to be also the future price. For non stationary prices, that is for any sequence of past prices where the equalities $p_{-1} = \ldots = p_{-T} = p$ do not hold simultaneously, we require a high dispersion of the expectations function along each of its arguments.

Consider, for example, the requirement that the expectations function is highly heterogeneous along the current price system. This requirement is in accordance with Postulate 1, in the sense that there is a very large number of admissible reactions of the expectations function to a current price change, if the parameterization of the expectations function satisfies:

Whatever $s \in P^{T+1}$, where $(p = p_{-1} = \ldots = p_{-T})$ does not hold, and whatever $\epsilon \in \mathcal{E}$, for any $(l+1)$-dimensional vector $\triangle$ close to the nul vector there exist $\epsilon' \in \mathcal{E}$ such that

$$\psi^{\epsilon'}(p, p_{-1}, \ldots, p_{-T}) = \psi^{\epsilon}(I_{\triangle}p, p_{-1}, \ldots, p_{-T}) \qquad (3)$$

This condition implies a high heterogeneity of the expectations function near any stationary price. For example, its heterogeneity along the current price system is such that:

> Whatever $p \in P$ and $\epsilon \in \mathcal{E}$, for any $(l+1)$-dimensional vectors $\triangle \neq 0$ and $\triangle' \neq 0$ close to the nul vector there exist $\epsilon' \in \mathcal{E}$ such that
>
> $$\psi^{\epsilon'}(I_{\triangle'}p, p, \ldots, p) = \psi^{\epsilon}(I_{\triangle}I_{\triangle'}p, p, \ldots, p)$$

This condition imposes a minor restriction in the sense that, if the class of expectations functions $\{\psi^{\epsilon} \mid \epsilon \in \mathcal{E}\}$ is large, then ϵ' will not be determined uniquely. Furthermore, this condition is specified more in terms of vectors on the price set than in terms of functions. It requires that the class of demand vectors $\{\psi^{\epsilon}(s) \in P \mid \epsilon \in \mathcal{E}\}$ is large given s. Again, a high dispersion on the price set does not necessarily mean a high dispersion on the set of the expectations *functions*. Hence, we additionally require that for $\epsilon, \epsilon^* \in \mathcal{E}$ close to each other the parameters $\epsilon', \epsilon^{*'} \in \mathcal{E}$ with $\psi^{\epsilon'}(s) = \psi^{\epsilon}(I_{\triangle}p, p_{-1}, \ldots, p_{t-T})$ and $\psi^{\epsilon^{*'}}(s) = \psi^{\epsilon^*}(I_{\triangle}p, p_{-1}, \ldots, p_{-T})$ can be chosen in such a way that

$$\| \psi^{\epsilon'}(s) - \psi^{\epsilon^{*'}}(s) \| = \| \psi^{\epsilon}(I_{\triangle}p, p_{-1}, \ldots, p_{-T}) - \psi^{\epsilon^*}(I_{\triangle}p, p_{-1}, \ldots, p_{-T}) \|$$

carries over

$$\| \psi^{\epsilon'} - \psi^{\epsilon^{*'}} \|_2 = \| \psi^{\epsilon} - \psi^{\epsilon^*} \|_2 \qquad (4)$$

where $\| \psi^{\epsilon} \|_2 = \sup_{q \in P^{T+1}} \| \psi^{\epsilon}(q) \|$. Note that $\| \, . \, \|_2$ is a norm of the space of expectations functions $\{\psi^{\epsilon} \mid \epsilon \in \mathcal{E}\}$. Hence, this second restriction affects the set of admissible functions of the subgroup. Clearly, Eq.3

and Eq.4 are fulfilled if the parameter space $\mathcal{E}$ as well as the class of expectations functions $\{\psi^\epsilon \in | \ \epsilon \in \mathcal{E}\}$ are large enough. It requires that for any current price system at given past price systems there exists in a neighborhood of the associated expected price vector in the space of expectations functions a continuum of heterogeneous expectations functions describing the learning process of other households. Furthermore, expectations functions are distinct in a continuous way, where continuity is defined with respect to the current price system. It follows from Eq.3 and Eq.4 that given s one can define mappings $\tau_\triangle$ from $\mathcal{E}$ onto $\mathcal{E}$ such that

$$\psi^{\tau_\triangle(\epsilon)}(s) = e^\gamma(I_\triangle p, p_{-1}, \ldots, p_{-T})$$

with $\tau_0(\epsilon) = \epsilon$. In addition to $\triangle$, $\tau_\triangle$ might depend on s despite the notation does not state it. One should always keep in mind this possible dependence. It seems to be a minor restrictions to require that the $\tau_\triangle$ be homeomorphisms from $\mathcal{E}$ onto $\mathcal{E}$.

Note that $\tau_\triangle(\epsilon)$ corresponds to ϵ if ψ^ϵ is invariant through the transformations under consideration, more precisely if expectations are independent of the current price system. As a consequence, we require that, whatever $\epsilon, \epsilon^* \in \mathcal{E}$ with $\epsilon \neq \epsilon^*$ then $\psi^\epsilon \neq \psi^{\epsilon^*}$. To conclude we require that the class of admissible expectations functions induced by the parameterization is large enough to be compatible with the following regularity conditions:

Assumption 19 *1. For a given $\triangle \leq 0$ the transformation $\tau_\triangle$ is a homeomorphism from $\mathcal{E}$ onto $\mathcal{E}$. It holds $\tau_0(\epsilon) = \epsilon$ for all $\epsilon \in \mathcal{E}$. Furthermore, $\tau_\triangle$ and its inverse function $\tau_\triangle^{-1}$ are twice continuously differentiable functions of $(\triangle, \epsilon)$.*

2. *There exists a constant $C > 0$ such that $\| \psi^{\tau_\triangle(\epsilon)} - \psi^\epsilon \|_2 \leq C\triangle$ forall $\epsilon \in \mathcal{E}$.*

3. *There is a $\delta > 0$ such that $\| \psi^{\tau_\triangle(\epsilon)} - \psi^{\tau_\triangle(\epsilon)} \|_2 = \| \psi^\epsilon - \psi^\epsilon \|_2$ holds for all $\epsilon, \epsilon^* \in \mathcal{E}$ with $\| \epsilon - \epsilon^* \| < \delta$.*

The density function ϕ of the distribution of the parameters ϵ induces a density function $\phi_\triangle$ of the distribution of the transformed parameters $\epsilon' = \tau_\triangle(\epsilon)$, where:

$$\phi_\triangle(\epsilon) = \det(\partial_\epsilon \tau_\triangle^{-1}(\epsilon)).\phi(\tau_\triangle^{-1}(\epsilon))$$

Again, for flatness of ϕ to possess any connection with the idea of heterogeneity of expectations function in accordance with Postulate 2. we require another restriction on the parameterization of demand. A "strong heterogeneity" defined by a flat distribution on the parameter space $\mathcal{E}$ means highly heterogeneous demand functions with respect to the current price system, if the demand function is highly heterogeneous in terms of its reaction to a current price change. This will be true if the structure of φ give information on whether the distribution of budget share functions is concentrated at some particular ψ^ϵ. The interpretable parameterization reflects the fact that ϕ and $\phi_\triangle$ are of comparable degree of heterogeneity. More precisely, there exists a constant $c < \infty$ such that for every sufficiently small $\triangle$

$$ch(\phi) \geq h(\phi_\triangle)$$

This property is implied by the following assumption

Assumption 20 *The functions* $\partial_{\epsilon_r} \tau_\triangle^{-1}(\epsilon)$ $\forall r$ *are bounded, furthermore,*

$$\partial_{\epsilon_r} det(\partial_\epsilon \tau_\triangle^{-1}(\epsilon)) \mid_{\triangle=0} = 0$$

To prove our main result we need indeed a slightly different assumption:

Assumption 21 *The functions* $\partial_{\triangle_r} \tau_\triangle^{-1}(\gamma)$ $\forall r$ *are bounded, furthermore,*

$$\partial_{\triangle_r} det(\partial_\epsilon \tau_\triangle^{-1}(\epsilon)) \mid_{\triangle=0} = 0$$

This assumption implies the determinant property that

Property 2 *There exists* $q > 0$ *such that*

$$qh(\phi) \geq \int_{\mathcal{E}} \mid \partial_{\triangle_r} \phi_\triangle(\epsilon) \mid d\epsilon = h^*(\phi)$$

Again, this property formalizes the intuition that when the density function of the parameter distribution is flat, that is $h(\phi)$ is small, then the density $\phi_\triangle$ is not highly affected by a small variation of $\triangle$. If $h(\varphi, \phi \times \rho)$ is close to zero then $h^*(\varphi, \phi \times \rho)$ is as well close to zero.

For a flat density of the parameter distribution to yield a high heterogeneity of the expectations function along past price systems the restrictions required on the parameterization of expectations are symmetrical. Again, to distinguish the different dimensions of heterogeneity of the expectations function we introduce a further notation. When the parameterization is restricted to allow for the interpretation of a flat density ϕ as a strong heterogeneity of the expectations function along the price system p_{t-j} $j = 0, 1, \ldots, T+1$, we denote as $m_{2,t-j} > 0$ the upper bound of

$$h^*(\phi, \varphi \times rho) = \int_{\mathcal{B}\times\Omega} \mid \partial_{\triangle_r}\phi_\triangle(\epsilon) \mid \varphi(\gamma)\rho(\omega)d\beta d\omega < m_{2,t-j}$$

where $\triangle$ denotes any small perturbation of the price system p_{t-j}. Under the assumptions 19 and 21, $m_{2,t-j}$ can be interpreted as a degree of heterogeneity of the expectations function along the price system p_{t-j}. As $m_{2,t-j}$ converges towards zero the expectations function becomes more heterogeneous in terms of its reaction to a variation of p_{t-j}.

1.3 Heterogeneity and Aggregate Learning Process

Consider a function describing households' characteristics. If it is highly dispersed in the population along its k-th argument, then intuitivelly the aggregate function must be less sensitive to the k-th argument. Effectively, negative individual impacts will compensate positive individual impacts, since they are almost equally likely, in such a way that the aggregate impact is negligible. The purpose of this subsection is to prove this result when applied to the expectations function and to the dependence of the demand function on the expected price vector.

We shall first make some remarks about the learning process. It is worth while noting that, as observed by Grandmont and Laroque (1991), the expectations functions considered in the present analysis are a formulation compatible with a specification where a household has a set of a priori beliefs about the dynamics of his environment indexed for instance by a vector of unknown parameters. The household revises in each period its estimate of the unknown parameters, in view of the past data, by using statistical techniques and use the model corresponding to the new estimates to form expectations. For predictions obtained by OLS regressions during the learning period, see, for example, Fourgeaud et al. (1986) in a rational expectations model or Brousseau and Kirman (1991) in a model of game theory. As suggested by Grandmont and Laroque (1990) the eigenvalues of the Jacobian of $\partial\psi^{\epsilon}$ reflect the prior beliefs of the household. They describe the local regularities that households are able to recognize from past history near the stationary state. If the storage memory T is large the eigenvalues can be distributed on the complex plane so as to approximate any distribution with a continuous density, which we interpret as the households' prior. Thus, there is no reason to exclude a priori that the expectations function ψ puts some positive weight on eigenvalues that are of specific value. However, restricting the set of admissible eigenvalue is less demanding when the *mean* expectations function is concerned. Effectively, the relative weight of the subgroup of households giving a positive probability to eigenvalues with, for example, a high modulus can be nul, so that none of the eigenvalue of the mean expectations function is of high modulus.

Indeed, our assumption of heterogeneity, that is a high dispersion of the expectations function with respect to all its arguments, implies that the aggregate parameters $C_j = \int_{\mathcal{E}} C_j^{\epsilon}\phi_y(\epsilon)d\epsilon$ and $D_j = \int_{\mathcal{E}} D_j^{\epsilon}\phi_o(\epsilon)d\epsilon\ \forall j$ are small. That is, at any stationary price the mean expectation mapping is almost not affected by a small change of one of its argument. This restricts the modulus of the eigenvalues of the mean learning process. Consider, for example, the derivative of the mean expectation mapping with respect to the current price system evaluated at $(I_{\triangle'}p, p, \ldots, p)$, where $\triangle'$ is a $(l+1)$-dimensional vector close to the nul vector.

$$\partial_{p_{t-k}} \int_{\mathcal{E}} \psi^{\epsilon}(I_{\triangle'}p, \ldots, p)\phi(\epsilon)d\epsilon$$

$$= \partial_{\triangle} \int_{\mathcal{E}} \psi^{\epsilon}(p, \ldots, I_{\triangle} I_{\triangle'} p, \ldots, p) \mathcal{P}^{-1} \phi(\epsilon) d\epsilon \mid_{\triangle=0}$$
$$= \int_{\mathcal{E}} (p, \ldots, \psi^{\epsilon}(I_{\triangle'} p, \ldots, p) \mathcal{P}^{-1} \partial_{\triangle} \phi_{\triangle}(\epsilon) d\epsilon) \mid_{\triangle=0}$$
$$= \int_{\mathcal{E}} \psi^{\epsilon}(p, \ldots, I_{\triangle'} p, \ldots, p) \mathcal{P}^{-1} \partial_{\triangle} \phi_{\triangle}(\epsilon) \mid_{\triangle=0} d\epsilon \quad \forall k$$

where $\mathcal{P}$ is the diagonal matrix with the vector p on the diagonal. Thus integrating over the whole set of households' characteristics[6]

$$\| \partial_{p_{t-k}} \int_{\mathcal{B} \times \Omega} \psi^{\epsilon}(p, \ldots, I_{\triangle'} p, \ldots, p) \phi(\epsilon) \varphi(\gamma) \rho(\omega) d\epsilon d\gamma d\omega \|$$
$$\leq l q_2 m_{2,t-k} \| \psi^{\epsilon}(p, \ldots, I_{\triangle} p, \ldots, p) \mathcal{P}^{-1} \| \quad \forall k$$

By assumption, the partial derivatives of the mean expectations function are continuous. Hence,

$$\lim_{\triangle' \to 0} \partial_{p_{t-k}} \int_{\mathcal{E}} \psi^{\epsilon}(I_{\triangle'} p, p, \ldots, p) \phi(\epsilon) d\epsilon = \partial_{p_t} \int_{\mathcal{E}} \psi^{\epsilon}(p, \ldots, p) \phi(\epsilon) d\epsilon \quad \forall k$$

and consequently,

$$\| C_k \| = \| \partial_{p_{t-k}} \int_{\mathcal{E} \times \Omega} \psi^{\epsilon}(p, \ldots, p) \phi(\epsilon) \varphi(\gamma) \rho(\omega) d\epsilon d\gamma d\omega \| \leq l q_2 m_{2,t-k} \quad \forall k$$

The norm of the Jacobian matrix $\partial \int_{\mathcal{B} \times \Omega} \psi^{\epsilon} d\epsilon$ becomes negligible when $h(\phi, \varphi \times \rho)$ is close to zero.

Note that this "insensitivity" of the mean expectations function is obtained through aggregation by heterogeneity. Nevertheless, the Jacobian matrix of a household expectations function is allowed to be any matrix. According to a well known result of dynamic theory if all eigenvalues of the Jacobian matrix of the learning process are of modulus lower than one (respectivelly higher than one) then the learning process described by ψ^{ϵ} is stable (respectivelly unstable), $\forall \epsilon \in \mathcal{E}$. This means, as

[6]The $(n \times n)$ matrix norm consider in this paper is the row sum norm, that is the norm of a matrix A is given by the formula:

$$\| A \| = \max_{1 \leq j \leq n} \sum_{k=1}^{n} | a_{jk} |$$

proved in Appendix A.12, that, at the household level, the learning process may be unstable (its Jacobian matrix may have a norm higher than one) while heterogeneity through aggregation leads to a stable aggregate learning process, as long as $\sum_{j=0}^{T} lq_2 m_{2;t-j} < 1$ (its Jacobian matrix has a norm lower than the unity).

However, the requirement that expectations are stationary leads to restrict the degree of heterogeneity. Furthermore, note that we can extend the assumption of stationarity of expectations to the extrapolation of any periodic orbit. Assuming perfect foresight along any possible sequence of temporary equilibria amounts to postulating that each trader knows beforehand the structural characteristics of the economy. In a decentralized economy such an information is typically not available. Households know that the economy may converge to a stationary state, or more generally to a periodic orbit, however they do not know precisely where these orbits may lie. Therefore, they try to extract from the past data such possible regularities and extrapolate them. If, for example, prices correspond to a sequence s_t that has period k, i.e. $p_{t-j} = p_{t-j-k}$ $\forall j$, then any household will extrapolate this regularity.

$$\begin{array}{llll} \forall \epsilon_y \in \mathcal{E}_y & \psi^{\epsilon_y}(p_t, \ldots, p_{t-T}) & = & p_{t+h-nk} \\ \forall \epsilon_o \in \mathcal{E}_o & \psi^{\epsilon_o}(p_{t-1}, \ldots, p_{t-T-1}) & = & p_{t-1+h-nk} \end{array} \quad (5)$$

for all n such that $1 \leq nk - h \neq T$. Of course, this assumption is consistent if and only if $k < T$. This assumption is a generalization of the stationarity assumption previously introduced. It implies some restrictions on the derivatives of the expectations functions. More precisely, one can prove the following lemma (see Grandmont and Laroque, 1990):

Lemma 1 *Assume that the expectations functions satisfy Eq.5 $\forall \epsilon \in \mathcal{E}$. Then*

$$\sum_{j=0}^{T} C_j^{\epsilon} \lambda^{-j} = \lambda I_l$$

$$\sum_{j=1}^{T+1} D_j^{\epsilon} \lambda^{1-j} = \lambda I_l$$

for every $\lambda = e^{\frac{2\Pi d}{k} i}$ *and* $d = 1, \ldots, k$.

Applying this lemma to the stationary state, that is for $k = 1$, we get

$$\sum_{j=0}^{T} C_j^{\epsilon} = I_l$$
$$\sum_{j=1}^{T+1} D_j^{\epsilon} = I_l$$

The above lemma implies as well that all k-th roots of unity are roots of the polynomials corresponding to the expectations functions, that is $z^{T+1} - \sum_{j=0}^{T} C_j^{\epsilon} z^{T-j} = 0$ (for ψ^{ϵ_y}) or $z^{T+1} - \sum_{j=1}^{T+1} D_j^{\epsilon} z^{T+1-j} = 0$ (for ψ^{ϵ_o}) for all $\epsilon \in \mathcal{E}$. Thus, the degrees of heterogeneity of the expectations function $m_{2,t-k}$, where $\| C_k \| \leq lq_2 m_{2;t-k}$, are restricted such that the parameters C_k satisfy as well the inequalities of Lemma 1. This implies that the aggregate learning process may be unstable. If, for example, households extrapolate a stationary price it holds $\sum_k C_k = I_l$, thus $\sum_k \| C_k \| \geq 1$ which implies that $\sum_{k=0}^{T} lq_2 m_{2;t-k} \geq 1$, that is the sufficient condition for the stability of the mean learning process does not hold. Hence, as far as the expectations function is concerned we cannot consider an extreme degree of heterogeneity, since the parameters $m_{2;t-j}$ are bounded away from zero. The highest dispersion allowed, when $m_{2;t-j} = \bar{m}_2 \ \forall j$, is described by the smallest parameter in accordance with the stationarity requirement, that is $\bar{m}_2 = \frac{1}{(T+1)lq_2} > 0$. However, we shall show that this restricted heterogeneity of the expectations function combined with an assumption of heterogeneity of the demand function ensures the stability of the dynamics of temporary equilibria with learning. Stability of the dynamics with learning is compatible with the instability of the aggregate learning process.

To conclude this section, we observe that the insensitivity of the mean expectations function obtained at the aggregate through heterogeneity may be reinforced by a symmetrical insensitivity of the aggregate demand function to a variation of the expected price vector. The global impact on market demand of a variation of expectations induced by a price change, more precisely the absolute value of the expression $\| \int_{\mathcal{E}} \partial_{p_t^e} f^{\beta} \partial_{p_{t-j}} \psi^{\epsilon} d\nu \| \ \forall j \in \{0, \ldots, T\}$ becomes very small when the degrees of heterogeneity increase, that is when m_1 or m_2 converges towards zero.

By assumption the set of households' characteristics is bounded, hence under the mean value theorem, there exists a household described by $(\tilde{\beta}_y, \tilde{\omega}_y)$ such that

$$\int_{\mathcal{B}\times\Omega} \partial_{p^e_{t+1}} f^{\beta} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_y = \partial_{p^e_{t+1}} f^{\tilde{\beta}_y} \mid_{\tilde{x}=p^T\tilde{\omega}_y} \int_{\mathcal{B}\times\Omega} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_y$$

Denote $\tilde{f}^y_i = \mid \sum_k p_i p_k \partial_{p^e_{t+1;k}} f_i^{\tilde{\beta}_y} \mid$ and $\tilde{f}^y = \max_i \tilde{f}^y_i$. $\tilde{f}^o$ is defined symmetrically. In the same way there exists an expectations function described by $\tilde{\epsilon}$ such that, whatever $r = 1, \ldots, T$:

$$\int_{\mathcal{B}\times\Omega} \partial_{p^e_{t+1}} f^{\beta} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_y = \partial_{p_{t-r}} \psi^{\tilde{\epsilon}_y} \int_{\mathcal{B}\times\Omega} \partial_{p^e_{t+1}} f^{\beta} d\nu_y$$

Denote $\tilde{\psi}^y_{r;i} = \mid \sum_k \partial_{p_{t-r;k}} \psi_i^{\tilde{\epsilon}_y} \mid$ and $\tilde{\psi}^y_r = \max_i \tilde{\psi}^y_{r;i}$. $\tilde{\psi}^o_{r+1}$ is defined symmetrically. We state the following lemma:

Lemma 2 *(i) Under assumptions 12 and 13 there exists a positive constant k such that:*

$$\| \int_{\mathcal{B}\times\Omega} \partial_{p^e_{t+1}} f^{\gamma} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_y + \int_{\mathcal{B}\times\Omega} \partial_{p^e_t} f^{\gamma} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_o \| \leq kh(\varphi, \phi\times\rho) \leq kq_1 m_1$$

(ii) and there exists a positive constant k′ such that

$$\| \int_{\mathcal{B}\times\Omega} \partial_{p^e_{t+1}} f^{\gamma} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_y + \int_{\mathcal{B}\times\Omega} \partial_{p^e_t} f^{\gamma} \partial_{p_{t-r}} \psi^{\epsilon} d\nu_o \| \leq k'h(\phi, \varphi\times\rho) \leq k'q_2 m_2$$

forall $r = 1, \ldots, T$

This means that as the degree of heterogeneity of demand behaviour along any price system increases market demand becomes less sensitive to a variation of expectations induced by a change of the current prices or by a change of any past prices. The heterogeneity of demand behaviour is induced by the heterogeneity of the demand function (along the expected price vector) or/and by the heterogeneity of the expectations function (along current prices and past prices).

Remark 5 Assume now that households have an unlimited memory so that their learning process is described by a time dependent function of the type:

$$\forall \epsilon \qquad p^e_{t+1} = \psi^\epsilon_t(p_t, \ldots, p_0; \bar{p}_{-1}, \ldots, \bar{p}_{-T})$$

where $\bar{p}_{-1}, \ldots, \bar{p}_{-T}$ are given but arbitrary conditions. The set of expectations functions is also time dependent, denote its corresponding parameter set $\mathcal{E}_t$. The Lemma 2 remains valid under the heterogeneity requirement that the *time dependent distribution* of expectations is sufficiently heterogeneous in *each period.*

It would be an interesting topic for future research to study whether the formulation in the text fot T large is a good approximation of the case where households have time-dependent expectations functions, when the functions ψ^ϵ_t do not change much eventually (in a sense to be made precise) as t goes to $+\infty$. The idea of the answer might be that such an approximation is legitimate at the aggregate (for the mean expectations function) under the heterogeneity requirement that the time-dependent expectations function is highly dispersed along each of its arguments in each period. Intuitivelly, the far distant past observations have a small influence on current forecasts which may even vanishes at the aggregate under heterogeneity, according to Lemma 2.

Remark 6 One may argue that heterogeneity of the expectations function should be defined in terms of normalized expected prices. Effectively, two distinct expected price vectors whose projections on the space of normalized prices $\Re_+ \times S$ are identical lead to the same vector of demands, according to our concept of demand (see assumptions 1 and 2). Hence, some dimension of the heterogeneity of expected prices in $\Re^l_{++}$ allows for non heterogeneous demand behaviours. However, it is easy to reformulate the definition of heterogeneity taking into account the price normalization. Denote $\psi^{*\epsilon}$ the projection of ψ^ϵ on $\Re_+ \times S$. The new definition of heterogeneity is obtained by sustituting in the previous assumptions ψ^ϵ by $\psi^{*\epsilon}$. The conlusions are not affected by such a consideration. Thus, we shall keep the previous formalization which is independent of the price normalization.

2 Structure of Market Short-Run Demand and Short-Run Stability from Heterogeneity

2.1 Heterogeneity and Monotonicity of Short-Run Market Demand

The purpose of this section is to underline that the result of Kneip for exogeneous expectations and for exogeneous incomes can be extended to endogeneous expectations, since the only property of household demand function it requires is its continuity and differentiability with respect to (p, x) which is not a more demanding assumption when expectations are endogeneous. Clearly, the assumption of heterogeneity considered by Kneip, that is a high dispersion of the demand function (when it includes the expectations function) along the current price system is implied by a high dispersion of the demand function along the current price vector and the vector of expected prices, and but not necessarily, by a high dispersion of the expectations function along the current price system. We state, here, a more specific result in the sense that heterogeneity of the demand function along its arguments (including the vector of expected prices) and the heterogeneity of the expectations function itself are explicitely distinguished. Let us introduce the desirability assumption:

Assumption 22 *Whatever the commodity h we have,* $p_h F_h > \varepsilon \bar{x}$.

Note that this assumption implies the two desirability assumptions 7 and 10.

Theorem 1 *Assume assumptions 1 through 14 and 18.*

(i) If assumptions 15 and 17 hold for the current price system and the vector of expected prices then a high enough degree of heterogeneity of the parameter distribution on Γ*, interpreted as a high enough heterogeneity in tendency of the demand function along the current price*

system and along the expected price system, ensures the monotonicity of market demand for exogeneous incomes. More precisely, if the degree of heterogeneity m_1 satisfies the inequality

$$m_1 < \frac{\varepsilon \bar{x}}{q_1 l(x_{\max} + l x_{y\max} \tilde{\psi}_0^y)}$$

where $x_{\max} = \max_{\omega \in \Omega} p^T \omega$ and $x_{y\max} = \max_{\omega_y} p^T \omega_y$, then the Jacobian matrix of market demand is diagonal dominant with negative entries, for exogeneous incomes.

(ii) If furthermore, assumptions 19 and 17 hold for the price system of any period, then the monotonicity of market demand is as well ensured by the inequality

$$q_2 m_{2;t} < \frac{\varepsilon \bar{x} - l x_{\max} q_1 m_1}{l^2 \tilde{f}^y}$$

The required degree of heterogeneity in tendency of the demand function to get monotonicity of market demand decreases with the increasing heterogeneity in tendency of the expectations function in the subgroup.

The first part of the theorem is directly implied by Kneip's result which states that for exogeneous expectations and exogeneous income when household demand is highly dispersed along the current price system market demand is monotone. Effectively, while expectations do matter at the household level, when household demand is highly heterogeneous along the expected price vector, aggregation tends to reduce the impact of a variation of expectations on demand. At the limit, market demand behaves as if the mean expectations were exogeneous. Another way to deduce the theorem from Kneip's result is to argue that the demand function he considers implicitely includes the expectations function. Hence, the heterogeneity of the demand function he retains, that is its high dispersion along the current price system is induced by a high dispersion of the demand function (for fixed expectations) along the vector of expected prices.

The monotonicity of market demand can be ensured by the only heterogeneity of household demand along the price vector and along the expected price vector whatever the shape of the expectations functions (as long as the Jacobian of the mean expectations function $\partial\psi^{\epsilon}$ has a bounded norm). However, the second part of the theorem states that heterogeneity of expectations helps to get the desired result. Note that to ensure the natural assumption of stationarity of expectations at any stationary temporary equilibrium, we have to restrict, as proved previously, the degree of heterogeneity such that $\sum_{j=0}^{T} lq_2 m_{2;t-k} > 1$. Hence, as long as households have a positive memory storage and the economy is assumed to be in equilibrium where households observe price regularities, expectations have to satisfy some consistency requirement and their degree of heterogeneity is reduced. This sheds light on the role of the heterogeneity of expectations and shows that one can get monotonicity of market demand without requiring an excessively high degree of heterogeneity of the demand function as long as the heterogeneity of the expectations function is non negligible.

Symmetrically, the dispersion of household demand with respect to the income level restricts the behaviour of market demand after a variation of income. Under the assumption that uncertainty lies only on the future price vector, a variation of income does not affect expectations. As a consequence the second result of Kneip is still valid, the assumption of a high dispersion of household demand along the income level still ensures that the aggregate matrix of income effect is positive semidefinite. The proof is not repeated since it is not affected by the endogeneity of expectations.

2.2 Heterogeneity and Negative Semidefiniteness of the Aggregate Slutsky Substitution Effect

In the multiperiod framework, there are no a priori reasons why the Slutsky substitution effect of the short-run demand should be negative in all circumstances. The current demand depends on current prices and current income, as well as on expected future prices. If these expectations were exogeneous, that is to say, if they do not depend on current prices, then one can show that the assumption of temporal utility maximisation implies that the Slutsky substitution matrix is negative semidefinite. But, typically, the expectations on future prices depend on current prices. In this case the assumption of temporal utility maximisation alone does not imply a non-positive Slutsky substitution effect. As long as one does not formulate specific assumption on the formation of expectations, the assumption of temporal utility maximization does not imply any useful property of the short-run demand function.

However, a consequence of Theorem 1 and of Kneip's result regarding the heterogeneity of the demand function along the income level is that if the demand function is highly dispersed along the any of its arguments the matrix of substitution effects $\bar{S}$ is negative definite. At the limit (for $q_1 m_1 = 0$ and a parameterization of demand which satisfies assumptions 15 and 17 for small perturbations of p, p^e and x the matrix $\mathcal{P}(\bar{S} + \bar{S}^T)\mathcal{P}$ is equal to

$$
\begin{aligned}
& -2 \int_{\mathcal{B}\times\Omega} x \begin{pmatrix} e_1^\beta(p,x) & & 0 \\ & \ddots & \\ 0 & & e_l^\beta(p,x) \end{pmatrix} \eta(\beta) d\beta \rho(x) dx \\
+ & \ 2 \int_{\mathcal{B}\times\Omega} x e^\beta(p,x) e^\beta(p,x)^T \eta(\beta) d\beta \rho(x) dx
\end{aligned}
$$

By continuity for m_1 close to zero the matrix $\mathcal{P}(\bar{S} + \bar{S}^T)\mathcal{P}$ is close to the above matrix. As a result a sufficient condition for $\bar{S}$ to be negative definite, when m_1 is close to zero, is that

1. the demand satisfies the budget identity, or that
2. $\int_{\mathcal{B}\times\Omega} e_i^\beta (1 - \sum_{j=1}^l e_j^\beta(p,x)) d\mu > 0 \qquad \forall i \in \{1,\ldots,l\}$. This condition is interpreted as the requirement that on average households save.

Since the model incorporates the market of money the assumption of the budget identity holding in each period, is not restrictive[7]. By the way, the following theorem holds:

[7]When working with preferences one has just to assume the monotonicity of preferences.

Theorem 2 *Assume assumptions 1 through 14 and 18. Under assumptions 15 and 17 holding for any argument of the demand function, a flat density of the parameter distribution formalized by a small parameter m_1, interpretable as high heterogneity in tendency of the demand function along each of its arguments, yields a matrix of substitution effect $\bar{S}(p)$ negative definite.*

Hence, rational aggregate behaviour is the consequence of a behavioural heterogeneity and a high endowment variation in the subgroup.

The assumption of a high dispersion in tendency of household demand along the current price system and along the expected price vector is a strong assumption; it implies that the Jacobian matrix of market demand is almost diagonal. It means that, at the limit (for $m_1 = 0$), markets are perceived at the aggregate as separate, on a given market only the own price affects demand. Nevertheless, this assumption can be avoided as proved below.

A simple way to avoid this assumption is to assume that the demand function is highly dispersed in tendency along the expected price vector (the parameter m_1 is small for a parameterization of demand which satisfies assumptions 15 and 17 for small variations of the expected price vector) and to introduce the requirement that:

Assumption 23 *The matrix of Slutsky substitution effects for fixed expectations is negative definite.*

This assumption is, for example, the outcome of the standard assumption of utility maximization under budget constraint when preferences are strictly monotone. Decompose the matrix of substitution effects of the subgroup $\bar{S}_\nu$ between the effect for fixed expectations $\bar{S}_\nu^{\cdot,\bar{\epsilon}}$ and the effect due to the variation of expectations $\bar{S}_\nu^{\bar{\gamma},\cdot}$.

$$\bar{S}_\nu = \bar{S}_\nu^{\cdot,\bar{\epsilon}} + \bar{S}_\nu^{\bar{\gamma},\cdot}$$

$\bar{S}_\nu^{\cdot,\bar{\epsilon}}$ is negative definite by assumption. The matrix $\bar{S}_\nu^{\bar{\gamma},\cdot}$ corresponds to the Jacobian matrix of demand resulting from a price change affecting demand through expectations $\mathcal{J}_{\nu}^{\bar{\gamma},\cdot}$, since at any date there is no uncertainty about the current income, and thereby a price variation does not affect income through expectations. According to the previous analysis $\mathcal{J}_{\nu}^{\bar{\gamma}}$ is close to zero when market demand is highly dispersed in tendency along the expected price vector, as long as the Jacobian matrix of the mean expectations function described by $\tilde{\epsilon}$ is bounded (see Lemma 2). Note that the required degree of heterogeneity of the demand function with respect to the expected price vector decreases as the degree of heterogeneity of the expectations function increases.

2.3 Uniqueness and Stability of Short-run Equilibria

The previous results can be applied to study the uniqueness and short-run stability of the equilibrium of a pure exchange economy. They imply that a high heterogeneity of household demand in average over the endowment distribution yields the monotonicity of market demand on the hyperplane $H(\bar{\omega}) = \{y \in \Re^l \mid y^T \bar{\omega} = 0\}$, which is a determinant property. The main hurdle to the monotonicity of market demand in an exchange economy is that the income is no longer exogeneous but equal to the product of the current price system and the vector of initial endowments. As a result, the income effect after a price change includes now the variation of demand induced by the variation of the nominal income. However, it was shown (see Maret, 1993) that a high dispersion of household demand along the income level implies that this extra component of the Jacobian matrix of market demand almost disappears. This result remains valid in the model incorporating expectations without any further assumption since income does not enter in the learning process. Combining this result with the previous analysis yields the conclusions that:

- For a high dispersion in tendency of the demand function along each of its arguments (including the vector of expected prices) the market demand of the pure exchange economy is monotone on the hyperplane $H(\bar{\omega})$, where the required degree of heterogeneity decreases as the degree of heterogeneity of the expectations function increases in tendency.

- Under the requirement that the aggregate matrix of substitution effect for exogeneous expectations is negative definite a high heterogeneity in tendency of the demand functuion along the income level and along the vector of expected prices yields a monotone market demand on the hyperplane $H(\bar{\omega})$.

It is a well known result (see for example Qhah, 1993) that this restricted monotonicity property of market demand ensures the uniqueness of the equilibrium and its stability for the standard tâtonnement processes described by:

$$(I) \qquad \partial_t p_i = \alpha_i Z_i$$

with $\alpha_i > 0$ for $i = 1, \ldots, l$, or

$$(II) \qquad \partial_t p_i = \beta_i p_i Z_i$$

with $\beta_i > 0$ for $i = 1, \ldots, l$.

The aggregate structure obtained through heterogeneity by aggregation ensures as well short-run stability for any tâtonnement process described by

$$(III) \qquad \partial_t p_i = G_i(Z_i(p))$$

where G_i is a sign preserving function of Z_i with $G_i(0) = 0$ and differentiable with $G_i' > 0$. However this result is obtained only for price normalizations which preserve the mean income or under the restrictive assumption of colinearity of initial endowments. This is not totally satisfactory, since once such restrictions are not required, the income effect induced by changes in the relative income distribution are shown to be worse enough to destroy any nice structure of market demand. This is true even in very specific cases like for example when households are assumed to have homothetic preferences, where excess demand is monotone for fixed income. Nevertheless, the main factor of instability is the direct dependence of income on prices due to the unrealistic definition of income as the nominal value of households initial endowments evaluated at the current price system. An answer to the stability question for any tâtonnement procedure of type (III) is, thereby, to keep the previous assumption of heterogeneity and to introduce a one-period lag in the income determination. In such a framework income is defined by the product of the vector of initial endowments times the price system of last period, thereby, it is independent of current endogeneous variables. Since a high dispersion in tendency of demand along the current price system and along the expected price system (which can be substituted by a high dispersion in tendency of the demand function along the income level and the expected price system and by Assumption 23) ensures the monotonicity of market demand for fixed income, it ensures the local asymptotic stability of the unique temporary equilibrium in each period, given past prices.

3 Dynamics of a Sequence of Temporary Equilibria

This gives some answers to short-run stability, we shall focus now on the stability issue as defined by the sequence of short-run equilibria. In the first subsection, we reintroduce the price dependence of income, that is income is determined by the scalar product of the current price system and the vector of initial endowments in non storable commodities. We show that the assumption that the demand function is highly heterogeneous in tendency with respect to all its arguments ensures the local asymptotic stability of the sequence of stationary equilibria. Furthermore, the required degree of heterogeneity of demand decreases as the heterogeneity of the expectations function increases in tendency. Nevertheless, the heterogeneity of expectations cannot eliminate any feature of instability in the model, despite there exist cases where the heterogeneity of expectations stabilizes the dynamical system. In the second subsection the one-period time-lag in the income determination is reintroduced. This ensures more stability in the short-run. Furthermore, heterogeneity is still proved to give enough structure to market demand to ensure the local asymptotic stability of stationary temporary equilibria.

The state of the economy is described at any date by the price vector p_t. The local dynamics of our model is defined by the map g from the open neighbourhood of a stationary equilibrium $U(\bar{s})$, where $\bar{s} = (\bar{p}, \ldots, \bar{p})$ and $\bar{p} \in W$, to the set of prices such that the assumptions of the first subsection hold, and is given by the equations, $s_t = g(s_{t-1})$ where g is a map from P^{T+2} onto P^{T+2}, or more precisely there exists a map π from P^{T+2} onto P such that,

$$\begin{aligned} q_t &= \pi(s_{t-1}) \\ q_{t-1} &= p_{t-1} \\ &\vdots \\ q_{t-T-1} &= p_{t-T-1} \end{aligned} \tag{6}$$

$(q_t, \ldots, q_{t-T})$ defines the new price vectors induced by s_{t-1}. Clearly $\bar{s}$ is a fixed point of this dynamical system.

In the sequel, we shall consider a map h which coincides with g on P and has a zero derivative along some direction transversal to P: thus if the Jacobian of h has its eigenvalues smaller than one, so has the Jacobian of g.

Denote Z the aggregate excess demand function. We first introduce the map H obtained by substituting the total monetary stock $-M$ to $\int_{B\times\Omega} f_1^{\gamma}(p, p_{-1}, x, \psi_{-1}^{\epsilon}(s_{-1}))d\nu_o$ in Z, we then solve in p, $H(s) = 0$, and we get h by substituting π in Eq.6 by the new expression of p_t. Denote $\tilde{H}$ the reduced matrix deduced from the matrix H by deleting the last row and the last column. Denote $\tilde{H}_u$ the Jacobian of $\tilde{H}$ with respect to the price system p_{-u} for $u = 1, \ldots, T+1$ at point $\bar{s}$. Stability of a long run equilibrium and convergence to self-fulfilling expectations depend in a complex manner on the interaction between the fundamental dynamics of the economy and the learning process of households. In our deterministic framework, a temporary equilibrium depends at each date on the last period price system p_{t-1} and on the forecasts made by the households at the current and last periods. The precise way in which these variables determine p_t depends upon the "fundamental" characteristics of the households (tastes, endowments, demographic structure, ...). The relation that links the current temporary equilibrium state to past states and current and past forecasts takes the form:

$$\tilde{H}(p_t, p_{t-1}, (p_t^{e,\epsilon})_{\epsilon\in\mathcal{E}_y}, (p_{t+1}^{e,\epsilon})_{\epsilon\in\mathcal{E}_o}) = 0 \tag{7}$$

$\tilde{H}$ is continuously differentiable and the matrices $A_0 = \partial_{p_t}\tilde{H}$, $B_0^\gamma = (\partial_{p_t^e}\tilde{H})^\gamma$ and $B_1^\gamma = (\partial_{p_{t+1}^e}\tilde{H})^\gamma$ are assumed to be invertible. Denote $A_1 = \partial_{p_{t-1}}\tilde{H}$. Eq.7 summarizes the structural characteristics of the system. To specify consistently the dynamics we need to describe how a temporary equilibrium at any date is obtained given any past history. To this end it is necessary to spell out how forecasts are made, that is, to specify how p^e are determined given past history. The expectations of young households (respectively of old households) are defined by the functions ψ^{ϵ_y} (respectively ψ^{ϵ_o}). By plugging these functions into Eq.7 we get the relation defining the temporary equilibrium at each date.

$$L(p_t, p_{t-1}, \ldots, p_{t-T-1}) = 0 \tag{8}$$

We assume that $A_0 + \int_{\mathcal{B}} B_1^\gamma C_0^\epsilon d\nu_y$ is invertible. From the implicit function theorem the above equation can then be solved uniquely in p_t near $(\bar{p}, \ldots, \bar{p})$. Specifically, there are open neighborhood $U(\bar{s})$ and $V(\bar{x}) \subset P$, and a continuously differentiable map $h : U(\bar{s}) \to V(\bar{x})$ such that (p_t, s_{t-1}) verifies Eq.8 if and only if

$$p_t = h(p_{t-1}, \ldots, p_{t-T-1})$$

The local temporary equilibrium dynamics are then well defined near the stationary state and are given by the above equation. We focus on the dynamics near stationary temporary equilibria and leave the study of the dynamics near cycles to future researches.

We denote $P_h(\lambda) = 0$ the characteristic equation associated to Eq.8 at the stationary state and assume that there is no characteristic root of modulus one[8]. A well known sufficient condition for the local asymptotic stability of $\bar{p} \in W$ is that every characteristic root λ of P_h satisfies $|\lambda| < 1$

[8]Note that ∂g has at least one eigenvalue of norm one; the one corresponding to the eigenvector $\bar{s}$, that is, $P_g(\lambda) = 0$ has at least one characteristic root of modulus one.

or equivalently that the eigenvalues of the matrix ∂h are all of modulus smaller than one. The Jacobian matrix ∂h can be written:

$$\partial h = \begin{pmatrix} -(\tilde{L}_0)^{-1}\tilde{L}_1 & \dots & \dots & -(\tilde{L}_0)^{-1}\tilde{L}_{T+1} \\ I_l & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I_l & 0 \end{pmatrix}$$

where I_l denote the identity matrix in $\Re^l$. To say that the eigenvalues of ∂h are of modulus smaller than one is equivalent to saying that $(\partial h - \mu I)$ is invertible for any μ with $\mid \mu \mid > 1$. Thus, a sufficient condition for stability of the dynamics with learning near $(\bar{p}, \dots, \bar{p})$ is the following (for a detailed proof see Fuchs and Laroque, 1986, p.1168):

Lemma 3 *A sufficient condition for any STE $\bar{p}$ in W to be locally asymptotically stable for the dynamics generated by Eq.6 is that*

$$\sum_{u=1}^{T+1} \| (\tilde{L}_0)^{-1}\tilde{L}_u \| < 1 \qquad (9)$$

The condition of Lemma 3 links all the price effects on demand effects through the expectation processes and effects through variation of tastes and risk aversion. It requires that market demand is more sensitive to a variation of current prices than to the variation of all past prices. Let us write $\tilde{L}_j$ in terms of the "fundamental derivatives" A_i and B_i $i = 0, 1$ and of the derivatives of the expectations functions C_j and D_j, for $j = 0, 1, \dots, T+1$.

$$\begin{aligned} \tilde{L}_0 &= A_0 + \int_{\mathcal{B}\times\Omega} B_1^\gamma C_0^\epsilon d\nu_y \\ \tilde{L}_1 &= A_1 + \int_{\mathcal{B}\times\Omega} B_0^\gamma D_1^\epsilon d\nu_o + \int_{\mathcal{B}\times\Omega} B_1^\gamma C_1^\epsilon d\nu_y \\ \tilde{L}_j &= \int_{\mathcal{B}\times\Omega} B_0^\gamma D_j^\epsilon d\nu_o + \int_{\mathcal{B}\times\Omega} B_1^\gamma C_j^\epsilon d\nu_y \qquad 1 \le j \le T \\ \tilde{L}_{T+1} &= \int_{\mathcal{B}\times\Omega} B_0^\gamma D_{T+1}^\epsilon d\nu_o \end{aligned}$$

Remark 7 The bounded memory is a determinant factor of the dynamics. However, in general the precise way the dynamics are perturbed by truncating the memory is not a priori obvious. First, one might hope that this would increase the possibility of obtaining stability of equilibria since this removes the direct influence of the initial conditions. Nevertheless, if for example a dynamical process with unlimited memory were to converge after a finite number of periods, then when households have finite memories the process might no longer be defined.

In the present analysis, the result of asymptotic stability remains valid when T converges to $+\infty$. If households have an unlimited memory and their expectations functions are specified by the formula given in Remark 5, then the sufficient condition for the stability of any stationary state in the dynamics with learning can be written:

$$\sum_{\tau=0}^{t-1} \| (\partial_{p_t} \tilde{L})^{-1} (\partial_{p_\tau} \tilde{L}) \| < 1$$

However, a direct consequence of Lemma 2 is that for high enough degrees of heterogeneity m_1 and m_2 small enough $\| \partial_{p_\tau} \tilde{L} \|$ is close to zero for any $\tau \leq t-1$. Hence, if there exist some degrees of heterogeneity m_1 and m_2 such that the inequality holds for $t-T \leq \tau \leq t-1$, then there exist $m_1' \leq m_1$ and $m_2' \leq m_2$ such that the inequality holds for $0 \leq \tau \leq t-1$. m_1' and m_2' denote the degree of heterogeneity of the expectations function with respect to all its arguments whose number increases in each period. The distribution of households' characteristics is time-dependent and the assumption of heterogeneity has to be reformulate for each period. If $m_1'^t$ and $m_2'^t$ denote the degrees of heterogeneity in the period t, then $m_1' = \min_t m_1'^t$ and $m_2' = \min_t m_2'^t$.

The dynamics with learning depend in a complex manner on the perfect foresight dynamics and on the dynamics of the learning process. We are interested in relating the above stability property with the stability or instability of the stationary state when households do not make forecasting errors. An intertemporal equilibrium of the perfect foresight dynamics is defined given the predetermined prices p_{t-1} as a sequence of states p_t and of expectations (p^e_t, p^e_{t+1}) for $t = 0, \ldots, \infty$ such that Eq.7 is satisfied and forecasts are correct at all dates, i.e. :

$$\begin{aligned} p^e_t &= p_t \\ p^e_{t+1} &= p_{t+1} \qquad \forall t \geq 0 \end{aligned}$$

A sequence of states corresponds to such an intertemporal equilibrium if and only if $\forall t \geq 0$:

$$H(p_t, p_{t-1}, (p^e_t)^\epsilon_{\epsilon\in\mathcal{E}_o}, (p^e_{t+1})^\epsilon_{\epsilon\in\mathcal{E}_o}) \mid_{\substack{p^e_t = p_t \\ p^e_{t+1} = p_{t+1}}} = 0$$

Under the assumption of perfect foresight this equation can be rewritten:

$$X(p_{t-1}, p_t, p_{t+1}) = 0$$

Since $H(\bar{p}, \ldots, \bar{p}) = 0$ and the matrix B_1 is invertible[9] the above equation defines consistently a local perfect foresight dynamics near the stationary state $\bar{p}$. From the implicit function theorem it can be solved near $\bar{p}$.

$$p_{t+1} = F(p_t, p_{t-1})$$

where F is defined in an appropriate neighborhood of the constant sequence $(\bar{p}, \bar{p}) \in P^2$. Again a sufficient condition for the stability of the dynamics is that any eigenvalue of the Jacobian matrix of F is of modulus smaller than one. The Jacobian of F evaluated at $(\bar{p}, \bar{p})$ is:

$$\begin{pmatrix} -(\partial_{p_{t+1}}\tilde{X})^{-1}\partial_{p_t}\tilde{X} & -(\partial_{p_{t+1}}\tilde{X})^{-1}\partial_{p_{t-1}}\tilde{X} \\ I_l & 0 \end{pmatrix}$$

where $(\partial_{p_{t+1}}\tilde{X})^{-1}\partial_{p_t}\tilde{X} = B_1^{-1}(A_0 + B_0)$ and $(\partial_{p_{t+1}}\tilde{X})^{-1}\partial_{p_{t-1}}\tilde{X} = B_1^{-1}A_1$. Therefore, a sufficient condition for the stability of the perfect foresight dynamics near $(\bar{p}, \bar{p})$ is that

$$\| (\partial_{p_{t+1}}\tilde{X})^{-1}\partial_{p_t}\tilde{X} \| + \| (\partial_{p_{t+1}}\tilde{X})^{-1}\partial_{p_{t-1}}\tilde{X} \| < 1 \tag{10}$$

[9] It might be that for one specific household $\det(\partial_{p^e_{t+1}} H^\epsilon) = 0$ if for example one commodity does not enter in the household demand function (because its consumption does not affect the household utility and/or the household is not endowed with this commodity). However, $\det(\partial_{p^e_{t+1}} H^\epsilon) = 0$ is valid at the aggregate as long as the desirability assumption and the requirement that the mean vector of endowments is strictly positive $\bar{\omega} >> 0$ hold.

As observed by Grandmont and Laroque (1990), one can easily make explicit the link between the two dynamics. For this purpose, we assume that all households are described by identical fundamental characteristics A_0, A_1, B_0 and B_1, they differ only in their way of processing information.

Finding an eigenvalue and an eigenvector of ∂h amounts to looking for a complex number and a complex l-dimensional vector $v \neq 0$ such that

$$\lambda^{T+1} v = \sum_{j=1}^{T+1} \lambda^{T+1-j} (\tilde{L}_0)^{-1} \tilde{L}_j v$$

Therefore, the characteristic equation of ∂h is

$$P_h(\lambda) = \det[\lambda^{T+1} I_l - \sum_{j=1}^{T+1-j} \lambda^{T+1-j} (\tilde{L}_0)^{-1} \tilde{L}_j]$$

which leads to

$$\begin{aligned} \det(\tilde{L}_0) P_h(\lambda) &= \det[A_0 \lambda^{T+1} + A_1 \lambda^T + B_1 (\int_{\mathcal{E}} \sum_{j=0}^{T} C_j^{\epsilon} d\nu_y) \lambda^{T+1-j} \\ &\quad + B_0 (\int_{\mathcal{E}} \sum_{j=1}^{T+1} D_j^{\epsilon} d\nu_o) \lambda^{T+1-j}] \end{aligned} \qquad (11)$$

By applying the same procedure to F we get,

$$\det B_1 P_X(\lambda) = \det[B_1 \lambda^2 + (A_0 + B_0)\lambda + A_1] \qquad (12)$$

The examination of the two equations above and the assumption that households are able to extrapolate some regularities from the past allow us to make the interaction between the two dynamics explicit.

Lemma 4 *If households extrapolate any orbit of period two and are described by identical fundamental characteristics A_0, A_1, B_0 and B_1, then it follows that*

$$\begin{aligned} det(\tilde{L}_0) P_h(1) &= det(B_1) P_X(1) \\ (-1)^{l(T+1)} det(\tilde{L}_0) P_h(-1) &= (-1)^l det(B_1) P_X(-1) \end{aligned}$$

3.1 Dynamics with Learning

We state the following theorem:

Theorem 3 *Assume assumptions 1 through 14, 18 and that the matrix* $A_0 + \int_{\mathcal{B}} B_1^\gamma C_0^\epsilon d\nu_y$ *is invertible.*

(i) Under the assumption that the restrictions on the parameterization 15 and 21 hold for any argument of the demand function, a high enough degree of heterogeneity of the parameter distribution, interpretable as a high enough degree of heterogeneity in tendency of the demand function along each of its arguments, such that

$$q_1 m_1 < \frac{\varepsilon \bar{x}}{(l+1)x_{\max} + 2l x_{o\,\max} + l x_{y\,\max}}$$

ensures that any stationary temporary equilibrium of the dynamics with learning is locally asymptotically stable for an open subset of initial conditions near the stationary price.

(ii) Under the further requirement that the restrictions on the parameterization 19 and 21 hold for the price system of any past period, the stability is also ensured by the following inequality:

$$q_1 m_1 (l x_{o\,\max} + (l+1)x_{\max}) + q_2 l(\tilde{f}^o + \tilde{f}^y)(T+1)m_2 < \varepsilon \bar{x}$$

Thus the required degree of heterogeneity in tendency of the demand function decreases as the degree of heterogeneity of the expectations function increases in tendency. $x_{o\,\max}$ *denotes the maximum income in the population of old households, that is* $x_{o\,\max} = \max_{\omega_o} p^T \omega_o$.

For $m_1 = 0$ the result (i) is trivially explained. As the degree of dispersion of demand along the expected price vector increases, market demand becomes insensitive to a variation of expectations. Hence at the limit, for $m_1 = 0$, market demand behaves as if the mean expectations were exogeneous. Furthermore, a high dispersion of the old household demand along the price system of the last period p_{t-1} leads as well at the limit to the insensitivity of market demand to p_{t-1}. Therefore, market demand depends only on current prices. Thus, the short-run local stability of the temporary equilibrium $\bar{p}$ ensures the local stability of the stationary temporary equilibrium of the dynamics with learning $(\bar{p}, \dots, \bar{p})$. Clearly, the sufficient condition for stability established in Lemma 3 holds. In the long run, the economy is never pushed from an equilibrium state.

For $m_1 > 0$ the result is less trivial. Now, past period prices do affect market demand. Nevertheless, for any initial price in a neighborhood of $(\bar{p}, \dots, \bar{p})$ the sequence $\{p_t\}$ converges towards $\bar{p}$. This seems to contradict the result of Grandmont and Laroque (1991) who prove that if traders are prepared to extrapolate a sufficiently large set of past growth rates, then the stationary state will be actually unstable. However, their result is valid under assumptions which cannot be put into question in a representative agent framework, but which are not obvious a priori once heterogeneity is introduced. It is a well known result that the representative agent model is not the result of aggregation under standard individual behaviour. This follows, partly, from the fact that aggregation through heterogeneity of household behaviour has a structurizing impact on aggregate characteristic functions. The structurizing effect might be such that the resulting aggregate concept diverges from the corresponding individual one, while the representative agent theory regards the two notions as identical. Thus, in Grandmont and Laroque's model aggregate behaviour is considered as though it was the behaviour of a single representative agent. Incidently, the norm of the aggregate impact of a change in expectations on market demand is assumed to be bounded away from zero and the derivatives of the mean expectations function are allowed to have very large values. However, once heterogeneity is introduced, while the individual impact of a variation of expectations on market demand is still bounded away from zero and each trader is still allowed to extrapolate a large set of past growth rates, these two conditions may not hold at the aggregate. A degree of heterogeneity of the demand functions in tendency along the expected price vector yields, indeed, to reduce the aggregate impact of a variation of expectations on market demand. Furthermore, a high degree of heterogeneity of the expectations function in tendency along each of its arguments yields a bounded norm of the Jacobian matrix of the mean expectations function. This underlines the well foundation of the criticism of the macroeconomic analysis based on the concept of the representative agent. As claimed by Hildenbrand (1983) this concept does not really simplify the analysis, but on the contrary, might be quite misleading.

Hence, the above theorem establishes that adaptative learning does not necessarily lead to endogeneous fluctuations, as it was suggested by Grandmont and Laroque, and is compatible with the convergence to self-fulfilling expectations, providing the learning process and the impact of expectations on demand are dispersed in the population. Nevertheless, it should be emphasized that our formulation being non linear local stability does not imply that the actual dynamics will be globally stable. Thus, for a large deviation from any STE it may be the case that dynamics evolve along a closed orbit. The above result is compatible with endogeneous fluctuations where forecasting errors never vanish.

Heterogeneity of the expectations function reinforce the stability property. With regards to the second part of the theorem it seems already clear that the heterogeneity of expectations cannot eliminate on its own any feature of instability in the model. This is explained by the fact that the heterogeneity of the demand function plays a determinant role, we shall come back later on this point.

We have already remarked that the heterogeneity of the expectations function tends to stabilize the dynamics of the learning process. However, as the degrees of heterogeneity $m_{2;t-j}$ $\forall j$ are restricted such that the inequalities of the Lemma 1 hold, the aggregate learning process may be unstable while the dynamics of temporary equilibria with learning is stable. In the next section, we even show that the stability of the dynamics with learning ensured by the assumptions of Theorem 3 (including the dispersion of household demand along the vector of expected prices) is compatible with the instablity of the dynamics of perfect foresight.

3.2 Dynamics of perfect foresight

In this subsection we assume that all present and future households know a priori the structure of the system and they coordinate their forecasts such that $p^e_{t+1} = p_{t+1}\ \forall t$.

Since the set of households' characteristics is bounded, there exists a household $(\tilde{\beta}_y, \tilde{\omega_y})$ such that:

$$\int_{\mathcal{B}\times\Omega} \mathcal{P} f^{\beta} \partial_{\triangle}\varphi_{y\triangle} \mid_{\triangle=0} \phi_y(\epsilon)\rho_y(\omega) d\beta d\omega$$
$$= \mathcal{P} f^{\tilde{\gamma}_y} \mid_{\substack{\tilde{x}=p^T\tilde{\omega} \\ p^e_{t+1}=\psi^{\tilde{\epsilon}}(s)}} \int_{\mathcal{B}\times\Omega} \partial_{\triangle}\varphi_{\triangle} \mid_{\triangle=0} \phi(\epsilon)\rho(\omega) d\beta d\omega \tag{13}$$

We introduce the following desirability assumption:

Assumption 24 *Whatever k there exists $\tilde{\varepsilon}$ such that:*

$$p_k f_k^{\tilde{\beta}_y} > \tilde{\varepsilon}\tilde{x}_y$$

Theorem 4 *Assume assumptions 1 through 14, 18 and that the matrix B_1 is invertible. Restrict the degree of heterogeneity of the demand function of the young household along expected prices by restricting the "flatness" of the density $\varphi_\triangle$ (where $\triangle$ refers to small variations of the expected price vector) in the following way*

$$\min_{i=1,\ldots,l} \int_{\mathcal{B}\times\Omega} \mid \partial_{\triangle_i}\varphi_{\triangle}(\gamma) \mid \phi(\epsilon)\rho(\omega) d\beta d\omega > k^y_{1,p^e} \tag{14}$$

where $\varphi_\triangle$ is the density function of the transformed parameter $\tau_\triangle(\gamma)$ of demand obtained through transformation $I_\triangle$ of the vector of expected prices. The restrictions on the parameterization of demand 15 and 17 are assumed to hold for any argument of the demand function and for both types of household, except for the expected price vector as far as the young household demand is concerned, m_1 denotes then the degree of heterogeneity. A sufficient condition for the local asymptotic stability of the perfect foresight dynamics at any stationary temporary equilibrium $\bar{p} \in W$ is given by the inequality

$$2\bar{x} + q_1 m_1((l+1)x_{\max} + 2l x_{o\,\max}) < \tilde{\varepsilon} l k^y_{1,p^e}\tilde{x_y}$$

Hence, this stability property is ensured if the demand function of both types of household is heterogeneous enough in tendency along any of its arguments, except along the expected price vector as far as the young household demand is concerned.

This result follows from the sufficient condition for stability 10 which imposes that market demand is more sensitive at the stationary state to a variation of future prices than to a variation of current prices or past prices. Hence, the demand function should be more heterogeneous along the current prices and past prices than along the vector of expected prices, since the heterogeneity along one argument implies that the aggregate function is less sensitive to a variation of this argument. Clearly, the above inequality is feasible only if the degree of dispersion of young household demand along the vector of expected prices satisfies in tendency:

$$k^{y}_{1,p^e} > \frac{2\bar{x}}{\tilde{\varepsilon} l \tilde{x}_y}$$

Therefore, the heterogeneity of young household demand along the vector of expected price tends to destabilize the dynamics of perfect foresight, while it tends to stabilize the dynamics with learning. This means that when households anticipate perfectly the price system, if they react in very heterogeneous ways to a small perturbation of the expected price vector, the dynamics will never converge towards a STE (starting in a neighborhood of this STE). On the other hand, when they already have very distinct expectations and react in very heterogeneous ways to a change of the expected price vector, the two types of heterogeneity may compensate each other so that market demand gets enough structure to ensure the local asymptotic stability of any STE. This theorem is in accordance with the result of Grandmont and Laroque (1990), which states that there are cicumstances under which the stability of a stationary state in the dynamics with learning implies its instablity in the dynamics of perfect foresight[10].

[10] Grandmont and Laroque require that households extrapolate from the past any cycle of period 2. Here we only require the stationarity of expectations, however the heterogeneity assumption gives enough structure to the aggegate to get the desired

The sufficient condition of stability of Theorem 4 requires also that the mean income of young households is high with respect to the mean income of the whole population. Intuitivelly, this should ensure a non negligible amount of saving, that is, in tendency households save in each period, as illustrated by the example in Section 4. Therefore, the stability of the dynamics with learning ensured by the heterogeneity of household demand combined with a restricted heterogeneity of the expectations function is not only compatible with the instability of the aggregate learning process, but as well with the instability of the dynamics of perfect foresight. The requirements for stability are far from the rational expectations assumption.

Remark 8 Here the dependence of income on current prices does not introduce a high level of instability in the long run. This effect appears only in the matrix $\tilde{Z}_0$ which is not determinant to get the previous result of stability as long as it remains finite.

3.3 Instability of the Perfect Foresight Dynamics versus Stability of the Dynamics with Learning

The purpose of this subsection is to underline the role of heterogeneity of expectations. We show that, despite the heterogeneity of expectations cannot eliminate any feature of instability in the model, there exists cases where it stabilizes the dynamics with learning. We consider a population where households are described by identical fundamental characteristics but differ in their way of processing information. For specific fundamental characteristics which determinate a high incentive to save and by the way solve the indeterminateness of the prefect foresight dynamics, a high enough heterogeneity of the expectations function along the current price system ensures the convergence of the dynamics with learning. More precisely,

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constraints.

Theorem 5 *Assume assumptions 1 through 14 and 18. Assume that the restrictions on the parameterization of the expectations function hold for the current pric system. If the fundamental characteristics of the model are such that the stationary state of the perfect foresight dynamics is a saddle point and such that* $detB_1$ *and* $detA_0$ *are of opposite sign, a high enough heterogeneity of the parameter distribution on* $\mathcal{E}$ *which can be interpreted as high enough heterogeneity of the expectations function along the current price system, such that* $m_{2,t}$ *is small enough to ensure that*

$$sgn[det(B_1^{-1}A_0 + \int_{\mathcal{E}} C_0^{\epsilon} d\nu_y)] = sgn[det(B_1^{-1}A_0)]$$

leads to the local asymptotic stability of the STE in the dynamics with learning.

To explain this result we recall that the heterogeneity of the expectations function tends to reduce the sensitivity to a variation of any endogeneous variable of the mean expectations function. Hence, after any price deviation from the stationary state the trajectory of the dynamics with learning evolves "not far from" the trajectory of the perfect foresight dynamics. Then, the above theorem formalizes the intuition that the required structure of the perfect foresight dynamics to get stability of the dynamics with learning should consequently be a stability of saddle point rather than an asymptotic stability. For the first structure, there exists only one stable trajectory near the stationary state, and thereby, any small price deviation from the stationary state in the dynamics with learning leads to stable trajectory not far from the stable trajectory of the perfect foresight dynamics. On the contrary, for the second structure, there exists an infinite number of stable trajectories of the perfect foresight dynamics, and thereby, any small price deviation from the stationary state in the dynamics with learning can lead to any kind of trajectories.

Note that the stationary state of the prefect foresight dynamics is a saddle point if one of the characteristic roots of the polynomial P_X of degree two is of modulus lower than one while the other is of modulus higher than one, that is ($P_X(1) < 0$ and $P_X(-1) > 0$) or ($P_X(1) > 0$ and

$P_X(-1) < 0$. This is ensured by

$$\det(A_1 + B_1 + A_0 + B_0)\det(A_1 + B_1 - (A_0 + B_0)) < 0$$

This inequality can be rewritten

$$\det[(A_1 + B_1)^2 - (A_0 + B_0)^2] < 0 \tag{15}$$

When the matrices A_i, B_i are reduced to parameters the above inequality means that short-run market demand is more sensitive to a variation of current prices than to a variation of past or expected prices. Recall that the matrices A_i, B_i denote the Jacobians of market demand, that is $A_0 = \partial_{p_t}\tilde{H}$, $A_1 = \partial_{p_{t-1}}\tilde{H}$, $B_0 = \partial_{p_t^e}\tilde{H}$ and $B_1 = \partial_{p_{t+1}^e}\tilde{H}$.

The determinant of a matrix is the product of its eigenvalues. Thus the requirement that $\det B_1$ and $\det A_0$ are of opposite sign implies the number of negative eigenvalues of B_1 differs by an odd amount of the one of A_0. There exist some directions of price changes to which market demand answers in opposite direction depending on whether the changes concern current prices or expected prices. Thus, the two restrictions on the fundamental characteristics of Theorem 5 imply some substitution effect between current and future consumption, but such that, market demand remains more sensitive to a change of the current price system. Even for high expected price the amount of saving is non negligible and the perfect foresight dynamics is perfectly locally determinate.

Remark 9 All the previous results of stability can be proved to be structural. Indeed, for the model under consideration, Fuchs and Laroque (1976) state the theorem.

Theorem 6 *Denote $\mathcal{U}$ an open and dense subset of the set of admissible economies. For any economy in $\mathcal{U}$ the behaviour of the trajectories of the dynamical system generated by our model, near stationary states and near a finite number of cycles, is preserved under small modifications of any exogeneous parameter. More precisely, any small change in the total monetary stock or any small change in households' characteristics does not affect the dynamical properties of stationary states and of cycles and in particular the stability properties of these trajectories.*

This means that for most economies any STE is even locally structurally stable under our assumption of heterogeneity, since the dynamics in the long run are but slightly affected by changes of the households' characteristics.

4 An Example

In this section we consider the simple case where the state variable p_t is a real number, that is the case where there is only one non storable commodity in the economy $l = 1$, by the way $\forall p \in P \; p_{t;2} = 1$. Denote $p_{t;1} = p_t$. We prove that heterogeneity as defined in Theorem 3 which ensures the local asymptotic stability of the dynamics with learning at any stationary prices $\bar{p} \in W$ implies that $\bar{p}$ is a saddle point of the dynamics with perfect foresight. $\tilde{Z}$ corresponds now to the excess demand of money, hence

$$\tilde{H} = \int_{\mathcal{B}\times\Omega} f_1^\gamma(p_t, x, p_{t+1}^e) d\nu_y - M$$

The partial derivatives of the function $\tilde{H}$ with respect to p_t, p_{t-1}, p_t^e and p_{t+1}^e, respectively, evaluated at the stationary equilibrium $\bar{p}$ denoted by A_0, A_1, B_0 and B_1, respectively, are now real numbers. Furthermore, since $\tilde{H}$ depends only on the aggregate demand of young households it follows that $B_0 = 0$, $A_1 = 0$ and $A_0 = \int_{\mathcal{B}\times\Omega} \frac{f^\gamma}{p} d\nu_y + \int_{\mathcal{B}\times\Omega} \frac{f_1^\gamma}{p} \partial_\triangle \varphi_{y\triangle}(\gamma) \mid_{\triangle=0} \phi_y(\epsilon)\rho_y(\omega) d\beta d\omega$. The two sufficient conditions for the local asymptotic stability of a stationary state of the dynamics with learning defined in Theorem 3 are written

$$q_1 m_1 \; < \; \frac{\varepsilon\bar{x}}{(1 + \sum_{j=0}^T \tilde{\psi}_j^y) x_{y\max}} \leq \frac{\varepsilon\bar{x}}{2x_{y\max}} \quad (16)$$

$$q_1 m_1 x_{y\max} + q_2 \tilde{f}_y (T+1) m_2 \; < \; \varepsilon\bar{x} \quad (17)$$

Note that the demands of the old households do not affect the dynamics, since mean demand for money of the population of old households is assumed to be identically equal to the opposite of the total amount of money in the economy. Theorem 3 states that when the young household demand is heterogeneous enough along the current price system and along the vector of expected prices (for our example, such that the parameter $q_1 m_1$ satisfies Eq.16) the dynamics with learning at any STE is locally asymptotically stable. Note that no dispersion of household demand along the income level is required, this is explained by the fact that a variation of the monetary price Δp does not affect demand through income since the young household does not inherit any endowment in money. The theorem states as well that the required degree of heterogeneity in tendency of the demand function decreases as the heterogeneity of the expectations function along each of its arguments increases in tendency. This is illustrated in our example by Eq.17.

The local perfect foresight dynamics are obtained by solving $\tilde{H}(p_t, p_{t-1}, p_t, p_{t+1}) = 0$ in p_{t+1} near the stationary state. The corresponding characteristic equation is:

$$P_X(\lambda) = \lambda^2 + B_1^{-1} A_0 \lambda = 0$$

The two roots of this equation are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= A_0/B_1 \end{aligned}$$

A stationary state is a saddle point if some of the characteristic roots are of modulus lower than one and the others are of modulus higher than one. Therefore, if $\mid A_0/B_1 \mid < 1$ the dynamics with perfect foresight is stable and if, on the other hand if $\mid A_0/B_1 \mid > 1$ the stationary state $(\bar{p}, \bar{p}, \bar{p})$ is a saddle point. More precisely,

Theorem 7 *(i) Assume assumptions 1 through 14, 18 and that $B_1 \neq 0$. Assume that the restrictions on the parameterization of young household demand hold for any of its arguments. A high enough degree of heterogeneity of the parameter distribution on Γ, which can be interpreted as a high enough degree of heterogeneity in tendency of the demand function of young households along each of its arguments, such that*

$$q_1 m_1 < \frac{\varepsilon \bar{x}}{2 x_{y\,\max}}$$

implies that any stationary state of the perfect foresight dynamics is a saddle point (and thus, the dynamics are locally determinate).

(ii) If we restrict now the heterogeneity of the young household demand function along the vector of expected prices by requiring the condition 14, while maintaining a high enough degree of heterogeneity along the other arguments such that

$$\frac{(1+q_1 m_1)x_{y\max}}{\varepsilon\tilde{x}_y} < k^y_{1,p^e}$$

then any STE of the perfect foresight dynamics is locally asymptotically stable. The household $(\tilde{\beta}_y, \tilde{\omega}_y)$ *is defined by the condition 13.*

Note that a high degree of heterogeneity such that Eq.16 holds leads both to the stability of the dynamics with learning and to the instability of the perfect foresight dynamics, where any stationary state is a saddle point.

According to Theorem 3 a high heterogeneity in tendency of the demand function along each of its arguments can lead to enough structure to ensure the local asymptotic stability of any STE whatever the heterogeneity of expectations. However, a high heterogeneity in tendency of the expectations function along any of its argument does not systematically lead to the local asymptotic stability of a STE. First, heterogeneity of expectations is restricted not to contradict the stationarity assumption of expectations. Secondly, independently of expectations, the dynamics with learning depend on the perfect foresight dynamics and on the local determinateness of the long run equilibrium (see Theorem 5). To illustrate the actual impact of the heterogeneity of expectations, we consider an even more specific example of economy. Assume, again, that all households have identical demand functions and identical vectors of endowments, but that they differ in their way of processing information. Furthermore, assume that any household is characterized by a utility function of the Cobb-Douglas type, that is

$$U(y_t, y_{t+1}) = y_t^{a_t} y_{t+1}^{a_{t+1}}$$

where $a_t + a_{t+1} = 1$, y_t (respectivelly y_{t+1}) is the current consumption (respectivelly the future consumption) of the non storable commodity. Denote b the current demand for money. The household demand function is computed by maximizing the utility function under the budget constraints:

$$\begin{aligned} y_t + p_t b &\leq \omega_t \\ y_{T+1} &\leq \omega_{t+1} + p^e_{t+1} b \end{aligned}$$

Simple computations give by induction

$$\begin{aligned} y_{t+1} &= \omega_{t+1} + p^e_{t+1} b \\ y_t &= a_t[\frac{p_t}{p^e_{t+1}}\omega_{t+1} + \omega_t] \\ b &= a_{t+1}\frac{\omega_t}{p_t} - a_t\frac{\omega_{t+1}}{p^e_{t+1}} \end{aligned}$$

$\tilde{H}$, that is the excess demand of money, is written:

$$\tilde{H} = (1 - a_t)\frac{\omega_t}{p_t} - \int_{\mathcal{E}} a_t \frac{\omega_{t+1}}{p^e_{t+1}} \, |_{p^e_{t+1} = \psi^\epsilon(s_t)} \, \phi_y(\epsilon) d\epsilon - M$$

Its derivatives are easily computed.

$$\begin{aligned} \tilde{H}_0 &= A_0 + B_1 \int_{\mathcal{E}} C^\epsilon_0 \phi_y(\epsilon) d\epsilon \\ &= -\frac{a_{t+1}}{p_t^2}\omega_t + \frac{a_t}{(p^e_{t+1})^2}\omega_{t+1} C_0 \\ \tilde{H}_j &= -\frac{a_t}{(p^e_{t+1})^2}\omega_{t+1} C_j \qquad \forall j = 1, \ldots, T \end{aligned}$$

Insofar as short-run stability is concerned, a sufficient condition of stability is trivially obtained by requiring that the derivative of the excess demand of money is strictly negative, that is $\partial_{p_t}\tilde{H} < 0$. This condition, when the derivative is evaluated at a stationary price, is implied by the inequality:

$$\int_{\mathcal{E}} \partial_{p_t} \psi^\epsilon \phi_y(\epsilon) d\epsilon < \frac{\omega_t}{a_t} \frac{a_{t+1}}{\omega_{t+1}}$$

which holds if

$$m_{2,t} < \frac{\omega_t}{a_t}\frac{a_{t+1}}{\omega_{t+1}} \tag{18}$$

For exogeneous expectations (or expectations which do not depend on current prices) it is a well known result that when households' demands are Cobb-Douglas the one-period equilibrium of the pure exchange economy is globally stable under any standard tâtonnement procedure of type (III). If now expectations are assumed to be dependent on the current price system stability can still be ensured by a high enough degree of heterogeneity of this dependency of expectations on current prices. Effectively if the degree of dispersion of the expectations function along the current price system satisfies Eq.18 the equilibrium is globally stable. Note that the required degree of heterogeneity decreases when the incentive to save increases (that is when $\frac{a_{t+1}}{\omega_{t+1}}$ increases more quickly than $\frac{a_t}{\omega_t}$).

For what regards long-run stability, the condition Eq.15 which implies that the STE is a saddle point of the prefect foresight dynamics is written

$$| B_1 |<| A_0 |$$

that is $\frac{a_t}{\omega_t} < \frac{a_{t+1}}{\omega_{t+1}}$. We state the following result:

Theorem 8 *Assume assumptions 1 through 14, and 18. We assume also in addition that the parameterization of the expectations function satisfies the restrictions 19 and 17 for the price system of any period.*

(i) If the incentive to save is high because of a preference for the future and/or a comparatively low amount of initial endowments at date $t+1$ such that $\frac{a_t}{\omega_t} < \frac{a_{t+1}}{\omega_{t+1}}$, that is the STE is a saddle point of the perfect foresight dynamics, then a high enough degree of heterogeneity of the parameter distribution on $\mathcal{E}$, which can be interpreted as a high enough degree of heterogeneity of the expectations function along any of its arguments, such that

$$q_2(\sum_{j=1}^{T} m_{2;t-j} + m_{2;t}) < \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}} \tag{19}$$

ensures that any STE of the dynamics with learning is locally asymptotically stable.

(ii) However if households' incentive to save is low because of a preference for the present and/or a comparativelly low amount of initial endowments at date t such that $\frac{a_t}{\omega_t} > \frac{a_{t+1}}{\omega_{t+1}}$, *that is the STE is locally asymptotically stable in the dynamics with learning, then,*

- *when* $C_0 > \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$ *the sufficient condition for stability holds if* $\sum_{j=1}^{T} \mid C_j \mid < C_0 - \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$ *which in turn holds if* $\sum_j m_{2;t-j} < k_{2,t} - \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$, *where* $k_{2,t-j} = \min_{r=1,\ldots,l} \int_{\mathcal{E}} \mid \partial_{\Delta_r}\phi_{\Delta}(\epsilon) \mid d\epsilon$ *where* Δ *is a vector of pertubations affecting the price system* p_{t-j}. *In this case a high enough heterogeneity of the expectations function such that the above inequality holds and compatible with the stationarity requirement ensures stability.*
- *when* $C_0 < \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$ *the sufficient condition for stability never holds.*

Furthermore, there exist quite high degree of heterogeneity of the expectations function such that

- *when* $C_0 < \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$, *if* $\mid C_T \mid + C_0 > \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$ *which holds if* $k_{2;t} + k_{2:t-T} > \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$
- *or when* $C_0 > \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$, *if* $\mid C_T \mid > C_0 - \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$ *which holds if* $k_{2;t-T} > m_{2;t} - \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$ *for which any STE of the dynamics with learning is locally asymptotically unstable.*

Heterogeneity of the expectations function does not ensure the stability of the stationary temporary equilibrium whatever the households' preferences, risk aversion and endowments. This heterogeneity requirement leads to the local asymptotic stability of any STE only if households have a high enough incentive to save determined by their fundamental characteristics. Note that this restriction on households' preferences implies that the stationary state of the perfect foresight dynamics is a saddle point. It tends as well to stabilize the short-run dynamics near a stationary state. This illustrates the previous result stating that if the perfect foresight dynamics are locally determinate around a stationary price and expectations are heterogeneous enough then the stationary state is locally asymptotically stable in the dynamics with learning. Without such an incentive to save it may be the case that despite quite heterogeneous expectations (such that $m_{2;t-j}$ is small but $k_{2;t-j} > 0$ such that the stationarity assumption of expectations holds) any stationary temporary equilibrium is unstable. The way the expected price vector affects household demand is determinant in the dynamics with learning. Even if the mean expected prices are only slightly affected by a small price deviation from the stationary price, the market demand is significantly affected so that the economy is pulled away from the STE because of the indeterminateness of the perfect foresight dynamics.

5 Dynamics with a Time Lag in the Income Determination

In this last section we introduce a time lag in the determination of income, that is income is defined by the product of the vector of initial endowments times the price vector of last period. As our basic working assumption we want to keep the assumption that household behaviour is heterogeneous enough in tendency to ensure the monotonicity of excess demand function for exogeneous income. More precisely, we require that the demand function is heterogeneous enough in tendency with respect to the current price system, with respect to expected price system and with respect to the income level. Therefore, in every period of time a tâtonnement in current prices is globally stable under the Walrasian adjustment process, and we are led to investigate whether the sequence of unique and globally stable short-run equilibria converges.

The motivation which inspired Hens and Hildenbrand (1993) to introduce this time lag in the determination of income

> 'is that in a private ownership economy, production serves as buffer to prevent an immediate pass through of price changes on income. Production needs time and income consists merely

out of wages and dividends, both being fixed before the period of consumption.'

The current income can be interpreted as the nominal value of a vector of outputs produced from a vector of inputs introduced in the production process at the previous period, thereby it is avalaible at the beginning of the current period, but evaluated at the price system of the previous period. This definition of income is a step closer to reality. Furthermore, theoretically this simple modification changes the stability properties of an exchange economy as suggested by Kirman (1989) (page 136). More precisely it should ensure more stability. as mentioned, it implies that under our basic assumption short-run equilibria are stable under the standard Walrasian tâtonnement and we can focus on the intertemporal stability. Furthermore, the previous results of stability of the iterative process defined by the sequence of short-run equilibria remain valid. We focus again on the stability of temporary stationary equilibria in the long run. Note, however, that other long run equilibrium trajectories may appear like cycles.

The dynamics is still described by the system Eq.6 where the fundamental derivatives A_0 and A_1 are affected by the modification of the income determination in the following way:

$$
\begin{aligned}
A_0 &= \int_{\mathcal{B}\times\Omega} \begin{pmatrix} \frac{f^{\gamma_y}}{p_1} & & & 0 \\ & \frac{f_2^{\gamma}}{p_2} & & \\ & & \ddots & \\ 0 & & & \frac{f_l^{\gamma}}{p_l} \end{pmatrix} d\nu \\
&+ \int_{\mathcal{B}\times\Omega} \begin{pmatrix} \frac{f_1^{\gamma_y}}{p_1} \\ \frac{f_2^{\gamma}}{p_2} \\ \vdots \\ \frac{f_l^{\gamma}}{p_l} \end{pmatrix} \partial_{\triangle}\varphi_{\triangle}(\gamma) \mid_{\triangle=0} \phi(\epsilon)\rho(\omega) d\beta d\omega
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 \\ \frac{f_2^\gamma}{p_2} \\ \vdots \\ \frac{f_l^\gamma}{p_l} \end{pmatrix} \partial_\triangle \varphi_\triangle(\gamma) \mid_{\triangle=0} \phi(\epsilon)\rho(\omega) d\beta d\omega \\
&- \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 & \frac{f_1^{\gamma y}\omega_2}{x} & \dots & \frac{f_1^{\gamma y}\omega_l}{x} \\ \vdots & \frac{f_2^\gamma \omega_2}{x} & \dots & \frac{f_2^\gamma \omega_l}{x} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \frac{f_l^\gamma \omega_2}{x} & \dots & \frac{f_l^\gamma \omega_l}{x} \end{pmatrix} d\nu \\
&+ \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 & \frac{f_1^{\gamma y}\omega_2}{x} & \dots & \frac{f_1^{\gamma y}\omega_l}{x} \\ \vdots & \frac{f_2^\gamma \omega_2}{x} & \dots & \frac{f_2^\gamma \omega_l}{x} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \frac{f_l^\gamma \omega_2}{x} & \dots & \frac{f_l^\gamma \omega_l}{x} \end{pmatrix} \partial_\triangle \varphi_\triangle(\gamma) \mid_{\triangle=0} \phi(\epsilon)\rho(\omega) d\beta d\omega
\end{aligned}
$$

where the first column of the two last matrices is equal to the nul vector since a variation of the price of money does not affect the nominal disposable income. We shall analyze the behaviour of market demand when household demand is highly dispersed along each of its arguments and when the expectations function is dispersed but in a restrictive sense to ensure the stationarity of expectations. We focus on the behaviour of market demand at the limit, that is for a parameter m_1 close to zero, however by continuity the results remain valid for a neighborhood of the limit point.

Theorem 9 *Assume assumptions 1 through 14 and 18. Income is the nominal value of the initial endowments evaluated at the price system of last period. The parameterization of the demand function satisfies the restrictions 15 and 17 for any argument of the demand function. A high degree of heterogeneity of the parameter distribution on Γ formalized by m_1 close to zero, which is interpreted as a high degree of heterogeneity in tendency of the demand function, implies that any STE of the dynamics with learning is locally asymptotically stable.*

As in the previous section a high heterogeneity in tendency of the demand function might lead to the instability of the perfect foresight dynamics.

Theorem 10 *Assume assumptions 1 through 14 and 18. Income is the nominal value of the initial endowments evaluated at the price system of the last period. The parameterization of the demand function is assumed to satisfy the restrictions 15 and 17 for any argument of the demand function. A high degree of heterogeneity of the parameter distribution on* Γ *formalized by a small* m_1*, interpreted as a high degree of heterogeneity in tendency of the demand function along any of its arguments. implies that the sufficient condition for stability of the perfect foresight dynamics does not hold.*

Appendix

A.1 Implication of Assumption 16

We prove that the property that there exists a constant $c > 0$ such that $ch(\varphi) \geq h(\varphi_\triangle)$ is implied by Assumption 16. We have

$$\begin{aligned}
\int_\Gamma \mid \partial_{\gamma_r}\varphi_\triangle(\gamma) \mid d\gamma &= \int_\Gamma \mid \partial_{\gamma_r}[\det(\partial_\gamma \tau_\triangle^{-1}(\gamma))\eta(\tau_\triangle^{-1}(\beta))] \mid d\gamma \\
&= \int_\Gamma \mid [\partial_{\gamma_r}\det(\partial_\gamma \tau_\triangle^{-1}(\gamma))]\varphi(\tau_\triangle^{-1}(\gamma)) \mid d\gamma \\
&\quad + \int_\Gamma \mid \det(\partial_\gamma \tau_\triangle^{-1}(\gamma))\partial_{\xi_r}(\xi) \mid_{\xi=\tau_\triangle^{-1}(\gamma)} \partial_{\gamma_r}\tau_\triangle^{-1}(\gamma) \mid d\gamma
\end{aligned}$$

Clearly, under Assumption 16 the first term is equal to zero and the second term admits $ch(\varphi)$ as an upper bound.

Symmetrically, we prove that assumption 17 implies Property 1.

A.2 Heterogeneity of the Demand Function along the Income Level Implied by the Heterogeneity along the Price System

The proof is achieved for the young household demand function, it is symmetrical for the old household demand function.

Clearly if the parameterization of demand satisfies assumptions 15, 16 and 17 for small price changes (current and expected) the assumption of homogeneity of demand implies that these assumptions hold for small income variations. Under assumption 15 which holds for small variations of the current price system and for small variations of the expected price system, whatever $p, p_{t+1}^e \in \Re_{++}^l$, $x \in \Re_+$ and $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that whatever $\triangle$ close to the nul vector

$$e^{\gamma'}(p, p_{t+1}^e, x) = e^{\gamma}(I_\triangle p, I_\triangle p_{t+1}^e, x) \tag{20}$$

Choose $I_\triangle$ such that

$$I_\triangle = \begin{pmatrix} 1+\triangle & & 0 \\ & \ddots & \\ 0 & & 1+\triangle \end{pmatrix} \tag{21}$$

where $\triangle$ is a parameter close to zero. Therefore, there exist $\gamma' \in \Gamma$ such that whatever $\triangle$ close to zero

$$e^{\gamma'}(p, p_{t+1}^e, x) = e^{\gamma}(p, p_{t+1}^e, (1+\triangle)^{-1}x) \tag{22}$$

Hence, assumptions 15, 16 and 17 are true when perturbations of income are concerned and yield to define the homeomorphisms $\tau_{\triangle^*}$, where $\triangle^*$ is a parameter given by $\triangle^* = \frac{\triangle}{1+\triangle}$ which converges towards zero when $\triangle$ converges towards zero. To conclude if the demand function is highly heterogeneous along the current price vector and the expected price vector it is as well highly heterogeneous along the income level.

A.3 Proof of Lemma 2

(i) Under the assumption that the set of households' characteristics is bounded we deduce from the mean value theorem that there exists a household described by $\tilde{\epsilon}$ such that

$$\int_{\mathcal{B}\times\Omega} B_0^\gamma D_r^\epsilon d\nu_o = \partial_{p_{t-r-1}}\psi^{\tilde{\epsilon}_o}\int_{\mathcal{B}\times\Omega}\partial_{p_t^e} f^\gamma d\nu_o$$

For any small perturbation $\triangle$ of the price system p_{t-r-1}, we have

$$\begin{aligned}
&\| \mathcal{P}\int_{\mathcal{B}\times\Omega} B_0^\gamma D_{r+1}^\epsilon d\nu_o \mathcal{P} \| \\
&=\| \mathcal{P}\partial_{p_{t-r-1}}\psi^{\tilde{\epsilon}_o}\int_{\mathcal{B}\times\Omega} f^\gamma \partial_\triangle \varphi_{o\triangle} \mid_{\triangle=0} \phi_o(\epsilon)\rho_o(\omega)d\beta d\omega \| \\
&= \int_{\mathcal{B}\times\Omega}\max_i[\sum_j \mid \sum_k \partial_{p_{t-r-1;k}}\psi_i^{\tilde{\epsilon}_o} p_k f_k^\gamma \partial_{\triangle_j}\varphi_{o\triangle} \mid_{\triangle=0}|]\phi_o(\epsilon)\rho_o(\omega)d\beta d\omega \\
&< x_{o\,\max}\max_i \int_{\mathcal{B}\times\Omega} \mid \sum_k \partial_{p_{t-r-1;k}}\psi^{\tilde{\epsilon}_o} \mid \sum_j \mid \partial_{\triangle_j}\varphi_{o\triangle} \mid_{\triangle=0}| \ \phi_o(\epsilon) \\
&\rho_o(\omega)d\beta d\omega
\end{aligned}$$

Denote $\tilde{\psi}_{r+1;i}^o =| \sum_k \partial_{p_{t-r-1;k}}\psi_i^{\tilde{\epsilon}_o} \mid$ and $\tilde{\psi}_{r+1}^o = \max_i \tilde{\psi}_{r+1;i}^o$. $\tilde{\psi}^y$ is defined symmetrically. Then the inequalities below hold:

$$\begin{aligned}
\| \mathcal{P}\int_{\mathcal{B}\times\Omega} B_0^\gamma D_{r+1}^\epsilon d\nu_o \mathcal{P} \| &< x_{o\,\max}\tilde{\psi}_{r+1}^o q_1 m_1 l \\
\| \mathcal{P}\int_{\mathcal{B}\times\Omega} B_1^\gamma C_r^\epsilon d\nu_y \mathcal{P} \| &< x_{y\,\max}\tilde{\psi}_r^y q_1 m_1 l
\end{aligned}$$

for $r = 0, \ldots, T$. ¶

(ii) Under the assumption that the set of households' characteristics is bounded we deduce from the mean value theorem that there exists a household described by $(\tilde{\beta}, \tilde{\omega})$ such that

$$\int_{\mathcal{B}\times\Omega} B_0^\gamma D_{r+1}^\epsilon d\nu_o = \partial_{p_t^e} f^{\tilde{\beta}_o} \mid_{\tilde{x}=p^T\tilde{\omega}} \int_{\mathcal{B}\times\Omega}\partial_{p_{t-r-1}}\psi^\epsilon d\nu_o$$

For any small perturbation $\triangle$ of the price system p_{t-r-1}, we have

$$\begin{aligned}
&\| \mathcal{P}\int_{\mathcal{B}\times\Omega} B_0^\gamma D_{r+1}^\epsilon d\nu_o \mathcal{P} \| \\
&=\| \mathcal{P}\partial_{p_t^e} f^{\tilde{\beta}_o}\int_{\mathcal{B}\times\Omega}\psi^\epsilon \partial_\triangle \phi_{o\triangle} \mid_{\triangle=0} \varphi_o(\gamma)\rho_o(\omega)d\beta d\omega \| \\
&= \int_{\mathcal{B}\times\Omega}\max_i[\sum_j \mid \sum_k p_i \partial_{p_{t;k}^e} f_i^{\tilde{\beta}_o}\psi_k^\epsilon \partial_{\triangle_j}\phi_{o\triangle} \mid_{\triangle=0}| \ \varphi_o(\gamma)\rho_o(\omega)d\beta d\omega \\
&< \max_i \tilde{f}_i^o \int_{\mathcal{B}\times\Omega}\sum_j \mid \partial_{\triangle_j}\phi_{o\triangle} \mid_{\triangle=0}| \ \varphi_o(\gamma)\rho_o(\omega)d\beta d\omega
\end{aligned}$$

where $\tilde{f}_i^o = | \sum_k p_i p_k \partial_{p^e_{t;k}} f_i^{\tilde{\beta}_o} |$ and $\tilde{f}^o = \max_i \tilde{f}_i^o$. $\tilde{f}_y$ is defined symmetrically. Then the inequalities below hold:

$$\begin{aligned} \| \mathcal{P} \int_{\mathcal{B}\times\Omega} B_0^\gamma D_{r+1}^\epsilon d\nu_o \mathcal{P} \| &< \tilde{f}^o q_2 m_2 l \\ \| \mathcal{P} \int_{\mathcal{B}\times\Omega} B_1^\gamma C_r^\epsilon d\nu_y \mathcal{P} \| &< \tilde{f}^y q_2 m_2 l \end{aligned}$$

for $r = 0, \ldots, T$. ¶

A.4 Proof of Theorem 1

(i) The monotonicity of market demand is translated mathematically by the negative semidefiniteness of the Jacobian matrix of market demand $\mathcal{J}_\mu$. Such a property is additive, therefore it remains to prove that the Jacobian matrix of each subgroup $\mathcal{J}_\nu$ is negative semidefinite. $\mathcal{J}_\nu$ is negative semidefinite if and only if $\mathcal{P}\mathcal{J}_\nu\mathcal{P}$ is negative semidefinite.

$$\begin{aligned} &\mathcal{P}\mathcal{J}_{\nu/x}\mathcal{P} \\ &= \int_{\mathcal{B}} (p_1 \partial_p \bar{f}_1^\beta(s,x), \ldots, p_l \partial_p \bar{f}_l^\beta(s,x))^T \mathcal{P} d\nu/x \\ &= \partial_\triangle \int_{\mathcal{B}} x\Big(\frac{e_1^\gamma(I_\triangle p, p^e_{t+1}, x)}{p_1(1+\Delta_1)}, \ldots, \frac{e_l^\gamma(I_\triangle p, p^e_{t+1}, x)}{p_l(1+\Delta_{l+1})}\Big)^T \mathcal{P}\eta_y(\beta) d\beta \\ &+ \partial_\triangle \int_{\mathcal{B}} x\Big(\frac{e_1^\gamma(I_\triangle p, p_{-1}, p^e_t, x)}{p_1(1+\Delta_1)}, \ldots, \frac{e_l^\gamma(I_\triangle p, p_{-1}, p^e_t, x)}{p_l(1+\Delta_{l+1})}\Big)^T \mathcal{P}\eta_o(\beta) d\beta \\ &+ x \int_{\mathcal{B}} \partial_\triangle [e_1^\gamma(p, \psi^{\epsilon_y}(I_\triangle p), x), \ldots, e_l^\gamma(p, \psi^{\epsilon_y}(I_\triangle p), x)]^T_{\triangle=0} \varphi_y(\gamma)\phi_y(\epsilon) d\gamma d\epsilon \\ &= -x \int_{\mathcal{B}} \begin{pmatrix} \bar{e}_1^\beta(s,x) & & 0 \\ & \ddots & \\ 0 & & \bar{e}_l^\beta(s,x) \end{pmatrix} \eta(\beta) d\beta \\ &+ x \int_{\mathcal{B}} \partial_\triangle [e_1^\gamma(I_\triangle p, p^e_{t+1}, x), \ldots, e_l^\gamma(I_\triangle p, p^e_{t+1}, x)]^T_{\triangle=0} \varphi_y(\gamma)\phi_y(\epsilon) d\gamma d\epsilon \\ &+ x \int_{\mathcal{B}} \partial_\triangle [e_1^\gamma(I_\triangle p, p_{-1}, p^e_t, x), \ldots, e_l^\gamma(I_\triangle p, p_{-1}, p^e_t, x)]^T_{\triangle=0} \varphi_o(\gamma)\phi_o(\epsilon) d\gamma d\epsilon \\ &+ x \int_{\mathcal{B}} \partial_\triangle [e_1^\gamma(p, x, \psi^{\epsilon_y}(I_\triangle p)), \ldots, e_l^\gamma(p, x, \psi^{\epsilon}(I_\triangle p))]^T_{\triangle=0} \varphi_y(\gamma)\phi_y(\epsilon) d\gamma d\epsilon \end{aligned}$$

where the superscript "T" refers to the transposed vector. The last matrix aggregated over endowments can be written:

$$\begin{aligned}&\int_{\mathcal{B}\times\Omega} x\partial_{p^e_{t+1}} e^\gamma(p,x,p^e_{t+1})\partial_{p_t}\psi^\epsilon d\nu_y\\&=\int_{\mathcal{B}\times\Omega} x\partial_\triangle e^\gamma(p,x,I_\triangle p^e_{t+1})\partial_{p_t}\psi^\epsilon d\nu_y\\&=\int_{\mathcal{B}\times\Omega} e^\gamma\partial_\triangle\varphi_{y\triangle}\mid_{\triangle=0}\partial_{p_t}\psi^\epsilon\phi_y(\epsilon)\rho_y(\omega)d\beta d\omega\end{aligned}$$

whose norm, according to Lemma 2 is bounded above by $x_{y\,\max}q_1m_1l\tilde{\psi}^y$. Thus, we get the following inequalities:

$$\begin{aligned}p_h^2\mid\partial_{p_h}F_h\mid &> \bar{x}\varepsilon_h - x_{\max}q_1m_1 - x_{y\,\max}q_1m_1l\tilde{\psi}_0^y\\ p_kp_h\mid\partial_{p_k}F_h\mid &< x_{\max}q_1m_1 + x_{y\,\max}q_1m_1l\tilde{\psi}_0^y\end{aligned}$$

From these inequalities we deduce that $p_h\mid\partial_{p_h}F_h\mid>\sum_{k\neq h}p_k\mid\partial_{p_k}F_h\mid$ if

$$\varepsilon_h\bar{x}-x_{\max}q_1m_1-x_{\max}q_1m_1l\tilde{\psi}_0^y>(l-1)x_{\max}q_1m_1+(l-1)lx_{y\,\max}q_1m_1\tilde{\psi}_0^y$$

This inequality is equivalent to

$$m_1q_1<\frac{\varepsilon\bar{x}}{l(x_{\max}+lx_{y\,\max}\tilde{\psi}_0^y)}$$

Symmetrically, the same inequality implies that $p_h\mid\partial_{p_h}F_h\mid>\sum_{k\neq h}p_k\mid\partial_{p_h}F_k\mid$.

(ii) Under the assumption that the set of households' characteristics is bounded the Lemma 2 states that the norm of $\mathcal{P}\int_{\mathcal{B}\times\Omega}\partial_{p^e_{t+1;k}}f_h^\beta\partial_{p_{t-j}}\psi^\epsilon d\nu_y\mathcal{P}$ is bounded above by $q_2m_{2,t-j}l\tilde{f}^y$. The following inequalities hold:

$$p_h^2\mid\partial_{p_h}F_h\mid > \bar{x}\varepsilon - x_{\max}q_1m_1 - \tilde{f}^ylq_2m_{2,t} \quad (23)$$

$$p_kp_h\mid\partial_{p_k}F_h\mid < \tilde{f}^ylq_2m_{2,t} + x_{\max}q_1m_1 \quad (24)$$

Therefore, it holds that $p_h\mid\partial_{p_h}F_h\mid>\sum_{k\neq h}p_k\mid\partial_{p_k}F_h\mid$ if

$$q_2m_{2,t}<\frac{\varepsilon\bar{x}-lx_{\max}q_1m_1}{l^2\tilde{f}^y}$$

The same inequality ensures that $p_h\mid\partial_{p_h}F_h\mid>\sum_{k\neq h}p_k\mid\partial_{p_h}F_k\mid$. ¶

A.5 Proof of Lemma 4

The lemma is directly deduced from Eq.11, Eq.12 and Lemma 1 which implies that if households extrapolate orbits of period two, then whatever ϵ: $\sum_{j=0}^{T} C_j^{\epsilon} = I_l$, $\sum_{j=1}^{T+1} D_j^{\epsilon} = I_l$, $\sum_{j=0}^{T}(-1)^j C_j^{\epsilon} = -I_l$ and $\sum_{j=1}^{T+1}(-1)^j D_j^{\epsilon} = I_l$. ¶

A.6 Proof of Theorem 3

(i) We first define some bounds to the term $\| \tilde{L}_0^{-1}\tilde{L}_1 \|$. A result of linear algebra is that if $f(A)$ is a norm of the matrix A then $f(M^{-1}AM)$ is as well a norm of A whatever the non singular matrix M. In the sequel we define the matrix norm by $g(A) = f(\mathcal{P}^{-1}A\mathcal{P})$ whatever the matrix A, where f is the row matrix norm and $\mathcal{P}$ is the diagonal matrix with the price system p on the diagonal. We have

$$\mathcal{P}^{-1}\tilde{L}_0\tilde{L}_1\mathcal{P} = (\mathcal{P}\tilde{L}_0\mathcal{P})^{-1}\mathcal{P}\tilde{L}_1\mathcal{P}$$

As a result

$$\| \mathcal{P}^{-1}\tilde{L}_0\tilde{L}_1\mathcal{P} \| \leq \| (\mathcal{P}\tilde{L}_0\mathcal{P})^{-1} \| \| \mathcal{P}\tilde{L}_1\mathcal{P} \|$$

where

$$\begin{aligned}
&\| \mathcal{P}A_0\mathcal{P} \| \\
&\geq \| \int_{\mathcal{B}\times\Omega} \begin{pmatrix} p_1 f_1^{\beta_y} & & & 0 \\ & p_2 f_2^{\beta} & & \\ & & \ddots & \\ 0 & & & p_l f_l^{\beta} \end{pmatrix} d\nu \| \\
&- \| \int_{\mathcal{B}\times\Omega} \begin{pmatrix} p_1 f_1^{\beta_y} \\ p_2 f_2^{\beta} \\ \vdots \\ p_l f_l^{\beta} \end{pmatrix} \partial_{\triangle_1}\varphi_{\triangle_1}(\gamma) \mid_{\triangle_1=0} \phi(\epsilon)\rho(\omega) d\beta d\omega \|
\end{aligned}$$

$$+ \parallel \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 & \frac{p_1 f_1^{\beta y} p_2 \omega_2}{x} & \dots & \frac{p_1 f_1^{\beta y} p_l \omega_l}{x} \\ \vdots & \frac{p_2 f_2^{\beta} p_2 \omega_2}{x} & \dots & \frac{p_2 f_2^{\beta} p_l \omega_l}{x} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{p_l f_l^{\beta} p_2 \omega_2}{x} & \dots & \frac{p_l f_l^{\beta} p_l \omega_l}{x} \end{pmatrix} d\nu \parallel$$

$$- \parallel \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 & \frac{p_2 f_2^{\beta y} p_2 \omega_2}{x} & \dots & \frac{p_2 f_2^{\beta y} p_l \omega}{x} \\ \vdots & \frac{p_2 f_2^{\beta} p_2 \omega_2}{x} & \dots & \frac{p_2 f_2^{\beta} p_l \omega_l}{x} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{p_l f_l^{\beta} p_2 \omega_2}{x} & \dots & \frac{p_l f_l^{\beta} p_l \omega_l}{x} \end{pmatrix} \partial_{\triangle_2} \varphi_{\triangle_2}(\gamma) \mid_{\triangle_2 = 0}$$

$$\phi(\epsilon)\rho(\omega) d\beta d\omega \parallel$$

where the two first terms give a lower bound to the impact of a variation of current prices for fixed income and the two last terms give a lower bound of the variation of demand induced by the income variation induced by this price change. Note "$\triangle_1$" is a vector and refers to perturbations of current prices and "$\triangle_2$" is a parameter and refers to perturbations of income variation. The first columns of the two last matrices are equal to the nul vector since a variation of the price of money does not affect the nominal income (the household does not have any endowment in money). Under the desirability assumption, we deduce:

$$\mathcal{P} A_0 \mathcal{P} \geq \varepsilon \bar{x} - q_1 m_1 l x_{\max} - q_1 m_1 x_{\max} \tag{25}$$

Whatever $j = 0, \dots, T$, we have according to Lemma 2

$$\begin{aligned} \parallel \mathcal{P} \int_{\mathcal{B}\times\Omega} B_1^{\gamma} C_j^{\epsilon} d\nu_y \mathcal{P} \parallel &< x_{y\max} \tilde{\psi}_j^y l q_1 m_1 \parallel \mathcal{P} \int_{\mathcal{B}\times\Omega} B_0^{\gamma} D_{j+1}^{\epsilon} d\nu_o \mathcal{P} \parallel \\ &< q_1 m_1 l x_{o\max} \tilde{\psi}_j^o \end{aligned}$$

$$\begin{aligned} \parallel \mathcal{P} A_1 \mathcal{P} \parallel &= \parallel \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 \\ \frac{p_2^2 f_2^{\beta_o}}{p_{t-1;2}} \\ \vdots \\ \frac{p_l^2 f_l^{\beta_o}}{p_{t-1;l}} \end{pmatrix} \partial_{\triangle} \varphi_{\triangle} \mid_{\triangle = 0} (\gamma) \phi(\epsilon) \rho(\omega) d\beta d\omega \parallel \\ &< l q_1 m_1 x_{o\max} \end{aligned}$$

Consequently the following inequalities hold:

$$\| \mathcal{P}(\tilde{L}_0)^{-1}\tilde{L}_1\mathcal{P} \| < \frac{lq_1m_1x_{o\,\max} + lq_1m_1x_{o\,\max}\tilde{\psi}_0^o + lq_1m_1x_{y\,\max}\tilde{\psi}_1^y}{\varepsilon\bar{x} - q_1m_1(l+1)x_{\max} - lq_1m_1x_{y\,\max}\tilde{\psi}_0^y}$$

$$\| \mathcal{P}(\tilde{L}_0)^{-1}\tilde{L}_j\mathcal{P} \| < \frac{q_1m_1lx_{o\,\max}\tilde{\psi}_j^o + q_1m_1lx_{y\,\max}\tilde{\psi}_j^y}{\varepsilon\bar{x} - q_1m_1(l+1)x_{\max} - lq_1m_1x_{y\,\max}\tilde{\psi}_0^y}$$

$$\| \mathcal{P}(\tilde{L}_0)^{-1}\tilde{L}_{T+1}\mathcal{P} \| < \frac{q_1m_1l\bar{x}_o\tilde{\psi}_{T+1}^o}{\varepsilon\bar{x} - q_1m_1(l+1)x_{\max} - lq_1m_1x_{y\,\max}\tilde{\psi}_0^y}$$

forall $j = 2, \ldots, T$. Thus Fuchs and Laroque's condition of local asymptotic stability of any STE can be written

$$q_1m_1 < \frac{\varepsilon\bar{x}}{(l+1)x_{\max} + lx_{0\,\max} + lx_{0\,\max}\sum_{j=0}^{T}\tilde{\psi}_j^o + lx_{y\,\max}\sum_{j=0}^{T}\tilde{\psi}_j^y}$$

Under the assumption of stationarity of expectations the heterogeneity of the expectations function is restricted such that $\sum_{j=0}^{T}\tilde{\psi}_j^y \geq 1$ and $\sum_{j=0}^{T}\tilde{\psi}_j^o \geq 1$. Thus, a sufficient condition for the stability of the dynamics with learning is that

$$q_1m_1 < \frac{\varepsilon\bar{x}}{(l+1)x_{\max} + 2lx_{0\,\max} + lx_{y\,\max}}$$

(ii) Under Lemma 2 the inequalities below hold:

$$\| \mathcal{P}\int_{\mathcal{B}\times\Omega} B_0^{\gamma}D_{r+1}^{\epsilon}d\nu_o\mathcal{P} \| < \tilde{f}^o q_2m_2l$$

$$\| \mathcal{P}\int_{\mathcal{B}\times\Omega} B_1^{\gamma}C_r^{\epsilon}d\nu_y\mathcal{P} \| < \tilde{f}^y q_2m_2l$$

for $r = 0, \ldots, T$. Thus, Fuchs and Laroque's sufficient condition for the local asymptotic stability of the dynamics with learning at any stationary equilibrium can be written:

$$q_1m_1(lx_{o\,\max} + (l+1)x_{\max}) + q_2m_2(T+1)l(\tilde{f}^o + \tilde{f}^y) < \varepsilon\bar{x}$$

The assumption of stationarity of expectations leads to restrict the heterogeneity of expectations in such a way that $l(T+1)q_2m_2 > 1$, thereby, the above inequality is feasible only if:

$$\frac{\varepsilon\bar{x}}{\tilde{f}^o + \tilde{f}^y} > 1$$

The satisfaction of this inequality depends on the degree of heterogeneity of household demand which affects the parameter $\tilde{f}_o$ and $\tilde{f}_y$. It is always true for a high dispersion of household demand since in this case (see Lemma 2) $\int_{\mathcal{B}\times\Omega} B_0^\gamma D_j^\epsilon d\nu_o$ and $\int_{\mathcal{B}\times\Omega} B_1^\gamma C_j^\epsilon d\nu_y$ are close to zero, this can only be explained by parameters $\tilde{f}_o$ and $\tilde{f}_y$ close to zero - since by the stationarity assumption the norm of the derivatives of the aggregate expectations function is bounded away from zero. ¶

A.7 Proof of Theorem 4

Using the same techniques as in the Proof of Theorem 3 we get the following inequalities:

$$\begin{aligned} \mathcal{P}A_0\mathcal{P} &\leq \bar{x} + lq_1 m_1 x_{\max} + \bar{x} + q_1 m_1 x_{\max} \\ \| \mathcal{P}A_1\mathcal{P} \| &< lq_1 m_1 x_{o\max} \\ \| \mathcal{P}B_0\mathcal{P} \| &< lq_1 m_1 x_{o\max} \end{aligned}$$

Clearly, to ensure the satisfaction of the sufficient condition of stability of the dynamics of perfect foresight one has to restrict the degree of the heteroegeneity of the young household demand function along the vector of expected prices. Denote $I_\triangle$ the matrix of perturbations affecting the expected price vector and $\eta_\triangle$ the density function of the transformed parameter $\tau_\triangle(\beta)$ where only expected prices have been affected. We require that

$$\max_{i=1,\ldots,l} \int_{\mathcal{B}\times\Omega} | \partial_{\triangle_i}\varphi_\triangle(\gamma) | \phi(\epsilon)\rho(\omega) d\beta d\omega > k^y_{1,p^e}$$

Therefore,

$$\begin{aligned} \| \mathcal{P}B_1\mathcal{P} \| &= \max_i [| p_i f_i^{\tilde{\beta}_y} | \int_{\mathcal{B}\times\Omega} \sum_j | \partial_{\triangle_j}\varphi_\triangle | \phi(\epsilon)\rho(\omega) d\beta d\omega \\ &\geq lk^y_{1,p^e}\tilde{\varepsilon}\tilde{x}_y \end{aligned}$$

As a result a sufficient condition for stability is given by the inequality:

$$\frac{2\bar{x} + (l+1)q_1 m_1 x_{\max} + 2lq_1 m_1 x_{o\max}}{\tilde{\varepsilon} l k^y_{1,p^e}\tilde{x}_y} < 1 \qquad ¶$$

A.8 Proof of Theorem 5

The stationary state is a saddle point of the perfect foresight dynamics. Therefore, from Lemma 4 we deduce that

$$\begin{aligned} P_h(1) &= -\text{sgn}(\frac{\det B_1}{\det \tilde{L}_0}) \\ (-1)^{(T+1)l} P_X(-1) &= -\text{sgn}(\frac{\det B_1}{\det \tilde{L}_0}) \end{aligned}$$

It is a well known result that if $P_h(1) > 0$ all eigenvalues of P_h are lower than one and if $(-1)^{(T+1)l} P_h(-1) > 0$ all eigenvalues of P_h are higher than -1. Therefore, the dynamics with learning are locally asymptotically stable at the STE if $\det\tilde{L}_0$ and $\det B_1$ are of opposite sign. Since $\det\tilde{L}_0 = \det B_1 \det(B_1^{-1} A_0 + \int_{\mathcal{E}} C_0^{\epsilon} d\nu)$, it follows that,

$$\text{sgn}(\det(B_1^{-1} A_0 + \int_{\mathcal{E}} C_0^{\epsilon} d\nu_y)) = \text{sgndet}(B_1^{-1} A_0) = -1 \qquad \P$$

A.9 Proof of Theorem 7

(i) The following inequalities hold:

$$\begin{aligned} \| \mathcal{P} A_0 \mathcal{P} \| &\geq \varepsilon \bar{x} - q_1 m_1 x_{y\max} \\ \| \mathcal{P} B_1 \mathcal{P} \| &\leq q_1 m_1 x_{y\max} \end{aligned}$$

Hence, a sufficient condition for $\bar{p} \in W$ to be a saddle point is given by the inequality

$$\frac{\varepsilon \bar{x} - q_1 m_1 x_{y\max}}{x_{y\max} q_1 m_1} > 1 \qquad \P$$

(ii) The following inequality holds:

$$\| \mathcal{P} A_0 \mathcal{P} \| \leq (1 + q_1 m_1) x_{y\max}$$

We restrict the degree of heterogeneity of the young household demand function along the current price system such that:

$$\min_{j=1,\ldots,l} \int_{\mathcal{B}\times\Omega} | \partial_{\triangle_j} \varphi_{\triangle}(\gamma) | \phi(\epsilon)\rho(\omega) d\beta d\omega > k^y_{1,p^e}$$

Therefore $\| \mathcal{P} B_1 \mathcal{P} \| > k^y_{1,p^e} \varepsilon \tilde{x}_y$. Hence, a sufficient condition for stability of the dynamics of perfect foresight at $\bar{p} \in P$ is that:

$$\frac{(1+q_1 m_1) x_{y\max}}{k^y_{1,p^e} \varepsilon \tilde{x}_y} < 1 \qquad \P$$

A.10 Proof of Theorem 8

(i) The standard sufficient condition of stability is written, for the present example:

$$\sum_{j=1}^{T} | C_j | < \left| \frac{a_{t+1}\omega_t}{a_t \omega_{t+1}} - C_0 \right|$$

The first part of the Theorem is easily deduced from this inequality.

(ii) The dynamics is described by the equation

$$p_t = h(p_{t-1}, \ldots, p_{t-T-1}) \tag{26}$$

The Jacobian matrix of $\partial \tilde{g}$ is written

$$\partial \tilde{g} = \begin{pmatrix} -(\tilde{L}_0)^{-1} \tilde{L}_1 & \ldots & \ldots & -(\tilde{L}_0)^{-1} \tilde{L}_{T+1} \\ 1 & & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix}$$

Note that $\tilde{L}_{T+1} = 0$. It implies that one of the eigenvalues of ∂h is null $\lambda_{T+1} = 0$. Finding one of the other eigenvalues of ∂h amounts to looking for a complex number λ such that:

$$\lambda^T v = - \sum_{j=1}^{T} \lambda^{T-j} (\tilde{L}_0)^{-1} \tilde{L}_j v$$

Thus the characteristic equation of Eq.26 is

$$P_h(\lambda) = \lambda^T + \sum_{j=1}^{T} \lambda^{T-j} (\tilde{L}_0)^{-1} \tilde{L}_j = 0$$

It is a well kown result that the constant term of this equation can be expressed in terms of the product of the eigenvalues, that is:

$$(\tilde{L}_0)^{-1}\tilde{L}_T = (-1)^T \prod_{i=1}^{T} \lambda_i$$

Hence,

$$\| (\tilde{L}_0)^{-1}\tilde{L}_T \| < 1 \tag{27}$$

implies that $| \prod_{i=1}^{T} \lambda_i | > 1$ which, in turn, implies that at least one eigenvalue has a modulus higher than one, that is, any stationary temporary equilibrium is unstable. A sufficient condition for instability of any STE is given by Eq.27. This inequality holds

- when $C_0 < \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$, if

$$| C_T | > \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}} - C_0$$

- when $C_0 > \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}}$, if

$$| C_T | > C_0 - \frac{a_{t+1}\omega_t}{a_t\omega_{t+1}} \qquad \P$$

A.11 Proof of Theorem 9

At the limit, for m_1 close to zero, $\mathcal{P}\tilde{L}_0\mathcal{P}$ is close to the diagonal matrix

$$\mathcal{P}\tilde{L}_0\mathcal{P} = \int_{\mathcal{B}\times\Omega} \begin{pmatrix} p_1 f_1^{\beta_y} & & & 0 \\ & p_2 f_2^{\beta} & & \\ & & \ddots & \\ 0 & & & p_l f_l^{\beta} \end{pmatrix} d\nu$$

$\mathcal{P}\tilde{L}_1\mathcal{P}$ is close to the matrix

$$\mathcal{P}\tilde{L}_1\mathcal{P} = \int_{\mathcal{B}\times\Omega} \begin{pmatrix} 0 & \frac{p_1 f_1^{\beta_y} p_2\omega_2}{x} & \dots & \frac{p_1 f_1^{\beta_y} p_l\omega_l}{x} \\ \vdots & \frac{p_2 f_2^{\beta} p_2\omega_2}{x} & \dots & \frac{p_2 f_2^{\beta} p_l\omega_l}{x} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{p_l f_l^{\beta} p_2\omega_2}{x} & \dots & \frac{p_l f_l^{\beta} p_l\omega_l}{x} \end{pmatrix}$$

while the matrices $\tilde{L}_j$ for $j = 2, \ldots, T+1$ almost disappear. As a result, the norm $\sum_{j=1}^{T+1} \parallel (\tilde{L}_0)^{-1}\tilde{L}_j \parallel$ is close to the norm of the product of the two above matrices. Under the assumption that the set of households' characteristics is bounded, there exists $\tilde{\omega}$ such that

$$\int_{\mathcal{B}\times\Omega} \frac{f_i^\beta \omega_j}{x} d\nu_y = \frac{\tilde{\omega}_j}{\tilde{x}} \int_{\mathcal{B}\times\Omega} f_i^\beta d\nu_y \qquad \forall i, j = 1, \ldots, l$$

This implies that $\sum_{j=1}^{T+1} \parallel (\tilde{L}_0)^{-1}\tilde{L}_j \parallel$ is close to the norm of the following matrix

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (0, \frac{p_2\tilde{\omega}_2}{\tilde{x}}, \ldots, \frac{p_l\tilde{\omega}_l}{\tilde{x}})$$

The norm of this matrix is $\sum_{j=2}^{l} \frac{p_j\tilde{\omega}_j}{\tilde{x}}$ which is strictly lower than one as long as $\tilde{\omega}_{l+1} > 0$. ¶

A.12 Stability of the Learning Process

The Jacobian matrix of the learning process is

$$L = \begin{pmatrix} C_0 & \ldots & \ldots & C_T \\ I_l & & o & O \\ & \ddots & & \vdots \\ 0 & & I_l & 0 \end{pmatrix}$$

The characteristic equation associate to the dynamics of the learning process is

$$I_l \lambda^{T+1} - \sum_{j=0}^{T} C_j \lambda^{T-j} =$$

The roots of this equation are the eigenvalues of the matrix L which are all of modulus lower than one if $(L - \mu I)$ is invertible forall μ such that $| \mu |> 1$. Thus, a sufficient condition for stability of the learning process is that

$$\sum_{j=0}^{T} \parallel C_j \parallel < 1 \qquad ¶$$

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