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Endogenous growth and time-to-build: the AK case*

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Abstract

In this paper, a continuous time AK model is fully analyzed under the time-to-build assumption. Existence and uniqueness of a balance growth path, as well as oscillatory convergence are proved. Moreover, the role of transversality conditions and capital depreciation are highlighted. Numerical simulations are also provided for different choices of the time-to-build delay.
Keywords: AK Model; Time-to-Build; D-Subdivision method.
JEL Classification: E00, E3, O40.

1 Introduction

Recently Boucekkine et al. [10], have studied the dynamics of an AK-type endogenous growth model with vintage capital. They find that vintage capital leads to oscillatory dynamics governed by replacement echoes consistently with previous results in Benhabib and Rustichini [5], and Boucekkine et al. [9]. In this paper, we propose an AK endogenous growth model under the assumption that capital takes time to become productive. In the literature, this assumption is often referred as "time-to-build".

Jevons [19], was one of the first to underline the empirical relevance of this assumption: "A vineyard is unproductive for at least three years before it is thoroughly fit for use. In gold mining there is often a long delay, sometimes even of five or six years, before gold is reached"[1]. The time dimension of capital was further studied by Hayek [17], who identified in the time of production one of the possible sources of aggregate fluctuations. Hayek's insight was formally confirmed for the first time by Kalecki [20], and afterward by Kydland and Prescott [21], who showed that it contributes to the persistence of the business cycle. In this paper, the time-to-build assumption is introduced by a delay differential equation for capital. Delay differential equations, and in general, functional differential equations are very interesting but, at the same time, quite complicated mathematical objects. Since the first contributions of Kalecki [20], Frisch and Holme [14], and, Belz and James [7], very few authors have used this mathematical instrument for modeling the time structure of capital. To our knowledge, the only works in (exogenous) growth theory introducing time-to-build in this way, are Rustichini [24], Asea and Zak [1], and Collard et al. [12]. All these papers find that for values of the delay coefficient which are sufficiently small, time-to-build is responsible for the oscillatory behavior of capital, output and investment.

In this paper, some theorems regarding the existence, uniqueness and shape of the general (continuous) solution of a linear delay differential equation with forcing term are presented in

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[1] Jevons [19], Chapter VII: Theory of Capital, page 225.

details, and a "new" method to prove stability, the D-Subdivision method, is introduced. This method is really useful since it let us count the number of roots (eigenvalues) having positive real part even if the dimension of the set of the roots is infinite. Taking into account this theoretical background, the existence of a unique balance growth path and the dynamic behaviors of the detrended variables are fully analyzed.

The paper is organized as follows. We firstly present the model setup in Section 2 and we derive the first order conditions by applying a variation of the Pontrjagin's maximum principle. In Section 3, we introduce some mathematical results on the theory of functional differential equations and the D-Subdivision method. Then the existence and uniqueness of the balance growth path is proved and the influence of a variation of the delay coefficient on the magnitude of the growth rate is fully analyzed and reported also in a picture. The transitional dynamics of the economy is reported in Section 5. The next section makes some considerations regarding the role of capital depreciation on the dynamic behavior of capital. In particular, we explain how the introduction of an hypothesis of depreciation "before use" let us extend the results to any choice of the time-to-build delay. A numerical example showing the dynamic behavior of the economy is reported in Section 6. Finally, in Section 7 there are some concluding remarks.

2 Problem Setup

We analyze a standard one sector AK model with time-to-build. To be precise we assume from now on that capital takes d years to become productive. Then the social planner solves the following problem

$$\max \int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$\dot{k}(t) = \tilde{A} k(t-d) - c(t) \tag{1}$$

given initial condition $k(t) = k_0(t)$ for $t \in [-d, 0]$ with $d > 0$. All the variables are per capita. The parameter $\tilde{A} = (A - \delta) e^{-\phi d} > 0$ depends on A, the productivity level, δ, the usual capital depreciation, and ϕ, the depreciation rate of capital before it becomes productive. From now on we refer to it as depreciation "before use". Given this capital depreciation structure, $k(t-d)e^{-\phi d}$ is net capital at the time it becomes productive. Observe that the lower d is, the higher is the net capital which is effectively employed in production. Moreover let us assume $\phi > 0$, which may be justified by referring to the depreciation in use literature (see Greenwood *& al.* [15], and Burnside and Eichenbaum [11])[2]. Finally, with no time-to-build the problem becomes a standard AK model. Following Kolmanovskii and Myshkis [22] it is possible to extend the *Pontrjagin's principle* to this optimal control problem. Then, the Hamiltonian for this system can be constructed:

$$\mathcal{H}(t) = \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} + \mu(t) \left[\tilde{A} k(t-d) - c(t)\right]$$

and its optimality conditions are

$$c(t)^{-\sigma} e^{-\rho t} = \mu(t) \tag{2}$$

$$\mu(t+d)\tilde{A} = -\dot{\mu}(t) \tag{3}$$

with the standard transversality conditions

$$\lim_{t\to\infty} \mu(t) \geq 0 \quad and \quad \lim_{t\to\infty} \mu(t) k(t) = 0$$

[2] Remembering Jevon's example, the introduction of the depreciation "before use" hypothesis, let us take into account the effect of exogenous elements (such as bad weather etc.) on the vineyard before it becomes productive.

From equations (2) and (3) we can get the *forward looking Euler-type equation*

$$\frac{\dot{c}\,(t)}{c(t)} = \frac{1}{\sigma}\left[\tilde{A}\left(\frac{c(t)}{c(t+d)}\right)^{\sigma} e^{-\rho d} - \rho\right] \tag{4}$$

Exactly as in the standard AK model, consumption growth does not depend on the stock of capital per person. However in our context the positive constant growth rate is not explicitly given by the Euler equation which is a nonlinear advanced differential equation in consumption. This difference is due to the fact that the real interest rate $r = \tilde{A}\left(\frac{c(t)}{c(t+d)}\right)^{\sigma} e^{-\rho d}$, which the household gets investing in capital, is weighted by the marginal elasticity of substitution between consumption at time t and consumption at time $t+d$. Before proceeding with the analysis of the BGP of our economy, we present in the next section the mathematical instruments which will be used to prove the main results and characteristic of the economy under study.

3 Some Preliminary Results

Before proceeding, let us evoke some theoretical results from functional differential analysis. Consider the general linear delay differential equation with forcing term $f(t)$:

$$a_0\dot{u}(t) + b_0 u(t) + b_1 u(t-d) = f(t) \tag{5}$$

subject to the initial or boundary condition

$$u(t) = \xi(t) \quad \text{with} \quad t \in [-d, 0]\,. \tag{6}$$

Theorem 1 ***(Existence and Uniqueness)*** *Suppose that f is of class C^1 on $[0,\infty)$ and that ξ is of class C^0 on $[-d,0]$. Then there exists one and only one continuous function $u(t)$ which satisfies (6), and (5) for $t \geq 0$. Moreover, this function u is of class C^1 on (d,∞) and of class C^2 on $(2d,\infty)$. If ξ is of class C^1 on $[-d,0]$, $\dot{u}$ is continuous at τ if and only if*

$$a_0\dot{\xi}(d) + b_0\xi(d) + b_1\xi(0) = f(d) \tag{7}$$

If ξ is of class C^2 on $[-d,0]$, $\ddot{u}$ is continuous at $2d$ if either (7) holds or else $b_1 = 0$, and only in these cases.

Proof. See Bellman and Cooke [6], , Theorem 3.1, page 50-51. ■

The function u singled out in this theorem is called the continuous solution of (5) and (6). Then in order to see the shape of this continuous solution the following theorem is useful:

Theorem 2 *Let $u(t)$ be the continuous solution of (5) which satisfies the boundary condition (6). If ξ is C^0 on $[-d,0]$ and f is C^0 on $[0,\infty)$, then for $t > 0$,*

$$u(t) = \sum_r p_r e^{z_r t} + \int_0^t f(s) \sum_r \frac{e^{z_r(t-s)}}{h'(z_r)} ds \tag{8}$$

where $\{z_r\}_r$ and $\{p_r\}_r$ are respectively the roots and the residue coming from the characteristic equation, $h(z)$, of the homogeneous delay differential equation

$$a_0\dot{u}(t) + b_0 u(t) + b_1 u(t-d) = 0 \tag{9}$$

Note: $p_r = \frac{p(z_r)}{h'(z_r)}$ where

$$p(z_r) = a_0\xi(0) + (a_0 z_r + b_0)\int_{-d}^{0} \xi(s) e^{-z_r s} ds$$

Proof. See Appendix A.1. ■

Since in our context it shall be fundamental to have real continuous general solution, we present here the following theoretical results.

Theorem 3 *The unique general continuous solution of problem (5) with boundary condition* $\xi : I \subset \mathbb{R} \to \mathbb{R}$ *and forcing term* $f : I \to \mathbb{R}$*, is a real function.*

Proof. See Appendix A.2. ■

Some considerations on these theorems are needed. We start with the last result. The important message of Theorem 3 is the following: if we assume a boundary condition and a forcing term which are real functions then also the general continuous solution must be real. Other considerations regard the proofs of Theorem 1 and 2: both of them are strictly related to the fact that all the roots of $h(z)$ lie in the complex z-plane to the left of some vertical line. That is, there is a real constant c such that all roots z have real part less then c. This consideration is in general no longer true for advanced differential equations which are characterized by CE with zeros of arbitrarily large real part. However as explained by Bellman and Cooke [6],[3] it is possible to write the solution of any advanced differential equation as a sum of exponentials using the finite Laplace transformation technique. Moreover observe that the characteristic equation of (5),

$$h(z) \equiv z + a + be^{-zd} = 0 \tag{10}$$

with $a = \frac{b_0}{a_0}$ and $b = \frac{b_1}{a_0}$, is a trascendental function with an infinite number of finite roots. Sometimes $h(z)$ is also called the characteristic quasi-polynomial. Asymptotic stability requires that all of these roots have negative real part. In order to help in the stability analysis we introduce two important mathematical results: the Hayes theorem and the *D-Subdivision method* or D-Partitions method. Hayes Theorem [18] in its more general formulation states the following:

Theorem 4 *The roots of equation* $pe^z + q - ze^z = 0$ *where* $p, q \in \mathbb{R}$ *lies to the left of* $\operatorname{Re}(z) = k$ *if and only if*

(a) $p - k < 1$

(b) $(p-k)\, e^k < -q < e^k \sqrt{a_1^2 + (p-k)^2}$

where a_1 *is the root of* $a = p \tan a$ *such that* $a \in (0, \pi)$*. If* $p = 0$*, we take* $a_1 = \frac{\pi}{2}$*.*

One root lies on $\operatorname{Re}(z) = k$ *and all the other roots on the left if and only if* $p - k < 1$ *and* $(p-k)\, e^k = -q$*.*

Two roots lies on $\operatorname{Re}(z) = k$ *and all the other roots on the left if and only if* $-q = e^k \sqrt{a_1^2 + (p-k)^2}$

Proof. See Hayes [18], page 230-231. ■

However this Theorem doesn't say anything about the sign of the real part of the roots of the trascendental function when the conditions *(a)* and *(b)* are not respected. For this reason the D-Subdivision method is now introduced (for more details on this method, El'sgol'ts and Norkin [13], or Kolmanovskii and Nosov [23]). Given a trascendental function like, for example, (10), this method is able to determine the number of roots having positive real part (for now on p-zeros) in accordance with the value of its coefficients (a and b in our specific case). This is possible since the zeros of a trascendental function are continuous functions of those same coefficients.

[3] Look at Chapter 6 page 197-205.

Definition 1 *Given the characteristic equation of a functional differential equation with constant coefficients, a D-Subdivision is a partition of the space of coefficients into regions by hypersurfaces, the points of which correspond to quasi-polynomials having at least one zero on the imaginary axis (the case $z = 0$ is not excluded).*

For continuous variation of the trascendental function coefficients the number of p-zeros may change only by passage of some zeros through an imaginary axis, that is, if the point in the coefficient space passes across the boundary of a region of the D-Subdivision. Thus, to every region Γ_k of the D-Subdivision, it is possible to assign a number k which is the number of p-zeros of the trascendental function. Among the regions of this partition are also found regions Γ_0 (if they exist) which are regions of asymptotic stability of solutions. Finally in order to clarify how the number of roots with positive real parts changes as some boundary of the D-Subdivision is crossed, the differential of the real part of the root is computed, and the decrease or increase of the number of p-zeros is determined from its algebraic sign. Since it becomes very useful later, we study, with the D-Subdivision method, the trascendental function (10).

First of all, observe that this equation has a zero root for $a + b = 0$. This straight line (see Figure 1) is one of the lines forming the boundary of the D-Subdivision. It is also immediately derived that the trascendental function (10) has purely imaginary roots if and only if

$$a + b\cos dy = 0, \quad y - b\sin dy = 0 \tag{11}$$

or

$$b = \frac{y}{\sin dy}, \quad a = \frac{-y\cos dy}{\sin dy} \tag{12}$$

The equations in parametric form (11) or (12) identify all the other D-Subdivision boundaries. To be precise there is one boundary for any of the following interval of y: $\left(0, \frac{\pi}{d}\right), \left(\frac{\pi}{d}, \frac{2\pi}{d}\right), \left(\frac{2\pi}{d}, \frac{3\pi}{d}\right), \ldots$. Moreover it is possible (and useful) to find the values of b for which the boundaries intercept the b-axis. The sequence of such b is $\left\{\ldots, -\frac{7\pi}{2d}, -\frac{3\pi}{2d}, 0, \frac{\pi}{2d}, \frac{5\pi}{2d}, \ldots\right\}$. Finally we show how p-zeros rises. In particular, when a crossing of C_l from Γ_0 to Γ_2 implies the rising of two p-zeros (that is, we focus on the interval $0 < y < \frac{\pi}{d}$). From (10) applying the implicit function theorem, we have that on C_l

$$\begin{aligned} dx &= -\operatorname{Re}\frac{da}{1 - bde^{-diy}} \\ &= -\operatorname{Re}\frac{da}{1 - bd(\cos dy - i\sin dy)} \\ &= \frac{(1 - bd\cos dy)\, da}{(1 - bd\cos dy)^2 + b^2d^2\sin^2 dy} \end{aligned}$$

We find that $\cos yd < 0$ for $bd > 1$. Therefore, upon crossing the boundary C_l from region Γ_0 into Γ_2, a pair of complex conjugate roots gain positive real parts. The analysis on the other boundaries of the D-Subdivision is completely analogous. Taking into account all of these results, we are now ready to study our model completely.

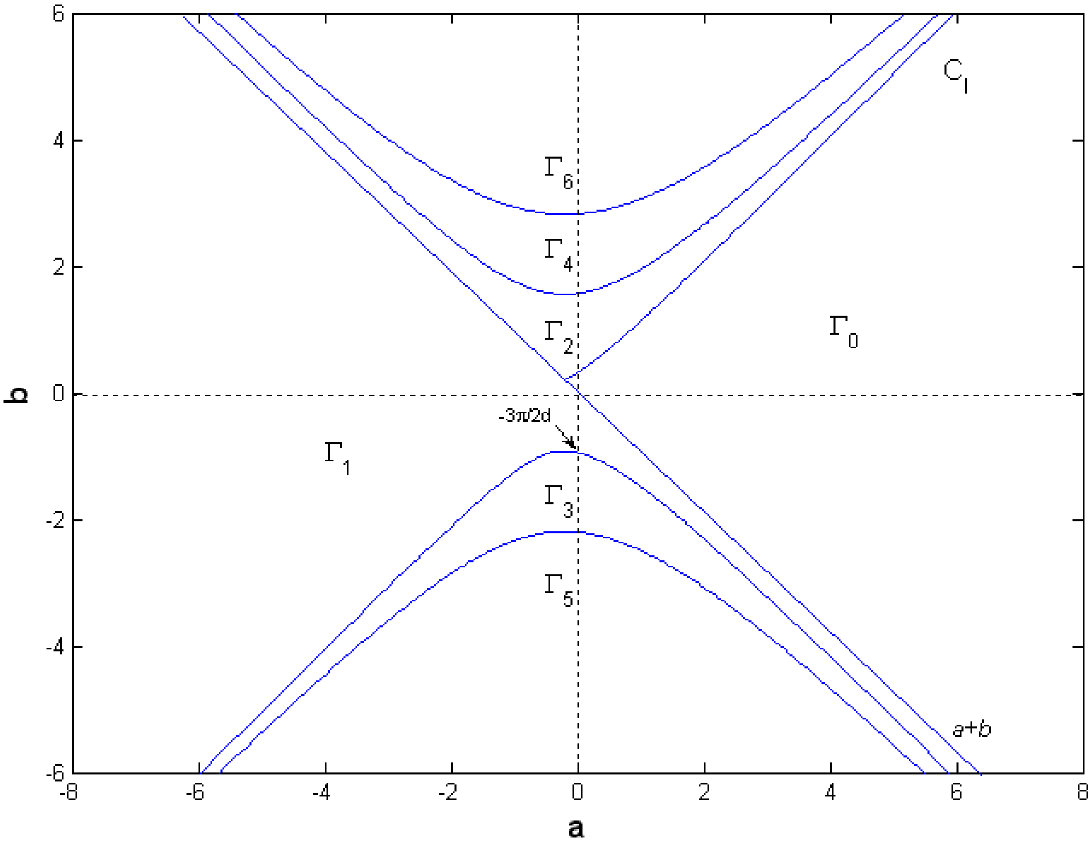

Figure 1: D-Subdivision for the trascendental function (10) assuming $d = 5$.

4 Balance Growth Path Analysis

In order to show the existence and uniqueness of the BGP, we now present some results regarding the roots of the characteristic equation of the law of motion of capital, of its shadow price, and of consumption. These results are presented and proved in Lemma 1 and Lemma 2, respectively. Some pictures are also provided in order to help the reader get the main message behind the maths. After that, the continuous solution of capital is rewritten as a sum of weighted exponential (Proposition 1) and then, following a very similar strategy as that used in the standard AK model, a unique balance growth path for consumption and capital is proved by checking the transversality condition. Very similar to this, is the requirement that for any exogenously given choice of the delay coefficient, the production function has to be sufficiently productive to ensure growth in consumption, but not so productive as to yield unbounded utility: $A \in (A_{\min}, A_{\max})$. On the other hand, it is possible to express the same requirement, given a certain level of technology, in term of the delay coefficient: $d \in (d_{\min}, d_{\max})$. Finally as in the standard case if $\sigma > 1$, then $A_{\max}$ is equal to plus infinity, while $d_{\min}$ is zero.

As anticipated in Lemma 1 we report some information on the roots of the CE of the law of motion of capital and its shadow price:

Lemma 1 *For any sufficiently high rate of depreciation "before use", ϕ, the following results* *hold:*

1) $\tilde{z}$ *is the unique root with positive real part of the CE of the law of motion of capital;*

2) $\tilde{s}$ *is the unique root with negative real part of the CE of the law of motion of shadow price.*

Proof. The characteristic equation of the law of motion of capital (1) is equal to the characteristic equation of its homogeneous part[4], namely

$$h(z) \equiv z - \tilde{A}e^{-zd} = 0 \tag{13}$$

It immediate to show that this equation has a unique positive real root $z_{\tilde{v}} = \tilde{z}$ which is also the highest among its roots. In particular, through the *D-Subdivision method* it is possible to prove that the trascendental equation (13) has an increasing number of p-zeros as d rises. On the other hand if we assume $\phi = \hat{\phi}$ sufficiently high,[5] it happens that $\tilde{A} < \frac{3\pi}{2d}$ for any choice of d and then a unique p-zero exists[6]. These facts can be easily observed in Figure 2. Finally, $\tilde{z} > \mathrm{Re}(z_v)$ occurs for any $v \neq \tilde{v}$ since all the roots of the CE of (1) in the detrended variables $\hat{x}(t) = x(t)e^{-\tilde{z}t}$ are negative. This is sufficient to prove result 1). Now observe that the CE of the shadow price law of motion (3) is

$$h(s) \equiv -s - \tilde{A}e^{sd} = 0 \tag{14}$$

then we can put in correspondence the roots of (13) and (14) through the transformation $z = -s$. From this consideration follows immediately that $\mathrm{Re}(s) = -\,\mathrm{Re}(z)$ and $\tilde{s} = -\tilde{z}$ is the root with the lowest real part of the characteristic equation of the law of motion of shadow price. ■

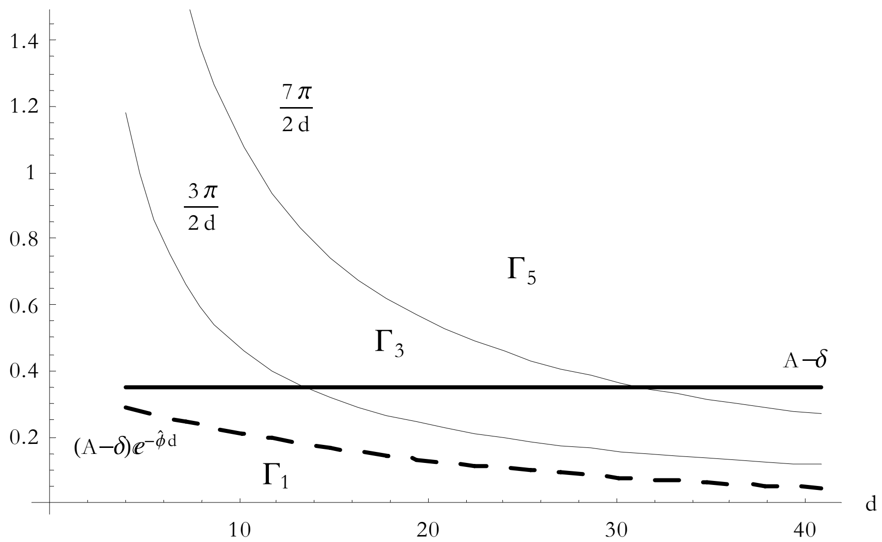

Figure 2: Number of p-zeros of (13) according to the choice of the delay coefficient.

Lemma 1 tells us that if we assume a sufficiently high depreciation "before use" rate, $\hat{\phi}$, then $\tilde{z}$ is the constant growth rate of capital and the unique p-zero of (13). Now it will be useful for proving a common growth rate of consumption and capital to show the following Lemma:

Lemma 2 *A positive and constant growth rate of consumption,* g_c*, always exists for* $A > A_{\min} = \delta + \rho e^{(\rho+\hat{\phi})d}$.

[4] The part of equation (1) not considering the forcing term $-bC(t)$.

[5] In the numerical simulation, reported in Section 7, we have assumed $\hat{\phi} \simeq 0.03$.

[6] This is also a consequence of the fact that $\tilde{A}$ converges to zero faster than $\frac{3\pi}{2d}$ as $d \to \infty$.

Proof. First of all observe that since the Euler equation (4) is a nonlinear ADE we cannot write directly its continuous general solution (Theorem 2 doesn't apply). However it is possible to overcome this fact by observing that the general continuous solution of consumption can be obtained indirectly by the first order condition (2). Considering the general continuous solution of the shadow price of capital $\mu(t) = \sum_m a_m e^{-z_m t}$, we have that

$$c(t) = \frac{1}{\left(\sum_m a_m e^{-\sigma\lambda_m t}\right)^{\frac{1}{\sigma}}} \tag{15}$$

where we have called

$$\lambda = \frac{1}{\sigma}(z - \rho) \tag{16}$$

From equation (15) we can derive that the basic solutions of (4) have exponential form, namely the basic solutions are $\left\{e^{\lambda_m}\right\}_m$; moreover taking into account (13) and (16) we can derive indirectly the characteristic equation[7] of (4)

$$h(\lambda) = \sigma\lambda + \rho - \tilde{A}e^{-(\sigma\lambda+\rho)d} \tag{17}$$

Using the Hayes theorem or the D-Subdivision method, a unique positive real root, $\lambda_{\tilde{m}} = g_c$ exists for A sufficiently large, namely $A > A_{\min} = \delta + \rho e^{(\rho+\phi)d}$. This is exactly the same condition of the standard AK model when the assumption of time-to-build is introduced. Observe also that in this context the same requirement can be expressed in term of the delay, $d < d_{\max} = \frac{1}{\rho+\phi}\log\frac{A-\delta}{\rho}$. Exactly as before, a unique p-zero exists if $\tilde{A}e^{-\rho d} < \frac{3\pi}{2d}$. It is obvious that, for $\phi = \hat{\phi}$, the inequality is always respected (see Figure 3) since $\hat{\phi}$ was sufficient to force $\tilde{A}$ to stay below $\frac{3\pi}{2d}$, and given that $(A-\delta)e^{-\hat{\phi}d}e^{-\rho d}$ is a product of functions which are positive and monotonic decreasing in d. As it will appear clear in Section 6 for any $\phi \in \left[0, \hat{\phi}\right)$, some economically implausible prediction may arise. Then, from now on, we focus on the case $\phi \geq \hat{\phi}$. Observe moreover that endogenous growth implies that consumption and capital have to grow at a positive rate over time. This implies that $\lim_{t\to\infty} c(t) = +\infty$; then given (15), we have to impose that

$$\lim_{t\to\infty} \frac{1}{\left(a_{\tilde{m}}e^{-\sigma g_c t} + \sum_{m\notin\tilde{m}} a_m e^{-\sigma\lambda_m t}\right)^{\frac{1}{\sigma}}} = +\infty \tag{18}$$

Using the properties of the limits[8], it is possible to rewrite (18) as

$$\frac{1}{\left(\underbrace{\lim_{t\to\infty} a_{\tilde{m}}e^{-\sigma g_c t}}_{\to 0} + \underbrace{\sum_{m\notin\tilde{m}} \lim_{t\to\infty} a_m e^{-\sigma\lambda_m t}}_{\to\infty}\right)^{\frac{1}{\sigma}}} = +\infty$$

Then it results that the relation (18) is satisfied if and only if $a_m = 0$ for any $m \neq \tilde{m}$. Taking into account this fact, the general continuous solution of consumption is

$$c(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c t}$$

[7] We have referred to equation (17) as the characteristic equation of the law of motion of consumption since gives us all the basic solutions.

[8] The following properties have been used: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, and $\lim_{x\to a}\left[\sum_i f_i(x)\right] = \sum_i \lim_{x\to a} f_i(x)$

■

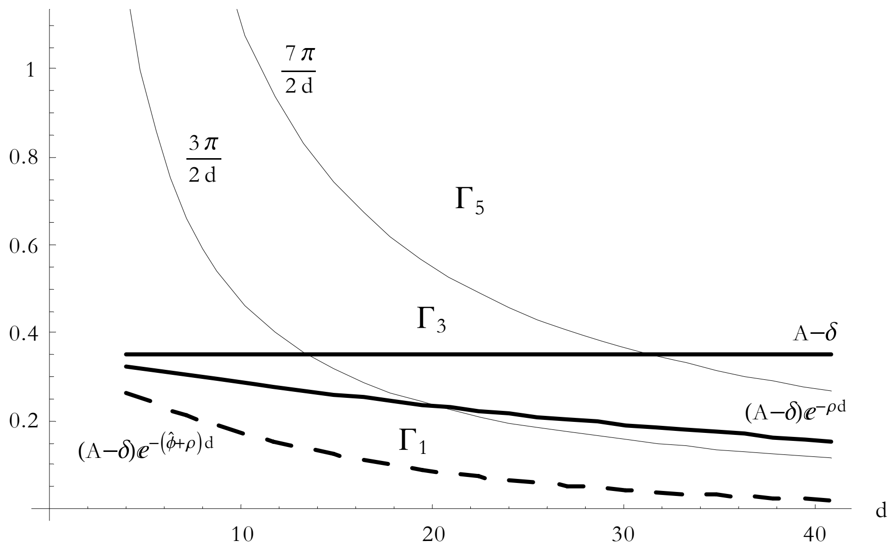

Figure 3: Number of p-zeros of (17) according to the choice of the delay coefficient.

Our objective is to prove that the growth rate of consumption and capital are the same $g = g_c$. However before proving it, we introduce the following Corollary of Theorem 2 which let us to rewrite the continuous solution of capital as a sum of weighted exponentials.

Proposition 1 *The solution of the law of motion of capital can be written as*

$$k(t) = \sum_v P_{\tilde{m},v} e^{g_c t} + \sum_v N_{\tilde{m},v} e^{z_v t} \tag{19}$$

where $P_{\tilde{m},v} = -\frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{(g_c - z_v)h'(z_v)}$ *and* $N_{\tilde{m},v} = n_v - P_{\tilde{m},v}$.

Proof. According to Theorem 2 and Lemma 2, the continuous general solution of consumption and capital are respectively

$$c(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c t} \tag{20}$$

$$k(t) = \sum_v n_v e^{z_v t} - \int_0^t c(s) \sum_v \frac{e^{z_v (t-s)}}{h'(z_v)} ds \tag{21}$$

Now the integral part of equation (21) is equal to

$$\int_0^t a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c s} \sum_v \frac{e^{z_v (t-s)}}{h'(z_v)} ds = \sum_v \frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{(g_c - z_v) h'(z_v)} \left(e^{g_c t} - e^{z_v t} \right)$$

and substituting in (21) after some algebra we get (19). ■

Some comments on equations (20) and (19) are needed. These equations are very close to the general solution form for consumption and capital in the usual framework, with ordinary differential equations; in particular $k(t)$ is a weighted sum of exponentials; however, this similarity can be found for systems of mixed functional differential equations *only* in the particular case of a single equation with forced term. In the most general cases there doesn't exist a theorem which lets us write the solution in this way[9]. Moreover, the continuous solution of the law of motion of consumption (20) and capital (19), are not the optimal solution exactly as it happens in the ordinary case. Before getting optimality, transversality conditions have to be checked. Using this corollary and TVC, we prove now the existence of a unique balance growth path for consumption and capital.

Proposition 2 *Consumption and capital have the same balanced growth path $g = g_c$. This growth rate is positive and yields bounded utility if $A \in (A_{\min}, A_{\max})$.*

Proof. As shown in Lemma 2, the growth rate of consumption g_c is a positive constant if $A > A_{\min}$. Given that, we have to distinguish two cases: $\tilde{z} \leq g_c$ and $\tilde{z} > g_c$. The first case is never possible. In fact, assume that $\tilde{z} \leq g_c$ then g_c is also the growth rate of capital as follows immediately by looking at equation (19). Then we can rewrite the characteristic equation of capital, after the transformation $\hat{k}(t) = k(t)e^{-g_c t}$, as

$$-we^{w} - g_c de^{w} + \tilde{A}de^{-g_c d} = 0 \tag{22}$$

where $w = zd$. Since g_c is the root having greater positive real part, all the roots of (22) must have negative real part which, from Hayes Theorem implies also that $g_c > \tilde{A}e^{-g_c d}$. However, this is never possible since it contradicts the positive sign of the consumption to output ratio at the balanced growth path

$$\frac{c(t)}{k(t)} = \tilde{A}e^{-g_c d} - g_c > 0$$

which can be obtained by dividing the law of motion of capital (1) by $k(t)$. Then the only possible case is $\tilde{z} = \sigma g_c + \rho > g_c$. This is exactly the requirement for having no unbounded utility: $(1-\sigma)\, g_c < \rho$. Then, before passing to the TVC we observe that if $\sigma > 1$, the utility is always bounded; on the other hand if $0 < \sigma < 1$ we need a condition on A such that the utility is bounded. Taking into account the CE (17) after some algebra this condition is $A < A_{\max} = \delta + \frac{\rho}{1-\sigma} e^{\left(\frac{\rho+\phi(1-\sigma)}{1-\sigma}\right)d}$ which is exactly the same condition for the standard AK model when the time-to-build parameter is equal to zero. Observe also that such a condition can be rewritten also in terms of the delay, $d > d_{\min} = \frac{1-\sigma}{\rho+(1-\sigma)\phi} \log \frac{(A-\delta)(1-\sigma)}{\rho}$. Now we show that the TVC

$$\lim_{t\to\infty} \mu\,(t)\, k(t) = 0 \tag{23}$$

implies necessarily a unique BGP which is g_c. In order to see this, we substitute the general continuous solutions of $\mu(t)$ and $k(t)$, into the TVC (23) and we get:

$$\lim_{t\to\infty} a_{\tilde{m}} e^{-\tilde{z}t} \left(\sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_v N_{\tilde{m},v} e^{z_v t} \right) = 0 \tag{24}$$

which is equal to

$$\lim_{t\to\infty} \left[a_{\tilde{m}} N_{\tilde{m},\tilde{v}} + \sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},\tilde{v}} e^{(z_v - \tilde{z})t} + \sum\nolimits_v P_{\tilde{m},v} e^{(g_c - \tilde{z})t} \right] = 0$$

now for $a_{\tilde{m}} \neq 0$, the second and third term in the parenthesis converge to zero since $z_v - \tilde{z} < 0$ for any v and $g_c - \tilde{z} < 0$. Then the TVC are respected if and only if

$$N_{\tilde{m},\tilde{v}} \equiv \frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{(g_c - \tilde{z})\, h'(\tilde{z})} + n_{\tilde{v}} = 0 \tag{25}$$

[9] Recently Asl and Ulsoy [2] have proved that a general solution form can be written for system of delay differential equations using Lambert function.

which implies

$$a_{\tilde{m}} = \left(\frac{1}{(\tilde{z} - g_c)\, h'(\tilde{z})\, n_{\tilde{v}}} \right)^{\sigma} \tag{26}$$

Observe that if we assume a constant boundary condition for capital, k_0, and for consumption, c_0, we can derive the following relation

$$c_0 = (\tilde{z} - g_c)\, e^{\tilde{z}d} k_0$$

which for $d = 0$ is exactly equal to the relation between c_0 and k_0 in the standard AK model (see Barro and Sala-i-Martin [4]). Concluding TVC holds if and only if condition (25) is verified. Given this condition, g_c is also the growth rate of capital since the continuous general solution of capital (21) can be rewritten as follows

$$k(t) = \sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_{v \neq \tilde{v}} N_{\tilde{m},v} e^{z_v t} \tag{27}$$

Then the optimal solution of capital (27) is asymptotically driven by g_c which implies a common growth rate with consumption. ■

This proposition provides evidence of how a unique balance growth path for consumption and capital can be proved to exist also in the case of time-to-build by checking to the transversality conditions. In fact, through condition (25), it is possible to rule out the eigenvalue coming from the characteristic equation of the law of motion of capital, having positive real part greater than g_c. Observe also that this fact and the assumption of the new structure of capital depreciation make all of these results hold for any choice of the delay in the interval $(d_{\min}, d_{\max})$ which guarantees the presence of endogenous growth and no unbounded utility.

Once we have shown that $g = g_c$ is the unique balanced growth path of consumption and capital, it is also interesting to see how different choices of the delay coefficient, d, and of the level of technology A affect it. These considerations are reported in the following corollary of Proposition 2:

Corollary 1 *Under $A \in (A_{\min}, A_{\max})$, $\frac{\partial g}{\partial d}$ and $\frac{\partial g}{\partial \phi}$ are negative while $\frac{\partial g}{\partial A}$ is positive.*

Proof. Under $A \in (A_{\min}, A_{\max})$, we have shown that g is the unique positive balance growth path for consumption and capital. The effect of a variation of d, ϕ, and A on g can be easily computed by applying the Implicit Function Theorem on the trascendental equation (17) which is always satisfied for $\lambda = g$. After some algebra we obtain that

$$\begin{aligned}
\frac{\partial g}{\partial d} &= -\frac{(A-\delta)(\sigma g + \rho + \phi)\, e^{-(\sigma g + \rho + \phi)d}}{\sigma + \sigma d (A-\delta)\, e^{-(\sigma g + \rho + \phi)d}} < 0 \\
\frac{\partial g}{\partial \phi} &= -\frac{d(A-\delta)\, e^{-(\sigma g + \rho + \phi)d}}{\sigma + \sigma d (A-\delta)\, e^{-(\sigma g + \rho + \phi)d}} < 0 \\
\frac{\partial g}{\partial A} &= \frac{e^{-(\sigma g + \rho + \phi)d}}{\sigma + \sigma d (A-\delta)\, e^{-(\sigma g + \rho + \phi)d}} > 0
\end{aligned}$$

These results are very intuitive; the negative relations between the time-to-build delay and the growth rate and between the depreciation "before use" and the growth rate are due, respectively, to the fact that an increase in the time-to-build delay increases the time to produce output and by the fact that a higher depreciation "before use" reduces the net capital. On the other hand, the positive effect of the productivity of capital is obvious and is present in the standard AK model as well. In Figure 4, we have reported the behavior of g as d rises (the decreasing curve) and the standard case with $d = 0$ for the following parametrization: $\sigma = 8$, $\rho = 0.02$, $A = 0.30$,

$\delta = 0.04$, and $\hat{\phi} = 0.04$. Given these values, d has to be in the interval $(0, 42.74)$ in order to have a positive balance growth path. ■

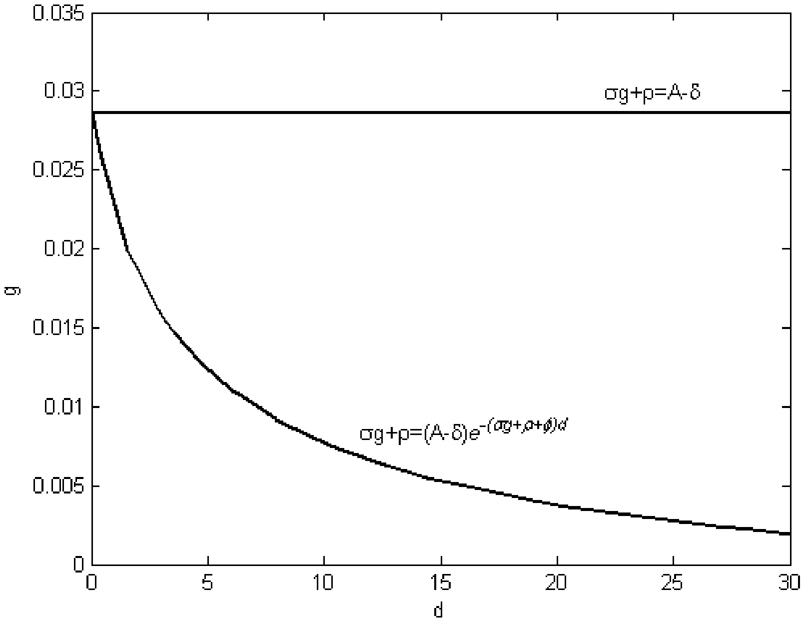

Figure 4: Behavior of the balance growth path, g, to variations of d.

5 Consumption and Capital Dynamics

In the previous section, we have proved the existence and uniqueness of the balance growth path. We have also shown the influence of the delay coefficient on the growth rate for a given level of technology. In this section, we focus on the dynamic behavior of the optimal detrended consumption and capital which let us to derive indirectly the behavior of detrended income and detrended investment.

Proposition 3 *Optimal detrended consumption is constant over time while optimal detrended capital path is unique and oscillatory converges to a constant.*

Proof. The optimal detrended solution of capital and consumption can be obtained by multiplying both sides of equations (27) and (20) by $e^{-g_c t}$

$$\hat{c}(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} \tag{28}$$

$$\hat{k}(t) = \sum_v P_{\tilde{m},v} + \sum_{v \neq \tilde{v}} N_{\tilde{m},v} e^{(z_v - g_c)t} \tag{29}$$

After calling $z = x + iy$ and $n = \alpha + i\beta$, and taking into account Theorem 3, the detrended solution for capital can be rewritten, as shown in Appendix A.3, in the following way

$$\hat{k}(t) = \alpha_{\tilde{v}} + 2\sum_{v \neq \tilde{v}} \Psi_{0,v} + 2\sum_{v \neq \tilde{v}} \left[(\alpha_v - \Psi_{0,v}) \cos yt - (\beta_v + \Psi_{1,v}) \sin yt\right] e^{(x_v - g_c)t} \tag{30}$$

where $\Psi_{0,v}, \Psi_{1,v} \in \mathbb{R}$ for any v. Finally, the asymptotic behavior of capital is equal to

$$\lim_{t \to \infty} \hat{k}(t) = \alpha_{\tilde{v}} + 2\sum_v \Psi_{0,v} \tag{31}$$

Expressions (30) and (31) tell us that the transition to the BGP is oscillatory due to the presence of the cosine and sine term, and that the convergence is guarantee by the fact that $x_v = \text{Re}(z_v) < g_c$ for any $v \neq \tilde{v}$. Finally, the uniqueness of the path is due to the fact that the residue $\{n_v\}_v$ are fixed by the boundary condition of capital while the residue $a_{\tilde{m}}$ is fixed by the transversality condition through the expression (26). ■

Moreover, taking into account the technology and the resources constraint of our economy, it follows immediately that output and investment have an oscillatory behavior. In the following section, we discuss the opportunity of introducing the depreciation "before use" hypothesis and the role which a choice of a $\phi \geq \hat{\phi}$ has in extending our results for all the feasible values of the delay. On the other hand, as it will appear clear soon, all the results obtained until now remain valid even for the extreme case $\phi = 0$ when an appropriate sub-interval of d is appropriately chosen.

6 Considerations on the depreciation "before use" hypothesis.

It is quite easily observable that all the results obtained until now remain valid in the specific case of $\phi = 0$ for a restricted interval of the time-to-build coefficient. However, for a sufficiently high choice of the delay, several technical problems arise. In particular, a general continuous solution as a sum of exponentials as in (19) can no longer be written and following that the validity of transversality conditions become extremely difficult to assess. Remember that in exogenous growth models, as in Asea and Zak [1], and Rustichini [24], the choice of the delay is crucial in avoiding the possible presence of capital divergence behavior. The introduction of the depreciation "before use" hypothesis, and in particular with a sufficiently high level of ϕ, has a crucial role in avoiding these problems and then in extending the results to the whole interval of the delay. To be precise, the introduction of depreciation "before use", depending inversely on the time-to-build parameter, is able to stabilize the capital equilibrium path by reducing net capital.

7 Numerical Exercise

Now, we present a numerical exercise. In this section we report only the result of our simulation while a detailed explanation of the computational methods is reported in Appendix A.4. The following parametrization of our economy has been chosen:

σ	ρ	δ	ϕ	d	A	d_{min}	d_{max}
0.8	0.02	0.05	0.03	20	0.3	7	50.51

Remember that if we have chosen $\sigma > 1$ the d_{min} should be equal to 0; in our case with $\sigma = 0.8$ a value of d less than d_{min} implies unbounded utility. On the other hand a value of the delay greater than d_{max} implies no endogenous growth since the highest root is close to 0.02 and taking into account our parametrization and relation (16), we have, that at the right of this value the growth rate of consumption is no longer positive. Moreover, observe that given this parametrization, the D-Subdivision method tells us that: in the case of no depreciation "before use" ($\phi = 0$), in the interval $d = \left[\tilde{d}_{\text{min}}, 18.85\right)$ we have only one root with positive real part; in the interval $d = [18.85, 43.98)$, three roots with positive real part, and finally in the interval $d = [43.98, \tilde{d}_{\text{max}}]$, five roots with positive real part. This fact is reported in Figure 5, where a subset of the infinite roots of the homogeneous part of (1) are numerically computed through the Lambert function. Figure 5 and Figure 6 shows the real parts of the roots in the x-axis and the imaginary parts in the y-axis:

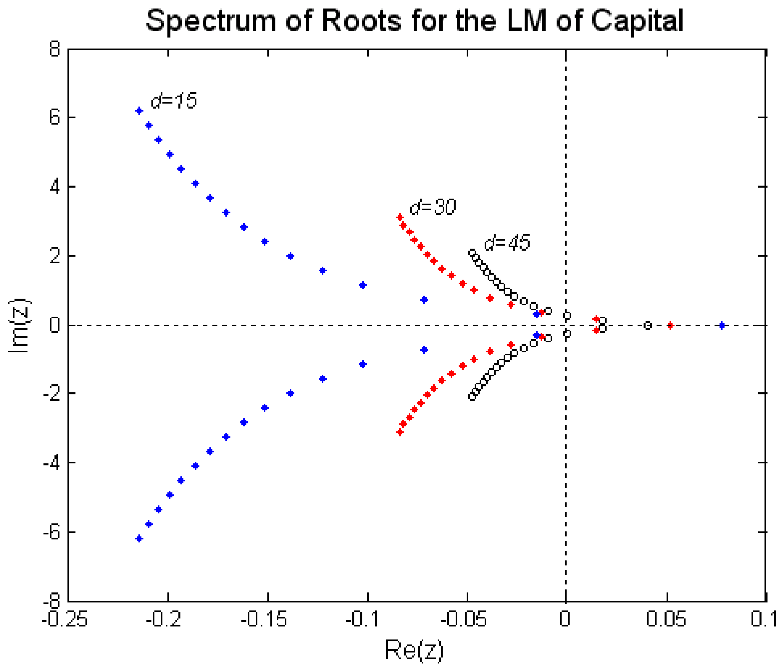

Figure 5: Number of p-roots for the law of motion of capital according to different choices of d.

This first graph of the spectrum is interesting, since it shows how an increase in the value of the time-to-build coefficients reduces the magnitude of the real part of the highest eigenvalue. Taking into account relation (16), this numerical result confirms Corollary 2. Now we show the effect of the introduction of a minimum degree of depreciation "before use" on the capital dynamics. In particular, through Figure 6, it is possible to observe how a choice of $\phi = 0.03$ forces the spectrum of roots of the law of motion of capital to have only one eigenvalue with positive real part even in the extreme case of a delay coefficient equal to $d_{\max} = 50$.

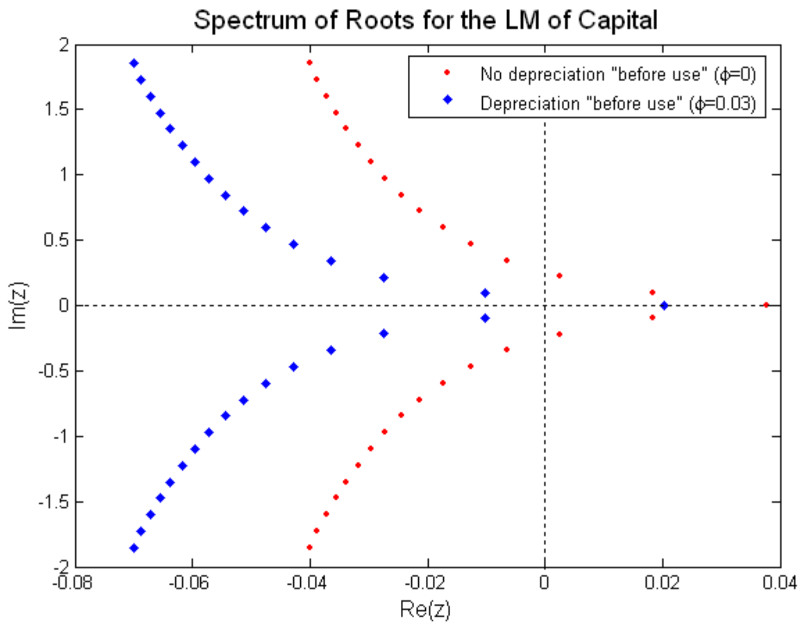

Figure 6: Variation in the number of roots with positive real part in the case $d_{\max} = 50$.

As we can expect, the presence of a positive depreciation "before use" rate reduces the growth rate of capital, and indirectly, through relation (16), of consumption. This effect is due to the fact that net capital is reduced and, indirectly, output, consumption, and investment. The next two figures show the dynamic behavior of detrended capital (equation (30)) over time. In the first case, Figure 7, we have studied the detrended capital dynamics given a constant initial value (boundary condition) of capital, k_0.

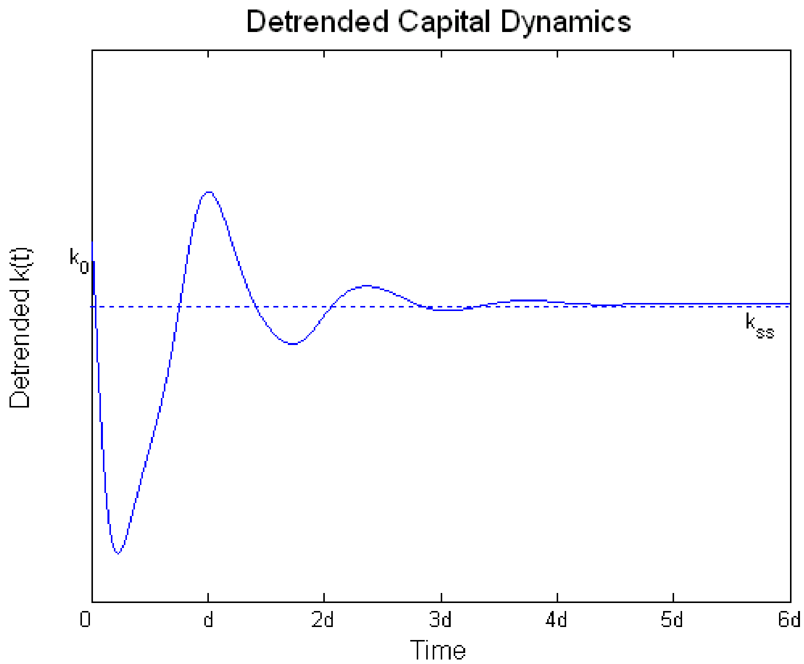

Figure 7: Dynamic behavior of detrended capital given a positive and constant boundary condition, k_0.

As it appears clear, the presence of time-to-build is able to generate oscillatory behavior of capital for a long interval of time. Taking into account the technology and the resources constraint of our economy both output and investment will have a similar dynamic behavior as capital. Observe that the oscillatory dynamic behavior of these variables is enhanced by a consumption smoothing effect. In fact from Proposition 3, we know that the social planner optimally chooses to have a constant detrended consumption while detrended capital bears most of the adjustment to the BGP. Finally we have reported in Figure 8, the capital dynamic behavior for different choices of the delay given the following parametrization

σ	ρ	δ	ϕ	d_1, d_2, d_3, d_4	A	$d_{\min}$	$d_{\max}$
8	0.02	0.05	0.03	0.5; 1; 5; 10	0.3	0	50.51

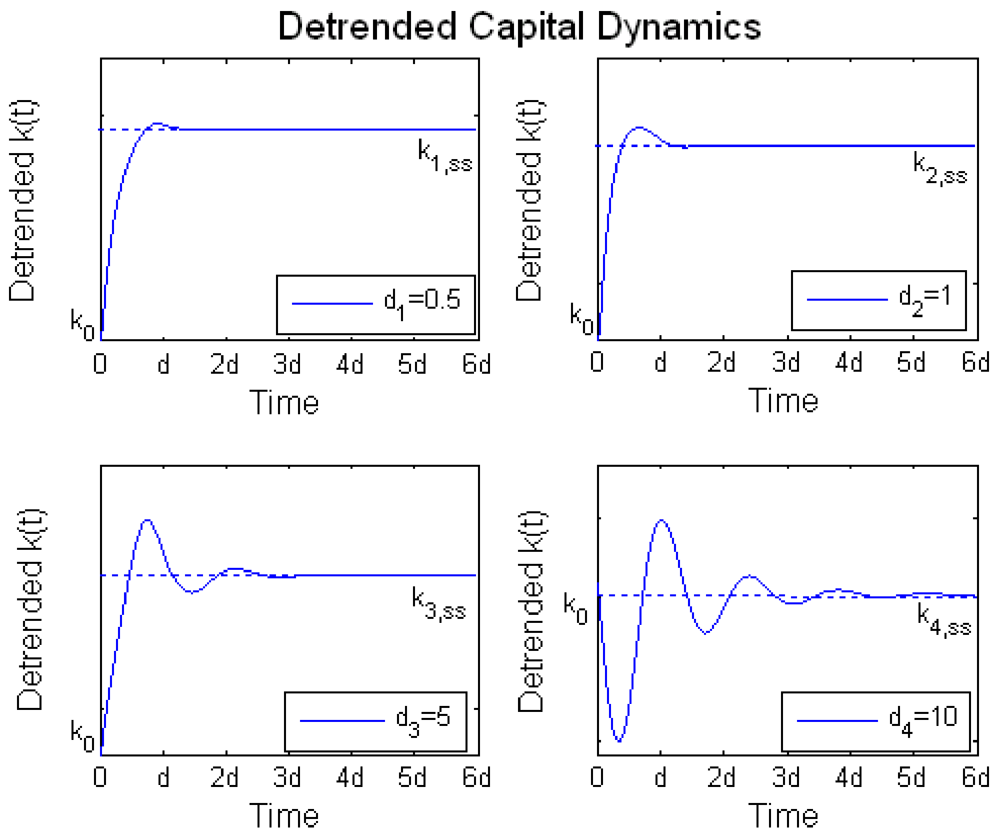

Figure 8: Capital dynamic behavior for different choices of the delay.

It is interesting to notice that the higher the choice of the delay is the more relevant is the oscillatory structure of capital dynamics. This fact has been reported in Figure 8, in the case of $\sigma = 8$ starting with values of the delay sufficiently close to zero and given a same boundary condition for capital k_0. Remember that variation in the choice of the delay have an influence on the value of the balance growth path. In particular, for Corollary 1, the higher is the delay, the lower is the balance growth path. This fact appears also in Figure 8, where with $k_{i,ss}$ and $i = 1, ..., 4$, we have indicated the different balance growth paths. Finally, the dynamic behavior of capital appears more and more smooth as d is close to zero: this dynamic behavior is consistent with what we aspect in the extreme case $d = 0$.

8 Conclusion

This paper has fully analyzed an AK endogenous growth model when the time-to-build assumption is introduced through a delay differential equation for capital. It has been proved the existence and uniqueness of the BGP and also that a unique optimal path of the detrended capital is oscillatory convergent to its steady state value while detrended consumption jumps directly on it as the usual case without delay. These results have been obtained through a careful analysis of the role of transversality conditions and the introduction of a new structure of capital depreciation, which takes into account the depreciation of capital before it becomes productive. This last assumption appears to be crucial in avoiding implausible economic predictions which may appear in this type of model for choices of the time-to-build coefficient sufficiently high. Finally the analysis of the model let us confirm that time-to-build can be considered a source of aggregate fluctuation for capital and output exactly as the vintage capital assumption.

A Appendix

A.1 Proof of Theorem 2.

The proof of this theorem is mainly based on Bellman and Cooke [6] (Section 3.9, page 73-75). The only relevant difference is that we assume a boundary condition defined in the interval $[-d, 0]$, and not in $[0, d]$. Given this difference we need an "auxiliary" function $x(t)$ having the following properties:

(a) $x(t) = 0 \qquad t < -d$;
(b) $x(-d) = a_0^{-1} e^{-sd}$;
(c) $x(t)$ is of class C^0 on $[0, \infty)$;
(d) $x(t)$ satisfies the equation

$$a_0 \dot{x}(t) + b_0 x(t) + b_1 x(t-d) = 0 \qquad for\ t > -d \tag{32}$$

Before showing that the Laplace transform of $x(t)$ is $h^{-1}(z)$, it is important to notice that it is possible to prove (see Bellman and Cooke [6]) the existence and uniqueness of $x(t)$ even if equation (32) doesn't respect theorem 1 since the boundary condition doesn't define a continuous function over $[-2d, d]$. We multiply each term of equation (32) by e^{-zt} and integrate with respect to t from $-d$ and ∞, we get

$$a_0 \int_{-d}^{\infty} \dot{x}(t) e^{-zt} dt + b_0 \int_{-d}^{\infty} x(t) e^{-zt} dt + b_1 \int_{-d}^{\infty} x(t-d) e^{-zt} dt = 0 \tag{33}$$

and integrating by part the first term and making the change of variables $t_1 = t - d$ in the last term, relation (33) can be rewritten

$$-1 + a_0 z \int_{-d}^{\infty} x(t) e^{-zt} dt + b_0 \int_{-d}^{\infty} x(t) e^{-zt} dt + b_1 e^{-zd} \int_{-d}^{\infty} x(t_1) e^{-zt_1} dt_1 = 0$$

from which follows immediately that the Laplace transform of $x(t)$ is

$$\int_{-d}^{\infty} x(t) e^{-zt} dt = h^{-1}(z) \tag{34}$$

Now, using the Laplace transform formula we get

$$x(t) = \int_{(c)} \frac{e^{zt}}{h(z)} dz \qquad for\ t > -d \tag{35}$$

For the *residue theorem* equation (35) is equivalent to

$$x(t) = \sum_r RES \left\{ \frac{e^{zt}}{h(z)}, z_r \right\}$$

and taking into account the formula $RES\left[\frac{\psi(a)}{\phi(a)}, \tilde{a}\right] = \frac{\psi(\tilde{a})}{\phi'(\tilde{a})}$ when $\psi(\tilde{a}) \neq 0$

$$x(t) = \sum_r \frac{e^{z_r t}}{h'(z_r)} \qquad for\ t > -d \tag{36}$$

Now for Theorem 3.7 of Bellman and Cooke [6], the general continuous solution, $u(t)$, of the delay differential equation with forcing term

$$a_0\dot{u}(t) + b_0u(t) + b_1u(t-d) = f(t) \tag{37}$$

which satisfies the initial or boundary condition $u(t) = \xi(t)$ with $t \in [-d, 0]$, is

$$u(t) = a_0\xi(0)\,x(t) + (a_0z_r + b_0)\int_{-d}^{0} \xi(s)x(t-s)\,ds + \int_{0}^{t} f(s)x(t-s)ds \tag{38}$$

and taking into account relation (36) we can rewrite (38) as

$$u(t) = \sum_r \frac{a_0\xi(0) + (a_0z_r + b_0)\int_{-d}^{0} \xi(s)e^{-z_rs}ds}{h'(z_r)}e^{z_rt} + \int_{0}^{t} f(s)\sum_r \frac{e^{z_r(t-s)}}{h'(z_r)}ds$$

which ie exactly equal to relation (8).

A.2 Proof of Theorem 3.

The proof is organized in two parts. In the first part, we show that the unique general solution of (5) with boundary condition (6)

$$u(t) = \sum_r p_re^{z_rt} + \int_{0}^{t} f(s)\sum_r \frac{e^{z_r(t-s)}}{h'(z_r)}ds \tag{39}$$

where the roots $\{z_r\}$ and the residues $\{v_r\}$ come respectively from the characteristic equation of the homogeneous part of (5)

$$h(z) = a_0z + b_0 + b_1e^{-zd} \tag{40}$$

and from the relation

$$p_r = \frac{p(z_r)}{h'(z_r)} = \frac{a_0\xi(0) + (a_0z_r + b_0)\int_{-d}^{0} \xi(s)e^{-z_rs}ds}{a_0 - b_1de^{-z_rd}} \tag{41}$$

can be rewritten as

$$\begin{aligned} u(t) \;=\; & \sum_{r=0}^{k} \varsigma_re^{x_rt} + \sum_{r=k}^{\infty}\left(a_re^{z_rt} + \bar{a}_re^{\bar{z}_rt}\right) + \\ & + \int_{0}^{t} f(s)\left[\sum_{r=0}^{k}\frac{e^{x_r(t-s)}}{h'(x_r)} + \sum_{r=k}^{\infty}\left(\frac{e^{z_r(t-s)}}{h'(z_r)} + \frac{e^{\bar{z}_r(t-s)}}{h'(\bar{z}_r)}\right)\right]ds \end{aligned} \tag{42}$$

where $\{x_r\}$ are real roots, $\{z_r\}$ are complex conjugate roots[10], $\{\varsigma_r\}$ are real constants, and $\{a_r\}$ are complex conjugate constants. In fact, from the D-Subdivisions method we know that (40) has at most two real roots and an infinite number of complex conjugate roots. From (41), it

[10] We have indicated the conjugate of a complex number a with $\bar{a}$.

appears also clear that the residues related to real roots are real while those related to complex roots are complex. Taking into account these results it is possible to split (39) as follows

$$u(t)=\sum_{r=0}^{k}\varsigma_r e^{x_r t}+\sum_{r=k}^{\infty}\left(a_r e^{z_r t}+c_r e^{\bar{z}_r t}\right)+\int_0^t f(s)\left[\sum_{r=0}^{k}\frac{e^{x_r(t-s)}}{h'(x_r)}+\sum_{r=k}^{\infty}\left(\frac{e^{z_r(t-s)}}{h'(z_r)}+\frac{e^{\bar{z}_r(t-s)}}{h'(\bar{z}_r)}\right)\right]ds$$

where $z=x+iy$ and $\bar{z}=x-iy$. We now show that $c_r=\bar{a}_r$ is always the case. This is equivalent to show that, given the expressions of a_r and c_r, the $\mathrm{Im}(c_r+a_r)=0$ and that the $\mathrm{Re}(c_r-a_r)=0$. We start by showing the first relation. In order to simplify the notation we omit from now on the r:

$$\begin{aligned}
a+c &= \frac{a_0\xi(0)+(a_0z+b_0)\int_{-d}^{0}\xi(s)e^{-zs}ds}{a_0-b_1de^{-zd}}+\frac{a_0\xi(0)+(a_0\bar{z}+b_0)\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds}{a_0-b_1de^{-\bar{z}d}} \\
&= \frac{\left[\xi(0)+\left(z+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-zs}ds\right]\left[a_0-b_1de^{-\bar{z}d}\right]}{\frac{1}{a_0}\left[a_0-b_1de^{-zd}\right]\left[a_0-b_1de^{-\bar{z}d}\right]}+ \qquad (43)\\
&\quad+\frac{\left[\xi(0)+\left(\bar{z}+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds\right]\left[a_0-b_1de^{-zd}\right]}{\frac{1}{a_0}\left[a_0-b_1de^{-zd}\right]\left[a_0-b_1de^{-\bar{z}d}\right]} \qquad (44)
\end{aligned}$$

where $\tilde{b}_0=\frac{b_0}{a_0}$. The denominator is always real since:

$$\begin{aligned}
\left[a_0-b_1de^{-zd}\right]\left[a_0-b_1de^{-\bar{z}d}\right] &= a_0^2-a_0b_1d\left(e^{-zd}+e^{-\bar{z}d}\right)+b_1^2d^2e^{-zd-\bar{z}d}= \\
&= a_0^2-a_0b_1de^{-xd}\left(e^{-iyd}+e^{iyd}\right)+b_1^2d^2e^{-2xd}
\end{aligned}$$

and taking into account that $e^{iy}+e^{-iy}=2\cos y$ while $e^{iy}-e^{-iy}=2i\sin y$ we have that

$$\left[a_0-b_1de^{-zd}\right]\left[a_0-b_1de^{-\bar{z}d}\right]=a_0^2-2a_0b_1de^{-xd}\cos yd+b_1^2d^2e^{-2xd}$$

which is real. Then we have also to show that the numerator of relation (43) is real.

$$Num=A+B$$

where

$$\begin{aligned}
A &= \xi(0)\left[a_0-b_1de^{-\bar{z}d}\right]+\xi(0)\left[a_0-b_1de^{-zd}\right] \\
&= 2a_0\xi(0)-2b_1d\xi(0)e^{-xd}\cos yd \\
&= 2\xi(0)\left[a_0-b_1de^{-xd}\cos yd\right]
\end{aligned}$$

which is real. On the other hand

$$B=\left[z+\tilde{b}_0\right]\left[a_0-b_1de^{-\bar{z}d}\right]\int_{-d}^{0}\xi(s)e^{-zs}ds+\left[\bar{z}+\tilde{b}_0\right]\left[a_0-b_1de^{-zd}\right]\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds \qquad (45)$$

Now observe that $\int_{-d}^{0} \xi(s)e^{-zs}ds = \underbrace{\int_{-d}^{0} \xi(s)e^{-xs}\cos ys\ ds}_{\alpha} - i\underbrace{\int_{-d}^{0} \xi(s)e^{-xs}\sin ys\ ds}_{\beta}$ while $\int_{-d}^{0} \xi(s)e^{-\bar{z}s}ds = \alpha + i\beta$. Taking into account this relation (45) is equivalent to

$$\begin{aligned}
B &= \left[z+\tilde{b}_0\right]\left[a_0 - b_1 d e^{-\bar{z}d}\right][\alpha - i\beta] + \left[\bar{z}+\tilde{b}_0\right]\left[a_0 - b_1 d e^{-zd}\right][\alpha + i\beta] \\
&= a_0\alpha\,(z+\bar{z}) - \alpha b_1 d\left(ze^{-\bar{z}d} + \bar{z}e^{-zd}\right) - ia_0\beta\,(z-\bar{z}) + i\beta b_1 d\left(ze^{-\bar{z}d} - \bar{z}e^{-zd}\right) + \\
&\quad +2\tilde{b}_0\alpha a_0 - \alpha\tilde{b}_0 b_1 d\left(e^{-\bar{z}d} + e^{-zd}\right) - i\tilde{b}_0 b_1\beta d\left(e^{-\bar{z}d} - e^{-zd}\right) \\
&= 2a_0\alpha x + 2a_0\beta y + 2\tilde{b}_0\alpha a_0 - \alpha b_1 d e^{-xd}\left[x\left(e^{iyd} + e^{-iyd}\right) + iy\left(e^{iyd} - e^{-iyd}\right)\right] + \\
&\quad +i\beta b_1 d e^{-xd}\left[x\left(e^{iyd} - e^{-iyd}\right) + iy\left(e^{iyd} + e^{-iyd}\right)\right] - \alpha\tilde{b}_0 b_1 d e^{-xd}\left(e^{iyd} + e^{-iyd}\right) + \\
&\quad -i\tilde{b}_0 b_1\beta d e^{-xd}\left(e^{iyd} - e^{-iyd}\right) \\
&= 2a_0\alpha x + 2a_0\beta y + 2\tilde{b}_0\alpha a_0 - 2\alpha b_1 d e^{-xd}\left[x\cos yd - y\sin yd\right] - 2\beta b_1 d e^{-xd}\left[x\sin yd + y\cos yd\right] \\
&\quad -2\alpha\tilde{b}_0 b_1 d e^{-xd}\cos yd + 2\tilde{b}_0 b_1\beta d e^{-xd}\sin yd
\end{aligned}$$

which is real. This is sufficient to prove that $a+c$ is real, given that it is a ratio of real numbers. Now we have to show that $\mathrm{Re}\,(a_r - c_r) = 0$.

$$a - c = \frac{\left[\xi(0) + \left(z+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-zs}ds\right]\left[a_0 - b_1 d e^{-\bar{z}d}\right]}{\frac{1}{a_0}\left[a_0^2 - 2a_0 b_1 d e^{-xd}\cos yd + b_1^2 d^2 e^{-2xd}\right]} + \\
- \frac{\left[\xi(0) + \left(\bar{z}+\tilde{b}_0\right)\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds\right]\left[a_0 - b_1 d e^{-zd}\right]}{\frac{1}{a_0}\left[a_0^2 - 2a_0 b_1 d e^{-xd}\cos yd + b_1^2 d^2 e^{-2xd}\right]}$$

the denominator as before is real. Then we have to show that the numerator is purely imaginary. As before, we split the numerator

$$Num = C + D$$

where

$$\begin{aligned}
C &= \xi(0)\left[a_0 - b_1 d e^{-\bar{z}d}\right] - \xi(0)\left[a_0 - b_1 d e^{-zd}\right] \\
&= -b_1 d\xi(0)e^{-xd}\left(e^{iyd} - e^{-iyd}\right) \\
&= 2ib_1 d\xi(0)e^{-xd}\sin yd
\end{aligned}$$

which is purely imaginary, and

$$\begin{aligned}
D &= \left[z+\tilde{b}_0\right]\left[a_0-b_1de^{-\bar{z}d}\right]\int_{-d}^{0}\xi(s)e^{-zs}ds-\left[\bar{z}+\tilde{b}_0\right]\left[a_0-b_1de^{-zd}\right]\int_{-d}^{0}\xi(s)e^{-\bar{z}s}ds \\
&= \left[z+\tilde{b}_0\right]\left[a_0-b_1de^{-\bar{z}d}\right][\alpha-i\beta]-\left[\bar{z}+\tilde{b}_0\right]\left[a_0-b_1de^{-zd}\right][\alpha+i\beta] \\
&= a_0\alpha\left(z-\bar{z}\right)-\alpha b_1d\left(ze^{-\bar{z}d}-\bar{z}e^{-zd}\right)-ia_0\beta\left(z+\bar{z}\right)+i\beta b_1d\left(ze^{-\bar{z}d}+\bar{z}e^{-zd}\right)-2i\tilde{b}_0a_0\beta+ \\
&\quad -\alpha\tilde{b}_0b_1d\left(e^{-\bar{z}d}-e^{-zd}\right)-i\tilde{b}_0b_1\beta d\left(e^{-\bar{z}d}+e^{-zd}\right) \\
&= 2i\Big[a_0\alpha y-a_0\beta x-\tilde{b}_0a_0\beta-\alpha b_1de^{-xd}\left(x\sin yd+y\cos yd\right)+\beta b_1de^{-xd}\left(x\cos yd-y\sin yd\right)+ \\
&\quad -\alpha\tilde{b}_0b_1de^{-xd}\sin yd-\tilde{b}_0b_1\beta de^{-xd}\cos yd\Big]
\end{aligned}$$

which is purely imaginary too. Then $\operatorname{Re}\left(a_r-c_r\right)=0$ since it is a ratio of a sum of purely imaginary numbers and a real number. This is sufficient to prove expression (42).

The second part of the proof consists in showing that (42) is a real function. We start by considering the first term

$$\sum_{r=0}^{k}\varsigma_re^{x_rt}+\sum_{r=k}^{\infty}\left(a_re^{z_rt}+\bar{a}_re^{\bar{z}_rt}\right) \tag{46}$$

Calling $a=\varsigma+i\omega$ we have that

$$\begin{aligned}
ae^{zt}+\bar{a}e^{\bar{z}t} &= \left(\varsigma+i\omega\right)e^{xt}e^{iyt}+\left(\varsigma-i\omega\right)e^{xt}e^{-iyt} \\
&= e^{xt}\left[\left(\varsigma+i\omega\right)\left(\cos yt+i\sin yt\right)+\left(\varsigma-i\omega\right)\left(\cos yt-i\sin yt\right)\right] \\
&= 2e^{xt}\left(\varsigma\cos yt-\omega\sin yt\right)
\end{aligned}$$

and then (46) becomes

$$\sum_{r=0}^{k}\varsigma_re^{x_rt}+2\sum_{r=k}^{\infty}e^{xt}\left(\varsigma\cos yt-\omega\sin yt\right)$$

which is a real function of t. Now we study the term

$$\sum_{r=k}^{\infty}\left(\frac{e^{z_r(t-s)}}{h'\left(z_r\right)}+\frac{e^{\bar{z}_r(t-s)}}{h'\left(\bar{z}_r\right)}\right)$$

After some boring algebra this can be rewritten as

$$\sum_{r=k}^{\infty}\frac{2\left\{a_0\cos\left[y_r\left(t-s\right)\right]-b_1de^{-x_rd}\cos\left[y_r\left(t-s+d\right)\right]\right\}}{a_0^2-2a_0b_1de^{-x_rd}\cos y_rd+b_1^2d^2e^{-2x_rd}}e^{x_r(t-s)}$$

which is a real function. Then it follows immediately that the general continuous solution (42) can be rewritten as

$$\begin{aligned}
u\left(t\right) &= \sum_{r=0}^{k}\varsigma_re^{x_rt}+2\sum_{r=k}^{\infty}e^{xt}\left(\varsigma\cos yt-\omega\sin yt\right)+ \\
&\quad +\int_0^t f(s)\left[\sum_{r=0}^{k}\frac{e^{x_r(t-s)}}{h'\left(x_r\right)}+\sum_{r=k}^{\infty}\frac{2\left\{a_0\cos\left[y_r\left(t-s\right)\right]-b_1de^{-x_rd}\cos\left[y_r\left(t-s+d\right)\right]\right\}}{a_0^2-2a_0b_1de^{-x_rd}\cos y_rd+b_1^2d^2e^{-2x_rd}}e^{x_r(t-s)}\right]ds
\end{aligned} \tag{47}$$

which is clearly a real function.$u:I\to\mathbb{R}$.

A.3 How to get expression (30) from (29).

First of all, observe that from Theorem 3 we can rewrite

$$\begin{aligned}\sum\nolimits_v P_{\tilde{m},v} &= -a_{\tilde{m}}^{-\frac{1}{\sigma}}\left[\frac{1}{(g_c-\tilde{z})\,h'(\tilde{z})}+\sum\nolimits_{v\neq\tilde{v}}\left(\frac{1}{(g_c-z_v)\,h'(z_v)}+\frac{1}{(g_c-\bar{z}_v)\,h'(\bar{z}_v)}\right)\right]\\ &= \alpha_{\tilde{v}}-a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_{v\neq\tilde{v}}\left(\frac{(g_c-\bar{z}_v)\,h'(\bar{z}_v)+(g_c-z_v)\,h'(z_v)}{(g_c-z_v)\,(g_c-\bar{z}_v)\,h'(z_v)h'(\bar{z}_v)}\right)\end{aligned}$$

Now calling $z = x + iy$ and $n = \alpha + i\beta$, and taking into account the shape of the characteristic equation we get after some algebra

$$\sum\nolimits_v P_{\tilde{m},v} = \alpha_{\tilde{v}} + 2\sum\nolimits_{v\neq\tilde{v}} \Psi_{0,v} \tag{48}$$

where

$$\Psi_{0,v} = -a_{\tilde{m}}^{-\frac{1}{\sigma}}\frac{g_c - x_v + \tilde{A}e^{-x_v d}\left\{\left[(g_c-x_v)\,x_v+y_v^2\right]\cos y_v d+\left[(g_c-x_v)\,y_v+x_v y_v\right]\sin y_v d\right\}}{(g_c^2-2g_c x_v+x_v^2+y_v^2)\left[1+\tilde{A}e^{-2x_v d}\,(x_v^2+y_v^2)+2\tilde{A}e^{-x_v d}\,(x_v\cos y_v d+y_v\sin y_v d)\right]}$$

Now we have to rewrite

$$\begin{aligned}\sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},v}e^{z_v t} &= \sum\nolimits_{v\neq\tilde{v}}(n_v - P_{\tilde{m},v})\,e^{z_v t}\\ &= \sum\nolimits_{v\neq\tilde{v}}\left(n_v e^{z_v t}+\bar{n}_v e^{\bar{z}_v t}\right)+a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_{v\neq\tilde{v}}\left(\frac{e^{z_v t}}{(g_c-z_v)\,h'(z_v)}+\frac{e^{\bar{z}_v t}}{(g_c-\bar{z}_v)\,h'(\bar{z}_v)}\right)\end{aligned}$$

which taking into account the results of the previous Appendix is equal to

$$2\sum\nolimits_{v\neq\tilde{v}}(\alpha_v\cos y_v t-\beta_v\sin y_v t)\,e^{x_v t}+a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_{v\neq\tilde{v}}\left(\frac{(g_c-\bar{z}_v)\,h'(\bar{z}_v)e^{z_v t}+(g_c-z_v)\,h'(z_v)e^{\bar{z}_v t}}{(g_c-z_v)\,(g_c-\bar{z}_v)\,h'(z_v)h'(\bar{z}_v)}\right)$$

which after some algebra and taking into account some trigonometric relations can be rewritten as

$$2\sum\nolimits_{v\neq\tilde{v}}\left[(\alpha-\Psi_{0,v})\cos yt-(\beta+\Psi_{1,v})\sin yt\right]e^{x_v t} \tag{49}$$

where

$$\Psi_{1,v} = a_{\tilde{m}}^{-\frac{1}{\sigma}}\frac{y_v + \tilde{A}e^{-x_v d}\left\{\left[(g_c-x_v)\,x_v+y_v^2\right]\sin y_v d-\left[(g_c-x_v)\,y_v+x_v y_v\right]\cos y_v d\right\}}{(g_c^2-2g_c x_v+x_v^2+y_v^2)\left[1+\tilde{A}e^{-2x_v d}\,(x_v^2+y_v^2)+2\tilde{A}e^{-x_v d}\,(x_v\cos y_v d+y_v\sin y_v d)\right]}$$

Finally taking into account relations (48) and (49) follows immediately the shape of the general continuous solution in (30).

A.4 Computational method

In order to obtain the spectrum of the roots from the law of motion of capital and its solution, we have used Lambert functions as proposed recently by Asl and Ulsoy [2]. A class of functions $W(s)$ are called Lambert functions if they satisfy the relation

$$W(s)e^{W(s)} = s \tag{50}$$

Then considering the characteristic equation of the law of motion of capital

$$-se^s + d\tilde{A} = 0 \tag{51}$$

with $s = zd$, and taking into account the definition of the Lambert function (50), we have that

$$W\left(d\tilde{A}\right)e^{W(d\tilde{A})} = d\tilde{A} \tag{52}$$

Now comparing (51) and (52), the solutions of the equation which describe the characteristic spectrum are

$$z = \frac{1}{d}W\left(d\tilde{A}\right)$$

In the most general form, the Lambert function is a complex function with infinite branches. Calculation of both the principal branch and the other branches can be presented in series form ([2] see for more details). Taking into account these results, we have used the MatLab programs (`Lambertww.m`, `Spectrum.m`, and `Solutions.m`) in order to derive the first $m = 16$ branches[11] and from them the corresponding roots. Then we have derived the roots of the characteristic equation of the law of motion of consumption through relation (16) and residue p_m through the relation (41). Observe that to any branch corresponds a particular solution for the delay differential equation. Finally, using the result in Theorem 3, namely the shape of the general continuous solution (30), it is possible to derive the general continuous solution.

[11] The results obtained in our analysis are invariant to a higher choice of m.

References

[1] Asea, P., and Zak, P. (1999). "Time to build and cycles". *Journal of Economic Dynamic and Control* 23, 1155-1175;

[2] Asl, F. M. and Ulsoy, A. G. (2003). "Analysis of a system of linear delay differential equations". *Journal of Dynamic System, Measurement and Control* 125, 215-223;

[3] Bambi, M. and Licandro, O. (2004). "Indeterminacy and Time-to-Build". *EUI Working Paper ECO2004-17*;

[4] Barro, R. J. and Sala-i-Martin, X. (2004). *Economic Growth. Second Edition.* The MIT Press;

[5] Benhabib, J. and Rustichini, A. (1991). "Vintage capital, investment and growth". *Journal of Economic Theory*, 55, 323-339;

[6] Bellman, R. and Cooke, K. (1963). *Differencial-difference equations*, New York Academic Press;

[7] Belz, M. H., and James, R. W. (1938). "The significance of the characteristic solutions of mixed difference and differential equations", *Econometrica* 6(4), 326-343;

[8] Bohm-Bawerk, Eugen von (1849). *Capital and interest.* South Holland Libertarian Press;

[9] Boucekkine, R., Germain, M., and Licandro, O. (1997). "Replacement echoes in the vintage capital growth model", *Journal of Economic Theory*, 74, 333-348;

[10] Boucekkine, R., Licandro, O., and Puch, L. (2005). "Vintage capital and the dynamics of the AK model", *Journal of Economic Theory*, 120, 39-72;

[11] Burnside, C., and Eichenbaum, M. (1996). "Factor-hoarding and the propagation of business cycle shocks". *American Economic Review*, 86, 1154-1174;

[12] Collard, F., Licandro, O., and Puch, L. (2004). "The short-run dynamics of optimal growth models with delays". EUI mimeo;

[13] El'sgol'ts, L. E., and Norkin, S. B. (1973). *Introduction to the theory and application of differential equations with deviating arguments.* Academic Press;

[14] Frisch, R., and Holme, H. (1935). "The characteristic solutions of a mixed difference and differential equation occuring in economic dynamics", *Econometrica* 3(2), 225-239;

[15] Greenwood, J., Hercowitz, Z., and Huffman, G.W. (1988). "Investment, capacity utilization, and the real business cycle", *American Economic Review*, 78(3), 402-417;

[16] Hale, J. K. and Lunel V. S (1993). *Introduction to functional differential equations.* Springer-Verlag;

[17] Hayek, F. A. (1940). *The pure theory of capital.* Routledge & Kegan Paul;

[18] Hayes, N. D. (1950). "Roots of the trascendental equation associated with a certain difference-differential equation", *Journal of the London Mathematical Society* 25, 226-232;

[19] Jevons, W. S. (1871). *The theory of political economy.* A. M. Kelly;

[20] Kalecki, M. (1935). "A macroeconomic theory of the business cycle", *Econometrica*, 3(3), 327-344;

[21] Kydland, F. E., and Prescott, E. C. (1982). "Time to build and aggregate fluctuations", *Econometrica*, 50(6), 1345-1370;

[22] Kolmanovskii, V. B., and Myshkis, A. (1998). *Introduction to the theory and aplications of functional differential equations.* Kluwer Academic Publishers, Dordrecht;

[23] Kolmanovskii, V. B., and Nosov, V. R. (1986). *Stability of functional differential equations.* Academic Press, Inc;

[24] Rustichini, A. (1989). "Hopf bifurcation for functional differential equation of mixed type", *Journal of Dynamics and Differential Equation*, 1(20), 145-177.