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Heterogeneity and Stability: Variations on Scarf's Processes

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Abstract

There are still problems in handling diversity in economics. The general equilibrium model itself lacks determinacy for a generic population of economic agents. In an outstanding contribution, Jean-Michel Grandmont (1992) argues that increasing behavioural heterogeneity makes aggregate expenditures more independent of prices. He conjectures that, in the aggregate, weak axiom of revealed preference, gross substitutability, uniqueness and stability of the Walrassian exchange equilibrium would prevail under "flat enough" distributions of characteristics. This note emphasises the specific nature of the distributions involved in Grandmont's framework and enhances that the strong macroeconomic regularities that he evidences cannot be considered as a general property of exchange markets.

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1. Introduction[1]

1.1. The difficulties arising from heterogeneity

Surprisingly enough, economic theory is still not able to deal correctly with heterogeneity. Handling diversity is nevertheless a central problem of economics, as it is presumably in all other social sciences. If people are indeed different, they are likely to be affected differently by their environment and to react differently facing an identical situation. Two main questions are thus raised. First, what are the criteria to be used in order to take a decision that concerns various people? This is the normative problem. Second, how does an economy that is compounded of different people work? This is the positive problem. Naturally, such a distinction is usually not possible as both issues are intrinsically connected.

A good example of the problems at hand and a clear outline of the limits of economic theory in handling diversity is given by the representative agent approach. Despite its widespread use, the model is, in general, both unjustified and misleading. (See A. P. Kirman - 1992). It is unjustified because *(i)* individual optimization does not engender collective rationality and conversely *(ii)* the fact that the community exhibits a certain rationality does not imply that individuals find themselves in an

[1]This paper is a part of my Ph.D. Thesis. It develops a suggestion by Alan Kirman. I would like to thank Jean-Michel Grandmont for his detailed comments on a first draft of this paper and for his kind encouragements. I met Reinhard John at a conference and he gave me an interesting reference for the interpretation of my results. I was then invited by Werner Hildenbrand in Bonn where I enjoyed an outstanding environment for research. No need to say that his deep understanding of the aggregation problem played a leading role for the present work. Special thanks to Kurt Hildenbrand for an illuminating example presented in this paper. Herbert Scarf and Rodolphe Dos Santos Ferreira provided interesting remarks at my thesis defence. I am indebted to David Cass, Benedetto Gui and Bruno Versaevel for both helpful discussions and editing help. This work was presented at the "Second International Workshop on Economics with Heterogeneous and Interacting Agents" at Ancona (It.), the 97' **E**conometric **S**ociety **E**uropean **M**eeting in Toulouse (Fr.), at the "7ème journées du SESAME" -**S**éminaires d'**E**tudes et de **S**tatistiques **A**ppliquées à la **M**odélisation en **E**conomie" in Lyon (Fr.) and at the departemental seminar of the THEMA (Fr.). All remaining errors are mine.

optimal situation from their own point of view. It is misleading because the reaction of the representative agent to some change may not be the same as the aggregate reaction of the individuals he is supposed to represent. Thus the description of the economy is not accurate. Moreover, even if the choices of the representative agent coincide with aggregate choices, the representative agent might "prefer" one of two situations where individuals would prefer the other (See S. C. Dow and S. Werlang - 1988). In other words the valuation of the various situations can differ and normative judgement is not possible. Some results derived by M. Jerison (1993) show that the use of such a representative agent is legitimate for welfare interpretations under very narrow conditions that are not verified for generic cases.

1.2. Problems in the understanding of competitive markets

As a decentralised process, the free market is argued to overcome the difficulties that one faces in taking a uniform and centralised decision regarding a heterogeneous population. In some sense, it escapes the problems raised by the theory of social choice by leaving each economic agent to make the necessary decisions. However the properties of the competitive market as a coordination device are somewhat unclear. Despite its early introduction in the literature, the simple existence of an equilibrium, *i.e.* the possibility of consistent decisions across several individuals, was only proved in full generality by K. J. Arrow and G. Debreu in the 50's. But there are no results with respect to the uniqueness and stability of this equilibrium. Thus the general equilibrium model, often considered as the benchmark of neoclassical economics, does not allow of definite conclusions because of its lack of determinacy.

1.3. The fiasco of the micro-foundations project

While a normative approach to the problem of heterogeneity does not conceal its intrinsic difficulties "from the beginning", the idea that explanations of social phenomena must be based on individuals has enjoyed a long life. It is already to be found in W. S. Jevons (1871) who affirmed that *"the general forms of the laws of economics are the same in the case of individuals and the nations."* [2]. J. S. Mill (1874) made the following observation:

> *Human beings in society have no properties but those which are derived from, and may be resolved into, the laws of nature in the individual man. In social phenomena the Composition of Causes is the universal law.*[3]

More recently, J. R. Hicks (1939) even asserted that *"the behaviour of a group of individuals or a group of firms obeys the same laws as the behaviour of a single unit."* [4]

However the problem is not that simple. On the contrary, in the words of K. J. Arrow (1968), *"a limit is set to the tendency implicit in price theory, particularly in its mathematical versions, to deduce all properties of aggregate behaviour from assumptions about individual economic agents."* [5]

Since the *"negative"* results of H. Sonnenschein (1972, 1973), R. Mantel (1974, 1976, 1977) and G. Debreu (1974), it has been well known that *no satisfactory theory links the characteristics of economic agents identified by microeconomic theory and the properties of aggregated systems used in macroeconomic theory*: Even with "strong individualistic assumptions",[6] the only restrictions imposed on aggregate demand functions are the continuity for all strictly positive prices, the homogeneity of

[2]W.S. Jevons, (1871), *"The Theory of Political Economy"*, A.M. Kelley, New York, (1965), p.16. *Reference from Rizvi (1994).*

[3]Mill, J. S. (1875), *"A system of logic"*, in Robson, J. M., (1974), *"Collected works of John Stuart Mill"*, Toronto, p.879. *Reference from Rizvi (1994).*

[4]Hicks, J.R. (1939), *"Value and Capital: an Inquiry into some Fundamental Principles of Economic Theory"*, Clarendon Press, Oxford, p.245. *Reference from Rizvi (1994).*

[5]Arrow, K. J., (1968), *"Economic Equilibrium"*, in *"International Encyclopedia of the Social Sciences"*, vol **4**, pp.376-89, Mac Milan, London. *Reference from Rizvi (1994).*

[6]*i.e.* preferences are assumed to be continuous, convex and to give rise to a monotone preorder.

degree 0 in prices and the fact that Walras' law is verified. (For a review of the microfoundations project, see S. Abu Turab Rizvi (1994)).
In economics, there is no hope of ever explaining social objects by realistic assumptions on individuals *exclusively.*

As a consequence, almost any macroeconomic feature can be explained by the general equilibrium model. The latter provides no indications as to the nature of market outcomes[7]. In particular, as already mentioned, the assumption of uniqueness and stability of the equilibrium has no theoretical justification. Accordingly, to refer generically to a *"natural"* state of the economy is meaningless. (See A. P. Kirman - 1989).

1.4. The necessity of macroeconomic assumptions

As a result, the competitive market representation *must* turn to additional assumptions[8] in order to ensure the drawing of unambiguous figures. In strict microfoundations, macroeconomic regularities are sought in the individual behaviour of agents, not in the social structure in which they find themselves. (*See* E. J. Nell- 1984). The failure of the microfoundations project has shown macroeconomics to be somewhat irreducible. But uniqueness and stability of the equilibrium are required for the determinacy of the analysis.
This is the origin of the *gross-substitutability assumption,* first introduced by A. Wald in 1936. The latter has the merit of both strong technical implications and clear economic interpretation. This enables the inference to be challenged and avoids purely *ad hoc* assumptions hidden under technical arguments. While microeconomics happens to be insufficient to explain all macroeconomic properties, macroeconomic assumptions are expected to be at least plausible and should always be submitted to some consistency examination including compatibility with microeconomic choice theory.

[7] Except the "Welfare theorems"

[8] By additional assumptions, it is meant that assumptions cannot bear on individual behaviour *only* but should go beyond strict micro theory.

1.5. Gross substitutability

What does "gross substitutability" mean? It means that when the price of one good rises, the excess demand of all other goods increases. It is nice because it is simple. Furthermore it is sufficient to make general equilibrium models follow the rules of partial equilibrium models.

The idea that gross substitutability might be assumed is intuitive when consumption is considered: if two goods are substitutes, then increasing the price of one would lead to an increase in the demand for the other one, hence in excess demand. The idea that gross substitutability may not hold is intuitive when production is considered: if two commodities are joint-products (complements), then increasing the price of the first would lead producers to increase the production of both, which would tend to decrease the excess demand for the second. Note that, even in a pure exchange economy, gross substitutability is not granted. As a result, the assumption that gross substitutability prevails as the combined response of choices by consumers and producers is far from being obvious. The sufficient conditions imposed on the nature of economic agents' preferences that would guarantee gross-substitutability are not known.[9] However, gross-substitutability, together with the weak axiom of revealed preference, is a sufficient condition for gross stability (and uniqueness of the equilibrium). This amply justifies the interest in this assumption and the conditions that allow such a property to hold.[10]

[9] As already mentioned, some work by H. Sonnenschein, R. Mantel and G. Debreu in the 70's has shown instead that the only properties of the excess demand function which can be deduced by requiring the economic agents to be well behaved (*i.e.* preferences are continuous, convex and give rise to a monotone preorder) are Walras'Law, homogeneity, and continuity (or differentiability).
R. Mantel (1976) has shown the result to hold for the (even stronger) assumption of homothetic preferences.

[10] Note however that gross-substitutability is not necessary to get the stability of the general equilibrium.

1.6. Assumptions on distribution

An innovative approach in trying to fill the gap between micro and macro was introduced by J.-M. Grandmont (1987) following W. Hildenbrand (1983) and others. The idea is that instead of introducing some restrictions on the form of individual demand (or any micro-variable) or making unwarranted macro assumptions, "nice" macro-features may result from aggregation by the means of assumptions made on the *distribution* of characteristics *among* individuals. This conjecture, already formulated by many like A. Wald, J. R. Hicks, K. J. Arrow and F. H. Hahn, seems to find confirmation in an article by J.-M. Grandmont (1992). He establishes under quite mild conditions that "pathological" market features are unlikely to arise at the aggregate level when there is "enough behavioural heterogeneity".

We shall attempt to show that it is too early to draw some definite conclusions. The assumptions of Grandmont's model are shown to be more restrictive than they might appear at first glance. They are nevertheless necessary. An example is provided in which one (questionable) assumption is relaxed. In this exchange economy, *no distribution* of characteristics over the space of vector parameters precludes the aggregate excess demand function from violating the gross substitutability condition. The result established by Grandmont is far from being a general property of exchange markets. We will show that, in the limit and unless "pathological cases", his parametrization lead in fact the economy to be exclusively made of "Cobb-Douglas agents".

1.7. Structure of the paper

The paper is organized as follows. In section two, Grandmont's model is sketched and some comments on the hypothesis are made. In particular it is emphasised that the aggregation process is likely to be ill-designed because some "behavioural types" might be over represented. First the equivalence classes generated by the transformation introduced by Grandmont do not always have the same dimension. This introduces some biases in the whole distribution of demand functions. Sec-

ond, this affine transformation on the commodity space does not always generate an infinite number of distinguishable equivalence classes. However, Grandmont introduces a continuous parametrization of the space of equivalent classes[11] that leads necessarily to an over-representation of some types of consumers in those cases. Thus the smoothing process in the aggregation can be double: within and in addition *over* the equivalence classes. As a consequence, while heterogeneity increases within each equivalence class when the dispersion of parameters increases, it is not clear that heterogeneity also increases in the overall economy.
The assumption that guarantees aggregate budget shares to be uniformly bounded away from zero is shown to be quite restrictive. It is not verified for very common demand functions, like those generated by CES-utility functions. In fact this assumption requires all the commodities to be "essential" for the economy because it should hold for all set of prices. Moreover it appears to bear upon the (partial) result of the aggregation itself.

In section three, the assumption that guarantees aggregate budget shares to be uniformly bounded away from zero is removed in order to study its importance for Grandmont's theorem. A set of "behavioural types" and a (discrete) distribution over it is provided. For *any* symmetric distribution of parameters over the transformation space, the latter generates an aggregate excess demand function that violates the gross-substitutability assumption in this exchange economy. It shows that the "flatness" of the distribution is not sufficient for guarantee "nice" properties at the aggregate level.
As one might suspect that this result to come from the fact the we shift from a continuous and infinite set of types to a discrete and finite set[12], a continuous distribution of types is presented that exhibits the same out-

[11] Each equivalent class is supposed to represent a "behavioural type".

[12] See the results of Kirman and Koch (1986) as opposed those of Hildenbrand (1983). The first shows that, *in a finite economy,* Sonnenschein-Debreu-Mantel's result is valid for essentially any distribution of income in which individuals have identical preferences; While the second, *in an atomless economy,* obtains a monotonicity property for the market demand by restricting the shape of the income distribution in the consumption sector.

comes. Finally, the number of equivalence classes present in the economy is shown to have no clear impact on the aggregate demand function.

In the last section before conclusion, a come back is made to *Grandmont's* [1992] paper in order to re-interpret the results. It is argued that the conjunction of the parametrization and the assumption about budget shares imply indirectly very specific distributions of demand functions. With the degree of "heterogeneity", this is in fact the number of "Cobb-Douglas like" demand functions that increases in the economy. Thus Grandmont's results might be read as follows: If there is a "sufficient number" of "Cobb-Douglas like" demands in the whole population, then gross-substitutability is guaranteed. In this perspective however, the aggregation process presented appears to be rather a "Law of large numbers" that conserve some micro-properties common to "the most" (demand functions). It is *not* the heterogeneity of the demand functions that generates[13] some structure at the macro-level and provides the results.

Finally, a brief conclusion is drawn that questions the directions to explore for further research. The investigation of how micro-properties of the demand functions combine in the process of aggregation is certainly a fascinating program. However, if *generic assumptions* with respect to the shape of the distribution of characteristics alone might not be sufficient to derive general results, there is a question about the adequacy of such an approach when those characteristics cannot be observed.

[13] by counterbalancing some agents' reactions to price changes by the opposite reaction of others.

2. The Grandmont's model:

For the analysis to be self-contained, the main hypotheses and results by *J.-M. Grandmont (1992)* are reported in the following.

2.1. Outline of the main results

Consider a pure exchange economy with $l \geq 2$ commodities. The characteristics of an individual agent are assumed to be described by a demand function $\xi(p, w)$ and an income level w.
There is a first set A of *"types"* of agents that is taken to be a separable metric space, to simplify matters. A distribution over types is then described by a probability measure μ on A. To each type a corresponds a demand function $\xi_a(p, w)$ and an income level $w_a > 0$.
For any vector $\alpha \in \Re^l$ define the 'α-transform' of a demand function $\xi(p, w)$:

$$\xi(\alpha, p, w) = e^\alpha \otimes \xi(e^\alpha \otimes p, w)$$

where $\otimes$ indicates the normal tensorial product: $a \otimes b = (a_1 b_1, ..., a_l b_l)$. A conditional distribution $f(\alpha \mid a)$ on the space $\Re^l$ of indexing parameters α is specified. It describes the distribution of agents over the equivalence class of type a generated by the α-transforms.
The overall distribution of characteristics in the economy is thus described by the marginal probability measure μ on A and the conditional densities $f(\alpha \mid a)$ on $\Re^l$. The assumptions of the model are as follows:

Assumption 1. *The income level $w_a > 0$ depends continuously on the type a. Per capita income is finite* i.e.,

$$\bar{w} = \int_A w_a \mu(da) < +\infty$$

then total market demand given by

$$X(p) = \int_A X(a, p, w_a) \mu(da)$$

_where $X(a,p,w_a)=\int_{\Re^l}\xi_a(\alpha,p,w)\,f(\alpha\,|\,a)\,d\alpha$, is well defined, non negative, continuous in p and satisfies $p.X(p)\equiv\bar{w}$.*

Assumption 2.
(1) The demand function $\xi_a(p,w)$ is continuous in (a,p,w).
(2) The conditional density $f(\alpha\,|\,a)$ is continuous in (α,a). It has partial derivatives $(\partial f/\partial\alpha_k)(\alpha\,|\,a)$, and they are continuous in (α,a). Moreover, for each type a, partial derivatives are uniformly integrable, i.e., *for every $k=1,...,l$*

$$m_k(a)=\int_{\Re^l}\left|\frac{\partial f}{\partial\alpha_k}(\alpha\,|\,a)\right|d\alpha<+\infty.$$

Assumption 3. *For every commodity k, $m_k(a)$ is bounded above by m_k for μ-almost every type a.*

Assumption 4. *For μ-almost every a, the conditional density $f(\alpha\,|\,a)$ is independent of a.*

Assumption 5. *For every commodity h, there exists $\varepsilon_h>0$, with $\sum_h\varepsilon_h\leq 1$, such that for all vectors of positive prices p*

$$p_h\int_A\xi_{ah}(p,w_a)\,\mu(da)\geq\varepsilon_h\bar{w}.$$

With these assumptions, *Grandmont (1992)* proves the following theorem:

Theorem 2.1. *Assume $(A1),(A2),(A3),(A4),(A5)$. Then $p_hX_h(p)\geq\varepsilon_h\bar{w}$ for every h and for every p. The price elasticity of aggregate demand satisfies*

$$\left|\frac{\partial\ln X_h}{\partial\ln p_k}(p)+\delta_{hk}\right|\leq m_k/\varepsilon_h$$

This implies in particular

1. *Total market demand for commodity h is a decreasing function of its own price, i.e., $(\partial X_h/\partial p_h)(p)<0$, if $m_h<\varepsilon_h$.*

2. *Assume* $m_h < \varepsilon_h$ *for every commodity* h *and let* $DD\,(m, \varepsilon)$ *be the set of prices* p *in* $\mathrm{Int}\Re^l_+$ *such that* $\sum_k (m_k/p_k) < \varepsilon_h/p_h$ *for every* $h = 1, \ldots.., l$. *Then the Jacobian matrix of total market demand is such that*

$$\frac{\partial X_h}{\partial p_h}(p) < 0, \quad \left|\frac{\partial X_h}{\partial p_h}(p)\right| > \sum_{k \neq h} \left|\frac{\partial X_h}{\partial p_k}(p)\right|$$

for every p *in* $DD\,(m, \varepsilon)$ *and has therefore a dominant diagonal on that set.*

3. *Assume* $m_k\, l < \varepsilon_h$ *for all commodities* h, k. *Then total demand has a negative quasi-definite Jacobian Matrix* , i.e., $\sum_{h,k} v_h\,(\partial X_h/\partial p_k)\, v_k < 0$ *for every* $v = (v_1, ..., v_l) \neq 0$ *and every price system* p *in* $\mathrm{Int}\Re^l_+$, *and is thus strictly monotone,* i.e., $(p - q)\,.\,([X\,(p) - X\,(q)]) < 0$ *whenever* $p \neq q$. *In particular, the weak axiom of revealed preference is satisfied in the aggregate, i.e.,* $p.X\,(q) \leq \bar{w}$, $X\,(q) \neq X\,(p)$ *implies* $q.X\,(p) > \bar{w}$

It is demonstrated further on that, in an exchange economy and under the given assumptions, both the weak axiom of revealed preference and the gross-substitutability assumption hold. As a consequence, the general equilibrium model would exhibit a unique and stable equilibrium. Therefore, the consequence of Grandmont's result is that, if there is "enough behavioural heterogeneity", *(i)* the competitive market appears to be an efficient coordination device, even for an heterogeneous population and *(ii)* the general equilibrium model appears to be a sound tool for policy analysis. By showing the limits of Grandmont's result, it is also the universality of such a view that is questioned.

2.2. Some remarks on the model

2.2.1. What is to be called a type?

A strong point of Grandmont's article is the fact that his result extends to a wide range of populations and is not confined, say, to populations for which agents' characteristics are derived from the α-transforms of a common (unique) demand function. Such a generality is obtained by the introduction of the set A of agents' *types*. At this point, a natural question arises as to the nature of a *type* in this model. In what do types differ? What is the economic meaning of an equivalence class?

To begin with the analysis, it is worth drawing attention to a rather technical element. The equivalence class of all α-transforms of a given demand function is claimed to be isomorphic to $\Re^l$ by the author. This is *generally* true, but *not always*. If the demand function is proportional to income and more generally if it has special features with respect to the α-transforms, the dimension of the equivalence class is less than l.[14] This apparently innocuous point has strong consequences in terms of the overall distribution of demand functions in the economy, as will be shown in what follows.

There is nothing remarkable in the assertion that without any specification on the set A, *i.e.* on the nature of the demand functions, the notion of *type* might lack significance. Depending on A, the number of equivalence classes may vary from infinity to one. Obviously, if the partition in equivalence classes is reduced to one element,[15] the introduction of a continuous density over A is not an adequate representation and leads to a double smoothing process in the aggregation process. An illustration of these points can be found in the following examples.

[14] See W. Trockel.

[15] This is the case when all the demand functions of the set A can be obtained as the α-transform of one another: all demand functions are part of the same (unique) equivalent class.

Example 2.1. *Let* $A = \left\{ a \in [0,1]^l \quad / \quad \sum a_i = 1 \right\}$ *and* $\xi_a(p,w)$ *be the demand function derived from the maximization of the Cobb-Douglas utility function* $U_a = \prod_i x_i^{a_i}$.
$\xi_a(p,w)$ *is invariant to any* α*-transform. Thus the market share of every commodity is a constant in all equivalence classes. Note however that the latter are all reduced to a single point.*

Example 2.2. *Let* $l = 2$, $A =]0,1[$ *and assume the demand to be derived from a C.E.S. utility function, say* $u(x_1,x_2) = \left(a x_1^{-\rho} + (1-a) x_2^{-\rho} \right)^{-\frac{1}{\rho}}$ *where* $\rho = \frac{1-\sigma}{\sigma} < 1$.

$$
\begin{aligned}
\xi_{a1}(p,w) &= \frac{a^{\sigma} p_1^{-\sigma} w}{a^{\sigma} p_1^{1-\sigma} + (1-a)^{\sigma} p_2^{1-\sigma}} \\
\xi_{a2}(p,w) &= \frac{(1-a)^{\sigma} p_2^{-\sigma} w}{a^{\sigma} p_1^{1-\sigma} + (1-a)^{\sigma} p_2^{1-\sigma}}
\end{aligned}
$$

The α*-transform is given by the formula:*

$$
\begin{aligned}
\xi_{a1}(\alpha,p,w) &= \frac{e^{\alpha_1(1-\sigma)} a^{\sigma} p_1^{-\sigma} w}{e^{\alpha_1(1-\sigma)} a^{\sigma} p_1^{1-\sigma} + e^{\alpha_2(1-\sigma)} (1-a)^{\sigma} p_2^{1-\sigma}} \\
\xi_{a2}(\alpha,p,w) &= \frac{e^{\alpha_2(1-\sigma)} (1-a)^{\sigma} p_1^{-\sigma} w}{e^{\alpha_1(1-\sigma)} a^{\sigma} p_1^{1-\sigma} + e^{\alpha_2(1-\sigma)} (1-a)^{\sigma} p_2^{1-\sigma}}
\end{aligned}
$$

and all the types belong to the same equivalence class.

$$
\xi_a(\alpha,p,w) = \xi_{a'}(\alpha',p,w)
$$

where α' *is obtained by a simple translation in* $\Re^2$*:*

$$
\begin{aligned}
\alpha_1' &= \alpha_1 - \frac{1}{\rho} \ln\left(\frac{a'}{a}\right) \\
\alpha_2' &= \alpha_2 - \frac{1}{\rho} \ln\left(\frac{1-a'}{1-a}\right)
\end{aligned}
$$

Now, in the words of Grandmont, "if the conditional densities $f(\alpha \mid a)$ are "flat", they are "close" to being invariant by an arbitrary translation."

Thus, as the distribution of α becomes more and more uniform, there is almost no difference between the populations generated by both types.

$$\begin{aligned} f(\alpha \mid a)\, \xi_a(\alpha, p, w) - f(\alpha' \mid a')\, \xi_{a'}(\alpha', p, w) &= \\ [f(\alpha \mid a) - f(\alpha' \mid a')]\, \xi_a(\alpha, p, w) &\rightarrow 0 \end{aligned}$$

Example 2.3. *Let $A = \left\{a \in [0,1]^l \quad / \quad \sum a_i = 1\right\}$ and $\xi_a(p, w)$ be obtained by the maximization of the utility function $U_a = \min\{a_i x_i\}$.*
If $a_i > 0$, for all i, then $\xi_{ai}(p, w) = \left(\frac{w}{a_i \sum_j p_j / a_j}\right)$. As previously, all the types $a_i > 0$ are part of the same equivalence class. The original type *that generates a given demand function is indistinguishable. However, if $a_h = 0$, for some h, then $\xi_{ai \neq h}(p, w) = \left(\frac{w}{a_i \sum_{j \neq h} p_j / a_j}\right)$ and the demand for the h commodity(ies) is zero at any price. The equivalence class is no more isomorphic to $\Re^l$ but to $\Re^{l-n}$ where n is the number of commodities for which the demand is zero. It is obvious that those types of consumers do have a characterisation invariant to the α-transforms, namely a zero-consumption of some goods.*
The number of "significant" types *is* $\sum_{k=0}^{l-1} C_l^k = 2^l - 1$ *which* is a finite number. *As a result, the parametrization of the set of types by a continuous parameter a is likely to over-represent some of the "significant" types, in particular the equivalence class in which consumers have a strictly positive demand for all goods.*

As indicated by Grandmont in a private communication, in order to derive precise results on the distribution of the demand functions in the "overall"[16] economy it would be interesting to study the dimension of the equivalence classes and to introduce an adequate topology over the set of "types" A. If the equivalence class of one type is reduced to one point[17], the corresponding demand function inherits a big weight in the total distribution of demand functions. The demand functions that belong to an equivalence class of higher dimension are to be considered with no

[16] "overall" in the sense that the distribution of demand function is considered over both types and α-parameters.
[17] This is the case when the function is a Cobb-Douglas.

individual influence on the total outcome as in any atomless economy. Furthermore, as already argued, the relative weights of some "types" in the "overall" distribution can be biased by the parametrization of A. It follows that the notion of "flatness" is (rather) ambiguous and is likely to lead to very specific distributional assumptions.

To sum up, the difficulty in a parametric model of demand is that heterogeneity of the parameter distribution has no *a priori* relationship with heterogeneity of demand behaviour. The class of parametrizations has to be restricted as pointed out in Kneip (1993). The transformation introduced by J.-M. Grandmont is clearly such that, for a given type a, when the density function of the parameter α becomes flatter, heterogeneity of the demand function increases, as shown by I. Maret (1994). However, the space of parameters and the space of demand functions are not always topologically equivalent as their dimension differs. This, together with the lack of "control" over the set of types A and the density function μ, explains that the latter property is not conserved by aggregation over the different "types" a. *The heterogeneity of demand behaviour in the whole economy does not always increase with the flatness of the distribution of the parameters* α.

2.2.2. Minimum budget share

There is one assumption in Grandmont's 92 paper that appears to bear neither upon distributional aspects nor upon some fundamentals of the model but in some sense rather on the (partial) result of aggregation. This is *Assumption 5* or "desiderability condition" that is, in my view, rather *a minimum budget share* condition. In what follows it is argued that this is a stronger assumption than it might seem to be.

From the economic point of view, *Assumption 5* implies that there is no saturation for any commodity. The fact that the budget share is bounded away from zero for all vectors of prices means that all the commodities in the economy are "essential". If all commodities are free except one, whose price goes to infinity, the expenditures devoted to this commodity remains bounded from below by a strictly positive quantity.

From a technical point of view, the assumption means that every single price is a *pole of degree one* of the demand function and moreover that the coefficient of the pole (a function of the other prices) is to be bounded away from zero.

The peculiarity of the functions that satisfy *Assumption 5* is emphasized further by the extension of the demand functions that do not verify it. One can establish that the assumption does *not* hold for any demand function generated by the maximization of CES utility functions (except the Cobb-Douglas). Neither does it hold for simple and presumably "well behaved" demand functions like, say, the one generated by the maximization of $U(x_1, x_2) = \sqrt{x_1 + 4} + 2\sqrt{x_2}$. Note however that this condition is not supposed to hold for every single function but for the aggregate demand function.

Despite its peculiarity, *Assumption 5* is necessary for Grandmont's result to hold. This will be shown in the next section.

3. Variations on Scarf's process

In what follows a generalisation of H. Scarf's process (1960) is presented. For the sake of comparability, the framework introduced by Grandmont is used. It is shown that, for the proposed set of demand function (that does not verify *Assumption 5* about aggregate budget shares), gross-substitutability of the excess demand is violated, whatever the "degree of heterogeneity" in this exchange economy.

3.1. A simple example

The nowadays classic Scarf example is fitted into the Grandmont framework. One knows the limits of such an example: sensitivity to the initial distribution of goods and violation of the desiderability assumption (When the price for one good goes to zero, the corresponding demand does not go toward infinity). Those limits can easily be removed to the cost of more intricate constructs that we attempt to avoid here. Moreover, we consider that the example still makes the point.

- Let $A = [-1, 1]$ and $\xi_a(p, w)$ be defined as follows:

$$\begin{aligned}
\xi_{a1}(p, w) &= \frac{a^2 (a-1)^2 w_a}{4(p_1 + p_2)} + \frac{a^2 (a+1)^2 w_a}{4(p_3 + p_1)} \\
\xi_{a2}(p, w) &= \frac{a^2 (a-1)^2 w_a}{4(p_1 + p_2)} + \frac{[1 - a^2 (1 + a^2)/2] w_a}{p_2 + p_3} \\
\xi_{a3}(p, w) &= \frac{a^2 (a+1)^2 w_a}{4(p_3 + p_1)} + \frac{[1 - a^2 (1 + a^2)/2] w_a}{p_2 + p_3}
\end{aligned}$$

  It can be verified that $\xi_a(p, w)$ is a demand function in the sense of Grandmont:

  Definition 3.1. *A demand function $\xi(p, w)$ is defined for all vectors of positive prices $p \in \operatorname{Int}\Re^l_+$ and all positive income $w > 0$,* *takes values in* $\Re^l_+$, *is homogeneous of degree* 0 *in* (p,w) *and satisfies Walras' Law,* i.e., $p.\xi(p,w) \equiv w.$

Note that $\xi_{-1}(p,w)$, $\xi_0(p,w)$ and $\xi_1(p,w)$ might be obtained by the maximization of the respective utility function $U_{-1} = \min\{x_1, x_2\}$, $U_0 = \min\{x_2, x_3\}$, $U_1 = \min\{x_3, x_1\}$. According to *Example 2.3,* these three demand functions of the set A generate three distinct two-dimensional equivalence classes by the α-transform. The other types still generate distinct equivalence classes that appear instead to be isomorphic to $\Re^3$.

- The income level that corresponds to type a is given by the formula $w_a = p.r_a$ where $r_a = \left(\frac{a^2(a-1)^2}{4}, 1 - \frac{a^2(1+a^2)}{2}, \frac{a^2(a+1)^2}{4}\right)$ is the initial endowment.
  A distribution over types is given by the probability measure μ on A. Let $\mu(-1) = \mu(0) = \mu(1) = \frac{1}{3}$, and $\mu(a) = 0$ otherwise. Note that we introduced a finite number of ("significant"[18]) types in the economy that appear to generate isomorphic equivalence classes.

- *Assumption 1* is verified: The income level $w_a > 0$ depends continuously on the type a. Per capita income is finite, *i.e.*,
$$\bar{w} = \int_A w_a \mu(da) = \sum_{a=\{-1,0,1\}} w_a \mu(a) = \frac{w_{-1} + w_0 + w_1}{3} < +\infty$$
  then total market demand, given by
$$X(p) = \int_A X(a,p,w_a)\mu(da) = \sum_{a=\{-1,0,1\}} X(a,p,w_a)\mu(a)$$
  is well defined, non-negative, continuous in p and satisfies $p.X(p) \equiv \bar{w}$. Indeed
$$X(p) = \frac{1}{3}\left(\frac{p_1}{p_1+p_2} + \frac{p_3}{p_3+p_1}, \frac{p_2}{p_2+p_3} + \frac{p_1}{p_1+p_2}, \frac{p_3}{p_3+p_1} + \frac{p_2}{p_2+p_3}\right)$$

[18]By "significant" types, it is meant that the equivalent classes generated by those types exhibit specific characteristics that make possible to retrieve the original type that generated the demand curve. Moreover, because equivalence classes are not reduced to one point, there is a proper aggregation on a population that is not reduced to the initial set A.

- *Assumption 2 - (1)* is satisfied: The demand function $\xi_a(p,w)$ is continuous in (a,p,w).
  Assumption 2 - (2) is not necessary: We do not impose any assumption on the continuity of the conditional density $f(\alpha \mid a)$, on the existence of its partial derivative $(\partial f/\partial \alpha_k)(\alpha \mid a)$ and their continuity. Obviously, nothing prohibits this assumption from holding.

- *Assumption 3* is not necessary: There is *no assumption on the "flatness"* of the conditional density.

- Grandmont's *Assumption 4* is imposed: the conditional density $f(\alpha \mid a)$ is independent of a.

- *Assumption 5 is not verified.* However, note that

$$p_h \int_A \xi_{ah}(p, w_a)\, \mu(da) > 0$$

  for all $p \in \ Int\Re^l_+$.

Proposition 3.1. *For the set of types $\{\xi_a(p,w), w_a\}$ introduced, the probability measure μ over A, and* any unconditional *density $f(\alpha)$, the aggregate excess demand function violates the gross substitutability assumption.*

Proof: *In Appendix.*

3.2. The continuous representation

One might think that previous result is due to the shift from a continuous to a discrete representation of types. This is not the case. The distribution over A was chosen because of the previous claim that distributional aspects should be checked carefully. Indeed, the distribution over equivalence classes would be biased if the equivalence classes were not of the same dimension or were over-represented. But the example can easily be extended to the continuous framework:

Define the demand and the initial endowment as follows:

	$\xi_{a1}(p,w)$	$\xi_{a2}(p,w)$
$-1 \leq a \leq -1/2$	$\frac{w_a}{p_1+p_2}$	$\frac{w_a}{p_1+p_2}$
$-1/2 \leq a \leq -1/4$	$-4\left(a+\frac{1}{4}\right)\frac{w_a}{p_1+p_2}$	$\frac{-4\left(a+\frac{1}{4}\right)w_a}{p_1+p_2} + \frac{\left[1+4\left(a+\frac{1}{4}\right)\right]w_a}{p_2+p_3}$
$-1/4 \leq a \leq 1/4$	0	$\frac{w_a}{p_2+p_3}$
$1/4 \leq a \leq 1/2$	$\frac{4\left(a-\frac{1}{4}\right)w_a}{p_3+p_1}$	$\frac{\left[1-4\left(a-\frac{1}{4}\right)\right]w_a}{p_2+p_3}$
$1/2 \leq a \leq 1$	$\frac{w_a}{p_3+p_1}$	0

	$\xi_{a3}(p,w)$	$(r_1,r_2,r_3)_a$
$-1 \leq a \leq -1/2$	0	$(1,0,0)$
$-1/2 \leq a \leq -1/4$	$\frac{\left[1+4\left(a+\frac{1}{4}\right)\right]w_a}{p_2+p_3}$	$(-(1+4a), 2+4a, 0)$
$-1/4 \leq a \leq 1/4$	$\frac{w_a}{p_2+p_3}$	$(0,1,0)$
$1/4 \leq a \leq 1/2$	$\frac{\left[1-4\left(a-\frac{1}{4}\right)\right]w_a}{p_2+p_3} + \frac{4\left(a-\frac{1}{4}\right)w_a}{p_3+p_1}$	$(0, 2-4a, 4a-1)$
$1/2 \leq a \leq 1$	$\frac{w_a}{p_3+p_1}$	$(0,0,1)$

Proposition 3.2. *For the set of types* $\{\xi_a(p,w), w_a\}$ *introduced, for* any *probability measure* μ *with support included in* $\left[-1,-\frac{1}{2}\right] \cup \left[-\frac{1}{4},\frac{1}{4}\right] \cup \left[\frac{1}{2},1\right] \subset A$ *and such that* $\int_{-1}^{-\frac{1}{2}} \mu(da) = \frac{1}{3}, \int_{-\frac{1}{4}}^{\frac{1}{4}} \mu(da) = \frac{1}{3}, \int_{\frac{1}{2}}^{1} \mu(da) = \frac{1}{3}$ *and for* any unconditional *density* $f(\alpha)$, *the aggregate demand function violates the gross substitutability assumption.*

Proof: *It follows from the fact that*

$$X(p) = \frac{1}{3}\left(\frac{p_1}{p_1+p_2} + \frac{p_3}{p_3+p_1}, \frac{p_2}{p_2+p_3} + \frac{p_1}{p_1+p_2}, \frac{p_3}{p_3+p_1} + \frac{p_2}{p_2+p_3}\right)$$

that is identical with previous example.

The (negative) results seem to be due to the very fact that *Assumption 5* does not hold any more. Note however that the number of *effective types* (distinct equivalence classes) is still finite.

3.3. The question of types

The impact (if there is one) of the number of *types* on the aggregate demand function is not clear. Because the aggregation of a set (finite or infinite) of demand functions derived from the maximization of Cobb-Douglas utility functions can be obtained from the maximization of a

Cobb-Douglas utility function, the number of *effective types* has clearly no importance for Grandmont's theorem. The converse, i.e. whether a set A that generates an infinite number of equivalence classes might exhibit "pathologies" when *Assumption 5* is not verified is still to be established.

It has just been proved that without *Assumption 5* and with a finite number of *effective* types (and whatever the number of types in the parametrization), there might be pathological cases. The following example shows that this extends to the case in which there is an infinite number of equivalence classes. It is now based on the class of examples given in the second part of the 1960 Scarf article. The demand is defined as follows for a first "class" of consumers[19]:

$$\begin{aligned}
\xi_{a1}(\alpha, p, w) &= \frac{bp_1^{-1/(1+a)}w}{bp_1^{a/(1+a)} + p_2^{a/(1+a)}} \\
\xi_{a2}(\alpha, p, w) &= \frac{p_2^{-1/(1+a)}w}{bp_1^{a/(1+a)} + p_2^{a/(1+a)}} \\
\xi_{a3}(\alpha, p, w) &= 0
\end{aligned}$$

with $b > \frac{a+1}{a-1}$ and $a > 1$. Again, one can verify that this is a demand function in the sense of Grandmont. Now, *all the types generate distinct equivalence classes:* $\xi_a(\alpha, p, w) = \xi_{a'}(\alpha', p, w)$ (if ever it is possible) would be obtained by a transformation that depends on prices.
This family of types is proved to violate gross substitutability in the appendix.

Hence an infinity of effective types is not sufficient to restore Grandmont's results.

[19] We define by permutation two other "class" of consumers.

4. Grandmont's Model Revisited

If heterogeneity is not sufficient a property to insure the gross-substitutability in an exchange economy, the reader would certainly ask us to answer two questions: What is the effect of heterogeneity in Grandmont's Framework? Where does the nice result come from? Some hints are provided in this section.

Arguably, as the distribution over α spreads, the relative weight in the aggregate demand of the "non-asymptotic" values must decrease. Heuristically, if the conditional densities are "flat" enough, one can understand that the aggregate demand behaves in a similar manner to the α-transformed functions when α is "big" in absolute value.
Let us consider a demand function that verifies *Assumption 5.* The demand curve is bounded from below by the surface $\left(\varepsilon_h \bar{w}/p_h\right)_{h=1..l}$. It is bounded from above by the budget constraint $\sum_h p_h \int_A \xi_{ah}\left(p, w_a\right) \mu\left(da\right) \leq \bar{w}$, thus by $\left(\bar{w}/p_h\right)_{h=1..l}$. The general idea is that if such inequalities hold for all vectors of positive prices, the demand functions are *"likely"* to be "similar to" Cobb-Douglas functions for "asymptotic"[20] values of the prices. With a fixed price, but with asymptotic values of the parameter α, the α-transform of the demand should behave in the same manner[21]; While "heterogeneity" increases, everything should happen as if more and more "Cobb-Douglas like" demand functions were added to the economy.

The precise meaning of these ideas is given in what follows while a formal proof is provided in the appendix. The following example clearly illustrates how this works.[22]

Consider the transformation T defined as follows:

$$e^{\alpha_1} = \rho \cos \theta$$
$$e^{\alpha_2} = \rho \sin \theta$$

where $\left(\alpha_1, \alpha_2\right) \in \Re^2$ and $\left(\rho, \theta\right) \in \Re_+ \times \left[0, \pi/2\right]$.

[20] more precisely $p_h \to 0^+$ and $p_h \to +\infty$.

[21] $\lim_{\alpha \to -\infty} p_h e^{\alpha_h} \to 0^+$, $\lim_{\alpha \to +\infty} p_h e^{\alpha_h} \to +\infty$

[22] I'm grateful to Kurt Hildenbrand who gave me this illuminating example.

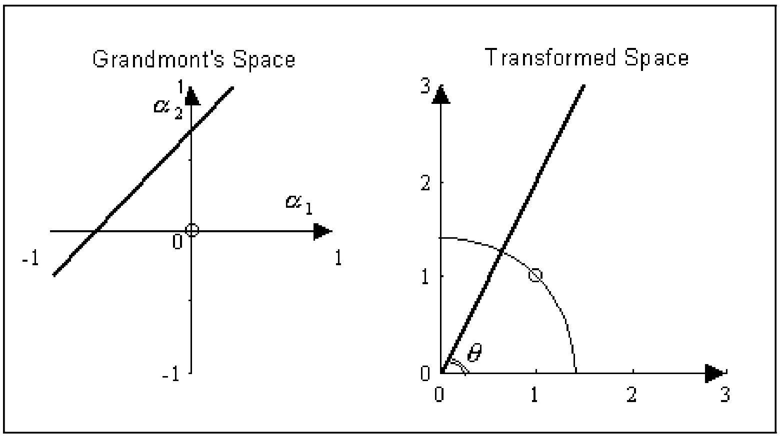

Figure 4.1: A Transformation in the Parameter Space.

The generating demand function $\xi_a(p, w)$ of the equivalent class a is represented by a circle O in both figures. This is to be found for $(\alpha_1, \alpha_2) = (0, 0)$ thus at the origin in Grandmont's space and for $(\rho, \theta) = (\sqrt{2}, \pi/4)$ or at the point $(1, 1)$ in the new space. The bold line represents all the points such that the ratio of the stretching coefficients $k(\theta) = e^{\alpha_2}/e^{\alpha_1} = \tan\theta$ is a constant.

The calculation of the demand can be obtained using following formulae:

$$\begin{aligned} X_a(p) &= \int_{\Re^2} \xi_a(\alpha, p) f(\alpha)\, d\alpha \\ &= 2\int_0^{\pi/2} \left[\int_0^{+\infty} \xi_a(\rho, \theta, p) f(\rho, \theta) \frac{d\rho}{\rho} \right] \frac{d\theta}{\sin\theta\cos\theta} \end{aligned}$$

In what follows, we will study the profile in the new parameter space of the density function

$$g(\rho, \theta) = \frac{2}{\rho \sin\theta \cos\theta} f(\rho, \theta)$$

in the case where heterogeneity increases in the sense of Grandmont, namely when $f(\alpha)$ is more and more "flat".

Define $I(\theta)$ as the weight given to the functions with a given "stretching ratio" $k(\theta) = e^{\alpha_2}/e^{\alpha_1}$:

$$I(\theta) = \frac{2}{\sin\theta\cos\theta} \left[\int_0^{+\infty} f(\rho, \theta) \frac{d\rho}{\rho} \right]$$

For the “standard” density function

$$f_N(\alpha) = \frac{\exp\left(-\frac{\alpha_1^2+\alpha_2^2}{N^2}\right)}{2\pi N^2}$$

one gets

$$\begin{aligned} m_N(i) &= \int_{\Re^2} \left| \frac{\partial f(\alpha)}{\partial \alpha_i} \right| d\alpha = \frac{1}{\sqrt{\pi} N} \\ I_N(\theta) &= \frac{\exp\left(-\frac{\ln^2(\tan\theta)}{2N^2}\right)}{\sqrt{2\pi} N \sin\theta \cos\theta} \end{aligned}$$

Consider now the weight $W_{\eta,N}$ given to the interval $A_\eta = \left[\eta, \frac{\pi}{2} - \eta\right]$:

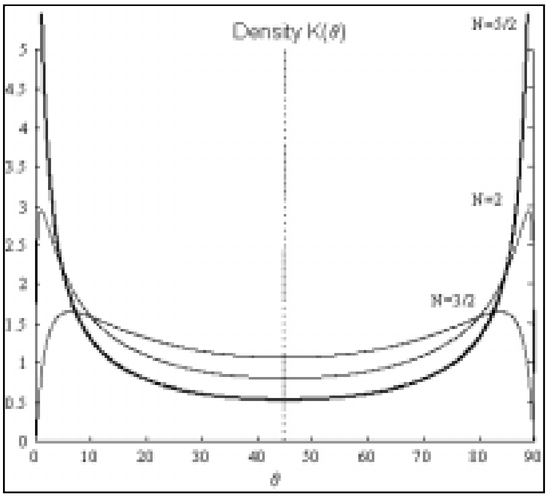

Figure 4.2: Density $I(\theta)$

$$W_{\eta,N} = \int_{\eta}^{\frac{\pi}{2}-\eta} \frac{2e^{-\frac{\ln^2(\tan\theta)}{2N^2}}}{\sqrt{2\pi} N \sin\theta \cos\theta} d\theta = \frac{1}{\sqrt{\pi}} \int_{\frac{\ln(\tan\eta)}{\sqrt{2}N}}^{\frac{\ln\left(\tan\left(\frac{\pi}{2}-\eta\right)\right)}{\sqrt{2}N}} e^{-v^2} dv$$

One can verify that $I_N(\theta)$ is a density function as $\lim_{\eta \to 0^+} W_{\eta,N} = 1$, any N. However, for any $\eta > 0$, $\lim_{N \to +\infty} W_{\eta,N} = 0$. As a result, when the density $f(\alpha)$ becomes more and more “flat” in the sense of Grandmont, the economy is almost only made up of functions $\xi_a(\alpha, p)$ such that θ

is "outside" A_η. This means that $e^{\alpha_1}/e^{\alpha_2}$ or $e^{\alpha_2}/e^{\alpha_1}$ is very low. This property, that can be generalised for any density function $f(\alpha)$ and in any dimension $l \geq 2$ leads to the following proposition:

Proposition 4.1. *For any $\eta > 0$, define the set of parameters A_η as follows:*

$$A_\eta = \{\alpha : |\alpha_i - \alpha_{j \neq i}| \leq \ln(1/\eta), \; all \; i\}.$$

For any smooth density $f(\alpha)$ over vectors α in $\Re^l$, let us define the density $f_\sigma(\alpha) = \frac{1}{\sigma} f\left(\frac{\alpha}{\sigma}\right)$. This density is associated with the "coefficient of flatness" $m(f_\sigma) = \sup_{k=1..l} m_k(f_\sigma)$ with

$$m_k(f_\sigma) = \int_{\Re^l} \left| \frac{1}{\sigma} \frac{\partial f}{\partial \alpha_k} \left(\frac{\alpha}{\sigma}\right) \right| d\alpha = \frac{1}{\sigma} \int_{\Re^l} \left| \frac{\partial f}{\partial \alpha_k}(\alpha) \right| d\alpha = \frac{1}{\sigma} m_k(f).$$

For any $\eta, \varepsilon > 0$ there exists a real $\sigma_{\eta,\varepsilon}$ such that for any $\sigma > \sigma_{\eta,\varepsilon}$, the density $f_\sigma(\alpha) = \frac{1}{\sigma} f\left(\frac{\alpha}{\sigma}\right)$ gives a weight to the set A_η that is lower than ε. Formally

$$\forall \varepsilon > 0, \; \exists \, \sigma_{\eta,\varepsilon} \in \Re_+^* \mid \forall \sigma > \sigma_{\eta,\varepsilon}, \quad \int_{A_\eta} f_\sigma(\alpha) \, d\alpha < \varepsilon.$$

As a result, when the "flatness" of $f(\alpha)$ increases, the whole weight is given to the functions $\xi_a(\alpha, p)$ such that $|\alpha_i - \alpha_{j \neq i}|$ goes to infinity (hence with a very large or an arbitrarily small "stretching ratio" $e^{\alpha_{j \neq i}}/e^{\alpha_i}$). Heuristically, when α_i goes to *minus* infinity, there is a stretching of the map $\xi_a(p, w)$ in the direction of p_i and the behaviour of the α-transformed function is the asymptotic behaviour of the generating function when p_i goes to zero. Conversely, when α_i goes to *plus* infinity, the map is compressed and the behaviour of the α-transformed function is the asymptotic behaviour of the generating function when p_i goes to infinity. In what follows, we will attempt to further specify this in order to outline the profile of these demand functions.

Consider the special case where the generating demand function is a C.E.S. The α-transformed function is given by the following formula

$$\xi_{a1}(\alpha, p) \;\; = \;\; \frac{a^\sigma e^{\alpha_1(1-\sigma)} p_1^{-\sigma} w}{a^\sigma e^{\alpha_1(1-\sigma)} p_1^{1-\sigma} + (1-a)^\sigma \, e^{\alpha_2(1-\sigma)} p_2^{1-\sigma}}$$

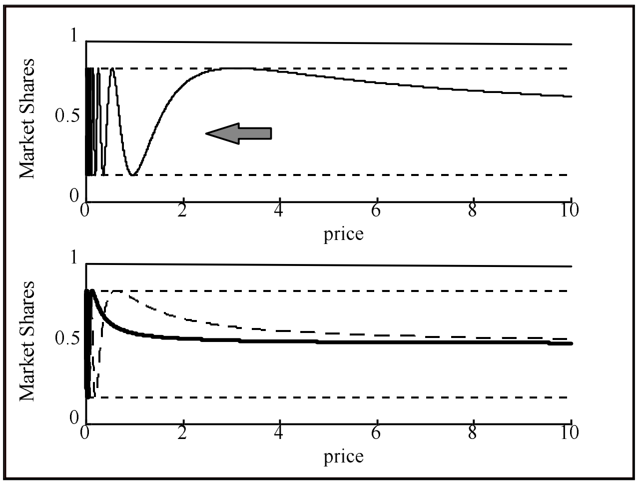

Figure 4.3: Impact of α when α_i goes to infinity

$$\xi_{a2}(\alpha, p) = \frac{(1-a)^{\sigma} e^{\alpha_2(1-\sigma)} p_2^{-\sigma} w}{a^{\sigma} e^{\alpha_1(1-\sigma)} p_1^{1-\sigma} + (1-a)^{\sigma} e^{\alpha_2(1-\sigma)} p_2^{1-\sigma}}$$

Thus when $|\alpha_2 - \alpha_1| \to +\infty$, the function $\xi_a(\alpha, p)$ is, in the limit, either $(\xi_{a1}, \xi_{a2})(p) = \left(\frac{w}{p_1}, 0\right)$ or $(\xi_{a1}, \xi_{a2})(p) = \left(0, \frac{w}{p_2}\right)$, namely two *degenerated* Cobb-Douglas. Remember that it was shown by Grandmont that these functions are the only ones that are invariant under α-transformation. Thus a Cobb-Douglas is like an atom in the economy generated by α-transformation. As the distribution of parameter α becomes "flatter", the number of "Cobb-Douglas like" demand functions increases and their behaviour dominates. This explains the nice behaviour of the aggregate.

More generally, two cases may arise. The expenditure functions $w_{ah} = p_h \xi_{ah}(p, w)$ are bounded from above by w and from below by zero. However the market shares do not always have a limit when p_h goes to infinity (and/or when p_h goes to zero). If there is a limit, like for the CES, the α-transformed function is reduced, in the limit, to a (possibly degenerated) Cobb-Douglas.

If the expenditure functions do not have a limit when the prices go to infinity (and/or when p_h goes to zero), then the market shares of the

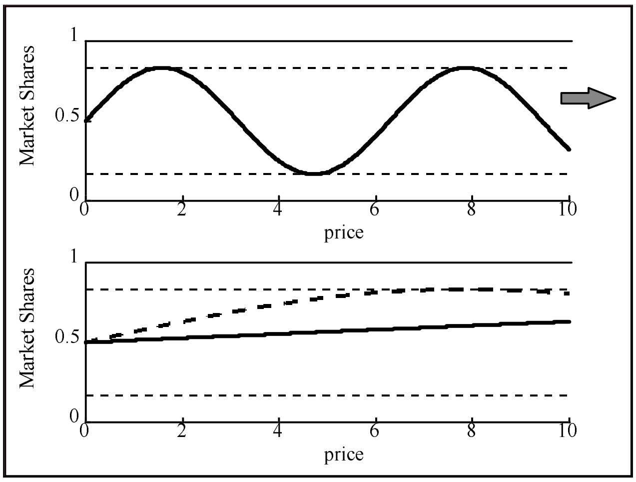

Figure 4.4: Impact of α when α_i goes to minus infinity

α-transformed functions *endlessly* "fluctuate". Formally

$$\begin{aligned} \exists\, \eta \;\; &> \;\; 0 \;\mid\; \forall \alpha, p \in \Re^l,\; \exists p' > p \in \Re^l \;\mid \\ & \qquad |w_{ah}(\alpha, p, w) - w_{ah}(\alpha, p, w)| > \eta \end{aligned}$$

Heuristically, these "fluctuating functions" are only likely to make a minimal contribution to the aggregate. Firstly, because the aggregation over the equivalent classes is likely to cancel out these variations and produce a smoother mean. Secondly, since the Cobb-Douglas are the only invariant of the α-transformation, the latter are more likely to impose their behaviour on the aggregate (provided there are some in the economy). Moreover, one might ask whether it is "reasonable" to consider a demand function such that the expenditures "fluctuate" when the prices go toward infinity (or toward zero).

As mentioned by Grandmont, *assumption 5* insures a non-vanishing aggregate demand, *i.e.* the presence of at least one non-degenerated Cobb-Douglas demand function in the limit. It is not quite clear whether this assumption could play another role in obtaining this result. One will observe that the diagonal dominance of the jacobian matrix of the demand can be derived from the small size of the coefficients of the expenditure elasticity matrix:

$$\frac{\partial \log \xi_h(\alpha, p)}{\partial \log p_k} + \delta_{hk} = \frac{p_k}{\xi_h(\alpha, p)} \frac{\partial \xi_h(\alpha, p)}{\partial p_k} + \frac{p_k}{p_h} \frac{\partial p_h}{\partial p_k} = \frac{p_k}{w_h} \frac{\partial w_h(\alpha, p)}{\partial p_k}$$

As a result, if $\xi_a(p)$ verifies *assumption 5,* the very fact that $\partial w_h / \partial p_k$ is "small" sufficies in order for any single function to be able to validate Grandmont's theorem. Nevertheless, it was not possible for us to derive general properties for the functions $\xi_h(\alpha, p)$: Even the additional assumption that $w_h(p)$ has a limit is not sufficient in order to be able to add further restrictions to the behaviour of its derivative $\partial w_h / \partial p_k$ hence for the theorem to be verified by every single function.

5. Conclusion

The transformation introduced by Grandmont is likely to over-represent some specific demand functions in the overall distribution. *Assumption 5* that guarantees aggregate budget shares to be uniformly bounded away from zero appears to be quite restrictive. However this assumption is necessary for Grandmont's theorem to hold: "Generic" heterogeneity does not suffice in order to guarantee nice aggregate properties. This suggests that it is not the "flatness" assumption *per se* that produces Grandmont's results but rather the conjunction of this assumption together with the specific form of the demand functions.
The economic interpretation of *Assumption 5* is that all commodities in the economy are "essential". In the present setup, its (technical) underlying consequence is the fact that, when the distribution of parameter α is "flat enough", there is a "large" number of demand functions in the overall population whose behaviour is "close to" the Cobb-Douglas demand. In the limit, and if "pathological" cases are excluded, the distribution of demand functions in the economy is reduced to a Cobb-Douglas. It is another matter, whether or not the underlying assumption ensuring this result is realistic.

In order to relax *Assumption 5,* that bears upon the (partial) result of aggregation, and to correct the bias induced by the parametrization, it would be in the logic of the distributional approach to investigate whether some properties can be derived from assumptions on both densities, over the set A of types and over the set of parameters α. This would require (i) a comprehensive topology to be introduced for the space of equivalence classes, and (ii) specific properties to be established for the corresponding demand functions. One should note that this is an attempt to restore regularity on the basis of a model which takes no account of the direct interaction between individuals. Distributional assumptions provide an alternative route to restoring "good" aggregate behaviour. Taking account of direct interaction may provide another. The problem with the distributional approach adopted here is that it is based on assumptions about characteristics of agents, which unlike income for example, cannot be observed. Thus the heterogeneity necessary to establish such results cannot be empirically tested. In any event, one has to reflect on the merits of such an approach. Clearly, one might be more confortable with a theory which can generate empirically testable propositions.

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7. Appendix

7.1. Violation of gross substitutability

The justification for reversing the order of the sign $\int$ and the linear operators $\sum$ and $\frac{\partial}{\partial p_i}$ is to be found in the dominated convergence theorem.

From Assumption 1,

$$X(a,p,w) = \int_{\Re^3} \xi_a(\alpha,p,w) f(\alpha)\, d\alpha$$

Thus

$$\begin{aligned} z_h(p) &= \sum_{a=\{-1,0,1\}} \int_{\Re^3} (\xi_{ah}(\alpha,p,p.r) - r_{ah}) f(\alpha)\, d\alpha \\ &= \int_{\Re^3} \sum_{a=\{-1,0,1\}} (\xi_{ah}(\alpha,p,p.r) - r_{ah}) f(\alpha)\, d\alpha \end{aligned}$$

The excess demand function for commodity 1 is:

$$\begin{aligned} z_1(p) &= \frac{1}{3}\int_{\Re^3} \left(\frac{p_1 e^{-\alpha_1}}{p_1 e^{-\alpha_1} + p_2 e^{-\alpha_2}} + \frac{p_3 e^{-\alpha_1}}{p_3 e^{-\alpha_3} + p_1 e^{-\alpha_1}} - 1 \right) f(\alpha)\, d\alpha \\ &= \frac{1}{3}\int_{\Re^3} \left(\frac{p_3 e^{-\alpha_1}}{p_3 e^{-\alpha_3} + p_1 e^{-\alpha_1}} - \frac{p_2 e^{-\alpha_2}}{p_1 e^{-\alpha_1} + p_2 e^{-\alpha_2}} \right) f(\alpha)\, d\alpha \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial z_1(p)}{\partial p_2} &= \frac{1}{3}\frac{\partial}{\partial p_2}\int_{\Re^3} \left(\frac{p_3 e^{-\alpha_1}}{p_3 e^{-\alpha_3} + p_1 e^{-\alpha_1}} - \frac{p_2 e^{-\alpha_2}}{p_1 e^{-\alpha_1} + p_2 e^{-\alpha_2}} \right) f(\alpha)\, d\alpha \\ &= \frac{1}{3}\int_{\Re^3} \frac{\partial}{\partial p_2} \left(\frac{p_3 e^{-\alpha_1}}{p_3 e^{-\alpha_3} + p_1 e^{-\alpha_1}} - \frac{p_2 e^{-\alpha_2}}{p_1 e^{-\alpha_1} + p_2 e^{-\alpha_2}} \right) f(\alpha)\, d\alpha \\ &= \frac{-1}{3}\int_{\Re^3} \frac{p_1 e^{-\alpha_1 - \alpha_2}}{(p_1 e^{-\alpha_1} + p_2 e^{-\alpha_2})^2} f(\alpha)\, d\alpha < 0 \end{aligned}$$

and gross substitutability is violated, any p.

7.2. With an infinity of types

The demand function for an individual belonging to the first category of types is:

$$
\begin{aligned}
\xi_{a1}(\alpha, p, w) &= \frac{bp_1^{-1/(1+a)} w}{bp_1^{a/(1+a)} + p_2^{a/(1+a)}} \\
\xi_{a2}(\alpha, p, w) &= \frac{p_2^{-1/(1+a)} w}{bp_1^{a/(1+a)} + p_2^{a/(1+a)}} \\
\xi_{a3}(\alpha, p, w) &= 0
\end{aligned}
$$

with $b > \frac{a+1}{a-1}$ and $a > 1$. The demand for the two other categories of types is obtained by permutating the role of the three commodities.
The α-transform of the first category of types is given by the formula:

$$
\begin{aligned}
\xi_{a1}(p, w) &= \frac{be^{-\alpha_1 a/(1+a)} p_1^{-1/(1+a)} w}{be^{-\alpha_1 a/(1+a)} p_1^{a/(1+a)} + e^{-\alpha_2 a/(1+a)} p_2^{a/(1+a)}} \\
\xi_{a2}(p, w) &= \frac{e^{-\alpha_2 a/(1+a)} p_2^{-1/(1+a)} w}{be^{-\alpha_1 a/(1+a)} p_1^{a/(1+a)} + e^{-\alpha_2 a/(1+a)} p_2^{a/(1+a)}} \\
\xi_{a3}(p, w) &= 0
\end{aligned}
$$

The demand for the first good over the three categories of types and given the parameter a is

$$
x_1(a, \alpha, p, w) = \frac{be^{\frac{-a\alpha_1}{1+a}} p_1^{\frac{a}{1+a}}}{be^{\frac{-a\alpha_1}{1+a}} p_1^{a/(1+a)} + e^{\frac{-a\alpha_2}{1+a}} p_2^{a/(1+a)}} + \frac{e^{\frac{-a\alpha_1}{1+a}} p_1^{-1/(1+a)} p_3}{be^{\frac{-a\alpha_3}{1+a}} p_3^{a/(1+a)} + e^{\frac{-a\alpha_1}{1+a}} p_1^{a/(1+a)}}
$$

The excess demand function for commodity 1 is:

$$
z_1(p) = \int_A \left(\int_{\Re^3} (x_1(a, \alpha, p, w) - 1) f(\alpha) \, d\alpha \right) \mu(a) \, da
$$

where

$$
\frac{\partial}{\partial p_2} \left((x_1(a, \alpha, p, w) - 1) \right)
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial p_2}\left(\frac{be^{\frac{-a\alpha_1}{1+a}}p_1^{\frac{a}{1+a}}}{be^{\frac{-a\alpha_1}{1+a}}p_1^{\frac{a}{1+a}}+e^{\frac{-a\alpha_2}{1+a}}p_2^{\frac{a}{1+a}}}+\frac{e^{\frac{\alpha_1}{1+a}}e^{-\alpha_3}p_1^{\frac{-1}{1+a}}p_3}{be^{\frac{-a\alpha_3}{1+a}}p_3^{\frac{a}{1+a}}+e^{\frac{-a\alpha_1}{1+a}}p_1^{\frac{a}{1+a}}}-1\right) \\
&= \frac{-\frac{a}{1+a}be^{\frac{-a\alpha_1}{1+a}}e^{\frac{-a\alpha_2}{1+a}}p_1^{\frac{a}{1+a}}p_2^{\frac{-1}{1+a}}}{\left(be^{-a\frac{\alpha_1}{1+a}}p_1+e^{-a\frac{\alpha_2}{1+a}}p_2\right)^2}
\end{aligned}
$$

Thus

$$
\frac{\partial z_1(p)}{\partial p_2} = \frac{\partial}{\partial p_2}\int_A\left(\int_{\Re^3} z_1(a,\alpha,p,w)\,f(\alpha)\,d\alpha\right)\mu(a)\,da < 0
$$

and gross substitutability is violated.

7.3. Density function in the transformed space

7.3.1. The example

The density function is:

$$
\begin{aligned}
f_N(\alpha) &= \frac{\exp\left(-\frac{\alpha_1^2+\alpha_2^2}{N^2}\right)}{2\pi N^2} \\
&= \frac{\exp\left(-\frac{1}{N^2}\left((\ln(\rho\cos\theta))^2+(\ln(\rho\sin\theta))^2\right)\right)}{2\pi N^2} \\
&= \frac{\exp\left(-\frac{1}{2N^2}\left(\ln^2(\rho^2\sin\theta\cos\theta)+\ln^2(\tan\theta)\right)\right)}{2\pi N^2} \\
&= f_N(\rho,\theta)
\end{aligned}
$$

The weight given to the functions with a “stretching ratio” $k(\theta)$ is:

$$
\begin{aligned}
I(\theta) &= \frac{1}{\sin\theta\cos\theta}\lim_{\substack{A\to0^+\\B\to+\infty}}\int_A^B\frac{2}{\rho}f_N(\rho,\theta)\,d\rho \\
&= \frac{1}{2\pi N^2}\frac{\exp\left(-\frac{\ln^2(\tan\theta)}{2N^2}\right)}{\sin\theta\cos\theta}\lim_{\substack{A\to0^+\\B\to+\infty}}\int_A^B\frac{2}{\rho}\exp\left(-\frac{\ln^2(\rho^2\sin\theta\cos\theta)}{2N^2}\right)d\rho
\end{aligned}
$$

$$= \frac{1}{\sqrt{2}\pi N} \frac{\exp\left(-\frac{\ln^2(\tan\theta)}{2N^2}\right)}{\sin\theta\cos\theta} \lim_{\substack{A\to 0^+ \\ B\to+\infty}} \int_{\ln(A^2\sin\theta\cos\theta)/\sqrt{2}N}^{\ln\left(B^2\sin\theta\cos\theta\right)/\sqrt{2}N} e^{-u^2} du$$

$$= \frac{\exp\left(-\frac{\ln^2(\tan\theta)}{2N^2}\right)}{\sqrt{2\pi}N\sin\theta\cos\theta}$$

By imposing $v = \frac{\ln(\tan\theta)}{\sqrt{2}N}$, one can verify that $I(\theta)$ is a density function on $[0, \pi/2]$:

$$\int_0^{\pi/2} \frac{2e^{-\frac{\ln^2(\tan\theta)}{2N^2}}}{\sqrt{2\pi}N\sin\theta\cos\theta} d\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v^2} dv = 1$$

However, the weight of the set $A_\eta = \left[\eta, \frac{\pi}{2} - \eta\right]$ is

$$W_{\eta,N} = \int_\eta^{\frac{\pi}{2}-\eta} \frac{2e^{-\frac{\ln^2(\tan\theta)}{2N^2}}}{\sqrt{2\pi}N\sin\theta\cos\theta} d\theta = \frac{1}{\sqrt{\pi}} \int_{\frac{\ln(\tan\eta)}{\sqrt{2}N}}^{\frac{\ln\left(\tan\left(\frac{\pi}{2}-\eta\right)\right)}{\sqrt{2}N}} e^{-v^2} dv$$

and

$$\lim_{N\to+\infty} W_{\eta,N} = \frac{1}{\sqrt{\pi}} \lim_{N\to+\infty} \int_{\frac{\ln(\tan\eta)}{\sqrt{2}N}}^{\frac{\ln\left(\tan\left(\frac{\pi}{2}-\eta\right)\right)}{\sqrt{2}N}} e^{-v^2} dv = 0$$

7.3.2. General Case

For any $\eta > 0$, define the set of parameters A_η as follows:

$$A_\eta = \{\alpha : |\alpha_i - \alpha_{j\neq i}| \leq \ln(1/\eta),\ all\ i\}$$

For any smooth density $f(\alpha)$ over vectors α in $\Re^l$, define the density $f_\sigma(\alpha) = \frac{1}{\sigma} f\left(\frac{\alpha}{\sigma}\right)$. This density is associated with the "coefficient of flatness" $m(f_\sigma) = \sup_{k=1..l} m_k(f_\sigma)$ with

$$m_k(f_\sigma) = \int_{\Re^l} \left| \frac{1}{\sigma} \frac{\partial f}{\partial \alpha_k}\left(\frac{\alpha}{\sigma}\right)\right| d\alpha = \frac{1}{\sigma} \int_{\Re^l} \left|\frac{\partial f}{\partial \alpha_k}(\alpha)\right| d\alpha = \frac{1}{\sigma} m_k(f)$$

In dimension one We will proof that $f(x) \leq m/2$:

$$\int_{\Re} \left| \frac{\partial f}{\partial \alpha}(\alpha) \right| d\alpha < +\infty \Rightarrow \forall \varepsilon > 0, \exists A \mid \int_{A}^{+\infty} \left| \frac{\partial f}{\partial \alpha_k}(\alpha) \right| d\alpha < \varepsilon$$

hence

$$\forall x, y \geq A, |f(y) - f(x)| \leq \int_{x}^{y} \left| \frac{\partial f}{\partial \alpha_k}(\alpha) \right| d\alpha \leq \int_{A}^{+\infty} \left| \frac{\partial f}{\partial \alpha_k}(\alpha) \right| d\alpha < \varepsilon.$$

and $f(\alpha)$ has a limit in $+\infty$. Because $\int_{\Re} f(\alpha)\, d\alpha < +\infty$, this limit is zero. *Idem* in $-\infty$.
Finally

$$\begin{aligned} m &= \int_{-\infty}^{+\infty} \left| \frac{\partial f}{\partial \alpha}(\alpha) \right| d\alpha = \int_{-\infty}^{x} \left| \frac{\partial f}{\partial \alpha}(\alpha) \right| d\alpha + \int_{x}^{+\infty} \left| \frac{\partial f}{\partial \alpha}(\alpha) \right| d\alpha \\ &\geq \left| \int_{-\infty}^{x} \frac{\partial f}{\partial \alpha}(\alpha)\, d\alpha \right| + \left| \int_{x}^{+\infty} \frac{\partial f}{\partial \alpha}(\alpha) \right| d\alpha = 2\,|f(x)| \end{aligned}$$

and the integral over a compact of a function that goes uniformly to zero goes itself to zero. Formally if $A_\eta = \{\alpha : |\alpha| \leq \ln(1/\eta)\}$

$$\forall \eta, \varepsilon > 0, \exists \sigma_{\eta,\varepsilon} \in \Re_{+}^{*} \mid \forall \sigma > \sigma_{\eta,\varepsilon}, \quad \int_{A_\eta} f_\sigma(\alpha)\, d\alpha \leq \frac{m}{2\sigma} < \varepsilon$$

In dimension $l > 1$ Without any loss of generality, assume $k \neq 1$. Define the transformation T_n^k that substitutes $\theta_k = \arctan\left(e^{\alpha_k}/e^{\alpha_1}\right)$ to α_k and let the other parameters unchanged. The Jacobi of the transformation is $(\sin\theta_k \cos\theta_k)^{-1}$. The computation of the demand is to be obtained according to the following formula:

$$\begin{aligned} X_a(p) &= \int_{\Re^l} \xi_a(\alpha, p)\, f(\alpha)\, d\alpha \\ &= \int_{\theta_k=0}^{\theta_k=\pi/2} \left[\int_{\alpha_{i\neq k} \in \Re} \xi_a(\alpha_{i\neq k}, \theta_k, p)\, f(\alpha_{i\neq k}, \theta_k, p)\, d\alpha_{i\neq k} \right] \frac{d\theta_k}{\sin\theta_k \cos\theta_k} \end{aligned}$$

In what follows, we will study the profile of the density function

$$I(\theta_k) = \frac{1}{\sin\theta_k \cos\theta_k} \left[\int_{\alpha_{i\neq k} \in \Re} f(\alpha_{i\neq k}, \theta_k, p)\, d\alpha_{i\neq k} \right]$$

By assumption,

$$\int_0^{\frac{\pi}{2}} I(\theta_k)\, d\theta_k = 1$$

However

$$\begin{aligned} f(\alpha) &= |f(\alpha)| = \left| \int_0^{\alpha_k} \frac{\partial f}{\partial \alpha_k}(\alpha_1, ..., \alpha_{k-1}, u_k, \alpha_{k+1}, ..\alpha_l)\, du_k \right| \\ &\leq \int_0^{\alpha_k} \left| \frac{\partial f}{\partial \alpha_k}(\alpha_1, ..., \alpha_{k-1}, u_k, \alpha_{k+1}, ..\alpha_l) \right| du_k \end{aligned}$$

Thus

$$\begin{aligned} \int_{\alpha_{i \neq k} \in \Re} f(\alpha)\, d\alpha_i &\leq \int_{\alpha_{i \neq k} \in \Re} \int_0^{\alpha_k} \left| \frac{\partial f}{\partial \alpha_k}(\alpha_1, ..., \alpha_{k-1}, u_k, \alpha_{k+1}, ..\alpha_l) \right| du_k d\alpha_i \\ &\leq \int_{\Re^l} \left| \frac{\partial f}{\partial \alpha_k}(\alpha) \right| d\alpha = m_k < +\infty \end{aligned}$$

and finally

$$\begin{aligned} \int_{A_\eta} f_\sigma(\alpha)\, d\alpha &\leq \int_{\arctan \eta}^{\arctan(1/\eta)} \left[\int_{\Re^{l-1}} f_\sigma(\alpha)\, d\alpha \right] \frac{d\theta_k}{\sin\theta_k \cos\theta_k} \\ &\leq \frac{m_k}{\sigma} \int_{\arctan \eta}^{\arctan(1/\eta)} \frac{d\theta_k}{\sin\theta_k \cos\theta_k} \\ &\leq \frac{m_k}{\sigma} \ln\left(1/\eta^2\right) \end{aligned}$$

For any $\eta, \varepsilon > 0$ there exist a real $\sigma_{\eta,\varepsilon}$ such that for any $\sigma > \sigma_{\eta,\varepsilon}$, the density $f_\sigma(\alpha) = \frac{1}{\sigma} f\left(\frac{\alpha}{\sigma}\right)$ gives a weight to the set A_η that is lower than ε. Formally

$$\forall \varepsilon > 0, \exists \sigma_{\eta,\varepsilon} \in \Re_+^* \mid \forall \sigma > \sigma_{\eta,\varepsilon}, \quad \int_{A_\eta} f_\sigma(\alpha)\, d\alpha < \varepsilon$$