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Abstract

We model the formation of friendships as repeated cooperation within a set of heterogeneous players. The model builds around three of the most important facts about friendship: friends help each other, there is reciprocity in the relationship and people usually have few friends. In our results we explain how similarity between people affects the friendship selection. We also characterize when the friendship network won't depend on the random process by which people meet each other. Finally, we explore how players' patience influences the length of their friendship relations. Our results match and explain empirical evidence reported in social studies on friendship. For instance, our model explains why troublesome subjects have few friends.

JEL Classification: C72, C73, Z13.

Keywords: Friendship, cooperative game, grim trigger strategy, social networks.

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1 Introduction

Social relationships represent one of the most basic needs of human beings. They arise quickly between subjects in any kind of environment and they condition the behavior of the subjects involved. Different degrees of social relationships can exists between individuals: family members, work mates, partners, friends, etc. Among all of them, friendship relations represent one of the most intriguing aspects of social relationships. While every person can identify his friends if asked, it is difficult to find a proper definition for what friendship means.

The most commonly mentioned characteristics of friendship relations are: helping, reciprocity and limited number of friends[1]. Mutual help in a friendship relation implies that friends help each other in case of necessity. The exchange or reciprocity means that a player expects from her friends a similar attitude to the one that he takes towards them. Finally, limited number of friends simply means that subjects don't have as many friends as the want since keeping the friendship relations takes time and effort.

The present paper presents a model based on these three facts: helping, reciprocity and limited number of friends, tries to reproduce some of the main characteristics of friendship. The interactions between a group of players are model in the following repeated setting: each period every player has to decide whether to perform an activity with each of the other players in the population (one activity per pair of people). We might think of this activity as going to the cinema, having a trip, doing sports together, etc. Each player is characterized by an exogenous degree of needing help (or having a problem) with each of the activities. Call this problem, for example, needing money, having an accident, being sad, etc. After both players decide to do the activity together, they have to decide simultaneously whether to help other or not. Helping involves a cost for the player who provides help but also a benefit for the person receiving help. In game-theoretical terms we model the activity that players can do together as a cooperation game of a class of prisoners' dilemma game. We define such game as the Helping Game. Each player will be able to provide help a limited number of times per time period. The degree to which a player needs help is exogenous, common knowledge and heterogeneous among the players. If two players are performing the activity and helping each other (playing the cooperative equilibrium) they are called friends. If they are doing the activity but not helping each other, they are called mates. Finally, if they are not performing the activity they will be called strangers.

As mentioned above, our aim is to construct a model that based on helping, reciprocity and limited number of friends, is able to explain some of the phenomena that we observe in the real-world friendship relations. From the preceding paragraph it is clear how we make use of the helping and the limited number of friendship. For implementing reciprocity, the strategies that we use for supporting cooperation (providing help) will be Grim Trigger. According to Grim Trigger strategies, a player will keep on providing help to another player as long as this other player is also providing help to her. Because of the fact that Grim Trigger strategies don't allow

[1]See, for example, Hruschka and Henrich (2004), Silk (2002), Hallinan (1979), de Vos and Zeggelink (1997), van de Bunt, van de Duijn and Snijders (1999) or Zeggelink (1995).

for forgiveness, in section 5.1 we check for the robustness of the results when instead players use Tit-for-Tat for supporting cooperation.

In our three most important results we explore the three following issues: role of similarity in friendship relations, uniqueness of equilibrium of the friendship network and length of the friendship relations. First, we manage to explain the role of similarity in friendship relations[2]. It has been reported in empirical studies that similar people (same hobbies, race, etc.) are more likely to have friendship relation although strong friendship relations between very different people can exist[3]. Section 3 suggests a solution to this fact. According to the model, three types of relationship between two people may exists: stranger, mates and friends. When two people are mates this means that they are having a relationship but their relationship is not strong enough to consider them friends. We find that similarity matters only if a 'mate' relationship between two people is possible. On the other hand, if a 'mates' relationship relation between two people is not possible, as in love-hate relationships, then similarity will play no role in determining if these two people can become friends.

Second, we show that in our model it is in general impossible to accurately predict the friendship relations that will prevail within a group of people in the long run. In particular we show that the equilibrium will depend at the order in which people meet each other. This order is model as a random process. In game-theoretical terms, the equilibrium is history dependent and the history follows a random process. What is interesting about the model we present is that we give two precise explanations to why the equilibrium may be history dependent. First,if people belonging to the group are not different enough in terms of their degree of needing help from each other, then a certain degree of substitutability between people will exist. In this case, the random process by which people meet each other will play a role in the final outcome of the process. Second, if it doesn't exist some kind of social rule by which agents punish those agents who 'betray' their friends, then the random process by which people meet each other will again play a crucial role in determining the final outcome.

Finally, it has been shown in many empirical studies that the length of the friendship relations increases with age[4]. That is, young people tend to have shorter relationships than old people. Our model manages to explain this fact in terms of patience. Comparing two groups of people, we expect to find that the friendship relations last longer in the group where agents are more patient.

Many sociological, physiological and anthropological papers have modeled the process of friendship formation. For example, in paper by Zeggelink (1995), friends have a fixed desired number of friends and each player is defined by a dichotomous variable (they are either type-1 player or type-2 player). Each player tries to have the desired number of friends and to maximize the similarity in his type with the type of his friends. The author performs simulations and

[2]Similarity in the friendship contexts is often refereed to as homophily.

[3]For example, Marmaros and Sacerdote (2004) found, using the volume of emails exchanged between students from Dartmouth college, that similarity in age, geographic closeness, race and interests increase the likelihood for two people to become friends.

[4]See, for example, Hallinan (1978).

finds that the players tend to group with the others of the same type. The taste for similarity is exogenously imposed whereas we make no assumption on this respect. In this respect, Hruschka and Henrich (2004) developed a model in which in each period players can choose with whom they want to play a prisoners' dilemma game. The model is focused on the evolutionary biological point of view of the cooperative relations. That is, they focus on the differences between the survival rates of cooperative players and selfish players.

The model presented is different also from the economic models of social networks pioneered by Jackson and Wolinsky (1995) and Bala and Goyal (2000). It differs from the first one in that in our model the payoff of the players is not determined uniquely by the state of the friendship network but also by the actions of the players against those with whom they don't share a friendship relation. The model presented, on the other hand, differs from Bala and Goyal (2000) in that when two players share a link, they then play a cooperative game and not a coordination game. To our knowledge, two papers examine the issue of social networks when players play a cooperative game. These are Lippert and Spagnolo (2005) and Vega-Redondo (2005). The first one focuses on the information transmission about the defectors in the network and on the different punishment mechanism for supporting cooperation. On the other hand, Vega-Redondo (2005) explores the amount of cooperation that will emerge in the network when the environment suffers from aggregate shocks to payoffs.

The rest of the paper is organized a follows. In Section 2 we develop the model. Section 3 explores the simplest case in which the population consists of only two players. Section 4 extends the model for more than two players. In Section 5 we discuss the robustness of the results and the assumptions as well as present some extensions. Finally, Section 6 concludes.

2 The Model

2.1 Informal Discussion

Time is discreet and denoted by $t = 1, 2, \ldots$. Each period a player, say i, is selected by nature. This player can make 'phone calls' to the players with whom she intends to join a relationship. There are three types of relationships, friends, mates and strangers, explained in more detail below. When player i calls player j, then players i and j decide simultaneously and non-cooperatively whether to join such a mutual relationship or not. Relationships carry a benefit to both players but also involve additional cooperation. In any relationship, each party needs some help were the degree of help needed differs among players. Part of the relationship is an observable decision of whether or not to cooperate in the sense of providing help. So when two players have decided to join a mutual relationship they then non-cooperatively simultaneously decide whether or not to help the other. When both decide to help the other then we speak of a friends, otherwise we speak of mates. If the relationship does not even arise because at least one of the two parties does not want to join then we speak of strangers.

The maximum number of friends that a player can have is limited to $m > 0$. This constraint reflects the fact that providing help is costly in terms of time and, hence, a player can only provide help a limited number of times per period.

We limit the set of possible strategies of each player as follows. Only in a period in which a player makes or receives phone calls she can change her plan of action. Otherwise, she plays as she decided at the time of the last phone call. Two types of plans of actions or strategies are only considered, the cooperative and the defective.

In the cooperative the player acts as in Grim Trigger. A friendship is suggested which means that first a relationship is suggested and then if the other agrees the player suggests to help the other when in need of help. The Grim Trigger plan also specifies what to do if the other does not want to be friends or even to be mates: If a friendship does not arise then the player chooses what ever is best for herself in the one shot-game. Depending on the payoffs, this can be to not accept any relationship or to suggest to form a relationship. Finally, the Grim Trigger plan protects for later defections and proceeds as in the case when a friendship does not even arise in the first place.

In the defective strategy the player rejects or breaks the friendship relation. If both players were having a friendship relation last period then she breaks it. The player does so by not providing help to the other player but still getting the benefits from the other player helping her. In case both players were not having a friendship relation, she rejects a possible friendship relation and plays what ever is best for her in the one-shot game.

2.2 Formal Presentation

Denote time intervals by $t = 1, 2, \ldots$. Assume a population $\mathcal{N}$ of n players. Every period each player faces a one shot game with each other. Denote by l the action for link (or suggest a relationship) and n the action for not link. Further, denote by H the action of providing help and by N the action of not providing help. We call Relationship Game to the one-shot game that all players $i \in \mathcal{N}$ and $k \in \mathcal{N}$ face on every period. The Relationship Game is shown in Figure 1.

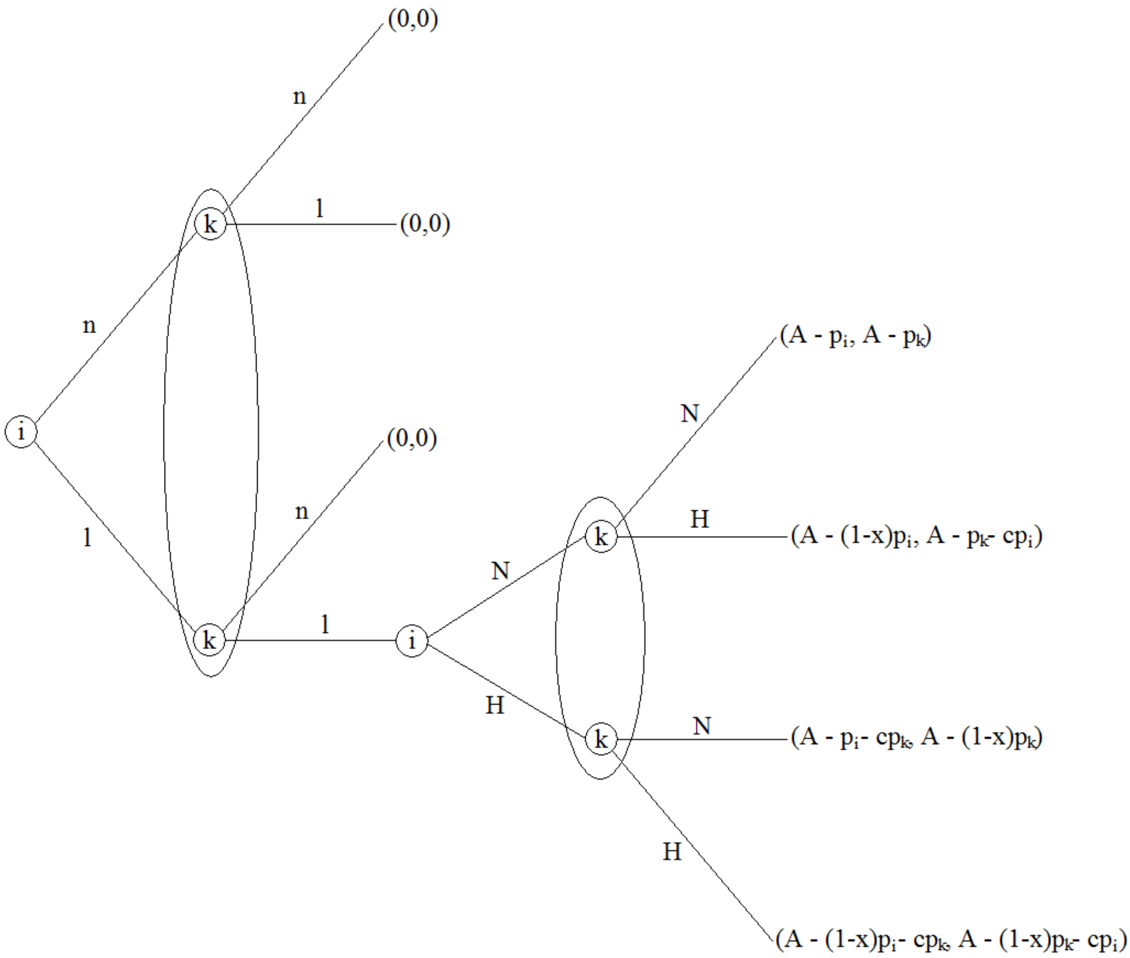
The payoff scheme works as follows. If one of the players decides not to link with the other, then they both get 0 payoff. If both players decide to link with each other their payoff is given in Table 1 (note that Table 1 is simply the normal-form representation of the sub-game that starts after both player played l).

Table 1: Helping Game

	H	N
H	$A-(1-x)p_i-cp_k, A-(1-x)p_k-cp_i$	$A-p_i-cp_k, A-(1-x)p_k$
N	$A-(1-x)p_i, A-p_k-cp_i$	$A-p_i, A-p_k$

The interpretation of the payoffs is as follows. If both players are not helping each other, then player i gets $A - p_i$ and player k gets $A - p_k$. Hence, they get a fix amount $A \in (0, 1)$

Figure 1: Relationship Game



minus the degree to which they need help. If player i is helping player k but not the other way around, then player i gets $A - p_i - cp_k$ and player k gets $A - p_k + xp_k$. That is, player i has to pay the cost cp_k with $c \in (0, 1)$ for helping player k and player k receives a benefit xp_k with $x \in (0, 1)$ because of being helped by player i. The payoffs in the other situations follows the same logic just described. We assume $A \geq 1 - x$ so that being helped without providing help is always weakly preferred to not being linked

The interpretation of the actions and payoffs of the Relationship Game should be clear with the following simple example.

Example 1. *Two players, called i and j, have to agree on going to the cinema together. If they don't agree they will then stay home and get 0 payoff. Otherwise, each of the two players enjoys going to the cinema and amount A. However, there are issues that keep the players away from enjoying fully the activity they are performing. For example, it may happen that the movie is in a language that player i doesn't understand and that the topic of the movie is not j's favorite one. This will make the joy of both players smaller than A. Nevertheless, each player has the opportunity of helping the other: j can explain what is going on in the movie to i and i can give* *j insights about why the topic of the movie might be considered interesting. Providing help is costly in terms of effort and time, but it makes the player enjoy the movie much more.*

Another way of interpreting p_j is as the probability of needing help. The following example illustrates the Helping Game when p_j is interpreted as the probability of needing help.

Example 2. *Two risk-neutral players, called i and j, are to decide whether to go fishing or to stay home. If they agree on going fishing together, each of them has a probability p_j of not catching any fish. The payoff of each of them is A if they catch at least one fish and $A - 1$ otherwise. When fishing, each player can offer help to the other: say, i can provide j with better worms and j can teach to i some sophisticated fishing techniques. Obviously, providing help increases the chances of the other player catching a fish. On the other hand, providing help decreases own's chances because of the resources (time, etc.) one has to spent.*

The following proposition characterizes the Nash equilibria of the Relationship Game between players i and k. Whenever we write $((l, H), (l, N))$ this means that player i plays (l, H) and player k plays (l, N).

Proposition 1. *In the Relationship Game for any players $i, k \in N$:*

- *Nash equilibria: For each $p_i, p_k \in (0, 1)$, $((n, \{H, N\}), (n, \{H, N\}))$ are Nash equilibria. If $p_i, p_k \leq A$ then $((l, N), (l, N))$ is also a Nash equilibrium.*
- *Sub-game-perfect Nash equilibria: For each $p_i, p_j \in (0, 1)$, $((n, N), (n, N))$ is a sub-game perfect Nash equilibrium. If $p_i, p_k \leq A$, then $((l, N), (l, N))$ is also a sub-game perfect Nash equilibrium.*

As mentioned above, if two players are playing (l, H) repeatedly against each other, we define them as friends. If they are playing (l, N) repeatedly against each other, they are mates. If two player play (n, N) repeatedly against each other, we say they are strangers.

We say that a player i *betrays* another player j if they were friends in the last period but in the current period i, still having a linked with j, doesn't provide help to her. That is, they both played (l, H) against each other in the past round but i switches to play (l, N). In the example above we have that player 4 betrayed player 1 and that player 1 betrays player 3.

As it is well known by the Folk theorem in repeated games, infinitely many strategies can form Nash equilibria. Hence, we shall restrict the strategy space of the agents to make the model tractable. In our model as already mentioned, players are only able to have two types of plans, the Cooperative Plan and the Defective Plan.

Cooperative Play according to Grim Trigger (defined below).

Defective Play (l, N) if you and the other player played (l, H) in the last round, play your weakly dominant strategy in the Relationship Game otherwise.

As it can be infer from the Relationship Game, whenever we write play your weakly dominant strategy it implies *play (n, N) if your degree of needing help is smaller than A, play (l, N) otherwise.*

Definition 1. *Define the Grim Trigger strategy for player* $i \in \mathcal{N}$ *played against any player* $k \in \mathcal{N}$ *as follows:*

- *If a play in any past period against k was either* $((l, H), (l, N))$ *or* $((l, N), (l, H))$*, then play your weakly dominant strategy.*
- *Otherwise, play* (l, H)*.*

Note that by they way we define the Grim Trigger strategies players are protected against possible deviations from the other player when both are playing (l, H). An alternative way of defining the Grim Trigger strategy will be to make the players to play (l, N) in case the other player did not play (l, H) in the past period. This way of defining the Grim Trigger will make helping more sustainable since the costs of deviating from (l, H) are higher. However, this way of defining the Grim Trigger will imply that a player could possibly be playing a strictly dominated action, which is something we don't want to have here. While making this suggested change in the strategies will change part of our results for the 2 player game, our results for the n Player game will be qualitatively the same.

In section 5.1 we check for the robustness of the results when instead of Grim Trigger players are allowed to use the Tit-for-Tat.

We constrain the agents to provide help at most $m \in \{1, ..., n-1\}$ times per period and, hence, each player can have at most m friends in a given period.

When each player is to decide with whom she can set up a friendship relation, she will do so in a pairwise fashion. This means that, if i is to decide whether she can set up a friendship relation with k, i will take this decision as if there were no more players in the population. That is, as if $\mathcal{N} = \{i, k\}$. However, i will still take into account the upper-bound m. Hence, if $m = 4$ and i has already 4 friends, she will take into account that before setting up a friendship relation with k she must break one of her already existing friendship relations. On the other hand, if $m = 4$ and i has 3 friends, i's decision of whether setting up a friendship relation with k will be taken as if $\mathcal{N} = \{i, k\}$.

We refer to a friendship relation between i and k as pairwise sustainable if the friendship relation is possible when $\mathcal{N} = \{i, k\}$. Thus, Proposition 2 in the next section, were we consider the two player case, is telling us which friendship relations may exists in equilibrium.

Players are allowed to revise (or update) their strategies in the following way. Each period a player, say i, is selected by nature. This player can make "phone calls" to the players with whom she wants to play the Cooperative Plan. Player i then updates her strategies as follows. She plays the Cooperative Plan with whom she calls and plays the Defection Plan with the rest. The players who get a call from i can update only the plan or strategy they are playing against i. However, if a player gets a call but is already providing help to m players, then she can switch to play the Defective Plan against one of her friends in order to be able to play the Cooperative Plan with the player that called her. The rest of players do not update their strategies in any manner[5].

3 Two-Player Game

Assume that the population $\mathcal{N}$ consists of only two players i and k.

Proposition 2. *A friendship relation between players i and k can be supported in the repeated Relationship Game when both players use Cooperative Plan if and only if the following holds:*

- *if $p_i, p_k \leq A$ then $\frac{c}{\delta x} \leq \frac{p_i}{p_k} \leq \frac{\delta x}{c}$,*
- *if $p_i, p_k \leq \frac{A}{1-x}$ but either $A < p_i$ or $A < p_k$, then $A - p_i + xp_i - \frac{c}{\delta}p_k \geq 0$ and $A - p_k + xp_k - \frac{c}{\delta}p_i \geq 0$.*

Proof. See Appendix A.1. □

From Proposition 2 we conclude what follows. First, if both p_i and p_k are smaller than a A, then the friendship's relation can only be supported if the relative difference between respective probabilities of needing help is sufficiently. A player with low probability of needing help (less than A) won't accept a friendship relations with a player whose probability of needing help, although being also smaller than A, is very different from his.

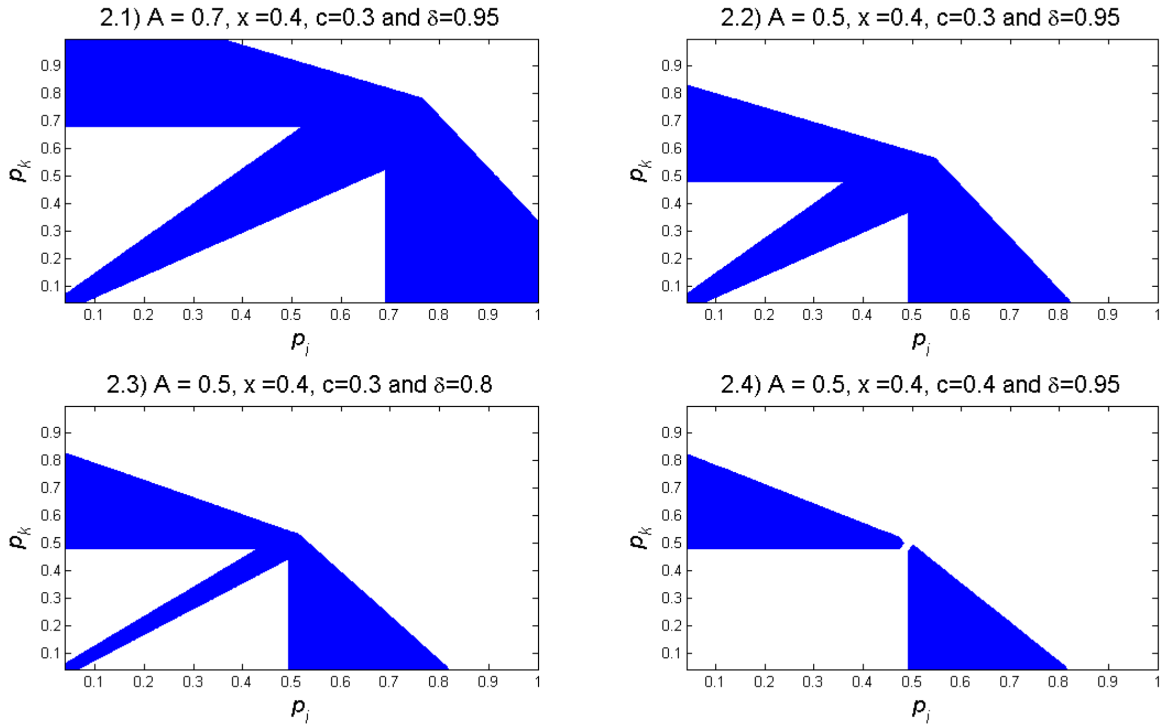
Second if at least one of the players needs help with a degree higher than A and both players' need of help is below $\frac{A}{1-x}$, then she no longer cares about the relative difference in probabilities of needing help but about their absolute values. In this case, as long as the inequalities $A - p_i + xp_i + \frac{c}{\delta}p_k \geq 0$ and $A - p_k + xp_k + \frac{c}{\delta}p_i > 0$ are satisfied, the friendship's relation can be supported. The relevant implication of this case is that player i cares now only about the balance between costs and benefits of the relationship instead of, as in the previous case, being similar to the other player.

To have a better understanding of implications of Proposition 1 we present Figure 2. It plots when, for a given value of the parameters A, x, c and δ a friendship relation is possible between players i and k. So if the coordinate (p_i, p_k) is shaded it is because for the given parameters a player whose probability of needing help is p_i can be a friend of a player whose probability of needing help is p_k and vice versa.

The common interesting feature of these graphs is the existence of a jump between the area when both p's are smaller than A and the area when one of the p's is higher than A. The intuition behind this result is that a player with a small degree of needing help will not want to be linked with a player with a much smaller degree of needing help. This may happen because he is afraid that this player may betray her (i.e. deviate from playing the Cooperative Plan) since this person may prefer her only has a mate. On the other hand, if their p's are close enough or the other player's degree of needing help is high, then he will be willing to have the friendship

[5]For a more formal definition of the dynamics the reader is referred to Appendix A2.

Figure 2: Friendship Relations



relation with that other person because she knows that: (1) she needs the other player as much as the other player needs her and since both are getting positive profits from the relation no one will have incentives to terminate it and, (2) if they are not having a relationship they won't be even mates and since they are both getting positive profits from the friendship relation the betrayal will make them not to be linked anymore. In other words, the loss in case of betraying the other is too high (they won't be even mates) in this second case.

Figure 2.4 (when $c > x$) merits a special attention. It shows that no friendship relation between players with low probability of needing help will arise. Since having a friendship relation is not very profitable in terms of x and c, the low-probability players will only like each other as a mate and not as a friend. This happens because the likelihood of betrayal is too high. For the same reason, the relationship between low probability players and moderate probability players will be possible. Players with a moderate probability of needing help won't want to have mates, only friends or strangers. Therefore, in this latter case, the cost of betrayal is very high. This makes the relationship more likely to be supported.

Proposition 2 states that the relevance of similarity for friendship selection differs dependently on the type of the players. For some pairs of players the similarity with their friends will matter and for some other players similarity will be irrelevant. The thing that will matter in this later case will be the balance between costs and benefits from the relationship.

One more thing is worth underlying. A player with a very low p may be "marginated" among the players with low p because needing "too little" help. Summing up the results of Proposition

2:

- If both p_i and p_k are small, the relevant thing for a player i is the relative difference between p_i and p_k.
- If either p_i or p_k is not small, the relevant thing a player i is the absolute value of p_i and p_k.

4 n-Player Game

For a clearer understanding of the dynamics of the model work when the population consists of more than two players, we present example 3. The example is drawn in Figure 3, were each node represents a player and a line between to players represents the fact that those two players are friends.

Example 3. *The example is conducted for* $N = \{1, 2, 3\}$, $p_1 = 0.4$, $p_2 = 0.45$, $p_3 = 0.55$, $A = 0.5$, $x = 0.6$, $c = 0.3$, $\delta = 0.7$ *and* $m = 1$. *In this setting all the possible friendship relations are pairwise sustainable. To check this one only has to apply the result in Proposition 2 to the present example (this and all the other computations are left to Appendix A.3).*

In the first period, player 1 *is selected by nature to make calls. Since* $m = 1$, *player* 1 *can only provide help to at most one player. Hence, because* $p_2 < p_3$, *player* 1 *prefers to have a friendship relation with* 2 *than with* 3. *Therefore, she will call player* 2 *and both of them will switch to play Grim Trigger against each other.*

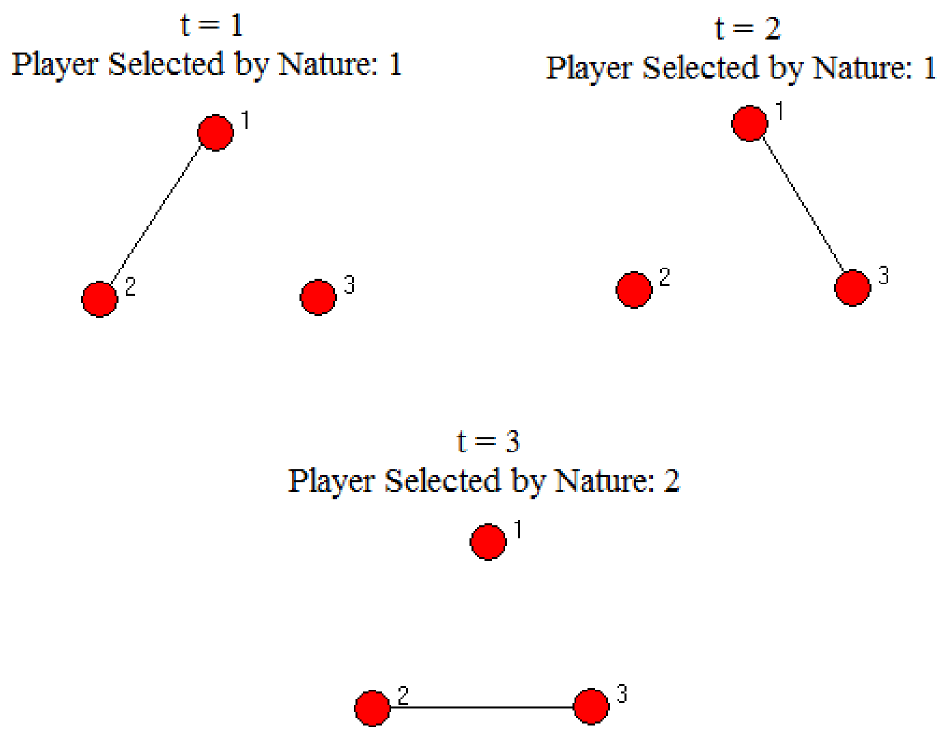
In the second period, player 1 *is again the one allowed to make calls. Since this time she has one friend, she makes different considerations. Because of the fact that* $m = 1$, *she now wonders if betraying* 2 *and setting up a relationship with* 3 *is better than keeping the friendship relation with* 2. *Betraying player* 2 *is profitable for player* 1 *because providing help is costly. Hence, it may happen that the lower payoff associated with a friendship relation with player* 3 *is compensated by the one-period gains from betraying player* 2. *This is exactly the case in this particular example. Therefore, player* 1 *will call player* 3 *and they will switch to play Grim Trigger against each other. Moreover, player* 1 *will play* (l, N) *against player* 2. *Note that player* 2 *didn't foresee player* 1*'s betrayal since all players consider each relationship pairwisely.*

In period 3, *nature selects player* 2. *Since player* 1 *betrayed player* 2, *the friendship relation between them is no longer possible. This is due to the unforgiveness property of the Cooperative Plan. Hence, player* 2 *will call player* 3. *Player* 3 *is in a similar situation than the one faced by player* 1 *in the second period. In this particular example, player* 3 , *as did player* 1, *finds the betrayal profitable. Hence, player* 3 *betrays player* 1 *to set up a friendship relation with player* 2.

An equilibrium has been reached. Since player 1 *betrayed player* 2 *and player* 3 *betrayed player* 1, *no new friendship relations can arise in the network.*

As we can already see, the nature (or chance) plays an important role in determining which friendship relations can arise. If the players selected by nature were 2, 2 and 3 in this order, the equilibrium would have had players 1 and 3 as the only friends. This result together with other important ones are presented in the next subsection.

Figure 3: Simulation



In the example above, the equilibrium in which players 2 and 3 are friends results only if 1 betrayed 2 (or vice versa) and 3 betrayed 1 (or vice versa). On the other hand, the equilibrium in which 1 and 3 are friends is possible only if 1 betrayed 2 (or vice versa) and 2 betrayed 3 (or vice versa). Therefore, the equilibrium in this case is history dependent.

The key to this result is that, if players 2 and 3 from the example above are similar enough, player 1 is not loosing much by having a relationship with player 3 instead of with player 2. This loss can be compensated by the one-shot profits from betraying 2 today. Hence, the existence of a certain degree of substitutability between friends creates history dependence in the equilibria. This fact will be exploited in Proposition 4.

The resulting equilibrium will depend on the order at which players are selected by nature. In a way we can think of the order at which players are selected by nature as of the order at which players in the population meet each other. The player selected by nature is the one who can have the initiative to meet new people via making phone calls.

The fact that the order at which people meet each other affect the long run friendship relations was reported in an empirical study by Cloninger (1986). Cloninger found that meeting new people may result in breaking old and strong friendship relations because of the novelty of having new friends. In their forthcoming article, Whitmeyer and Yeingst refer to this characteristic of the friendship relations as *fickleness.*

Proposition 3. *The system converges with probability* 1 *to an equilibrium network architecture that can be history dependent.*

Proof. See Appendix A.1. □

Proposition 4 shows that convergence to equilibrium is always guaranteed although the equilibrium can depend on the order at which players are selected by nature. This fact represents an important feature of friendship relations. The friendship relations that emerge in the real world are the result of a complex process of interactions between individuals in which unpredictable events may play a crucial role in the final outcome. The order of meeting people has important effects on one's long run relationships. However, as we shall see below, there are some situations in which the convergence of the process to a uniquely determined equilibrium is guaranteed. These situations are: 1) when the players in the population are different enough from each other (Proposition 5) and, 2) when there exists some type of social rule by which betrayers are punished (Proposition 9).

Proposition 4. *If* $A > c\delta$ *then there exists a* $\varepsilon > 0$ *such that if* $\min_{i,k\in\mathcal{N},i\neq k} |p_i - p_k| > \varepsilon$*, then the friendship network converges with probability* 1 *to a unique network architecture.*

Proof. See Appendix A.1. □

Therefore, when players in the population are different enough, the process will converge to a unique equilibrium. In other words, the process has only one equilibrium that is not history dependent and the process will always converge to it. As mentioned before, there exists a certain degree of substitutability between friends. Hence, if players are different enough, no player will want to betray a friend to set up a relationship with a higher-degree-of-needing-help player. Once the substitutability between friends is eliminated, we can successfully predict the long term friendship relations that will arise within the population. Note that conditions in Proposition 4 do not rule out the case where players with $p < A$ can have friendship relations between each other. Hence, even when players are different enough so that the friendship network converges to a unique equilibrium, similarity may play a role.

Another interesting feature of the model is that, when subjects are more patient, the length of the friendship relations will tend to be longer. This is formally stated in the following proposition.

Proposition 5. *Ceteris paribus, the length of the friendship relations depends positively on* δ*.*

Proof. See Appendix A.1. □

When a player is to betray another one, she has to consider the fact that the betrayal will yield her a higher current payoff but possibly a lower future payoff (consider, for instance, the first betrayal on the particular case above). Hence, as subjects are more patient, they will less likely betray other player to set up a friendship relation with a higher degree of needing help player. Note that the decision of a player to betray one of her friends and to set up a friendship relation with a lower-degree-of-needing-help player is independent of δ.

This result seems to match empirical findings. In a sample with children from the forth and the sixth grade Hallinan (1978) found that the length of the friendship relations was considerably longer among the children from the sixth-grade than among the children from the fourth grade. So, if we consider that the discount factor decreases with the age[6], the result stated in the proposition matches the empirical result concerning friendship relations between children.

5 Discussion on the modeling and the assumptions

Concerning the way we model social interactions, we propose a repeated setting in which players are friends when they are helping each other. Many empirical studies show how important the exchange of help between friends is. For example, Walker (1995) interviewed 52 working- and middle-class subjects and found that one of the main functions of the friends was to provide help. She found that among the working-class this help was based on providing goods and services such as borrowing or lending small amounts of money or helping in finding a job. In turn, helping among middle-class was based on emotional and intellectual support.

In the study with 185 dutch students, Buunk and Prins (1998) found that in the relationship with their best friend, the 73.6% of the subjects considered as reciprocal. In our paper, players' reciprocity is translated into Grim Trigger: I help you as long as you help me. The Cooperative Plan is restrictive as it doesn't allow for forgiveness, which is a standard feature of friendship. We believe that using Grim Trigger is not so far from reality as betraying a friend is something very severe that is difficult to forgive. Betraying a friend causes direct and conscious harm, which is different to, for instance, having a small argument with a friend. Nevertheless, as a robustness check,section 5.1 presents the results for the case in which players instead use the Tit-for-Tat strategy, that allows for forgiveness.

Finally, we assumed that each period, one player is allowed to make calls to the others. This simply reflects the fact that real-world relationship don't happen instantaneously, rather, they are the result of a 'meeting people' process. Furthermore, in each period, not all players are allowed to change the strategy they are currently playing. Instead, only the player selected by nature and those players to whom she calls are allowed to change their strategies. One can interpret this as if changing strategy was costly in terms of effort.

To sum up, the main assumptions made are:

1. Each player can offer help at most m times per period

2. Players consider each relationship pairwisely.

[6]See, for example, Read and Read (2004).

Concerning point 1, empirical studies have show the number of friends to be usually between 5 and 15[7]. This number is usually variable with each person. We abstract from this situation by assuming that all players can have the same number of friends. Concerning point 2. It has been repeatedly reported by empirical and experimental papers that real agents do not normally behave fully rational. Point 2 presents a bound on the rationality of players which seems plausible to us.

5.1 Tit-for-Tat

In this section we check for the robustness of our results when players, instead of using Grim Trigger, use the Tit-for-Tat. Because of our way of modeling, we can not use the standard definition of Tit-for-Tat. The problem arises due to of the possibility of not being linked. We define Tit-for-Tat as follows. If player k betrays player i, player i will offer help to the other player again only if k offers help to i and at the same time i doesn't help k. Hence, after a betrayal, the friendship relation can be reestablished only if the betrayer 'pays back' to the betrayed for the harm done. Formally,

Definition 2. *Define the Tit-for-Tat strategy for player $i \in \mathcal{N}$ played against any player $k \in \mathcal{N}$ as follows:*

- *If i never betrayed k and k never betrayed i, play (l, H).*
- *Otherwise:*
  1. *If the play in the past period against k was $((l, N), (l, H))$ or $((l, H), (l, H))$ then play (l, H).*
  2. *If the play in the past period against k was $((n, \{H, N\}), (l, H))$ then play (l, N).*
  3. *Otherwise, play your weakly dominant strategy.*

The following result shows that the conditions for supporting friendship under Grim Trigger (Proposition 1) and Tit-for-Tat are the same up to small differences.

Proposition 6. *Under Tit-for-Tat strategies, a friendship relation between players i and k can be supported in the repeated game if and only if the following holds:*

- *if $p_j \leq \frac{A}{1-x}$ and $p_{-j} \leq A$ then $\frac{p_j}{p_{-j}} \geq \frac{c}{\delta x}$*
- *if $p_j \leq \frac{A}{1-x}$ and $A < p_{-j} \leq \frac{A}{1-x}$ then $\frac{1}{1+\delta}(A - p_j) + xp_j - \frac{c}{\delta}p_{-j} \geq 0$*

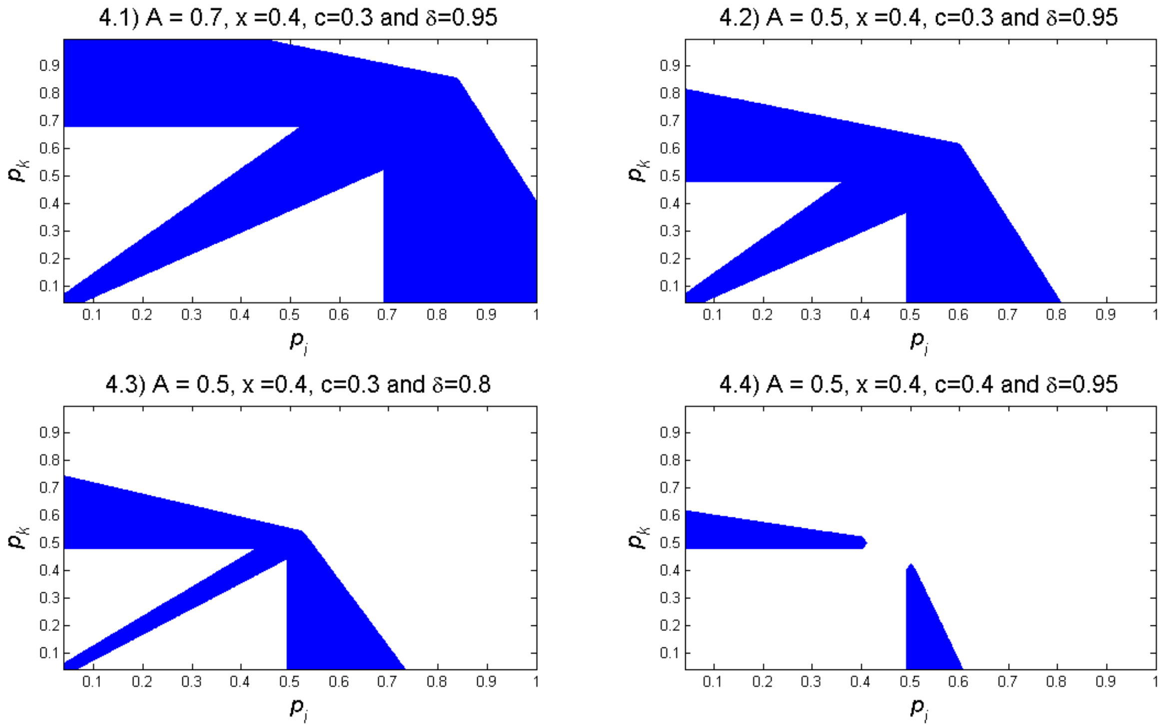
for $j \in \{i, k\}$ and $-j \in \{i, k\} \setminus \{j\}$.

Proof. See Appendix A.1. □

[7] See, for example, Franzen (2000).

When both p_i and p_k are below A both Grim Trigger and Tit-for-Tat strategies yield the same conditions for supporting friendship. When either $p_i > A$ or $p_k > A$, the condition is slightly different between the two settings. Figure 4 is a counterpart of Figure 2 for the case of Tit-for-Tat strategies formulation. As it can be verified, both figures are very close.

Figure 4: Tit-for-Tat



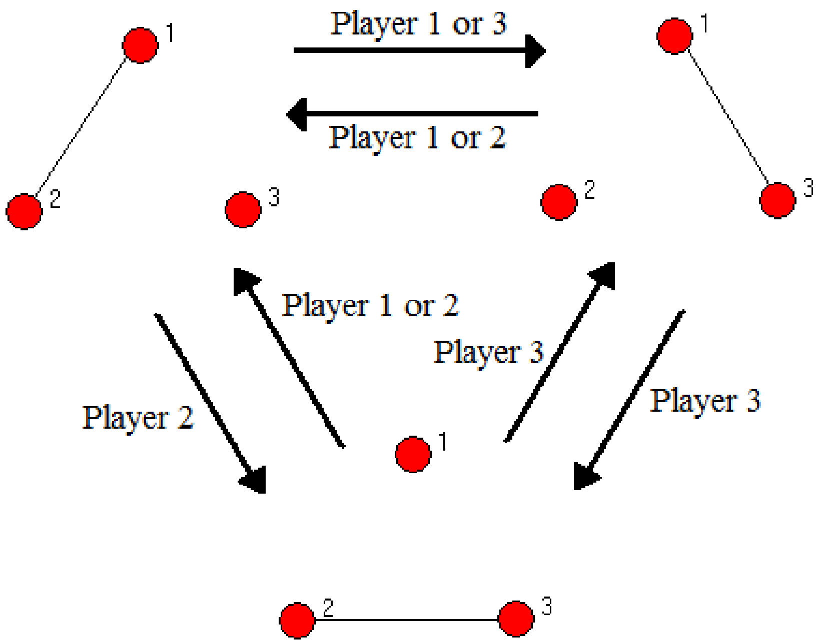
Under Tit-for-Tat strategy convergence to equilibrium is not guaranteed, that is, the friendship network may cycle between different configurations forever. The reason why this happens is that, because of the discount factor, it may be in some players interest to continuously betray each other and became friends again. In Figure 5 we present an example of a situation in which the social network never reaches an equilibrium. The parameters used are the same as in example 3. Computations are presented in Appendix A.4.

Remark 1. *Under Tit-for-Tat strategies, convergence to an equilibrium is not guaranteed.*

5.2 Social Punishment

Now we analyze a different issue. In the model presented above, if a player betrays another, the rest of players in the population doesn't react to the betrayal. We may think that if an agent is betraying her friends, it is less likely that new agents will want to set up a friendship relation with her. We explore a situation in which, if a player betrays another, all players will automatically switch to play (n, N) against the betrayer. Each player knows this fact when considering whether to betray one of her friends or not. We call this punishment mechanism

Figure 5: Simulation - Tit-for-Tat



as the Social Punishment. This punishment mechanism may seem to be a bit too strong but we are not interested here in studying the effects of different punishment mechanism. Kandori (1992) undertakes this issue. Here we restrict ourselves to the most basic and simple mechanism of social punishment.

In addition to this, we add one plan to the two plan players have on their disposal. The Friendly Ending Plan is now available for the players. By friendly ending we mean that the player who wants to break the friendship relation switches to play (n, N) instead of betraying the other player by playing (l, N).

F. Ending Play (n, N) if you and the other player played (l, H) in the last round, play your weakly dominant strategy in the Relationship Game otherwise.

Proposition 7. *Assume Social Punishment. For all A, x, c there exists a $\hat{\delta} > 0$ such that if $\delta > \hat{\delta}$ and $p_i \neq p_j$ $\forall i, j \in \mathcal{N}$, then the friendship network converges with probability 1 to a unique network architecture.*

Proof. See Appendix A.1. □

Without the Social Punishment, we only need each player to be different enough ensure convergency to a unique equilibrium. On the other hand, with Social Punishment, we need each to be sufficiently patient and not to be equal to anybody else in the population.

The following result for the case of Social Punishment deserves attention. Define a *component* as the set of players who are all friends with each other and with no player outside the component.

Proposition 8. *Consider an equilibrium situation. If* $p_i \neq p_j \ \forall i, j \in \mathcal{N}$, $\delta > \bar{\delta}$ *and there exists Social Punishment, then there exists no component of* $m+1$ *or more players in which all of the players have degree of needing help bigger than* A.

Proof. See Appendix A.1. □

Proposition 8 implies that players with high probability of needing help can not form big groups of friendship. This fact is not present without the Social Punishment mechanism. Without Social Punishment a betrayal in an early period between people with high probability of needing help may make possible the existence in equilibrium of a component of more than $m+1$ players.

Liu and Chen (2003), using a sample of 296 eighth grade students in Sanghai, found that children who had only one or two friends had lower scores in social and school competence and higher scores on learning problems and loneliness that those same scores of children who belonged to bigger friendship groups. Nevertheless, to our knowledge, the psychological literature doesn't seem to agree on the causality between the likelihood of having problems and number of friends relationship. In another study, using data from the National Longitudinal Study of Adolescent Health (Add Health), Ueno (2005) found that the number of friends was the strongest predictor of the depressive symptoms. So, empirical studies do not clarify whether being more likely to have problems implies smaller friendship network or the other way around. However, the correlation between likelihood of having problems and size of the friendship network seems clear and it is also found in the current model.

6 Conclusions

We have presented a model of friendship selection between a group of players. Each player can decide with whom of the other players in the group she wants to set up a friendship relation. The results of the paper state under which conditions friendship can arise between players. We find that when there are only two players, the decision of being friends between players whose degree of needing help is low depends on the relative difference between their degrees of needing help: the bigger the difference the less likely they are to become friends. For players whose degree of needing help is high, we find that rather than caring about the relative difference in degrees of needing help, they look only at the absolute level of these values

When we move to analyze the case of a group of more than two people, we find that it is in general impossible to predict which friendship relations will be present in equilibrium. We present two explanations to why this is happening. These are the existence of a certain degree of substitutability between friends and the non-existence of a social mechanism to punish the players betraying friends. We also find that the length of the friendship relations positively depends on the patience of the players.

The model presented here differs mainly from some existing models in psychology, anthropology and sociology in that it is solved analytically and in the fact no assumption in the taste for friends are made. Moreover, it differs from the existing models of social networks in that: there exists heterogeneity between players, the cooperation game that players play in the network is micro-founded in friendship relations and the strategies of each player can be different in each one of the cooperative games that they play on each period.

The results found in the paper seem to match the findings reported in many empirical studies of friendship selection. In our opinion, the value of the paper lies in the fact that it gives a precise non-trivial explanation to some of the phenomena we find in the friendship relations among humans. Possible extensions of the model may include a more general setting in which the degrees of needing help are unknown but players can learn them or allowing for a more flexible dynamic setting with respect to how players change their strategies.

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Appendix

A.1 Proofs

Proposition 2. *A friendship relation between players i and k can be supported in the repeated Relationship Game when both players use Cooperative Plan if and only if the following holds:*

- *if $p_i, p_k \leq A$ then $\frac{c}{\delta x} \leq \frac{p_i}{p_k} \leq \frac{\delta x}{c}$,*
- *if $p_i, p_k \leq \frac{A}{1-x}$ but either $A < p_i$ or $A < p_k$, then $A - p_i + xp_i - \frac{c}{\delta}p_k \geq 0$ and $A - p_k + xp_k - \frac{c}{\delta}p_i \geq 0$.*

Proof. The payoff to player i when both players play (l, H) equals $\frac{1}{1-\delta}(A - p_i + xp_i - cp_k)$. If player i deviates at time t from this strategy, according to the definition of the Cooperative Plan three things can happen:

Case 1. *If $p_i \leq A$ and $p_k \leq A$ then the most profitable deviation for player i is to play (l, N). This is weakly better for her than to play (n, N) if $p_i \leq A$. According to the Cooperative Plan, in the period after this deviation occurs, player k would switch to play (l, N) forever because $p_k < A$. Then, the payoff of the deviation for player i equals $(A - p_i - xp_i) + \frac{\delta}{1-\delta}(A - p_i)$. Hence, the increase of payoff for player i from the deviation is weakly negative if and only if:*

$$\frac{p_i}{p_k} \geq \frac{c}{\delta x} \tag{1}$$

Case 2. *If $p_i \leq \frac{A}{1-x}$ but $A < p_k$, then $A - p_k < 0$ and $A - p_i > 0$. In this case the best deviation for player i is to play (l, N) as shown before. But this time, however, because $p_k > A$ player k will switch to play (n, N) forever after player's i deviation occurs. The payoff of deviation from (l, H) for player i is given by $A - p_i - xp_i$. Hence, the increase of payoff for player i from the deviation is weakly negative if and only if:*

$$A - p_i + xp_i - \frac{c}{\delta}p_k \geq 0 \tag{2}$$

Case 3. *If $\frac{A}{1-x} \geq p_i > A$, then the best deviation for player i is to play (l, N) if the other player is playing (l, H) and to play (n, N) if the other players is not playing (l, H). So if they are both playing the Cooperative Plan and no deviation has occurred, if i deviates from (l, H), the action she will play is (l, N) (because $A - p_i + xp_i \geq 0$). Next period after the deviation the action played is (n, N). Hence, i's payoff is the same as in the previous case, so his incentives to deviate are the same as in the previous case. Therefore, equation (2) gives us the condition for player i to support the Friendship's Equilibrium.*

Grouping the results of cases 1, 2 and 3 and their equivalent for player k gives us the result stated in the proposition. ☐

Proposition 3. *The system converges with probability* 1 *to an equilibrium network architecture that can be history dependent.*

Proof. The fact that the equilibrium network architecture may be history dependent was already shown in example 3. Moreover, the system will always converge to an equilibrium because of the following. Given that players are using Grim Trigger as the strategy for supporting cooperation, if one player betrayed one of her friends then they won't become friends ever again. Hence, the process will eventually get to a point in which no player will want to betray her friends nor to change the strategy she is currently playing against the other players. Once this happens, the process has reached an equilibrium. □

Proposition 4. *If $A > c\delta$ then there exists a $\varepsilon > 0$ such that If $\min_{i,k\in\mathcal{N},i\neq k} |p_i - p_k| > \varepsilon$, then the friendship network converges with probability 1 to a unique network architecture.*

Proof. As mentioned earlier, the process is not ergodic because of the substitutability between players. That is, given the order in which players are selected by nature, it may happen that a player betrays one of her friends to set up a friendship relation with a third one whose degree of needing help is higher. As we show in the next paragraph, this will never happen if players are different enough. If players are different enough, then the unique equilibrium can be constructed in a fashion that we will specify below.

The increase in the profit for player i from betraying a friend, say player k, for setting up a friendship relations with another player j with $p_k < p_j$ is at most $c\left(p_k - \frac{\delta}{1-\delta}(p_j - p_k)\right) + \frac{1}{1-\delta}(A - xp_i + p_i)$. Hence, if $c(p_k - \delta p_j) + A - (1-x)p_i) < 0$ for all $i, j, k \in \mathcal{N}$, that is, if $-c\delta + A < 0$ and p_j and p_k are different enough, then the profits from betraying will be negative and the only betrays that will occur will be those in which one player betrays another for setting up a friendship relation with a third one that has smaller degree of needing help.

For constructing the equilibrium network when players are different enough we proceed as follows. Take the player with the lowest degree of needing help in the population, say i. Define the combination of relationships between i and the rest of the players that maximize i's payoff for a given m. This combination of friendship relations, call it f_i is uniquely determined if all the players are different from each other. Now take the player that has the second lowest probability of needing help in the group, say k. Define the combination of relationships between k and the rest of the players that maximize k's payoff for a given m and considering that the friendship relation prescribed by f_i have to hold. Continue in this fashion until the player with the highest degree of needing help. This result in a friendship network $F = f_1 \cup f_2 \cup \cdots \cup f_N$.

It is clear that if players are different enough F is an equilibrium network as no player can improve her situation by betraying a friend for setting up a friendship relation with a different player. F is indeed the unique equilibrium network, this follows from the fact that in any network configuration different than F, there exists at least one player that can improve her situation by changing her current strategy. To see this, consider a network configuration different than F. Take the player with the lowest probability that has her friendship relations different that what F prescribes. If she breaks her links and offers links to the players with whom she should be linked according to F, this links will be accepted and she will improve her payoff (by construction of the network F).

Now we show that the process converges with probability one to the network F. To do so we only have to consider the fact that (1) for any network different from F there is a positive probability from moving to a different network, (2) once the network F is reached, the process remains there forever, and (3), for any network there is a positive probability of reaching the network configuration F in a finite number of steps. Statements (1) and (2) where proved in the preceding paragraphs (by showing that F is the unique equilibrium network). To show (3) it is enough to notice that at any point in time there is a positive probability that the players allowed to revise their strategy in each period are ordered from the one with the lowest probability of needing help to the one with the highest probability of needing help. But if so, the network that the process reaches is exactly F, as we wanted to show. □

Proposition 5. *Ceteris paribus, the length of the friendship relations depends positively on δ.*

Proof. For any friendship relation between two players, the chances that one of the players betrays the other negatively depend on the discount factor. This is so because when a player betrays other player, she is increasing her present payoff for a possible decrease of her future payoff. Hence, the higher the discount factor, the less likely a player will betray one of her friends. Therefore, the speed at which the friendship relations change for a given set of parameters and a given population depend negatively on the discount factor. □

Proposition 6. *Under Tit-for-Tat strategies, a friendship relation between players i and k can be supported in the repeated game if and only if the following holds:*

- *if $p_j \leq \frac{A}{1-x}$ and $p_{-j} \leq A$ then $\frac{p_j}{p_{-j}} \geq \frac{c}{\delta x}$*
- *if $p_j \leq \frac{A}{1-x}$ and $A < p_{-j} \leq \frac{A}{1-x}$ then $\frac{1}{1+\delta}(A - p_j) + xp_j - \frac{c}{\delta}p_{-j} \geq 0$*

for $j \in \{i, k\}$ and $-j \in \{i, k\} \setminus \{j\}$.

Proof. We structure the proof of this result similarly to the proof of Proposition 1 but with the only difference that once a player betrays the other, it may be in her interest to play (l, H) in the next round after betrayal so as to bring the helping situation back. The payoff of player i that the situation in which both players play (l, H) forever equals $\frac{1}{1-\delta}(A - p_i + xp_i - cp_k)$. If player i deviates at time t from this strategy, according to the definition of the Tit-for-Tat strategy two things can happen:

Case 1. *If $p_i \leq \frac{A}{1-x}$ and $p_k \leq A$ then the best deviation for player i is to play (l, N). In the period after this deviation occurred, player k will switch to play (l, N) because $p_k \leq A$. If player i plays then (l, N) or (n, N) forever, we are in the same case as Grim Trigger, i.e. the case in which the condition 1 has to hold. On the other hand, if player i plays (l, H) from after the period she deviated on, player k will also come back to playing (l, H). Note that for player i there is no difference between trying to bring back helping immediately after he betrayed player k T periods after the betrayal has taken place. The payoff of the deviation for player i equals*

$(A - p_i + xp_i) + \delta(A - p_i - cp_k) + \frac{\delta^2}{1-\delta}(A - p_i - xp_i - cp_k)$. *Hence, the increase of payoff for player i from the deviation is weakly negative if and only if:*

$$\frac{p_i}{p_k} \geq \frac{c}{\delta x} \tag{3}$$

Note that in this case the condition for friendship to be possible is the same as in the case with Grim Trigger.

Case 2. *If* $p_i \leq \frac{A}{1-x}$ *but* $A < p_k$, *then* $A - p_k < 0$ *and* $A - p_i > 0$. *In this case the best deviation for player i is to play* (l, N) *as shown before. But this time, however, because* $p_k > A$ *player k will switch to play* (n, N) *after player's i deviation occurs. If player i plays then* (l, N) *forever, we are in the same case as Grim Trigger, i.e. the case in which the condition 2 has to hold. On the other hand, imagine that player i plays* (l, H) *from after the period she deviated on. Then, according to the Tit-for-Tat strategy, player k will play first* (l, N) *and then* (l, H) *forever. The payoff of deviation from* (l, H) *for player i is given by* $A - p_i + xp_i + \frac{\delta^2}{1-\delta}(A - p_i - cp_k) + \frac{\delta^3}{1-\delta}(A - p_i + xp_i - cp_k)$. *Hence, the increase of payoff for player i from the deviation is weakly negative if and only if:*

$$\frac{1}{1+\delta}(A - p_i) + xp_i - \frac{c}{\delta}p_k > 0 \tag{4}$$

Note that condition 4 is stronger than condition 2.

Grouping the results of cases 1, 2 and 3 and their equivalent for player k gives us the result stated in the proposition. □

Proposition 7. *Assume Social Punishment. For all A, x, c there exists a* $\hat{\delta} > 0$ *such that if* $\delta > \hat{\delta}$ *and* $p_i \neq p_j$ $\forall i, j \in \mathcal{N}$, *then the friendship network converges with probability* 1 *to a unique network architecture.*

Proof. If a player i is to betray another player under Social Punishment setting, she knows that from the moment of her betrayal on she will get a payoff of 0 forever. Hence, when deciding whether to betray the player takes into account the present period increase in her profits with the future decrease in her payoff. Hence, if player i is patient enough she won't be interested in betraying any of her friends ever. This result combined with the fact that all players are different shows that, using the same arguments as in Proposition 4, a unique network exists and that the system converges to it with certainty. □

Proposition 8. *Consider an equilibrium situation. If* δ *is high enough,* $p_i \neq p_j$ $\forall i, j \in \mathcal{N}$ *and there exists Social Punishment, then there exists no component of* $m + 1$ *or more players in which all of the players have probability of needing help bigger than A.*

Proof. By contradiction. Take a group of $k > m + 1$ players among which all have their degree of needing help bigger than A. Take the $m + 1$ players of the component with the lowest degree of needing help. Because $k > m + 1$, at least one of them won't be linked with the other m (if not these $m + 1$ players will form a closed component which by assumption is not the case). Take a player among the $m + 1$ with the lowest degree of needing help in the component who is not linked with the other m with the lowest degree of needing help in the component. If she makes calls to the players with whom she is not linked and have the lowest degree of needing help in the component, the calls will result in new friendships. Note that this won't happen if some player have a degree of needing help smaller than A. Hence, the initial situation was not an equilibrium. □

A.2 Dynamics

The dynamics of the model work as follows.

1. At $t = 0$ each player is playing the strategy "play (n, N) against all players in all the rounds".
2. In period t for $t = 1, 2, \ldots$ the following of events sequence takes place:
   (a) A player $i \in \mathcal{N}$ is selected by nature. This player can make calls to the other players.
   (b) Every player $k \in \mathcal{N}_{-i}$ plays, in case she gets a call from i, according to one of the possible schemes:
      i. In case cooperation between k and i is not pairwisely sustainable. Then k plays the same strategy she played last period against all the players in the population.
      ii. In case cooperation between k and i is pairwisely sustainable and that player k is providing help less than m times. Then player k plays Grim Trigger with i and plays the same strategy she played last period against the rest of players.
      iii. In case cooperation between k and i is pairwisely sustainable and that k is providing help exactly m times. Let j be the player with the highest probability of needing help among those who k is currently helping to. If further that the discounted present value of the profits from playing the Cooperative strategy with i plus playing the Defective strategy j are higher than the profits of player k from playing the Cooperative strategy against j. Then player k switches to play the Cooperative strategy with i, the Defective strategy with j and plays the same strategy she played last period against the rest of players. Otherwise, k plays the same strategy she played last period against all the players in the population.
   (c) Player i, the one selected by nature in the current period, makes calls to the rest of players and changes her current strategy against the other players. She does so knowing that the players who get a call will react as stated in step b. She makes the calls and changes her strategy in a way as to maximize her present value payoff myopically, i.e.
      i. she will decide whom to call and play the Cooperative strategy with
      ii. she will play the strategy the Defective strategy with the players she doesn't give a call to.
   (d) The players who get a call from i play accordingly to step b.
   (e) All other players don't change strategy.

A.3 Example: A Simple Case

Example 3 is conducted for $N = \{1, 2, 3\}$, $p_1 = 0.4$, $p_2 = 0.45$, $p_3 = 0.55$, $A = 0.5$, $x = 0.6$, $c = 0.3$, $\delta = 0.7$ and $m = 1$. First we check that all friendship relations are possible. To do so we only have to apply Proposition 2 to the present example. Players 1 and 2 can be friends because $\frac{0.3}{0.6\times0.7} \leq \frac{0.4}{0.45} \leq \frac{0.6\times0.7}{0.3}$. Players 1 and 3 can be friends because $0.5 - 0.4 + 0.6 \times 0.4 - \frac{0.3}{0.7} \times 0.55 = 0.104 \geq 0$ and $0.5 - 0.55 + 0.6 \times 0.55 - \frac{0.3}{0.7} \times 0.4 = 0.108 \geq 0$. Finally, players 2 and 3 can be friends because $0.5 - 0.45 + 0.6 \times 0.45 - \frac{0.3}{0.7} \times 0.55 = 0.084 \geq 0$ and $0.5 - 0.55 + 0.6 \times 0.55 - \frac{0.3}{0.7} \times 0.45 = 0.087 \geq 0$.

In period 1, player 1 is selected by nature. Since she can set up friendship relations with the other two players but she is constrained to have at most 1 friendship relation player 1 will choose to call player 2. This is true simply because $p_2 < p_3$ and hence, the stream of payoffs for player 1 is higher if she sets up a friendship relation with player 2. In particular, the stream of payoffs if player 1 sets up a relationship with player 2 equals $\frac{1}{1-0.7}(0.5 - 0.4 + 0.6 \times 0.4 - 0.3 \times 0.45) = 0.683$. On the other hand, if player 1 sets up a friendship relation with player 3, her stream of payoffs equals $\frac{1}{1-0.7}(0.5 - 0.4 + 0.6 \times 0.4 - 0.3 \times 0.55) = 0.583$. Player 2 will respond to the call of player 1 by switching to play Grim Trigger with her since $\frac{0.3}{0.6\times0.7} \leq \frac{0.4}{0.45} \leq \frac{0.6\times0.7}{0.3}$ holds.

In period 2, player 1 is again selected by nature. Now her decision is whether or not to betray player 2. In this example, player 1 will switch to the strategy "play (l, N) if you and the other player played (l, H) in the last round, play your weakly dominant strategy in the Relationship Game otherwise" against player 2 and will call player 3 and play Grim Trigger against her. That is, player 1 will betray player 2. The next period after this deviation occurs both player 1 and player 2 will switch to play (l, N) against each other (because $p_1 < p_2 < A$). To see that player 1 will betray player 2 and call player 3 and play Grim Trigger against her, we consider her payoff with this change of her strategies. This payoff equals to $(0.5 - 0.4 + 0.6 \times 0.4) + \frac{0.7}{1-0.7}(0.5 - 0.4) + \frac{1}{1-0.7}(0.5 - 0.4 + 0.6 \times 0.4 - 0.3 \times 0.55) = 1.156$. On the other hand, if player 1 keeps her friendship relation with player 2, she will get a payoff equal to: $\frac{1}{1-0.7}(0.5 - 0.4 + 0.6 \times 0.4 - 0.3 \times 0.45) = 0.683$. Hence, player 1 will betray player 2 and set up a friendship relation with player 3.

In period 3, player 2 is selected by nature. She will call player 3 instead of player 1 because the betrayal that happened in period 2 now makes the friendship between player 1 and 2 impossible forever. In this example, we have that, in response to player's 2 call, player 3 will betray player 1 to set up a relationship with player 2 even though to profit of player 3 is higher if she has a friendship relation with player 1. The stream of payoffs of player 3 from betraying player 1 by setting up a relationship with player 2 equals: $(0.5 - 0.55 + 0.6 \times 0.55) + \frac{1}{1-0.7}(0.5 - 0.55 + 0.6 \times 0.55 - 0.3 \times 0.45) = 0.763$. On the other hand, the stream of payoffs of player 3 if she keeps her friendship relation with player 1 equals: $\frac{1}{1-0.7}(0.5 - 0.55 + 0.6 \times 0.55 - 0.3 \times 0.4) = 0.533$. Hence, player 3 will betray player 1.

After period 3, the network is in equilibrium. No player can increase her profit by changing the strategy as it can be easily verified.

A.4 Example: Nonexistence of Equilibrium under Tit-for-Tat

We now show with an example that under Tit-for-tat there may not exist equilibrium. We use the same set of parameters as in example 3. That is, $N = \{1, 2, 3\}, p_1 = 0.4, p_2 = 0.45, p_3 = 0.55, A = 0.5, x = 0.6, c = 0.3, \delta = 0.7$ and $m = 1$.

First, we check that a friendship relation is possible between any two players in the group. For doing so, we only have to apply Proposition 6 to the present example in the same fashion as we applied Proposition 2 in Appendix 3.A.3.

We show now that, in this particular case, the process will never converge no matter how nature selects the players. As we show in the proceeding paragraph, all friendship relations are possible. Now we show that for any given friendship relation in this group, there is always a profitable betrayal independent of the history of past play. Imagine that players 1 and 2 are friends and both players betrayed player 3 recently. If one of these two players wants to set up a friendship relation with player 3, they will have to first 'pay back' and offer help to player 3. Imagine that player 2 is selected by nature, she will betray player 1 and switch to play (l, H) against player 3 if and only if $(0.5 - 0.45 + 0.6 \times 0.45) + 0.7(0.5 - 0.45 - 0.3 \times 0.55) + \frac{0.7}{1-0.7}(0.5 - 0.45) \cdots + \frac{0.7^2}{1-0.7}(0.5 - 0.45 + 0.6 \times 0.45 - 0.3 \times 0.55) > \frac{1}{1-0.7}(0.5 - 0.45 + 0.6 \times 0.45 - 0.3 \times 0.4)$. This inequality holds true. Note that because $p_1 < p_2$ and $p_3 < A$, if player 2 finds profitable to retake her friendship with player 3 so will player 1 . Also note that if player 1 (2) finds profitable to betray 2 (1) to retake her friendship relation with 3, it is straight-forward to show that since $p_3 > p_2 > p_1$, if player 1 (2) is having a friendship relation with 3, she will find profitable to betray player 3 and to retake (or start) a friendship relation with 2 (1). Also, in this example player 3 finds profitable to betray player 1 (2) to retake her friendship relation with player 2 (1). However, this is not needed for the result we want to show.

So we have that for any history of past play (or, more intuitive, history of past betrayals), there always exists at least one player that can increase her profit by changing strategy. Hence, the process never converges to an equilibrium.

A.5 Formal Definitions of the Sets of Strategies

Let i and k stand for the two typical elements of $\mathcal{N}$ and let $\mathcal{N}_{-i} = \mathcal{N} \setminus \{i\}$. Define the set of actions in the Relationship Game as $A = \{l, n\} \times \{H, N\}$ and let A_i be the set of actions of each player i against every other player in the Relationship Game, $A_i = (A_{ij})_{j \ in \mathcal{N}_{-i}}$ with $A_{ij} \in A$. Let H_{ik}^t be the set of all possible histories between players i and k till the beginning of time $t \geq 0$. Hence, we have that $\left(h_{ik}^1, \ldots, h_{ik}^{t-1}\right) \in H_{ik}^t$ for $t \geq 1$ and $H_{ik}^0 = \varnothing$ with $h_{ik}^s \in \{A_{ik} \times A_{ki}\}$ for $s \in \{1, \ldots, t-1\}$. Define $H_i^t = (H_{ij}^t)_{j \in \mathcal{N}_{-i}}$.

Let L^t be the sequence of players selected by nature till time t, hence $L^t = (l^\tau)_{\tau=1}^t$ with $l^\tau \in \mathcal{N}$. Define the set of strategies of each player i against player k given the players selected by nature each period and set of all possible histories *between i and all the other players* as Σ_{ik}. Hence, if $\sigma_{ik} \in \Sigma_{ik}$ then:

$$\sigma_{ik} : \cup_{t=0}^{\infty} \left\{L^t \times H_i^t\right\} \to A$$

Therefore, a strategy σ_{ik} is a plan that maps all the possible histories and all the possible combinations of players selected by nature into the set of actions. We make use of pairwise strategies. That is, if player i is to decide which action to take against player k, i will only consider the past history between i and k as if $\mathcal{N} = \{i, k\}$. Formally, denote the pairwise set of strategies for each player i against player k given the players selected by nature in every period and set of all possible histories *between players i and k* by Σ_{ik}^p. Hence, if $\sigma_{ik}^p \in \Sigma_{ik}^p$ then:

$$\sigma_{ik}^p : \cup_{t=0}^{\infty} \left\{ L^t \times H_{ik}^t \right\} \to A$$

We are using the superscript p to refer to the fact that the strategy is pairwise. For each i define $\Sigma_i = (\Sigma_{ij})_{j \in \mathcal{N}_{-i}}$ and $\Sigma_i^p = \left(\Sigma_{ij}^p \right)_{j \in \mathcal{N}_{-i}}$.

We write $\pi_{ik}(\sigma_i, \sigma_{-i})$ as the discounted present value payoff for player i when he plays the Relationship game against player k when i's strategy is σ_i and the rest of players are playing a strategy σ_{-i}. Define the best response of each player i as $\sigma_i^{BR} = (\sigma_{ij}^{BR})_{j \in \mathcal{N}_{-i}}$ where:

$$\begin{aligned} \sigma_{ij}^{BR} \in \; & \arg \max_{\sigma_{ij}^p \in \Sigma_{ij}^p} \pi_{ij}(\sigma_{ij}^p, \sigma_{-i}) \\ & st : \# \left\{ \sigma_i^p : \sigma_i^p \in (l, H) \right\} \leq m \end{aligned}$$

Put in words, each player maximizes her payoff taking each relationship pairwisely subject to the constraint of not offering help more than m times.

Finally, we reduce the strategy space to the case in which, for each player i, her strategy against every player k consists on either the Cooperative strategy or the Defective strategy. For every i, k let $\hat{\Sigma}_i^p$ be this strategy space.