Department of Economics

Three Essays on Behavioral and Applied Game Theory

Andrea Gallice

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

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Examining Board:

Prof. Pascal Courty, EUI (Co-supervisor)
Prof. Dorothea Kuebler, Humboldt Universität
Prof. Marco Mariotti, Queen Mary, University of London
Prof. Karl Schlag, EUI (Supervisor)

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DEDICATION

Alla mia famiglia, a Denise ed a Martina e Luca.

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CONTENTS

Part I

Introduction

This dissertation is composed by three independent chapters. Their common denominator is that they all deal with game theory, either contributing to this branch of microeconomics from a theoretical point of view (Chapter 1), or using a game theoretic approach in analyzing a specific problem (Chapters 2 and 3).

Game theory is defined as the study of strategic interactions among (economic) agents. Even though, like everyone, I have been involved in strategic interactions since the moment of my birth, I officially met game theory in the spring of 1997 during my undergraduate course in microeconomics at the University of Torino. Accompanied by the classical story of a criminal undecided between confessing or not his crime, the payoff matrix of a Prisoner's Dilemma was drawn on the board by the instructor. I must say that it looked much more fun than previous topics and I immediately felt attracted by the issue. Moreover, having already quite a tradition of evenings spent with friends in playing Risk and other table games, I found myself easily acquainted with the kind of strategic reasoning the instructor was explaining. The choice to proceed with my education in economics and to specialize in game theory possibly boils down to that two hours class.

In the subsequent years things obviously got more serious and my approach towards game theory became more conscious and mature. Still I kept on being fascinated mainly by two aspects of the theory. The first one concerns the generality of game theory. In fact virtually every situation involving more than a single agent (not only in economics, but also in politics, biology, history...) can be framed and analyzed in a game theoretic setting. The second aspect, which is more critical, is related to the solution concepts used in game theory and to their relevance in adequately capturing agents' behavior.

Consider for instance the concept of Nash equilibrium, possibly the cornerstone of game theory. The intuition behind Nash equilibrium ("do what is better for you assuming that the others will do the same") is simple and sharp, its formal definition is precise and neat, the proof of its existence is very elegant. Therefore I initially thought the Nash equilibrium to be a never failing tool. It was then quite a disappointment to progressively discover that this is not the case. Luckily this disappointment gradually evolved into a stimulus to broaden a bit the perspective.

In particular I found quite disturbing to discover that the predictive power of the Nash equilibrium often happens to be considerably low. A large number of studies and experiments show in fact that in many strategic situations the actual behavior of economic agents is not correctly captured by the Nash conjecture. Indeed the problem of the Nash equilibrium is to be too elegant and precise to properly anticipate something as hectic as human behavior can be. In fact, the Nash analysis is based on the assumption of perfect rationality of the agents, a hypothesis which is rarely met by real and heterogeneous players. Nowadays the Nash equilibrium idea remains the leading concept of game theory but, when it is used to solve strategic interactions that happen for real, its indication should be interpreted more in a normative way (about how fully rational agents would act) rather than in a positive one (about how boundedly rational players actually act). This consideration opened the way to new important contributions. To provide tools which are able to decently approximate, anticipate and predict individuals' actual behavior became the common goal of an important and fruitful area of research. Indeed the reward is high given that the study of this problem can have (and it already had) a significant impact in many different fields. These range, to name a few, from the issue of mechanisms design to the setting of incentive schemes, from the writing and enforcement of new laws and rules to the analysis of phenomena like asset markets, speculation and betting.

Behavioral game theory is an important pillar of this stream of research and, in the last few years, it proved to be very successful in narrowing the gap between theory and experimental evidence. Basically behavioral game theory enriches pure game theory by adding elements which are typical of the human nature: limited rationality, heuristic decisions, psychological regularities, feelings and emotions. As it is true in all the realm of economic modeling, researchers working in this area face a rigid trade off. Indeed here, given individuals' heterogeneity and all the possible ways with which their behavior can depart from the hypothesis of full rationality, this problem is even more exacerbated. On one hand the performance of a predictive tool can be improved increasing its complexity and sophistication. On the other, the tool should still be parsimonious and general enough. Obviously a good model is the one which finds an adequate balance between complexity and performance.

The first paper of this dissertation ("*Predicting One Shot Play in 2x2 Games using Beliefs based on Minimax Regret*") falls into the behavioral game theory area and tries to cope with this trade off. It initially focuses on simple games (actually as simple as possible, i.e. two players with two strategies each) that are called Matching Pennies. These games model those situations in which players' interests are always in contrast so that each agent wants to outguess the opponent. As an example consider the situation of a penalty kick between a football player and a goalkeeper where they both have to simultaneously decide if to shoot/jump left or right. This class of games is characterized by the existence of a unique Nash equilibrium in mixed strategies. This means that a rational player should randomize between his two pure strategies according to a certain probability distribution. The empirical evidence of this equilibrium ranges from being very high in some cases to be totally misleading in others. In the paper I propose a procedure that reconciles and explains this inconsistency. In particular I use a concept normally used in decision theory, the minimax regret, to approximate the beliefs of each player about what the opponent will do. I justify this instrumental use of the minimax regret through an axiomatic approach, i.e. I show that this concept is characterized by a list of properties that a beliefs function should have. Then I let the players choose the strategy that maximizes their expected payoff given these conjectured beliefs. In other words I let the players best respond to their own beliefs, a behavior which is consistent with some recent empirical evidences (see for instance Nyarko and Schotter, 2002). When compared with existing experimental evidences my procedure is able to correctly predict the actual choices of the vast majority of the players, outperforming by far the performance of the Nash prediction. The procedure is general enough, in the sense that it can be applied to any 2x2 game, but it is still very simple to implement. It constitutes a relevant contribution because, on one hand, it sheds some more light on the behavior of inexperienced and boundedly rational agents and, on the other hand, it can be used for predictive purposes.

As I mentioned before one of the characteristics I like about game theory, and indeed a quality which contributed in making it so popular and successful, is the fact that a game theoretic approach can be used to model and analyze an extremely wide spectrum of problems. Indeed we can say that wherever there is interaction there is space for strategic behavior and thus for game theory... and interactions are basically everywhere!! We act strategically with people we know (shall I call my girlfriend or shall I wait for her to call?) as well as with people we do not know (how shall I dress for an interview?) or even with collective subjects (how can I improve my chances to find a job given the market situation?). Examples are endless also because agents' interactions can be simultaneous or sequential, one off events or repeated over time, they can take place under certain or uncertain conditions. Moreover players can be individuals, as in the above examples, as well as countries, firms, parties, animals... Applied game theory models and analyses a problem of interest focusing on its strategic aspects and it uses game theoretic techniques to solve it. The second chapter of my dissertation falls in this frame.

The second chapter of this thesis ("*Education, Dynamic Signaling and Social Distance*") focuses on a robust tendency that took place in European countries in the last decades, namely the constant and rapid growth in the average level of education of the population. As I explain in the introduction of the paper, there are many plausible reasons likely to have simultaneously contributed to such a trend. I propose a new one which combines theories of signaling with theories of social distance.

Models of signaling are used to deal with problems of asymmetric information. The idea is that, when some characteristics of an individual cannot be directly observed by the others, this agent may have incentives in adopting behaviors that may reveal them. In the original signaling model (Spence, 1973) the level of education is assumed to convey information about the individual's unobservable and innate level of productivity. Therefore the individuals (the informed parties), in entering into the labor market with a certain level of schooling, are sending a signal to the employers (the uninformed parties) who can then screen them relying on their observable characteristics.

Social distance theories are applied to study many phenomena like, just to name a few, consumption patterns, the provision of effort in the workplace or the development of fads and fashions. The idea is that individuals' choices are often affected by the ones of some reference group (neighbors, colleagues, peers...). Indeed to consider social distance issues in economics acknowledges the fact that strong human sentiments like pride, esteem, shame or acceptance do indeed affect people's behavior.

Going back to the paper its main intuition is that the level of education of an individual can be a signal which is sent not only to the labor market but also to the rest of the society or, at least, to some reference group. In fact a higher level of education can facilitate social interactions and lead to a better social position. It follows that social distance considerations may influence individuals' educational choices. In particular I consider how the existence of conformist and status seeking agents can affect educational dynamics, under the assumption that both individuals and firms are myopic. I show that the presence of individuals that care about their relative level of education, even if minimal, is likely to lead to an increasing trend in average education which will progressively involve also individuals characterized by independent preferences. The presence of status seeking individuals is not necessary to trigger such a dynamic, even though it reinforces the effect. I also briefly analyze firms' behavior in front of such an increasing trend. I show that as average education increases signals get less informative, firms are more likely to be disappointed by the workers' real productivity and thus they will raise their educational requirements to effectively screen the individuals. All these dynamics are in line with recent trends and other stylized facts about education.

In the categorization between theoretical and applied work the third chapter of this thesis ("*Preempting versus Postponing: the Stealing Game*") falls somewhere in the middle. In fact it presents a theoretical analysis of a particular game that, even though it is not applied to any specific problem, can mimic situations likely to arise in economics, politics or biology. The game is about a small number of agents that fight over a divisible resource of constant size. As an example this resource can be a market, the electorate or a territory. The twist is that peaceful mechanisms (market, bargaining, negotiation...) are not implementable such that the only way through which the players can increase their share is by stealing parts of the resource held by the opponents. Moreover this ability to steal is proportional to the player's current holding of the resource (i.e. their size). The question is then about the optimal timing of players' move. Is it better to preempt the rivals and steal them part of their holdings as soon as possible (but possibly facing their retaliation) or, at the opposite, is it better to wait and best respond to what the others did (but possibly being preempted and weakened)? The paper shows that the answer to this question changes according to how payoffs are defined (we study two payoff formulations) and to the number of players participating in the game. It is shown that in general players always want to preempt their rivals. But there are situations in which players may prefer to postpone their move. This last result shares some similarities with the analysis of so called truels (Kilgour and Brams, 1997). The paper provides therefore an example of a timing game (i.e. a game in which players have to decide when to make their move) in which, for a given payoff structure, optimal strategies change as a function of the number of participants. This is an aspect of timing games which so far has been neglected.

To conclude let me comment on the order of the chapters in this dissertation, an order which is not chronological. In fact the first paper I have been working on during my PhD is the one about education and social distance (Chapter 2). Almost simultaneously I started thinking of a strategic situation then later would have evolved in the material presented in Chapter 3. Still when I properly focused on a preliminary version of the game I got puzzled by its (mixed strategy) Nash equilibrium which I found very counterintuitive. This made me think a lot about the relevance of (some) Nash equilibria and pushed me towards investigating the issue with more attention. What started as a critique of the mixed strategy Nash equilibrium concept gradually evolved in a richer and more constructive project whose result is the paper in Chapter 1. I let to this paper the "honor" to open my dissertation because this is the one that gave me more pleasure and opportunity to learn in thinking about it and writing it down.

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Part II

Chapters

CHAPTER 1

PREDICTING ONE SHOT PLAY IN 2X2 GAMES USING BELIEFS BASED ON MINIMAX REGRET

Abstract

We wish to establish a formal model for predicting one shot play of inexperienced agents in 2x2 matching pennies games. We do this by axiomatically describing players' beliefs and then by assuming that players best respond to these beliefs. We find the minimax regret criterion to be the simplest functional form satisfying our axioms and thus we let agents play as if their opponents were playing according to minimax regret. When compared with existing experimental evidences about one shot matching pennies games, the procedure correctly indicates the choices of the vast majority of the players. Applications to other classes of 2x2 games are also explored.

Keywords: predictions, minimax regret, beliefs, matching pennies, experiments.

JEL classification: C72, C91.

1.1 Introduction

Consider the situation faced by individuals who are involved in a one shot 2x2 game, possibly as subjects of a controlled experiment. The players do not have any knowledge of game theory and they never played before the specific game they are facing. In addition, given that the interaction is not repeated, they cannot expect to learn and improve on their performance over time.

How should these players decide which strategy to play? Irrespective of the specific game under consideration, the rational approach for dealing with such a decision under uncertainty (in a spirit similar to Savage, 1954) is the following:

1 - each player forms a belief about what his opponent will play.

2 - each player chooses the strategy which best responds to this belief.

Interpreting it as a heuristics and acknowledging players' heterogeneity, this procedure cannot be expected to describe the behavior of all the agents. Single individuals may in fact use different decisional processes or they may incur in computational errors in choosing the best response. Still, in spite of this noise, the fact that in simple strategic situations the majority of individuals behave coherently with their beliefs finds confirmation in some recent papers[1]. Nyarko and Schotter (2002) study a 60 times repeated 2x2 game and find that around 75% of the players do indeed best respond to their stated beliefs. For the case of 3x3 games, Rey-Biel (2004) considers 10 one shot games and finds a similar rate of compliance while 55% is the percentage found by Costa-Gomes and Weizsäcker (2005) using data about 14 (more complex) one shot games. It is therefore a conservative guess to expect that, in one shot 2x2 games, at least half of the individuals play consistently with their beliefs. We want to capture the behavior of this majority of players.

The first part of this paper is focused on the process of beliefs formation. The goal is to find a function that may describe players' beliefs. In looking for this function we adopt an axiomatic approach. First, we list desirable properties that, according to us, should characterize a belief function. Then, we check existing concepts and criteria commonly used in game theory and decision theory to see which of them, if any, fulfills all the requirements.

We find the minimax regret criterion (originally proposed by Savage, 1951) to be the unique candidate to obey all the axioms. Therefore we propose minimax regret as a proxy for players' beliefs and we claim that the majority of players play "as if" they were best responding to these approximated beliefs.

This conjecture is tested in the second part of the paper. The predictions stemming from the suggested procedure (best respond to beliefs equal to the minimax regret distribution of the opponent) are compared with experimental evidences about different versions of 2x2 one shot matching pennies games. To forecast players' choices in this class of games can be particularly problematic, also because the Nash prediction is often misleading (see for instance Ochs, 1995 and Goeree and Holt, 2001). Our procedure proves to be an effective way to identify the strategies which are more likely to be played. In fact it correctly predicts the actual choices of around 80% of the players.

In a later section the same procedure is also applied to other classes of 2x2 games and its relationship with the Nash prediction is explored. An interesting result is that the procedure selects a single outcome even in games that have multiple Nash equilibria such that it contributes to the debate on equilibrium selection (see Straub, 1995 and Haruvy and Stahl, 2004).

This paper thus aims to get some insights into the way people actually behave in simple

[1]These papers elicit players' beliefs using a proper quadratic scoring rule, such that for the players "telling the truth" is optimal.

strategic interactions and it is motivated by experimental results that traditional theory fails to explain. Therefore, despite of the initial theoretic axiomatic approach, the paper places itself in the behavioral game theory literature. Behavioral game theory enriches pure game theory by adding elements which are typical of the human nature: limited rationality, heuristic decisions, psychological regularities, feelings and emotions. In the last few years it proved to be successful in narrowing the gap between theory and experimental data[2]. Camerer (2003) provides a very rich overview of the aims, the methods, the empirical evidence and the findings of this rapidly growing area of research.

A number of studies that focus on how people play one shot simultaneous games and investigate the issue of beliefs formation are more related to this paper. Stahl and Wilson (1995) and Costa-Gomes et al. (2000) test the existence and relative importance of various archetypes of players that differ in the prior they have about the degree of sophistication of their opponents. Results indicate that the majority of the individuals behave as if they were performing one or two steps of strategic thinking. A similar result is also found by Camerer et al. (2004) with experiments about market entry games, Nash demand games and stag hunt games.

There are also various papers that are more specifically focused on the experimental study of matching pennies (MP in what follows) games. Games of this family have in fact been extensively used to test the validity of the Nash prediction and to study the issues of individuals' learning and adaptive dynamics. These questions stimulated much research with important contributions by Mookherjee and Sopher (1994), Ochs (1995), McKelvey and Palfrey (1995), Erev and Roth (1998), McKelvey et al. (2000), Tang (2001) and Goeree et al. (2003). The natural design of these experiments consisted in letting subjects repeatedly play the same version of a MP game. A different question is to study how agents behave in front of a single interaction: in fact in this case players cannot learn over time and their behavior is not affected by inter temporal considerations. Less work has been carried out to study individuals' play in one shot MP games[3] with a unique Nash equilibrium in mixed strategies, the reason possibly being the fact that agents' behavior is too erratic to reach some general conclusions. Our paper is focused on one shot games mainly for three reasons: first, we claim our theory to be able to capture the behavior of inexperienced players; second, as just mentioned, one shot individuals' play has been less

[2] For instance, and as already briefly mentioned, the concept of Nash equilibrium is sometimes too "radical" and it may lead to conclusions which are often rejected by experimental evidences.

[3] A notable exception is Goeree and Holt (2004) that presents a model of iterated noisy introspection for one shot interactions which is then tested over a large number of games.

investigated; third, we think that enough real life situations are more likely to be similar to one off events rather than to repeated interactions.

The paper is structured as follows: Section 2 lists the axioms we feel characterize a beliefs function. Section 3 shows that minimax regret satisfies all the axioms being at the same time very simple in its functional form. In Section 4 other candidate functions are shown to under perform the minimax regret; in particular it is shown that various proposals connected with the concept of mixed strategy Nash equilibrium cannot be expected to mimic players' beliefs. Section 5 formalizes the procedure that we claim is able to capture the behavior of the majority of individuals. Section 6 uses existing experimental evidences about MP games to test the validity of our predictions. In Section 7 the procedure is applied, as a robustness check, to other classes of 2x2 games. Section 8 concludes.

1.2 An axiomatic approach to belief formation

The aim of this section is to provide an axiomatic description of players' beliefs. These beliefs will be later used as the starting point for a procedure that selects the strategies most likely to be played in one shot 2x2 games, with our main interest being matching pennies games. We tackle the issue of beliefs formation in a very simplified framework. Still the axioms and the results that follow can be easily restated for the more general case.

Consider the following 2x2 game, where player $i \in \{A, B\}$ can choose between strategies H_i and T_i.

1)

	H_B	T_B
H_A	$a, \cdot$	$b, \cdot$
T_A	$c, \cdot$	$d, \cdot$

We model the beliefs of player B about what player A will play. This is the reason why the payoff matrix is incomplete and only the payoffs of player A appear. Whilst keeping in mind the example of an inexperienced boundedly rational player, B's beliefs are considered as being just a function of player A's payoffs. This implies that player B realizes A does respond to changes in his own payoffs. This own payoff effect is a robust feature of games played in experiments (for clear evidences of this effect in MP games see, among others, Ochs, 1995 and Goeree and Holt, 2001). The fact that we focus on B's beliefs implies no loss of generality given that a similar analysis can be done for what concerns A' beliefs.

As a preliminary requirement and in order to simplify the exposition we postulate that B's beliefs obey to the following axiom.

[A1] Invariance to adding a constant to a column.

This axiom requires that the beliefs of player B about what A will play in Game 1 should be the same as in a modified game where a constant is added to both the payoffs a and c or to both the payoffs b and d. Such an operation does not change the relative attractiveness of the two strategies from A's point of view and thus should not modify B's beliefs. Assuming that $a, b, c, d > 0$, we can then subtract $\min\{a, c\}$ to the first column and $\min\{b, d\}$ to the second one and define $x = \max\{a, c\} - \min\{a, c\}$ and $y = \max\{b, d\} - \min\{b, d\}$ such as to deal with a simpler game. Ignoring the trivial case in which a strategy is strictly dominated and the equally trivial case in which $x = y = 0$, we are left with the two cases captured in Game 2 (the meaning of the expressions $g_H(x, y)$ and $g_T(x, y)$ will be shortly explained) and Game 2′.

2)

		H_B	T_B
$g_H(x,y)$	H_A	$x, \cdot$	$0, \cdot$
$g_T(x,y)$	T_A	$0, \cdot$	$y, \cdot$

2′)

	H_B	T_B
H_A	$0, \cdot$	$y, \cdot$
T_A	$x, \cdot$	$0, \cdot$

In Game 2′ it is like if the payoffs associated with the two rows were switched with respect to Game 2. Assuming that $x \neq y$, B's beliefs about A playing strategy H_A in 2 have to be equal to B's beliefs about A playing strategy T_A in 2′. This leads to the statement of the following axiom:

[A2] Consistency with rows or columns switch.

The axiom indicates that the beliefs of player B consistently react to the payoff structure of the game and they do not depend on strategies labels. In particular A2 not only requires that, whenever $x \neq y$, B's beliefs about A playing strategy H_A in 2 must be equal to B's beliefs about A playing T_A in 2′. It also requires that B's beliefs remain the same in front of a column switch. In fact a column switch does not change the relative attractiveness of strategies H_A and T_A from player A's point of view.

These considerations imply that we can restrict our attention to Game 2 without loss of generality. Therefore we indicate with:

- $g_H(x, y)$ the belief of player B about player A playing strategy H_A in Game 2.
- $g_T(x, y)$ the belief of player B about player A playing strategy T_A in Game 2′.

According to us, a belief function must also obey to the following axioms:

[A3] Consistency with probability distribution: $g_H(x,y) \geq 0$, $g_T(x,y) \geq 0$ and $g_H(x,y) + g_T(x,y) = 1$ for any x and any y.

Axiom 3 states a very basic property a belief function must have, namely that it has to identify a meaningful and complete probability distribution. Note that, because of the relationship $g_H(x,y) + g_T(x,y) = 1$, a single probability is enough to define the entire distribution. Therefore A3 allows us to focus just on $g_H(x,y)$. Combining A3 with A2 we can also derive the relation $g_H(x,y) + g_H(y,x) = 1$ which we will later use.

[A4] Symmetry: $g_H(x,x) = \frac{1}{2}$, for any x.

[A5] Dominance: $g_H(0,y) = 0$, $g_H(x,0) = 1$, for any $x > 0$ and any $y > 0$.

Axioms 4 and 5 restrict the behavior of the beliefs function for some peculiar values of the payoffs x and y. A4, which is partly derived from A3, states that the function has to assign a uniform prior to player B whenever A's strategies look the same. Even if A may still have idiosyncratic preferences over his two pure strategies, these cannot be anticipated by B. Axiom 5 implies that players are able to recognize a weakly dominated strategy and that they assign a null probability to the event of the opponent playing such a strategy. The same holds a fortiori for strictly dominated strategies. A5 is thus in line with basic rationality assumptions. In 2x2 one shot games the majority of players seem to be able to recognize and eliminate dominated strategies (Roth, 1995).

[A6] Continuity: $g_H(x,y)$ is a continuous function of both x and y.

[A7] Monotonicity: $g_H(x_1,y) > g_H(x_2,y)$ if $x_1 > x_2$, $g_H(x,y_1) < g_H(x,y_2)$ if $y_1 > y_2$.

[A8] Homogeneity of degree zero: $g_H(kx,ky) = g_H(x,y)$, for any $k > 0$.

These three axioms list some more general properties that must characterize the function g_H. Continuity (A6) is required since there are no evident reasons for B's beliefs to jump in a discrete way given small changes in the arguments of the function. The monotonicity axiom (A7) defines the sign of the already mentioned own payoff effect. It states that players believe their opponent are attracted by strategies that look better. This implies that if the payoffs associated with strategy H_A increase so does the probability that player B assigns to the event of A playing that strategy. The axiom therefore requires

$g_H(x,y)$ to be strictly increasing in x and decreasing in y. This requirement is in line with a large experimental evidences (among others Ochs, 1995; Goeree and Holt, 2001; Goeree et al., 2003). Players' beliefs have thus to respond to any change in the payoff structure with the exception of a rescaling of the payoffs given that such a transformation would not modify the relative attractiveness of the strategies. The homogeneity of degree zero axiom (A8) formalizes this requirement.

1.2.1 Some useful properties of the beliefs function

If a function satisfies axioms A1-A8 then it also fulfills some other more specific requirements. An important relation is easily obtained. Consider Game 2 and start from the equality stated by A3, $g_H(x,y)+g_T(x,y)=1$. To avoid triviality we assume that at least a payoff is strictly positive and, using A2, we can actually assume that $y>0$. Now, because of homogeneity of degree zero (A8) every argument of g_H and of g_T can be multiplied by $\frac{1}{y}>0$. Let $z:=\frac{x}{y}$ with $z\geq 0$ to get $g_H(z,1)+g_T(z,1)=1$. Invoking again A8 multiply the arguments of g_T by $\frac{1}{z}$ to get $g_H(z,1)+g_T(1,\frac{1}{z})=1$. Because of A2 and A3 (remember the relation $g_H(x,y)+g_H(y,x)=1$) the term $g_T(1,\frac{1}{z})$ is equivalent to $g_H(\frac{1}{z},1)$ so that $g_H(z,1)+g_H(\frac{1}{z},1)=1$ must hold.

Let $f_{g_H}(z):=g_H(z,1)$ for any $z\geq 0$ such that the last relation can be rewritten as:

$$f_{g_H}(z)=1-f_{g_H}\left(\frac{1}{z}\right) \tag{1}$$

Lemma 1.1 focuses on the relationship between the functions g_H and f_{g_H}.

Lemma 1.1 *a) If $f_{g_H}:[0,\infty]\rightarrow[0,1)$ s.t. $f_{g_H}(z)=1-f_{g_H}\left(\frac{1}{z}\right)$, $f_{g_H}(0)=0$, $f_{g_H}(1)=\frac{1}{2}$, f_{g_H} is strictly increasing on $[0,1]$ and homogeneous of degree 0 then $g_H(x,y):=f_{g_H}(\frac{x}{y})$ satisfies axioms A1-A8.*

b) If g_H satisfies axioms A1-A8 then f_{g_H} defined by $f_{g_H}(z):=g_H(z,1)$ is such that $f_{g_H}(z)=1-f_{g_H}\left(\frac{1}{z}\right)$, $f_{g_H}(0)=0$, $f_{g_H}(1)=\frac{1}{2}$ and f_{g_H} is strictly increasing on $[0,1]$.

Proof. a) Given that $f_{g_H}(z):=g_H(z,1)$, then if f_{g_H} is homogeneous of degree 0 we can write $f_{g_H}(z):=g_H(kz,k)$ for any $k>0$. Therefore the original belief function g_H can be reconstructed starting from $f_{g_H}(z)=1-f_{g_H}\left(\frac{1}{z}\right)$ and following backwards the steps used to get (1) such that $g_H(x,y)$ satisfies the axioms.

b) The proof replicates the steps used to derive (1). Transformations applied to g_H in order to get f_{g_H} are innocuous, therefore the axioms are still valid and they just need to be restated. $f_{g_H}(0) = 0$ indicates f_{g_H} obeys dominance, $f_{g_H}(1) = \frac{1}{2}$ refers to symmetry and the requirement of f_{g_H} being strictly increasing on $[0, 1]$ is equivalent to the monotonicity axiom about g_H. ■

Relation 1 implies that player B assigns the same probability to the event of player A playing strategy H_A in Game 3 (belief indicated by $f_{g_H}(z)$) and to the event of A playing strategy T_A in Game 4 (belief indicated by $1 - f_{g_H}(\frac{1}{z})$) shown below.

3)

		H_B	T_B
$f_{g_H}(z)$	H_A	$z, \cdot$	$0, \cdot$
	T_A	$0, \cdot$	$1, \cdot$

4)

		H_B	T_B
	H_A	$\frac{1}{z}, \cdot$	$0, \cdot$
$1 - f_{g_H}(\frac{1}{z})$	T_A	$0, \cdot$	$1, \cdot$

Notice that in games 3 and 4 only the variable z appears. Moreover, because of (1), to know $f_{g_H}(z)$ means to know $f_{g_H}\left(\frac{1}{z}\right)$. Given the one to one relation between any $z \in [0, 1]$ and the reciprocal $\frac{1}{z} \in [1, \infty)$, the analysis of a function that obeys (1) can be restricted to the partial domain $z \in [0, 1]$. It is therefore much more practical to study the beliefs function f_{g_H} that refers to Game 3 rather than the original function g_H in Game 2.

In what follows our main object of study will be the function f_{g_H}, i.e. the beliefs of player B about player A playing strategy H_A in Game 3.

Some other properties of f_{g_H} are implied by the axioms and by (1). In particular the function f_{g_H} is continuous both at $z = 1$ ($\lim_{z \to 1^-} f_{g_H}(z) = \lim_{z \to 1^-} f_{g_H}\left(\frac{1}{z}\right) = \frac{1}{2}$ by Lemma 1.1) and at $z = 0$ ($\lim_{z \to 0} f_{g_H}(z) = 0$ and $f_{g_H}(0) = 0$ by Lemma 1.1) and $\lim_{z \to 0} f_{g_H}(\frac{1}{z}) = 1$ ($\lim_{z \to 0} f_{g_H}(z) = 0$ and A3). Moreover, as mentioned before, there is a one to one relationship between any z and the correspondent $\frac{1}{z}$ and therefore between $f_{g_H}(z)$ and $f_{g_H}\left(\frac{1}{z}\right)$.

$$f_{g_H} : [0, 1] \to \left[0, \frac{1}{2}\right] \Leftrightarrow f_{g_H} : [1, \infty) \to \left[\frac{1}{2}, 1\right) \tag{2}$$

The following propositions are particularly important in the task of describing as precisely as possible the beliefs function because they investigate the curvature of f_{g_H}. Proposition 1.2 does not require f to be differentiable, but if differentiability is assumed then sharper results can be proven (Proposition 1.3).

Proposition 1.2 f_{g_H} *is not convex.*

Proof. For any $z \in (0,1)$ take the linear combination between z and $\frac{1}{z}$ such that $\hat{\alpha}z + (1-\hat{\alpha})\frac{1}{z} = 1$ so that $\hat{\alpha} = \frac{1}{z+1}$. Then, by A2, we know that $f_{g_H}(\hat{\alpha}z + (1-\hat{\alpha})\frac{1}{z}) = f_{g_H}(1) = \frac{1}{2}$. Compare it with $\hat{\alpha}f_{g_H}(z) + (1-\hat{\alpha})f_{g_H}\left(\frac{1}{z}\right)$ which, using (1), can be expressed as $\frac{1}{z+1}f_{g_H}(z) + \frac{z}{z+1}\left[1 - f_{g_H}(z)\right]$ and thus as $f_{g_H}(z)\left(\frac{1-z}{1+z}\right) + \frac{z}{z+1}$. We show that the relation defining concavity holds: $f_{g_H}\left(\hat{\alpha}z + (1-\hat{\alpha})\frac{1}{z}\right) > \hat{\alpha}f_{g_H}(z) + (1-\hat{\alpha})f_{g_H}\left(\frac{1}{z}\right)$ which in this specific case means $\frac{1}{2} > f_{g_H}(z)\left(\frac{1-z}{1+z}\right) + \frac{z}{z+1}$. This last condition simplifies to $f_{g_H}(z) < \frac{1}{2}$ which is satisfied given that $z \in (0,1)$ and the monotonicity axiom. It follows that the function $f_{g_H}(z)$ is concave at least over part of its domain and thus it cannot be strictly convex (and not even linear). ■

Proposition 1.3 *If f_{g_H} is twice differentiable in the neighborhood of $z = 1$ then f_{g_H} is strictly concave at $z = 1$.*

Proof. Differentiate twice with respect to z the relation $f_{g_H}(z) = 1 - f_{g_H}\left(\frac{1}{z}\right)$ to get the first (3) and second (4) derivatives.

$$f'_{g_H}(z) = \frac{1}{z^2}f'_{g_H}\left(\frac{1}{z}\right) \tag{3}$$

$$f''_{g_H}(z) = -\frac{2}{z^3}f'_{g_H}\left(\frac{1}{z}\right) - \frac{1}{z^4}f''_{g_H}\left(\frac{1}{z}\right) \tag{4}$$

Evaluating (4) at $z = 1$ we get $f''_{g_H}(1) = -f'_{g_H}(1)$. Given that, because of monotonicity, $f'_{g_H}(1) > 0$ it follows that $f''_{g_H}(1) < 0$ and the function is strictly concave at $z = 1$. ■

1.2.2 Bounds on the function f_{g_H}

The axioms and the derived properties imply a rather specific behavior of the beliefs function. A graphical description of the bounds that restrict f_{g_H} appears in Figure 1. In what follows we let $z \in [0,1]$ and therefore $\frac{1}{z} \in [1, \infty)$.

Axioms 4 and 5, when restated according to Lemma 1.1, provide the starting point: given that $f_{g_H}(0) = 0$ and $f_{g_H}(1) = \frac{1}{2}$ the function has to pass through points $a = (0,0)$ and $b = \left(1, \frac{1}{2}\right)$. However, this is not enough to identify the function given that f_{g_H} is not linear (see proof of Proposition 1.2). Still the fact that $f_{g_H}(1) = \frac{1}{2}$ together with the monotonicity axiom implies that $f_{g_H}(z) \in \left[0, \frac{1}{2}\right]$ and $f_{g_H}\left(\frac{1}{z}\right) \in \left[\frac{1}{2}, 1\right)$.

For what concerns the curvature of the function we know that, assuming the function to be twice differentiable (as we do), f_{g_H} has to be strictly concave at $z = 1$ (Proposition 1.3). Note that $z = 1$, given its mirroring properties captured by (1), is the only peculiar point of the domain, i.e. the point in which a change in the sign of the second derivative may have been expected. Moreover f_{g_H} has also to be concave at least in some part of the domain for $\frac{1}{z} \to \infty$ because it is strictly increasing but bounded above by 1. Because of these two facts and in order to find a function which is as simple as possible[4], we require f_{g_H} to be strictly concave over all the domain.

The concavity requirement provides a lower bound for the function in the interval $[0, 1]$. In fact for any $z \in [0, 1]$ then $f_{g_H}(z) \geq \frac{1}{2}z$ has to hold where $\frac{1}{2}z$ is the equation of the line that connects points $a = (0, 0)$ and $b = \left(1, \frac{1}{2}\right)$. Since $f_{g_H}(z) = 1 - f_{g_H}(\frac{1}{z})$ has to be always valid, this lower bound becomes an upper bound in the interval $[1, \infty)$ where $f_{g_H}(\frac{1}{z}) \leq 1 - \frac{1}{2}z$ has to hold.

Finally notice that for any $\frac{1}{z} \in [1, \infty)$, and given that $f_{g_H}(\frac{1}{z}) < 1$, the condition $f_{g_H}(\frac{1}{z}) < \frac{1}{z}$ holds and the same condition holds for any $z \in \left(\frac{1}{2}, 1\right]$ as well. We require this condition to be valid also for any $z \in \left(0, \frac{1}{2}\right]$ and thus $f_{g_H}(z) < z$, for any $z \neq 0$. This last assumption implies two things: 1) $f_{g_H}(z)$ increases less than proportionally with respect to z and 2) f_{g_H} approaches the upper limit 1 not too slow. In fact, given that $f_{g_H}(z) \leq z$, the lower bound $f_{g_H}(\frac{1}{z}) \geq 1 - z$ in the interval $\frac{1}{z} \in [1, \infty)$ is directly derived from (1).

Figure 1 provides a graphical representation of the results of this section. The four thin lines define the two corridors in which the function f_{g_H} has to develop with the additional constraints that f_{g_H} has to pass through points $(0, 0)$ and $\left(1, \frac{1}{2}\right)$ and be strictly increasing. These restrictions do not identify a unique function. Indeed any function that stays within the bounds could be used to approximate players' beliefs as stated by the following proposition.

Proposition 1.4 *For any f_{g_H} s.t. $f_{g_H}(z) = 1 - f_{g_H}(\frac{1}{z})$, f is strictly increasing on $[0, 1]$ and $\frac{1}{2}z \leq f_{g_H}(z) \leq \min\left\{z, \frac{1}{2}\right\}$, for any $z \in [0, 1]$ then g_H defined as $g_H(x, y) := f_{g_H}(z)$ satisfies the axioms.*

Proof. If f_{g_H} satisfies (1), it is strictly increasing on $[0, 1]$ and it stays within the bounds then it means that f_{g_H} satisfies Lemma 1.1 and thus $g_H(x, y) := f_{g_H}(z)$ satisfies the axioms. ■

[4] Given that the beliefs retrieved through f_{g_H} will be used to predict the outcome of 2x2 games, a simple functional form is a valuable quality. Indeed our aim is to identify the simplest function among those that satisfy all the axioms and the derived properties.

Among all these possible functions we now turn our attention to the one which appears in bold in Figure 1. This function is the minimax regret.

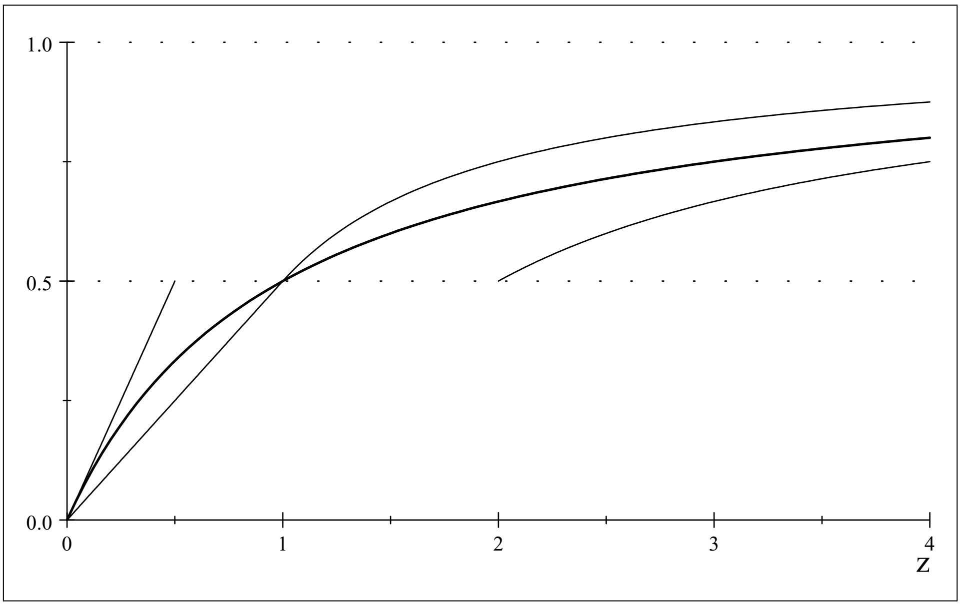

Figure 1: lower and upper bounds for f_{g_H} vs. the minimax regret proposal.

1.3 The proposed belief function: the minimax regret

In what follows we present a specific belief function which is based on minimax regret. More precisely we claim that the beliefs of player i about what player j will play can be approximated by the minimax regret of player j. We start introducing the concept of minimax regret and then we show that the proposed belief function satisfies all the axioms.

Minimax regret, originally proposed by Savage (1951), is a concept which found its main applications as a selection criterion in decision theory (starting with Milnor, 1954). More recently minimax regret has also been used in modeling the behavior of subjects with limited rationality (for instance Bergermann and Schlag, 2005, for the case of boundedly rational monopolists) as well as a way to deal with missing data in econometrics (Manski, 2005) and it also appears in the artificial intelligence literature (Brafman and Tennenholtz, 2000). The minimax regret criterion prescribes a player who has to make a decision under uncertainty to choose the action that minimizes his expected regret. The regret is defined as the difference between the best payoff player A could have got if he knew what his opponent (a player or Nature) had played and the payoff the player actually got. In fact the first step to compute the minimax regret consists in building the regret matrix which captures these differences. In the specific case of Game 3, which remains the game under study, and given that $z > 0$, the regret matrix is given by R_3:

3) $f_{g_H}(z)$		H_B	T_B
	H_A	$z, \cdot$	$0, \cdot$
	T_A	$0, \cdot$	$1, \cdot$

R_3)				
	H_A	$0, \cdot$	$1, \cdot$	(p_r)
	T_A	$z, \cdot$	$0, \cdot$	$(1 - p_r)$

Strategy H_A attains minimax regret in pure strategies for any $z > 1$ while strategy T_A attains minimax regret for $z \in [0, 1)$. Taking this specification as a belief function would clearly be unsatisfactory since the pure version of the minimax regret fails both the continuity and the monotonicity axioms. The use of mixed strategies solves this problem. Note that, by construction, in any 2x2 game, the regret matrix contains at least two zeros, a feature that makes the computation of the minimax regret very easy.

To find the minimax regret means to find the probability distribution (defined by $\hat{p}_r$, where the index r indicates regret) that equalizes the expected regret of the two strategies so that player A is indifferent in playing H_A or T_A. This optimal $\hat{p}_r$ solves $p_r\,(1) = (1 - p_r)\,z$ so that $\hat{p}_r = \frac{z}{z+1}$. According to the conjecture of this paper $f_{g_H} = \hat{p}_r$ should hold and thus:

$$f_r = \frac{z}{z+1}$$

Once again, referring to Game 3 above, this means that player B approximately believes player A to play strategy H_A with probability $\frac{z}{z+1}$ and strategy T_A with the complementary probability of $\frac{1}{z+1}$. This candidate function obeys all the axioms and the derived properties and assumptions as shown by the following proposition.

Proposition 1.5 *The minimax regret belief function f_r:*

i) satisfies axioms A1-A8,

ii) satisfies (1), and

iii) is differentiable and strictly concave.

Proof. i) Trivial for the first few axioms. In particular: $f_r(z) \in [0, 1]$ for any z (A3), $f_r(1) = \frac{1}{2}$ (A4), $f_r(0) = 0$ (A5), f_r is continuous (A6) as well as strictly increasing (A7) in z and it is homogeneous of degree zero in the payoffs (A8). For what concerns A2 simple computations show that f_r is indeed insensitive to column switch (3′) and sensitive to row switch (3″) as shown below.

3′)		H_B	T_B	
	H_A	$0,\cdot$	$z,\cdot$	$f_r(z) = \frac{z}{z+1}$
	T_A	$1,\cdot$	$0,\cdot$	

3″)		H_B	T_B	
	H_A	$0,\cdot$	$1,\cdot$	$f_r(z) = \frac{1}{z+1}$
	T_A	$z,\cdot$	$0,\cdot$	

ii) Compute $f_r\left(\frac{1}{z}\right) = \frac{\frac{1}{z}}{\frac{1}{z}+1} = \frac{1}{z+1}$ and verify that $f_r(z) + f_r\left(\frac{1}{z}\right) = \frac{z}{z+1} + \frac{1}{z+1} = 1$.

iii) The second derivative is given by $f_r''(z) = \frac{-2}{(z+1)^3}$ which is defined and strictly negative for any $z \in [0, \infty]$. ■

The minimax regret proposal is effective also in games that have strictly dominated strategies. In fact in these cases the function f_r assigns to a player a null probability to the event of his opponent playing a strictly dominated strategy.

As explained in the previous section we cannot claim the minimax regret to be the unique function that satisfies all the axioms and the additional requirements. It is however an advantage that an already existing concept (though normally used for different purposes) may be used to approximate players' beliefs. In this way in fact there is no need to invoke new definitions or *ad hoc* formulas. Moreover the following proposition underlines an appreciable feature of f_r, namely that, in the family of functions that are a ratio of linear functions, it is the unique one to obey all the axioms and derived properties.

Proposition 1.6 *If g_H satisfies A1-A8 and $f_{g_H}(z) = \frac{az+b}{cz+d}$ then $f_{g_H}(z) = \frac{z}{z+1} = f_r(z)$.*

Proof. If g_H satisfies A1-A8 then, because of Lemma 1.1b, $f_{g_H}(0) = \frac{b}{d} = 0$ so that $b = 0$ and $f_{g_H}(1) = \frac{a}{c+d} = \frac{1}{2}$ so that $2a = c+d$. Moreover $f_{g_H}(z) + f_{g_H}(\frac{1}{z}) = \frac{az}{cz+d} + \frac{a}{c+dz} = 1$ also holds and the last expression can be restated as $(z^2+1)(ad-cd) + z(2ac - c^2 - d^2) = 0$ for any z. Consider for instance the cases of $z = \frac{1}{2}$ and $z = \frac{1}{4}$. If $z = \frac{1}{2}$ then 1) $\frac{5}{4}(ad-cd) + \frac{1}{2}(2ac - c^2 - d^2) = 0$. If $z = \frac{1}{4}$ then 2) $\frac{17}{16}(ad-cd) + \frac{1}{4}(2ac - c^2 - d^2) = 0$. Subtract 2) from 1) to get 3) $\frac{1}{4}(2ac - c^2 - d^2) = -\frac{3}{16}(ad-cd)$. Substitute 3) in 2) to get $\frac{17}{16}(ad-cd) - \frac{3}{16}(ad-cd) = \frac{14}{16}(ad-cd) = 0$, i.e. $ad - cd = 0$. This condition is verified if $d = 0$ or if $a = c$. If $d = 0$, and given that $2a = c+d$, then $f_{g_H}(z) = \frac{1}{2}$ for any z which does not satisfy Lemma 1.1b because f_{g_H} would not be strictly increasing. If $a = c$, and given that $2a = c+d$, then we must have $a = c = d$. This simplifies to our formulation: $f_{g_H}(z) = \frac{az+b}{cz+d} = \frac{az}{cz+d} = \frac{az}{az+a} = \frac{z}{z+1} = f_r(z)$. ■

1.4 Other candidate concepts

In the previous section the mixed version of the minimax regret has been shown to obey all the axioms and derived properties required for approximating players' beliefs in a 2x2 game, being at the same time very simple in its functional form.

In this section we check the compliance to the axioms of other existing concepts commonly used in game theory and decision theory. Two categories are recognizable among these candidates. The first one collects proposals that are connected with the concept of mixed strategy Nash equilibrium (subsections 1.4.1 and 1.4.2), the second one considers criteria which mainly find application in decision theory, namely the maxmin (1.4.3) and the Laplace (1.4.4) criteria. Subsection 1.4.5 considers the hypothesis that beliefs may be captured by a logit specification. In line with what has been done for the minimax regret we keep on referring to Game 3, considering how these candidate functions perform in approximating the beliefs of player B on what A will play. The table that appears in section 1.4.6 summarizes the results.

1.4.1 Mixed strategy Nash equilibrium of player A

According to this hypothesis player B believes player A randomizes over H_A and T_A following the probability distribution the mixed strategy Nash equilibrium (*msne* in what follows) attaches to player A. At first glance this may seem a good candidate given that mixed equilibria, out of many different interpretations[5], have also been considered as mimicking players' beliefs. However this proposal does not even pass the most basic requirements. First there is an issue of existence; in a 2x2 game a well defined *msne* exists only when there are no strictly dominant strategies. Second, and more important, the *msne* of player A depends by construction only on player B's payoffs given that the mix adopted by A has to make B indifferent among his strategies[6]. In other words, this criterion does not capture any own payoff effect: no matter how the payoffs of player A could change, A's distribution in the mixed equilibrium (and thus B's beliefs) remains the same as far as B's payoffs remain fixed.

[5] Cfr. section 3.2 in the book "A course on game theory" by Osborne and Rubinstein (1994, MIT press).

[6] Indeed the predictive power of such a beliefs formulation would be null. In fact any strategy in the support of the mixed equilibrium of B would be a best response.

1.4.2 Mixed strategy Nash equilibrium of player B

This alternative would imply that the probability distribution that the *msne* attaches to player B could be considered as B's beliefs about what A would play. The *msne* of player B still suffers from the problem of nonexistence in the presence of dominant strategies but it is indeed a function of the payoffs of player A. In order to assess the performance of this proposal, we apply it to Game 3, which is reproduced below.

3)

		(q)	$(1-q)$	
		H_B	T_B	
$f_{g_H}(z)$	H_A	$z,\cdot$	$0,\cdot$	(p)
	T_A	$0,\cdot$	$1,\cdot$	$(1-p)$

3″)

		(q)	$(1-q)$
		H_B	T_B
$f_{g_H}(z)$	H_A	$0,\cdot$	$1,\cdot$
	T_A	$z,\cdot$	$0,\cdot$

The probability distribution of the *msne* is defined by the $\hat{q}$ that solves $\hat{q}z + (1-\hat{q})\,0 = \hat{q}0 + (1-\hat{q})\,1$, i.e. $\hat{q} = \frac{1}{1+z}$. Then the beliefs of player B about A playing H_A should be captured by:

$$f_1 = \frac{1}{1+z}$$

This specification does not obey the monotonicity axiom since f_1 is decreasing in z, the opposite behavior with respect to the one prescribed by Axiom 7. To see how misleading this interpretation could be, consider as an example the case in which $z = 9$. In such a situation the *msne* of player B proposal would imply that player B believes A will play H_A with a probability of 0.1, clearly a counter intuitive indication. Indeed letting players best respond to these beliefs would lead to predictions which are often totally in contrast with experimental results.

To solve for this problem one may be tempted to approximate the beliefs of player B with the complement to 1 of f_1. The functional form for the beliefs function would then be $f_2 = 1 - \hat{q}$, where again $\hat{q}$ defines the probability distribution of the *msne* of player B. If applied to Game 3, this proposal leads to the following functional form:

$$f_2 = \frac{z}{1+z}$$

which indeed satisfies the monotonicity axiom. Actually this function identifies the same beliefs indicated by the minimax regret[7] and so, apparently, it should obey all the

[7] Gallice (2006) shows that in any 2x2 game where a well defined *msne* exists its probability distibution is either the same or the mirror image of the minimax regret distribution of the other player.

axioms. However it fails axiom 2. Consider for instance Game 3$''$ above, a game in which the payoffs of the two rows have been inverted with respect to Game 3. The probability distribution of the mixed equilibrium of B is again defined by $\hat{q} = \frac{1}{1+z}$. It follows that player B's beliefs on A playing H_A would then be the same as before, i.e. $f_2 = \frac{z}{z+1}$, a violation of Axiom 2 for any $z \neq 1$.

1.4.3 Maxmin of player A

Are the strategies selected by the maxmin criterion a credible candidate for approximating players' beliefs? The pure version of the maxmin criterion predicts a player to choose the strategy which guarantees him the highest minimum payoff. As in the case of the minimax regret, the pure strategy formulation does not obey the continuity and monotonicity axioms. Allowing for mixed strategies the maxmin criterion assumes the player to mix over his strategies such as to maximize the expected minimum. Referring to Game 3 this would imply $f_3 = \hat{p}$ where $\hat{p}$ is such that $\hat{p}z = (1 - \hat{p})$ and thus

$$f_3 = \frac{1}{1+z}$$

In the context of Game 3 this alternative leads to the same functional form indicated by f_2. Therefore this specification does not satisfy the monotonicity axiom. More in general, in 2x2 games where all the payoffs are different from zero, the probability distributions implied by the maxmin criterion and by the *msne* are usually different. However they both continue to fail the monotonicity requirement.

1.4.4 Laplace

According to this possibility player B believes player A chooses the strategy to play following the Laplace criterion. This criterion assumes a player to best respond to uniform priors. The strategy to be chosen is then the one which has the highest sum of payoffs. In the case of Game 3, this means:

$$f_4 = \begin{cases} 1 \ \text{ if } \ z > 1 \\ 0 \ \text{ if } \ z < 1 \\ 0.5 \ \text{ if } \ z = 1 \end{cases}$$

Clearly this criterion fails both the continuity and the strict monotonicity axioms.

1.4.5 Logit rule

If the beliefs of player B about player A playing H_A in Game 3 were approximated using a logit rule, they would take the following analytical form:

$$f_5 = \frac{e^z}{e^z + e^1}$$

It is easy to see that this formulation fails Axiom 5 (dominance) given that $f_5(0) \neq 0$ as well as Axiom 8 (homogeneity of degree 0) given that $\frac{e^{kz}}{e^{kz}+e^k} \neq \frac{e^z}{e^z+e^1}$.

1.4.6 A summary

Table 1 summarizes the compliance to the axioms of the candidate functions which have been considered till now. The axioms are identified as: invariance to adding a constant to a column (1), consistency with rows or columns switch (2), consistency with probability distribution (3), symmetry (4), dominance (5), continuity (6), monotonicity (7), homogeneity of degree zero (8).

Criteria\Axioms	1	2	3	4	5	6	7	8
Minimax regret of pl. A (f_r)	y	y	y	y	y	y	y	y
Msne of pl. A	n	n	n	n	n	n	n	n
Msne of pl. B (f_1)	n	n	y[8]	y	n	y	n	y
1-Msne of pl. B (f_2)	n	n	y[9]	y	y	y	y	y
Maxmin (f_3)	n	y	y	y	n	y	n	y
Laplace (f_4)	y	y	y	y	y	n	n	y
Logit (f_5)	n	y	y	y	n	y	y	n

Table 1: compliance to the axioms of the candidate functions.

1.5 A procedure to forecast outcomes

We now present a procedure that indicates the outcomes most likely to be played by inexperienced agents involved in 2x2 games. The procedure simply computes players' best responses to the conjectured minimax regret beliefs. Let $S_i = \{H_i, T_i\}$ be the strategy

[8] The axiom is satisfied if there are no strictly dominant strategies.

[9] As before.

space and $u_i(s_i, s_j)$ the payoffs of the game for $i, j \in \{A, B\}$. The unique minimax regret distribution is given by $\{(\hat{p}_A H_A + (1-\hat{p}_A)T_A), (\hat{p}_B H_B + (1-\hat{p}_B)T_B)\}$ where $\hat{p}_i$ defines the probability with which player i should play strategy H_i in order to minimize his expected regret. With a slightly different notation with respect to the previous sections where only B's beliefs were considered, define now as $\beta_i = (\theta, 1-\theta)$ the beliefs player i holds on player j playing strategies H_j with probability θ and strategy T_j with probability $1-\theta$. $BR_i(\beta_i)$ is the best reply function of player i. It uses i's beliefs as an input and provides as an output the strategy i must choose in order to maximize his expected payoff.

The procedure

1. Compute the minimax regret distribution for the two players and retrieve $\hat{p}_A$ and $\hat{p}_B$.

2. Assign the following beliefs to the two players:

   - $\beta_A = (\hat{p}_B, (1-\hat{p}_B))$
   - $\beta_B = (\hat{p}_A, (1-\hat{p}_A))$

3. Let the two players choose the strategy to play according to BR_i:

   - $BR_i(\beta_i) = \begin{cases} \{H_i\} \text{ iff } u_i(H_i|\beta_i) > u_i(T_i|\beta_i) \\ \{T_i\} \text{ iff } u_i(H_i|\beta_i) < u_i(T_i|\beta_i) \\ \{0.5H_i + 0.5T_i\} \text{ iff } u_i(H_i|\beta_i) = u_i(T_i|\beta_i) \end{cases}$

The procedure provides a forecast in three simple steps: it is enough to compute the minimax regret, use its probability distributions to approximate players' beliefs and choose for each player the strategy that best responds to these beliefs. In case the two strategies lead to the same expected payoff we assume players to uniformly randomize. Again we do not claim this procedure to be consciously used by players. What we claim is that, on average, the procedure is operationally valid, i.e. the majority of individuals play the game "as if" they were applying it.

1.6 Experimental evidences for matching pennies games

We apply the proposed procedure to MP games for which experimental results are available from other studies[10]. Given that the procedure aims to capture the behavior of inexperienced players the ideal data to test our conjecture come from experiments in which subjects played just once a single game (data are reported in Table 2, Section 1.6.1). Then, as a matter of comparison, we also consider data about the first round of repeated games provided that players were randomly matched in each round so that inter temporal effects are minimized (data appear in Table 4, Section 1.6.2).

1.6.1 One shot games

The first three games in Table 2 ($GH1$, $GH2$ and $GH3$) and the correspondent experimental results are taken from Goeree and Holt (2001). Each game was played only once by a different pool of 50 subjects. In the original paper the authors use these games to evaluate the predictive power of the mixed strategy Nash equilibrium. The last three games appear in Goeree and Holt (2004) who took them from Guyer and Rapoport (1972). In the original experiment 214 subjects were asked to play in a random order 244 games belonging to different typologies. Note two things about these last three games. First, the payoff structure is more complex and second, despite of the fact that games were one shot, the huge number of strategic situations that the players had to face makes the experiment less reliable for our purposes.

The last four columns of Table 2 are the important ones: in the fourth to last column we report $BR_i(\beta_i)$, the prediction of the procedure. The third to last column presents the experimental results in the form $a/b\ S_i$, where a is the number of players that chose strategy $S_i \in \{H_i, T_i\}$ and $b = 0.5N$ is the total number of row or column players.

The second to last column shows the hit rate which measures the performance of the prediction in forecasting actual behavior. The hit rate is a simple summary statistics which counts the number of hits, i.e. the proportion of player that chose the forecasted strategy. Therefore it ranges between 0% (all misses) and 100% (all hits). It is described in Verbeek (2004) and used for instance in Gneezy and Guth (2003). We present the hit rate only for the cases for which the procedure selects a single strategy. Still notice that, in games in which the procedure indicates that subjects should uniformly randomize (like in $GH1$), the actual data match a uniform distribution even though the population is small.

[10] With respect to the original papers strategies will be renamed in order to be consistent with previous sections.

Finally in the last column of Table 2 we test for the validity of our conjecture, i.e. we test the hypothesis of the procedure being able to ex ante predict the strategies that are overplayed. Therefore we compare the actual data of the experiments with the alternative situation of a uniform distribution of play. Using a one side test, we thus test if the proportion of players that plays $BR_i(\beta_i)$ is significantly greater than 50%. To do so we use the Fisher's exact probability test which calculates the difference between the data observed and an alternative data distribution. Therefore when our procedure selects a single strategy we would expect the null hypothesis (observed data not being significantly different from the uniform distribution) to be rejected. In other words we would expect the (one sided) p-value that appears in the last column to be below 5%. An asterisk next to the p-value indicates that this is the case.

Game N		H_B	T_B	Notes	Procedure selects	Exper. results	Hit rate	Fisher p-values
$GH1$	H_A	$80, 40$	$40, 80$	1 shot	$\frac{1}{2}H_A+\frac{1}{2}T_A$	12/25 H_A	-	-
50	T_A	$40, 80$	$80, 40$		$\frac{1}{2}H_B+\frac{1}{2}T_B$	12/25 H_B	-	-
$GH2$	H_A	$320, 40$	$40, 80$	//	H_A	24/25 H_A	96%	∗0.04%
50	T_A	$40, 80$	$80, 40$		T_B	21/25 T_B	84%	∗1.6%
$GH3$	H_A	$44, 40$	$40, 80$	//	T_A	23/25 T_A	92%	∗0,18%
50	T_A	$40, 80$	$80, 40$		H_B	20/25 H_B	80%	∗3,6%
$GR4$	H_A	$24, 5$	$5, -10$	1 shot	H_A	91/107 H_A	85%	∗0%
214	T_A	$26, 9$	$-10, 26$	244 g.	H_B	85/107 H_B	79%	∗0%
$GR5$	H_A	$15, 5$	$5, -10$	//	H_A	82/107 H_A	77%	∗0%
214	T_A	$26, 9$	$-10, 26$		H_B	81/107 H_B	76%	∗0,01%
$GR6$	H_A	$9, 5$	$5, -10$	//	$\frac{1}{2}H_A+\frac{1}{2}T_A$	74/107 H_A	-	-
214	T_A	$26, 9$	$-10, 26$		T_B	32/107 T_B	30%	0,17%

Table 2: the hit rate of the procedure in one shot matching pennies games.

To have a better feeling of how the procedure works in practice consider a couple of examples. Game $GH1$ is a symmetric matching pennies game. The minimax regret is obviously $\frac{1}{2}H_i + \frac{1}{2}T_i$, for any $i \in \{A, B\}$ and thus the procedure assigns uniform beliefs to both players. Both strategies therefore lead to the same expected payoff and the procedure predicts all outcomes to be equally likely. Even if the population is quite small, actual frequencies confirm that the distributions of players' choices are as uniform as possible.

Things are different when the game is asymmetric like for instance in game $GH2$ where the payoff for players A in the outcome (H_A, H_B) has been modified. In these cases the minimax regret distribution remains the same for players B $\left(\frac{1}{2}H_B + \frac{1}{2}T_B\right)$ but it is now different for players A $\left(\frac{7}{8}H_A + \frac{1}{8}T_A\right)$. It follows that, according to our conjecture, an A player still has uniform beliefs about B while B's beliefs change. The procedure then selects strategy H_A as the most likely choice for players A. This strategy has an expected value of $\frac{1}{2}(320) + \frac{1}{2}(40) = 180$ which is larger than the expected value of T_A, $\frac{1}{2}(40) + \frac{1}{2}(80) = 60$. Strategy H_A was actually chosen by 24 out of the 25 row players. The mechanism is the same for players B; according to the conjecture they strongly believe (probability of $\frac{7}{8}$) that their opponents will play H_A. The procedure thus selects T_B as B's most likely strategy given that $\frac{7}{8}(80) + \frac{1}{8}(40) = 75 > 45 = \frac{7}{8}(40) + \frac{1}{8}(80)$. Strategy T_B was indeed chosen by 84% of B players.

The prediction of the procedure is confirmed also in Game $GH3$ where strategies T_A and H_B are the selected ones and the hit rate is again very high. The hit rate remains above 75% and the p-values are in line with our conjecture also in games $GR4$ and $GR5$ while results are less good in the case of Game $GR6$ in which the procedure failed to predict that column players over played strategy H_B. Again we stress that the last three games use data collected more than thirty years ago (1972), that they have a more complex structure involving also a substantially negative payoff and that the design of the experiment does not exactly fit our ideal framework of a single non repeated interaction.

Considering only the games where the procedure indicates a single outcome ($GH2$, $GH3$, $GR4$, $GR5$), the procedure correctly predicts the choices of 81% of the players. Note that in these cases the outcome selected is clearly not an equilibrium since a generic player A would always like to deviate. It may then seem that somehow players A act with a lower degree of rationality in comparison with players B[11]. However the behavior of the majority of both classes of players is consistent with the archetype of individuals that play as if they were best responding to the conjectured minimax regret beliefs.

[11] This obviously cannot be the case given the large number of subjects and the random allocation of players to roles.

1.6.1.1 A comparison with Nash equilibrium and maxmin prediction

We briefly present the mixed Nash equilibrium and the maxmin predictions (see Table 3) to show that they underperform our procedure. We just consider the first three games of Table 2 (Goeree and Holt, 2001). Both the *msne* and the maxmin prediction indicate a uniform distribution of choices for Game $GH1$, a prediction that matches the actual data. But performances become disappointing in the case of asymmetric games. In games $GH2$ and $GH3$ the Nash equilibrium is very good in capturing players B's proportions but, because of the already mentioned "no own payoff effect", it totally fails to predict that players A over play strategy H_A. The maxmin proposal fails in capturing the behavior of both classes of players.

Game N		H_B	T_B	Nash eq. prediction	Maxmin prediction	Exper. results
$GH1$	H_A	$80, 40$	$40, 80$	$\frac{1}{2}H_A+\frac{1}{2}T_A$	$\frac{1}{2}H_A+\frac{1}{2}T_A$	$12/25\ H_A$
50	T_A	$40, 80$	$80, 40$	$\frac{1}{2}H_B+\frac{1}{2}T_B$	$\frac{1}{2}H_B+\frac{1}{2}T_B$	$12/25\ H_B$
$GH2$	H_A	$320, 40$	$40, 80$	$\frac{1}{2}H_A+\frac{1}{2}T_A$	$\frac{1}{8}H_A+\frac{7}{8}T_A$	$24/25\ H_A$
50	T_A	$40, 80$	$80, 40$	$\frac{1}{8}H_B+\frac{7}{8}T_B$	$\frac{1}{2}H_B+\frac{1}{2}T_B$	$21/25\ T_B$
$GH3$	H_A	$44, 40$	$40, 80$	$\frac{1}{2}H_A+\frac{1}{2}T_A$	$\frac{10}{11}H_A+\frac{1}{11}T_A$	$23/25\ T_A$
50	T_A	$40, 80$	$80, 40$	$\frac{10}{11}H_B+\frac{1}{11}T_B$	$\frac{1}{2}H_B+\frac{1}{2}T_B$	$20/25\ H_B$

Table 3: the Nash and maxmin prediction in the GH games.

For what concerns the Nash equilibrium, the authors (Goeree and Holt, 2001) present the results for Game $GH1$ as supportive of the *msne* prediction, while they show the results of games $GH2$ and $GH3$ as evidences of its failure. Therefore they write that "*The Nash analysis seems to work only by coincidence, when the payoff structure is symmetric and deviation risks are balanced*"[12].

Analyzing the same results through the lens of our conjecture, it seems indeed that the fact that the Nash analysis works in game $GH1$ may be the result of a coincidence. But

[12] Goeree, J. & Holt, C. (2001), "Ten Little Treasures of Game Theory and Ten Intuitive Contradictions", *American Economic Review*, Vol. 91, pp. 1419.

this coincidence has an explanation. In symmetric MP games the probability distributions implied by the *msne* and by the minimax regret always coincide[13]. Still, as soon as you move to asymmetric cases, individuals' behavior is by far better captured by our behavioral model rather than by the Nash prediction.

1.6.2 First round of games with random matching

We now consider individuals' play in the first round of repeated games. We restrict our attention to games in which players were randomly matched each round such as to minimize inter temporal strategic effects (still players knew they were going to face a repeated game). Table 4, which has the same structure as Table 2, reports data about MP games played in this way. The first game has been studied by Nyarko and Schotter (2002). It has been played for 60 rounds and over four treatments to investigate the issue of beliefs learning. Game NSa reports the data of treatment 4 (random matching and no belief elicitation) which, in the original paper, served as a control treatment. Game NSb is equal to the previous one but the number of players is larger because data from treatment 4 and treatment 1 (random matching and belief elicitation) are pooled together. The last four games (MPW) have been studied by McKelvey, Palfrey and Weber (2000)[14]. In the original paper these games were played 50 times and data were used to test a version of the quantal response equilibrium that allows for heterogeneity among subjects.

The hit rate of the procedure remains well above 50% in many cases. Still the procedure seems to be less successful than in the cases of one shot games (see Table 2). Notice also that, mainly because of the small number of players involved, the p-values are often not significant at a 5% level, as indicated by the absence of the symbol $*$. Finally note that the procedure seems to work better in anticipating the behavior of B players (with an overall hit rate of 77% in all the MPW games, p-value of 0%) rather than the one of A players (hit rate of 55%, p-value of 8%). Even if these games are not directly comparable with the ones which were played one shot (see Table 2), the view that individuals' behavior is different (though still similar) between one shot interactions and first round of repeated games seems to be confirmed.

[13]See the example in the appendix; $k = 1$ identifies the unique point for which the functions for the minimax regret and for the *msne* intersect.

[14]I thank Roberto Weber for having sent me the original data set.

Game n		H_B	T_B	notes	Procedure selects	Exper. results	Hit rate	Fisher p-values
NSa	H_A	$6,2$	$3,5$	1^{st} round	$\frac{1}{2}H_A+\frac{1}{2}T_A$	$8/15\ H_A$	-	-
30	T_A	$3,5$	$5,3$	random m.	T_B	$13/15\ T_B$	87%	∗4,7%
NSb	H_A	$6,2$	$3,5$	1^{st} round	$\frac{1}{2}H_A+\frac{1}{2}T_A$	$11/29\ H_A$	-	-
58	T_A	$3,5$	$5,3$	r.m.+bel.el.	T_B	$23/29\ T_B$	79%	∗2,5%
$MPWa$	H_A	$9,0$	$0,1$	1^{st} round	H_A	$20/36\ H_A$	55%	16,7%
72	T_A	$0,1$	$1,0$	random m.	T_B	$29/36\ T_B$	80%	∗0,5%
$MPWb$	H_A	$9,0$	$0,4$	//	H_A	$14/24\ H_A$	58%	19,4%
48	T_A	$0,4$	$1,0$		T_B	$21/24\ T_B$	88%	∗0,5%
$MPWc$	H_A	$36,0$	$0,4$	//	H_A	$13/24\ H_A$	54%	21,8%
48	T_A	$0,4$	$4,0$		T_B	$17/24\ T_B$	71%	8,1%
$MPWd$	H_A	$4,0$	$0,1$	//	H_A	$6/12\ H_A$	50%	31,6%
24	T_A	$0,1$	$1,0$		T_B	$8/12\ T_B$	75%	23,3%

Table 4: the procedure in first round of repeated MP games with random matching.

1.7 The procedure in other games

The same procedure which until now has been applied only to MP games can also be used with other games. In fact the procedure provides predictions for all 2x2 games and it is good to know that the procedure selects a meaningful outcome, i.e. an outcome which has some theoretical foundations and which is confirmed by experimental evidences. In this respect the content of this section can be seen as a robustness check of our conjecture.

In any 2x2 game the steps to select the strategies more likely to be chosen by inexperienced players remain the same: compute the minimax regret, use its probability distribution to approximate players' beliefs and choose the pure strategies that best respond to these beliefs. Simple examples of a game with a single dominant strategy (SD), prisoner's dilemma

(PD), pure coordination games (PC), stag-hunt games (SH) and symmetric (BS) and asymmetric (aBS) battle of the sexes games are shown in Table 5. The claim about the effectiveness of the prediction still refers to one shot interactions.

Game				Minimax regret	Best response	Procedure selects	Notes
		H_B	T_B				
SD	H_A	$3,1$	$1,0$	$1H_A{+}0T_A$	H_A	(H_A, H_B)	Unique
	T_A	$1,0$	$0,2$	$\frac{1}{3}H_B{+}\frac{2}{3}T_B$	H_B		NE
PD	H_A	$3,3$	$0,5$	$0H_A{+}1T_A$	T_A	(T_A, T_B)	Unique
		$5,0$	$1,1$	$0H_B{+}1T_B$	T_B		NE
PC	H_A	$2,2$	$0,0$	$\frac{1}{3}H_A{+}\frac{2}{3}T_A$	T_A	(T_A, T_B)	Pareto dominant
	T_A	$0,0$	$4,4$	$\frac{1}{3}H_B{+}\frac{2}{3}T_B$	T_B		NE
SH	H_A	$2,2$	$3,0$	$\frac{2}{3}H_A{+}\frac{1}{3}T_A$	H_A	(H_A, H_B)	Risk dominant
	T_A	$0,3$	$4,4$	$\frac{2}{3}H_B{+}\frac{1}{3}T_B$	H_B		NE
BS	H_A	$3,1$	$0,0$	$\frac{3}{4}H_A{+}\frac{1}{4}T_A$	$\{H_A, T_A\}$	$(\cdot, \cdot)$	All outcomes
	T_A	$0,0$	$1,3$	$\frac{1}{4}H_B{+}\frac{3}{4}T_B$	$\{H_B, T_B\}$		equally likely
aBS	H_A	$5,1$	$0,0$	$\frac{5}{6}H_A{+}\frac{1}{6}T_A$	H_A	(H_A, H_B)	Payoff dominant
	T_A	$0,0$	$1,3$	$\frac{1}{4}H_B{+}\frac{3}{4}T_B$	H_B		NE

Table 5: the procedure applied to other classes of 2x2 games.

In games that have at least a Nash equilibrium (NE) in pure strategies, if the procedure selects a single outcome, then this outcome is always a NE of the game (SD, PD, PC, SH and aBS). However it may be the case that the procedure does not select any outcome (or better it selects them all), even if pure Nash equilibria exist. This is what happens in the case of symmetric battle of the sexes (BS) where the expected payoffs of the two strategies conditional on the conjectured beliefs are equal. Indeed, because of the tension between the preferences of the two players, experimental data confirm quite a dispersed distribution of choices. The situation is different in the asymmetric version of the game

(aBS) where, again in accordance with experimental evidences, the procedure selects the payoff dominant equilibrium. The reason is that a generic player B correctly believes that his opponent has stronger incentives in playing his preferred strategy.

Notice that the conjectured beliefs of the players sometimes happen to be incorrect in the sense that they are not in line with the strategies selected by the opponent (SD, BS, aBS). For instance in the SD game the row player expects his opponent to be biased toward playing strategy T_B but indeed, according to the procedure, player B plays strategy H_B. We do not perceive this to be a problem. In fact we axiomatized the beliefs of inexperienced, unsophisticated and boundedly rational players and therefore the possibility that in some cases the procedure allocates to players incorrect beliefs was embedded in our model since the beginning. What matters is that the players actually play as if they were best responding to the minimax regret beliefs such that the prediction of the procedure is in line with existing evidences. In the case of the SD game for instance, player A chooses his strictly dominant strategy (which is a best response to any possible belief) and player B best responds choosing H_B.

It is also interesting to note how the procedure performs in the case of coordination games. In accordance with theory, intuition and experimental results the Pareto dominant NE is the outcome selected in pure coordination games (PC). More controversial is the indication in stag hunt games (SH) where players face a trade off between an unsafe, but potentially more rewarding, strategy and a safer one. Such games have therefore two Nash equilibria in pure strategies: a Pareto dominant one (more rewarding) and a risk dominant one (less risky). The latter is the one indicated by the procedure. For this class of games the experimental evidence is mixed (see for instance Harsanyi and Selten, 1988; Straub, 1995; Haruvy and Stahl, 2004) but there is a prevailing consensus indicating indeed the risk dominant equilibrium. With this respect and because of its capacity to select a single outcome, the suggested procedure can also be considered as a tool for equilibrium selection in games with multiple equilibria.

1.8 Conclusion

2x2 one shot games remain a fundamental tool for modeling strategic interactions. These games capture the simplest relations (the number of players and strategies is minimal) but still they can be used to describe an uncountable number of situations. In fact many real life interactions take place among two subjects and many decisions are binary in nature.

No wonder therefore that the study of 2x2 games has always attracted a lot of attention.

Theory provides elegant tools to individuate the equilibria of these games; these equilibria have often been given not only a normative interpretation (about how fully rational players should play) but also a positive one (about how, less rational, real players are indeed expected to play). The empirical relevance of these predictions may be weak, particularly for games in which players' interests are always in contrast (matching pennies). As a consequence to predict players' behavior in one off interactions remains a problematic issue.

This paper introduced a simple procedure to be used for forecasting the outcome of 2x2 one shot games. Using an axiomatic approach, we justified the use of minimax regret as a way to approximate the beliefs of inexperienced individuals. Then we let players behave as if they were responding to these conjectured beliefs.

A nice feature of the procedure is that, in selecting the strategies more likely to be played, it considers all the payoffs of the game. In fact the beliefs of generic player i are mimicked by the minimax regret probability distribution of the opponent j and thus they depend on the payoffs of the latter. Still, in computing best responses, also the payoffs of player i are taken into account. Traditional concepts like the Nash equilibrium and maxmin strategies do not display this "all payoff" effect being functions of only half of the payoffs of the game.

Indeed, when compared with existing experimental evidences about one shot matching pennies games, our procedure proved to be an effective tool in anticipating the moves of the vast majority of the players. Far from having fully solved the problem (for instance the performance of the procedure seems to be much lower in games with more than two strategies), we think that this paper may contribute to the study of individuals' behavior in one shot games.

1.9 Appendix

1.9.1 Minimax regret beliefs: an example

Here we apply our conjecture about players' beliefs to a game that encompasses the cases of a matching pennies game and of a game with a dominant strategy. The example is meant to show how simple is the process to approximate beliefs, under stressing once more the inadequacy of proposals connected with the concept of mixed strategy Nash equilibria. Consider the following game where $k \in (-\infty, \infty)$.

$a)$

	H_B	T_B
H_A	$k, -1$	$-1, 1$
T_A	$-1, 1$	$1, -1$

$\underset{k\in(-1,\infty)}{R_1} =$

	H_B	T_B
H_A	$0, 2$	$2, 0$
T_A	$k+1, 0$	$0, 2$

$\underset{k\in[-\infty,-1)}{R_2} =$

	H_B	T_B
H_A	$-1-k, 2$	$2, 0$
T_A	$0, 0$	$0, 2$

For $k \in (-1, \infty)$ Game $a)$ is a matching pennies game. With $k = 1$ the game is in its standard version, with $k \neq 1$ the game is asymmetric. In both cases the regret matrix is given by R_1. The minimax regret mixed strategy for player A is given by $(\hat{p}_A H_A + (1 - \hat{p}_A) T_A)$ where $\hat{p}_A = \frac{1+k}{3+k}$ is the probability attached to strategy H_A and thus our candidate to approximate B's beliefs about A playing that strategy. The function for $\hat{p}_A$ appears as the bold concave curve in Figure 2 which focuses on the conjectured beliefs of player B about what A will play[15].

For $k \in (-\infty, -1]$ the game has a different structure since strategy H_A is dominated by T_A. The dominance is weak for $k = -1$ and strict otherwise. The regret matrix is given by R_2 and the minimax regret attaches probability 0 to A playing H_A. In Figure 2 this appears as the bold line that lies on the horizontal axis for $k \leq -1$.

The other two functions (thin lines) that appear in Figure 2 depict, respectively, the probability that the mixed strategy Nash equilibrium assigns to player A playing strategy H_A $\left(\frac{1}{2}\right)$ and to player B playing strategy H_B $\left(\frac{2}{3+k}\right)$. The figure thus highlights the already mentioned problems of these two concepts. The *msne* of player A does not respond to a

[15] For any $k \in (-\infty, \infty)$ the minimax regret of player B is $\left(\frac{1}{2}H_B + \frac{1}{2}T_B\right)$. According to our interpretation this implies that player A believes player B is equally likely to play any of his strategies no matter the specific value of k.

change in A's payoff (in this case k) while the *msne* of player B does respond to a change in k but not in the desired direction. Note also that the functions for the minimax regret and for the *msne* of the two players intersect just once. The intersection happens for the unique k (in this case $k = 1$, symmetric game) for which all the three functions reach a value of $\frac{1}{2}$.

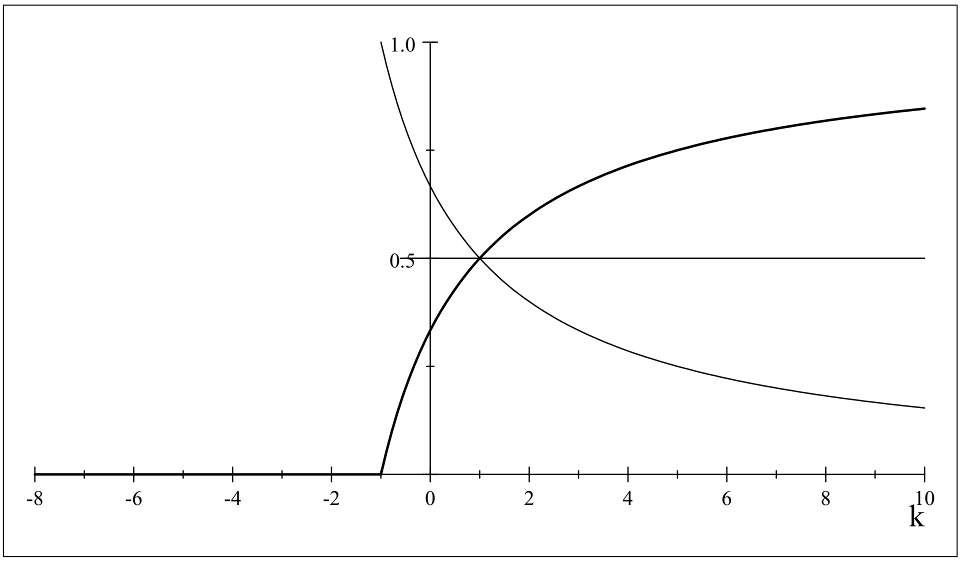

Figure 2: beliefs approximation through minimax regret in Game a.

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CHAPTER 2

EDUCATION, DYNAMIC SIGNALING AND SOCIAL DISTANCE

Abstract

The paper enriches a standard signaling model (Spence, 1973) with social preferences defined over educational achievements. In particular it considers the effects that the presence of conformist and status seeking individuals have on educational dynamics. Under very reasonable assumptions regarding the composition of the society, the model endogenously displays a growing average level of schooling. As education rises, signals get noisy and potentially misleading such that firms increase their educational requirements in order to adjust their screening process. All these dynamics are in line with recent trends and other stylized facts about education.

JEL classification: I20, D70, D82.

Keywords: education, signaling, status seeking, conformist behavior.

2.1 Introduction

In the last decades average educational level has been constantly increasing in almost every European country. Eurostat (2001) reports for instance that in 2001 over 75% of 25 to 29 year old had completed at least an upper secondary education compared with 52% of people aged 50-64. Or yet (Eurostat, 2005) that the total number of graduates in the EU25 increased by more than 30% between 1998 and 2003. The causes that have contributed in shaping such a massive phenomenon are obviously manifold: favorable sociopolitical situation, increasing wealth, sustained unemployment rates, rapid technological progress. In addition to these explanations (and possibly to many others which have not been mentioned), this paper shows that recent educational dynamics are easily accounted for by a simple signaling model (à la Spence, 1973) in which at least some of the individuals care about their relative position in the educational distribution.

There are two kinds of reasons because of which people may care about their relative level of education with respect to the rest of the society. First, there are material reasons connected with the probability of finding an adequate job in the labor market. It is a fact that, together with the increase of the general level of education, or better as a consequence to it, the "value" of a given level of schooling has decreased. Holding a bachelor's degree two generations ago was really a valuable asset as well as an effective signal to send to the labor market. Nowadays the same degree is much less informative and applicants, in order to impress the employers, need higher qualifications.

Second, there are social and psychological reasons. If education is a signal that workers send to the labor market, it is as well a signal individuals send to the rest of the society. Education may in fact affect an individual's social position. In many social contacts there is indeed a favorable bias towards more educated individuals and people are usually willing to interpret education as a proxy for respected qualities like brightness, knowledge and persistence. It is for instance common to feel admiration for those who got more than one degree while, at the opposite, individuals with a low level of education are often stigmatized, no matter their effective abilities. Different individuals may be more or less sensitive to these social influences[1] and they may react to them in different ways. Nevertheless, even in this second case, the signaling effect of a given level of schooling changes as average education in the society increases.

Therefore an individual who has to decide about his level of schooling must consider not only his absolute level but also, and possibly mainly, his relative one. This means that individuals' preferences about education are interdependent, a feature which is usually neglected by standard models of educational choices. An example that clarifies this double signaling effect (towards the labor market and towards the rest of the society) is the one of a student who gets a high grade in an important exam. His perception about this achievement is not independent from the average grade of the class. In fact, if the average is low, the signal the student sends to the teacher is more informative and thus it is more likely to help him in his scholastic career. At the same time the student sends a stronger signal also to his colleagues and his utility may be affected by his specific social preferences. For instance the student's ego could be gratified or, at the opposite, the student could feel some embarrassment in standing out so clearly.

This paper focuses on the choice about which level of education to reach, a choice that each individual has to make considering his innate characteristics as well as his ambitions. It combines a standard model of signaling (Spence, 1973) with theories of social distance. More precisely the paper analyzes educational dynamics in a population in which heterogeneous agents have different productivity as well as different attitudes about their

[1] For a specific empirical evidence about the consequences that social influences like peer effects and band wagon effects may have on educational choices see Cipollone and Rosolia (2004).

educational achievements. Some individuals, possibly the majority, are not influenced by social distance considerations (independent individuals), some try to differentiate themselves and reach a high status (status seeking individuals), some others adopt a more conformist behavior (conformist individuals). Moreover we depart from a standard rational analysis assuming that individuals are myopic in anticipating future levels of education in the society and that firms may hold erroneous beliefs about workers' real productivity.

The presence of non selfish[2] agents considerably modifies standard results of static signaling models. In particular, under very weak assumptions about the composition of the society, the model displays a growing level of average education. For instance education is likely to increase even in a purely conformist society provided that the average productivity level is not too low. It is shown anyway that the existence of agents with interdependent preferences, though necessary in the context considered, is not sufficient to trigger some positive educational dynamics. The model also rationalizes some other stylized facts. In fact, as average education increases, the signals the workers send to the labor market get less informative and potentially harmful for firms' profitability so that employers become progressively more demanding in setting their educational requirements. To the best of our knowledge, this paper is the first one to consider the effects that interdependent preferences defined over educational achievements may have on educational dynamics.

The paper is organized as follows: Section 2 reviews the relevant literature; Section 3 introduces and solves a repeated signaling game enriched with concerns about social distance. Section 4 studies the dynamics of the average educational level in the society. Section 5 analyzes the behavior of the firms. Section 6 concludes.

2.2 Related literature

Three are the elements combined in this paper. In fact, in order to provide an (additional) explanation for the *increase in education* of the last decades, we combine theories of *signaling* and theories of *social distance and interdependent preferences*. This section briefly reviews the relevant literature in these three areas.

Increase in education. Various are the reasons that can explain the recent substantial boost in education in developed countries. A first and pretty obvious explanation is that people study more because, in doing so, they expect to get higher wages. Indeed the positive relationship between schooling and earnings is generally confirmed (see, among

[2]Non selfish players are agents whose behavior is affected by the one of some other individual or reference group. See for instance Fehr and Fischbacher (2002).

many, Ashenfelter et al. (1999) or Psacharopoulos and Patrinos (2004)) by the estimation of Mincerian equations[3]. Related to wage expectations is the literature that focuses on the effects that skill biased technological progress had on the wage distribution (see for instance Heckman et al., 1998). The idea is that the wage gap between unskilled and skilled workers widened considerably, providing incentives to pursue a higher level of education.

From a social point of view the increase in education can be seen as an investment in human capital[4] which has beneficial effects on economic growth (Barro, 2001). Indeed many government policies have been designed with the scope of raising the educational level in the society: increase and diversification of the supply of education, subsidies and other incentives, increase of the years of compulsory schooling. On the demand side of the market for education, individuals face a trade off between studying and working. This trade off is affected by labor market conditions. For instance in periods of high unemployment the opportunity cost of proceeding in schooling is lower. Empirical evidences about this relationship appear in Giannelli and Monfardini (2000) and Dellas and Sakellaris (2003).

Signaling. The concept of signaling has been introduced by the seminal paper by Spence (1973) and rapidly became, from then on, an important branch of the highly dynamic economics of information. Riley (2001) presents a very rich overview of all the applications that signaling models found. Signaling is an attempt to solve problems of asymmetric information. In such a situation the informed agent may have incentives to adopt behaviors (signals) intended to reveal some of his unobservable characteristics to the uninformed party. In Spence's model the signal is the level of schooling an individual acquires. In fact education is assumed to convey information about unobservable and innate productivity.

Signaling models often predict a multitude of equilibria. In some of these equilibria individuals with different characteristics send different signals (separating equilibria), in others they adopt the same behavior such that the signal is uninformative and individuals remain indistinguishable (pooling equilibria). In order to get more precise predictions various concepts of equilibrium refinements have been proposed. The intuitive criterion (Cho and Kreps, 1987) proved to be the most successful one (for an example of this solution concept see section 3.1 below).

Social distance and interdependent preferences. Day to day life is plenty of evidences about individuals' preferences being somehow interdependent. Indeed to consider interde-

[3]After Jacob Mincer who was the first one to study the relation between labor income and schooling in 1958.

[4]The book by Becker (1993) offers a deep analysis of the effects that education, training and on-the-job experience may have on an individual's human capital.

pendent preferences means to acknowledge that sentiments like pride, esteem, shame or acceptance do affect agents' choices. The investigation of the effects that issues of social distance have on individuals' behavior started with the study of preferences defined over consumption. In fact people may care not only about their absolute level of consumption but also about their relative level. Individuals' decisions are then affected by social pressures. Duesenberry (1949) includes the average level of consumption into the utility function that individuals have to maximize so that a player increases his utility if she is able to "beat the average". A slightly different methodology has been followed by Pollak (1976) who models preferences that depend on other people's *past* consumption such that agents' behavior is not strategic and the model becomes analytically more tractable. From a more theoretical point of view the incentives that move status seeking individuals were first described in Frank (1985). Robson (1992) considers the situation in which people are interested in the ordinal rank they occupy in the distribution of wealth while Hopkins and Kornienko (2004) study the case in which utility is affected by the amount of general consumption as well as by the consumption of a particular good which defines the status.

When preferences are defined over the relative amount of consumption or wealth, it is natural to assume a tendency towards a status seeking behavior. Still, in other contexts, a more conformist attitude can be the rule rather than the exception. The classic work about conformism is Jones (1984) who studies examples of social influence like behaviors in a college environment, in the army and in the workplace. Focusing on the last example, Jones analyses the effort workers decide to exert. Given that extreme behaviors are stigmatized, slowest (fastest) members of the working group feel the pressure to speed up (slow down). Moreover new workers imitate the behavior of older colleagues. Evidences of these kinds of peer pressures come also from experiments (Falk and Ichino, 2003) while a more theoretical analysis of this tendency towards conformism is given by Bernheim (1994).

To sum up both status seeking and conformist individuals care about the social distance between them and some reference group. Because of this fact, externalities arise in both cases and thus the social efficiency of the final outcome is not ensured. Using simple formulations of utility functions, Akerlof (1997) shows that status seeking behavior usually leads to overindulge[5] in the status-producing activity (ex. over consumption). The outcome deriving from conformist behavior can instead range from underprovision to overprovision.

[5]In an older paper, Akerlof (1976) describes the so called rat-race mechanism. The probability of winning a prize increases with the effort the agent exerts; given that everyone tries to beat the others this mechanism leads to an overprovision of effort.

2.3 The model

2.3.1 A basic signaling model

Consider a population of potential workers[6]. Some of them, in proportion $\alpha_h \in (0,1)$, have a high productivity ($\theta_h = 2$), while the others, in proportion $\alpha_l = 1 - \alpha_h$, have a low productivity ($\theta_l = 1$). At least two firms compete in order to hire them. These firms are not able to distinguish between the two classes of workers given their observable characteristics. Then, if no signals were available, firms would offer a wage equal to the average productivity $\bar{w} = \bar{\theta} = 2\alpha_h + \alpha_l$, with $\bar{w} \in (1,2)$. A more efficient outcome can be achieved through signaling.

Assume that firms have some beliefs about workers' productivity, i.e. they think that there exists a certain level of education $\tilde{e}$, such that, if worker i acquires a level of education $e_i \geq \tilde{e}$, then the candidate must be highly productive. On the other side if $e_i < \tilde{e}$ then it must be the case that the individual has a low productivity. Given these beliefs and the specific costs of education (high types have a cost which is half of the cost of low types), the utility functions of the workers take the following form:

$$U_h(e) = 2 - \frac{1}{2}e^2 \qquad\qquad U_l(e) = 1 - e^2$$

Optimal educational levels have to be found subject to two incentive compatibility constraints which require high productivity individuals not to have any incentive in pretending to be low productivity types and vice-versa:

$$2 - \frac{1}{2}\tilde{e}^2 \geq 1 \qquad\qquad 1 \geq 2 - \tilde{e}^2$$

The constraints are satisfied by $\tilde{e} \in \left[1, \sqrt{2}\right]$. With any of these levels of education, different types of agents choose different levels of education (we restrict our attention to separating equilibria). In particular low productivity workers choose $e_l = 0$ and high productivity workers choose $e_h = \tilde{e}$. Among all these perfect Bayesian equilibria, the intuitive criterion (Cho and Kreps, 1987) selects the least cost separating equilibrium: low productivity workers get the minimum level of education ($e_l^* = 0$) while high productivity workers choose $e_h^* = 1$. This is the lowest possible level of education which cannot be profitably mimicked by low types. It follows that the average level of education in the society is given by $\bar{e} = \alpha_h$.

[6] The model is almost identical to the one presented by Spence (1973) and it provides the starting point for the analysis in the next sections.

2.3.2 A richer framework: signaling and social distance

Spence's signaling game covers only one period of time, i.e. it models the choices of a single generation of workers. Therefore its results (we focus on the least cost separating equilibrium, abstracting from other equilibria and off equilibrium play) are essentially static. We enrich the standard model such that it may display some dynamics in the average level of education of the society.

First notice that a repeated version of the game (subsequent cohorts of workers facing the one shot signaling game) would not be enough to create this effect. In fact, assuming average productivity to be equal among different generations, average education would remain constant over time at the level $\bar{e}_t = \alpha_h$. This is due to the fact that, in the standard model, firms' beliefs about workers productivity are self confirming[7] so that neither workers nor firms have any incentive to modify their strategies. In other words firms' initial beliefs about applicants' productivity match their actual productivity once they are hired so that firms have no reason to change their priors. Therefore any stage of the repeated game would be identical to (and independent from) the previous one.

This section introduces a repeated signaling game where the different stages are linked to each other because what players do does depend on what their predecessors did. This intertemporal link breaks the stationarity of the original model. In particular workers' behavior will not be constant over time, firms' beliefs will often happen to be incorrect and, as a result, the game will frequently be out of equilibrium.

What we add to Spence's original model is the presence in the population of some individuals that care about their relative educational level. This means that there are agents for which the level of education to acquire is a signal to send not only to the labor market but also to the rest of the society.

The basic structure of our stylized model consists in a signaling game repeated for two periods. In the first one ($t = 1$), considerations about social distance play no role and this stage exactly resembles a standard signaling model. In the second period ($t = 2$) a new generation of workers is involved in the same signaling game but now the choices of some of the players are influenced by social preferences. In particular, aside to standard individuals with independent preferences, we analyze the effects of two archetypes of agents: conformist individuals and status seeking individuals. We underline that players in the first and in the second period are different individuals belonging to different cohorts of workers.

[7] Indeed the original paper postulates these self-confirming beliefs of the firms exactly to avoid studying a non stationary system (cfr. Spence, 1973, pp. 360).

The link between the two periods is provided by the assumption that the players that in $t = 2$ care about their social position in the educational ladder are myopic. This means that their educational choice is influenced by the average level of education that emerged in $t = 1$. Forward looking individuals should base their decision about obtaining a certain degree on the average level that will arise at the time they will enter in the labor market. Nevertheless individuals' predictive ability may not be refined enough to adopt such a strategic behavior. The current average can then be used as a proxy for the future level. This may actually be a good approximation if education moves slowly or if the time gap between today's decision of getting a certain degree and tomorrow's entry in the labor market is short. An individual considering the idea of enrolling in a one year master may use for instance the current average level of education for evaluating his future relative position. Another, more practical, consideration is based on how people get informed about the available educational choices. Usual sources of information are provided by comments of older friends and by the reading of student guides and statistics about the likelihood of getting a job with a certain degree. All these information describe the current situation not the future one. Finally a similar approach has been used, though in different contexts, in other studies like in the already mentioned Pollak (1976) and Jones (1984).

2.3.3 The composition of the society

There are two subsequent cohorts of potential workers that have to choose which level of education to acquire before entering the labor market. Every cohort is formed by many individuals and every individual is characterized by a certain level of productivity and by some social preferences. Both the productivity level and the social preferences are assumed to be innate and fixed. In fact, prior to any decision of any players, two simultaneous random moves of Nature determine the composition of each cohort. As in the standard model the population is split into two classes according to the (low or high) productivity. The other move of Nature defines the social preferences with respect to the average education in the society: some individuals simply do not have social preferences (lower index i, for independent), some are characterized by a conformist behavior (lower index c), some others are status seeking (lower index s).

Needless to say that both moves of Nature are just a drastic approximation of a continuum space of different productive abilities and social preferences combinations. The population of potential workers, whose size is normalized to 1, is thus partitioned in this way:

	low productivity (l)	high productivity (h)
independent (i)	α_{il}	α_{ih}
conformist (c)	α_{cl}	α_{ch}
status seeking (s)	α_{sl}	α_{sh}

Table 1: the composition of the society at every period.

Being proportions of a population it follows that, for any $k \in \{i, c, s\}$ and $j \in \{l, h\}$, $\alpha_{kj} \in [0, 1]$ and $\sum_k \sum_j \alpha_{kj} = 1$. Once that Nature has moved the following game takes place:

- $t = 1$: a first cohort of workers plays the signaling game. Social preferences do not play any role.
- $t = 2$: a new cohort of workers plays the signaling game. Social preferences do play a role.

Indeed the signaling game in $t = 1$ just provides a starting point upon which the model develops in $t = 2$. Two assumptions underpin the analysis of the second period:

1) The size of the population and the proportions α_{kj} remain constant.

This means that the composition of this new cohort of individuals exactly resembles the one of the previous period. Therefore any movement in average education will be endogenously created by the model and it will not be due to population growth, changes in the productivity level or in the social preferences of the society.

2) In $t = 2$ firms maintain the same hiring strategy that they adopted in $t = 1$.

In other words firms keep offering a wage $w = 1$ to any individual whose educational level is such that $0 \leq e^* < 1$ while they offer $w = 2$ to all those players with $e^* \geq 1$.

An important note about the second point. This assumption implies that firms keep the same beliefs they held in $t = 1$. On one side this seems reasonable given that in $t = 1$ (standard signaling game) these beliefs happen to be correct such that the hiring process is perfectly effective in screening workers belonging to different productivity classes. On the other hand it means that firms are not forward looking such that their beliefs may not be confirmed by workers' actual behavior in $t = 2$ if these individuals behave differently with respect to their predecessors. This last consideration, together with the fact that also workers are myopic (in anticipating future average education, as introduced in section 2.3.2) makes the analysis that follows depart from a standard rational one.

The goal of our analysis is to study the dynamics of the average level of education in the society. Therefore we indicate with $\bar{e}_t$ the weighted average of the levels of education chosen by each class at time t.

$$\bar{e}_t = \sum_k \sum_j \alpha_{kj} e^t_{kj} \qquad k \in \{i, c, s\}, \, j \in \{l, h\}$$

2.3.4 Time $t = 1$

As we saw in section 2.3.1, low productivity workers get a level of education equal to 0, while high productivity individuals choose a level of 1. Optimal levels of education are thus $e^*_{il} = e^*_{cl} = e^*_{sl} = 0$ and $e^*_{ih} = e^*_{ch} = e^*_{sh} = 1$. The average at the end of $t = 1$ is $\bar{e}_1 = \alpha_{ih} + \alpha_{ch} + \alpha_{sh} = \sum_k \alpha_{kh}$. Notice, once again, that in the first period social preferences do not matter.

2.3.5 Time $t = 2$

At the beginning of the second period a new cohort of potential workers faces its educational choice. Each agent at $t = 2$ has to maximize a utility function specific to his class and resembling his preferences. Table 2 reports the utility functions of the six categories of agents (subscripts are omitted whenever unnecessary). The wage level is left in the implicit form w and it will be determined in accordance with the individual's optimal educational choice and with the hiring policy of the firms.

	low productivity	high productivity
independent	$U_{il} = w - e^2$	$U_{ih} = w - \frac{1}{2}e^2$
conformist	$U_{cl} = w - e^2 - (e - \bar{e}_1)^2$	$U_{ch} = w - \frac{1}{2}e^2 - (e - \bar{e}_1)^2$
status seekers	$\begin{cases} U_{sl} = w - e^2 - 2(e - \bar{e}_1)^2 & \text{if } e < \bar{e}_1 \\ U_{sl} = w - e^2 + 2\bar{e}_1(e - \bar{e}_1) - \frac{1}{2} & \text{if } e > \bar{e}_1 \end{cases}$	$\begin{cases} U_{sh} = w - \frac{1}{2}e^2 - 2(e - \bar{e}_1)^2 & \text{if } e < \bar{e}_1 \\ U_{sh} = w - \frac{1}{2}e^2 + 2\bar{e}_1(e - \bar{e}_1) - \frac{1}{2} & \text{if } e > \bar{e}_1 \end{cases}$

Table 2: utility functions of the six classes of individuals.

Independent players are not affected by their relative position in the educational distribution so that they are characterized by standard utility functions à la Spence. These players behave as the ones involved in the standard signaling model and their optimal choices are $e^*_{il} = 0$ (so that $w_{il} = 1$) and $e^*_{ih} = 1$ ($w_{ih} = 2$).

Utility functions for the players with interdependent preferences are slightly more complex. Beside the part which captures the trade off between wage and cost of education there is an extra term which models the social preferences. In accordance with many classic contributions in the field[8] this last term enters additively in the utility functions. For the reasons which have been explained before, this "social term" is a function of $\bar{e}_1$, the average level of education in the first period. In line with the utility functions presented in Spence's original article (1973), this additional part appears as an explicit function which is as simple as possible.

Conformist individuals have a preference for being close to the average, i.e. they do not want to stand out from the mass. The social term in the utility function assigns an increasing penalty whenever their level of education differs from the average level $\bar{e}_1$. The utility functions of status seeking individuals are more problematic. A differentiation has to be drawn between two cases. If the level of education is below $\bar{e}_1$ then status ambitions are frustrated. In this case the utility function resembles the one of conformist individuals but with a higher penalty for falling behind. If, on the contrary, the educational level is above $\bar{e}_1$, then individual's utility is increasing in this distance. In this second case the function has to be corrected in some way. Consider for example the *sh*-class. The natural formulation for the case $e > \bar{e}_1$ would be $U_{sh} = w - \frac{1}{2}e^2 + (e - \bar{e}_1)^2$. However this is a convex function with no internal maximum. To have a meaningful solution the utility function takes the form presented in Table 2.

We already know the optimal choices of individuals with independent preferences. The following subsections are devoted to a more specific analysis of the behavior of each of the last four classes. After having found optimal choices it will be possible to analyze educational dynamics between the two periods.

2.3.5.1 The "conformist + low productivity" class (*cl*)

Individuals belonging to this class face the problem $\max_e U_{cl}$ where

$$U_{cl}(e) = w - e^2 - (e - \bar{e}_1)^2$$

which has solution $\hat{e}_{cl} = \frac{1}{2}\bar{e}_1$. Since $\bar{e}_1 \in [0, 1]$ it follows that $\hat{e}_{cl} \in \left[0, \frac{1}{2}\right]$ confirming that agents of this category get the low wage $w = 1$. Firms are in fact able to correctly categorize them as low productivity workers. Is it always optimal for an individual belonging to this class to choose $\hat{e}_{cl}$? To answer this question the utility the agent gets choosing $\hat{e}_{cl}$ (function A) is compared with the utility stemming from some alternative educational levels.

[8] Among others: Pollak (1976), Jones (1984), Bernheim (1994), Akerlof (1997).

A) $U_{cl}\left(\frac{1}{2}\bar{e}_1\right) = 1 - \frac{1}{2}\left(\bar{e}_1\right)^2$ C) $U_{cl}\left(1\right) = -\left(\bar{e}_1\right)^2 + 2\bar{e}_1$

B) $U_{cl}\left(0\right) = 1 - \left(\bar{e}_1\right)^2$ D) $U_{cl}\left(\frac{2}{3}\bar{e}_1\right) = 1 - \frac{5}{9}\left(\bar{e}_1\right)^2$

Functions B) and C) consider the choices which were made by players of the previous cohort. Function D) depicts the utility a cl-player would get if he mimics agents belonging to the "conformist + high productivity" class[9]. The following figures provide a graphical analysis of the situation.

Fig. 1.a: Utility of the cl-class.

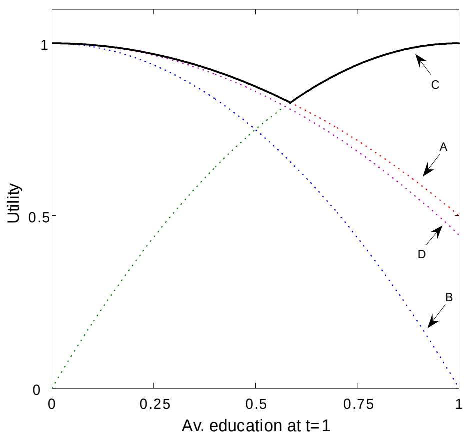


Fig. 1.b Education of the cl-class.

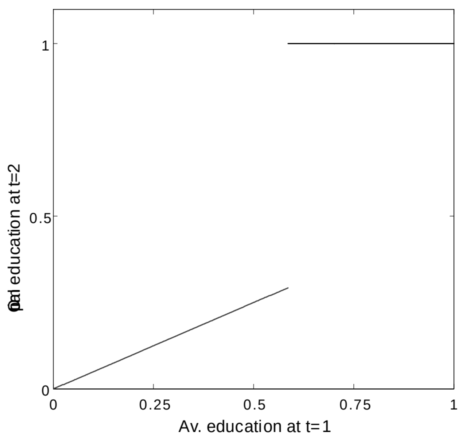


Figure 1.a plots the utility functions for $\bar{e}_1 \in [0, 1]$. From the upper contour set of these functions, the optimal level of education e^*_{cl} as a function of $\bar{e}_1$ can be easily derived (Fig. 1.b). Up to a critical value of $\bar{e}_1$ ($\cong 0.586$), the optimal choice for a cl-individual is to get a level of education equal to half of the average level. If, at the opposite, the average level of education in $t = 1$ is greater than the critical value, then it is more convenient to "jump" to $e^*_{cl} = 1$. The intuition for such a behavior is clear. In fact, as far as the average level is low, a cl-agent can conform to it without investing too much in costly education. At the opposite if $\bar{e}_1$ is high enough it is then better to choose $e^*_{cl} = 1$ despite of the higher cost. In doing so the worker could be farther away from $\bar{e}_1$ than playing the previous strategy[10]. Still, sending the signal $e^*_{cl} = 1$, he is able to make firms believe that he is highly productive such that he will be offered the high wage $w = 2$.

[9] The optimal level for the ch-class is solved in the next subsection.

[10] Consider the case in which $\bar{e}_1 = 0.6$. Choosing $e_{cl} = \frac{1}{2}\bar{e}_1$ the distance from the average would be equal to 0.3; choosing $e_{cl} = 1$ the distance is larger being equal to 0.4.

2.3.5.2 The “conformist + high productivity” class (ch)

Every worker belonging to this class has to solve $\max_{e} U_{ch}$ where:

$$U_{ch}(e) = w - \frac{1}{2}e^2 - (e - \bar{e}_1)^2$$

The solution is $\hat{e}_{ch} = \frac{2}{3}\bar{e}_1$ so that $\hat{e}_{ch} \in \left[0, \frac{2}{3}\right]$. Therefore a problem of effectiveness of the signal arises. Firms believe these workers to be little productive and they offer them the low wage $w = 1$. As in the previous subsection, the utility a worker enjoys from choosing $\hat{e}_{ch}$ is compared with the utility he would get choosing some other particular levels of education.

A) $U_{ch}\left(\frac{2}{3}\bar{e}_1\right) = 1 - \frac{1}{3}\left(\bar{e}_1\right)^2$

B) $U_{ch}(1) = \frac{1}{2} - \left(\bar{e}_1\right)^2 + 2\bar{e}_1$

C) $U_{ch}(0) = 1 - \left(\bar{e}_1\right)^2$

D) $U_{ch}\left(\frac{1}{2}\bar{e}_1\right) = 1 - \frac{3}{8}\left(\bar{e}_1\right)^2$

Fig. 2.a: Utility of the ch-class.

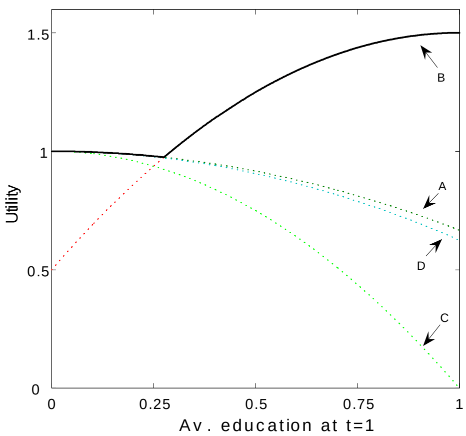
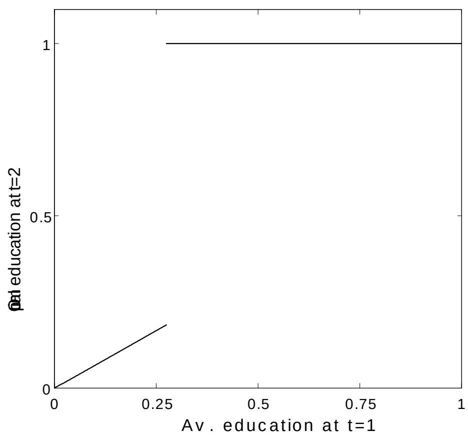


Fig. 2.b Education of the ch-class.



Figures 2.a and 2.b show that the optimal strategy is to maintain a level of education $e^*_{ch} = 1$ as far as the average level in $t = 1$ is not too low (the critical value is $\bar{e}_1 \cong 0.275$). In doing so a worker effectively signals his productivity and gets the high wage. Below the critical value it is instead more convenient to choose $e^*_{ch} = \frac{2}{3}\bar{e}_1$. When the average is very low the pressures to conform are stronger than the incentives to signal the real productivity. Notice that there are no incentives in mimicking the behavior of the cl-class given that curve D) is below some other curves for any $\bar{e}_1 \neq 0$. Still it could be the case that the cl-class and the ch-class are pooled together at a level $e^* = 1$.

2.3.5.3 The "status seeking + low productivity" class (sl)

The analysis of the behavior of agents with status seeking preferences is more complex given the fact that two different utility functions have to be considered. If the agent's optimal choice is below $\bar{e}_1$ then $\max_e U_{sl}$ has to be solved with:

$$U_{sl}(e) = w - e^2 - 2(e - \bar{e}_1)^2 \qquad \text{if } e < \bar{e}_1$$

which is similar to the one faced by the cl-class (apart from a double cost for "falling behind" the average) and it has solution $\hat{e}_{sl} = \frac{2}{3}\bar{e}_1$. Clearly it is always the case that $\hat{e}_{sl} < \bar{e}_1$ and thus the solution can be accepted. At the same time $\hat{e}_{sl} < 1$ and the worker receives the low wage $w = 1$.

If, on the contrary, the optimal choice is greater or equal than $\bar{e}_1$ then the problem is $\max_e U_{sl}$ with :

$$U_{sl}(e) = w - e^2 + 2\bar{e}_1(e - \bar{e}_1) - \tfrac{1}{2} \qquad \text{if } e \geq \bar{e}_1$$

It is easy to see the positive effect of an increasing distance $(e - \bar{e}_1)$. This effect is amplified if the average level in $t = 1$ was high. Still status concerns provide a positive additional utility only if $2\bar{e}_1(e - \bar{e}_1) > \frac{1}{2}$, i.e. only if the agent is able to reach a certain degree of differentiation with respect to the rest of the society. The utility function is concave and the maximization problem has solution $\hat{e}_{sl} = \bar{e}_1$. The comparison among different choices is depicted in Figure 3.a. Figure 3.b shows the optimal level of education for a generic sl-player.

A) $U_{sl}\left(\frac{2}{3}\bar{e}_1\right) = 1 - \frac{2}{3}\left(\bar{e}_1\right)^2$

B) $U_{sl}\left(\bar{e}_1\right) = \frac{1}{2} - \left(\bar{e}_1\right)^2$

C) $U_{sl}\left(0\right) = 1 - 2\left(\bar{e}_1\right)^2$

D) $U_{sl}\left(1\right) = \frac{1}{2} - 2\left(\bar{e}_1\right)^2 + 2\bar{e}_1$

Fig. 3.a: Utility of the sl-class.

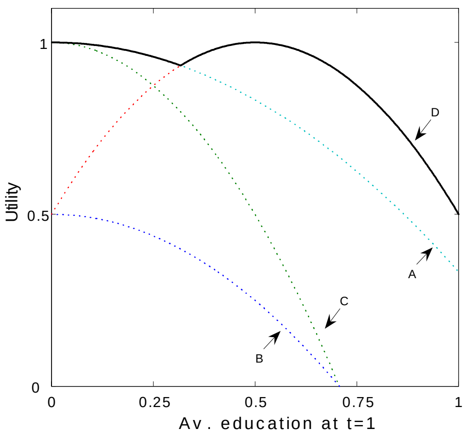
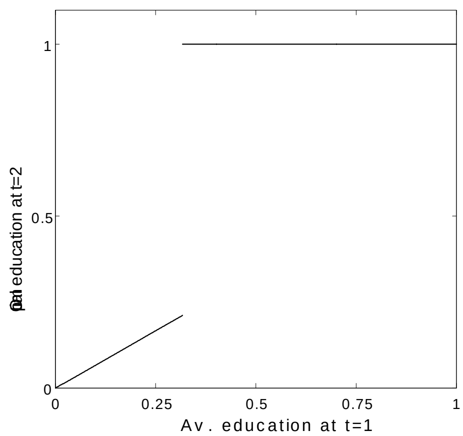


Fig. 3.b Education of the sl-class.



In comparison with the cl-class, the sl-class follows $\bar{e}_1$ in a closer way ($\frac{2}{3}\bar{e}_1$ vs. $\frac{1}{2}\bar{e}_1$) when average education is low and it jumps before to the value $e^*_{sl} = 1$ (the critical value at which this happens is approximately equal to 0.317).

2.3.5.4 The "status seeking + high productivity" class (sh)

The last class of individuals is characterized by utility functions:

$$U_{sh}(e) = w - \tfrac{1}{2}e^2 - 2(e - \bar{e}_1)^2 \qquad \text{if } e < \bar{e}_1$$
$$U_{sh}(e) = w - \tfrac{1}{2}e^2 + 2\bar{e}_1 (e - \bar{e}_1) - \tfrac{1}{2} \qquad \text{if } e \geq \bar{e}_1$$

The maximization of the first function leads to $\hat{e}_{sh} = \frac{4}{5}\bar{e}_1$ which verifies the condition but it leads to the low wage $w = 1$. The second function has solution $\hat{e}_{sh} = 2\bar{e}_1$ which can also be accepted. Now, as far as $\bar{e}_1 \geq \frac{1}{2}$, players receive the high wage $w = 2$. If, on the other hand, $\bar{e}_1 < \frac{1}{2}$, players get a level of education $\hat{e}_{sh} < 1$, they do not signal their real productivity and firms offer them the low wage. Figures 4.a and 4.b illustrate peculiar utility functions and the optimal choice e^*_{sh}.

A) $U_{sh}\left(\frac{4}{5}\bar{e}_1\right) = 1 - \frac{2}{5}\left(\bar{e}_1\right)^2$

B) $U_{sh}\left(2\bar{e}_1\right) = \frac{3}{2}$ with $\bar{e}_1 \geq \frac{1}{2}$

C) $U_{sh}\left(2\bar{e}_1\right) = \frac{1}{2}$ with $\bar{e}_1 < \frac{1}{2}$

D) $U_{sh}\left(1\right) = 1 - 2\left(\bar{e}_1\right)^2 + 2\bar{e}_1$

E) $U_{sh}\left(0\right) = 1 - 2\left(\bar{e}_1\right)^2$

Fig. 4.a: Utility of the sh-class.

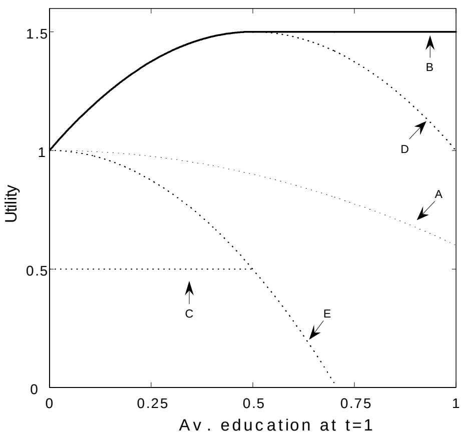
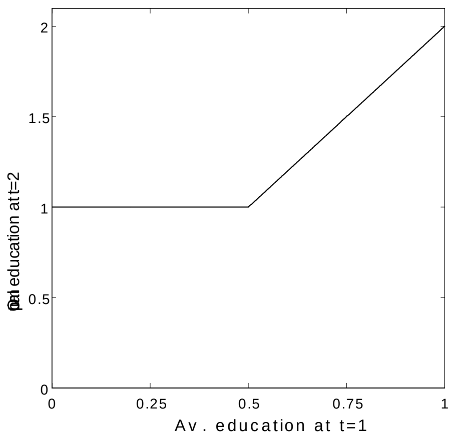


Fig. 4.b Education of the sh-class.



It is optimal for a generic sh-player to keep a level of education $e^*_{sh} = 1$ as far as $\bar{e}_1 \leq 0.5$. In this case the social distance between him and the average level is high enough to satisfy his status aspirations. On the contrary, if $\bar{e}_1 > 0.5$ it is then better to choose $e^*_{sh} = 2\bar{e}_1$. In this second case a sh-player realizes that his status is menaced so that he reaches a higher level of education in order to keep the distance from the rest of the society.

2.4 Education dynamics

Having solved for the optimal behavior of the six classes of agents it is now possible to focus on $\bar{e}_2$, the average level of education at time $t = 2$. The following table summarizes the behavior of the six classes of individuals and it provides a useful starting point for computing $\bar{e}_1$ and $\bar{e}_2$. We use the letter λ to indicate the critical value at which players' optimal choices present discontinuities.

class	prop.	product. θ	e^*_1	e^*_2 for $\bar{e}_1 \leq \lambda$	e^*_2 for $\bar{e}_1 > \lambda$	λ
il	α_{il}	1	0	0	0	–
ih	α_{ih}	2	1	1	1	–
cl	α_{cl}	1	0	$\frac{1}{2}\bar{e}_1$	1	0.586
ch	α_{ch}	2	1	$\frac{2}{3}\bar{e}_1$	1	0.275
sl	α_{sl}	1	0	$\frac{2}{3}\bar{e}_1$	1	0.317
sh	α_{sh}	2	1	1	$2\bar{e}_1$	0.500

Table 3: optimal choices at $t = 1$ and at $t = 2$.

At any period of time the average level of education is given by:

$$\bar{e}_t = \sum_k \sum_j \alpha_{kj} e^t_{kj} \qquad k \in \{i, c, s\}, \, j \in \{l, h\}$$

and thus in $t = 1$ it was $\bar{e}_1 = \alpha_{ih} + \alpha_{ch} + \alpha_{sh}$.

Notice that the two cohorts of players with independent preferences are characterized by a constant level of education. In fact $e^*_{il} = 0$ and $e^*_{ih} = 1$ in both periods. The model therefore predicts average education to remain constant in a society in which no player is influenced by issues of social distance in educational achievements[11], as captured by the following proposition:

Proposition 2.1 *If $\alpha_{il} + \alpha_{ih} = 1$ then $\bar{e}_2 = \bar{e}_1 = \alpha_{ih}$. Therefore the presence of individuals with interdependent preferences is a necessary condition to observe a non constant average level of education.*

Average education can change only if the optimal choices of the four classes with interdependent preferences do differ in different periods. The average level of education in $t = 2$ is a weighted average of the six classes' optimal choices. These choices are functions of $\bar{e}_1$. Moreover $\bar{e}_1$ is a function of the α_{kj}, the proportions of the different classes. It follows that $\bar{e}_2$ is a discontinuous function of the α_{kj}. More precisely the function takes the following form:

$$\bar{e}_2 = \begin{cases} \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch}\left(\frac{2}{3}\bar{e}_1\right) + \alpha_{sl}\left(\frac{2}{3}\bar{e}_1\right) + \alpha_{sh} & 0.000 \leq \bar{e}_1 \leq 0.275 \\ \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch} + \alpha_{sl}\left(\frac{2}{3}\bar{e}_1\right) + \alpha_{sh} & 0.275 < \bar{e}_1 \leq 0.317 \\ \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch} + \alpha_{sl} + \alpha_{sh} & 0.317 < \bar{e}_1 \leq 0.500 \\ \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch} + \alpha_{sl} + \alpha_{sh}\left(2\bar{e}_1\right) & 0.500 < \bar{e}_1 \leq 0.586 \\ \alpha_{ih} + \alpha_{cl} + \alpha_{ch} + \alpha_{sl} + \alpha_{sh}\left(2\bar{e}_1\right) & 0.586 < \bar{e}_1 \leq 1.000 \end{cases}$$

The study of $\bar{e}_2$ in its general form is not viable because it involves too many unknowns. In order to get more precise intuitions about the effect that social preferences may have on educational dynamics we study what happens in some specific population. First, we consider the cases of populations with homogeneous productivity. Then, in addition to the two classes of individuals with independent preferences which we assume to be always present (possibly covering the vast majority of the population), we consider the contribution given by certain combinations of classes of individuals characterized by interdependent preferences.

[11] This would be the case of a standard signaling game repeated over time.

The change in $\bar{e}_t$ from $t = 1$ to $t = 2$ is due to the introduction of social preferences and thus it captures their effect on impact. A flavor of longer term dynamics ($t > 2$) is given by the study of the function $\bar{e}_t(\bar{e}_{t-1})$ which is assumed to mimic the function $\bar{e}_2(\bar{e}_1)$.

1) Population with homogeneous productivity

In a society in which all the workers have low productivity the average education in the first period is $\bar{e}_1 = 0$. Because of this, agents' optimal choices equal zero also in the subsequent periods. There is no growth in education, despite of the presence of possibly many status seeking agents. This case is enough to prove the following proposition:

Proposition 2.2 *If $\alpha_{il} + \alpha_{cl} + \alpha_{sl} = 1$ then $\bar{e}_t = 0$ for any t. Therefore the presence of individuals with interdependent preferences is not sufficient to observe a non constant average level of education.*

If, at the opposite, all the workers have high productivity ($\alpha_{ih} + \alpha_{ch} + \alpha_{sh} = 1$) then $\bar{e}_1 = 1$ and $\bar{e}_2 = \alpha_{ih} + \alpha_{ch} + 2\alpha_{sh} = 1 + \alpha_{sh}$. The rate of growth of average education between the first and the second period is given by the proportion of status seekers (α_{sh}). For $t > 2$ the function is given by $\bar{e}_t = \alpha_{ih} + \alpha_{ch} + \alpha_{sh}(2\bar{e}_{t-1})$ and education keeps increasing being pulled by the sh-class.

2) Subpopulation with homogeneous social preferences

2.a) Only conformist individuals

A society with independent and conformist individuals implies the restrictions $\alpha_{il} + \alpha_{ih} + \alpha_{cl} + \alpha_{ch} = 1$ and $\bar{e}_1 = \alpha_{ih} + \alpha_{ch}$. The function $\bar{e}_2(\bar{e}_1)$ and, most importantly, the difference in the average educational levels between the two periods ($\Delta\bar{e} = \bar{e}_2 - \bar{e}_1$) are:

$$\bar{e}_2 = \begin{cases} \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch}\left(\frac{2}{3}\bar{e}_1\right) \\ \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch} \\ \alpha_{ih} + \alpha_{cl} + \alpha_{ch} \end{cases} \qquad \Delta\bar{e} = \begin{cases} \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{ch}\left(\frac{2}{3}\bar{e}_1 - 1\right) & 0.000 \leq \bar{e}_1 \leq 0.275 \\ \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) & 0.275 < \bar{e}_1 \leq 0.586 \\ \alpha_{cl} & 0.586 < \bar{e}_1 \leq 1.000 \end{cases}$$

The change in average education ($\Delta\bar{e}$) is solely a function of the proportions of conformist individuals (in line with Proposition 2.1). Assuming $\alpha_{cl} > 0$, then average education increases from $t = 1$ to $t = 2$ (though with different speed) provided that $\bar{e}_1 > 0.275$. In other words at least 27.5% of the population has to be characterized by a high level of productivity. Education is therefore likely to grow even in a society in which there are just independent and conformist individuals.

Proposition 2.3 *If $\alpha_{il} + \alpha_{ih} + \alpha_{cl} + \alpha_{ch} = 1$ and $\bar{e}_1 > 0.275$ then $\bar{e}_2 > \bar{e}_1$ and education increases even in a purely conformist society. Therefore the presence of status seeking individuals is not necessary to observe education growing over time.*

If $\bar{e}_1 \leq 0.275$, then average education could move in both directions depending on the specific composition of the society. Consider for instance the case of $\alpha_{il} = 0.5$, $\alpha_{ih} = 0.2$, $\alpha_{cl} = 0.25$ and $\alpha_{ch} = 0.05$ such that $\bar{e}_1 = 0.25$. Optimal choices in $t = 2$ are $e^*_{il} = 0$, $e^*_{ih} = 1$, $e^*_{cl} = 0.125$ and $e^*_{ch} = 0.166$ and they imply $\bar{e}_2 = 0.239$, i.e. average education decreased. Vice versa a different population that still has the same average level $\bar{e}_1 = 0.25$ could display a growing level of education. For instance with $\alpha_{il} = 0.3$, $\alpha_{ih} = 0.2$, $\alpha_{cl} = 0.45$ and $\alpha_{ch} = 0.05$ we would have $\bar{e}_2 = 0.264$ and thus a positive $\Delta\bar{e}$.

Obviously for any $\bar{e}_1 \in (0, 0.275)$ there exists a specific relationship between the two proportions of conformist players such that education remains constant over time (assuming firms' behavior not to change). This relation is found setting $\Delta\bar{e} = 0$ such that $\tilde{\alpha}_{cl} = \gamma\tilde{\alpha}_{ch}$ where $\gamma = \frac{1-\frac{2}{3}\bar{e}_1}{\frac{1}{2}\bar{e}_1}$ and $\gamma > 1$, for any $\bar{e}_1 \in (0, 0.275)$. Proportions of the kind $\tilde{\alpha}_{cl}$ and $\tilde{\alpha}_{ch}$ define a stationary outcome of the game.

Proposition 2.4 *For certain intervals of $\bar{e}_1$ and for appropriate values of the α_{kj}, interior steady state levels such that $\bar{e}_t \in (0, 1)$ and $\bar{e}_t = \bar{e}_{t-1}$ for any $t > 1$ exist.*

These interior outcomes are not stable. In fact any $\alpha_{cl} > \tilde{\alpha}_{cl}$ implies $\bar{e}_2 > \bar{e}_1$ and education keeps increasing so to eventually reach the (stationary and stable) outcome at $\bar{e}_t = \alpha_{cl} + \alpha_{ch} + \alpha_{ih}$.

Notice that we used the terminology "outcome" instead of "equilibrium" to define these situations. In fact these points are stationary provided that firms do not modify their beliefs (and thus their hiring strategy) even for $t > 2$. Still it may be the case that firms have incentives to actually change their behavior in which case these outcomes will not be equilibria. Section 2.5 will elaborate more on this point.

To analyze the properties of these outcomes we need the following definition:

Definition 2.5 *An outcome of the signaling game is called:*

- perfectly separating if $e^*_{kl} \neq e^*_{yh}$ for any $k \in \{i, c, s\}$ and any $y \in \{i, c, s\}$.

- partly separating if $e^*_{kl} = e^*_{yh}$ for some $k \in \{i, c, s\}$ and some $y \in \{i, c, s\}$.

In a perfectly separating outcome (like the equilibrium of the game in $t = 1$) players with different productivity send different signals, i.e. they choose different levels of education. This is not the case of the stationary and stable outcome mentioned above (such that $\bar{e}_t = \alpha_{cl} + \alpha_{ch} + \alpha_{ih}$) which is indeed partly separating. In fact conformist individuals with low productivity choose $e^*_{cl} = 1$, the same signal which is sent by the high productivity classes (ih and ch)[12]. As we will see partly separating outcomes easily arise in populations where conformist and/or status seeking individuals are present. These last considerations are captured by the following proposition.

Proposition 2.6 *The presence of agents with interdependent preferences ($\alpha_{il} + \alpha_{ih} < 1$) can turn perfectly separating outcomes in partly separating ones.*

Section 2.5 analyses the effects that the dynamics described by Proposition 2.6 can have on the hiring policies of the firms.

2.b) Only status seeking individuals

Such a restriction implies $\alpha_{il} + \alpha_{ih} + \alpha_{sl} + \alpha_{sh} = 1$ and $\bar{e}_1 = \alpha_{ih} + \alpha_{sh}$. The function $\bar{e}_2(\bar{e}_1)$ and the relative change in the average educational level are given by:

$$\bar{e}_2 = \begin{cases} \alpha_{ih} + \alpha_{sl}\left(\frac{2}{3}\bar{e}_1\right) + \alpha_{sh} \\ \alpha_{ih} + \alpha_{sl} + \alpha_{sh} \\ \alpha_{ih} + \alpha_{sl} + \alpha_{sh}(2\bar{e}_1) \end{cases} \quad \Delta\bar{e} = \begin{cases} \alpha_{sl}\left(\frac{2}{3}\bar{e}_1\right) & 0.000 \leq \bar{e}_1 \leq 0.317 \\ \alpha_{sh} & 0.317 < \bar{e}_1 \leq 0.500 \\ \alpha_{sh}(2\bar{e}_1 - 1) & 0.500 < \bar{e}_1 \leq 1.000 \end{cases}$$

The change in education between the two periods is never negative. In fact $\Delta\bar{e}$ can be equal to zero only if $\alpha_{sl} = 0$ or $\alpha_{sh} = 0$.

Proposition 2.7 *If $\alpha_{il} + \alpha_{ih} + \alpha_{sl} + \alpha_{sh} = 1$ then $\bar{e}_t \geq \bar{e}_{t-1}$. Education cannot decrease in a society with no conformist individuals.*

If $\alpha_{sl} > 0$ and $\alpha_{sh} > 0$ then education grows and it will reach the unique stationary outcome at the maximum level $\bar{e}_t = \alpha_{ih} + \alpha_{sl} + 2\alpha_{sh}$. As expected, the rat race effect among ambitious agents pushes average education up. Even in this case, and in line with Proposition 2.6, the signal that agents send can be misleading. In fact status seeking agents with low productivity can be pooled together with highly productive workers.

[12] Similarly if $\alpha_{cl} < \tilde{\alpha}_{cl}$ then $\bar{e}_2 < \bar{e}_1$ and education approaches its minimum at $\bar{e}_t = \alpha_{ih}$. This is a stable outcome which is also partly separating given that ch-individuals fail to signal their real productivity.

3) Conformist, low productivity + status seeking, high productivity

This case captures the situation in which agents' social preferences are correlated with their innate productivity Low type individuals learn that they cannot shine and they develop a taste for conformism. High type individuals realize they have the talent and the potential to be above the average and thus they adopt a more status seeking behavior. If this is the case then $\alpha_{il} + \alpha_{ih} + \alpha_{cl} + \alpha_{sh} = 1$ and $\bar{e}_1 = \alpha_{ih} + \alpha_{sh}$. The function for $\bar{e}_2$ can then be formalized as:

$$\bar{e}_2 = \begin{cases} \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{sh} \\ \alpha_{ih} + \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{sh}\left(2\bar{e}_1\right) \\ \alpha_{ih} + \alpha_{cl} + \alpha_{sh}\left(2\bar{e}_1\right) \end{cases} \quad \Delta\bar{e} = \begin{cases} \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) & 0.000 \le \bar{e}_1 \le 0.500 \\ \alpha_{cl}\left(\frac{1}{2}\bar{e}_1\right) + \alpha_{sh}\left(2\bar{e}_1 - 1\right) & 0.500 < \bar{e}_1 \le 0.586 \\ \alpha_{cl} + \alpha_{sh}\left(2\bar{e}_1 - 1\right) & 0.586 < \bar{e}_1 \le 1.000 \end{cases}$$

Education grows over time given that $\Delta\bar{e}$ is positive. For low levels of $\bar{e}_1$, the growth is driven by individuals of the cl-class that try not to fall too behind. Then growth in education becomes even faster given that also status seeking individuals, feeling their status menaced, reach higher levels of schooling. Many outcomes are partly separating because cl-individuals are indistinguishable from ih-individuals for any $\bar{e}_1 > 0.586$. Still the cl and sh are never pooled together.

4) The general case

When all the six classes of workers are present (the two with independent preferences and the four with interdependent ones) only a qualitative solution is possible. Consider first the uniform distribution defined by $\alpha_{kj} = \frac{1}{6}$ with $k \in \{i, c, s\}$ and $j \in \{l, h\}$ such that $\bar{e}_1 = \alpha_{ih} + \alpha_{ch} + \alpha_{sh} = 0.5$. Optimal choices in $t = 2$ are $e^*_{il} = 0$, $e^*_{ih} = 1$, $e^*_{cl} = 0.25$, $e^*_{ch} = 1$, $e^*_{sl} = 1$ and $e^*_{sh} = 1$ which imply $\bar{e}_2 = 0.71$ and thus an important increase in education. Given this example, it is easy to forecast that, under very weak assumptions regarding the α_{kj}, average education increases over time. In fact, in order to have the opposite result, namely a decreasing level of education, the composition of the society has to be strongly biased towards low productive workers ($\bar{e}_1 < 0.275$ is a necessary condition for average education to decrease) and, on top of that, the majority of individuals characterized by high productivity has to follow a conformist behavior.

As a numerical example consider the case in which $\alpha_{il} = 0.6$, $\alpha_{ih} = 0.05$, $\alpha_{cl} = 0.1$, $\alpha_{ch} = 0.15$, $\alpha_{sl} = 0.05$, $\alpha_{sh} = 0.05$. The resulting average education at time $t = 1$ will be $\bar{e}_1 = 0.25$. Optimal choices at $t = 2$ are $e^*_{il} = 0$, $e^*_{ih} = 1$, $e^*_{cl} = 0.125$, $e^*_{ch} = 0.166$, $e^*_{sl} = 0.166$ and $e^*_{sh} = 1$ so that $\bar{e}_2 = 0.146$, with $\bar{e}_2 < \bar{e}_1$. In the third period we have $\bar{e}_3 = 0.127$ and in the long run $\bar{e}_t$ approaches the minimum $\bar{e}_t = 0.1$.

2.4.1 Summary

The analysis of the previous sections highlighted the effects that interdependent preferences can have on educational dynamics. In particular it showed that the presence of individuals that care about their relative position in the educational distribution is necessary but not sufficient for observing an average level of education that evolves over time. Indeed it is the combination of social preferences and productivity levels that may trigger the inter-temporal dynamics. The assumptions needed for the model to display a growing level of education are not particularly challenging. For instance average education is likely to increase also in a society in which, aside to standard agents with independent preferences, there are just a few conformist individuals. Therefore the presence of status seeking individuals is not necessary, even though it can surely contribute.

2.5 Firms' behavior

The previous section focused on the change in average education between $t = 1$ and $t = 2$. A key assumption that we adopted is that firms do not modify their hiring strategy. This hiring strategy is based on firms' initial beliefs which are confirmed by individuals' behavior in $t = 1$. From then on firms are myopic and they keep on offering the high wage to any worker with $e^* \geq 1$ and the low wage to any worker with $e^* < 1$. Even if a formal discussion of a longer game is outside the scope of the paper, it is easy to understand that this assumption would become more and more restrictive as new cohorts of workers enter in the labor market. This is so because firms' beliefs may easily happen to be incorrect for $t > 1$. Indeed this is already evident in $t = 2$.

The model showed in fact that in $t = 2$ workers with different productivity often choose the same educational level. In these situations the signals that firms observe become less informative because they are no more perfectly correlated with workers' productivity. In fact, from the firms' point of view, agents with the same e^* are ex-ante indistinguishable. It follows that firms can fail to properly discriminate workers according to their real productivity.

Table 3 shows that, already in the second period, signals are precise and firms' beliefs are correct only in a short internal interval.

Interval of $\bar{e}_1$	Classes pooled together	Firms' beliefs	Effects on profits
$[0.000, 0.275]$	ch, sl at $e^* = \frac{2}{3}\bar{e}_1$, ih, sh at $e^* = 1$	incorrect	positive
$(0.275, 0.317]$	ih, ch, sh at $e^* = 1$	correct	null
$(0.317, 0.500]$	ih, ch, sl, sh at $e^* = 1$	incorrect	negative
$(0.500, 0.586]$	ih, ch, sl at $e^* = 1$	incorrect	negative
$(0.586, 1.000]$	ih, cl, ch, sl at $e^* = 1$	incorrect	more negative

Table 3: consequences on firms' profits in $t = 2$.

The last column of Table 3 focuses on the relation between firms' beliefs and firms' profitability[13]. In particular it shows that when beliefs are incorrect then profits are negatively affected in the vast majority of cases. For very low levels of $\bar{e}_1$ (first row) firms actually gain because they pay the low wage to conformist individuals with high productivity ($w_{ch} < \theta_{ch}$). But if $\bar{e}_1 > 0.317$ then firms are overpaying some low productivity worker. In fact $w_{sl} > \theta_{sl}$ in the interval $(0.317, 0.586]$ and both $w_{cl} > \theta_{cl}$ and $w_{sl} > \theta_{sl}$ in the interval $(0.586, 1.000]$. This leads to the following proposition:

Proposition 2.8 *For high levels of $\bar{e}_t$ firms are more likely to be disappointed by workers' real productivity.*

Notice that Proposition 2.8 is in line with the initial intuition of the paper. The average level $\bar{e}_t$ is low when just a few individuals have a high level of education. Then these few agents stand out clearly and they send a very effective signal to employers. At the opposite a high $\bar{e}_t$ indicates that a considerable fraction of the population has reached high levels of schooling. In this case signals get less informative and potentially harmful for firms' profitability.

Still employers will learn workers' real productivity on the workplace and they will realize if their beliefs and their screening process are inaccurate. If this is the case then firms will eventually change their hiring strategy. In particular, as average education grows, they will increase the minimum level of education they believe to be effective in discriminating between workers with different productivity. Such a behavior seems to be consistent with various anecdotic evidences. For instance minimum educational levels required to get certain jobs have been increasing in the last decades, both in the public and in the private sector.

[13]Firms break even whenever the wage they pay equals the real productivity of the workers. If firms pay higher wages they incur into losses.

The fact that firms increase the discriminating level will affect the choices of subsequent cohorts of individuals. Because of economic incentives and social distance considerations, different categories of workers will progressively acquire this new higher level of education which over time will again lose its separating power. According to this argument workers and firms are involved in a strategic interaction in which both reaction functions are positively sloped such that an escalation in the level of education takes place. A mechanism of this kind is likely to have contributed to the rise of average education in Europe.

2.6 Concluding remarks

The dynamics implied by the combination of theories of signaling and social distance are in line with educational trends of the last decades. In fact, under very reasonable assumptions about the composition of the society, the proposed model displays an increasing average level of schooling. For instance the presence of status seeking individuals is not a necessary condition. Education can easily go up even in a purely conformist society provided that there are enough high productivity individuals. The consideration that it is likely that, at least for some agents, the level of schooling may be a way to reach a respectable social position makes the result more robust.

The model also rationalizes the fact that, as average education grows, signals get progressively less informative. This makes firms' beliefs no more self confirming and it breaks the stability of Spence's original model. Our model suggests that firms react to an increasing level of education with an increase in their degrees requirements, a move which then affects the choices of subsequent cohorts of workers. The large and growing number of new postgraduate degrees that are activated every year provides an indirect evidence of the initial intuition. Today a bachelor's degree is by far a less effective signal with respect to 30 years ago and people, in order to stand out, need something more. The social welfare implications of the model are not positive. Asymmetric information about workers' productivity and externalities stemming from interdependent social preferences suggest a tendency towards overprovision of education. This indeed seems to be the direction towards which many developed countries are moving.

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CHAPTER 3

PREEMPTING VERSUS POSTPONING: THE STEALING GAME

Abstract

We present a game in which players steal from each other shares of a homogeneous and perfectly divisible pie (market). The ability to steal from a rival is assumed to be proportional to the aggressor's current share and players can steal once over the entire game. We analyze the situation in which players simply want to maximize their final share as well as the situation in which players also care about their ranking. We study how the incentives to preempt or to follow the rivals change under the two payoffs specifications and with the number of players involved in the game.

Keywords: pie allocation, timing games.

JEL classification: C72, C73.

3.1 Introduction

I own part of a pie. You own the remaining part so that there is no free pie around. The problem is that we both want more, actually we both want as much pie as possible. The pie cannot be traded and there are no judges who can settle down our conflict. Negotiation or bargaining mechanisms cannot work either because they may be illegal, not enforceable or simply not in our interest given that no one is willing to give up his holdings. Indeed the only way we can meet our objective is by trying to steal part of the other's pie.

Examples of situations that fit this description abound. For instance animals of a given species fight for food, territory and for mating possibilities. Or yet, though in a less physical way, firms compete to get more customers and enlarge their market share, political parties struggle to conquer more voters and shareholders race for the control of the firm.

In all these cases the potential of stealing parts of a rival's pie depends on the aggressor's strength which is positively correlated with his current holding of the pie. In fact a predator with a larger territory can better feed itself and it can thus be fitter when fighting against a rival; a firm with a larger market share is likely to have higher revenues so that it can organize more expensive advertising campaigns; a party with many voters is usually able to get more funds to be used in future campaigns.

To capture a stylized version of situations of this kind we present what we call the Stealing Game. The game models the strategic situation in which a few players fight among themselves over a homogeneous and perfectly divisible pie (market) of constant size. The game is repeated over two periods and players have to decide how much, from who and when to steal part of the pie with the constraint that players can steal only once over the entire game. The key assumption is that the larger is a player, the larger is the portion of market that he can steal from a rival. Therefore if a player suffers an attack his market share goes down and so does the effectiveness of his counterattack.

We study the game under two payoffs specifications. In both cases players' utility is positively related with their market share at the end of the game. With *proportional* payoffs players are simply interested in maximizing their final holdings of the pie. In the *winner takes all* case players want to be the market leader such that each agent cares not only about his absolute share but also about his relative one.

The main goal of the paper is to solve for the timing strategy of the agents, i.e. to find out when is the best moment for a player to behave aggressively and steal part of the pie owned by the rivals. We show that there exists a trade off between preempting or following the opponents. In fact a player who preempts the rivals can use a perfectly effective attack but he is then subject to the possible retaliation of those who postponed their move. At the opposite a player who decides to wait can best respond to what the others did but he faces the risk of being preempted and weakened. This trade off has different solutions depending on how payoffs are defined and on the number of agents involved in the game. With proportional payoffs players always want to preempt their rivals. At the opposite, with the winner takes all payoff formulation and three agents participating in the game, there are equilibria such that players prefer to postpone their action.

The Stealing Game falls therefore in different categories of timing games commonly studied in economics. In general it belongs to the class of preemption games. These are games in which postponing the move hurts; a famous example is given by the centipede game introduced by Rosenthal (1981). The case with three players has instead some features typical of a war of attrition (Maynard Smith, 1974), a strategic situation where preempting the others hurts. A general characteristic of timing games[1] is that optimal timing strategies

[1] More recent literature has focused in generalizing former results (Bulow and Klemperer, 1999), in providing a unified framework to study preemption games and wars of attrition (Park and Smith, 2003) or in testing experimentally some of the theoretical results (Brunnermeier and Morgan, 2005).

depend on the payoff structure and not on the number of participants. The Stealing Game provides instead an example of a game in which, for a given payoff structure, optimal timing strategies change as a function of the number of players.

From a strategic point of view the Stealing Game shares some similarities with the Colonel Blotto game (see Binmore, 1992 and Roberson, 2006). This is a game where two players have to decide how to allocate their army on different territories, each territory is won by the player who sent there the largest force and the overall winner is the player who conquers the majority of territories. In the Colonel Blotto game players have therefore to decide where to allocate a limited resource. In our game agents have instead to decide when to use it. Moreover our game is not restricted to two players. Indeed the case with three players is reminiscent of another well known game, the so called Truel which in its original formulation (Kilgour, 1972) depicts the situation of a gun duel among three players.

Finally notice that the paper does not take the point of view of an external social planner who wants to implement a particular allocation of the pie among different claimants (on this topic see among others Brams and Taylor, 1996, Bag, 1996 and Tasnadi, 2003). In our case the final allocation is the result of the actions of selfish pie maximizers players with no interventions of any superior authority.

The paper is organized as follows. We start introducing and solving the Stealing game in its one shot version (Section 3.2). Then, in Section 3.3, we analyze the two periods game and study how the equilibria of the game are affected by the payoff specification and by the number of players. Section 3.4 concludes.

3.2 The Stealing Game

We consider a game in which $N = \{1, .., n\}$ players fight over a fully covered market whose size is fixed and normalized to one. We indicate with $\Pi_i^t \in [0, 1]$ the share of the market that player $i \in N$ holds at time t. The condition $\sum_i \Pi_i^t = 1$ has to hold for any t. Players start in $t = 0$ from the symmetric position defined by $\Pi_i^0 = \frac{1}{N}$ for all i and Π_i^T indicates i's market share at the end of the game.

We study the game under two different payoffs specifications which will be shortly defined. In both specifications players' utility is (weakly) increasing in Π_i^T. During the game the only way a player can increase his market share is by stealing part of the share which belongs to one of the opponents. This stealing is what we call an attack, i.e. if player A attacks player B this means that player A steals part of player B's pie. An attack therefore increases the aggressor's share of the pie and decreases the victim's one.

First, we briefly study the one shot version of the game where players do not have to decide when to use their attack. Results of the one shot case will then simplify the analysis of the two stages game.

3.2.1 One period game

In the one shot simultaneous game the strategy space of the players is defined by $S_i = N_{-i} \times [0, \alpha \Pi_i^0]$ with $N_{-i} := \{1, ..., n\} \setminus \{i\}$. We restrict our attention to pure strategies. Therefore a typical element of the strategy space is given by $s_i = (x_i, y_i)$ where $x_i \in N_{-i}$ indicates the choice of player i about which rival to attack and $y_i \in [0, \alpha \Pi_i^0]$ indicates the intensity with which player i attacks. The parameter α measures the proportionality between a player's current size (Π_i^0) and the maximum intensity of his attack. We set $\alpha \in \left(0, \frac{1}{N-1}\right)$. The upper bound of the interval ensures that there is no excess demand such that every player gets the amount he wants to steal no matter the opponent he attacks.

With this simplification player i's final share (Π_i^1) takes the following form:

$$\Pi_i^1 = \Pi_i^0 + y_i - \sum_{j:x_j=i} y_j \tag{1}$$

In words player i's final share is the result of three components: his initial share plus the amount of the pie he steals from an opponent minus the amount he is stolen by the other players. Notice that a player can attack a single rival but he may suffer multiple attacks.[2]

As already mentioned we consider two possible payoffs formulations:

1. $u_i = \Pi_i^1$ (The *proportional* case)

[2]Without the upper bound on α, player i's final shares could take the form:

$$\Pi_i^1 = \Pi_i^0 + y_i - \min\left\{\Pi_i^0, \sum_{j:x_j=i} y_j\right\} \qquad \text{if} \quad \sum_{j:x_j=x_i} y_j \le \Pi_{x_i}^0$$

$$\Pi_i^1 = \Pi_i^0 + \frac{y_i}{\sum_{j:x_j=x_i} y_j}\Pi_{x_i}^0 - \min\left\{\Pi_i^0, \sum_{j:x_j=i} y_j\right\} \qquad \text{if} \quad \sum_{j:x_j=x_i} y_j > \Pi_{x_i}^0$$

The condition discriminates between the cases in which player i is able or not to fully get the amount y_i. In the second case aggressors share the available pie proportionally to the intensity of their attack. This more general formulation of Π_i^1 would add other strategic considerations and we left its analysis to future research.

2. $u_i = \begin{cases} \dfrac{1}{\sum_j \left(\mathbf{1}_{\{\Pi_j^1 = \Pi_i^1\}}\right)} & \text{if } \Pi_i^1 \geq \Pi_j^1 \text{ for any } j \neq i \\ 0 & \text{otherwise} \end{cases}$ (The *winner takes all* case)

In the *proportional* case players are simply interested in their final holding of the market and each player gets what he owns. In the *winner takes all* (WTA) case the player that holds the largest final share (the leader) gets the entire market while the others gets nothing. If there is more than a leader then the market is equally shared among the winners.

We now analyze the one shot game under the two payoffs specifications. Equilibria of the game are labelled by $\hat{s}_k$ where $k \in \{p, w\}$ refers to the payoff specification (p for the proportional case and w for the winner takes all case).

3.2.2 The proportional case

Proposition 3.1 defines the equilibria of the one shot game when $u_i = \Pi_i^1$ where Π_i^1 is defined by (1).

Proposition 3.1 *Any strategy such that a player uses his attack with less than full intensity is strictly dominated. The Nash equilibria of the game with proportional payoffs are* $\hat{s}_p = (\hat{x}_i, \alpha\Pi_i^0)_i$ *with* $\hat{x}_i \in N_{-i}$.

Proof. The result directly follows from the analysis of Π_i^1. Player i can only affect the second term of (1) and $\hat{y}_i = \alpha\Pi_i^0$ strictly dominates any other $\tilde{y}_i$. Players are indifferent as to who to attack so that $\hat{x}_i \in N_{-i}$. ■

Therefore in equilibrium each player steals as much pie as possible, it does not matter from who. Figure 1 provides some examples for the case with three players. An arrow which goes from player i to player j indicates that i attacks j.

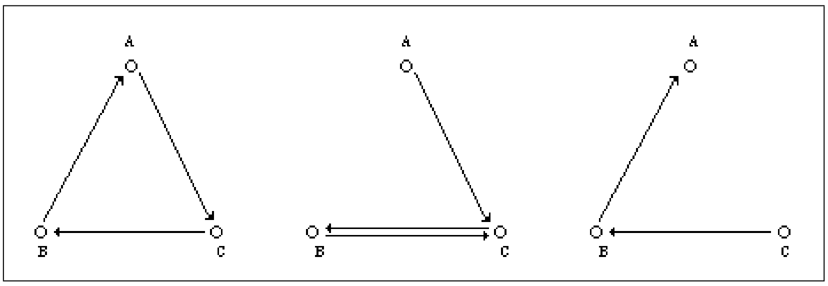

Figure 1: the one shot game with $N = 3$. The first two situations depict two NE, the third does not because player A would be better off attacking someone.

Given that each player is indifferent as to who to attack it follows that also the fact of being attacked is a matter of (bad) luck. To find an expression for players' expected payoff in equilibrium[3] we indicate with $x_j(i)$ the probability with which player j uses his attack against player i. Being a probability we have that $x_j(i) \in \{0,1\}$ for any i and any j (because we only consider pure strategies), $\sum_{i \neq j} x_j(i) = 1$ and $x_j(i) = 1 \Leftrightarrow x_j = i$. Then player i's payoff is given by $E\left(\Pi_i^1\right) = \frac{1}{N}\left(1 + \alpha - \sum_{j \neq i} \hat{x}_j(i)\alpha\right)$ where the initial condition and optimal strategies have been substituted in (1).

The lower bound on the payoffs (a player is attacked by all the rivals) is given by $\Pi_i^1 = \frac{1}{N}(1 - (N-2)\alpha)$. This is the situation of player C in Fig. 1.b who concludes the game with $\Pi_C^1 = \frac{1}{3}(1 - \alpha)$. The upper bound (a player receives 0 attacks) is given by $\Pi_i^1 = \frac{1}{N}(1 + \alpha)$, which is the case of player A in Fig. 1.b $\left(\Pi_A^1 = \frac{1}{3}(1 + \alpha)\right)$. Different market allocations can thus arise in equilibrium. Some of them are symmetric (Fig. 1.a), some others are not (Fig. 1.b).

3.2.3 The winner takes all case

Under the WTA payoffs specification the game has a different set of equilibria.

Proposition 3.2 *The Nash equilibria of the game with WTA payoffs are* $\hat{s}_w = (\hat{x}_i, \alpha\Pi_i^0)_i$ *with* $\hat{x}_i \in N_{-i}$ *and such that* $\Pi_i^1 = \frac{1}{N}$ *for any* i.

Proof. In any of the $\hat{s}_w$ profiles $u_i = \frac{1}{N}$. Take any of them and consider player i's possible deviations. If i deviates to $y_i < \alpha\Pi_i^0$ then obviously $u_i = 0$. If i instead of attacking j deviates and attacks k then player j would not receive any attack such that $\Pi_j^1 = \frac{1}{N}(1 + \alpha)$. Player i's share would still be $\Pi_i^1 = \frac{1}{N}$ but now $u_i = 0$ given that $\Pi_j^1 > \Pi_i^1$. Therefore $\hat{s}_w$ are Nash equilibria. ■

In any of the $\hat{s}_w$ equilibria each player uses his attack with maximum intensity and suffers a single attack which is equally strong so that $u_i = \Pi_i^1 = \frac{1}{N}$ for all i and market allocations in equilibrium are symmetric. The fact that under this payoff formulation players care not only about their absolute share (as in the proportional case) but also about their relative one implies stricter requirements on strategy profiles in order to qualify as equilibria.

[3] Actual payoffs depend on realized market share rather than on expected one. Still given linearity and the fact that the number of attacks suffered in $t = 1$ cannot be anticipated, players base their strategies on expected payoffs.

In fact for any $N > 2$ we have that $\hat{s}_w \subset \hat{s}_p$, i.e. the equilibria of the *winner takes all* case are a subset of the equilibria of the *proportional* case. As an example consider again the case with 3 players depicted in Figure 1. The equilibria of the *WTA* version are just the two circles (one of them is depicted in Fig. 1.a). Fig. 1.b is now no more an equilibrium because player B could move from $u_B = 0$ to $u_B = \frac{1}{3}$ if he attacks A.

3.3 Two periods game (T=2)

We now extend the game over a two periods time horizon. We assume that only one attack is available to each player during the entire game[4]. The temporal structure of the game is similar to the endogenous timing games à la Van Damme and Hurkens (1996). Still in our game the action space of the players may differ from one period to the other. In fact if a player is attacked in $t = 1$ then the effectiveness of his counterattack is weakened. The rules of the game are as follows:

Period 1: players simultaneously choose if to act or to wait till the second period. If a player decides to act then he has to choose who to attack and with which intensity.
Period 2: players who waited in $t = 1$ are fully informed about the actions taken by all the opponents. Then they simultaneously move deciding who to attack and by how much.
Payoffs: payoffs are now related to Π_i^2, i.e. players' share at the end of the game.

With this new time component players' action space is now tridimensional. A strategy will now specify when, from who and how much to steal and these components may be conditional on the opponents' behavior. The action space is indicated by $A_i = A_i^1 \times A_i^2$ (superscripts refer to the period) where:

- $A_i^1 = \{N_{-i} \times [0, \alpha\Pi_i^0]\} \cup \{\phi\}$
- $A_i^2 = N_{-i} \times [0, \alpha\Pi_i^1]$ if $a_i^1 = \phi$

With $\Pi_i^0 = \frac{1}{N}$ for any i (players start from a symmetric situation) and $\Pi_i^1 = \Pi_i^0 + y_i^1 - \sum_{j:x_j=i} y_j^1$. The final share Π_i^2 is given by:

$$\Pi_i^2 = \Pi_i^1 + y_i^2 - \sum_{j:x_j=i} y_j^2 \tag{2}$$

[4]If players had one attack available at every period then the trivial solution for any repeated game would be to use the attack with maximum intensity at every period.

Therefore payoffs are defined as:

1. $u_i = \Pi_i^2$ (The *proportional* case)

2. $u_i = \begin{cases} \dfrac{1}{\sum_j \left(\mathbf{1}_{\{\Pi_j^2=\Pi_i^2\}}\right)} & \text{if } \Pi_i^2 \geq \Pi_j^2 \text{ for any } j \neq i \\ 0 & \text{otherwise} \end{cases}$ (The *winner takes all* case)

We indicate with $x_i^t \in N_{-i}$ the choice of player i about who to attack in period $t \in \{1,2\}$ and with $y_i^t \in [0, \alpha\Pi_i^0]$ the intensity of the attack. We use the notation $s_i^1 = (x_i^1, y_i^1)$ to refer to a strategy such that player i is active in $t = 1$. Similarly we indicate with $s_i^2 = (\phi, x_i^2, y_i^2)$ a strategy such that player i is active in $t = 2$. From the previous section we know that agents always use the attack with maximum intensity and therefore we can set $\hat{y}_i^1 = \alpha\Pi_i^0 = \alpha\frac{1}{N}$ and $\hat{y}_i^2 = \alpha\Pi_i^1$. As before we consider $\alpha \in \left(0, \frac{1}{N-1}\right)$ and, given the complexity of players' strategy spaces, we restrict our attention to the cases with $N \in \{2, 3, 4\}$.

To sum up we deal with a two stages game of complete but imperfect information given that in each stage players' moves are simultaneous[5]. We look for the subgame perfect Nash equilibria (SPNE) of the game and we will use the notion of trembling hand perfection (Selten, 1975) to refine the results in case of multiple equilibria. A strategy profile qualifies as a SPNE if it constitutes a Nash equilibrium in every subgame. Trembling hand perfection requires the equilibrium strategy of any player to be robust to the possibility that the opponents may make small mistakes in implementing their strategies. Therefore equilibria such that players choose weakly dominated strategies do not pass this requirement. In our game players are often indifferent over the components of a dimension of their strategy space (namely the "who to attack" dimension) such that they have multiple best responses. This means that agents cannot properly anticipate the opponents' behavior and they have to consider the possibility of trembles and mistakes.

Our main interest lies in solving for the timing decision of the players under the two payoffs specifications. There are various strategic considerations that may shape players' decision about when to use their attack. These considerations depend on the payoffs specification as well as on the number of players involved in the game.

[5] To be more precise this is a stochastic game (see Maskin and Tirole, 2001 and Herings and Peeters, 2003) because players can influence the strategy space of themselves and of the opponents through their current moves.

There are anyway some general features. For instance a player that uses the attack in $t = 1$ is sure to steal the maximum amount $\left(\alpha\frac{1}{N}\right)$ from his victim. Still he must then remain inactive in $t = 2$ and others players may retaliate against him. On the other side a player that postpones his move can best respond to what happened in $t = 1$. But if the player is attacked in $t = 1$ then he has no chance to catch up with his initial share. In fact i's share would fall to $\Pi_i^1 = \frac{1}{N}(1-\alpha)$, the amount he can steal in $t = 2$ becomes $\alpha\left[\frac{1}{N}(1-\alpha)\right]$ and his final share is $\Pi_i^2 = \frac{1}{N}(1-\alpha^2) < \frac{1}{N}$.

3.3.1 The game with two players

The analysis of the two players case is very simple since each player has only one opponent he can steal from. The game has a unique equilibrium under both payoff specifications. Call the two players A and B and their strategies s_i^1 and s_i^2 with $s_i^1 = \left(j, \alpha\frac{1}{2}\right)$ and $s_i^2 = (\phi, j, \alpha\Pi_i^1)$ for $i, j \in \{A, B\}$. The first matrix illustrates the game with proportional payoffs. The second matrix refers to the winner takes all payoffs specification.

		B	
		s_B^1	s_B^2
A	s_A^1	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}(1+\alpha^2), \frac{1}{2}(1-\alpha^2)$
	s_A^2	$\frac{1}{2}(1-\alpha^2), \frac{1}{2}(1+\alpha^2)$	$\frac{1}{2}, \frac{1}{2}$

		B	
		s_B^1	s_B^2
A	s_A^1	$\frac{1}{2}, \frac{1}{2}$	$1, 0$
	s_A^2	$0, 1$	$\frac{1}{2}, \frac{1}{2}$

Proposition 3.3 *With $N = 2$ the strategy profile $\hat{s}^1 = \left(j, \alpha\frac{1}{2}\right)_i$ is the unique equilibrium of the game both with proportional and WTA payoffs.*

Proof. For both players the preempting strategy s_i^1 strictly dominates the postponing strategy s_i^2 under both payoffs specifications. ■

3.3.2 The game with three players

Different kinds of equilibria arise if three players participate in the Stealing game. In particular there exists equilibria such that players use their attack in $t = 1$ as well as equilibria such that players postpone their moves to $t = 2$. Only some of these equilibria are trembling hand perfect.

3.3.2.1 The proportional case

With proportional payoffs ($u_i = \Pi_i^2$) players never have strict incentives to postpone the use of their attack to the second period. In fact in doing so players only risk to be preempted and weakened in $t = 1$. The possibility of best responding in $t = 2$ does not have any value given that players simply care about their own market share and there are no strategic issues related with the final ranking in the market. Therefore Proposition 3.4 comes at no surprise.

Proposition 3.4 *With $N = 3$ and proportional payoffs the strategy profiles $\hat{s}_p^1 = \left(\hat{x}_i^1, \alpha\frac{1}{3}\right)_i$ with $\hat{x}_i^1 \in N_{-i}$ are the unique trembling hand perfect equilibria of the game.*

Proof. In the appendix. ■

The proof of Proposition 3.4 also shows that equilibria that are asymmetric in the timing decision do exist but they are not trembling hand perfect. For instance if player C is absolutely sure that A and B steal from each other in $t = 1$ then C is indifferent as to when to attack. It follows that the strategies $s_A = \left(B, \alpha\frac{1}{3}\right), s_B = \left(A, \alpha\frac{1}{3}\right), s_C = \left(\phi, A, \alpha\frac{1}{3}\right)$ identify a subgame perfect equilibrium of the game. Still if player C is unsure about the opponents' behavior or if he fears that A or B can make mistakes in implementing their strategies then to attack in $t = 1$ becomes the unique best response.

There are also equilibria in which all the agents use their attack in $t = 2$. Still players have no convincing reasons to follow the strategies that sustain these equilibria (i.e. punish the deviations). This is due to the fact that agents only care about their absolute share. In other words they are exclusively interested in how much to steal and not in who to steal from. Players are indifferent as to who to attack (actions are absolutely equivalent) such that there are no incentives to retaliate against a player who deviated and used his attack in $t = 1$. An example in the appendix clarifies the situation. Moreover the mere possibility that the opponents may tremble and use their attack in $t = 1$ makes a player prefer to attack in the first period in order not to be preempted.

3.3.2.2 The winner takes all case

With the WTA payoffs specification the game has four strict equilibria: the two circles with all the three players using their attack in $t = 1$ and the two circles with all the three players using their attack in $t = 2$. These are the unique equilibria of the game.

Proposition 3.5 *With $N = 3$ and with WTA payoffs the game has four strict equilibria:*
- *the two profiles $\hat{s}_w^1 = \left(\hat{x}_i^1, \alpha\frac{1}{3}\right)_i$ with $\hat{x}_i^1 \in N_{-i}$ and such that $\Pi_i^2 = \frac{1}{3}$ for any i;*
- *the two profiles $\hat{s}_w^2 = \left(\phi, \hat{x}_i^2, \alpha\frac{1}{3}\right)_i$ with $\hat{x}_i^2 \in N_{-i}$ and such that $\Pi_i^2 = \frac{1}{3}$ for any i.*

Proof. First consider $\hat{s}_w^1$. If player i postpones his attack to $t = 2$ then $\Pi_i^2 = \frac{1}{3}(1 - \alpha^2)$ such that $u_i = 0$ because there exists a j such that $\Pi_j^2 > \Pi_i^2$. Then consider $\hat{s}_w^2$. If player i deviates to $s_i^1 = \left(x_i^1, \alpha\frac{1}{3}\right)$ with $x_i^1 \in N_{-i}$ then there is a player j such that $\Pi_j^2 > \Pi_i^2$ and therefore $u_i = 0$. The proof that also other possible deviations are harmful is analogous to the proof of Proposition 3.2. ■

The interesting aspect is that there are strict equilibria such that all the players postpone their moves. In fact the possibility of best responding in $t = 2$ is now worthwhile because players can act in such a way as to be the leader of the market at the end of the game. Indeed, as Proposition 3.5 states, a player who deviates from $\hat{s}_w^2$ is sure to finish with $u_i = 0$. Consider the profile such that players are inactive in the first period and then in $t = 2$ A attacks B, B attacks C and C attacks A. Assume that player A deviates and attacks a player (say C) in $t = 1$. Then players B and C have to decide who to attack in the subgame that takes place at $t = 2$ (payoffs appear in the order u_A, u_B, u_C).

$t = 1$ — A attacks C

$t = 2$		C	
		att. A	att. B
	att. A	$0, 1, 0$	$0, 1, 0$
	att. C	$0, 1, 0$	$1, 0, 0$

It is now a dominant strategy for B to attack A who holds $\Pi_A^1 = \frac{1}{3}(1+\alpha)$. The intuition is clear: A used his attack and he cannot move anymore while C has been weakened ($\Pi_C^1 = \frac{1}{3}(1 - \alpha)$) and he cannot catch up with his initial share of $\frac{1}{3}$ (indeed $u_C = 0$ in all the outcomes). At the opposite B can still steal $\alpha\frac{1}{3}$ with the added advantage of having the chance to best respond to A's deviation. If B attacks A in $t = 2$ then B is sure to finish the game as the market leader. There are no subgame perfect equilibria for which A's deviation is profitable.

Notice that the equilibria $\hat{s}_w^1$ and $\hat{s}_w^2$ lead to the same payoff $u_i = \frac{1}{3}$ such that they are Pareto equivalent. But the strategy to wait and attack in $t = 2$ is in general less risky and more rewarding. In fact the above example shows that if player i finds himself in the situation of being the only player who moved in $t = 1$ then i has no chances to win the market. At the opposite if player i happens to be the only one who saved his attack for $t = 2$ then there is at least a possibility (the two rivals attacked each other in $t = 1$) that i will finish the game as the largest shareholder. Therefore the equilibria $\hat{s}_w^2$ seem to be more likely to arise. No one wants to break the initial symmetric situation because of the fear of the retaliation. This result is reminiscent of the analysis of so called truels (gun duels among three players) which shows that, under certain assumptions, the best strategy a player can adopt is to shoot in the air instead that against an opponent (see Kilgour, 1972 and Kilgour and Brams, 1997).

3.3.3 The game with four players

The four players game is analogous to the three players game apart from an important difference. In fact with four players there are no equilibria in which all the players postpone their moves. Therefore the fact that with the WTA payoff specification there exist equilibria in which all the players postpone their attack appears to be a peculiarity of the three players case.

3.3.3.1 The proportional case

Under proportional payoffs the game with four players is characterized by equilibria that have the same structure as those found in the three players case. As before players only risk to be preempted if they postpone their move. This risk is not worth taking given that there are no advantages in having the chance to best respond in $t = 2$.

Proposition 3.6 *With $N = 4$ and proportional payoffs the strategy profiles $\hat{s}_p^1 = \left(\hat{x}_i^1, \alpha\frac{1}{4}\right)_i$ with $\hat{x}_i^1 \in N_{-i}$ are the unique trembling hand perfect equilibria of the game.*

Proof. In the appendix. ■

3.3.3.2 The winner takes all case

Only the strategy profiles such that all the players use their attack in $t = 1$ and $u_i = \frac{1}{4}$ qualify as equilibria of the game.

Proposition 3.7 *With $N = 4$ and WTA payoffs the strategy profiles $\hat{s}_w^1 = \left(\hat{x}_i^1, \alpha\frac{1}{4}\right)_i$ with $\hat{x}_i^1 \in N_{-i}$ and such that $\Pi_i^2 = \frac{1}{4}$ are the unique trembling hand perfect equilibria of the game.*

Proof. Consider $\hat{s}_w^1$. If player i postpones his attack to $t = 2$ then $\Pi_i^2 = \frac{1}{4}(1-\alpha^2)$ such that $u_i = 0$ because there exists a j such that $\Pi_j^2 > \Pi_i^2$. Now consider profiles of the kind $s_w^2 = \left(\phi, x_i^2, \alpha\frac{1}{4}\right)_i$ with $x_i^2 \in N_{-i}$ and such that $\Pi_i^2 = \frac{1}{4}$ for any i and assume that a player deviates and attacks an opponent j in $t = 1$. In the subgame that follows (see below) the two players with $\Pi_i^1 = \frac{1}{4}$ (those who were not attacked) will attack each other letting the player who deviated as the market leader. ■

Start from the following candidate profile for being an equilibrium in $t = 2$: player A attacks B, B attacks C, C attacks D and D attacks A such that $\Pi_i^2 = u_i = \frac{1}{4}$. If A deviates and attacks an opponent (say B) in $t = 1$ then B, C and D have to move in the subgame which takes place in the second period. We know that player B cannot catch up with his initial share and $u_B = 0$ no matter his move. Indeed B's strategies are absolutely equivalent such that he is indifferent as to who to attack. His move, from the other players' point of view, is totally unpredictable. Focusing on the players who still have chances to be the market leader (C and D) we can avoid to model B's move. For both C and D to attack B is a dominated action given that B cannot win any more. Therefore we can simplify the subgame considering only the undominated strategies of C and D (payoffs appear in the order u_A, u_B, u_C, u_D).

$t = 1$ — A attacks B

$t = 2$

		D: att. A	D: att. C
C	att. A	$0, 0, \frac{1}{2}, \frac{1}{2}$	$0, 0, 0, 1$
	att. D	$0, 0, 1, 0$	$1, 0, 0, 0$

This subgame has three Nash equilibria but only the one in which C and D attack each other is trembling hand perfect. In fact in the other two equilibria a player uses a weakly dominated strategy which cannot be a best response to a totally mixed strategy of the opponent. In the equilibrium such that C and D attack each other player A conquers the entire market, therefore his deviation is profitable and the profiles in which all the players postpone their attack are not equilibria of the entire game.

The difference with the three players case can be easily explained. In the three players game there is only an agent who can take advantage of the situation in which a unique player used his attack in $t = 1$. With four players two are the agents who can exploit such a situation and a free rider problem arises between them. In the above example in fact players C and D attack each other in the hope that the other attacks A.

3.4 Summary and Discussion

The paper presented a new class of games in which a few symmetric players have to decide when to steal part of the pie that belongs to the competitors. The amount of pie that can be stolen is assumed to be proportional to the aggressor's size and players' utility is increasing in their final share of the pie. This simple game captures therefore a stylized version of strategic interactions that may occur in biology, business and politics.

We studied the game under two different payoffs specifications. With *proportional* payoffs players simply want to maximize their final share. With *winner takes all* (WTA) payoffs players want to be the largest shareholder in the market. Strategic considerations are more complex in the second case given that players are also interested in how their share compare with those held by the opponents.

The paper focused on solving for the optimal timing strategies of the players. In particular it investigated how a player's choice of preempting or following the rivals changes under the two payoffs specifications and in response to the number of players involved in the game. The main results are that:

- The equilibria of the WTA case are a subset of the equilibria of the proportional case.
- Equilibrium market allocations are always symmetric in the WTA case while they can be asymmetric in the proportional case.
- With proportional payoffs agents always want to preempt the rivals.
- With WTA payoffs agents want to preempt the rivals when the game is played by two or four players. In the three players case there are equilibria such that players postpone their moves.

Focusing on the last point, the Stealing game therefore provides an example of a timing game in which, for a given payoff structure, optimal timing strategies change according to the number of participants. This is an interesting aspect of timing games that has been so far neglected and that possibly requires further and more general research.

3.5 Appendix

3.5.1 Proof of Propositions 3.4 and 3.6

Because of Prop. 3.1 and given that nothing can happen in $t = 2$, the expected share $\left(E(\tilde{\Pi}_i^2)\right)$ associated with the profiles $\hat{s}_p^1 = \left(\hat{x}_i^1, \alpha\frac{1}{3}\right)_i$ with $\hat{x}_i^1 \in N_{-i}$ is given by:

$$E(\tilde{\Pi}_i^2) = \frac{1}{N}\left(1 + \alpha - \sum_{j \neq i} x_j^1(i)\alpha\right)$$

where, as in Section 2, $x_j^t(i) \in \{0, 1\}$ indicates the probability with which player j attacks player i in $t \in \{1, 2\}$. If player i deviates and saves his attack for the second period then i's share at the end of $t = 1$ is $E(\breve{\Pi}_i^1) = \frac{1}{N} - \sum_{j \neq i} x_j^1(i)\alpha\frac{1}{N}$. Player i will then use his (possibly weaker) attack in $t = 2$ and his payoff, which we indicate with $E(\breve{\Pi}_i^2)$, is:

$$E(\breve{\Pi}_i^2) = \frac{1}{N} - \sum_{j \neq i} x_j^1(i)\alpha\frac{1}{N} + \alpha\left(\frac{1}{N} - \sum_{j \neq i} x_j^1(i)\alpha\frac{1}{N}\right)$$

The difference between $E(\tilde{\Pi}_i^2)$, the payoff of the proposed equilibrium, and $E(\breve{\Pi}_i^2)$, the payoff stemming from the deviation, is given by $E(\tilde{\Pi}_i^2) - E(\breve{\Pi}_i^2) = \alpha^2\frac{1}{N}\sum_{j \neq i} x_j^1(i)$. Given that $\alpha^2\frac{1}{N} > 0$ and $\sum_{j \neq i} x_j^1(i) \geq 0$ it follows that this difference cannot be negative and $\hat{s}_p^1$ are equilibria of the game. Considering that players are indifferent as to who to attack and that they may tremble and make mistakes then $\sum_{j \neq i} x_j^1(i) > 0$ for any i and $E(\tilde{\Pi}_i^2) - E(\breve{\Pi}_i^2) > 0$ such that to use the attack in $t = 1$ is a strict best response. Therefore equilibria that are asymmetric in the timing decision are not trembling hand perfect.

3.5.2 Example of the proportional case with 3 players

The possible equilibrium outcomes when all the players postpone their move are the ones indicated in Proposition 3.1. In fact the situation, given that nothing happened in $t = 1$, is analogous to the one shot game analyzed in Section 3.2.2. This is a connected set of Nash equilibria such that we can check if it is profitable to deviate and attack someone in $t = 1$ from any of these profiles without loss of generality.

Take the profile in which players are inactive in $t = 1$ and then in $t = 2$ A attacks B, B attacks C and C attacks A such that $u_i = \frac{1}{3}$. Assume that player A deviates and attacks C in $t = 1$. Then in the second period players B and C have to simultaneously use their attack. The subgame which takes place in $t = 2$ and follows A's deviation is captured by the following matrix. Payoffs appear in the order u_A, u_B, u_C.

		C	
		att. A	att. B
B	att. A	$\frac{1}{3}(1-\alpha+\alpha^2)$, $\frac{1}{3}(1+\alpha)$, $\frac{1}{3}(1-\alpha^2)$	$\frac{1}{3}$, $\frac{1}{3}(1+\alpha^2)$, $\frac{1}{3}(1-\alpha^2)$
	att. C	$\frac{1}{3}(1+\alpha^2)$, $\frac{1}{3}(1+\alpha)$, $\frac{1}{3}(1-\alpha-\alpha^2)$	$\frac{1}{3}(1+\alpha)$, $\frac{1}{3}(1+\alpha^2)$, $\frac{1}{3}(1-\alpha-\alpha^2)$,

There exists a strategy profile such that player A has no incentive to deviate, namely the profile under which B and C retaliate against A. This is a Nash equilibrium of the above subgame but actually all the four outcomes are Nash equilibria. In fact both B and C are indifferent among their two strategies given that these are absolutely equivalent. Players do not have any advantage in having the possibility to best respond. Therefore there are no obvious reasons to expect players B and C to change their original plan of action (such that A would have $u_A = \frac{1}{3}(1+\alpha^2)$ and his deviation would be profitable).

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