European University Institute

Department of Economics

Some Essays in Growth Theory

Mauro Bambi

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

Florence, March 2007

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Part I

Introduction

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This thesis deals with economics growth theory. The purpose of this introduction is to give an idea of the issues that are a subject of my thesis, avoiding technical details. A technical summary of all chapters is provided below as abstracts of the three thesis chapters.

By growth theory I refer to that field of economic theory which studies through mathematical models, the mechanisms behind the growth of the main macroeconomic variables, like capital, consumption, and output. These economic mechanisms depend crucially on the set of hypothesis which the model is built on. Relax some of these assumptions, or change some of them, in order to increase the explicatory power of the model is one of the main objective of the theoretic macroeconomist.

The first two chapters of my thesis can be read exactly under this view point. More precisely, the common question behind these two papers can be summarized as follows "What are the implications in term of economic growth when capital takes time to becomes productive?" To this purpose, it is worth noting that in almost all the economic models the factors of production become productive istantaneously even if a large number of economists have often underlined the importance of the time dimension of some of them, like, for example, capital. For this reason the economic implications of the introduction of a delay in production, or time to build, has been studied for an endogenous AK growth model in Chapter 1 and for an exogenous Benhabib Farmer model (1994) in Chapter 2.

The main results of the first model regards the dynamics of capital. Differently form the prediction of the standard AK model, capital doesn't jump immediately to its balanced growth path but oscillatory converges to it. This result, exactly as others obtained by doing different assumptions on the time dimension of capital, like for example vintage capital, are closer to the empirical evidences underlined by Jones (1995) and McGrattan (1998).

In the second chapter of the thesis, the time to build assumption is introduced in a Benhabib and Farmer model (1994). A first interesting question is to catch the influence of the introduction of a delay in production on the presence of local indeterminacy. According to our results, local indeterminacy is preserved but is conditioned to suitable choices of the level of externalities and of the delay coefficient: a higher time to build coefficient has to be related to a higher level of externalities in order to generate optimizing multiple equilibrium paths. Moreover a complex structure of capital induces more complicated dynamics of the macroeconomic variables. Economic cycle and region of local instability may rise in this context. Finally, it is worth noting that exactly as in Chapter 1, capital, investment and output display oscillatory convergence.

The Benhabib and Farmer model (1994) is studied in the last Chapter, too. Differently from the previous two chapters, in Chapter 3, the model is not modified in any of its assumptions but some predictions in term of welfare are highlighted. Assuming a government which is able to pin down expectations on one of the optimizing multiple equilibrium paths, it can be relevant to understand what of these paths is the best one from a welfare point of view. This is exactly the purpose of the last chapter where numerous economic considerations on the mechanism at work for different level of the externalities are also underlined.

Abstratcs of Thesis Chapters

Abstract of Chapter 1: Endogenous growth and Time-to-Build: the AK-case

In this paper, a continuous time AK model is fully analyzed under the time-to-build assumption. Existence and uniqueness of a (real) balance growth path, as well as oscillatory convergence are proved. Moreover, the role of transversality conditions and capital depreciation are highlighted. Numerical simulations are also provided for different choices of the time-to-build delay.

Keywords: AK Model; Time-to-Build; D-Subdivision method.

JEL Classification: E00, E3, O40.

Abstract of Chapter 2: (In)determinacy and Time-to-Build

This paper generalizes Benhabib and Farmer model (1994), by allowing for a strictly positive time-to-build of capital. The introduction of a time-to-build delay yields a system of mixed functional differential equations. We develop an efficient strategy to fully describe the dynamic properties of our economy; in the simpler case of no or "mild" externalities, the dynamic behavior of the economy around the steady state is of "saddle-path" type. On the other hand, "high" externalities leads to a more complex dynamics; according to the choice of the delay coefficient, local indeterminacy, Hopf bifurcation and local instabiity may rise.

Keywords: Indeterminacy; Time-to-Build; Mixed Functional Differential Equations

JEL Classification: E00, E3, O40.

Abstract of Chapter 3: Welfare Ranking of Equilibrium Paths in One-Sector Growth Models with Non-Convex Technologies

We consider a business cycle model with productive externalities and an aggregate non-convex technology *à la* Benhabib and Farmer, which exhibit indeterminacy of the steady state and multiplicity of deterministic equilibria. The aim of the paper is to rank these different equilibria according to the initial values of consumption using both linear approximation methods when the initial conditions lay in the region of stability (in the sense of Lyapunov) and simulation methods for initial conditions outside this region. We finally study the implications of such a ranking in terms of smoothness of the (second best) optimal solution and show that maximizing welfare consumption and labor paths are all the smoother than the level of increasing returns is low.

Keywords: Increasing returns, Local indeterminacy, Welfare analysis

JEL classification: E32, E4, H61, O42, O47.

Part II

Chapters

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CHAPTER 1

ENDOGENOUS GROWTH AND TIME-TO-BUILD: THE AK-CASE

1.1 Introduction

Recently Boucekkine et al. [5], have studied the dynamics of an AK-type endogenous growth model with vintage capital. They find that vintage capital leads to oscillatory dynamics governed by replacement echoes consistently with previous results in Benhabib and Rustichini [5], and Boucekkine et al. [9]. In this paper, we propose an AK endogenous growth model under the assumption that capital takes time to become productive. In the literature, this assumption is often referred as "time-to-build".

Jevons [19], was one of the first to underline the empirical relevance of this assumption: "A vineyard is unproductive for at least three years before it is thoroughly fit for use. In gold mining there is often a long delay, sometimes even of five or six years, before gold is reached"[1]. The time dimension of capital was further studied by Hayek [17], who identified in the time of production one of the possible sources of aggregate fluctuations. Hayek's insight was formally confirmed for the first time by Kalecki [20], and afterward by Kydland and Prescott [6], who showed that it contributes to the persistence of the business cycle. In this paper, the time-to-build assumption is introduced by a delay differential equation for capital. Delay differential equations, and in general, functional differential equations are very interesting but, at the same time, quite complicated mathematical objects. Since the first contributions of Kalecki [20], Frisch and Holme [14], and, Belz and James [7], very few authors have used this mathematical instrument for modeling the time structure of capital. To our knowledge, the only works in (exogenous) growth theory introducing time-to-build in this way, are Rustichini [9], Asea and Zak [1], and Collard et al. [12]. All these papers find that for values of the delay coefficient which are sufficiently small, time-to-build is responsible for the oscillatory behavior of capital, output and investment.

In this paper, some theorems regarding the existence, uniqueness and shape of the general (continuous) solution of a linear delay differential equation with forcing term are presented in details, and a "new" method to prove stability, the D-Subdivision method, is introduced.

[1] Jevons [19], Chapter VII: Theory of Capital, page 225.

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This method is really useful since it let us count the number of roots (eigenvalues) having positive real part even if the dimension of the set of the roots is infinite. Taking into account this theoretical background, the existence of a unique balance growth path and the dynamic behaviors of the detrended variables are fully analyzed.

The paper is organized as follows. We firstly present the model setup in Section 1.2 and we derive the first order conditions by applying a variation of the Pontrjagin's maximum principle. In Section 1.3, we introduce some mathematical results on the theory of functional differential equations and the D-Subdivision method. Then the existence and uniqueness of the balance growth path is proved and the influence of a variation of the delay coefficient on the magnitude of the growth rate is fully analyzed. The transitional dynamics of the economy is reported in Section 1.5. The next section makes some considerations regarding the role of capital depreciation on the dynamic behavior of capital and the possibility of Hopf bifurcation. A numerical example showing the dynamic behavior of the economy is reported in Section 1.6. Finally, in Section 1.7 there are some concluding remarks.

1.2 Problem Setup

We analyze a standard one sector AK model with time-to-build. To be precise we assume from now on that capital takes d years to become productive. Then the social planner solves the following problem

$$\max \int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$\dot{k}(t) = \tilde{A}k(t-d) - c(t) \tag{1.1}$$

given initial condition $k(t) = k_0(t)$ for $t \in [-d, 0]$ with $d > 0$. All the variables are per capita. The parameter $\tilde{A} = (A - \delta)e^{-\phi d} > 0$ depends on the productivity level, A, the usual capital depreciation rate, δ,, and the depreciation rate of capital before it becomes productive, ϕ. From now on we refer to the last one as depreciation "before use". Given this capital depreciation structure, $k(t-d)e^{-\phi d}$ is net capital at the time it becomes productive. Observe that the lower d is, the higher is the net capital which is effectively employed in production. Moreover let us assume $\phi \leq \delta$, which may be justified by referring to the depreciation in use literature (see Greenwood *& al.* [15], and Burnside and Eichenbaum [11]). Finally, with no time-to-build the problem becomes a standard AK model.

Following Kolmanovskii and Myshkis [3] it is possible to extend the *Pontrjagin's principle* to this optimal control problem. Then, the Hamiltonian for this system can be constructed:

$$\mathcal{H}(t) = \frac{c(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \mu(t)\left[\tilde{A}k(t-d) - c(t)\right]$$

and its optimality conditions are

$$c(t)^{-\sigma} e^{-\rho t} = \mu(t) \tag{1.2}$$

$$\mu(t+d)\tilde{A} = -\dot{\mu}(t) \tag{1.3}$$

with the standard transversality conditions

$$\lim_{t\to\infty} \mu(t) \geq 0 \quad \textit{and} \quad \lim_{t\to\infty} \mu(t) k(t) = 0$$

From equations (1.2) and (1.3) we can get the *forward looking Euler-type equation*

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[\tilde{A}\left(\frac{c(t)}{c(t+d)}\right)^{\sigma} e^{-\rho d} - \rho\right] \tag{1.4}$$

Exactly as in the standard AK model, consumption growth does not depend on the stock of capital per person. However in our context the positive constant growth rate is not explicitly given by the Euler equation which is a nonlinear advanced differential equation in consumption. This difference is due to the fact that the real interest rate $r = \tilde{A}\left(\frac{c(t)}{c(t+d)}\right)^{\sigma} e^{-\rho d}$, which the household gets investing in capital, is weighted by the marginal elasticity of substitution between consumption at time t and consumption at time $t+d$. Before proceeding with the analysis of the BGP of our economy, we present in the next section some theoretical results from functional differential analysis which will be used to prove the main results and characteristics of the economy under study.

1.3 Some Preliminary Results

Consider the general linear delay differential equation with forcing term $f(t)$:

$$a_0\dot{u}(t) + b_0 u(t) + b_1 u(t-d) = f(t) \tag{1.5}$$

subject to the initial or boundary condition

$$u(t) = \xi(t) \quad \text{with} \quad t \in [-d, 0]. \tag{1.6}$$

Theorem 1.1 ***(Existence and Uniqueness)*** *Suppose that f is of class C^1 on $[0,\infty)$ and that ξ is of class C^0 on $[-d,0]$. Then there exists one and only one continuous function $u(t)$*

which satisfies (1.6), and (1.5) for $t \geq 0$. *Moreover, this function* u *is of class* C^1 *on* (d, ∞) *and of class* C^2 *on* $(2d, \infty)$. *If* ξ *is of class* C^1 *on* $[-d, 0]$, $\dot{u}$ *is continuous at* τ *if and only if*

$$a_0 \dot{\xi}(d) + b_0 \xi(d) + b_1 \xi(0) = f(d) \tag{1.7}$$

If ξ *is of class* C^2 *on* $[-d, 0]$, $\ddot{u}$ *is continuous at* $2d$ *if either (1.7) holds or else* $b_1 = 0$, *and only in these cases.*

Proof. See Bellman and Cooke [8], , Theorem 3.1, page 50-51. ■

The function u singled out in this theorem is called the continuous solution of (1.5) and (1.6). Then in order to see the shape of this continuous solution the following theorem is useful:

Theorem 1.2 *Let* $u(t)$ *be the continuous solution of (1.5) which satisfies the boundary condition (1.6). If* ξ *is* C^0 *on* $[-d, 0]$ *and* f *is* C^0 *on* $[0, \infty)$, *then for* $t > 0$,

$$u(t) = \sum_r p_r e^{z_r t} + \int_0^t f(s) \sum_r \frac{e^{z_r (t-s)}}{h'(z_r)} ds \tag{1.8}$$

where $\{z_r\}_r$ *and* $\{p_r\}_r$ *are respectively the roots and the residue coming from the characteristic equation,* $h(z)$, *of the homogeneous delay differential equation*

$$a_0 \dot{u}(t) + b_0 u(t) + b_1 u(t-d) = 0 \tag{1.9}$$

Note: $p_r = \frac{p(z_r)}{h'(z_r)}$ *where*

$$p(z_r) = a_0 \xi(0) + (a_0 z_r + b_0) \int_{-d}^{0} \xi(s) e^{-z_r s} ds$$

Proof. See Appendix A.1. ■

Since in our context it shall be fundamental to have real continuous general solution, we present here the following theoretical results.

Theorem 1.3 *The unique general continuous solution of problem (1.5) with boundary condition* $\xi : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ *and forcing term* $f : I \to \mathbb{R}^+$, *is a real function.*

Proof. See Appendix A.2. ■

Some considerations on these theorems are needed. We start with the last result. The important message of Theorem 1.3 is the following: if we assume a boundary condition and a forcing term which are real functions then also the general continuous solution must be real. Other considerations regard the proofs of Theorem 1.1 and 1.2: both of them are strictly related to the fact that all the roots of $h(z)$ lie in the complex z-plane to the left of some vertical line. That is, there is a real constant c such that all roots z have real part less then c. This consideration is in general no longer true for advanced differential equations which are characterized by CE with zeros of arbitrarily large real part. However as explained by Bellman and Cooke [8],[2] it is possible to write the solution of any advanced differential equation as a sum of exponentials using the finite Laplace transformation technique. Moreover observe that the characteristic equation of (1.5),

$$h(z) \equiv z + a + be^{-zd} = 0 \tag{1.10}$$

with $a = \frac{b_0}{a_0}$ and $b = \frac{b_1}{a_0}$, is a transcendental function with an infinite number of finite roots. Sometimes $h(z)$ is also called the characteristic quasi-polynomial. Asymptotic stability requires that all of these roots have negative real part. In order to help in the stability analysis we introduce two important mathematical results: the Hayes theorem and the *D-Subdivision method* or D-Partitions method. Hayes Theorem [2] in its more general formulation states the following:

Theorem 1.4 *The roots of equation* $pe^z + q - ze^z = 0$ *where* $p, q \in \mathbb{R}$ *lie to the left of* $\operatorname{Re}(z) = k$ *if and only if*

(a) $p - k < 1$

(b) $(p-k)\,e^k < -q < e^k\sqrt{a_1^2 + (p-k)^2}$

where a_1 *is the root of* $a = p\tan a$ *such that* $a \in (0, \pi)$. *If* $p = 0$, *we take* $a_1 = \frac{\pi}{2}$.

One root lies on $\operatorname{Re}(z) = k$ *and all the other roots on the left if and only if* $p - k < 1$ *and* $(p-k)\,e^k = -q$.

Two roots lies on $\operatorname{Re}(z) = k$ *and all the other roots on the left if and only if* $-q = e^k\sqrt{a_1^2 + (p-k)^2}$

Proof. See Hayes [2], page 230-231. ■

However this Theorem doesn't say anything about the sign of the real part of the roots of the transcendental function when the conditions *(a)* and *(b)* are not respected. For

[2] Look at Chapter 6 page 197-205.

this reason the D-Subdivision method is now introduced (for more details on this method, El'sgol'ts and Norkin [13], or Kolmanovskii and Nosov [7]). Given a transcendental function like, for example, (1.10), this method is able to determine the number of roots having positive real part (for now on p-zeros) in accordance with the value of its coefficients (a and b in our specific case). This is possible since the zeros of a transcendental function are continuous functions of those same coefficients.

Definition 1.5 *Given the characteristic equation of a functional differential equation with constant coefficients, a D-Subdivision is a partition of the space of coefficients into regions by hypersurfaces, the points of which correspond to quasi-polynomials having at least one zero on the imaginary axis (the case $z = 0$ is not excluded).*

For continuous variation of the transcendental function coefficients the number of p-zeros may change only by passage of some zeros through an imaginary axis, that is, if the point in the coefficient space passes across the boundary of a region of the D-Subdivision. Thus, to every region Γ_k of the D-Subdivision, it is possible to assign a number k which is the number of p-zeros of the transcendental function. Among the regions of this partition are also found regions Γ_0 (if they exist) which are regions of asymptotic stability of solutions. Finally in order to clarify how the number of roots with positive real parts changes as some boundary of the D-Subdivision is crossed, the differential of the real part of the root is computed, and the decrease or increase of the number of p-zeros is determined from its algebraic sign. Since it becomes very useful later, we study, with the D-Subdivision method, the transcendental function (1.10).

First of all, observe that this equation has a zero root for $a + b = 0$. This straight line (see Figure 1.1) is one of the lines forming the boundary of the D-Subdivision. It is also immediately derived that the transcendental function (1.10) has purely imaginary roots if and only if

$$a + b\cos dy = 0, \quad y - b\sin dy = 0 \tag{1.11}$$

or

$$b = \frac{y}{\sin dy}, \quad a = \frac{-y\cos dy}{\sin dy} \tag{1.12}$$

The equations in parametric form (1.11) or (1.12) identify all the other D-Subdivision boundaries. To be precise there is one boundary for any of the following interval of y: $\left(0, \frac{\pi}{d}\right), \left(\frac{\pi}{d}, \frac{2\pi}{d}\right), \left(\frac{2\pi}{d}, \frac{3\pi}{d}\right), \ldots$. Moreover it is possible (and useful) to find the values of b for which the boundaries intercept the b-axis. The sequence of such b is $\left\{\ldots, -\frac{7\pi}{2d}, -\frac{3\pi}{2d}, 0, \frac{\pi}{2d}, \frac{5\pi}{2d}, \ldots\right\}$. Finally we show how p-zeros rises. In particular, when a crossing of C_l from Γ_0 to Γ_2 implies the rising of two p-zeros (that is, we focus on the interval $0 < y < \frac{\pi}{d}$). From (1.10) applying the implicit function theorem, we have that on C_l

$$\begin{aligned} dx &= -\operatorname{Re}\frac{da}{1-bde^{-diy}} \\ &= -\operatorname{Re}\frac{da}{1-bd\left(\cos dy - i\sin dy\right)} \\ &= \frac{\left(1-bd\cos dy\right)da}{\left(1-bd\cos dy\right)^2+b^2d^2\sin^2 dy} \end{aligned}$$

We find that $\cos yd < 0$ for $bd > 1$. Therefore, upon crossing the boundary C_l from region Γ_0 into Γ_2, a pair of complex conjugate roots gain positive real parts. The analysis on the other boundaries of the D-Subdivision is completely analogous. Taking into account all of these results, we are now ready to study our model completely.

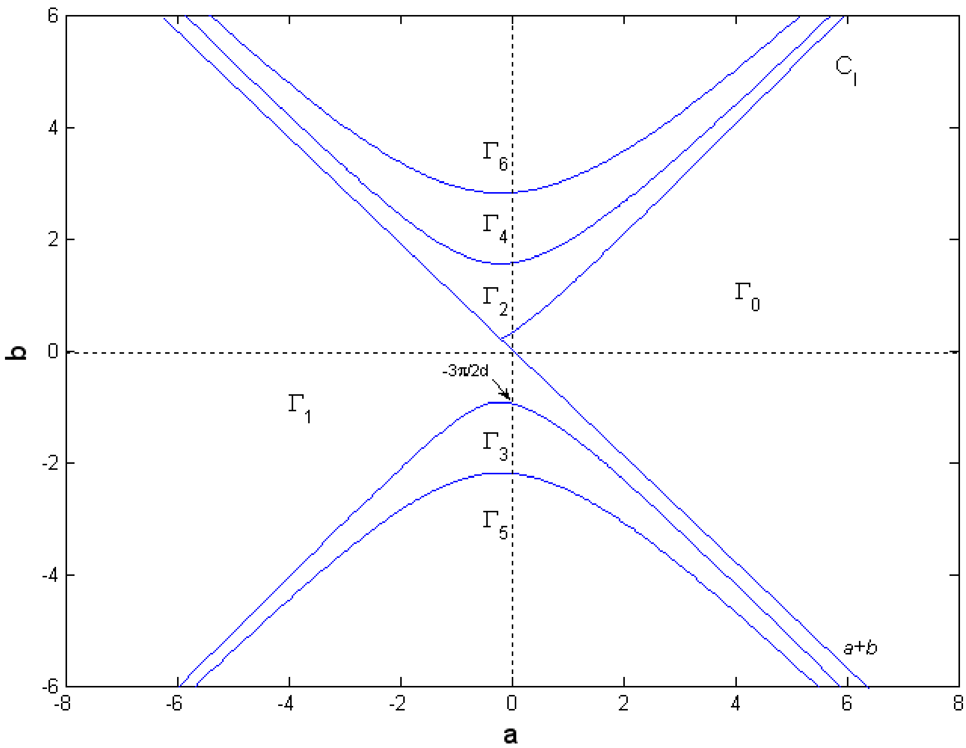

Figure 1.1: D-Subdivision for the trascendental function (1.10) assuming $d = 5$.

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1.4 Balance Growth Path Analysis

In order to show the existence and uniqueness of the BGP, we now present some results regarding the roots of the characteristic equation of the law of motion of capital, of its shadow price, and of consumption. These results are presented and proved in Lemma 1.6 and Lemma 1.7, respectively. Some pictures are also provided in order to help the reader get the main message behind the math. After that, the continuous solution of capital is rewritten as a sum of weighted exponentials (Corollary 1.8) and then, following a very similar strategy as that used in the standard AK model, a unique balance growth path for consumption and capital is proved by checking the transversality conditions. Very similar to this, is the requirement that for any exogenously given choice of the delay coefficient, the production function has to be sufficiently productive to ensure growth in consumption, but not so productive as to yield unbounded utility: $A \in (A_{\min}, A_{\max})$. On the other hand, it is possible to express the same requirement, given a certain level of technology, in term of the delay coefficient: $d \in (d_{\min}, d_{\max})$. Finally as in the standard case if $\sigma > 1$, then $A_{\max}$ is equal to plus infinity, while $d_{\min}$ is zero.

As anticipated in Lemma 1.6 we report some information on the roots of the CE of the law of motion of capital and its shadow price:

Lemma 1.6 *For any sufficiently high rate of depreciation "before use", ϕ, the following results hold:*

1) $\tilde{z}$ is the unique root with positive real part of the CE of the law of motion of capital;

2) $\tilde{s}$ is the unique root with negative real part of the CE of the law of motion of shadow price.

Proof. The characteristic equation of the law of motion of capital (1.1) is equal to the characteristic equation of its homogeneous part[3], namely

$$h(z) \equiv z - \tilde{A}e^{-zd} = 0 \tag{1.13}$$

It is immediate to show that this equation has a unique positive real root $z_{\tilde{v}} = \tilde{z}$ which is also the highest among its roots. In particular, through the *D-Subdivision method* it is possible to prove that the transcendental equation (1.13) has an increasing number of p-zeros as d rises. On the other hand if we assume $\phi = \hat{\phi}$ sufficiently high,[4] it happens that $\tilde{A} < \frac{3\pi}{2d}$ for any choice of d and then a unique p-zero exists[5]. These facts can be easily observed in

[3] The part of equation (1.1) not considering the forcing term $-bC(t)$.

[4] In the numerical simulation, reported in Section 7, we have assumed $\hat{\phi} \simeq 0.03$.

[5] This is also a consequence of the fact that $\tilde{A}$ converges to zero faster than $\frac{3\pi}{2d}$ as $d \to \infty$.

Figure 1.2. Finally, $\tilde{z} > \mathrm{Re}(z_v)$ occurs for any $v \neq \tilde{v}$ since all the roots of the CE of (1.1) in the detrended variables $\hat{x}(t) = x(t)e^{-\tilde{z}t}$ are negative. This is sufficient to prove result 1). Now observe that the CE of the shadow price law of motion (1.3) is

$$h(s) \equiv -s - \tilde{A}e^{sd} = 0 \tag{1.14}$$

then we can put in correspondence the roots of (1.13) and (1.14) through the transformation $z = -s$. From this consideration follows immediately that $\mathrm{Re}(s) = -\mathrm{Re}(z)$ and $\tilde{s} = -\tilde{z}$ is the root with the lowest real part of the characteristic equation of the law of motion of shadow price. ■

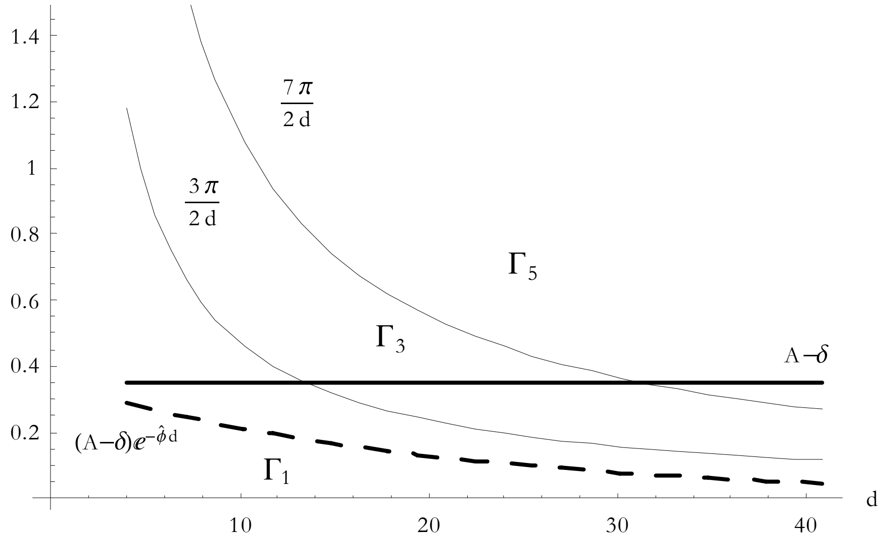

Figure 1.2: Number of p-zeros of (1.13) according to the choice of the delay coefficient.

Lemma 1.6 tells us that if we assume a sufficiently high depreciation "before use" rate, $\hat{\phi}$, then $\tilde{z}$ is the constant growth rate of capital and the unique p-zero of (1.13). Now it will be useful for proving a common growth rate of consumption and capital to show the following Lemma:

Lemma 1.7 *A positive and constant growth rate of consumption, g_c, always exists for $A > A_{\min} = \delta + \rho e^{(\rho+\hat{\phi})d}$.*

Proof. First of all observe that since the Euler equation (1.4) is a nonlinear advanced differential equation we cannot write directly its continuous general solution (Theorem 1.2

doesn't apply). However it is possible to overcome this fact by observing that the general continuous solution of consumption can be obtained indirectly by the first order condition (1.2). Considering the general continuous solution of the shadow price of capital $\mu(t) = \sum_m a_m e^{-z_m t}$, we have that

$$c(t) = \frac{1}{\left(\sum_m a_m e^{-\sigma\lambda_m t}\right)^{\frac{1}{\sigma}}} \tag{1.15}$$

where we have called

$$\lambda = \frac{1}{\sigma}(z - \rho) \tag{1.16}$$

From equation (1.15) we can derive that the basic solutions of (1.4) have exponential form, namely the basic solutions are $\left\{e^{\lambda_m}\right\}_m$; moreover taking into account (1.13) and (1.16) we can derive indirectly the characteristic equation[6] of (1.4)

$$h(\lambda) = \sigma\lambda + \rho - \tilde{A}e^{-(\sigma\lambda+\rho)d} \tag{1.17}$$

Using the Hayes theorem or the D-Subdivision method, a unique positive real root, $\lambda_{\tilde{m}} = g_c$ exists for A sufficiently large, namely $A > A_{\min} = \delta + \rho e^{(\rho+\phi)d}$. This is exactly the condition for endogenous growth in the standard AK model when the assumption $d = 0$ is relaxed. Observe also that in this context the same requirement can be expressed in term of the delay, $d < d_{\max} = \frac{1}{\rho+\hat{\phi}} \log \frac{A-\delta}{\rho}$. Exactly as before, a unique p-zero exists if $\tilde{A}e^{-\rho d} < \frac{3\pi}{2d}$. It is obvious that, for $\phi = \hat{\phi}$, the inequality is always respected (see Figure 1.3) since $\hat{\phi}$ was sufficient to force $\tilde{A}$ to stay below $\frac{3\pi}{2d}$, and given that $(A-\delta)\, e^{-\hat{\phi}d} e^{-\rho d}$ is a product of functions which are positive and monotonic decreasing in d. Some considerations on the choice of ϕ lower than $\hat{\phi}$ are reported in Section 1.6. Then, from now on, we focus on the case $\phi \geq \hat{\phi}$. Now, endogenous growth implies that consumption and capital have to grow at a positive rate over time. This implies that $\lim_{t\to\infty} c(t) = +\infty$; then given (1.15), we have to impose that

$$\lim_{t\to\infty} \frac{1}{\left(a_{\tilde{m}} e^{-\sigma g_c t} + \sum_{m \notin \tilde{m}} a_m e^{-\sigma\lambda_m t}\right)^{\frac{1}{\sigma}}} = +\infty \tag{1.18}$$

[6]We have referred to equation (1.17) as the characteristic equation of the law of motion of consumption since gives us all the basic solutions.

Using the properties of the limits[7], it is possible to rewrite (1.18) as

$$\frac{1}{\left(\underbrace{\lim_{t\to\infty} a_{\tilde{m}} e^{-\sigma g_c t}}_{\to 0} + \underbrace{\sum_{m\notin\tilde{m}} \lim_{t\to\infty} a_m e^{-\sigma\lambda_m t}}_{\to\infty}\right)^{\frac{1}{\sigma}}} = +\infty$$

Then it results that the relation (1.18) is satisfied if and only if $a_m = 0$ for any $m \neq \tilde{m}$. Taking into account this fact, the general continuous solution of consumption is

$$c(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c t}$$

■

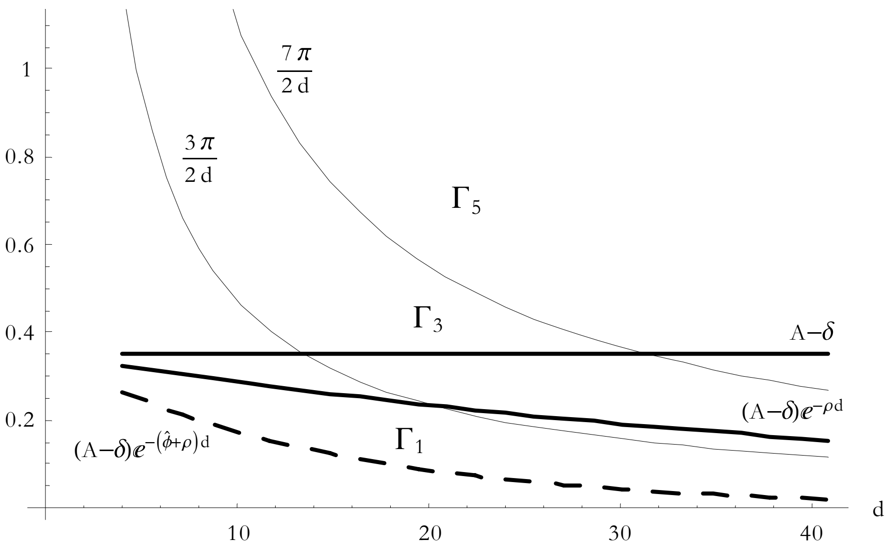

Figure 1.3: Number of p-zeros of (1.17) according to the choice of the delay coefficient.

Our objective is to prove that the growth rate of consumption and capital are the same $g = g_c$. However before proving it, we introduce the following Corollary of Theorem 1.2 which let us to rewrite the continuous solution of capital as a sum of weighted exponentials.

[7]The following properties have been used: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$, and $\lim_{x\to a} [\sum_i f_i(x)] = \sum_i \lim_{x\to a} f_i(x)$

Corollary 1.8 *The solution of the law of motion of capital can be written as*

$$k(t) = \sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_v N_{\tilde{m},v} e^{z_v t} \tag{1.19}$$

where $P_{\tilde{m},v} = -\frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{(g_c - z_v) h'(z_v)}$ *and* $N_{\tilde{m},v} = n_v - P_{\tilde{m},v}$.

Proof. According to Theorem 1.2 and Lemma 1.7, the continuous general solution of consumption and capital are respectively

$$c(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c t} \tag{1.20}$$

$$k(t) = \sum\nolimits_v n_v e^{z_v t} - \int_0^t c(s) \sum\nolimits_v \frac{e^{z_v (t-s)}}{h'(z_v)} ds \tag{1.21}$$

Now the integral part of equation (1.21) is equal to

$$\int_0^t a_{\tilde{m}}^{-\frac{1}{\sigma}} e^{g_c s} \sum\nolimits_v \frac{e^{z_v (t-s)}}{h'(z_v)} ds = \sum\nolimits_v \frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{(g_c - z_v) h'(z_v)} \left(e^{g_c t} - e^{z_v t} \right)$$

and substituting in (1.21) after some algebra we get (1.19). ∎

Some comments on equations (1.20) and (1.19) are needed. These equations are very close to the general solution form for consumption and capital in the usual framework, with ordinary differential equations; in particular $k(t)$ is a weighted sum of exponentials; however, this similarity can be found for systems of mixed functional differential equations *only* in the particular case of a single equation with forced term. In the most general cases there doesn't exist a theorem which lets us write the solution in this way[8]. Moreover, the continuous solution of the law of motion of consumption (1.20) and capital (1.19), are not the optimal solution exactly as it happens in the ordinary case. Before getting optimality, transversality conditions have to be checked. Using this corollary and TVC, we prove now the existence of a unique balance growth path for consumption and capital.

Proposition 1.9 *Consumption and capital have the same balanced growth path* $g = g_c$. *This growth rate is positive and yields bounded utility if* $A \in (A_{\min}, A_{\max})$.

Proof. As shown in Lemma 1.7, the growth rate of consumption g_c is a positive constant if $A > A_{\min}$. Given that, we have to distinguish two cases: $\tilde{z} \leq g_c$ and $\tilde{z} > g_c$. The first case

[8] Recently Asl and Ulsoy [2] have proved that a general solution form can be written for system of delay differential equations using Lambert function.

is never possible. In fact, assume that $\tilde{z} \leq g_c$ then g_c is also the growth rate of capital as follows immediately by looking at equation (1.19). Then we can rewrite the characteristic equation of capital, after the transformation $\hat{k}(t) = k(t)e^{-g_c t}$, as

$$-we^{w} - g_c de^{w} + \tilde{A}de^{-g_c d} = 0 \tag{1.22}$$

where $w = zd$. Since g_c is the root having greater positive real part, all the roots of (1.22) must have negative real part which, from Hayes Theorem implies also that $g_c > \tilde{A}e^{-g_c d}$. However, this is never possible since it contradicts the positive sign of the consumption to output ratio at the balanced growth path

$$\frac{c(t)}{k(t)} = \tilde{A}e^{-g_c d} - g_c > 0$$

which can be obtained by dividing the law of motion of capital (1.1) by $k(t)$. Then the only possible case is $\tilde{z} = \sigma g_c + \rho > g_c$. This is exactly the requirement for having no unbounded utility: $(1-\sigma) g_c < \rho$. Then, before passing to the TVC we observe that if $\sigma > 1$, the utility is always bounded; on the other hand if $0 < \sigma < 1$ we need a condition on A such that the utility is bounded. Taking into account the CE (1.17) after some algebra this condition is $A < A_{\max} = \delta + \frac{\rho}{1-\sigma} e^{\left(\frac{\rho+\phi(1-\sigma)}{1-\sigma}\right)d}$ which is exactly the same condition for the standard AK model when the time-to-build parameter is equal to zero. Observe also that such a condition can be rewritten also in terms of the delay, $d > d_{\min} = \frac{1-\sigma}{\rho+(1-\sigma)\phi} \log \frac{(A-\delta)(1-\sigma)}{\rho}$. Now we show that the TVC

$$\lim_{t\to\infty} \mu(t)\, k(t) = 0 \tag{1.23}$$

implies necessarily a unique BGP which is g_c. In order to see this, we substitute the general continuous solutions of $\mu(t)$ and $k(t)$, into the TVC (1.23) and we get:

$$\lim_{t\to\infty} a_{\tilde{m}} e^{-\tilde{z}t} \left(\sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_v N_{\tilde{m},v} e^{z_v t} \right) = 0 \tag{1.24}$$

which is equal to

$$\lim_{t\to\infty} \left[a_{\tilde{m}} N_{\tilde{m},\tilde{v}} + \sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},\tilde{v}} e^{(z_v-\tilde{z})t} + \sum\nolimits_v P_{\tilde{m},v} e^{(g_c-\tilde{z})t} \right] = 0$$

now for $a_{\tilde{m}} \neq 0$, the second and third term in the parenthesis converge to zero since $z_v - \tilde{z} < 0$ for any v and $g_c - \tilde{z} < 0$. Then the TVC are respected if and only if

$$N_{\tilde{m},\tilde{v}} \equiv \frac{a_{\tilde{m}}^{-\frac{1}{\sigma}}}{(g_c - \tilde{z})\, h'(\tilde{z})} + n_{\tilde{v}} = 0 \tag{1.25}$$

which implies

$$a_{\tilde{m}} = \left(\frac{1}{(\tilde{z} - g_c)\, h'(\tilde{z})\, n_{\tilde{v}}} \right)^{\sigma} \tag{1.26}$$

Observe that if we assume a constant boundary condition for capital, k_0, and for consumption, c_0, we can derive the following relation

$$c_0 = (\tilde{z} - g_c)\, e^{\tilde{z}d} k_0$$

which for $d = 0$ is exactly equal to the relation between c_0 and k_0 in the standard AK model (see Barro and Sala-i-Martin [4]). Concluding TVC holds if and only if condition (1.25) is verified. Given this condition, g_c is also the growth rate of capital since the continuous general solution of capital (1.21) can be rewritten as follows

$$k(t) = \sum\nolimits_v P_{\tilde{m},v} e^{g_c t} + \sum\nolimits_{v \neq \tilde{v}} N_{\tilde{m},v} e^{z_v t} \tag{1.27}$$

Then the optimal solution of capital (1.27) is asymptotically driven by g_c which implies a common growth rate with consumption. ■

This proposition provides evidence of how a unique balance growth path for consumption and capital can be proved to exist also in the case of time-to-build by checking to the transversality conditions. In fact, through condition (1.25), it is possible to rule out the eigenvalue coming from the characteristic equation of the law of motion of capital, having positive real part greater than g_c. Observe also that this fact and the assumption of the new structure of capital depreciation make all of these results hold for any choice of the delay in the interval $(d_{\min}, d_{\max})$ which guarantees the presence of endogenous growth and no unbounded utility.

Once we have shown that $g = g_c$ is the unique balanced growth path of consumption and capital, it is also interesting to see how different choices of the delay coefficient, d, and of the level of technology A, affect it. These considerations are reported in the following corollary:

Corollary 1.10 *Under $A \in (A_{\min}, A_{\max})$, $\frac{\partial g}{\partial d}$ and $\frac{\partial g}{\partial \phi}$ are negative while $\frac{\partial g}{\partial A}$ is positive.*

Proof. Under $A \in (A_{\min}, A_{\max})$, we have shown that g is the unique positive balance growth path for consumption and capital. The effect of a variation of d, ϕ, and A on g can be easily computed by applying the Implicit Function Theorem on the transcendental equation

(1.17) which is always satisfied for $\lambda = g$. After some algebra we obtain that

$$\begin{aligned}
\frac{\partial g}{\partial d} &= -\frac{(A-\delta)(\sigma g+\rho+\phi)\, e^{-(\sigma g+\rho+\phi)d}}{\sigma+\sigma d\,(A-\delta)\, e^{-(\sigma g+\rho+\phi)d}} < 0 \\
\frac{\partial g}{\partial \phi} &= -\frac{d\,(A-\delta)\, e^{-(\sigma g+\rho+\phi)d}}{\sigma+\sigma d\,(A-\delta)\, e^{-(\sigma g+\rho+\phi)d}} < 0 \\
\frac{\partial g}{\partial A} &= \frac{e^{-(\sigma g+\rho+\phi)d}}{\sigma+\sigma d\,(A-\delta)\, e^{-(\sigma g+\rho+\phi)d}} > 0
\end{aligned}$$

These results are very intuitive; the negative relations between the time-to-build delay and the growth rate and between the depreciation "before use" and the growth rate are due, respectively, to the fact that an increase in the time-to-build delay increases the time to produce output and by the fact that a higher depreciation "before use" reduces the net capital. On the other hand, the positive effect of the productivity of capital is obvious and is present in the standard AK model as well. ■

1.5 Consumption and Capital Dynamics

In the previous section, we have proved the existence and uniqueness of the balance growth path. We have also shown the influence of the delay coefficient on the growth rate for a given level of technology. In this section, we focus on the dynamic behavior of the optimal detrended consumption and capital which let us to derive indirectly the behavior of detrended income and detrended investment.

Proposition 1.11 *Optimal detrended consumption is constant over time while optimal detrended capital path is unique and oscillatory converges to a constant.*

Proof. The optimal detrended solution of capital and consumption can be obtained by multiplying both sides of equations (1.27) and (1.20) by $e^{-g_c t}$

$$\hat{c}(t) = a_{\tilde{m}}^{-\frac{1}{\sigma}} \tag{1.28}$$

$$\hat{k}(t) = \sum\nolimits_{v} P_{\tilde{m},v} + \sum\nolimits_{v\neq\tilde{v}} N_{\tilde{m},v} e^{(z_v-g_c)t} \tag{1.29}$$

After calling $z = x+iy$ and $n = \alpha+i\beta$, and taking into account Theorem 1.3, the detrended solution for capital can be rewritten, as shown in Appendix A.3, in the following way

$$\hat{k}(t) = \alpha_{\tilde{v}} + 2\sum\nolimits_{v\neq\tilde{v}} \Psi_{0,v} + 2\sum\nolimits_{v\neq\tilde{v}} \left[(\alpha_v - \Psi_{0,v})\cos yt - (\beta_v + \Psi_{1,v})\sin yt\right] e^{(x_v-g_c)t} \tag{1.30}$$

where $\Psi_{0,v}, \Psi_{1,v} \in \mathbb{R}$ for any v. Finally, the asymptotic behavior of capital is equal to

$$\lim_{t\to\infty} \hat{k}(t) = \alpha_{\tilde{v}} + 2\sum\nolimits_{v} \Psi_{0,v} \tag{1.31}$$

Expressions (1.30) and (1.31) tell us that the transition to the BGP is oscillatory due to the presence of the cosine and sine term, and that the convergence is guarantee by the fact that $x_v = \mathrm{Re}\,(z_v) < g_c$ for any $v \neq \tilde{v}$. Finally, the uniqueness of the path is due to the fact that the residue $\{n_v\}_v$ are fixed by the boundary condition of capital while the residue $a_{\tilde{m}}$ is fixed by the transversality condition through the expression (1.26). ■

Moreover, taking into account the technology and the resources constraint of our economy, it follows immediately that output and investment have an oscillatory behavior. In the following section, we discuss the opportunity of introducing the depreciation "before use" hypothesis and the role which a choice of a $\phi \geq \hat{\phi}$ has in extending our results for all the feasible values of the delay. On the other hand, as it will appear clear soon, all the results obtained until now remain valid even for the extreme case $\phi = 0$ when an appropriate sub-interval of d is appropriately chosen.

1.6 Considerations on the Depreciation "Before Use" Hypothesis

It is quite easily observable that all the results obtained until now remain valid in the specific case of $\phi = 0$ for a restricted interval of the time-to-build coefficient. As we have seen, the introduction of depreciation "before use", depending inversely on the time to build parameter, is able to extend the previous analytical results to the whole, feasible, interval of the delay. On the other hand, when $\phi = 0$, several technical problems may arise for a sufficiently high choice of the delay. In particular, a general continuous solution as a sum of exponentials as in (1.19) can no longer be written. This implies that the validity of transversality conditions becomes extremely difficult to assess. Another relevant difference is that Hopf bifurcation may rise in the interval $d \in \left(0, \tilde{d}_2\right]$ with $\tilde{d}_2$ the value of the delay under which the curve $(A - \delta)e^{-\rho d}$ intersects $\frac{3\pi}{2d}$ in Figure 1.3. In order to show why Hopf bifurcation may rise, we write again the detrended solution for capital (1.29) in the following way:

$$\hat{k}(t) = \sum\nolimits_v P_{\tilde{m},v} + \sum\nolimits_{v \neq \tilde{v}} N_{\tilde{m},v} e^{w_v t}$$

where $w_v = z_v - g_c$. Then according to Kolmanovskii and Myshkis ([3], Chapter 3, page 183) the following proposition holds

Proposition 1.12 *Hopf bifurcation rises if it exists a* $d^* \in \left[d_{\min}, \tilde{d}_2\right]$ *such that*

1) for $d < d^*$ *all the roots have (after transversality conditions) negative real part;*

2) $w_v(d) = \pm i y_0$ *with* $y_0 > 0$ *and* $v = 1, 2$*;*

3) $\left.\frac{\partial\,\mathrm{Re}\, w_{1,2}(d)}{\partial d}\right|_{d=d^*} > 0$ *and* $\left.\mathrm{Re}\, w_v\,(d)\right|_{d=d^*} < 0$ *for* $v > 2$

However studying the presence of such w_1 and w_2 is not analytically but only numerically tractable[9] since w_1 and w_2 can be computed only estimating the roots of the characteristic equation of capital having the second highest real part. In the next section we present a numerical example in order to help the reader to get the main messages behind all these results.

1.7 Numerical Exercise

In this section we report only the results of our simulations while a detailed explanation of the computational methods is reported in Appendix A.4. Moreover all the Figures are reported at the end of the Chapter, before the Bibliography.

The following parametrization of our economy has been chosen:

σ	ρ	δ	ϕ	d	A	$d_{\min}$	$d_{\max}$
0.8	0.02	0.05	0.03	20	0.3	7	50.51

Remember that if we have chosen $\sigma > 1$ the $d_{\min}$ should be equal to 0; in our case with $\sigma = 0.8$ a value of d less than $d_{\min}$ implies unbounded utility. On the other hand a value of the delay greater than $d_{\max}$ implies no endogenous growth[10]. Moreover, observe that given this parametrization, the D-Subdivision method tells us that: in the case of no depreciation "before use" ($\phi = 0$), in the interval $d = \left[\tilde{d}_{\min}, 18.85\right)$ we have only one root with positive real part; in the interval $d = [18.85, 43.98)$, three roots with positive real part, and finally in the interval $d = [43.98, \tilde{d}_{\max}]$, five roots with positive real part. This fact is reported in Figure 1.4, where a subset of the infinite roots of the homogeneous part of (1.1) are numerically computed through the Lambert function. Figure 1.4 and Figure 1.5 shows the real parts of the roots in the x-axis and the imaginary parts in the y-axis.The first graph of the spectrum is interesting, since it shows how an increase in the value of the time-to-build coefficients reduces the magnitude of the real part of the highest eigenvalue. Taking into account relation (1.16), this numerical result confirms Corollary 1.10. Now we show the effect of the introduction of a minimum degree of depreciation "before use" on the capital dynamics. In particular, through Figure 1.5, it is possible to observe how a choice of $\phi = 0.03$ forces the spectrum of roots of the law of motion of capital to have only one eigenvalue with positive real part even in the extreme case of a delay coefficient equal to $d_{\max} = 50$.As we can expect, the presence of a positive depreciation "before use" rate reduces the growth

[9] Following Bellman and Cooke , it is possible to (...)

[10] In fact the highest root of the homogeneous part of (1.1) is close to 0.02 and taking into account our parametrization and relation (1.16), we have, that at the right of this value the growth rate of consumption is no longer positive.

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rate of capital, and indirectly, through relation (1.16), of consumption. This effect is due to the fact that net capital is reduced and, indirectly, output, consumption, and investment. The next two figures show the dynamic behavior of detrended capital (equation (1.30)) over time. In the first case, Figure 1.6, we have studied the detrended capital dynamics given a constant initial value (boundary condition) of capital, k_0.As it appears clear, the presence of time-to-build is able to generate oscillatory behavior of capital for a long interval of time. Taking into account the technology and the resources constraint of our economy both output and investment will have a similar dynamic behavior as capital. Observe that the oscillatory dynamic behavior of these variables is enhanced by a consumption smoothing effect. In fact from Proposition 1.11, we know that the social planner optimally chooses to have a constant detrended consumption while detrended capital bears most of the adjustment to the BGP. Finally we have reported in the Figure 1.7 the capital dynamic behavior for different choices of the delay.It is interesting to notice that the higher the choice of the delay, the more relevant is the oscillatory structure of capital dynamics. This fact has been reported in Figure 1.7, in the case of $\sigma = 8$ starting with values of the delay sufficiently close to zero and given a same boundary condition for capital k_0. Remember that variation in the choice of the delay have an influence on the value of the balance growth path. In particular, for Corollary 1.10, the higher is the delay, the lower is the balance growth path. This fact appears also in Figure 1.7, where with $k_{i,ss}$ and $i = 1, ..., 4$, we have indicated the different balance growth paths. The dynamic behavior of capital appears more and more smooth as d is close to zero: this dynamic behavior is consistent with what we aspect in the extreme case $d = 0$. Finally we study numerically the same economy when the depreciation "before use", ϕ, is assumed to be zero. In this case, we have $d_{\min} = 9.16$ while $d_{\max} = 126.3$[11]. As explained in the previous section, the dynamic of the economy is fully described by the sign behavior of $\text{Re}(w_v) = z_v - g_c$ with $v = 1, 2$. Then we have reported in Figure 1.8 the transitional dynamics of the economy according to the choice of d and the value of $\text{Re}(w_v)$:As it appears clear the economy is locally determinate in the interval of the delay $d \in (d_{\min}, d^*)$ while locally unstable in $(d^*, d_{\max})$. Moreover d^* induces an Hopf bifurcation since all the requirement in Proposition 1.12 are satisfied.

1.8 Conclusion

This paper has fully analyzed an AK endogenous growth model when the time-to-build assumption is introduced through a delay differential equation for capital. It has been proved

[11]Referring to Figure 3, we have that for $A = 0.3$ the curve $(A - \delta)e^{-\rho d}$ is always under the curve $\frac{3\pi}{2d}$ and then $\tilde{d}_2$ tends to infinity. However this is not in general true.

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the existence and uniqueness of the BGP and also that a unique optimal path of the detrended capital is oscillatory convergent to its steady state value while detrended consumption jumps directly on it as the usual case without delay. These results have been obtained through a careful analysis of the role of transversality conditions and the introduction of a new structure of capital depreciation, which takes into account the depreciation of capital before it becomes productive. This last assumption appears to be crucial in avoiding implausible economic predictions (like local instability of the equilibrium) which may appear in this type of model for choices of the time-to-build coefficient sufficiently high. Finally the analysis of the model let us confirm that time-to-build can be considered a source of aggregate fluctuation for capital and output exactly as the vintage capital assumption.

1.9 Appendix A: Some Proofs

Proof of Theorem 1.2. The proof of this theorem is mainly based on Bellman and Cooke [8] (Section 3.9, page 73-75). The only relevant difference is that we assume a boundary condition defined in the interval $[-d, 0]$, and not in $[0, d]$. Given this difference we need an "auxiliary" function $x(t)$ having the following properties:

(a) $x(t) = 0 \qquad t < -d$;

(b) $x(-d) = a_0^{-1}e^{-sd}$;

(c) $x(t)$ is of class C^0 on $[0, \infty)$;

(d) $x(t)$ satisfies the equation

$$a_0\dot{x}(t) + b_0 x\,(t) + b_1 x\,(t-d) = 0 \qquad for\ t > -d \tag{1.32}$$

Before showing that the Laplace transform of $x(t)$ is $h^{-1}(z)$, it is important to notice that it is possible to prove (see Bellman and Cooke [8]) the existence and uniqueness of $x(t)$ even if equation (1.32) doesn't respect theorem 1.1 since the boundary condition doesn't define a continuous function over $[-2d, d]$. We multiply each term of equation (1.32) by e^{-zt} and integrate with respect to t from $-d$ and ∞, we get

$$a_0 \int_{-d}^{\infty} \dot{x}\,(t)\,e^{-zt}dt + b_0 \int_{-d}^{\infty} x\,(t)\,e^{-zt}dt + b_1 \int_{-d}^{\infty} x\,(t-d)\,e^{-zt}dt = 0 \tag{1.33}$$

and integrating by part the first term and making the change of variables $t_1 = t - d$ in the last term, relation (1.33) can be rewritten

$$-1 + a_0 z \int_{-d}^{\infty} x\,(t)\,e^{-zt}dt + b_0 \int_{-d}^{\infty} x\,(t)\,e^{-zt}dt + b_1 e^{-zd} \int_{-d}^{\infty} x\,(t_1)\,e^{-zt_1}dt_1 = 0$$

from which follows immediately that the Laplace transform of $x(t)$ is

$$\int_{-d}^{\infty} x\,(t)\,e^{-zt}dt = h^{-1}(z) \tag{1.34}$$

Now, using the Laplace transform formula we get

$$x\,(t) = \int_{(c)} \frac{e^{zt}}{h\,(z)}dz \qquad for\ t > -d \tag{1.35}$$

For the *residue theorem* equation (1.35) is equivalent to

$$x\,(t) = \sum_{r} RES\left\{\frac{e^{zt}}{h\,(z)}, z_r\right\}$$

and taking into account the formula $RES\left[\frac{\psi(a)}{\phi(a)},\tilde{a}\right]=\frac{\psi(\tilde{a})}{\phi'(\tilde{a})}$ when $\psi(\tilde{a})\neq 0$

$$x(t)=\sum_r \frac{e^{z_r t}}{h'(z_r)} \qquad for\ t>-d \tag{1.36}$$

Now for Theorem 3.7 of Bellman and Cooke [8], the general continuous solution, $u(t)$, of the delay differential equation with forcing term

$$a_0\dot{u}(t)+b_0u(t)+b_1u(t-d)=f(t) \tag{1.37}$$

which satisfies the initial or boundary condition $u(t)=\xi(t)$ with $t\in[-d,0]$, is

$$u(t)=a_0\xi(0)x(t)+(a_0z_r+b_0)\int_{-d}^{0}\xi(s)x(t-s)\,ds+\int_0^t f(s)x(t-s)ds \tag{1.38}$$

and taking into account relation (1.36) we can rewrite (1.38) as

$$u(t)=\sum_r \frac{a_0\xi(0)+(a_0z_r+b_0)\int_{-d}^{0}\xi(s)e^{-z_rs}ds}{h'(z_r)}e^{z_rt}+\int_0^t f(s)\sum_r\frac{e^{z_r(t-s)}}{h'(z_r)}ds$$

which ie exactly equal to relation (1.8).

Proof of Theorem 1.3. The proof is organized in two parts. In the first part, we show that the unique general solution of (1.5) with boundary condition (1.6)

$$u(t)=\sum_r p_re^{z_rt}+\int_0^t f(s)\sum_r p_re^{z_r(t-s)}ds \tag{1.39}$$

where the roots $\{z_r\}$ and the residues $\{v_r\}$ come respectively from the characteristic equation of the homogeneous part of (1.5)

$$h(z)=a_0z+b_0+b_1e^{-zd} \tag{1.40}$$

and from the relation

$$p_r=\frac{p(z_r)}{h'(z_r)}=\frac{a_0\xi(0)+(a_0z_r+b_0)\int_{-d}^{0}\xi(s)e^{-z_rs}ds}{a_0-b_1de^{-z_rd}} \tag{1.41}$$

can be rewritten as

$$\begin{aligned} u(t) &= \sum_{r=0}^{k} \varsigma_r e^{x_r t} + \sum_{r=k}^{\infty} \left(a_r e^{z_r t} + \bar{a}_r e^{\bar{z}_r t} \right) + \\ &+ \int_0^t f(s) \left[\sum_{r=0}^{k} \frac{e^{x_r(t-s)}}{h'(x_r)} + \sum_{r=k}^{\infty} \left(\frac{e^{z_r(t-s)}}{h'(z_r)} + \frac{e^{\bar{z}_r(t-s)}}{h'(\bar{z}_r)} \right) \right] ds \end{aligned} \tag{1.42}$$

where $\{x_r\}$ are real roots, $\{z_r\}$ are complex conjugate roots[12], $\{\varsigma_r\}$ are real constants, and $\{a_r\}$ are complex conjugate constants. In fact, from the D-Subdivisions method we know that (1.40) has at most two real roots and an infinite number of complex conjugate roots. From (1.41), it appears also clear that the residues related to real roots are real while those related to complex roots are complex. Taking into account these results it is possible to split (1.39) as follows

$$u(t) = \sum_{r=0}^{k} \varsigma_r e^{x_r t} + \sum_{r=k}^{\infty} \left(a_r e^{z_r t} + c_r e^{\bar{z}_r t} \right) + \int_0^t f(s) \left[\sum_{r=0}^{k} \frac{e^{x_r(t-s)}}{h'(x_r)} + \sum_{r=k}^{\infty} \left(\frac{e^{z_r(t-s)}}{h'(z_r)} + \frac{e^{\bar{z}_r(t-s)}}{h'(\bar{z}_r)} \right) \right] ds$$

where $z = x + iy$ and $\bar{z} = x - iy$. We now show that $c_r = \bar{a}_r$ is always the case. This fact can be proved by taking into account the following properties of complex numbers

i) let $z = \frac{u}{v}$ with u and v two complex numbers..Then $\bar{z} = \frac{\bar{u}}{\bar{v}}$;

ii) let $z = uv$ with u and v two complex numbers.Then $\bar{z} = \bar{u}\bar{v}$;

iii) let z be a complex number. Then $\overline{e^z} = e^{\bar{z}}$;

and observing that $\bar{p}(z) = p(\bar{z})$, and $\bar{h}'(z) = h'(\bar{z})$.

The second part of the proof consists in showing that (1.42) is a real function. We start by considering the first term

$$\sum_{r=0}^{k} \varsigma_r e^{x_r t} + \sum_{r=k}^{\infty} \left(a_r e^{z_r t} + \bar{a}_r e^{\bar{z}_r t} \right) \tag{1.43}$$

Calling $a = \varsigma + i\omega$ we have that

$$\begin{aligned} a e^{zt} + \bar{a} e^{\bar{z}t} &= (\varsigma + i\omega) e^{xt} e^{iyt} + (\varsigma - i\omega) e^{xt} e^{-iyt} \\ &= e^{xt} \left[(\varsigma + i\omega)(\cos yt + i \sin yt) + (\varsigma - i\omega)(\cos yt - i \sin yt) \right] \\ &= 2 e^{xt} (\varsigma \cos yt - \omega \sin yt) \end{aligned}$$

and then (1.43) becomes

$$\sum_{r=0}^{k} \varsigma_r e^{x_r t} + 2 \sum_{r=k}^{\infty} e^{xt} (\varsigma \cos yt - \omega \sin yt)$$

[12] We have indicated the conjugate of a complex number a with $\bar{a}$.

which is a real function of t. Now we study the term

$$\sum_{r=k}^{\infty}\left(\frac{e^{z_r(t-s)}}{h'(z_r)}+\frac{e^{\bar{z}_r(t-s)}}{h'(\bar{z}_r)}\right)$$

After some boring algebra this can be rewritten as

$$\sum_{r=k}^{\infty}\frac{2\left\{a_0\cos\left[y_r(t-s)\right]-b_1de^{-x_rd}\cos\left[y_r(t-s+d)\right]\right\}}{a_0^2-2a_0b_1de^{-x_rd}\cos y_rd+b_1^2d^2e^{-2x_rd}}e^{x_r(t-s)}$$

which is a real function. Then it follows immediately that the general continuous solution (1.42) can be rewritten as

$$\begin{aligned} u(t) &= \sum_{r=0}^{k}\varsigma_r e^{x_r t}+2\sum_{r=k}^{\infty}e^{xt}(\varsigma\cos yt-\omega\sin yt)+ \qquad (1.44)\\ &+\int_0^t f(s)\left[\sum_{r=0}^{k}\frac{e^{x_r(t-s)}}{h'(x_r)}+\sum_{r=k}^{\infty}\frac{2\left\{a_0\cos\left[y_r(t-s)\right]-b_1de^{-x_rd}\cos\left[y_r(t-s+d)\right]\right\}}{a_0^2-2a_0b_1de^{-x_rd}\cos y_rd+b_1^2d^2e^{-2x_rd}}e^{x_r(t-s)}\right]ds \end{aligned}$$

which is clearly a real function.$u: I\rightarrow\mathbb{R}$.

1.10 Appendix B: How to get expression (1.30) from (1.29).

First of all, observe that from Theorem 1.3 we can rewrite

$$\begin{aligned} \sum\nolimits_v P_{\tilde{m},v} &= -a_{\tilde{m}}^{-\frac{1}{\sigma}}\left[\frac{1}{(g_c-\tilde{z})\,h'(\tilde{z})}+\sum\nolimits_{v\neq\tilde{v}}\left(\frac{1}{(g_c-z_v)\,h'(z_v)}+\frac{1}{(g_c-\bar{z}_v)\,h'(\bar{z}_v)}\right)\right]\\ &= \alpha_{\tilde{v}}-a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_{v\neq\tilde{v}}\left(\frac{(g_c-\bar{z}_v)\,h'(\bar{z}_v)+(g_c-z_v)\,h'(z_v)}{(g_c-z_v)\,(g_c-\bar{z}_v)\,h'(z_v)h'(\bar{z}_v)}\right) \end{aligned}$$

Now calling $z=x+iy$ and $n=\alpha+i\beta$, and taking into account the shape of the characteristic equation we get after some algebra

$$\sum\nolimits_v P_{\tilde{m},v}=\alpha_{\tilde{v}}+2\sum\nolimits_{v\neq\tilde{v}}\Psi_{0,v} \qquad (1.45)$$

where

$$\Psi_{0,v}=-a_{\tilde{m}}^{-\frac{1}{\sigma}}\frac{g_c-x_v+\tilde{A}e^{-x_vd}\left\{\left[(g_c-x_v)\,x_v+y_v^2\right]\cos y_vd+\left[(g_c-x_v)\,y_v+x_vy_v\right]\sin y_vd\right\}}{(g_c^2-2g_cx_v+x_v^2+y_v^2)\left[1+\tilde{A}e^{-2x_vd}\left(x_v^2+y_v^2\right)+2\tilde{A}e^{-x_vd}\left(x_v\cos y_vd+y_v\sin y_vd\right)\right]}$$

Now we have to rewrite

$$\begin{aligned} \sum\nolimits_{v\neq\tilde{v}}N_{\tilde{m},v}e^{z_vt} &= \sum\nolimits_{v\neq\tilde{v}}\left(n_v-P_{\tilde{m},v}\right)e^{z_vt}\\ &= \sum\nolimits_{v\neq\tilde{v}}\left(n_ve^{z_vt}+\bar{n}_ve^{\bar{z}_vt}\right)+a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_{v\neq\tilde{v}}\left(\frac{e^{z_vt}}{(g_c-z_v)\,h'(z_v)}+\frac{e^{\bar{z}_vt}}{(g_c-\bar{z}_v)\,h'(\bar{z}_v)}\right) \end{aligned}$$

which taking into account the results of the previous Appendix is equal to

$$2\sum\nolimits_{v\neq\tilde{v}}\left(\alpha_v\cos y_v t-\beta_v\sin y_v t\right)e^{x_v t}+a_{\tilde{m}}^{-\frac{1}{\sigma}}\sum\nolimits_{v\neq\tilde{v}}\left(\frac{\left(g_c-\bar{z}_v\right)h'(\bar{z}_v)e^{z_v t}+\left(g_c-z_v\right)h'(z_v)e^{\bar{z}_v t}}{\left(g_c-z_v\right)\left(g_c-\bar{z}_v\right)h'(z_v)h'(\bar{z}_v)}\right)$$

which after some algebra and taking into account some trigonometric relations can be rewritten as

$$2\sum\nolimits_{v\neq\tilde{v}}\left[\left(\alpha-\Psi_{0,v}\right)\cos yt-\left(\beta+\Psi_{1,v}\right)\sin yt\right]e^{x_v t} \tag{1.46}$$

where

$$\Psi_{1,v}=a_{\tilde{m}}^{-\frac{1}{\sigma}}\frac{y_v+\tilde{A}e^{-x_v d}\left\{\left[\left(g_c-x_v\right)x_v+y_v^2\right]\sin y_v d-\left[\left(g_c-x_v\right)y_v+x_v y_v\right]\cos y_v d\right\}}{\left(g_c^2-2g_c x_v+x_v^2+y_v^2\right)\left[1+\tilde{A}e^{-2x_v d}\left(x_v^2+y_v^2\right)+2\tilde{A}e^{-x_v d}\left(x_v\cos y_v d+y_v\sin y_v d\right)\right]}$$

Finally taking into account relations (1.45) and (1.46) follows immediately the shape of the general continuous solution in (1.30).

1.11 Appendix C: Computational method

In order to obtain the spectrum of the roots from the law of motion of capital and its solution, we have used Lambert functions as proposed recently by Asl and Ulsoy [2]. A class of functions $W(s)$ are called Lambert functions if they satisfy the relation

$$W(s)e^{W(s)}=s \tag{1.47}$$

Then considering the characteristic equation of the law of motion of capital

$$-se^s+d\tilde{A}=0 \tag{1.48}$$

with $s=zd$, and taking into account the definition of the Lambert function (1.47), we have that

$$W\left(d\tilde{A}\right)e^{W(d\tilde{A})}=d\tilde{A} \tag{1.49}$$

Now comparing (1.48) and (1.49), the solutions of the equation which describe the characteristic spectrum are

$$z=\frac{1}{d}W\left(d\tilde{A}\right)$$

In the most general form, the Lambert function is a complex function with infinite branches. Calculation of both the principal branch and the other branches can be presented in series form ([2] see for more details). Taking into account these results, we have used the MatLab programs (`Lambertww.m`, `Spectrum.m`, and `Solutions.m`) in order to derive the first $m=16$

branches[13] and from them the corresponding roots. Then we have derived the roots of the characteristic equation of the law of motion of consumption through relation (1.16) and residue p_m through the relation (1.41). Observe that to any branch corresponds a particular solution for the delay differential equation. Finally, using the result in Theorem 1.3, namely the shape of the general continuous solution (1.30), it is possible to derive the general continuous solution.

[13]The results obtained in our analysis are invariant to a higher choice of m.

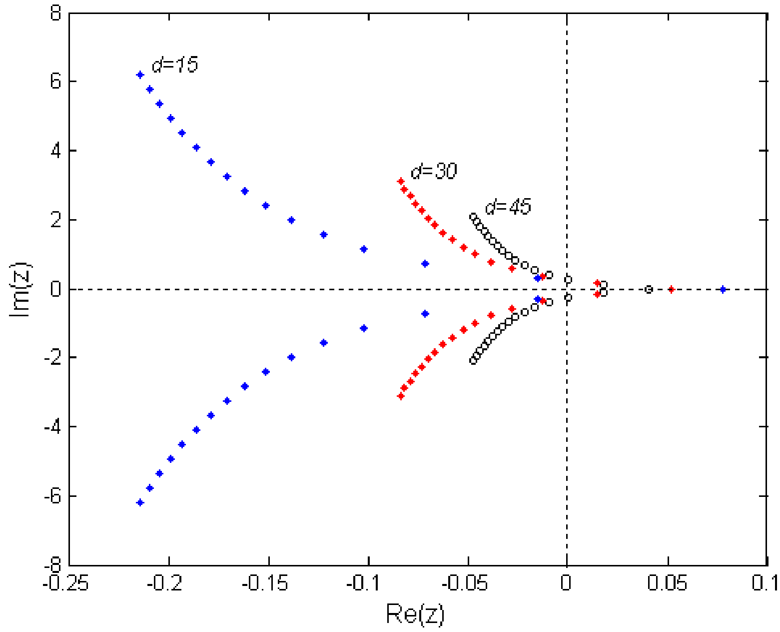

Figure 1.4: Spectrum of roots for the law of motion of capital (1.1).

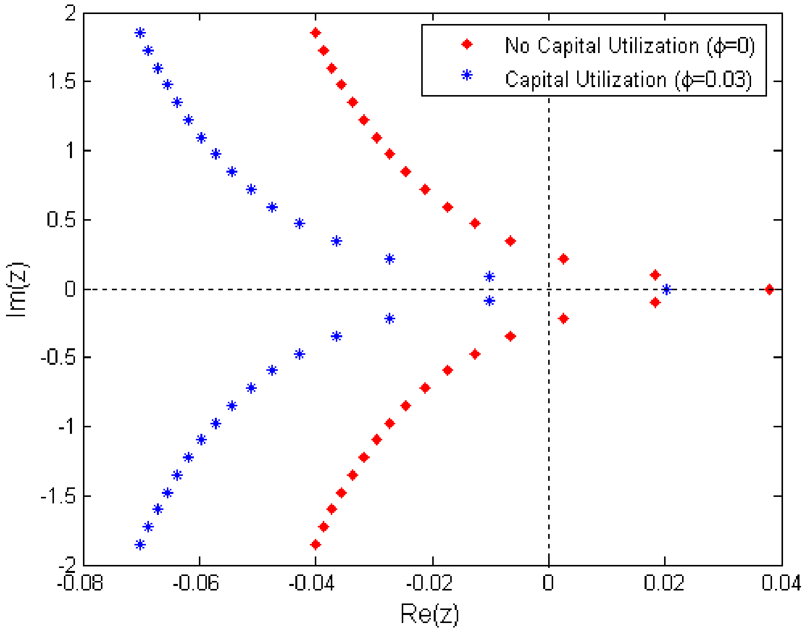

Figure 1.5: Spectrum of roots for capital (1.1) in the case $d_{\max} = 50$.

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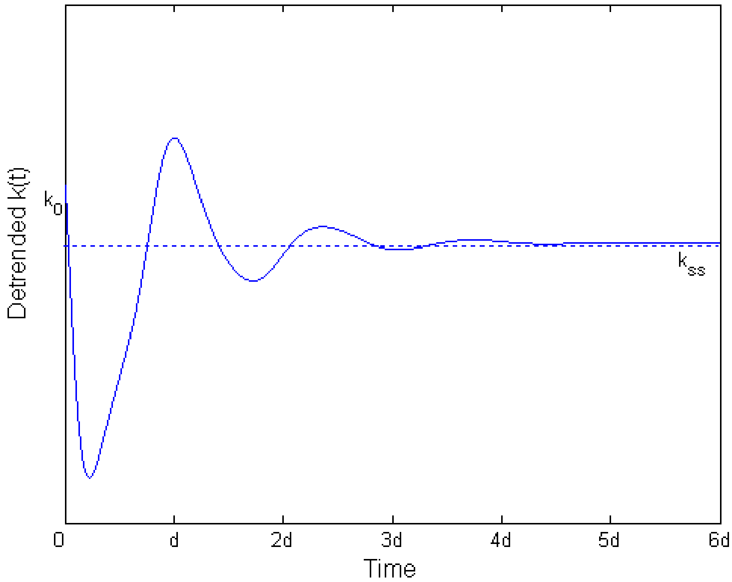

Figure 1.6: Dynamic behavior of detrended capital.

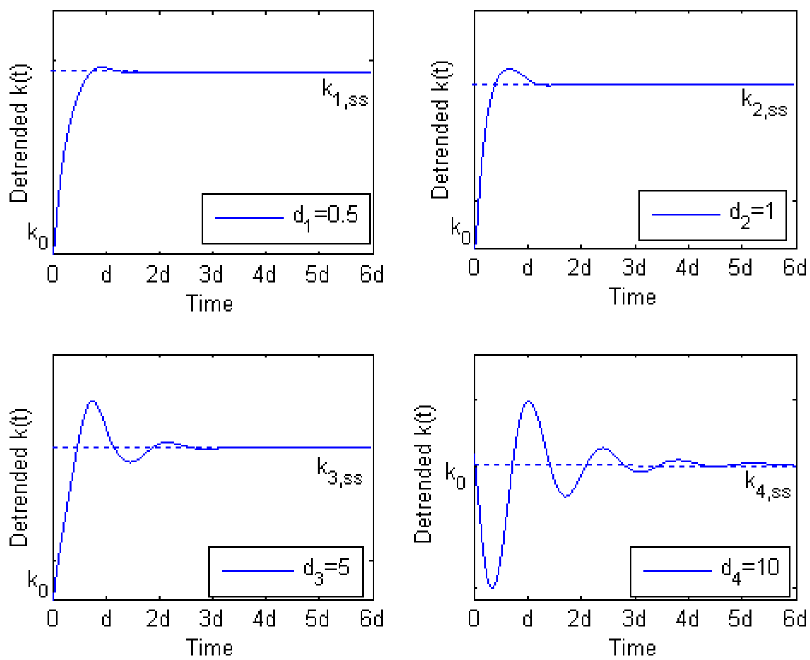

Figure 1.7: Capital dynamic behavior for different choices of the delay.

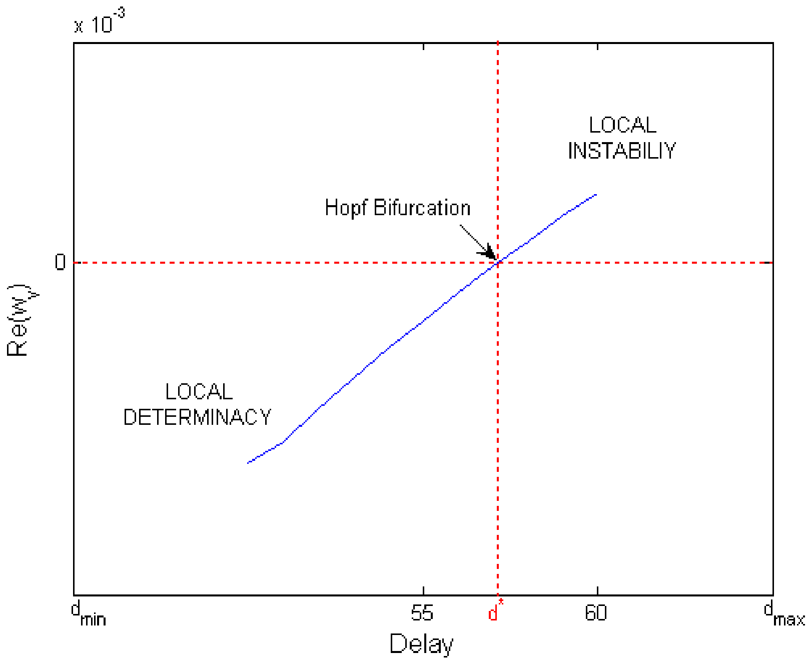

Figure 1.8: Transitional dynamics when $\phi = 0$.

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CHAPTER 2

(IN)DETERMINACY AND TIME-TO-BUILD.

2.1 Introduction

This paper is an extension of Benhabib and Farmer [6] under the time-to-build assumption that new capital goods become productive with some delay. The main concern is to understand how the dynamic properties of a neoclassical growth economy with production externalities change by the introduction of a time-to-build delay, as well as variations on its magnitude. In particular, we are interested in capturing the influence of time-to-build on the existence of local indeterminacy.

The implications of time-to-build has long been analyzed by economists (s.a. Bohm-Bawerk [9]), who have conjectured that production lags may induce cycles in output (see also Kalecki [21]) and account for the persistence of output fluctuations. In their seminal paper, Kydland and Prescott [25] argue that time-to-build, in the sense that investment projects need more than one period to be completed, strongly contributes to the persistence of the business cycle. Asea and Zak [1] propose a continuum time optimal growth model with a time-to-build delay and show that the optimal path may converge to the steady state, eventually by oscillations, or even (Hopf) cycle around it. Consequently, they show that the dynamics can be intrinsically oscillatory due (entirely) to the time-to-build technology.[1]

Local indeterminacy is a concept strictly related to the dynamics, and in particular to the stability properties of the equilibrium in an infinite horizon economy. In a two dimension dynamic general equilibrium model, with one control and one state, there is local indeterminacy when a steady state is not (locally) a saddle path, as usual, but a stable node or a stable focus.[2] In these cases, the equilibrium is said to be locally indeterminate since for any given initial condition for the state variable there exists a continuum of initial levels of the control (or co-state), each of which associated to a different equilibrium path. Kehoe and Levine [22] argue that in pure exchange economies with infinitely lived consumers, equilibria are generically determinate. However, from the beginning of the nineties, infinitely lived

[1] Asea and Zak [1] use delayed differential equations to rigorously analyze the implications of time-to-build delays. See also Collard et al [12]. A rigorous proof of the existence of cycles in an optimal growth model with time-to-build was done by Rustichini [27].

[2] In continuous time, the eigenvalues lie respectively, in $\mathbb{R}^- / \{0\}$, and in the left of the imaginary axis. In discrete time, the eigenvalues are real and inside the unit circle, and complex and inside the unit circle, respecively.

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agent models with some degree of increasing returns have been shown to exhibit multiple equilibria, indeterminacy, and the possibility of sunspots. Benhabib and Farmer [6] (hereafter BF) add increasing returns to the one sector neoclassical growth model and show that the equilibrium may be locally indeterminate[3].

In a discrete time Benhabib-Farmer framework, Hintermaier [19] analyses the existence of indeterminacy for different time frequencies. He shows that the conditions for the existence of indeterminacy are stronger the lower is the time frequency. At the limit, when the time frequency goes to infinite, or the period length goes to zero, he obtains the same conditions than in BF. As it is standard in discrete general equilibrium models, Hintermaier assumes that capital produced at time t becomes productive at time $t+1$. This is a one period time-to-build assumption. Consequently, by reducing the frequency of the economy the time-to-build becomes longer and longer.

The introduction of adjustment costs in the BF model, has been shown by Kim [23] to increase the required degree of increasing returns for indeterminacy to rise; Herrendorf and Valentinyi [18], starting with a two sector model characterized by mild sector-specific externalities, extend this result both in the case of total and of sector's specific capital adjustment costs.

In this paper, we extend BF by assuming that capital produced at time t becomes productive at time $t+\tau$, where $\tau > 0$ is a time-to-build delay. The analysis focuses, first, on the effect of the time to build in a Ramsey model with endogenous labour supply and then in a Benhabib Farmer model. It is possible to show that local indeterminacy of the steady state depends crucially on the level of externalities but also on the choice of the delay coefficient.

The paper is organized as follows. Section 2.2 describes the time-to-build economy. In section 2.3 we analyze the dynamics of the model and we present the major theoretical results; section 2.4 concludes.

2.2 Time-to-Build

We model time-to-build in the simplest possible way by assuming, as suggested by Kalecki [21], that capital goods produced at time t become operative at time $t+\tau$, the time-to-build delay τ being strictly positive[4]. This assumption is appended to the dynamic general

[3]The empirically plausibility of the BF model has been extensively discussed in the literature, since an implausible high level of externalities are required to the equilibrium be indeterminate. Benhabib and Nashimura [8] and Benhabib and Perli [7] propose more general models where the conditions for indeterminacy are plausible.

[4]Kalecki refers to the parameter τ as *"gestation period"* of any investment. This period starts with the investment orders and finished with the deliveries of finished industrial equipments.

equilibrium model with externalities proposed by Benhabib and Farmer [6].

2.2.1 Firm's Problem

Markets are perfectly competitive and there is a continuum of measure one of identical firms using a Cobb-Douglas technology that transforms labor N and capital K into output Y:

$$Y(t) = A(t)K(t-\tau)^a N(t)^b.$$

As said before, the time-to-build assumption imposes that at time t firms use capital goods produced at time $t-\tau$. The state of technology is $A(t) = \bar{K}(t-\tau)^{\alpha-a}\bar{N}(t)^{\beta-b}$, where $1 > \alpha > a > 0$, and $\beta > b > 0$. As in BF, no-tradeable externalities come from the economy-wide capital average $\bar{K}$, and the economy-wide labor average $\bar{N}$. *Constant returns to scale* at the firm level requires $a+b=1$. There are, however, *increasing returns to scale* at the aggregate level, since $\alpha+\beta > 1$. The aggregate technology, after substitution of $\bar{K}$ by K and $\bar{N}$ by N, can be written as

$$Y(t) = K(t-\tau)^\alpha N(t)^\beta. \tag{2.1}$$

Under the time-to-build assumption, the representative firm faces the following static profit maximization problem:

$$\max_{N(t),K(t)} A(t)K(t-\tau)^a N(t)^b - w(t)N(t) - [r(t)+\delta]\, K(t-\tau).$$

where $w(t)$ is the wage rate, $\delta > 0$ is the depreciation rate and $r(t)+\delta$ is the rental rate of capital.

From the first order conditions, we get

$$bY(t) = w(t)N(t) \tag{2.2}$$

$$aY(t) = [r(t)+\delta]\, K(t-\tau). \tag{2.3}$$

Constant private returns to scale imply that factors of production receive a fixed share of output and profits are zero, which is consistent with perfect competition.

2.2.2 Consumer's Problem

The economy is inhabited by a continuum of measure one of infinitely lived households, with preferences depending positively on consumption C and negatively on employment N.

Households are assumed to own the capital stock. The representative household faces the following infinite horizon problem:

$$\max \int_0^\infty \left\{ \log C(t) - \frac{N(t)^{1-\chi}}{1-\chi} \right\} \mathrm{e}^{-\rho t} dt,$$

$$\text{s.t.} \quad \dot{K}(t) = r(t)K(t-\tau) + w(t)N(t) - C(t), \tag{2.4}$$

given initial conditions $K(t) = \xi(t)$, for $t \in [-\tau, 0]$. Parameter $\chi \leq 0$ while $\rho > 0$. This dynamic optimization problem differs from the standard consumers problem mainly because the budget constraint (2.4) is not an ordinary differential equation but a delayed differential equation. From the time to build assumption, consumers rent at time t the capital stock produced at $t - \tau$ and they build new capital which will be available at $t + \tau$. Consequently, initial conditions $\xi(t)$ need to be specified in order to identify the relevant history of the state variable K.

Following Kolimanovskii and Myshkis [24], the Hamiltonian associated to this problem is

$$\mathcal{H}(t) = \left\{ \log C(t) - \frac{N(t)^{1-\chi}}{1-\chi} \right\} \mathrm{e}^{-\rho t} + \lambda(t) \left[r(t)K(t-\tau) + w(t)N(t) - C(t) \right],$$

and the associated optimal conditions are

$$\frac{1}{C(t)} \mathrm{e}^{-\rho t} = \lambda(t) \tag{2.5}$$

$$\frac{1}{N(t)^{\chi}} \mathrm{e}^{-\rho t} = \lambda(t)w(t) \tag{2.6}$$

$$\lambda(t+\tau)r(t+\tau) = -\dot{\lambda}(t) \tag{2.7}$$

and, as shown by Boucekkine *et al* [11], the standard transversality conditions

$$\lim_{t\to\infty} \lambda(t) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)K(t) = 0$$

holds. The main difference with respect to a standard optimal control problem is in equation (2.7). The fundamental trade off is between consuming today, whose marginal value is given by $\lambda(t)$, and consuming at $t + \tau$, with marginal value $\lambda(t + \tau)$. From (2.5) and (2.6) we get the standard *intratemporal substitution condition between consumption and labor*

$$\frac{C(t)}{N(t)^{\chi}} = w(t). \tag{2.8}$$

From (2.5) and (2.7), we get the *forward-looking Euler-type condition*:

$$\frac{\dot{C}(t)}{C(t)} = \frac{C(t)}{C(t+\tau)} \mathrm{e}^{-\rho\tau} r(t+\tau) - \rho, \tag{2.9}$$

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where the real interest rate, which the household get at time $t+\tau$ by investing in capital today, is weighted by the marginal elasticity of substitution between consumption at t and consumption at $t+\tau$. It reflects the fact that investment allows households to substitute current consumption by consumption at time $t+\tau$.

2.3 Analysis of the Dynamics

In order to reduce the problem to a nonlinear functional differential equations (FDEs) system,[5] we proceed in the following way. Firstly, we use equations (2.2) and (2.3) to substitute w and r into (2.4), (2.8) and (2.9). Secondly, we substitute N from (2.8). Finally, we substitute Y from (2.1) in (2.4) and (2.9). After making a logarithmic transformation of K and C, we get a delayed differential equation (DDE) for capital

$$\dot{k}(t) = \mathrm{e}^{k(t-\tau)-k(t)} \left\{ \mathrm{e}^{\lambda_0+\lambda_1 k(t-\tau)+\lambda_2 c(t)} - \delta \right\} - \mathrm{e}^{c(t)-k(t)}, \tag{2.10}$$

and an advanced differential equation (ADE) for consumption

$$\dot{c}(t) = \mathrm{e}^{-\rho\tau+c(t)-c(t+\tau)} \left\{ a \, \mathrm{e}^{\lambda_0+\lambda_1 k(t)+\lambda_2 c(t+\tau)} - \delta \right\} - \rho, \tag{2.11}$$

where

$$\lambda_0 = -\tfrac{\beta \log b}{\beta+\chi-1}, \ \lambda_1 = \tfrac{(\alpha-1)(\chi-1)-\beta}{\beta+\chi-1}, \quad \text{and} \quad \lambda_2 = \tfrac{\beta}{\beta+\chi-1}.$$

Small capital letters refer to variables in logarithms. We can immediately observe the following:

Remark 2.1 *The FDEs system (2.10)-(2.11) becomes the differential system in Benhabib and Farmer [6]*

$$\begin{aligned} \dot{k}(t) &= e^{\lambda_0+\lambda_1 k(t)+\lambda_2 c(t)} - \delta - e^{c(t)-k(t)} \\ \dot{c}(t) &= a \ e^{\lambda_0+\lambda_1 k(t)+\lambda_2 c(t)} - \delta - \rho. \end{aligned}$$

when the time-to-build assumption is ruled out, i.e. $\tau \to 0$.

Moreover, we can prove some relevant relations between the signs of λ_2, $\lambda_1+\lambda_2$, and $1+\lambda_1$.

Lemma 2.2 *The following relations holds:*

$$sign(\lambda_2) = sign\,(\lambda_1+\lambda_2) = -sign(1+\lambda_1) \tag{2.12}$$

$$\lambda_1 = -\alpha\lambda_2 + \alpha - 1 \tag{2.13}$$

[5]See Hale and Lunel [16].

Proof. See Appendix A.1. ■

Finally let us give the following definition of an equilibrium path in a functional differential equation context.

Definition 2.3 *An equilibrium path is any trajectory* $\varphi(t) = \{c(t), k(t)\}$ *that solves the two autonomous mixed differential equations (2.10)-(2.11) subject to the boundary condition* $k(t) = \log(\xi(t))$, *for* $t \in [-\tau, 0]$, *and the transversality conditions*

$$\lim_{t\to\infty} e^{-c(t)} e^{-\rho t} \geq 0 \quad \text{and} \quad \lim_{t\to\infty} e^{k(t)-c(t)} \; e^{-\rho t} = 0. \tag{2.14}$$

2.3.1 Steady State Analysis

Under the usual assumption that at steady state $\dot{k}(t) = \dot{c}(t) = 0$, implying $c(t) = c(t+\tau) = c_s$ and $k(t) = k(t-\tau) = k_s$, from (2.10) and (2.11), we get

$$k_s = \frac{1}{\lambda_1 + \lambda_2} \left(\log[A] - \lambda_2 \log[A - \delta] - \lambda_0\right) \tag{2.15}$$

$$c_s = \log[A - \delta] + k_s, \tag{2.16}$$

where $A \equiv \dfrac{\delta + \rho \; e^{\rho\tau}}{a}$.

Since k_s and c_s are natural logarithms, they may have either positive or negative sign.

Remark 2.4 *Equations (2.15)-(2.16) are identical to those obtained by Benhabib and Farmer [6], when* $\tau = 0$.

Moreover, as expected the following result holds:

Proposition 2.5 *The time-to-build delay* τ *affects negatively both* k_s *and* c_s.

Proof. See Appendix A.2. ■

The economy is more inefficient the larger the time-to-build delay is, implying that the steady state values of capital and consumption are smaller.

2.3.2 Stability Analysis

Let first linearize the system (2.10)-(2.11) around its steady state and compute the Jacobian. As shown in Bellman and Cooke [4] (page 337-339), the solution of the linearized system will have the same properties of the nonlinearized one for sufficiently small perturbations. After some algebra,[6], and taking into account the relation (2.13), our linearized system around the steady state is

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} [\alpha(1-\lambda_2)A-\delta]\ \mathrm{e}^{-z\tau} & (\lambda_2-1)\,A+\delta \\ a\,[(1-\lambda_2)\alpha-1]\,A\mathrm{e}^{-\rho\tau} & \rho-[a(1-\lambda_2)A-\delta]\ \mathrm{e}^{z\tau}\mathrm{e}^{-\rho\tau} \end{pmatrix} \begin{pmatrix} k(t) \\ c(t) \end{pmatrix}. \tag{2.17}$$

The characteristic equation associated to (2.17) describes completely the spectrum of the eigenvalues $Z_\infty = \{z_r\}_r$ associated to the FDEs system. Let us call $\mathrm{Re}(Z_\infty)$ the set of the real parts of the eigenvalues; and with Z_∞^k and Z_∞^c the sets of all the eigenvalues coming, respectively, from the characteristic equation of the linearized law of motions of capital and consumption.

Before proceeding, let us remember that all the theoretical results on functional differential analysis are presented in Chapter 1. Now, using Theorem 1.2, we show how it is possible to write explicitly the general continuous solution of a system of functional differential equations when the Jacobian is triangular.

Theorem 2.6 *Consider the linearized system of functional differential equations*

$$\dot{u}(t) \simeq J(u^*)u(t) \tag{2.18}$$

with $u : \mathbb{R} \to \mathbb{R}^2$ *and* $J(u^*)$ *an upper triangular Jacobian evaluated around the steady state* $u^* \in \mathbb{R}$ *Then the general continuous solution of this system is*

$$u_1(t) = \sum_v n_v e^{\lambda_v t} \tag{2.19}$$

$$u_2(t) = \sum_r \Gamma_r e^{z_r t} + \sum_v \Upsilon_v e^{\lambda_v t} \tag{2.20}$$

where $\{z_r\}_r$ *and* $\{\lambda_v\}_v$ *are the zeros, respectively, of the characteristic equations* $h(z)$ *and* $h(\lambda)$ *of the homogenous part of the two equations.*

Proof. If the Jacobian is upper triangular then the characteristic equation associated to it is

$$h = \begin{vmatrix} h(\lambda) & 0 \\ c & h(z) \end{vmatrix}$$

then the spectrum of the roots of the system is exactly the union of the spectrum of the roots coming from the homogenous part of the two equations. Moreover since the triangular

[6] See Appendix A.3 for technical details.

assumption, it is also possible to write directly the solution of the linear functional differential equations without forcing term

$$u_1(t) = \sum_v n_v e^{\lambda_v t} \tag{2.21}$$

Taking into account the result (1.8) in Theorem 1.2, the other solution is

$$u_2(t) = \sum_r p_r e^{z_r t} + c \int_0^t u_1(s) \sum_r \frac{e^{z_r(t-s)}}{h'(z_r)} ds \tag{2.22}$$

which after substituting (2.21) in (2.22), we get

$$u_2(t) = \sum_r \left(p_r - \sum_v \frac{c n_v}{(\lambda_v - z_r) h'(z_r)} \right) e^{z_r t} + \sum_v \sum_r \frac{c n_v}{(\lambda_v - z_r) h'(z_r)} e^{\lambda_v t}$$

Then if we call $\Gamma_r = p_r - \sum_v \frac{c n_v}{(\lambda_v - z_r) h'(z_r)}$ and $\Upsilon_v = \sum_r \frac{c n_v}{(\lambda_v - z_r) h'(z_r)}$ we obtain exactly (2.19).
■

Observe that the requirement of a triangular matrix is crucial in the context of functional differential equations since it is never possible, given the presence of (infinite) complex roots to transform (through a change of variables) a non triangular into a triangular Jacobian. Moreover, we underline till now that the requirement of triangularity is important both in checking transversality conditions explicitly and in writing the general continuous solution of the main variables of our economy[7] in closed form. In the following section, we study a Ramsey model with endogenous labour supply with time to build. In order to do that, we proceed as follows: first we study an "auxiliary" Jacobian which is the original Jacobian when one of the coefficient out of the main diagonal has been replaced by a zero. Then we extend the results by considering small variation of that coefficient from zero.

2.3.3 The Ramsey model with time to build and endogenous labor supply

The Ramsey model is simply a special case of the Benhabib Farmer model when there are no externalities, namely $\alpha = a$ and $\beta = b$.

Proposition 2.7 *If* $\tau \in \left[0, \frac{3\pi}{2[a(1-\lambda_2)A-\delta]}\right)$ *then the equilibrium of a Ramsey model with endogenous labor supply exists and is unique.*

[7]If the assumption of triangularity is ruled out is still possible to prove the existence and uniqueness of the general continuous solution but not the explicit shape of it.

Proof. Consider the "auxiliary" Jacobian[8]

$$\tilde{J} = \begin{pmatrix} [a(1-\lambda_2)A-\delta]\ \mathrm{e}^{-z\tau} & (\lambda_2-1)\,A+\delta \\ 0 & \rho-[a(1-\lambda_2)A-\delta]\ \mathrm{e}^{z\tau}\mathrm{e}^{-\rho\tau} \end{pmatrix}$$

whose characteristic equation is

$$\tilde{h}(z) = \underbrace{\left[z-[a(1-\lambda_2)A-\delta]\ \mathrm{e}^{-z\tau}\right]}_{A} \cdot \underbrace{\left[z-\rho+[a(1-\lambda_2)A-\delta]\ \mathrm{e}^{z\tau}\mathrm{e}^{-\rho\tau}\right]}_{B}$$

The spectrum of roots of $\tilde{h}(z)$ is given by all the roots of A and B. Consider first A. By applying the D-Subdivision method we find that for $\tau \in \left[0, \frac{3\pi}{2[a(1-\lambda_2)A-\delta]}\right)$, the spectrum of roots of A is characterized by all the roots with negative real part but one positive, call it $z_{\tilde{r}}$. In particular, observe that

$$a(1-\lambda_2)A-\delta = -\lambda_2\left(\rho+\delta e^{\rho\tau}\right)+\rho+\left(e^{\rho\tau}-1\right)\delta > 0$$

since $\lambda_2 < 0$ and $e^{\rho\tau} > 1$. Now look at B. B is equal to A after the transformation $w = -z+\rho$. Then, B have all the roots with positive real part but one negative, call it $\lambda_{\tilde{v}}$, in the considered interval of τ. Moreover, taking into account Theorem 2.6, we can write the solutions as

$$\begin{aligned} k(t) &\simeq \sum_r \Gamma_r e^{z_r t} + \sum_v \Upsilon_v e^{\lambda_v t} \\ c(t) &\simeq \sum_v n_v e^{\lambda_v t} \end{aligned}$$

Now we have to check the transversality conditions (2.14), in order to get optimality. Taking into account the previous consideration on the spectrum of roots, and assuming for now only one positive root coming from the law of motion of capital, we have that transversality conditions hold if and only if

$$n_v = 0 \quad \forall v \neq \tilde{v} \tag{2.23}$$

$$n_{\tilde{v}} = [(1-\lambda_2)\,A-\delta]\,p_{\tilde{r}}\left(z_{\tilde{r}}-\lambda_{\tilde{v}}\right)h'\left(z_{\tilde{r}}\right) \tag{2.24}$$

where the last requirement, which is equivalent to $\Gamma_{\tilde{r}} = 0$, is fundamental in order to rule out the root with positive real part coming from the law of motion of capital. Then we can

[8] We consider lower triangularity otherwise the transversality condition should be verified only in the case that all the roots coming from the law of motion of capital have negative real part.

write the optimal general solution

$$k(t) = \sum_{r \neq \tilde{r}} p_r e^{z_r t} + \sum_r \frac{[(1-\lambda_2) A - \delta]^2 p_{\tilde{r}} (z_{\tilde{r}} - \lambda_{\tilde{v}}) h'(z_{\tilde{r}})}{(z_r - \lambda_{\tilde{v}}) h'(z_r)} e^{\lambda_{\tilde{v}} t} \quad (2.25)$$

$$c(t) = [(1-\lambda_2) A - \delta] p_{\tilde{r}} (z_{\tilde{r}} - \lambda_{\tilde{v}}) h'(z_{\tilde{r}}) e^{\lambda_{\tilde{v}} t} \quad (2.26)$$

and since the residues $\{p_r\}$ and $\{n_v\}$ are uniquely determined by the boundary condition of capital and the transversality condition through (2.23), (2.24), we have that the equilibrium is locally determinate.

How these results change for a small variation of the zero coefficient[9] in $\tilde{J}$? In that case the new characteristic equation is

$$h(z) = \tilde{h}(z) - a\varepsilon [A - a(\delta + \varepsilon)] \mathrm{e}^{-\rho\tau}$$

Since ρ is usually assumed small, the $\tilde{h}(z)$ can be considered an "almost" even function and then any small shift of the x-axis let the number of roots having positive and negative real part invariant[10] and then the dynamic behavior of the economy. ■

From an economic point of view is also really interesting to observe how Hopf bifurcation may rise in this context, confirming the prediction in Asea and Zak [1] and Rustichini [27].

Remark 2.8 *Hopf bifurcation rises when* $\tau = \frac{3\pi}{2[a(1-\lambda_2)A-\delta]}$.

Proof. According to the D-Subdivision method when $\tau = \tau^* = \frac{3\pi}{2[a(1-\lambda_2)A-\delta]}$ two roots which have negative real part in the interval $\tau \in \left(0, \frac{3\pi}{2[a(1-\lambda_2)A-\delta]}\right)$ become purely imaginary and then positive in the interval $\tau \in \left(\frac{7\pi}{2[a(1-\lambda_2)A-\delta]}, \infty\right)$. Then in τ^* we have all the roots with negative real part but two purely imaginary, since all the (other) roots with positive real part are ruled out by transversality condition through (2.23) and (2.24). Then according to Kolmanovskii and Myshkis ([24], Chapter 3, page 183) we have Hopf bifurcation since all the following conditions are verified

a) if $\tau < \tau^*$ all the roots have (after transversality condition) negative real part;

b) $z_{1,2}(\tau)|_{\tau=\tau^*} = \pm i w_0, \ w_0 > 0$;

c) Through the D-Subdivision method follows immediately that

$$\left.\frac{d \operatorname{Re} z_{1,2}(\tau)}{d\tau}\right|_{\tau=\tau^*} > 0, \qquad \operatorname{Re} z_j(\tau)|_{\tau=\tau^*} < 0 \quad (j > 2)$$

■

[9] It is easy to check that a Ramsey model have a lower triangular Jacobian when a value of χ equal to zero is assumed. That is the so-called Gary Hansen model [17].

[10] A similar argument is invoked by Rustichini [27].

2.3.4 The Benhabib and Farmer model with time to build

Now we focus on the Benhabib-Farmer model. The analysis in the case of "mild" externalities, namely $\beta \in (b, 1-\chi)$, leads to results very close to those obtained in the previous section for a continuity argument. On the other hand, we cannot use the D-Subdivision method in order to study the dynamics in the case $\beta \in [1-\chi, \infty)$ since $\beta = 1-\chi$ is a discontinuity point for the characteristic equation of system (2.17)[11].

Then we proceed as follows. Given any transcendental characteristic equation $\Delta(z)$, by expanding all exponents in Taylor series we obtain

$$\tilde{\Delta}(z) = \sum_{j=0}^{\infty} a_j z^j \tag{2.27}$$

As observed by Kolmanovskii and Myshkis [24] (Chapter 4, page 240-241) the zeroes of $\Delta(z)$ coincide with those of $\tilde{\Delta}(z)$;[12] hence we can deduce the stability properties of our system (2.17) by making a n-order Taylor approximation of the exponential terms in $h(z)$ and then studying the sign of the roots of the polynomial $\tilde{h}(z)$ of degree j. In the following, we propose a numerical exercise where we study the stability properties of our system for different values of the marginal product of labor, β, and the delay coefficient, τ, given an approximation of order $n = 8$ and then a characteristic polynomial of degree $j = 14$.[13] Moreover, we have assumed capital's share, a, at 0.34, labor's share, b, at 0.66, marginal product of capital, α, at 0.83, the discount rate at 0.02, the depreciation rate at 0.05, and the parameter χ at -0.25. Given this parametrization, it is possible to calculate the full spectrum of the eigenvalues which are the zeros of the j-order polynomial obtained by Taylor expanding the exponents in $h(z)$. Moreover since the only state variable is capital, the economy will face local determinacy when the number of roots having negative and positive real part are equal. On the other hand if the number of roots having negative real part or positive real part is higher then the equilibrium will be local indeterminate or local unstable respectively. In Appendix A4, we have reported the spectrum of roots according to different parametrization of the marginal product of labor and the delay coefficient. In the following graph we have summarized the dynamics properties of the equilibrium of the economy.

[11] Remember that the D-Subivision method requires that all the coefficients of the characteristic equation vary continuously, otherwise it may be that a change in the sign of some roots happen without passing through zero. This is exactly what happens in the Benhabib Farmer model as clearly shown in Figure 2 and 3 of their article [6].

[12] To be precise, let $\Delta(z) = e^{-zh_{pq}}\tilde{\Delta}(z)$ where $h_{pq} = \max_{l,j} h_{lj}$. In the case under analysis $\Delta(z) = e^{-z\tau}\tilde{\Delta}_1(z)\, e^{z\tau}\tilde{\Delta}_2(z) + c = \tilde{\Delta}(z)$, where $c \in \mathbb{R}$.

[13] The critical values under which there is a change in stability have been controlled for a higher choice of n. In particular we have tried with $n = 12$ and then $j = 22$.

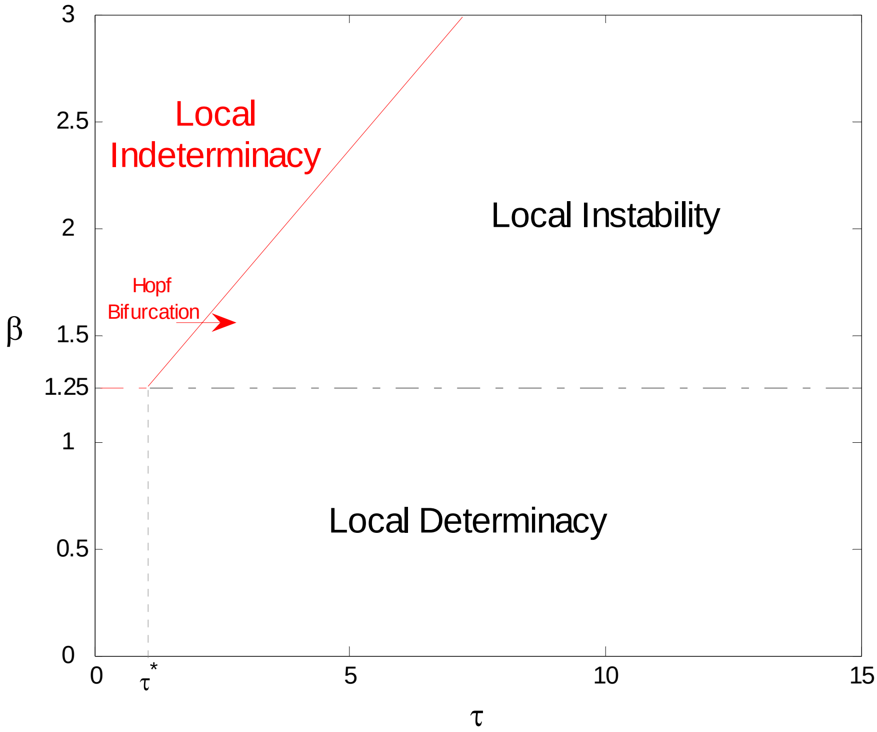

Figure 2.1: Dynamic behavior of the economy for different choice of (τ, β).

From Figure 2.1, some considerations rise. First of all, the local determinacy for the case of "mild" externalities is confirmed for any choice of the delay coefficient between 0 and 15. Given a value for the marginal product of capital higher than $1 - \chi$, the presence of local indeterminacy is confirmed even in the case of time to build but it depends crucially on the choice of the delay coefficient. In particular, given a value of the marginal product of capital higher than $1 - \chi$, the equilibrium is locally indeterminate in the interval $\tau \in [0, \tilde{\tau})$ with $\tilde{\tau}$ close to τ^* when β is closed to $1 - \chi$. On the other hand, if the delay coefficient $\tau \in (\tilde{\tau}, \infty)$ this is no more since a couple of conjugate complex roots, having negative real part, becomes positive. It is also worth noting that $\tau = \tilde{\tau}$ is an Hopf bifurcation point since all the requirements in Remark 2.8 are respected.[14]

Moreover, we have studied how different choice in the marginal product of capital affects the dynamics of the economy by changing the sign (in a no-continuous manner) of some of the roots of the spectrum reported in Appendix A4. In particular, we report in Figure 2.2,

[14]Observe that such requirements are not respected when we pass from the region of local determinacy to the region of local instability since the changing in the sign of roots happens in a not continuous way.

the roots which modify their sign when we set β in a neighbors of $1-\chi$ for a choice of τ equals to 0.01.

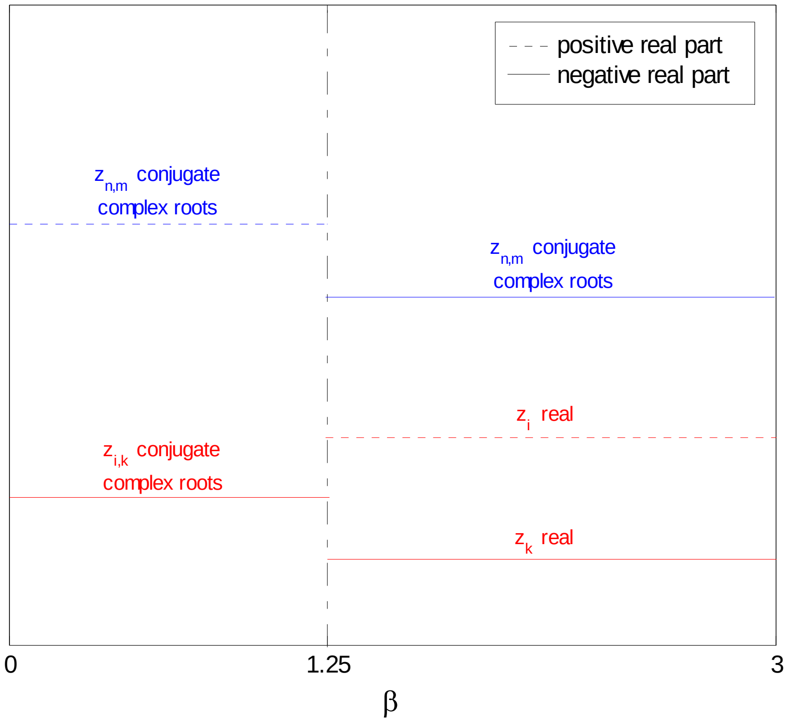

Figure 2.2: Behavior of the roots changing their sign at $\beta = 1-\chi$ and $\tau < \tilde{\tau}$.

Figure 2.2 shows what happens to some roots around the critical value $\beta = 1-\chi$ for a choice of $\tau \in [0, \tilde{\tau})$. In particular, a couple of conjugate complex roots, $z_{i,k}$ split in one positive, z_i, and one negative, z_k, root while another couple $z_{n,m}$ change the sign of their real part (from positive to negative). All the signs of the other roots remain unchanged. It is also possible to observe that for higher choices of τ, the changing in sign of the roots $z_{n,m}$ happens for choices of β to the right of $1-\chi$. This is the reason according to which we can display local instability as reported in Figure 2.2.

2.4 Conclusions

We have studied a Benhabib Farmer model in order to analyze the effect of the time to build assumption on the dynamic behavior of the economy. In a first moment, we have focused on a simpler Ramsey model with endogenous labor supply, and we have proved that the dynamic behavior of the economy around the steady state remains of *"saddle-path"* type. By a continuity argument, the same dynamic behavior is displayed by a Benhabib and Farmer model when "mild" externalities are assumed. As explained previously, the same argument cannot be adopted for "higher" externalities, and then the dynamic behavior of the economy is studied numerically. Presence of *local indeterminacy*, *Hopf bifurcation* and even *local instability* appear strictly related to the choice of the marginal product of capital and the delay coefficient.

Bambi, Mauro (2007), Some Essays in Growth Theory
European University Institute 10.2870/23610

2.5 Appendix A: Some proofs

Proof of Lemma 2.2. We start with the case $\lambda_2 > 0$. We can observe immediately that

$$\lambda_2 > 0 \Longleftrightarrow \beta > 1 - \chi.$$

But then given the assumptions $\alpha \in (0,1)$ and $\chi \leq 0$, follows immediately that

$$\begin{aligned} \lambda_2 &> 0 \Longleftrightarrow \lambda_1 + \lambda_2 = \tfrac{(\alpha-1)(\chi-1)}{\beta+\chi-1} > 0, \\ \lambda_2 &> 0 \Longrightarrow 1 + \lambda_1 = \tfrac{\alpha(\chi-1)}{\beta+\chi-1} < 0. \end{aligned}$$

Now we analyze the case $\lambda_2 < 0$.We can observe immediately that

$$\lambda_2 < 0 \Longleftrightarrow \beta < 1 - \chi.$$

But then given the assumptions $\alpha \in (0,1)$ and $\chi \leq 0$, follows immediately that

$$\begin{aligned} \lambda_2 &< 0 \Longleftrightarrow \lambda_1 + \lambda_2 = \tfrac{(\alpha-1)(\chi-1)}{\beta+\chi-1} < 0, \\ \lambda_2 &< 0 \Longrightarrow 1 + \lambda_1 = \tfrac{\alpha(\chi-1)}{\beta+\chi-1} > 0. \end{aligned}$$

and then we have proven all the relations between $\lambda_2, \lambda_1 + \lambda_2$ and $1 + \lambda_1$. Moreover since we can write λ_1 as follows:

$$\lambda_1 = \tfrac{\alpha(\chi-1)}{\beta+\chi-1} - 1 = \lambda_1 = \tfrac{\alpha(\chi-1+\beta-\beta)}{\beta+\chi-1} - 1 = -\alpha\lambda_2 + \alpha - 1,$$

then we'll have that if

$$\begin{aligned} \lambda_2 &\in \left[\tfrac{\alpha-1}{\alpha}, +\infty\right) \Longrightarrow \lambda_1 \leq 0, \\ \lambda_2 &\in \left(-\infty, \tfrac{\alpha-1}{\alpha}\right) \Longrightarrow \lambda_1 > 0, \end{aligned}$$

Proof of Proposition 2.1.We need to prove that both $\frac{dk_s}{d\tau}$ and $\frac{dc_s}{d\tau}$ are negative. First of all we'll have that:

$$\frac{dk_s}{d\tau} = \frac{A'(\tau)}{\lambda_1 + \lambda_2} \left\{ \frac{(1-\lambda_2) A(\tau) - \delta}{A(\tau)[A(\tau) - \delta]} \right\}$$

now since $A(\tau) > 0$, $A'(\tau) = \frac{\rho^2}{a} e^{\rho\tau} > 0$ and $A(\tau) - \delta \overset{(2.16)}{>} 0$ then $sign\left(\frac{dk_s}{d\tau}\right)$ depends exclusively on λ_2. If $\lambda_2 < 0$ then $(1-\lambda_2) A(\tau) - \delta > 0$ but for Lemma1, $\lambda_1 + \lambda_2 < 0$ and then $\frac{dk_s}{d\tau} < 0$. On the other hand if $\lambda_2 > 0$, since $1 - \lambda_2 < 0$, we'll have that $(1-\lambda_2) A(\tau) - \delta < 0$ but this time $\lambda_1 + \lambda_2 > 0$ and then $\frac{dk_s}{d\tau} < 0$.

Now we'll study the *sign* $\left(\frac{dc_s}{d\tau}\right)$ in order to do that we put (2.15) into (2.16) and then we take the derivative respect to τ :

$$\frac{dc_s}{d\tau} = \frac{A'(\tau)}{\lambda_1+\lambda_2}\left\{\frac{(1+\lambda_1)\,A(\tau)-\delta}{A(\tau)\,[A(\tau)-\delta]}\right\}$$

as before the *sign* $\left(\frac{dc_s}{d\tau}\right)$ depends exclusively on λ_2. In fact if $\lambda_2>0$, since $1+\lambda_1 \overset{L1}{<} 0$, we'll have $(1+\lambda_1)\,A(\tau)-\delta<0$ but $\lambda_1+\lambda_2 \overset{L1}{>} 0$ and then $\frac{dc_s}{d\tau}<0$. On the other hand suppose that $\lambda_2<0$, if we prove that $(1+\lambda_1)\,A(\tau)-\delta>0$ since $\lambda_1+\lambda_2 \overset{L1}{<} 0$ then $\frac{dc_s}{d\tau}<0$. In order to prove that $(1+\lambda_1)\,A(\tau)-\delta>0$ we distinguish the following two cases:

$$\begin{aligned}
\lambda_2 &\in \left(-\infty,\tfrac{\alpha-1}{\alpha}\right) \overset{L1}{\Longrightarrow} \lambda_1>0 \Longrightarrow A(1+\lambda_1)-\delta > A-\delta \overset{(2.16)}{>} 0\\
\lambda_2 &\in \left(\tfrac{\alpha-1}{\alpha},0\right) \overset{L1}{\Longrightarrow} A(1+\lambda_1)-\delta>0
\end{aligned}$$

where the last relation is obtained by studying the limit case $\lambda_2\to 0^-$. In fact if

$$\lambda_2\to 0 \overset{L1}{\Longrightarrow} 1+\lambda_1\to\alpha \Longrightarrow \Pi_2\to\alpha A-\delta > aA-\delta=\rho e^{\rho\tau}>0.$$

2.6 Appendix B: Linearization around the steady state

We show how to obtain the Jacobian starting from the *DDE for capital* and the *ADE for consumption.* In order to simplify the algebra we rewrite the two functional differential equations as follows:

$$\begin{aligned}
\dot{k}(t) &= e^{f(k(t),k(t-\tau))}\left\{e^{g(k(t-\tau),c(t))}-\delta\right\}-e^{h(k(t),c(t))},\\
\dot{c}(t) &= e^{v(c(t),c(t+\tau))}\left\{ae^{\tilde{g}(k(t),c(t+\tau))}-\delta\right\}-\rho,
\end{aligned}$$

and we'll use the following notation:

$$e^{\lambda_0+\lambda_1 k_s+\lambda_2 c_s} = \frac{\delta+\rho e^{\rho\tau}}{a}\equiv A, \tag{2.28}$$

$$e^{c_s-k_s} = \frac{\delta+\rho e^{\rho\tau}}{a}-\delta\equiv A-\delta. \tag{2.29}$$

Now we calculate the following derivative[15]:

$$\begin{aligned}\frac{\partial \dot{k}(t)}{\partial k(t)} &\equiv \left[\tfrac{\partial}{\partial k(t)} f(k(t), k(t-\tau))\right] e^{f(k(t),k(t-\tau))} \left\{e^{g(k(t-\tau),c(t))} - \delta\right\} \\ &\quad + e^{f(k(t),k(t-\tau))} \left[\tfrac{\partial}{\partial k(t)} g(k(t-\tau), c(t))\right] e^{g(k(t-\tau),c(t))} \\ &\quad - \left[\tfrac{\partial}{\partial k(t)} h(k(t), c(t))\right] e^{h(k(t),c(t))} \\ &= \left(e^{-z\tau} - 1\right) e^{k(t-\tau)-k(t)} \left\{e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} - \delta\right\} + \\ &\quad e^{k(t-\tau)-k(t)} \lambda_1 e^{-z\tau} e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} + e^{c(t)-k(t)},\end{aligned}$$

and then

$$\left.\frac{\partial \dot{k}(t)}{\partial k(t)}\right|_{s.s.} = \left(e^{-z\tau} - 1\right)\left(e^{\lambda_0 + \lambda_1 k_s + \lambda_2 c_s} - \delta\right) + \lambda_1 e^{-z\tau} e^{\lambda_0 + \lambda_1 k_s + \lambda_2 c_s} + e^{c_s - k_s},$$

and taking into account the relations (2.28) and (2.29) we'll have finally:

$$\left.\frac{\partial \dot{k}(t)}{\partial k(t)}\right|_{s.s.} = e^{-z\tau}\left(A - \delta + \lambda_1 A\right). \tag{2.30}$$

Now we search for

$$\begin{aligned}\frac{\partial \dot{k}(t)}{\partial c(t)} &\equiv e^{f(k(t),k(t-\tau))} \left[\tfrac{\partial}{\partial c(t)} g(k(t-\tau), c(t))\right] e^{g(k(t-\tau),c(t))} - \left[\tfrac{\partial}{\partial c(t)} h(k(t), c(t))\right] e^{h(k(t),c(t))} \\ &= \lambda_2 e^{\lambda_0 + \lambda_1 k(t-\tau) + \lambda_2 c(t)} - e^{c(t)-k(t)},\end{aligned}$$

and then in steady state we get:

$$\left.\frac{\partial \dot{k}(t)}{\partial c(t)}\right|_{s.s.} = \lambda_2 A - A + \delta. \tag{2.31}$$

Now we pass to find

$$\begin{aligned}\frac{\partial \dot{c}(t)}{\partial k(t)} &\equiv e^{v(c(t),c(t+\tau))} \left[\tfrac{\partial}{\partial k(t)} \tilde{g}(k(t), c(t+\tau))\right] a e^{\tilde{g}(k(t),c(t+\tau))} \\ &= e^{-\rho\tau + c(t) - c(t+\tau)} a \lambda_1 e^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t+\tau)},\end{aligned}$$

and then in steady state we get:

$$\left.\frac{\partial \dot{c}(t)}{\partial k(t)}\right|_{s.s.} = e^{-\rho\tau} a \lambda_1 A. \tag{2.32}$$

[15] We search for a solution of type $c(t) = k(t) = e^{zt}$ and then we have the following relations $k(t-\tau) = e^{z(t-\tau)}$ and $c(t+\tau) = e^{z(t+\tau)}$

At last we calculate:

$$\begin{aligned}\frac{\partial \dot{c}(t)}{\partial c(t)} &\equiv \left[\frac{\partial}{\partial c(t)} v(c(t), c(t+\tau))\right] e^{v(c(t),c(t+\tau))} \left\{ae^{\tilde{g}(k(t),c(t+\tau))} - \delta\right\} + \\ & e^{v(c(t),c(t+\tau))} \left[\frac{\partial}{\partial c(t)} \tilde{g}(k(t), c(t+\tau))\right] ae^{\tilde{g}(k(t),c(t+\tau))} \\ &= \left(1 - e^{z\tau}\right) e^{-\rho\tau + c(t) - c(t+\tau)} \left\{ae^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t+\tau)} - \delta\right\} + \\ & e^{-\rho\tau + c(t) - c(t+\tau)} a\lambda_2 e^{\lambda_0 + \lambda_1 k(t) + \lambda_2 c(t+\tau)},\end{aligned}$$

and then in steady state we get

$$\left.\frac{\partial \dot{c}(t)}{\partial c(t)}\right|_{s.s.} = -(aA - \delta - a\lambda_2 A)e^{-\rho\tau}e^{z\tau} + (aA - \delta)e^{-\rho\tau}, \tag{2.33}$$

and then taking into account (2.30),(3.2),(2.32), and (2.33) we can construct the Jacobian (2.17).

The trace and the determinant of (2.17) are given by[16]:

$$Tr(J) = (A - \delta + \lambda_1 A)e^{-z\tau} - (aA - \delta - a\lambda_2 A)e^{-\rho\tau}e^{z\tau} + (aA - \delta)e^{-\rho\tau}, \tag{2.34}$$

$$\begin{aligned}Det(J) &= (A - \delta + \lambda_1 A)(aA - \delta)e^{-\rho\tau}e^{-z\tau} - (A - \delta)(aA - \delta - a\lambda_2 A)e^{-\rho\tau} \\ &+ \lambda_1 A(1 - a)\delta e^{-\rho\tau}.\end{aligned} \tag{2.35}$$

2.7 Appendix C: Roots of $\tilde{\Delta}(z)$

All the numerical results are obtained using MatLab and given the parametrization reported in Section 2.3.4. We report in the following only a subset of the whole numerical simulations for space reasons. More tables available under request.

[16]As we expected, we can obtain the same BF results for trace and determinant just assuming the delay equal to zero.

Delay coefficient τ=0.01

β=1.1	β=1.24	β=1.26	β=1.3	β=1.4	β=1.6	β=3
0.90	10.23	381.15	616.28	788.31	925.61	1155.5
817.01±431.78i	539.12±279.12i	609.44	778.39	918.9	1041	1285.5
704.13±530.18i	363.57±391.83i	345.99±487.98i	434.88±626.89i	507.74±743.46i	570.49±845.37i	695.14±1050.9i
100.39±877.98i	82.848±576.65i	199.76±609.69i	283.8±753.53i	357.95±876.85i	420.87±981.47i	530.05±1164i
-104.55±1022.9i	**-89.60±664.48i**	**-166.74±546.69i**	**-231.69±665.6i**	**-294.87±772i**	**-352.53±867.84i**	**-471.38±1065.7i**
-568.1±466.94i	**-281.12±375.76i**	**-394.48±563.21i**	**-495.38±723.54i**	**-576.28±854.84i**	**-642.68±963.96i**	**-755.52±1151.5i**
-949.33±505.48i	**-619.93±323.14i**	**-19,56**	**-27,91**	**-0.86**	**-0.34**	**-0.03±0.02i**
-0.017	**-0.018**	**-0.018**	**-0.018**	**-0.019**	**-0.02**	
		-238.9	**-480.12**	**-643.09**	**-776.2**	**-1032.9**
		-701.18	**-894.96**	**-1052.3**	**-1182.3**	**-1404.6**

Delay coefficient τ=0.5

0.49	0.90	1.03	1.43	3.64	6.05	9.28
9.04±4.64i	6.02 ± 3.04i	6.75	8.59	10.10	11.41	14.02
5.32 ± 7.4846i	3.01 ± 8.28i	4.01 ± 5.29i	4.95 ± 6.82i	5.79 ± 8.06i	6.50 ± 9.12i	7.88 ± 11.26i
1.5188 ± 9.57i	1.67 ± 5.98i	2.80 ± 8.72i	2.81 ± 9.54i	3.14 ± 10.51i	3.58 ± 11.44i	4.52 ± 13.12i
-1.63 ± 11.20i	**-1.31 ± 7.04i**	0.56 ± 5.56i	0.57 ± 4.48i	**-0.61 ± 2.34i**	**-3.10 ± 10.43i**	**-3.99 ± 12.19i**
-4.01 ± 7.55i	**-2.92 ± 8.42i**	**-2.84 ± 8.56i**	**-2.74 ± 8.94i**	**-2.8 ± 9.65i**	**-7.32 ± 10.37i**	**-8.56 ± 12.32i**
-10.48 ± 5.42i	**-6.91 ± 3.51i**	**-4.53 ± 6.14i**	**-5.66 ± 7.86i**	**-6.58 ± 9.23i**	**-0.78**	**-0.034 ± 0.02i**
-0.016	**-0.017**	**-0.017**	**-0.017**	**-0.018**	**-0.019**	
		-7.77	**-9.86**	**-11.56**	**-12.94**	**-15.31**
					-3.04	**-7.61**

Delay coefficient τ=1

β=1.1	β=1.24	β=1.26	β=1.3	β=1.4	β=1.6	β=3
0.35	0.49	0.52	0.59	0.90	1.92	3.64
4.07 ± 2.08i	2.72 ± 1.37i	3.04	3.87	4.55	5.14	6.30
2.17 ± 3.74i	1.47 ± 4.19i	1.83 ± 2.39i	2.24 ± 3.05i	2.62 ± 3.62i	2.94 ± 4.09i	3.57 ± 5.05i
0.76 ± 4.25i	0.81 ± 2.76i	1.42 ± 4.30i	1.38 ± 4.53i	1.44 ± 4.91i	1.58 ± 5.29i	1.94 ± 6.01i
-0.77 ± 5.02i	**-0.66 ± 3.19i**	0.22 ± 2.75i	0.33 ± 2.62i	0.18 ± 1.92i	**-1.43 ± 4.87i**	**-1.73 ± 5.60i**
-1.68 ± 3.89i	**-1.45 ± 4.23i**	**-1.43 ± 4.26i**	**-1.39 ± 4.35i**	**-1.37 ± 4.57i**	**-0.37 ± 1.06i**	**-3.88 ± 5.53i**
-4.73 ± 2.43i	**-3.12 ± 1.58i**	**-2.06 ± 2.76i**	**-2.57 ± 3.53i**	**-2.98 ± 4.15i**	**-3.32 ± 4.66i**	**-0.042 ± 0.008i**
-0.02	**-0.02**	**-0.02**	**-0.02**	**-0.02**	**-0.02**	**-2.76**
		-3.51	**-4.45**	**-5.21**	**-5.83**	**-6.90**

Delay coefficient τ=5

0.12	0.13	0.13	0.13	0.13	0.14	0.22
0.64 ± 0.32i	0.42 ± 0.21i	0.48	0.60	0.71	0.80	0.99
0.31 ± 0.81i	0.29 ± 0.84i	0.29 ± 0.38i	0.36 ± 0.47i	0.41 ± 0.55i	0.46 ± 0.63i	0.56 ± 0.78i
0.17 ± 0.63i	0.14 ± 0.47i	0.28 ± 0.85i	0.28 ± 0.86i	0.27 ± 0.88i	0.27 ± 0.92i	0.29 ± 1.01i
-0.13 ± 0.77i	**-0.13 ± 0.52i**	0.023 ± 0.50i	0.045 ± 0.55i	0.06 ± 0.55i	0.067 ± 0.51i	**-0.005 ± 0.28i**
-0.29 ± 0.83i	**-0.28 ± 0.84i**	**-0.28 ± 0.84i**	**-0.28 ± 0.85i**	**-0.28 ± 0.86i**	**-0.27 ± 0.87i**	**-0.28 ± 0.95i**
-0.75 ± 0.38i	**-0.49 ± 0.25i**	**-0.33 ± 0.44i**	**-0.41 ± 0.55i**	**-0.48 ± 0.65i**	**-0.53 ± 0.73i**	**-0.62 ± 0.87i**
-0.01	**-0.01**	**-0.01**	**-0.01**	**-0.01**	**-0.01**	**-0.01**
		-0.56	**-0.71**	**-0.82**	**-0.93**	**-1.09**

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CHAPTER 3

WELFARE RANKING OF NON-MONOTONIC PATHS IN ONE-SECTOR GROWTH MODELS WITH NON-CONVEX TECHNOLOGY.

3.1 Introduction

Despite the concavity of the utility function, Christiano and Harrison [3] have established that increasing volatility of labor may raise welfare in economies with non-convex technology sets *à la* Benhabib and Farmer [1]. In absence of any productive externality, fluctuations in consumption and labor are welfare-diminishing compared to a smooth consumption/investment plan when the utility function is concave. However, in the presence of productive externality, the welfare loss implied by fluctuations may be more than compensated by the gain inherited from the increasing returns to scale: for a given capital stock, by bunching hard work, agents are able to increase the average level of consumption without raising the average level of labor. When dis-utility of labor does not raise disproportionately compared to the additional utility procured by consumption, this "bunching" effect dominates the first negative "concavity" effect and makes the agents better-off. Thus, when the steady state equilibrium is locally indeterminate, that is when there is multiplicity of deterministic equilibria around the steady state, stochastic sunspot equilibria may be welfare-improving.

In the literature the possibility of stabilizing an economy characterized by local indeterminacy has been analyzed in such a framework by Guo and Lansing [5].[1] However, no much attention has been dedicated to the choice of the best equilibrium path on which stabilize the economy. It is clear, from Christiano and Harrison's estimates, that a stabilizing policy can make the agents worse-off when expectations are pinned down on a suboptimal path. From Pareto's criterion viewpoint, any (decentralized) deterministic equilibrium path of Benhabib and Farmer's economy is not efficient as long as agents do not internalize the externality of production. Nevertheless, from a welfare viewpoint, these deterministic equi-

[1] Economic policy constructed to stabilize the economy by minimizing the variance of output have also been analyzed in models in which the level of externality required to get indeterminacy is less stringent than in the current framework. See for instance Guo and Harrison [4] and Sims [2005].

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libria do not display the same level of utility: the optimization programme fails to determine which of them provides the maximum amount of welfare since they all satisfy the first order conditions and the transversality condition. Thus, when agents jump from one path to another, the stochastic equilibrium so obtained may increase their welfare provided they leave a welfare-dominated deterministic path for a welfare-improving deterministic path.

In this paper a welfare ranking of the different deterministic equilibria in an exogenous growth model with non-convex technology and presence of local indeterminacy is studied. In the continuity of Christiano and Harrison [3] who determined that a stochastic equilibrium may be welfare-improving for agents, we look for the conditions under which a change in the deterministic path chosen by the agents is welfare-improving. The starting value of consumption and the speed of capital accumulation (or equivalently the monotonicity of the consumption/investment plan) determine simultaneously the desirability of a change in the equilibrium path. Actually, these two components allow us to establish which one of the two effects described by Christiano and Harrison dominates according to the level of increasing returns to scale. Since all eigenvalues have strictly negative real part, the model exhibits local indeterminacy, that is a region of stability in which equilibrium paths converge to the steady state. According to the Grobman-Hartman theorem, this local stability implies the preservation of the topological properties of the system under linearization in a neighborhood of the steady state. Moreover, for a given initial stock of capital, Russell and Zecevic [6] determined the range of values of initial consumption lying in the region of attraction when a Benhabib and Farmer model [1] is considered. Then, taking into account these analytical results we have proposed two approaches. In the first one, we restrict the analysis to the equilibria converging monotonically to the steady state when the lowest level of increasing returns required to get indeterminacy is chosen. Then, we determine analytically the optimal starting value of consumption within this set of deterministic equilibria using a linear approximation of the dynamical system and the utility function around the steady state. In the other approach, we continue a local analysis but through numerical methods we are able to enlarge the range of initial conditions and the possible values of the externalities: these changes let us to consider also paths in the neighborhood of the steady state which do not converge monotonically to the steady state and are able to determine more precisely the value of the optimal starting condition of consumption and the optimal behavior of the consumption/investment plan within the attracting set.

Finally, the aim of the paper is to rank the different deterministic equilibria in terms of welfare according to the initial level of consumption in the neighborhood of the local indeterminacy steady state when a stabilization policy is introduced. In particular we will assume

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that the government can pin down expectations on one of the deterministic equilibria; this can be done by fixing the rental rate on capital or the real wage at any period (see Saïdi [8]).Then, it will be shown that the (decentralized) optimal welfare equilibrium displays a path all the less monotonic and an initial level of consumption all the higher since increasing returns to scale are high. Bunching hard work in the very first periods makes capital accumulation faster. In the next periods, agents can benefit from the high level of capital stock by maintaining a high level of consumption but decreasing labor significantly. When increasing returns are high enough, reaching the optimal capital stock requires few time, which explains the non-monotonicity of the equilibria during the first periods. However, when increasing returns are close to the condition of indeterminacy, bunching hard work in the first periods is not sufficient to accumulate a sufficient amount of capital stock, which would require large levels of labor and a loss of welfare that next periods consumption cannot offset. Thus, when increasing returns to scale are not high enough, a (second best) optimal policy should pin down expectations such that agents would rather smooth their consumption and labor paths and accumulate progressively in order to maximize their welfare.

In the second section, we will present briefly the main characteristics of Benhabib and Farmer's model, including uniqueness of the steady state equilibrium and the condition for indeterminacy. In section 3, we will assume this condition satisfied and specify the set of monotonic consumption paths for any values of the parameters. These results will be helpful in establishing the welfare ranking of section 4 when we use a linear approximation of the utility function and will be confirmed by the more general simulation method. Section 5 will conclude.

3.2 Model Setup

3.2.1 Agents' behavior

In this paper we analyze the welfare properties of different equilibrium paths of Benhabib and Farmer's model [1]. This deterministic continuous-time model with infinitely lived agents is characterized by increasing social returns to scale due to externality in the aggregate production function. However, the representative firm is assumed not to take into account the externality of production and then faces a Cobb Douglas production function with constant returns to scale at the micro-level.

Formally:

$$Y(t) = A(t)K(t)^a L(t)^b \qquad \text{with} \quad 0 < a < 1, \quad \text{and} \quad a + b = 1, \tag{3.1}$$

$$A(t) = \bar{K}(t)^{a\gamma_a} \bar{L}(t)^{b\gamma_b} \qquad \text{with} \quad \gamma_a, \gamma_b > 0, \tag{3.2}$$

where $\bar{K}$ and $\bar{L}$ represent the average economy-wide levels of capital and labor. In equilibrium, $K = \bar{K}$ and $L = \bar{L}$ and by making the parameters substitutions $\alpha = a(1+\gamma_a)$ and $\beta = b(1+\gamma_b)$, we get the aggregate production function:

$$Y(t) = K(t)^{\alpha} L(t)^{\beta},$$

which obviously exhibits increasing returns to scale. In the same time, the economy is populated by a large number of identical consumers. As usual, firms maximize profit, which breaks down because of the constant returns, while the representative consumer, owner of the firms, faces the following optimal control problem:

$$\int_0^{\infty} \left(\log C(t) - \frac{L(t)^{1-\chi}}{1-\chi} \right) e^{-\rho t} dt,$$

subject to:

$$\dot{K}(t) = (r(t) - \delta) K(t) + w(t)L(t) - C(t).$$

3.2.2 Dynamical system and Steady state equilibrium

From the first order conditions and after some algebra, Benhabib and Farmer obtain the following two nonlinear ordinary differential equations system:

$$\dot{k} = e^{\mu_0 + \mu_1 k + \mu_2 c} - \delta - e^{c-k} \quad (3.3)$$

$$\dot{c} = a e^{\mu_0 + \mu_1 k + \mu_2 c} - \delta - \rho \quad (3.4)$$

where $x = \ln X$, $\mu_0 = \frac{-\beta \ln b}{\beta + \chi - 1}$, $\mu_1 = \frac{(\chi-1)(\alpha-1)-\beta}{\beta+\chi-1}$ and $\mu_2 = \frac{\beta}{\beta+\chi-1}$. It is worth noting that the system represents the global dynamics of the economy.

Taking into account such dynamics, we determine the steady state of the system:

Remark 3.1 *The steady state values of labor and consumption are respectively:*

$$k_s = \frac{1}{\mu_1 + \mu_2} \left[\log \frac{\rho + \delta}{a} - \mu_2 \log \frac{\rho + \delta (1-a)}{a} - \mu_0 \right]$$

$$c_s = \log \frac{\rho + \delta (1-a)}{a} + k_s$$

$$l_s = \frac{c_s - \alpha k_s - \log(b)}{\beta + \chi - 1}$$

where the last one can be obtained by the labor demand and labor supply equations.

Benhabib and Farmer show that, under the condition $\beta - 1 + \chi > 0$, that is if the aggregate labor demand curve is upward sloping and steeper than the labor supply curve, the steady state equilibrium is indeterminate. In the neighborhood of such an equilibrium, there exists a continuum of paths converging to it and then satisfying the first order conditions of the optimal control programme, including the transversality condition. In this framework, perfect foresight hypothesis, which usually leads to a unique equilibrium path, cannot discriminate between the different paths: agents are allowed to switch from one path to another at any period. However, in terms of welfare, these paths are not equivalent.

3.3 Local analysis

The results of this section are closely related to the classical Grobman-Hartman theorem that states that, around an hyperbolic equilibrium, the flow of a nonlinear differential equation is topologically conjugate via an homeomorphism to the flow of its linear approximation. It is clear from Benhabib and Farmer [1] that no eigenvalues crosses zero as the determinant changes sign and the steady state becomes stable[2]. Then, the stationary equilibrium remains hyperbolic even for the minimum degree of externality necessary for local indeterminacy. In this section, after having linearly approximated the dynamics for capital and consumption, we describe qualitatively the different equilibrium paths in term of monotonicity and we study both analytically and numerically the welfare rank of the different equilibrium paths.

Finally the (second) best equilibrium path in term of welfare is selected through a stabilization policy *à la* Saïdi [8] which is able to coordinate over time the agents on a given deterministic path (see Appendix for more details).

3.3.1 Linearization

We proceed to a first order approximation of equations (3.3) and (3.4) around the deterministic equilibrium and express the general solution in terms of deviation of the two variables $k(t)$ and $c(t)$ from their steady state values k_s and c_s, i.e. $\tilde{x}(t) = \ln X(t) - \ln X_s$. We get:

$$\begin{bmatrix} \tilde{k}(t) \\ \tilde{c}(t) \end{bmatrix} \simeq \begin{bmatrix} \eta_1 v_{11} & \eta_2 v_{12} \\ \eta_1 v_{21} & \eta_2 v_{22} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} \tag{3.5}$$

with

$$V = [\xi_1 : \xi_2] = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} (1+\mu_1)\Psi - \delta - \lambda_1 & (1+\mu_1)\Psi - \delta - \lambda_2 \\ (1-\mu_2)\Psi - \delta & (1-\mu_2)\Psi - \delta \end{bmatrix}, \tag{3.6}$$

[2] To be precise the change in the stability of the equilibrium is related to the presence of a discontinuity in the value of one of the eigenvalues as a function of the externality, namely $\lambda_i(\gamma_b)$ with λ_i the i-eigenvalue.

where $\Psi \equiv (\rho + \delta)/a$ and ξ_1 and ξ_2 are eigenvectors associated to the eigenvalues λ_1 and λ_2, which can be obtained after computing the Jacobian of the system formed by equations (3.3) and (3.4). Moreover, given a starting point $[K(0), C(0)]$, we apply Cramer's rule and deduce:

$$\begin{aligned} \eta_1 &= \frac{v_{22}\tilde{k}(0) - v_{12}\tilde{c}(0)}{v_{11}v_{22} - v_{12}v_{21}} \\ \eta_2 &= \frac{v_{11}\tilde{c}(0) - v_{21}\tilde{k}(0)}{v_{11}v_{22} - v_{12}v_{21}}. \end{aligned}$$

3.3.2 Monotonicity of the equilibrium paths

In order to understand the economic implications of the welfare ranking of the equilibrium paths in term of consumption smoothness over time, we study in this subsection the conditions on $c(0)$ under which the path is monotonic. It is worth noting that monotonicity can only appear when eigenvalues are real. In the following, we will assume without loss of generality that $\lambda_1 < \lambda_2 < 0$.

Under the condition $\beta - 1 + \chi > 0$ the stable manifold has dimension 2. We call stable arms the two paths such that:

$$\tilde{c}(t) = \eta_i v_{2i} e^{\lambda_i t}, \quad i = \{1, 2\}.$$

As shown in the Appendix, the starting log-values of consumption on the stable arms for a given initial stock of capital $K(0)$ are:

$$c_{0,\xi_1} = c_s + \tilde{k}(0)\frac{v_{21}}{v_{11}} \tag{3.7}$$

$$c_{0,\xi_2} = c_s + \tilde{k}(0)\frac{v_{22}}{v_{12}}. \tag{3.8}$$

The following proposition holds:

Proposition 3.2 *For a given initial stock of capital $K(0) < K_s$ (resp. $K(0) > K_s$), there exists a strictly positive (resp. negative) ε^* such that for $c(0) \in [c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$ (resp. $[c_{0,\xi_1}, c_{0,\xi_2} - \varepsilon^*]$) equilibrium paths of consumption are monotonic.*

Proof. Monotonicity of consumption paths occurs provided the equation $d\tilde{c}(t)/dt = 0$ has no solution. This means that there is no $t \in \mathbb{R}^+$ such that:

$$\eta_1 z_1 v_2^1 e^{z_1 t} + \eta_2 z_2 v_2^2 e^{z_2 t} = 0. \tag{3.9}$$

Using the fact that $v_{21} = v_{22}$:

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{v_{22}\tilde{k}(0) - v_{12}\tilde{c}(0)}{v_{21}\tilde{k}(0) - v_{11}\tilde{c}(0)} \frac{\lambda_1}{\lambda_2}\right). \tag{3.10}$$

A solution exists if and only if $E \equiv \frac{v_{22}\tilde{k}(0) - v_{12}\tilde{c}(0)}{v_{21}\tilde{k}(0) - v_{11}\tilde{c}(0)} > 0$. Assume, for instance, that $c(0) = c_{0,\xi_2} + \varepsilon$. In this case relation E becomes

$$E = \frac{-v_{12}^2\varepsilon}{[v_{12}v_{21} - v_{11}v_{22}]\tilde{k}(0) - v_{11}v_{12}\varepsilon},$$

where $v_{ij} < 0$ for any $i, j = \{1, 2\}$ (as shown in Appendix).

If $\varepsilon > 0$, equation (3.10) has a solution if and only if $[v_{12}v_{21} - v_{11}v_{22}]\tilde{k}(0) - v_{11}v_{12}\varepsilon < 0$, that is for:

$$\begin{aligned} \varepsilon &> \left[\frac{v_{21}}{v_{11}} - \frac{v_{22}}{v_{12}}\right]\tilde{k}(0) \\ &> c_{0,\xi_1} - c_{0,\xi_2}, \end{aligned}$$

or equivalently for:

$$c(0) > c_{0,\xi_1}.$$

If $\varepsilon < 0$, equation (3.10) has a solution if and only if $[v_{12}v_{21} - v_{11}v_{22}]\tilde{k}(0) - v_{11}v_{12}\varepsilon > 0$, that is for:

$$\begin{aligned} \varepsilon &< \left[\frac{v_{22}}{v_{12}} - \frac{v_{21}}{v_{11}}\right]\tilde{k}(0) \\ &< c_{0,\xi_2} - c_{0,\xi_1}, \end{aligned}$$

or equivalently for:

$$c(0) < c_{0,\xi_2} - \varepsilon^*,$$

with $\varepsilon^* \equiv c_{0,\xi_1} - c_{0,\xi_2}$, which is positive (resp. negative) according to Appendix provided $K(0) < K_s$ (resp. $K(0) > K_s$).

Thus consumption paths have a monotonic behavior if and only if $c_0 \in [c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$ for $K(0) < K_s$ and $c_0 \in [c_{0,\xi_1}, c_{0,\xi_2} - \varepsilon^*]$ for $K(0) > K_s$. ■

A specific case with $K(0) < K_s$ is reported in Figure 3.1.

3.3.3 Reformulation of the optimization programme

For a given initial stock of capital $K(0)$, the optimal paths of consumption and capital can be computed using equations (3.3) and (3.4). Then, optimal path of labor can be computed using the following first order condition:

$$(\beta - 1 + \chi)l(t) = c(t) - ak(t) - \ln b. \tag{3.11}$$

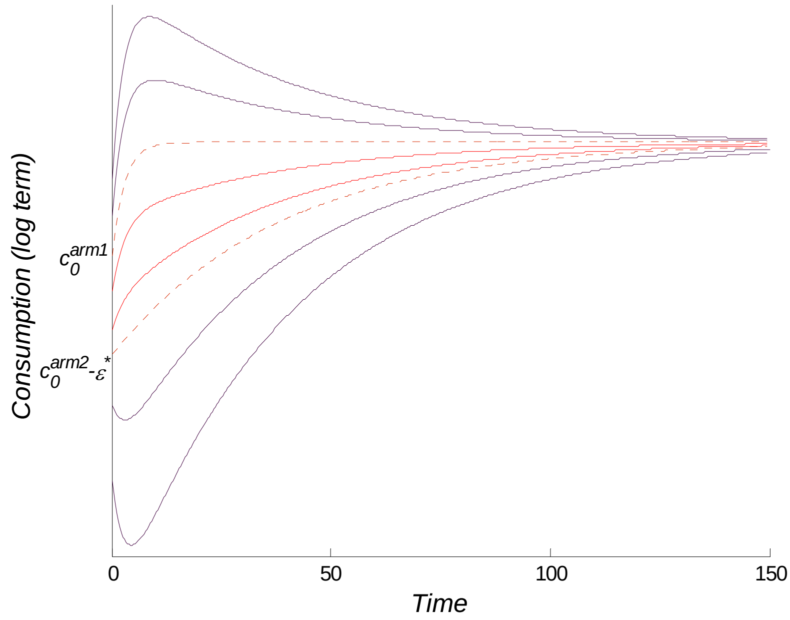

Figure 3.1: Consumption dynamic behavior around the equilibrium.

Consequently, once the initial level of consumption $C(0)$ has been chosen, agent's welfare can be derived. Searching the path making the agents better off consists of determining the initial level of consumption that maximizes welfare:

$$\max_{C_0} \int_0^\infty \left(\log C(t) - \frac{L(t)^{1-\chi}}{1-\chi} \right) e^{-\rho t} dt. \tag{3.12}$$

Since $U(c_s, l_s)$ is a constant, it must be noticed that our optimization programme (3.12) can be rewritten as:

$$\max_{C_0} \int_0^\infty \tilde{U}(c(t), l(t)) e^{-\rho t} dt, \tag{3.13}$$

where:

$$\tilde{U}(c(t), l(t)) = U(c(t), l(t)) - U(c_s, l_s),$$

and:

$$U(c(t), l(t)) = c(t) - \frac{e^{l(t)(1-\chi)}}{1-\chi}.$$

3.4 Paths Ranking of deterministic paths

When the dynamics is constrained to be linear, an analytical approach can be used to determine approximatively within the set of possible values the initial level of consumption maximizing welfare. However, the approximation error implied by such a technique requires us to work in a small neighborhood of the steady state

The technique used by Benhabib and Farmer and which consists in linearizing the dynamical system and checking whether eigenvalues have negative real parts ensures that the stationary equilibrium is asymptotically stable, that is locally attractive.

Moreover, Russell and Zecevic [6] has shown that it is possible to evaluate the region of attraction for the Benhabib Farmer model: given the initial level of capital, it can be computed the largest interval of values of the initial conditions of consumption such that the stability properties of the linearized and nonlinearized system remain invariant.

Taking into account these results, we propose an analytical and a purely numerical approach in order to make a welfare rank. In the first case we focus on the minimum degree of externalities and on the set of monotonic paths; then the analysis is enlarged to any value of the externalities and on all the equilibrium paths.

3.4.1 Welfare ranking of monotonic paths

3.4.1.1 Approximation method for $\beta - 1 + \chi$ close to zero

Applying total differentiation around the steady state $[c_s, l_s]$ to equation (3.13), we get the difference in welfare units between a given state and the steady state:

$$\begin{aligned} \tilde{U}(c(t), l(t)) &= \tilde{c}(t) - e^{l_s(1-\chi)}\tilde{l}(t) \qquad (3.14)\\ &= \phi_1\left(\eta_1 v_{11}e^{\lambda_1 t} + \eta_2 v_{12}e^{\lambda_2 t}\right) + \phi_2\left(\eta_1 v_{21}e^{\lambda_1 t} + \eta_2 v_{22}e^{\lambda_2 t}\right), \qquad (3.15) \end{aligned}$$

with $\phi_1 \equiv \frac{ae^{l_s(1-\chi)}}{\beta-1+\chi}$ and $\phi_2 \equiv 1-\frac{e^{l_s(1-\chi)}}{\beta-1+\chi}$. If we rearrange equation (3.15) and skip all (constant) terms in $\tilde{k}(0)$, the optimization programme becomes:

$$\max_{C_0} \int_0^\infty \left[-v_{12}\frac{\phi_1 v_{11} + \phi_2 v_{21}}{v_{11}v_{22} - v_{12}v_{21}}e^{\lambda_1 t} + v_{11}\frac{\phi_1 v_{12} + \phi_2 v_{22}}{v_{11}v_{22} - v_{12}v_{21}}e^{\lambda_2 t}\right]\tilde{c}(0)e^{-\rho t}dt \qquad (3.16)$$

with $c_0 \in [c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$ and k_0 very close and on the left respect to k_s. Let F be the term in brackets. It is straightforward that if $F > 0$ (resp. $F < 0$) the optimal value of $c(0) < c_s$ is c_{0,ξ_1} (resp. $c_{0,\xi_2} - \varepsilon^*$) since $c_{0,\xi_2} < c_{0,\xi_1} < c_s$.

With real eigenvalues, F becomes:

$$F = -\frac{v_{12}(\phi_1 v_{11} + \phi_2 v_{21})}{(\rho - \lambda_1)(v_{11}v_{22} - v_{12}v_{21})} + \frac{v_{11}(\phi_1 v_{12} + \phi_2 v_{22})}{(\rho - \lambda_2)(v_{11}v_{22} - v_{12}v_{21})}.$$

For the lowest levels of increasing returns insuring indeterminacy, that is for $\beta - 1 + \chi$ close to 0, it can be shown that F tends to $\bar{F}$ (look at the Appendix) with:

$$\bar{F} = -\frac{e^{l_s(1-\chi)}}{(\beta - 1 + \chi)(\rho - \bar{\lambda}_2)} \left(1 - a\frac{v_{12}}{v_{22}}\right). \tag{3.17}$$

Then the following proposition holds:

Proposition 3.3 *For $K(0) < K_s$, $\tilde{U}(c(t), l(t))$ is always strictly positive for monotonic paths of consumption with absolute maximum at $c_{0,\xi_2} - \varepsilon^*$ when β tends to $1 - \chi$. For $K(0) > K_s$, $\tilde{U}(c(t), l(t))$ is always strictly negative for monotonic paths of consumption with absolute maximum at c_{0,ξ_1} when β tends to $1 - \chi$.*

Proof. Assume that $k(0) < k_s$. According to Proposition 1 monotonic paths are such that $c(0) \in [c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$. Since $\frac{v_{12}}{v_{22}} = 1 + \frac{(\mu_1+\mu_2)\Psi-\lambda}{(1-\mu_2)\Psi-\delta} < 1$ and $1 - a > 0$, it is straightforward to see that $\bar{F} < 0$. Then representative agent's welfare can be maximized by minimizing $\tilde{c}(0)$, that is for $C(0) = c_{0,\xi_2} - \varepsilon^*$. According to equation (3.16), since $\bar{F} < 0$ and $\tilde{c}(0) < 0$, representative agent's welfare is strictly positive.

Assume that $k(0) < k_s$. According to Proposition 1 monotonic paths are such that $c(0) \in [c_{0,\xi_1}, c_{0,\xi_2} - \varepsilon^*]$. It has been showed that $\bar{F} < 0$ then agent's welfare is negative and can be maximized by maximizing $\tilde{c}(0)$, that is for $C(0) = c_{0,\xi_1}$. ■

3.4.1.2 Economic arguments

A government which wants to maximize welfare and is able to pin down expectations on a given path through a stabilization policy, has an incentive to coordinate consumers' expectations on a $c(0)$ as far as possible to c_s given an initial capital $k(0)$ on the left hand side of its steady state value, and on a $c(0)$ as close as possible to c_s when $k(0)$ is on the right hand side of its steady state value.

For $k(0) < k_s$ and $\beta - 1 + \chi$ close to zero, log-deviations of consumption, capital and labor are negative and evolve monotonically (approximately) at the same rate $\bar{\lambda}_2$:

$$\begin{aligned}
\tilde{c}(t) &\sim \tilde{c}(0)e^{\bar{\lambda}_2 t} \\
\tilde{k}(t) &\sim a\frac{v_{12}}{v_{22}}\tilde{c}(0)e^{\bar{\lambda}_2 t} \\
\tilde{l}(t) &\sim \frac{1}{\beta - 1 + \chi}\left(1 - a\frac{v_{12}}{v_{22}}\right)\tilde{c}(0)e^{\bar{\lambda}_2 t}.
\end{aligned}$$

Log-deviation of welfare at the initial state is positive and decreases monotonically to zero as t tends to infinity. Instantaneous utility remains higher for the lowest levels of consumption (and labor) at any time.

In the presence of increasing returns to scale, agents have two alternatives for their consumption/investment plans. Either for large levels of increasing returns to scale they start with the highest level of consumption, investment and labor, accumulate rapidly capital then benefit from this accumulation for the rest of the time (since labor can decrease faster than consumption). Whereas for smaller levels of increasing returns, high levels of consumption and investment require extremely high levels of labor, which deteriorates welfare compared to more balanced levels of consumption. This last alternative is the one computed above: agents are better off when they choose an initial level of consumption equal to $c_{0,\xi_2} - \varepsilon^*$.

It would be interesting to enlarge the set of possible initial conditions in order to check whether or not the agents have interest to choose a non-monotonic equilibrium path, whose starting value of consumption would be higher or lower than c_{0,ξ_1} and $c_{0,\xi_2} - \varepsilon^*$, respectively. This is the objective of the next subsection.

3.4.2 Welfare-ranking of non-monotonic equilibria

Until now our analysis has focused on the set of monotonic paths. In this section, we relax the linear approximation of the utility function. Derivations are more complex and require to switch to the numerical analysis. In the same time, we can consider a larger set of initial conditions, including the trajectories that do not converge monotonically to the steady state, and compute formally the (second best) optimal initial level of consumption which may lay outside the range $[c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$. Then, we draw some qualitative predictions on the relation between the initial level of consumption and the level of increasing returns. Especially, it will be shown that the higher the increasing returns to scale the higher the welfare maximizing initial level of consumption. And then, according to Proposition 2, we can conclude that the higher the level of increasing returns the less smooth the maximizing welfare paths of consumption, labor and investment.

3.4.2.1 Simulation methods

Now we are interested in understanding the effect of a change in the initial level of consumption on the welfare for paths which are not monotonic. In Proposition 1, we have observed that according to the choice of $C(0)$ consumption converges more or less monotonically to its steady state value. This implies that different feasible equilibrium trajectories present different degrees of consumption smoothness.

Our problem is to identify what is the best among the welfare optimizing equilibrium trajectories. Moreover, we are interested in understanding if this trajectory has a high degree of consumption smoothness respect the others. In order to solve our problem we proceed as follows. First, we substitute approximated log-value of consumption into the utility function:

$$W = \frac{c^*}{\rho} - \frac{\eta_1 v_{21}}{\lambda_1 - \rho} - \frac{\eta_2 v_{22}}{\lambda_2 - \rho} - \int_0^\infty \frac{L(t)^{1-\chi}}{1-\chi} e^{-\rho t} dt.$$

Then, for different initial values of consumption, we compute the level of labor $L(t)$ at any period using equation (3.3), (3.4) and (3.11), and compute numerically the agent's welfare W.We have parameterized the economy as follows: capital's share, a, at 0.34, marginal product of capital, α, at 0.83, the discount rate at 0.02, the depreciation rate at 0.05, and the parameter χ at -0.25. Moreover we have studied the dynamics starting from an initial value of capital $k_0 = k^* - k^*/100$ and considered initial value of consumption as percentage variation of its steady state value. All the choices of the initial values are checked to be in the attraction set[3]. In Figure 3.2, we have sketched the results for the minimum level of productive externality γ_b satisfying the condition for indeterminacy, which implies $\beta = 1.251$. It must be noticed that for readability purpose a zero value has been imposed to any negative welfare values.

These results confirm Proposition 2 which predicts that within the set of monotonic paths, the maximizing welfare equilibrium starts with an initial level of consumption $c(0) = c_{0,\xi_2} - \varepsilon^*$. It is also interesting to notice that even if we enlarge the range of initial conditions, the maximum welfare is reached by agents when they choose a path with the highest degree of consumption smoothness.

However, for a choice of $\beta = 1.66$, that is when the economy faces a higher level of externalities, the maximum welfare is reached for an initial level of consumption outside the range $[c_{0,\xi_2} - \varepsilon^*, c_{0,\xi_1}]$, meaning that the optimal path is non-monotonic and the degree of consumption smoothness lower. It is clear, from Figure 3.4, that the maximizing welfare path's degree of consumption smoothness decreases as the level of increasing returns to scale raises.

3.4.2.2 Economic arguments

In optimal growth model à la Benhabib and Farmer with social increasing returns to scale and productive externalities, Christiano and Harrison [3] distinguish two effects affecting the consumption/investment plans. For a given technological coefficient (a given productive externality), the concavity of the utility function prevents from fluctuations which deteriorate

[3]Taking into account table1 in Russell and Zecevic (1998) it is, for example, possible to observe that c_0 may be chosen in the interval $\left(c^{low}_{-34\%}, c^{\max}_{103\%}\right)$ when $\beta = 1.26$.

welfare. This "concavity effect" leads to choose monotonic equilibria and smooth consumption over time so as to maximize agent's welfare. However, when the externality varies with the average levels of capital and labor, increasing returns to scale appear at the aggregate level. It may be welfare improving to bunch hard work in the first periods to boost capital accumulation in order to benefit from higher productive externalities in the future for lower levels of labor. When this "bunching effect" dominates the "concavity effect", agents bring forward a part of their labor supply, raising consumption at any period and decreasing labor after a while. On Figure 3.5, we pictured the optimal paths of capital, consumption and welfare for $\beta = 1.99$.It is worth noting that, when paths are monotonic, capital, consumption and labor lay below their steady state value forever. Here, this is no longer true: consumption and capital remain at any tile above their steady state values whereas labor remains below its steady state value after a while. It can be easily seen how agents accumulate the capital stock during the first periods, which erodes gradually afterwards. When increasing returns to scale are not sufficient, accumulating this comfortable maximum amount of capital requires to pay a stringent tribute in terms of dis-utility of labor that the increase in consumption cannot compensate. When the level of increasing returns is close to the minimum value to get indeterminacy of the steady state, there is no level of comfortable capital stock such that the "bunching effect" dominates the "concavity effect": far from accelerating capital accumulation, agents are better-off when they smooth consumption and labor over time. Finally, as increasing returns become more and more important, the "bunching effect" increases and offsets the "concavity effect": the welfare maximizing initial level of consumption as well as the maximum amount of capital stock raise.

In that extend, the linear method gives results that are particular cases of what has been found with the numerical analysis. As the productive externality increases, the maximizing welfare initial level of consumption moves away from $c_{0,\xi_2} - \varepsilon^*$, passes through c_{0,ξ_1} then keeps raising in the range of non-monotonic paths.[4]

It is also clear on the simulations above that the loss in welfare for an agent maintaining $c_{0,\xi_2} - \varepsilon^*$ as a starting level of consumption is increasing with the level of increasing returns. As this level goes up the “bunching effect" raises and more than offsets the “concavity effect". A higher level of increasing returns to scale makes capital accumulation larger for the same amount of worked hours or equivalently allows the representative agent to raise consumption without raising labor: welfare must go up (Figure 3.6, red line). Finally, the difference of utility between the optimal path and the path starting with a level of consumption of

[4]It must be noticed that the values of $c_{0,\xi_2} - \varepsilon^*$ and c_{0,ξ_1} are also increasing as the level of increasing returns gets larger.

$c_{0,\xi_2} - \varepsilon^*$ increase exponentially. Then it is clear that a benevolent planner would have to use its stabilization policy according to the magnitude of the externality: when non-convex technology set is assumed it may be welfare reducing to pin down expectations of the agents on a monotonic path respect to a non-monotonic one.

3.5 Conclusion

In this paper, we have proved that in a one-sector growth model with non-convex technology and productive externalities it is possible to rank the different equilibrium paths according to the initial value of consumption when the steady state is indeterminate. In the continuity of Christiano and Harrison's simulations, we have showed that welfare-improvement of stochastic sunspot equilibria is all the more powerful in the earlier periods of time since they condition the long run behavior of consumption and labor either by accelerating capital accumulation when the level of increasing returns is high (for a given elasticity of labor) or by decelerating the accumulation when it is low. Large fluctuations are then likely to be welfare-diminishing in the last case where the "concavity effect" dominates the "bunching effect". It can be inferred that progressive taxes able to pin down expectations as those developed by Guo and Lansing [5] are more likely to be welfare-diminishing compared to any stochastic equilibrium when increasing returns are large since they smooth consumption and labor and decelerate capital accumulation, as shown previously by using a stabilization policy *à la* Saïdi. Our analysis raises a question that deserve further investigations. Can we say something about the nature of the social planer's allocation? All the equilibria we considered are inefficient since the agents do not internalize the externality of production. In this case, the maximizing welfare deterministic equilibrium is more or less monotonic according to the aggregate level of increasing returns. Christiano and Harrison present an example of monotonic social planer's allocation while for different values of the externalities Dupor and Lenhert [2002] and Saïdi [8] show that this allocation is discontinuous and cycling. It can be conjectured that there is a close relationship between the monotonicity of the first best allocation and of the decentralized optimal solution.

3.6 Appendix A: Stabilization policy

Assume that the stationary equilibrium is indeterminate and that the government aims at coordinating the expectations on a deterministic indeterminate path characterized by the initial level of consumption and labor $(\bar{C}_0, \bar{L}_0)$. The expected rate of returns on capital is $\bar{r}_0 \equiv \alpha K_0^{a-1} \bar{L}_0^b$. The economic policy consists in subsidizing or taxing production such that the rate of returns on capital equals $\bar{r}_0$ by fixing a tax rate τ_0 (possibly negative) at the first period. Firms maximize their profit Π_0:

$$\Pi_0 = (1 - \tau_0) Y_0 - r_0 K_0 - w_0 L_0,$$

with:

$$\tau_0 = 1 - \bar{r}_0 / r_0.$$

Since K_0 and $\bar{r}_0$ are predetermined, the equality of the after-tax rental rate of capital to the after-tax productivity of capital determines the quantity of labor at time 0:

$$L_0 = (\bar{r}_0 / \alpha K_0^a)^{1/b} = \bar{L}_0.$$

Simultaneously, the couple (K_0, L_0) determines the equilibrium value of the first period after-tax real wage satisfying the second first order condition of profit maximization:

$$w_0 = (1 - \tau_0) \frac{(1 - \alpha) Y_0}{L_0}. \tag{3.18}$$

Finally, the first order condition (respect to labor) determines consumption at time 0, that is $\bar{C}_0$, which in turn determines the variation of the capital stock $\dot{K}_0$ via the law of motion of capital. It is straightforward to show that, by iteration, fixing the after-tax rental rate of capital at each period allows to determine the triple (K_t, L_t, C_t) at any time t.

3.7 Appendix B: Slopes of the stable arms

The Jacobian matrix of the system formed by equation (3.3) and (3.4) is:

$$J = \begin{pmatrix} (1 + \mu_1)\Psi - \delta & (\mu_2 - 1)\Psi + \delta \\ a\mu_1 \Psi & a\mu_2 \Psi \end{pmatrix}$$

where $\Psi \equiv (\rho + \delta)/a$, $\mu_0 = \frac{-\beta \ln b}{\beta + \chi - 1}$, $\mu_1 = \frac{(\chi - 1)(\alpha - 1) - \beta}{\beta + \chi - 1}$ and $\mu_2 = \frac{\beta}{\beta + \chi - 1}$. Let $\xi_i = (v_{1i}, v_{2i})^T$, $i = \{1, 2\}$, the eigenvectors of the system defined such that:

$$\begin{pmatrix} (1 + \mu_1)\Psi - \delta - \lambda_i & (\mu_2 - 1)\Psi + \delta \\ a\mu_1 \Psi & a\mu_2 \Psi - \lambda_i \end{pmatrix} \begin{pmatrix} v_{1i} \\ v_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.19}$$

The slope of the stable arm associated to ξ_i at the stationary equilibrium is v_{2i}/v_{1i}. We want to show that:

$$\frac{v_{21}}{v_{11}} > \frac{v_{22}}{v_{12}} > 0,$$

or equivalently that the slope of the stable arm associated to ξ_2 is steeper that the slope of the stable arm associated to ξ_1 at the stationary equilibrium.

According to system (3.19), notice first that:

$$\frac{v_{2i}}{v_{1i}} = -\frac{(1+\mu_1)\Psi - \delta - \lambda_i}{(\mu_2 - 1)\Psi + \delta}. \tag{3.20}$$

Moreover, when Benhabib and Farmer's condition for indeterminacy is satisfied, that is when $\beta - 1 + \chi > 0$, $\mu_2 - 1 > 0$ and $1 + \mu_1 < 0$. Since the trace is equal to the sum of the two eigenvalues, the following relation holds for any $i, j = \{1, 2\}$ with $i \neq j$:

$$\begin{aligned} sign\left\{\frac{dv_2^i}{dv_1^i}\right\} &= sign\left\{-(1+\mu_1)\Psi + \delta + \lambda_i\right\} \\ &= sign\left\{a\mu_2\Psi - Trace(J) + \lambda_i\right\} \\ &= sign\left\{a\mu_2\Psi - \lambda_j\right\}. \end{aligned}$$

Under Benhabib and Farmer's condition for indeterminacy, both $a\mu_2\Psi$ and $-\lambda_j$ are positive.

Finally since $\lambda_1 < \lambda_2$ it follows immediately from equation (3.20) that the slope of the stable arm associated to ξ_2, v_{22}/v_{12}, is steeper than the slope of the stable arm associated to ξ_1, v_{21}/v_{11}. If we assume to start with an initial stock of capital lower (resp. greater) than its steady state value, $\tilde{k}(0) < 0$ (resp. $\tilde{k}(0) > 0$) and from equations (3.7) and (3.8) it is easily deduced that $c_{0,\xi_1} > c_{0,\xi_2}$ (resp. $c_{0,\xi_1} < c_{0,\xi_2}$).

3.8 Appendix C: Solution of some limits

The trace and determinant of the Jacobian matrix J are the following:

$$\begin{aligned} Tr(J) &= (\rho+\delta)\frac{\rho+\delta(1-a)}{a}\frac{(1-\alpha)(1-\chi)}{\beta-1+\chi} \\ Det(J) &= -(\rho+\delta)(1+\gamma)\frac{a-\chi}{\beta-1+\chi} - \delta \end{aligned}$$

When the condition for indeterminacy holds, one can see immediately that $Tr(J)$ tends to $-\infty$ and $Det(J)$ tends to $+\infty$ as $\beta - 1 + \chi$ tends to zero. Moreover the two limits have the same "order" of convergence. Now consider the following limits:

$$\lim_{\beta\to 1-\chi} \lambda_1 = \lim_{\beta\to 1-\chi} \frac{Tr(J) - |Tr(J)|\sqrt{1 - 4\frac{Det(J)}{Tr(J)^2}}}{2} = \lim_{\beta\to 1-\chi} \frac{Tr(J) - |Tr(J)|}{2} = -\infty \tag{3.21}$$

Multiplying and dividing by $Tr(J) - |Tr(J)|\sqrt{1 - 4\frac{Det(J)}{Tr(J)^2}}$, we get:

$$\begin{aligned}\lim_{\beta \to 1-\chi} \lambda_2 &= \lim_{\beta \to 1-\chi} \frac{Tr(J) + |Tr(J)|\sqrt{1 - 4\frac{Det(J)}{Tr(J)^2}}}{2} \\ &= \lim_{\beta \to 1-\chi} \frac{2 \det J}{trJ - |trJ|\sqrt{1 - 4\frac{\det J}{(trJ)^2}}} \\ &= \lim_{\beta \to 1-\chi} \frac{Det(J)}{Tr(J)} = -\frac{(1-\alpha)[\rho + \delta(1-\alpha)]}{\alpha - a}.\end{aligned}$$

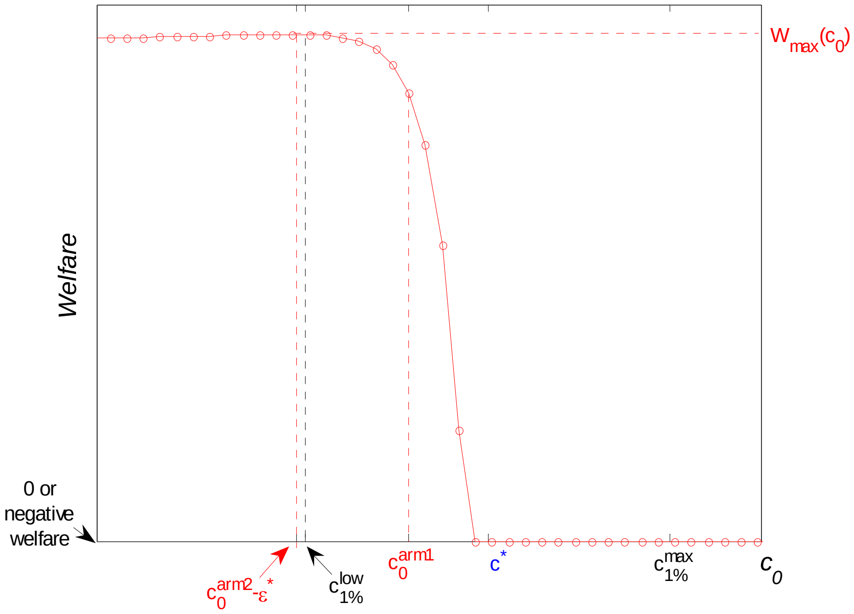

Figure 3.2: Welfare analysis when minimum degree of externalities for local indeterminacy

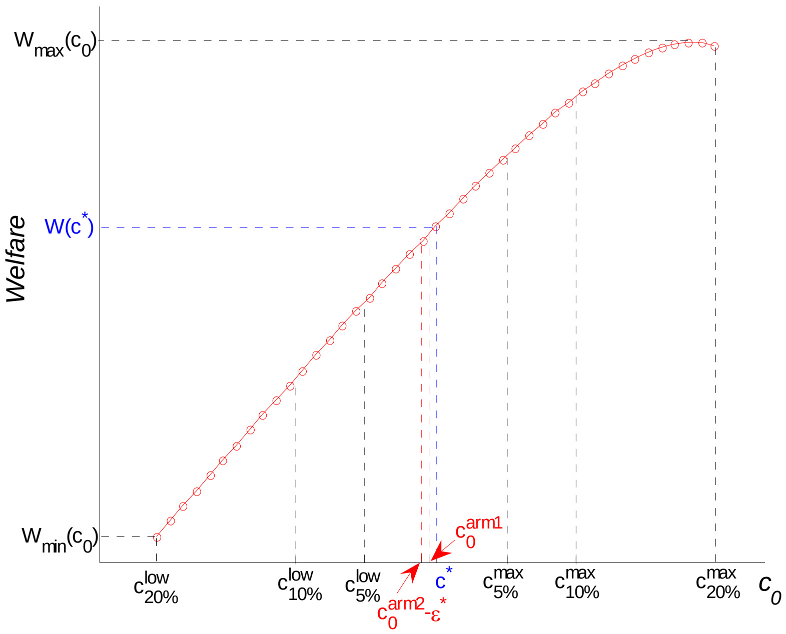

Figure 3.3: Welfare analysis when $\beta = 1.66$

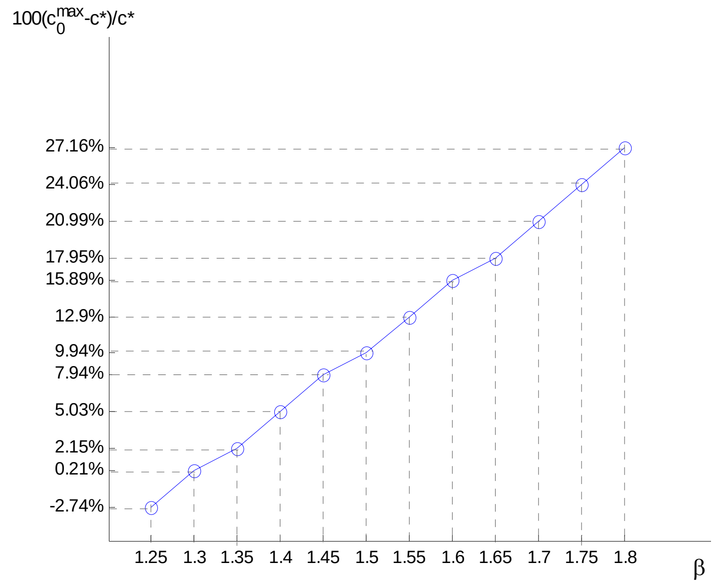

Figure 3.4: Initial level of consumption maximizing welfare according to β.

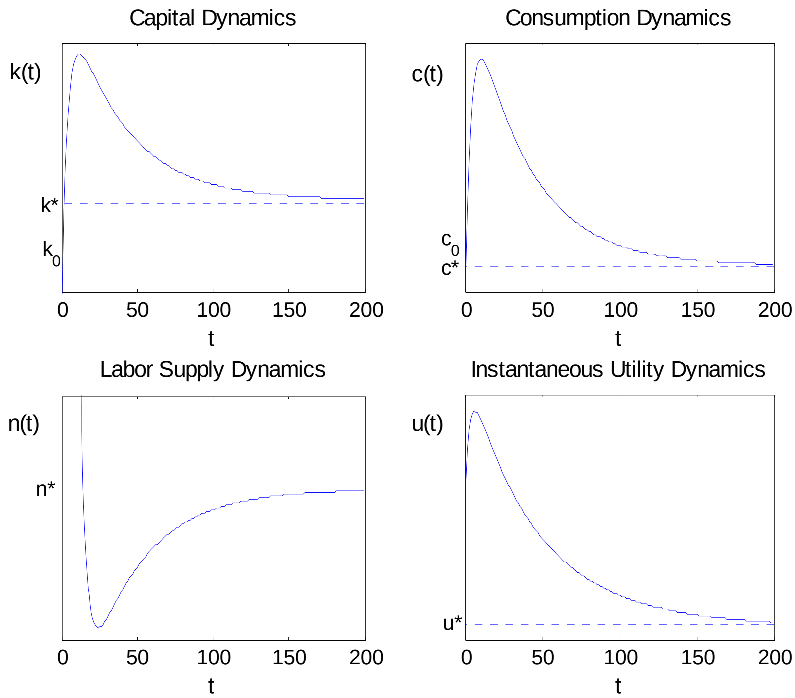

Figure 3.5: Consumption, capital and labor path maximizing welfare.

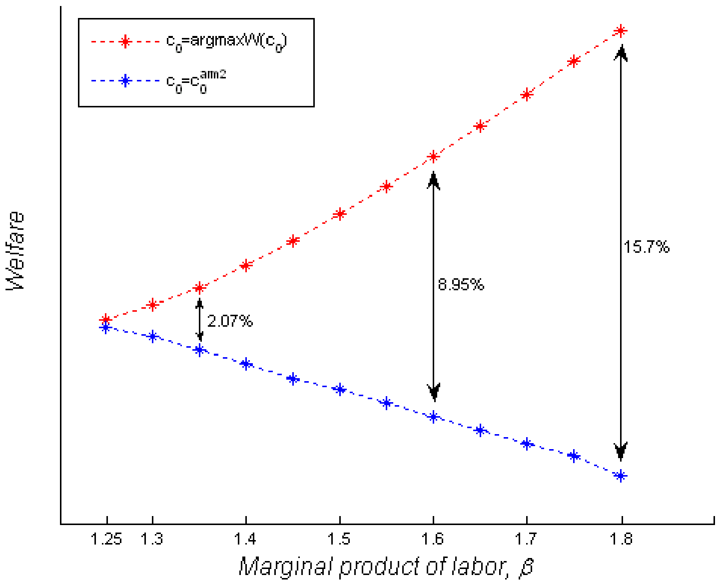

Figure 3.6: Welfare gap.

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