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Abstract

In a variety of contexts – ranging from public goods provision to information collection – a player's well-being depends on own action as well as on the actions taken by his or her neighbors. We provide a framework to analyze such strategic interactions when neighborhood structure, modeled in terms of an underlying network of connections, affects payoffs. We provide results characterizing how the network structure, an individual's position within the network, the nature of games (strategic substitutes versus complements and positive versus negative externalities), and the level of information, shape individual behavior and payoffs.

JEL classification: D85, C72, L14, Z13.

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1 Introduction

In a diverse range of social and economic interactions – including public goods provision, job search, political alliances, trade, friendships, and information collection – an agent's well being depends on her own actions as well as on the actions taken by other agents in close proximity; i.e., her neighbors. For example, the decision of an agent of whether or not to buy a new product, or to attend a meeting, is often influenced by the choices of his or her friends and acquaintances (be they social or professional). The empirical literature identifying the effects of agents' neighborhood patterns (i.e., their social network) on behavior and outcomes has grown over the past several decades. The literature is much too vast to survey here; influential works include Coleman (1966), Foster and Rosenzweig (1995), Granovetter (1994), Glaeser, Sacerdote, Scheinkman (1996) and Topa (2001). The emerging empirical evidence motivates the theoretical study of network effects. We would like to understand how the pattern of social connections shape the choices that individuals make and the payoffs they can hope to earn. We would also like to understand how changes in the network matter as this would tell us how individuals would like to shape the networks in which they are located.

Attempts at the study of these basic questions have been thwarted by a fundamental theoretical problem: even the simplest games played on networks have multiple equilibrium which display a bewildering range of possible outcomes. We know from the literature on global games that the introduction of incomplete information can sometimes resolve the problem of multiplicity as well as provide interesting and novel economic intuitions.[1] Recently, this approach has faced the critique that the equilibrium selection achieved depends very much on the type of incomplete information that is assumed, see Weinstein and Yildiz (2007). In the context of network games, we have a natural and measurable dimension – the number of connections of an individual, viz. the degree – and while a player knows his own degree he typically has incomplete information on the degrees of others.[2] Starting from these considerations we formulate a model in which players have private and incomplete

[1] Starting with the seminal work of Carlsson and van Damme (1993), there is now an extensive literature on global games. For a survey of this work see Morris and Shin (2003).

[2] The measurement of the degree distribution of networks has been the subject of extensive research in recent years; for a summary of the main findings, see Jackson and Rogers (2007) and Newman (2002). There is also a literature which explores the knowledge of individuals about the network, see e.g., Kumbasar, Romney, and Batchelder (1994).

information about the network. This leads us to defining a class of incomplete information games, where we study the symmetric Bayes-Nash equilibria.

There are two other aspects of our framework which we would like to mention here. One, individuals are allowed to have beliefs about degrees of their neighbors which depend on their own degree.[3] We capture correlations in the degrees of neighbors through the concept of positive and negative association, which is a generalization of the notion of affiliation commonly used in economic models in which a group of agents has heterogenous and randomly determined valuations. Different associations exhibited by well known models of random networks are briefly discussed in Appendix A. Practically all the degree distributions studied in the empirical and theoretical literature fall into one of the scenarios on degree association considered.

Two, the actions of others can affect a player's payoffs in different ways and we would like to develop a general understanding of how the payoff relations interact with the network structure. We focus on the two canonical types of interplay that social interaction might take: strategic complements and strategic substitutes.[4] These two types of strategic interaction allow for a wide class of games, and include as special cases most of the applications the literature has tackled.

Our analysis yields a number of insights which address the basic questions posed earlier in the introduction. We now present a simple example to convey these insights.

An example: Consider an n-player game, $n \geq 2$, with two actions $X = \{0, 1\}$ where we may interpret 1 as acquiring information (or providing any local and discrete public good) and 0 as not acquiring it. A player is identified with a node of a network, and the links between the nodes reflect social interactions. We shall denote a network by g; a link between players i and j is denoted by $g_{ij} \in \{0, 1\}$, taking value 1 if the link exists and 0 otherwise. The set of players with whom player i has a link in network g is denoted by $N_i(g)$ and let us define $k_i(g) = |N_i(g)|$. A player's

[3] Empirical work suggests that social networks exhibit rich patterns of correlations across the degrees of neighbors; for a survey of this work see e.g., Jackson and Rogers (2007) and Newman (2002).

[4] For instance, strategic complements arise whenever the benefit that an individual obtains from buying a product or undertaking a given behavior is greater as more of his partners do the same. This might be due to direct effects of having similar or compatible products (such as in the case of computer operating systems), peer pressures (as in the case of drug use), and so forth. The strategic substitutes case encompasses many scenarios that allow for free riding or have a public-good structure of play, such as costly experimentation or information collection. Formal definitions of these games are given in Section 2.

payoff depends on the sum of own action x_i and the actions of his neighbors. Specifically, denoting $x_{N_i(g)} = \sum_{j \in N_i(g)} x_j$ and $y_i = x_i + x_{N_i(g)}$, let us suppose that gross payoffs are equal to 1 if $y_i \geq 1$, and 0 in the alternative case. This is therefore a game of strategic substitutes. The cost to choosing 0 is 0 while the cost to choosing 1 is c; we suppose that $0 < c < 1$.[5] This game may be viewed as a game of local public-good provision. The ensuing discussion of this setup for the case of complete information draws on the analysis of Bramoullé and Kranton (2007).

First we observe that, since $c < 1$, in any Nash equilibrium, $y_i \geq 1$, for every player $i \in N$. We turn next to the relation between network connections and actions. There is a very rich set of equilibria in this game. To see this consider a star network and note that there exist two equilibria, one in which the center chooses 1 and the spokes choose 0, and a second equilibrium in which the spoke players choose 1 while the center chooses 0. Figure 1 illustrates these possibilities. In the first equilibrium, the center earns less than the spoke players while in the second equilibrium it is the other way round. Thus even in the simplest networks there exist multiple equilibria and the relation between network connections, equilibrium actions and payoffs exhibits no systematic pattern.

We now turn to the effects of adding links in a network. The effects depend very much on the details of the network and where the link is added. To see this, start with two stars with 3 spoke players each. Fix an equilibrium in which the two centers exert action 1 while the spoke players all choose 0. The aggregate payoff in this equilibrium is $8 - 2c$. *First* add a link between a center of one star and a spoke of the other star. In the new network the old action profile still constitutes an equilibrium. *Second*, add a link between the centers of the two stars. Now the old profile of actions is no longer an equilibrium. However, there is an equilibrium in which the spokes of the 'stars' choose 1 and the centers choose 0. In this equilibrium there is a clear change in profile of actions, and the aggregate payoffs are given by $8 - 6c$. It follows that aggregate payoffs decline upon the addition of links. Figure 2 illustrates these outcomes. Thus even in the simplest network, the addition of a link can have very different effects depending on where the link is added.

Now we relax the assumption that players know the complete network and suppose instead that they know only their own degree; moreover, suppose that their information about the rest of the

[5] This is known as the best-shot game; for a more detailed presentation of this game, see section 2.

network is summarized in a belief about the degree distribution of their neighbors. Also assume, for expositional simplicity, that the beliefs about the neighbors' degrees are independent of own degree. A player's strategy is a mapping from degree to actions. We are now in a framework of Bayesian games of incomplete information. Our results concern the existence and properties of symmetric Bayes-Nash equilibrium.

Propositions 1 and 2 imply that there exists a symmetric equilibrium and that such an equilibrium is monotone decreasing; moreover, we show that the equilibrium has a threshold property.[6] These results, taken together with Proposition 3, yield a simple relation between network connections and actions: *under incomplete network information, better connected players choose lower actions and earn higher payoffs.*

We then turn to the effects of adding links to the network, and Proposition 4 tells us that an increase in the degree of neighbors lowers their likelihood of choosing action 1 and this raises the threshold degree for action 1. This yields a simple and clear result on the effects of adding links in a network: *under incomplete network information, the addition of links raises the threshold but at the same time lowers the probability of action* 1 *from each neighbor.* ■

This example illustrates how the introduction of incomplete network knowledge yields a number of insights and facilitates the systematic study of the basic questions we posed at the start of the introduction.

Our formal results generalize these insights in a number of dimensions: we allow for games with continuous actions as well as discrete actions and cover a general class of games with strategic complements as well as strategic substitutes. We allow for general patterns of correlations – both positive as well as negative correlation – between the degrees of neighbors and we also allow for players having more extensive information about the network, as for instance when they know their own degrees as well as the degrees of their neighbors and so on.

Related literature: Our paper is a contribution to the study of games played on networks. In recent years, a number of papers have studied specific games played on a network. For instance,

[6] In a threshold equilibrium, there exists some $t \in \{1, 2, ...n-1\}$ such that players with degree $k_i < t$ choose 1, while players with degree $k_i > t$, choose 0, and the degree t players possibly mix between 1 and 0.

decisions to undertake criminal activity (Ballester, Calvó-Armengol, and Zenou (2006)), public-good provision (Bramoullé and Kranton (2007)), decisions on buying a product (Galeotti (2005)), and research collaboration among firms (Goyal and Moraga-Gonzalez (2001)) have been studied using specific formulations (of both network and payoff structure) and under complete information.[7] We would also like to mention Galeotti and Vega-Redondo (2005) and Sundararajan (2006) who study games with incomplete network knowledge in specific contexts. The principal contribution of our paper is the development of a general framework for the study of games played on networks when players have incomplete information about the network. This theoretical innovation in turn permits us to consider fairly general class of games with strategic complements and strategic substitutes. In particular, our framework includes as special cases practically all the papers mentioned above. Our approach also allows naturally for general patterns of association across the degrees of neighbors, and this is important as empirical work suggests that real world networks display such features. To the best of our knowledge, our paper is the first attempt to incorporate general patterns of degree association in the study of network games.[8]

There is also a literature in computer science which examines games played on a network; see e.g., the model of "graphical games" as introduced by Kearns, Littman, and Singh (2001), also analyzed by Kakade, Kearns, Langford, and Ortiz (2003), among others.[9] The graphical-games literature has focused on finding efficient algorithms to compute Nash equilibria in two-action games played on networks. Here, we allow for far more general games and information structures. Importantly, our focus is on the structure of equilibria and its interaction with the underlying network, rather than with the computation of particular equilibria.

The rest of the paper is organized as follows. Section 2 develops the general framework for the study of games played on networks. Section 3 presents results on the existence and monotonicity of

[7] In particular, regular networks (in which all players have the same degree) and core-periphery structures (the star network is a special case of such structures) have been extensively used in the literature.

[8] Recent work by Jackson and Yariv (2007) follows up on the approach introduced in this paper. It examines the multiplicity of equilibria of games on networks with incomplete information, but with a binary action model and a different formulation of payoffs, and obtains complementary results.

[9] There are also models of equilibria in social interactions, where players care about the play of certain other groups of players. See Glaeser and Scheinkman (2003) for an overview.

equilibria. Section 4 takes up the study of the effects of network changes on equilibrium behavior and payoffs. While the analysis in Sections 3 and 4 focuses on a setting in which players know their own degree and have some beliefs about the rest of the network, Section 5 takes up the issues that arise when players have deeper knowledge about the network. Section 6 concludes. As indicated, Appendix A presents a summary of well known models of random network formation and shows how the resulting degree distributions satisfy the properties we assume. All the proofs are presented in Appendix B.

2 The Model

This section presents our theoretical framework: the networks, the payoffs, the information available to agents, and the definition of equilibria.

Networks: The connections between a finite set of players $N = \{1, \ldots n\}$ are described by a network; this network is represented by a matrix $g \in \{0, 1\}^{n \times n}$, with $g_{ij} = 1$ implying that i's payoff is affected by j's behavior. We follow the convention of setting $g_{ii} = 0$ for all i.

Let $N_i(g) = \{j | g_{ij} = 1\}$ represent the set of direct neighbors of i. For any integer $d \geq 1$, $N_i^d(g)$ denotes the *d-neighborhood* of i in g; that is, all the players that can be reached from i by directed paths of length no more than d. Inductively $N_i^1(g) = N_i(g)$ and $N^d(g) = N_i^{d-1}(g) \cup (\cup_{j \in N_i^{d-1}(g)} N_j(g))$. The *degree*, $k_i(g)$, of player i is the number of i's direct connections:

$$k_i(g) = |N_i(g)|.$$

Strategies and Payoff Functions: Each player i takes an action x_i in X, where X is a compact subset of $[0, 1]$. Without loss of generality, we assume throughout that $0, 1 \in X$. We consider both discrete and connected action sets X. The payoff of player i when the profile of actions is $x = (x_1, ..., x_n)$ is given by:

$$v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i. Thus the payoff of a player depends on her own action and the actions that her direct neighbors take.

Note that the payoff function depends on the player's degree but not on her identity. Therefore, any two players who have the same degree have the same payoff function. We will also assume that v_k depends on the vector $x_{N_i(g)}$ in an anonymous way, so that if x' is a permutation of x (both k-dimensional vectors) then $v_k(x_i, x) = v_k(x_i, x')$ for any x_i. If X is a connected action set then v_k is taken to be continuous in all arguments and concave in own action.

We now turn to the relation between players strategies and their payoffs. First, we define effects of others strategies on the marginal returns of a player. We shall say that a payoff function exhibits *strategic complements* if it satisfies increasing differences: for all k, $x_i > x_i'$, and $x \geq x'$: $v_k(x_i, x) - v_k(x_i', x) \geq v_k(x_i, x') - v_k(x_i', x')$. Analogously, we shall say that a payoff function exhibits *strategic substitutes* if it satisfies decreasing differences: for all k, $x_i > x_i'$, and $x \geq x'$: $v_k(x_i, x) - v_k(x_i', x) \leq v_k(x_i, x') - v_k(x_i', x')$. These notions are said to apply strictly if the payoff inequalities are strict whenever $x \neq x'$.

We also keep track of the effects of others' strategies on a player's payoffs. We shall say that a payoff function exhibits *positive externalities* if for each v_k, and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$. Analogously, we shall say that a payoff function exhibits *negative externalities* if for each v_k, and for all $x \geq x'$, $v_k(x_i, x) \leq v_k(x_i, x')$. Correspondingly, the payoff function exhibits *strict* externalities (positive or negative) if the above payoff inequalities are strict whenever $x \neq x'$.

We now present some economic examples to illustrate the scope of our framework.

Example 1 *Payoffs Depend on the Sum of Actions*
Player i's payoff function when he chooses x_i and his k neighbors choose the profile $(x_1, ..., x_k)$ is:

$$v_k(x_i, x_1, ..., x_k) = f\left(x_i + \lambda \sum_{j=1}^{k} x_j\right) - c(x_i), \tag{1}$$

where $f(\cdot)$ is non-decreasing and $c(\cdot)$ is a "cost" function associated with own effort. The parameter $\lambda \in \mathbb{R}$ determines the nature of the externality across players' actions. This example exhibits (strict) strategic substitutes (complements) if, assuming differentiability, $\lambda f''$ is negative (positive).

The case where f is concave, $\lambda = 1$, and $c(\cdot)$ is increasing and linear corresponds to the case of information sharing as a local public good studied by Bramoullé and Kranton (2007), where actions are strategic substitutes. In contrast, if $\lambda = 1$, but f is convex (with $c'' > f'' > 0$), we obtain a model with strategic complements, as proposed by Goyal and Moraga-Gonzalez (2001) to study collaboration among firms. In fact, the formulation in (1) is general enough to accommodate a good number of further examples in the literature such as human capital investment (Calvo-Armengol and Jackson (2005)), crime networks (Ballester, Calvó-Armengol, and Zenou (2005)), some coordination problems (Ellison (1993)), and the onset of social unrest (Chwe (2000)). ■

An interesting special case of the above example is the Best-Shot game.

Example 2 *"Best-Shot" Public Goods Games*
The Best-Shot game is a good metaphor for many situations in which there are significant spillovers between players' actions. Here $X = \{0, 1\}$ and we may interpret 1 as acquiring information (or providing any local and discrete public good) and 0 as not acquiring it. We suppose that $f(0) = 0$, $f(x) = 1$ for all $x \geq 1$, so that acquiring one piece of information suffices. Costs, on the other hand, are assumed to satisfy $0 = c(0) < c(1) < 1$ so that no individual finds it optimal to dispense with the information but prefers one of his neighbors to gather it. This is a game of strategic substitutes and positive externalities.[10] ■

In the above examples, a player's payoffs depend on the sum of neighbors strategies and all of them satisfy the following general property.

Property A $v_{k+1}(x_i, (x, 0)) = v_k(x_i, x)$ for any $(x_i, x) \in X^{k+1}$.

Under Property A, adding a link to a neighbor who chooses action 0 is payoff equivalent to not having an additional neighbor. The above discussion clarifies that many economic examples studied so far satisfy Property A. There is however a prominent case where the payoffs violate Property A: this arises when payoffs depend on the average of the neighbors' strategies. Our framework allows a consideration of such games as well.

Example 3 *Payoffs Depend on the Average of Neighbors' Actions*

[10] For instance, consumers learn from relatives and friends (Feick and Price, 1987), in research and development, innovations often get transmitted between firms, and similarly in agriculture, experimentation is often shared amongst farmers (Foster and Rosenzweig, 1995, Conley and Udry, 2005). For a discussion of best shot games, see Hirshleifer (1983).

Let $X = \{0, 1\}$. Player i's payoff function when he chooses x_i and his k neighbors choose the profile $(x_1, ..., x_k)$ is:

$$v_k(x_i, x_1, ..., x_k) = x_i f\left(\frac{\sum_{j=1}^{k} x_j}{k}\right) - c(x_i), \tag{2}$$

where $f(.)$ is an increasing function. This is a game of strategic complements and positive externalities. ■

Information Structure, association and domination: The scenario considered throughout most of the paper is one where players know their own degree as well as the network-formation mechanism at work. In particular, the resulting network may display inter-neighbor degree correlations. Let the degrees of the neighbors of a player i of degree k_i be denoted by $\mathbf{k}_{N(i)}$, which is a vector of dimension k_i. The information a player i of degree k_i has regarding the degrees of his or her neighbors is captured by a distribution $P(\mathbf{k}_{N(i)} \mid k_i)$. Throughout, we model players' beliefs symmetrically across all players. This means that the information structure is given by a family of anonymous conditional distributions $\mathbf{P} \equiv \left\{\left[P(\tilde{\mathbf{k}} \mid k)\right]_{\tilde{\mathbf{k}} \in \mathbb{N}^k}\right\}_{k \in \mathbb{N}}$. In some of our results, we shall also need to refer to the underlying unconditional degree distribution, which will be denoted simply by $P(\cdot)$.

We model strategic interaction among players located on a network as a Bayesian game à la Harsanyi with a type space consisting of the potential degrees of a player. A strategy for player i is a mapping $\sigma_i : \{0, 1, ..., n-1\} \to \Delta(X)$, where $\Delta(X)$ is the set of distribution functions on X.

We emphasize that our framework allows for correlation between neighbors' degrees and this means that the conditional distributions concerning neighbors' degrees can in principle vary with a player's degree. In our setting, matters are complicated by the fact that players of different degrees have different numbers of neighbors, and the degrees of the neighbors will generally be correlated with each other.[11] To deal with this issue, we adapt a standard definition of "association" from the statistics literature to allow for comparisons across vectors of different dimensions (see Esary, Proschan, and Walkup (1967)).

Given a player with degree k_i, enumerate the degrees of i's neighbors as $\mathbf{k}_{N(i)} = (k_1, k_2, \ldots k_{k_i})$.

[11] For instance, if the degrees of i and j are perfectly correlated, and the degrees of j and l are correlated, then the degrees of i and l will also be correlated

Next, consider a function $f : \{0, 1, \ldots, n-1\}^m \to \mathbb{R}$ where $m \leq k_i$. Let

$$E_{P(\cdot|k_i)}[f] = \sum_{\mathbf{k}_{N(i)}} P(\mathbf{k}_{N(i)} \mid k_i) f(k_1, \ldots k_m).$$

This simply fixes some subset of $m \leq k_i$ of i's neighbors, and then takes the expectation of f operating on their degrees.

We shall say that $\mathbf{P}$ exhibits *positive association* if, for all $k' > k$, and any non-decreasing $f : \{0, 1, \ldots, n-1\}^k \to \mathbb{R}$.

$$E_{P(\cdot|k')}[f] \geq E_{P(\cdot|k)}[f].$$

Analogously, $\mathbf{P}$ exhibits *negative association* if the reverse inequality holds for each $k' > k$ and non-decreasing f.

Association is used to keep track of correlation patterns of groups of random variables, given the complicated interdependencies that might be present. Positive association embodies the idea that higher levels of one variable (in this case a player's degree) correspond to higher levels of all other variables (in this case the player's neighbors' degrees).[12]

Independence across neighbors' degrees is a special case of association and is satisfied in the classical Erdös-Rényi model of random networks; our results therefore cover this important class of networks. However, the recent empirical literature on networks has shown that technological and information networks such as the internet and the world wide web exhibit negative degree correlation while social networks such as the co-author network and company board director network exhibit positive degree correlations; see e.g., Newman (2002). This work has motivated the development of dynamic models of network formation which yield patterns of degree association. Appendix A discusses four prominent models: the Erdos-Renyi model, the configuration model, the Barabasi-Albert model of preferential attachment, and the Jackson-Rogers-Vazquez model of local linking. The configuration model exhibits negative association, while the preferential attachment model and the local linking model lead to positive association.

[12] We note that association is a weaker notion than affiliation that requires association for all conditional variables as well (and is not satisfied in many network models).

We are also interested in comparisons of behavior across different networks, in particular in the effects of adding and redistributing links. We use the notion of dominance to study the idea of adding links. We shall say that $\mathbf{P}'$ *dominates* $\mathbf{P}$ if for all k, and any non-decreasing $f : \{0, 1, \ldots, n-1\}^k \to \mathbb{R}$

$$E_{P'(\cdot|k)}[f] \geq E_{P(\cdot|k)}[f].$$

The concept of dominance is a generalization of stochastic dominance relationships adapted to vectors and families of distributions.

Equilibrium: We focus on symmetric equilibria, i.e., equilibria in which all players follow the same strategy. We will also be interested in tracking how strategies vary with degree. Noting that σ may involve mixed strategies we say that σ is *non-decreasing* if $\sigma(k')$ first order stochastic dominates $\sigma(k)$ for each $k' > k$. Similarly, we say that σ is *non-increasing* if the domination relationship is reversed.

Given a player i of degree k_i let $\mathrm{d}\psi_{-i}(\sigma, k_i)$ denote the probability density over $x_{N_i(g)} \in X^{k_i}$ induced by the beliefs $P(\cdot \mid k_i)$ held by i over the degrees of her neighbors composed with the strategies played via σ. Let

$$U(x_i, \sigma, k_i) = \int_{x_{N_i(g)} \in X^{k_i}} v_{k_i}(x_i, x_{N_i(g)}) \mathrm{d}\psi_{-i}(\sigma, k_i)$$

the expected payoff to a player i with degree k_i when other players use strategy σ and i chooses action x_i. A Bayesian equilibrium is a (Bayesian) Nash equilibrium of this game, defined in the standard fashion.

In obtaining our results on equilibrium monotonicity the key property of expected payoffs that we will exploit is a form of degree complements/substitutes. We say that expected payoffs exhibit *degree complementarity* if

$$U(x_i, \sigma, k_i) - U(x_i', \sigma, k_i) \geq U(x_i, \sigma, k_i') - U(x_i', \sigma, k_i')$$

whenever $x_i > x_i'$, $k_i > k_i'$, and σ is non-decreasing. Analogously, payoffs exhibit *degree substitution* if the inequality above is reversed in each case and σ is taken to be non-increasing.

Are there conditions on the primitives – payoff functions and beliefs – under which degree complements obtains? Recall that Property A says that $v_{k+1}(x_i,(x,0)) = v_k(x_i,x)$ for any $(x_i,x) \in X^{k+1}$. We next observe that under Property A, strategic complements of $v_k(.,.)$ and positive association of $\mathbf{P}$ ensure degree complementarity. To see why this is true consider a strategy σ which is non-decreasing and suppose that $k' = k+1$. Now observe that

$$\begin{aligned}
& U(x_i,\sigma,k) - U(x_i',\sigma,k) \\
= & \int_{x\in X^k}[v_k(x_i,x) - v_k(x_i',x)]\mathrm{d}\psi_{-i}(\sigma,k) \\
= & \int_{x\in X^k}[v_{k'}(x_i,(x,0)) - v_{k'}(x_i',(x,0))]\mathrm{d}\psi_{-i}(\sigma,k) \\
\leq & \int_{x\in X^k}[v_{k'}(x_i,(x,0)) - v_{k'}(x_i',(x,0))]\mathrm{d}\psi_{-i}(\sigma,k') \\
\leq & \int_{x\in X^{k'}}[v_k(x_i,(x,x_{k+1})) - v_k(x_i',(x,x_{k+1}))]\mathrm{d}\psi_{-i}(\sigma,k') \\
= & U(x_i,\sigma,k') - U(x_i',\sigma,k').
\end{aligned}$$

where the second equality follows from Property A, the first inequality follows from positive association, σ being non-decreasing and strategic complements, while the second inequality follows from strategic complements. Analogous considerations establish that under Property A, strategic substitutes of $v_k(.,.)$ and negative association of $\mathbf{P}$ ensure degree substitutability.

While Property A (taken along with the corresponding properties on $\mathbf{P}$) is sufficient to establish degree complementarity and substitutability, it is not necessary. The following discussion, which builds on example 3, illustrates this point.

Example 4 *Degree complements and substitutes without Property A.*

Suppose that payoffs are as in example 3. In addition let $\mathbf{P}$ be such that players' degrees are independent (for example, as in an Erdös-Rényi random network). Let Y_m be a random variable that has a binomial distribution with m draws each with probability $\sum_k \frac{kP(k)}{\langle k\rangle}\sigma(k)$, where $\langle k\rangle = E_P[k]$ (when degrees are independent, $\frac{kP(k)}{\langle k\rangle}$ captures the probability that a random neighbor is of degree k). Then the expected payoffs to a player i are given by:

$$U(x_i,\sigma,k_i) = E\left[x_i f\left(\frac{Y_{k_i}}{k_i}\right)\right] - c(x_i),$$

and thus

$$U(1,\sigma,k_i) - U(0,\sigma,k_i) = E\left[f\left(\frac{Y_{k_i}}{k_i}\right)\right] - c(1) + c(0).$$

Note that $\frac{Y_{k'}}{k'}$ is a mean-preserving spread of $\frac{Y_k}{k}$ when $k' < k$. Thus, if f is concave, we have degree complementarity, while if f is convex then degree substitution obtains. ■

3 Equilibrium: existence and monotonicity

We start by showing existence of an equilibrium which is in monotone strategies. We then obtain conditions under which all equilibria are monotone. Finally, we turn to the relation between network degree and equilibrium payoffs and identify the conditions under which payoffs increase/decrease with network degree thereby clarifying the contexts in which network connections are advantageous and disadvantageous, respectively.

Proposition 1 *There exists a symmetric equilibrium. If the game has degree complements, then the equilibrium strategy σ can be chosen in pure strategies. If there is degree complementarity (substitution) then there is a symmetric equilibrium that is non-decreasing (non-increasing).*

The proof of existence of equilibrium is standard and omitted.[13] The proof of existence of a pure-strategy equilibrium under degree complements follows along the lines of other strategic complements arguments (e.g., see Milgrom and Shannon (1994)). With regard to monotonicity, we first exploit the degree complements/substitutes property to show that for a player faced with a monotone strategy played by the rest of the population, there always exists a monotone best-reply. Then, since the set of monotone strategies is convex and compact, existence follows from standard arguments (see, e.g., Milgrom and Shannon (1994) or van Zandt and Vives (2007)).

We elaborate on two aspects of this result. *First*, we discuss whether *every* symmetric equilibrium is monotone. Consider a game with action set $X = \{0, 1\}$ and payoffs $v_k(x_i, x_1, ..., x_k) = x_i \sum_{j \in N_i(g)} x_j - cx_i$, where $0 < c < 1$. Note that this example satisfies Property A and the underlying game displays strategic complements. Now suppose that there is perfect degree correlation so that players are connected to others of the same degree. It is then clear that *any* symmetric pure strategy profile defines an equilibrium. This example, suggests that the possibility of non-monotone equilibria is related to the correlation in degrees. This point is highlighted by the following result.

[13] For example, see Theorem 2 in Jackson, Simon, Swinkels and Zame (2002), and note that the games here are a special case where communication is unnecessary as the outcome is single-valued.

Proposition 2 *Suppose that payoffs satisfy Property A and that the degrees of neighboring nodes are independent. Then, under strict strategic complements (substitutes) every symmetric equilibrium is monotone increasing (decreasing).*

The key point to note here is that, under independence, degree k and degree $k' = k + 1$ players have the same beliefs about the degree of each of their neighbors. If the $k + 1^{\text{th}}$ neighbor is choosing 0 then under Property A the degree k' player will choose the same best response as the degree k player; if the $k + 1^{\text{th}}$ neighbor chooses a positive action then strict complementarities implies that the degree k' player best responds with a higher action.[14]

The *second* issue is whether the nature of degree correlation – positive association under strategic complements, or negative association under strategic substitutes – is essential for existence of monotone equilibria. Consider a special case of Example 1 in which $X = [0, 1]$, $f(y) = \gamma y + \alpha y^2$, $y = x_i + \sum_{j \in N_i(g)} x_j$, and $c(x_i) = \beta x_i^2$ for some $\gamma, \alpha, \beta > 0$. This game exhibits strategic complements. Next suppose that the unconditional degree distribution satisfies $P(1) = P(2) = \varepsilon$ and $P(\bar{k}) = 1 - 2\varepsilon$ for some small ε and a given large $\bar{k}$. Further suppose that $P(\bar{k} \mid 1) = P(2 \mid 2) = 1$, i.e., all agents with degree 1 are connected to those of degree $\bar{k}$ and all those of degree 2 are connected among themselves. Note that this pattern of connections violates positive association. It is now possible to verify that if $\beta > \alpha$ then *every* equilibrium is interior; moreover if $\bar{k}$ is large enough and ε sufficiently small then σ satisfies $\sigma_2 < \sigma_1 < \sigma_{\bar{k}}$ and is not monotone.

A recurring theme in the study of social structure in economics is the idea that social connections create personal advantages. In our framework the relation between degrees and payoffs is the natural way to study network advantages. Let us consider games with positive externalities and positive association, and look at a player with degree $k + 1$. Suppose that all of his neighbors follow the monotone increasing equilibrium strategy, but his $k + 1^{\text{th}}$ neighbor chooses the minimal 0 action. Property A implies that our $(k + 1)$ degree player can assure himself an expected payoff which is at least as high as that of any k degree player by simply using the strategy of the degree k player. These considerations lead us to state the following result.

[14] The strictness is important for the result. For instance, if players were completely indifferent between all actions, then non-monotone equilibria are clearly possible.

Proposition 3 *Suppose that payoffs satisfy Property A. If* $\mathbf{P}$ *is positively associated, then in every non-decreasing symmetric equilibrium, the expected payoffs are non-decreasing (non-increasing) in degree if the game displays positive externalities (negative externalities). If* $\mathbf{P}$ *is negatively associated, then in every non-increasing symmetric equilibrium, the expected payoffs are non-decreasing (non-increasing) in degree if the game displays positive externalities (negative externalities).*

We emphasize that under positive externalities, players with more neighbors earn higher payoffs irrespective of whether the game exhibits strategic complements or substitutes (under the appropriate monotone equilibrium). These network advantages are specially striking in games with strategic substitutes and negative association: here higher degree players exert lower efforts but earn a higher payoff as compared to their less connected peers.

4 The effects of changing networks

Our interest is in understanding how changes in a network – such as the addition/deletion or links or the redistribution of links away from a regular network to highly unequal distributions which characterize empirically observed networks – affect the behavior and welfare of players. We start with games of strategic substitutes and then take up games with strategic complements.

4.1 Games with strategic substitutes

Our analysis will focus on binary action games; an attractive feature of binary-action games with strict strategic substitutes and negative association is that there is a unique non-increasing symmetric equilibrium strategy σ and it is fully characterized by a threshold. That is, there exists some *threshold* $t \in \{1, 2, ...\}$ such that $\sigma(k_i) = 1$ for $k_i < t$, $\sigma(k_i) = 0$ for all $k_i > t$, and for $k_i = t$ the induced $\sigma(t)$ may be a mixture over 0 and 1.

What is the effect of adding links on equilibrium behavior? We first observe that the best response of a player depends on the actions and hence the expectations concerning the degrees of his neighbors. Thus the effects of addition of links must be studied in terms of the change in the degree distribution of the neighbors.[15] We therefore approach the addition of links in terms of an increase in the degrees

[15] Indeed, it is important to note that the relationship between two underlying (unconditional) degree distributions

of a neighbor. In our context of decreasing strategies, this means a fall in his action (on average), which, from strategic substitutes, suggests that the best response of the player in question should increase. However, this increase in action of every degree may come into conflict with the expectation that neighbors must be choosing a lower action, on average. The following result clarifies how this tension is resolved.

Proposition 4 *Suppose that $X = \{0, 1\}$, payoffs satisfy property A, strict strategic substitutes, and $\mathbf{P}'$ exhibits negative association. If $\mathbf{P}$ dominates $\mathbf{P}'$, then in the unique non-increasing symmetric equilibrium, the threshold under $\mathbf{P}$ is at least as large as the threshold under $\mathbf{P}'$, i.e., $t \geq t'$. However, for the threshold degree type, t, the probability that a neighbor chooses 1 is lower under $\mathbf{P}$.*

This result clarifies that an increase in threshold for choosing 1 is consistent with equilibrium behavior because each of the neighbors is more connected and chooses 1 with a lower probability (in spite of an increase in the threshold). The best shot game helps to illustrate the effects of dominance shifts in degrees which are derived in the above result.

Example 5 *Effects of increasing degrees in a best-shot game.*

Consider the best shot game discussed in the introduction and described in example 2. Set $c = 25/64$. Suppose that degrees take on values $1, 2$ and 3 and that the degrees of neighbors are independent. Let us start with initial beliefs $\tilde{\mathbf{P}}'$ which assigns probability one-half to players having degree 1 and 2. In the unique symmetric equilibrium, degree 1 players choose 1 with probability 1, while degree 2 players choose 1 with probability 0. In this equilibrium, the probability that a neighbor of a degree 2 player chooses action 1 is 1/2.

Now consider a dominance shift to $\tilde{\mathbf{P}}$, so that neighboring players are believed to have degrees 2 and 3 with probability one-half each. It can be checked that the unique equilibrium involves degree

does not imply a similar relation for the conditional distribution over neighbors' degrees, even under independence. As an illustration consider a case where degrees of neighbors are independent. Consider two degree distributions P and P', where P' assigns one half probability to degrees 2 and 10 each, while distribution P assigns one half probability to degrees 8 or 10 each. Clearly P FOSD P'. As mentioned above, when neighboring degrees are independent, the probability of having a link with a node is (at least roughly, depending on the process) proportional to the degree of that node, so that for all k, $P(k'|k) = k'P(k')/\sum P(l)l$. Let $\tilde{P}(k')$ be the neighbor's degree distribution. Under $\tilde{P}'$, the probability that a neighbor has degree 10 is 5/6, while under $\tilde{P}$, the same probability is 5/9. Thus, $\tilde{P}$ does not FOSD $\tilde{P}'$.

2 players choosing action 1 with probability 3/4 while degree 3 players choose 1 with probability 0. In this equilibrium, the probability that a neighbor of a degree 2 player chooses action 1 is 3/8.

The dominance shift in the beliefs from $\tilde{\mathbf{P}}'$ to $\tilde{\mathbf{P}}$ leads to an increase in the threshold from 1 to 2. However, the threshold degree 2 has lower expectation of action 1 under $\tilde{\mathbf{P}}$ as compared to $\tilde{\mathbf{P}}'$. ■

We now turn to the effects on welfare. The expected welfare is assessed by the expected payoff of a randomly chosen player (according to the prevailing degree distribution). Observe that dominance shifts in the interaction structure lower the expected probability that a randomly selected neighbor of a t-degree player (the threshold player under $\mathbf{P}$) chooses 1. If the degrees of neighbors are independent, then the average effort of a randomly selected neighbor of a player i does not depend on i's degree, and a similar property of lower expected action from each neighbor would hold for all degrees. However, if there is negative association, matters are more complicated, and it is possible that the overall effect of a first order shift in degree distribution can be positive for some degrees and negative for others.

Proposition 4 compares behavior across networks when there is an increase in the density of links in the sense of domination. However, there are many cases where we might be interested in comparing networks when there is not a clear cut domination relation. We now develop a result on the effect of *arbitrary* changes in the degree distribution.

For simplicity, we focus on the case where degrees of neighbors are independent. Let $\mathbf{P}$ and $\mathbf{P}'$ be two different sets of beliefs and suppose that $\tilde{F}$ and $\tilde{F}'$ are the corresponding induced cumulative distribution functions of the degree distributions, respectively. Let t and t' stand for the threshold types defining the (unique) threshold equilibria under $\mathbf{P}$ and $\mathbf{P}'$, respectively.

Proposition 5 *Consider a binary-action game with Property A, strict strategic substitutes and independence across neighbors' degrees. If* $\tilde{F}(t') \leq \tilde{F}'(t'-1)$ *then* $t \geq t'$. *Moreover, the probability that any given neighbor chooses* 1 *falls.*

The key issue here is the change in the probability mass relative to the threshold. If the probability of degrees equal or below the threshold goes down then the probability of action 1 decreases and from strategic substitutes, the best response of threshold type t must still be 1. In other words, the threshold rises weakly.

The contribution of Proposition 5 is that it allows us to examine the effect of *any change of the degree distribution*. A natural and important example of such changes is increasing the polarization of the degree distribution by shifting weights to the ends of the support of the degree distribution, as is done under a mean preserving spread (MPS) of the degree distribution. In particular, the above results can be directly applied in the case of strong MPS shifts in the degree distributions. Focusing on the unconditional beliefs (taken to coincide with the unconditional degree distributions because of independence), we say that $P(\cdot)$ is a *strong MPS* of $P'(\cdot)$ if they have the same mean and there exists L and H such that $P(k) \geq P'(k)$ if $k < L$ or $k > H$, and $P(k) \leq P'(k)$ otherwise. Proposition 5 implies that, in the context of binary-action games, the equilibrium effects of any such change can be inferred from the relative values of the threshold t, and L and H.

4.2 Games with strategic complements

This section studies the effects of changes in the network on equilibrium behavior and payoffs in games with strategic complements. From Proposition 1 we know that equilibria are increasing in degree in games with strategic complements. As we shift weight to higher degree neighbors each player's highest best response to the original equilibrium profile would be at least as high as the supremum of his original strategy's support. We can now iterate this best response procedure. Since the action set is compact, this process converges and it is easy to see that the limit is a (symmetric) non-decreasing equilibrium which dominates the original one. The following result summarizes this argument.

Proposition 6 *Suppose that payoffs satisfy strict strategic complements, Property A, and* $\mathbf{P}'$ *exhibits positive association. If* $\mathbf{P}$ *dominates* $\mathbf{P}'$*, then for every non-decreasing equilibrium* σ' *under* $\mathbf{P}'$ *there exists a non-decreasing equilibrium* σ *under* $\mathbf{P}$ *which dominates it.*

The proof is straightforward and omitted. Consider next the effect of a dominance shift in the social network on welfare. Recall that the expected welfare is assessed by the expected payoff of a randomly chosen player. Naturally, it must depend on whether the externalities are positive or negative. Suppose, for concreteness, that they are positive and let $\mathbf{P}$ dominate $\mathbf{P'}$. Then, from Proposition 6, we know that for every monotone equilibrium $\boldsymbol{\sigma'}$ under $\mathbf{P'}$ there exists a monotone equilibrium σ under $\mathbf{P}$ in which players' actions are all at least as high. Hence, the expected payoff of each player is higher under $\mathbf{P}$. However, since expected payoffs are non-decreasing in the degree of a player, to assess welfare it is also important to consider the relation between the corresponding unconditional degree distributions $P(\cdot)$ and $P'(\cdot)$. For example, let us further assume that $P(\cdot)$ FOSD $P'(\cdot)$. Then, the above considerations imply that the *ex-ante* expected payoff of a randomly chosen player must rise when one moves from $\mathbf{P'}$ to $\mathbf{P}$. We summarize this argument in the following result. For a monotone increasing strategy profile σ under $\mathbf{P}$, define $W_{\mathbf{P}}(\sigma)$ to be the expected payoff of a node picked at random (under $P(\cdot)$).

Proposition 7 *Suppose that payoffs satisfy Property A, strict strategic complements and the degrees of neighbors exhibit positive association. Suppose* $\mathbf{P}$ *dominates* $\mathbf{P}'$*, and* $P(\cdot)$ *FOSD* $P'(\cdot)$*. For any non-decreasing equilibrium* σ' *and the corresponding expected payoff* $W_{\mathbf{P}'}(\sigma')$ *under* $\mathbf{P}'$*, there exists a non-decreasing equilibrium* σ *under* $\mathbf{P}$ *such that* $W_{\mathbf{P}}(\sigma) \geq W_{\mathbf{P}'}(\sigma')$.

The proof follows from the arguments above and is omitted. Propositions 6 and 7 pertain to dominance shifts in the conditional degree distributions. We conclude with a result on the effects of arbitrary changes in the degree distribution.

Proposition 8 *Consider a binary-action game with Property A, strict strategic complements and independence across neighbors' degrees. If* $\tilde{F}(t') \leq \tilde{F}'(t'-1)$ *then there is an equilibrium with corresponding threshold type* $t \leq t'$*. Moreover, the probability that any given neighbor chooses* 1 *rises.*

The proof for this result follows along the lines of the proof of Proposition 5 and is omitted. We conclude by observing that the strategic structure of payoffs has an important effect: recall from Subsection 4.1 that in the case of strategic substitutes, the probability that any neighbor chooses 1 falls when network connectivity grows; by contrast in games of strategic complements the addition of links leads to an increase in the probability that a neighbor chooses action 1.

5 Deeper Network Information

So far we have focused on the case where players only know their own degree and best respond to the anticipated actions of their neighbors based on the (conditional) degree distributions. We now investigate the implication of increasing the information that players possess about their local networks. As a natural first step along these lines, we examine situations where a player knows not only how many neighbors he has, but also how may neighbors each of his neighbors has. The arguments we develop in this section extend in a natural way to general radii of local knowledge. Indeed, in the limit, as this radius of knowledge grows, we arrive at complete knowledge of the arrangement of degrees in the network.[16]

Formally, the common type space $\mathcal{T}$ of every player i consists of elements of the form $(k; \ell_1, \ell_2, ..., \ell_k)$ where $k \in \{0, 1, 2, ..., n-1\}$ is the degree of the player and ℓ_j is the degree of neighbor j $(j = 1, 2, ..., k)$, where (in an anonymous setup where the identity of neighbors is ignored) we may assume without loss of generality that neighbors are indexed according to decreasing degree (i.e., $\ell_j \geq \ell_{j+1}$). Given the multi-dimensionality of types in this case, the question arises as to how one should define monotonicity. In particular, the issue is what should be the order relationship $\succeq$ on the type space underlying the requirement of monotonicity. For the case of strategic complements, it is natural to say that two different types, $t = (k; \ell_1, \ell_2, ..., \ell_k)$ and $t' = (k'; \ell'_1, \ell'_2, ..., \ell'_{k'})$, satisfy $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \geq \ell'_u$ for all $u = 1, 2, ..., k'$. On the other hand, for the case of strategic substitutes, we write $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \leq \ell'_u$ for all $u = 1, 2, 3, ...k'$. Given any such (partial) order on $\mathcal{T}$, we say that a strategy σ is monotone increasing if for all t_i, $t'_i \in \mathcal{T}$, $t_i \succeq t'_i \Rightarrow \sigma(t_i)$ FOSD $\sigma(t'_i)$. The notion of a monotonically decreasing strategy is defined analogously.

We first illustrate the impact of richer knowledge on the nature of equilibria. It is easier to see the effects of deeper network information in the simpler setting where the degrees of the neighbors are independent and so, for expositional simplicity, we will assume independence of neighbors' degrees in this section. Recall, from Proposition 2 that under degree independence all symmetric equilibria are

[16] For results on this limit case, see the earlier version of this paper, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006).

monotone increasing (decreasing) in the case of strategic complements (substitutes). The following example shows that greater network knowledge introduces non-monotone equilibrium even if degrees are independent.

Example 6 *Non-monotone Equilibria with Knowledge of Neighbors' Degrees*

Consider a setting where nodes have either degree 1 or degree 2, as given by the corresponding probabilities $P(1)$ and $P(2)$. Suppose that the game is binary-action with $X = \{0, 1\}$ and displays strategic complements. Specifically, suppose that the payoff of a player only depends on his own action x_i and the sum $\bar{x}$ of his neighbors' actions as given by a function $v(x_i, \bar{x})$ as follows: $v(0, 0) = 0$, $v(0, 1) = 1/2$, $v(0, 2) = 3/4$, $v(1, 0) = -1$, $v(1, 1) = 1$, $v(1, 2) = 3$.

It is readily seen that, for any P with support on degrees 1 and 2, the following strategy σ defines a symmetric equilibrium: $\sigma(1; 1) = 1$; $\sigma(1; 2) = 0$; $\sigma(2; \ell_1, \ell_2) = 0$ for any $\ell_1, \ell_2 \in \{1, 2\}$. Here, two players that are only linked to each other both play 1, while all other players choose 0. ■

Similar non-monotonic equilibrium examples can be constructed for games with strategic substitutes. These observations leave open the issue of whether there exist *any* suitably increasing or decreasing monotone equilibria. The following result shows that a monotone equilibrium always exists if players have deeper network information.

Proposition 9 *Suppose that payoffs satisfy Property A and players know their own degree and the degrees of their neighbors. Under strategic complements (strategic substitutes) there is a symmetric equilibrium that is monotone increasing (decreasing).*

The proof of the proposition, which appears in Appendix B, extends naturally the ideas mentioned for the proof of Proposition 2, i.e., the best-reply to a monotone strategy can be chosen monotone and the set of all monotone strategies is compact and convex. A direct implication of the result is that there is always an equilibrium that, on average across the types $(k; \ell_1, \ell_2, ..., \ell_k)$ consistent with each degree k, prescribes an (average) action that is monotone in degree. Equipped with the above monotonicity result, it is also possible to recover most of the insights obtained earlier under the assumption that players only know their own degree.

6 Concluding Remarks

Empirical work suggests that the patterns of social interaction have an important influence on economic outcomes. These interaction effects have however been resistant to systematic theoretical study: even in the simplest examples games on networks have multiple equilibria which possess very different properties. The principal innovation of our paper is the introduction of the idea that players have incomplete network knowledge: in particular we focus on an easily measurable aspect of networks, the number of personal connections/degree, and suppose that players know their own degree but have incomplete information concerning the degree of others in the network. This formulation allows us to develop a general framework for the study of games played on networks. Borrowing from the statistics literature, we propose a general concept of association across degrees of neighbors: this allows us to study the classical case of independent degrees (as in the models of Erdős and Renyí), as well as positive and negative degree correlations which are observed in empirical networks. Moreover, our framework allows payoffs to exhibit strategic substitutes as well as strategic complements, and covers almost all the economic applications the literature has so far studied.

The analysis of this framework yields a number of powerful and intuitively appealing insights with regard to the effects of location within a network as well as the effects of changes in networks. These results also clarify how the basic strategic features of the game – as manifest in the substitutes and complements property – combine with different patterns of degree association to shape behavior and payoffs.

In this paper we have focussed on the degree distribution in a network. The research on social networks has identified a number of other important aspects of networks, such as clustering, centrality and proximity, and in future work it would be interesting to bring them into the model.

7 Appendix A: Network formation mechanisms

In order to illustrate the generality of our assumptions on the underlying network structure we describe below some of the most heavily studied (theoretically and empirically) network formation processes, and the association characteristics they exhibit.[17]

(i) Poisson Networks

We start with the canonical model in the modern theory of complex networks. This is the model independently proposed by Solomonoff and Rapoport (1951), Gilbert (1959), and Erdös and Rényi (1960), and later intensively studied by the latter two authors. Given a finite set of nodes N, every possible link g_{ij} is assumed to form independently with a fixed probability p. Suppose the social network is actually formed as posited above. Then, it is easy to see that the degree distribution of a node picked at random is binomial (and, as n approaches ∞, Poisson). Here, for any finite n, the degree of nodes i and j, conditional on being connected, is independent, since the degree of j is the realization of $n-2$ Bernoulli random variables (the other links of j) that are completely independent of the realization of the other links of i.

(ii) The Configuration model

Following Bender and Canfield (1978) and Bollobás (1980), in the configuration model the degrees of nodes are fixed in advance, and nodes are connected in a manner realizing a pre-specified degree distribution. It is clear that this results in a family of conditional degree distributions exhibiting negative association. (As an example, suppose that there are four nodes, two of them with degree two and the others with degree one. Then, being a degree-two node makes it more likely that a link points to a degree-one node, and vice versa.) Clearly, such a negative association is a consequence of the finiteness of nodes and independence obtains as we take n to infinity.

(iii) The Barabási-Albert model

Barabási and Albert (1999) propose a dynamic model in which nodes enter in sequence, say, according to their index $i = 1, 2, .., n$. At the time of entry, every node i creates a fixed number of links, say m, to separate incumbents (i.e., nodes that entered before). For each such link, any incumbent node j is chosen with a probability proportional to j's current degree – this is what is generally known as *preferential attachment*. For large n, it can be shown that, with very high

[17] For an extensive discussion of network formation processes see Vega-Redondo (2007).

probability, the degree distribution $P_n(\cdot)$ resulting from the process is approximately "scale-free" with a decay parameter equal to 3. That is, it satisfies:

$$P_n(k) \simeq Ak^{-3} \qquad (k = m, m+1, m+2, ...) \tag{3}$$

for some suitable normalizing factor A.

Positive association of degrees in this model follows as a special case of the Jackson-Rogers-Vasquez model, described below.

(iv) The Jackson-Rogers-Vazquez model

The model described in Jackson and Rogers (2007) and Vasquez (2003) proceeds as follows: nodes enter in sequence and a new node chooses m_r nodes uniformly at random, and then links to each one of them with an independent probability p_r, and then also searches the neighborhoods of these 'parent' nodes to find m_n additional nodes, linking to each one of them with an independent probability p_n. At one extreme, with $p_r = 0$, we get the preferential attachment model, while at the other extreme with $p_n = 0$ we get the uniformly random attachment model as in a growing version of the Erdös-Rényi world. This model exhibits positive association. A first order stochastic dominance relation of the distribution of a single neighboring node conditional on degree is proven in Theorem 4 in Jackson and Rogers (2007), and a straightforward extension of that result establishes positive association (under a mean-field approximation).

8 Appendix B: Proofs of results

Proof of Proposition 2: We present the proof for the case of strategic complements. The proof for the case of strategic substitutes is analogous and omitted. Let $\{\sigma_k^*\}$ be the strategy played in a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial strategy with all degrees choosing action 0 with probability 1, the claim follows directly. Therefore, from now on, we shall assume that the equilibrium strategy is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \mathbf{supp}(\sigma_{k'}^*)$.

Consider any $k \in \{0, 1, ..., n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that

$x_{k'} \geq x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma^*_{k'})$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, the assumption of (strict) strategic complements implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, ..., x_{l_k}, x_s) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}) - v_k(x, x_{l_1}, ..., x_{l_k})$$

for any x_s, with the inequality being strict if $x_s > 0$. Then, averaging over all types, the fact that the degrees of any two neighboring nodes are stochastically independent random variables together with the fact that $x_k > 0$ implies that

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > U(x_k, \sigma^*, k) - U(x, \sigma^*, k).$$

On the other hand, note that from the choice of x_k,

$$U(x_k, \sigma^*, k) - U(x, \sigma^*, k) \geq 0$$

for all x. Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > 0$$

for all $x < x_k$. This in turn requires that if $x_{k+1} \in \mathbf{supp}(\sigma^*_{k+1})$ then $x_{k+1} \geq x_k$, which of course implies that σ^*_{k+1} FOSD σ^*_k. Iterating the argument as needed, the desired conclusion follows, i.e., $\sigma^*_{k'}$ FOSD σ^*_k whenever $k' > k$. ■

Proof of Proposition 3: We present the proof for positive externalities. The proof for negative externalities is analogous and omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. First, suppose positive association and let σ^* be a (non-trivial) monotone increasing equilibrium strategy. Suppose $x_k \in \mathbf{supp}(\sigma^*_k)$ and $x_{k+1} \in \mathbf{supp}\left(\sigma^*_{k+1}\right)$. Property A implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, 0) = v_k(x_k, x_{l_1}, ..., x_{l_k})$$

for all $x_{l_1}, ..., x_{l_k}$, and since the payoff structure satisfies positive externalities, it follows that for any

$x > 0$,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}).$$

We now have to consider two cases. First, assume positive association and consider a monotone increasing equilibrium, then looking at expected utilities we obtain that:

$$U(x_k, \sigma^*, k+1) \geq U(x_k, \sigma^*, k).$$

Since σ^*_{k+1} is a best response in the network game being played,

$$U(x_{k+1}, \sigma^*, k+1) \geq U(x_k, \sigma^*, k)$$

and the result follows. Second, observe that the case of negative association and monotone decreasing equilibrium strategy can be proven using analogous arguments. This concludes the proof. ■

Proof of Proposition 4: Under the maintained hypotheses there exists a unique non-increasing symmetric equilibrium with a threshold property. Suppose that this equilibrium σ' has threshold t' under $\mathbf{P}'$. The assumptions that $\mathbf{P}$ dominates $\mathbf{P}'$ for all k and that players choose a monotone decreasing strategy imply that the equilibrium threshold under $\mathbf{P}$ cannot be lower than t'. To see this, suppose that in the equilibrium under $\mathbf{P}$, σ, the threshold $t < t'$. We now show that this yields a contradiction. In equilibrium σ' under $\mathbf{P}'$, for the threshold degree t' the expected payoffs from action 1 are higher than the expected payoffs from action 0. With a slight abuse of notation with regard to mixed and pure strategies, we now write this as follows:

$$\begin{aligned}
0 &\leq U(1, \sigma', t') - U(0, \sigma', t') \\
&= \textstyle\sum_{t'_{N(i)}} P'(t'_{N(i)}|t') v_{t'}(1, \sigma'(\eta_1), ..., \sigma'(\eta_{t'})) - \sum_{t'_{N(i)}} P'(t'_{N(i)}|t') v_{t'}(0, \sigma'(\eta_1), ..., \sigma'(\eta_{t'})) \\
&= \textstyle\sum_{t'_{N(i)}} P'(t'_{N(i)}|t')[v_{t'}(1, \sigma'(\eta_1), ..., \sigma'(\eta_{t'})) - v_{t'}(0, \sigma'(\eta_1), \sigma'(\eta_2), ..., \sigma'(\eta_{t'}))] \\
&\leq \textstyle\sum_{t'_{N(i)}} P(t'_{N(i)}|t')[v_{t'}(1, \sigma'(\eta_1), ..., \sigma'(\eta_{t'})) - v_{t'}(0, \sigma'(\eta_1), ..., \sigma'(\eta_{t'}))] \\
&< \textstyle\sum_{t'_{N(i)}} P(t'_{N(i)}|t')[v_{t'}(1, \sigma(\eta_1), \sigma(\eta_2), ..., \sigma(\eta_{t'})) - v_{t'}(0, \sigma(\eta_1), ..., \sigma(\eta_{t'}))] \\
&= U(1, \sigma, t') - U(0, \sigma, t')
\end{aligned}$$

where the first inequality follows from the hypotheses that $\mathbf{P}$ dominates $\mathbf{P}'$, σ' is non-increasing and $v_{t'}$ satisfies the strategic substitutes property, while the second inequality follows from the hypothesis that $t < t'$ and $v_{t'}$ satisfies the strict strategic substitutes property. This however implies that for threshold degree t' action 1 yields strictly higher expected payoffs as compared to action 0. ■

Proof of Proposition 5: Suppose that $\tilde{F}(t') \leq \tilde{F}'(t'-1)$ but, contrary to what is claimed, $t < t'$. Then, under $\mathbf{P}$, the probability that any of the neighbors chooses action 1 is bounded above by $\tilde{F}(t')$ and, therefore, by $\tilde{F}'(t'-1)$. Given the hypothesis that t' is the threshold under $\mathbf{P}'$, the assumption of strategic substitutes now generates a contradiction with the optimality of actions of degree t' in an equilibrium under $\mathbf{P}$, and this completes the proof. ■

Proof of Proposition 9: Let us consider first the case of strategic complements and denote by $\sum^m$ the set of monotone strategies. The proof is based on the following two claims:

Claim 1: For any player i, if all other players $j \neq i$ use a common strategy $\sigma \in \sum^m$ there is always a strategy $\sigma_i \in \sum^m$ that is a best response to it.

Claim 2: A symmetric equilibrium exists in the strategic-form game where players' strategies are taken from $\sum^m$.

To establish Claim 1, consider a player i and let $t_i, t_i' \in \mathcal{T}$ such that $t_i' \succeq t_i$, where $\succeq$ is the partial order applicable to the case of strategic complements (see Section 5). For any $\sigma \in \sum^m$ chosen by every $j \neq i$, let $BR(\sigma, t_i)$ be the set of best-response strategies of player i to σ when his type is t_i. Let us assume that $\sigma(t_j) \neq 0$ for some $t_j \in \mathcal{T}$. (Otherwise, the desired conclusion follows even more directly, since the best-response correspondence is unaffected by being connected to a player whose strategy chooses action 0 uniformly.) By definition, for every $x_{t_i} \in BR(\sigma, t_i)$, we must have that

$$\forall x \in X, \qquad U(x_{t_i}, \sigma, t_i) - U(x, \sigma, t_i) \geq 0$$

Then, since $t_i' \succeq t_i$, the assumption of (strict) strategic complements implies that

$$\forall x \leq x_{t_i}, \qquad U(x_{t_i}, \sigma, t_i') - U(x, \sigma, t_i') > 0. \tag{4}$$

This follows from a two-fold observation:

(i) From Assumption A, if $t_i = (k, \ell_1, \ell_2, ..., \ell_k)$ and $t_i' = (k', \ell_1', \ell_2', ..., \ell_k')$ and $t_i' \succeq t_i$ we can think of t_i involving k' neighbors with all neighbors indexed from $k+1$ to k' (if any) choosing the action 0;

(ii) From strict strategic complements, since $\ell_u' \geq \ell_u$ the probability distribution over actions corresponding to each of his neighbors under t_i, $u = 1, 2, ..., k$, is dominated in the FOSD sense by the corresponding neighbor under t_i'.

Let us now make use of (4) in the case where x_{t_i} is the highest best response by type t_i to σ. Then, it follows that any $x_{t_i'} \in BR(\sigma, t_i')$ must satisfy:

$$x_{t_i'} \geq \sup\{x_{t_i} : x_{t_i} \in BR(\sigma, t_i)\},$$

which establishes Claim 1.

To prove Claim 2, we can simply invoke the concavity postulated for each payoff function $v_k(\cdot, x)$ for any given $x \in X^k$ and the fact that the set of monotone strategies is compact and convex. To see the latter point, note that the monotonicity of a strategy σ is characterized by the condition:

$$\forall t_i, t_i' \in \mathcal{T}, \qquad t_i' \succeq t_i \Rightarrow \sigma(t_i') \text{ FOSD } \sigma(t_i). \tag{5}$$

Clearly, if two different strategies σ and σ' satisfy (5), then any convex combination $\hat{\sigma} = \lambda\sigma + (1-\lambda)\sigma'$ also satisfies it.

Finally, to prove the result for the case of strategic substitutes, note that the above line of arguments can be applied unchanged, with the suitable adaptation of the partial order used to define monotonicity. In this second case, as explained in Section 5, we say that $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \leq \ell_u'$ for all $u = 1, 2, ..., k'$. ■

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