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CHAOTIC ECONOMIC DYNAMICS

by

Richard GOODWIN *

* Università di Siena

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European University Institute
Badia Fiesolana
- 50016 San Domenico (Fi) -
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PART I: CAPITALISM AS CREATIVE, CHAOTIC EVOLUTION BY STRUCTURAL CHANGE

Why is economics like the weather: because both are highly irregular if not chaotic, thus making prediction unreliable or even impossible. Lorenz, a pupil of the great George Birkhoff, opened up one of the more active parts of current mathematical analysis, by proposing an endogenous meteorological theory, which, without external disturbance, could produce chaotic time series solutions. This suggests to me that there will have to be a thorough reformulation of much econometrics. It opens up a wide range of problems and possibilities for the use of such models in empirical economics. There are two possible ways of explaining the complexities of economic time series - exogenous shocks and endogenous models which can produce irregular solutions. There is no question that in analyzing economic time series one must assume irregular shocks, but surely also one must seek out plausible nonlinear models to test, models capable of independently producing erratic behaviour, and then separate out the two components. I propose to give a brief, somewhat superficial, account of such nonlinear models followed by illustrative examples.

There are two broad types of dynamical equation systems: continuous time and discrete time; both have frequently been used in economics. The latter arise because

there are significant time lags in an economy. The trouble is that these occur in the context of economic activity which is substantially continuous, so that one should formulate mixed difference-differential systems, a procedure the complications of which place it beyond the scope of this paper.

Although most economic time series arrive in discrete time, this is irrelevant since the reality is continuous: this implies different equations. Simple one and two dimensional systems only give erratic behaviour if they have exogenous, time dependent parameters, e.g. economies subject to balance of payments effects. It appears that for differential systems endogenous chaos can only occur in models of 3 or more dimensions.

The weird and wonderful world that nonlinear dynamics opens up is so complicated that it is highly desirable to keep to low dimensional models, which I shall do. Yet there are a very large number of dimensions to all economies. Many dimensions usually mean very complicated behaviour, which makes the testing for structure difficult or impossible. Thus even with many, well-defined, constituent cycles, if some of the periods are incommensurate, the solutions will never repeat.

To indicate the complicated nature of a multidimensional system embodying structural change, I propose to give a brief example with both price and quantity changes in first one and then a second change in structural parameters. The maximum number of dimensions which my

computer will handle is 4. so that I shall consider a two sector economy.

I start with an economic potential defined as

$$V(p, q) = \langle p_1, p_2 \rangle \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} - \langle p_1, p_2; w \rangle \begin{bmatrix} a_{11} & a_{21} & | & a_{c1} \\ a_{12} & a_{22} & | & a_{c2} \\ \hline a_{l1} & a_{l2} & | & a_{l1} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} - \langle B_1, B_2 \rangle \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} - \langle p \rangle \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

where B are constant nominal costs per unit of output and A all real demands not dependent on output. For simplicity I set $a_{l1} = 0$.

First it is essential to determine the fixed, or equilibrium, points:

$$\hat{p} = \langle w a_{l1} \rangle + \langle B \rangle [I - a]^{-1} \rangle$$

$$\hat{q} = [I - a]^{-1} \left\{ \begin{Bmatrix} a_{c1} \\ a_{c2} \end{Bmatrix} + \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} A \right\}$$

Changes in B and in A are considered as exogenous to this simple linear production system. Our principal interest for the moment is the effect of changes in a_{ij} , in a_l , and in a_c . These changes may or may not be exogenous, but in any case they change the character of the dynamical behaviour of the system in response to changes in B and in A.

To find the dynamics, we take the slopes of the potential first with respect to p and then with respect to q. Taking account of consumption a_c , we have
Gradient $V_p = [I - a] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} - 1 * \begin{Bmatrix} a_{c1} \\ a_{c2} \end{Bmatrix} - \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$ The simplest dynamical adjustment is that output slides down the hills of demand - output, thus

$$\dot{q} = -\text{Grad } V_p \text{ and down the hills of value thus}$$

$$\dot{p} = -\text{Grad } V_q$$

To consider technical progress and the problem of whether or not it leads to technological unemployment, I shall take a somewhat Schumpeterian view, without the implicit assumption of full employment. In the above potential, there is the assumption that consumption occurs in fixed amounts per employed person (and that no profits are consumed). This assumption is quite unjustified for consumption, unlike production. Therefore, for this problem I take a constant nominal wage rate and assume that all personal income is spent on consumption in a fixed proportion, thus

$w = s_1 * w + s_2 * w$, so that (ignoring personal services),

$s_2 = 1 - s_1$. Real consumption then becomes inversely related to prices, thus: $cons_1 + cons_2 = s_1 / p_1 + (1 - s_1) / p_2$. We can then formulate our system of 4 differential equations.

$$\dot{q}_1 = (a_{11} q_1 + a_{12} q_2 + (a_{21} q_1 + a_{22} q_2) S / P_1 + A_1 - q_1) \epsilon_1$$

$$\dot{q}_2 = (a_{21} q_1 + a_{22} q_2 + (a_{11} q_1 + a_{12} q_2) (1 - S) / P_2 + A_2 - q_2) \epsilon_2$$

$$\dot{p}_1 = (a_{11} P_1 + a_{21} P_2 + a_{11} + B_1 - P_1) Y_1$$

$$\dot{p}_2 = (a_{12} P_1 + a_{22} P_2 + a_{12} + B_2 - P_2) Y_2$$

I propose to use this set of equations to analyze the nature of economic growth: to do this I propose to utilize the complementary theories of Schumpeter and Keynes. It was originally Schumpeter who posed the problem in its most basic form: one of his formulations states that a "... sequence of phenomena leads up to a new neighbourhood of equilibrium, in which enterprise will start again. This new neighbourhood of equilibrium is characterized, as compared to the one that preceded it, by a 'greater' social product

of a different pattern, new production functions, equal sum total of money incomes, a minimum (strictly zero) rate of interest, zero profits, zero loans, a different system of prices and a lower level of prices, the fundamental expression of the fact that all the lasting achievements of the particular spurt of innovation have been handed to consumers in the shape of increased real incomes". (p. 137, Business Cycles). This would have said it all, had he only written "greater potential social product". Implicit is the assumption of full employment: but how could he do so in the middle of the great depression of the 1930's and after Keynes? Logically, but unfortunately, he rejected totally the General Theory. One should note that in his description of the new equilibrium he includes zero profits, zero innovational investment; had he included continuing high investment, his conclusion would be more acceptable. The system of equations above assumes that output is adjusted to approach demand. Technological innovation means necessarily lowered inputs per unit of output, so that for any level of output, demand is reduced, leading to reduction of demand and hence of output. Consequently the simple dynamics mean a falling to a lowered output, and this is the first case I shall consider but not the only one.

In agreement with Schumpeter, the money wage, w , is constant and may be conveniently set equal to unity. All earned income is spent for consumer goods in the fixed proportions, s_1 and $1-s_1$. Eschewing more sophisticated and complicated analysis, I take real consumption per employee = $s_1/p_1 + (1-s_1)/p_2$, employment is $l = a_{11}*q_1 + a_{12}*q_2$. In

order merely to indicate the nature of the problem, I assign somewhat arbitrary values to all parameters, leaving the 4 variables to be determined.

With all variables initially in equilibrium assume that a dramatic innovation permits a reduction of a_{12} from 0.20 to 0.10. The resulting trajectories are shown in Chart 1 for 70 periods with the following percentage changes:

q_1 -22.6%

q_2 -14.5%

p_1 -4.4%

p_2 -19.5%

Though the time required to complete the absorption of the innovation differs, it is clear that the effects are completed well before 70. Total employment fell from 1.36 to 0.67, i.e. -49.0%. This result shows how wrong Schumpeter was: though these particular numbers prove nothing, the qualitative result does show that if one reduces demand, output will fall. All variables fell less than employment, some dramatically less. The rise in real income softens but cannot, in any realistic context, reverse the decline. The effects of changes in prices derive from the following percentage changes:

q_1/p_1 -19.1%

q_2/p_1 -10.7%

q_1/p_2 -3.9%

q_2/p_2 -6.2%

Such large changes in the value system will, in general, provoke consequent further changes in production processes.

To simulate this I have assumed that at $t=30$, sector 1 cuts its use of good 2 by 50%, i.e. a_{21} drops from 0.10 to 0.05. The resulting compound evolution is illustrated also in Chart 1. The results are most interesting: in the previous case all motions were in the same direction, but here there is one dramatic exception, q_1 . With both parametric changes operating the percentage changes from $t=30$ to 70 were as follows:

q_1 +1.56%

q_2 -5.94%

p_1 -9.68%

p_2 -0.29%

If one looks carefully at the dynamics of the problem one can see the reason; also one can see how complex is the dynamical behaviour of this extremely over-simplified system. Such an exercise can give one an insight into the enormous complexity of an economy with thousands of sectors and millions of agents, all in varying degrees different. The results even in an almost-linear system can be quite erratic, independently of any random shocks. Thus in spite of a decline of 50% in the parameter at time $t=30$, but because of the special behaviour of sector 1, employment only fell by 2.66% from $t=30$ to 70, whereas it fell by 46.3% from $t=0$ to $t=30$.

This simple simulation will, I hope, help to convince you, if you are not already convinced, that aggregative models, e.g. using GNP, price indices, etc., will usually mask the reality, however helpful they may be to illuminate the nature of problems in economics. I say this in spite of

the fact that I am now going to discuss chaotic attractors, by means of aggregative models! I do this because the problem is so new, so difficult, so incompletely understood, that multidimensional models are out of the question.

Dynamical systems display themselves as attractors and/or repellers (sinks and sources, or basins and hills). There are motions in to or out of fixed points (or equilibria). There are also attractors/repellers for fixed rates of change, or also other well defined motions such as periodics. The interesting and difficult cases are the chaotics (more soberly called aperiodics) which are not yet, and possibly never will be, fully understood, precisely because they are ever-changing. Nevertheless there are chaotic attractors, where the motion follows a recognizable pattern, though always changing within the pattern.

In linear systems there are transients which dissipate, leaving the steady-state: for nonlinear systems, it is not so simple, and initial conditions may produce distinctly different long-runs. Frequent random shocks may create for linear, non-chaotic systems a result that may look much like the behaviour of an undisturbed chaotic system. Given a statistic produced by a chaotic dynamical system which is also subject to exogenous disturbances, if one can correctly guess or otherwise discover the correct dynamical model, then one can extract the proper deterministic-chaotic component from the statistic, leaving the residual as the irregularity due to the shocks. This fact surely makes it urgent for econometricians to consider a wide range of promising chaotic dynamical models.

In a sense it all began with Poincaré about a century ago, who treated what would now be called bifurcation. A bifurcation is a change in the qualitative nature of solutions to a system, and is related to the concept of structural stability. The change may be produced either by the variables of the system itself or by exogenous variation of the system's parameters. The change may consist of a shift from repeller to attractor or from a fixed point to a fixed motion. These are to be sharply distinguished from changes in magnitude or duration of the existing solution.

Consequently it becomes important to determine the domain of attraction or repulsion: this yields a kind of catchment area for any particular type of motion. A single system can have more than one catchment area. Thus a differential limit cycle will ordinarily have an unstable (repeller) fixed point but a global attractor, so that between the two must lie at least one closed boundary which constitutes a stable fixed motion, or limit cycle.

In order to clarify the fundamental paradox of chaotic attractors, I propose first to consider a familiar type of cyclical, differential system. Like meteorology, economics can often yield good short-term prediction (not, however, for the recent crash), not so good for medium-term and more or less worthless long-term. The great mathematician John von Neumann once maintained that, given sufficient funds for global weather data collection and a mega-computer, he could provide detailed, accurate prediction of weather over time and space. He based himself on the fact that, by contrast with economics, we know very well the laws of fluid

dynamics. Hence, given the initial conditions globally, we could compute the flow in space and time of the weather. It was good sound methodology and I long believed he was right, though his assertion remained untested because of the scale of costs. I now think that empirically his conclusion was wrong in principle. To understand why he was wrong will, I hope, illuminate how fundamental for applied mathematics was the discovery of chaos and why it has unleashed a flood of new mathematical work. Mathematically, of course, von Neumann made no mistake: if we have a correct model, then, given the initial conditions, the sequence of events is necessarily calculable and therefore any future event must be predictable exactly - the entire logical basis of mathematics rests on this.

The rot began with another great mathematician. Poincaré, who, faced with some unpredictable astronomical irregularities, spoke of homoclinic tangles, or non-wandering sets, and doubted that they would ever be understood. The great American mathematician, George Birkhoff, carried Poincaré's work forward by developing ergodic theory. It was a pupil of his, E.N. Lorenz, in the 1960's, who really set the subject in motion - significantly by computer simulation of atmospheric irregularities. His system really posed the basic paradox: three nonlinear differential equations with specific initial conditions yielded unpredictable results. This seeming blunt challenge to mathematical logic has produced an explosion of theorizing, which goes a long way, but not all the way, to resolving the paradox.

Is this merely a storm in a mathematical tea cup, or does it raise a serious problem of wide application - in particular to economics? I urge you to think about it: how serious it is remains an open question. Scientists have, by taking certain types of issue, produced stunningly successful results. But a large area of natural phenomena are beyond the reach of natural science - the fall of a leaf, or turbulent flow of air, etc. In particular, as we all know, economic time series are highly erratic, so we must, it seems to me, take these new lines of investigation seriously. I remind you that probabilistic analysis is a confession of partial ignorance.

To come more specifically to economics: the great depression gave rise to two theories, the Kalecki and the Hansen-Samuelson models. Both were linear but such models are incapable of explaining the continued existence of oscillations. Frisch misled a generation of investigators by resolving the problem with exogenous shocks, whereas already in the 1920's van der Pol had shown (as Frisch should have known) that a particular form of nonlinear theory was the appropriate solution. His solution leads to a limit cycle.

For differential systems to give rise to chaotic attractors, the dimension must be 3 or more. With a cycle model formulated as two first order equations, a time-varying parameter then raises the dimension to three. Suppose that there is a simple sinusoidal investment function influencing x , giving the system as $\dot{x} = \phi(x, t)$. x has a natural period of $T_N = 2/\omega_N$. If the forcing function

has a period $T_t = T_N$, the result is a torus with the point in state space returning to any position once each cycle. By taking T_t shorter and shorter one gets a succession of points, which can be registered on a Poincaré section.

Proceeding thus one can get a continuous curve for the conjunction of the two cycles. This complicated and confusing experimental result has been much clarified by a Tübingen mathematician, Rössler. One uses a Möbius strip, top to bottom, sides reversed, giving a one-sided surface. For an appropriate T_t , one can analyze the result with a 3d model.

Following a trajectory, an outer loop expands to the right and upward; then folds over and descending contracts to an inner loop and so forth. I give a 2d projection which shows plainly the two cycles (the simulation is from a related model). It will be noticed that the lines tend to get

grouped with empty spaces in between; if the simulation is continued these bands become dense, i.e. ergodic. This result was first predicted by Birkhoff and first simulated by Shaw.

If the simulation is carried on long enough the dark bands become solid black. In fact they represent flow lines at different levels thus.

In my 3d drawing they are represented as a flat surface but in fact they are very thin layers, like French pastry. The loops get folded and arrive on different close layers thus.

They are called Smale Horseshoes. Astonishingly enough these experimental results have been found to conform to a branch of abstract number theory - Cantor Sets, also now called fractals. Take a line; divide it into three equal parts; then do the same repeatedly with the two end parts. The result is like Smale horseshoes.

With the foregoing superficial account in mind, one can attempt to resolve the paradox that a deterministic system can be unpredictable. The multiple surfaces expand, fold over and contract. Consequently one sees that points initially close together and increasingly diverge over time, both in direction and distance travelled. Therefore since one can never determine an exact initial condition in applied work, one increasingly loses accuracy. By contrast mathematically one can state exact initial conditions and calculate precisely the resultant flow, but this is not so for empirical analysis. Secondly on each circuit the flow goes on a different level so that the resultant dense bands defeat precise detection of subsequent positions.

Chaotic attractors introduce a remarkable, awkward new conception. Thus originally there was the fixed point

attractor, or stable equilibrium, sensu strictu. Poincaré generalized the conception of equilibrium to include equilibrium motion in the form of limit cycles. But for chaotic attractors there is neither a fixed point nor a fixed motion. This so violently generalizes the notion of equilibrium that the original conception is more or less lost. Can a system which never repeats itself over time still be considered in 'equilibrium'? It is a bounded motion, a non-wandering set.

We left Schumpeter impaled on a contradiction: his innovation produced decline instead of growth. The aim of his theory was precisely the reverse and that, in fact, is its superiority to all other dynamical and cyclical theories. Our perverse results arose because A, the investment, remained constant. There was some justification for this since his innovational investment had returned to zero on completion, but he, of course, escaped our dilemma by assuming full employment at the new, higher productivity. Our results are in conflict not only with Schumpeter but with the economic history of capitalism. The economy is characterized by growth, not decline, but interrupted by spectacular collapses as the 1880's, the 1930's and now. Therefore particular attention must be given to innovations, or technical progress. This is most interestingly done in a discrete-time, nonlinear model: it is a striking fact that such a very simple model can give rise to almost the entire gamut of erratic or chaotic solutions. There are many significant time lags in economics. e.g. breeding time for

livestock, trade union wage negotiations, or construction time for new capacity, which is the one I shall consider.

To keep matters simple, I take the case of a uniform lag, thus keeping to one dimension.

Consider the case of a group of related, innovatory methods of production or new products. There is considerable agreement amongst economists and economic historians that the introduction of a successful innovation is well represented by the logistic function, either

$$\dot{x} = ax(1-x/\bar{x}), \text{ or}$$

$$x_{t+1} = ax_t(1-x_t/\bar{x})$$

where $a > 0$ and $\bar{x}$ is the maximum level for the innovation.

Initially it appears in a rather crude form and is unfamiliar; then as it gains acceptance it reaches its maximum rate of expansion. After that there is a slow deceleration of the process of absorption as it approaches the complete integration into the economy. The resulting curve has the shape of an elongated s with a non-negative slope. Making a change of variable to nondimensional form, with $\bar{x} = 1$, the result is

$$x_{t+1} = ax_t(1-x_t) = ax_t - ax_t^2.$$

The result is that x maps into a unimodal, or 'one hump' function with its maximum at $1/2$. For $a \leq 3$, it is an attractor, and for small a it is a monotonic attractor. Its fixed point where $x_{t+1} = f(x_t)$ is $x^* = 1-1/a$. As larger a 's are considered, the max point rises above the 45 degree line and a stable oscillation arises where $-1 < f'(x) < a$, as is exemplified in the accompanying figure. For $a = 3$, $f'(x) =$

-1 and the result is a neutral cycle in the neighbourhood of $x = 2/3$.

For larger values of a , $f'(x)$ becomes < -1 and the fixed point generates an unstable cycle. This is illustrated in an enlarged section in the second figure.

As a is increased towards 4 and $f'(x)$ sinks further below -1 , one finds a bewildering collection of all forms of behaviour - oscillators, both attractors and repellers, as well as aperiodic motion, both stable and unstable, all mixed together, in a word, chaos. This is illustrated in the following diagram:

As a approaches 4 the variety of succeeding bifurcations leads to the conclusion that it is helpful to write $f(x,a)$ since the behaviour depends on both the variable and the parameter. This behaviour has been the object of close study and some, but incomplete, light has been shed on it. I quote from Thompson and Stewart commenting on this one dimensional mapping: "The most ambitious task would be to detect the sets of values of a for which the motion is periodic, ergodic or mixing (chaotic) respectively. In this way it would be possible to distinguish ..... Unfortunately this is in general impossible, even for very simple maps such as the logistic map, so we must content ourselves with more generic results on the size of the subsets. The fundamental result in this field is the following:

Choosing a value of the parameter a in the prescribed range there is a non-zero probability of mixing behaviour and a non-zero probability of ergodic behaviour. Since it is also clear that periodic motions occur with positive probability we can say that the parameter range, or control space, is partitioned into three subsets whose Lebesgue measures are all positive."

Difference equations lend themselves naturally to being investigated sequentially or iteratively and in this way much can be learned of how and why and when they bifurcate and thus mix types of solution. Therefore it is helpful to look at the iterates of the function. If $x_{t+1} = f(x_t)$, $f(x_0) = x_1$

$f(x_1) = x_2 = f(f(x_0)) = f^2(x_0) = f(x_1)$, hence $f()$ is a different function from $f^2(x_0)$.

Hence the first iterate of $f(x)$ with the appropriate variable must equal or pass through the same point as $f^2(x_0)$.

$$f^3(x_0) = x_3$$

$$f^m(x_0) = x_m$$

The study of these iterative maps has been intense in recent years and has yielded some insight into irregular or chaotic behaviour. The number of iterates increases very rapidly as a approaches 4; beyond that point arrive the complications of Cantor sets.

For $a > 3$ the number of periods doubles successively as a increases (see below). The doubling can be seen as a function of the parameter a .

logistically and its effect on demand is multiplied by m , the output-capital ratio. The saturation level for any one burst of innovations is n . The current demand for goods, both for input and consumption is d per unit of output, q . Thus demand consists of current demand plus demand from the innovational investment, with output adjusting by a proportion, f , to any discrepancy between output and demand, thus

$$q_{t+1} = q_t + f(m(akt(1 - kt/n) - kt) - (1-d)q_t),$$

$$k_{t+1} = akt(1 - kt/n).$$

Using vaguely plausible parameters, the following simulation results:

Initial output is taken to be in equilibrium, so that as Δk rises, so also does output, with a modest lag. But then as innovation reaches its saturation and Δk drops to towards zero, output falls back to its initial level. Therefore innovation, by itself, stimulates a boom but also a subsequent collapse. Even this conceals its effect on

employment (taking the innovation to be labour-saving). If inputs of labour per unit of output have fallen by 20%, so also will employment have fallen.

For the following innovative boom, one may assume a lower employment requirement, but also lower prices, higher real income and consumption. If real consumption per unit of output rises by less than the corresponding fall in labour inputs, there will be a decline in demand per unit of output. Consequently in the succeeding cycle the economy will relapse to a lower level of output and an even lower still level of employment. As the second wave of innovation is exhausted, output returns, not to its previous level but somewhat more than 2.6% lower than the initial value (with employment even lower), though this is not easily seen in the accompanying graph of the simulation.

Hence we see that innovation alone is not the explanation for a rising trend to output and employment, if with Keynes, we believe that output is controlled by effective demand. The only basis for the opposite

conclusion is that of the naive view that there is always market clearing for labour, i.e. full employment.

As Schumpeter maintained, at the end of an innovatory burst, a new one will commence (because of the exponential growth of new ideas). Thus a second boom and collapse is initiated and so on for as long as capitalism persists and technological conceptions continue to grow at around 4%. It is important to note that innovation is crucial to cycle theory because, in depression, with generalized excess capacity, there is no other basis for the new investment necessary to generate a new expansion. Hence Schumpeter is rescued by Keynes (though he would never have admitted it).

k goes to zero but output does not sink back to its previous level. The explanation for this must be sought in the continuing multiplier effects of exogenous real demands, which may continue after innovatory investment has ceased. In this curious fashion, Schumpeter's pungent dictum is validated: the cycle is simply the form in which growth takes place. Hence, as we all know, capitalism is creative, but in a chaotic way, generating both growth and unemployment. This has never been more evident than now, with the cybernetic revolution in technology in full swing.

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II: CLASSICAL DYNAMICS: THE CORN ECONOMY

Economists in the early nineteenth century were witnessing the dramatic onset of technological change. They emphasized the dynamic evolution of an economy in contrast to the neoclassicals, who concentrated on a more sophisticated market analysis, with the result that they lost an adequate picture of the functioning of the whole economy.

Since agriculture was still dominant, it is useful to keep to a single lag, the annual solar cycle, thus giving a realistic case of strict 'period' analysis. I shall discuss an isolated, one-crop (grain) economy.

Given a significant technological improvement in production, output, from a given amount of land and labour, rises and accrues to the producers as a profit. They then bid for more labour, raising the real wage. This increases the demand for food, raising its price. To increase production less fertile land is activated, the price of grain rises, partially offsetting the rise in wages. Malthusian doctrine predicted a fall in death rate, population increase, leading eventually to a return to the initial 'subsistence' wage. Likewise profits first rise, then sink back to their initial value. Thus all the gain from technology accrues to the rentiers. After over a century of technological progress, we know that this perceptive classical analysis was not realized.

There are two distinct roles in agriculture: the producer and the landowner, though they are often combined in one person. Rent goes to landowners and wages to workers. and I shall assume, for simplicity, that both are entirely consumed. Logically distinct is the producer, who pays rent and wages and owns the product and the profit, all of which is accumulated (saving only a portion changes the numbers but not the nature of the solution).

Consider the case of plenty of land of the same average quality, existing because of improved transport. This yields a constant technology with seed and rent a constant proportion, $\bar{a}$, of output, y , as is also labour employed, $l = \bar{a}y$. The wage fund is wl , with a surrounding, indigenous, population supplying labour at a constant $\bar{w}$. Then one can state a dynamical Say's Law: $ay_t + waly_t = y_t - 1$, from which $\Delta y/y = 1/(\bar{a} + \bar{a}\bar{w}) - 1$. Thus the result is close to Marx and von Neumann's constant growth rate.

If there is full employment with a constant growth rate of the labour force, $N(t) = 1(t)$, with $\Delta N/N = \bar{g}_N = \Delta y/y$, then $y_t = (a + w\bar{a}l)(1 + \bar{g})y_t$, so that the real wage is determined as $w = 1/\bar{a}l (1/(1 + \bar{g}_N) - \bar{a})$. In a completely competitive market, if w were higher than this, there would be unemployment and w would fall, and if it were lower, demand would exceed supply and w would rise. In this way the classical economists posed a most basic dynamical problem, without, however, getting the right answer.

There were some serious mis-specifications.

1) Labour Supply. It was not constant, nor was it infinitely elastic in the Malthusian way, thus making the

real wage constant. The real wage was slow to rise at first, but in the second half of the century began to rise and has risen for over a century.

2) Technical Progress. They could have had little reason to foresee the high, steady, exponential improvement in the techniques of production, which out-stripped the growth of labour force. Thus rising wages could lead to exponential growth in labour supply but at a lower rate than technology. Thus wages did not fall, growth did not decelerate, profits hence remained steady and rents were not the ultimate recipients of technical progress. The high wages were both effect of technology and also cause in the sense that they led to a labour-saving bias in technology, thus yielding a high growth rate of labour productivity.

To put a complicated issue very simply: what I am proposing is that we follow the example of the Classical economists in searching for a basic constraint on growth, but that we substitute labour for 'land' (i.e. natural resources). By virtue of technical progress, there has been a huge growth of output and of employment without coming up against effective shortages of land or natural resources. What has happened is that various shortages of labour have been resolved repeatedly by labour-saving innovations. This does not imply that it will always be so: we may run short of fuel or we may so reduce the need for labour, by computers, that it ceases to be a constraint.

Thus with a fixed technology and real wage, there is a constant growth rate if Say's Law holds. Thus if $a = 0.2$ and $a_1 = 0.7$ with a unit real wage, the growth rate is 11%

per year. The profit rate is $(y_{t+1} - y_t) / y_t$ and is equal to the growth rate. In stagnant economies all or most of the profit (or rent) is consumed and the wage is conventional, presumably determined by historical evolution. If this wage is high enough to lead to growth in the labour force, one finds the common situation in the LDC's of unemployed surplus population.

Now consider a somewhat different aspect of an isolated agricultural economy. Assume that the corn production is embedded in a wider economy so that both consumers and producers can switch to alternative goods. Suppose that the demand for corn is a nonlinear quadratic:

$$q_t = -a p_t + b + c q_t^2 + d, \text{ and that output is linear:}$$

$$q_{t+1} = d p_t - e.$$

Measuring in deviations from the fixed point (constants zero),

$$q_{t+1} = - (d/a) q_t + (c/a) q_t^2.$$

Such an equation can easily become a chaotic attractor, depending on the values of the parameters and on initial values as well. In view of the many varieties of supply and/or demand curves, it is not unrealistic to investigate various parameter values. With $d/a = 1.05$ and $c/a = 0.60$, a discrete limit cycle results, as is shown in Fig. 1, for 40 years and with various initial values.

Fig. 1

Fig. 2

Fig. 3

From an inspection of Figures 2 and 3, one finds that the limiting values, from inside and from outside, are not identical. Unlike differential limit cycles, these motions approach not a closed curve but an annular band. Once the point has reached the limit band, which it does in finite time, unlike the differential asymptotic approach, it will remain there indefinitely unless disturbed. Different initial conditions result in somewhat different behaviours, so that there is a slight scattering, which destroys precise periodicity. Of course an approximate periodicity is guaranteed by the market structure under the influence of the solar period.

Nonlinear supply and demand curves can easily give rise to a cubic form for $q_{t+1} = f(q_t)$. The dramatic consequences which follow from such a situation were totally missed by all the earlier investigators of lagged markets: the whole concept of deterministic chaos was unknown, in addition to which, in the absence of computers, one could not easily investigate the many possible consequences of all combinations of parameter values.

In Fig. 4 is given the behaviour of the first iterate around the fixed point for the equation

$$q_{t+1} = -2.5q_t + 3.5q_t^2.$$

This is highly erratic behaviour indeed and it all occurs in the complete absence of any exogenous shocks.

Fig. 4

In Fig. 5 is given for the same equation the results with the parameters altered to -3.0 and +4.0. Evidently the result is truly chaotic, deterministic behaviour from one of the simplest of all nonlinear dynamical models. Of especial interest is the fact that though it is ever-varying in shape, it never leaves a definite region. One can consider it as a stupendous generalization of a stable motion, as that is a generalization of a stable fixed point. This is why it is called an attractor and it constitutes a non-wandering set of points.

To see just how irregular such a model can be, I reproduce in Fig. 6 the resulting time series for the years 0 to 20, 80 to 120, and 120 to 160. From statistical analysis there is, as far as I know, no way of determining the structure of the model. More particularly, if this model had been subject to exogenous shocks, there is as yet no way to separate the deterministic from the shock effects. Until now such a series would have been regarded as dominated by shocks.

Fig. 5

Fig. 6

III: CLASSICAL VON NEUMANN

There is a remarkable direct line of descent from Ricardo to Marx to John von Neumann. Marx acknowledged his debt to Ricardo, but von Neumann knew no Marx and had no interest in the Marxian analysis: why, therefore, is there a connection? In my view it is because John von Neumann saw more deeply than others into the nature of the economic system and therefore arrived at an analysis with certain parallels with Marx. Be that as it may, he had some conversations with Kaldor in the summer of 1931, in the course of which he rejected Neoclassical economics and outlined his conception of how to formulate economics, stating that it should be analyzed like a slave economy, something like ancient China! This is the connection with Marx and Ricardo: a fixed real wage.

In 1932 he gave a seminar on his theory at Princeton, with no evident effect. Then he repeated his proposals at the Karl Menger seminar in Vienna, and from that was published "Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes" in 1938: it is arguably the greatest single advance in economic theory this century.

With its stunning generality and power, it establishes the existence of the best techniques of production to achieve maximum outputs of all goods at the lowest possible prices, outputs growing at the highest possible rate. Thus he transformed general equilibrium from statics to dynamics. Given the real wage, even with a constant technology, there necessarily results a surplus in

production (this is the Marxian connection), so that if there is to be dynamical equilibrium of supply and demand, cost and price, the economy must grow at a determinate constant rate with a constant rate of profit/interest. He dealt with n goods, m processes, and n prices, but I can save you, and me, much mental stress by dealing with one good and one price, and one real wage - our corn economy (this is the Ricardo connection). He implicitly applied it to a modern economy, but it really only fits a simple agricultural society. Furthermore, for obscure reasons, he formulated it as a lagged system, but this really only fits the common solar cycle, and hence the Classical dynamic. With seed, rent and wages constant proportions of output, the growth rate is determined, if, as in von Neumann, all the profit/surplus is invested in growing output. The available land of equal fertility is unlimited as is the supply of labour, as in von Neumann. In effect one produces employment by producing the real wage. The result is thus the same as in the previous analysis (II).

Along with its greatness, the von Neumann model has two crucial, basic defects: The infinite supply of labour and the constancy of technology are in glaring contradiction with the realities of the modern, industrialized economy. It is urgent that these two defects be removed in order to apply the theory to contemporary society. The first defect I shall deal with mainly in this section, reserving technological complications for part IV.

The total labour supply grows at a constant rate, g_N : it is therefore unproduced, a rent good like land, and its

price is determined by scarcity, not cost. If the wage is low in relation to productivity, then profit and growth rates are high, leading after a time to rising wage rates and falling profit and growth rates. This is a self-regulating, feedback mechanism. It is the simplest way to explain the extraordinary fact that labour supply, employment and output have multiplied themselves many times over in a century without serious misalignment. Labour productivity rises at a constant rate: the real wage may rise more rapidly or less rapidly yielding either a rising or falling unit labour cost, u . This in turn will alter the growth rate of output and employment.

The model, with Say's Law, gives

$$y_{t+1} = y_t / (\bar{a} + u_{t+1}), \text{ where } u_{t+1} = (a/w)_{t+1}$$

i.e. unit wage cost, proportional to the share of wages.

$v_t = l_t / N_t = a y_t / N_t$. Let $\Delta w/w = f(v) = b(1/(1-v) - c)$, $b > 0$, $c > 0$. b alters the sensitivity of the wage to the employment ratio, v .

$$u_{t+1} = (1 + f(v) - \bar{g}_a) u_t, \text{ where } \bar{g}_a = - \Delta a / a.$$

$$\Delta v/v = \Delta 1/1 - \bar{g}_N = \Delta y/y - (\bar{g}_a + \bar{g}_N). \text{ Hence}$$

$$v_{t+1} = (1/\bar{a} + u_{t+1}) - (\bar{g}_a + \bar{g}_N) v_t,$$

$$u_{t+1} = (1 + \bar{b}(1/(1 - v_t) - \bar{c}) - \bar{g}_a) u_t,$$

along with y_{t+1} .

Assuming a 'period' of one year, Fig. 1 gives for reasonable parameters the phase portrait of u and v over 50 years. They evidently oscillate with a constant periodicity which is moderately stable dynamically. There remains the problem of the inapplicability of a single lag system to an industrial economy. Perhaps the best justification lies in the fact that it is the simplest way to see the basic logic of a system. It is strange that von Neumann chose to use a lag system in preference to the more familiar differential type.

Fig. 2 shows what is not evident in the phase portrait - the sense in which a nonlinear difference system produces endogenously erratic behaviour. One sees that each cycle is different, though only moderately; from statistical analysis one could not predict turning points, rate of growth or decline, except very roughly.

In Fig. 3 are simulated just over two complete cycles covering 26 years. It shows that the system generates cyclical growth, with each cycle being moderately different from the other: each peak is different, each trough also, and no two depressions are alike. The long run average of such a system is steady-state growth at 3%: it grows enough on the average to pay a 2% increase each year in the real wage and to provide employment for 1% more employees. Thus some of the time the economy is growing faster than at 3% and at other times necessarily less, and similarly with

wages, part of the time above 2% to be balanced by less than 2% at other times.

The growth rate equals the profit rate, which is first too high and then too low. With a perfect capital market this

Fig. 1

Fig. 2

Fig. 3

would also be the behaviour of the rate of interest and the return on investment. Thus all that elaborate, traditional capital theory is unnecessary and gives the wrong answers. This is, of course, a system with no durable capital goods, which von Neumann treats as joint products. Durable goods complicate enormously the problem but also help to obscure certain central aspects of the problem.

Von Neumann's system was dynamically unstable; his fixed point growth was an unstable equilibrium, so that the slightest deviation from it would lead to ever further departure. This is serious since an unstable point is the one place no actual system would ever be found. The foregoing model corrects this fault, only to create another. Since it is stable, in the long run it will cease to oscillate. Therefore it cannot, by itself, explain the existence of cycles.

What is required is a stable, but not asymptotically stable system, i.e. a discrete limit cycle, a closed annular band within which the system will remain forever unless disturbed. This means a system which is unstable around its fixed point (equilibrium steady state growth) and hence will grow in its amplitude of oscillation, but which is stable in its outer region and will experience diminishing amplitude there. A high employment rate, v , means a high growth rate of wage cost or wage share and low v means a low one. A high growth rate of wages means a deceleration of growth and leads to a decline in the employment rate: the resulting low v means a low growth rate of wages and restores growth and ultimately brings us back to a high v .

To illustrate clearly the nature of the problem, consider the simplest formulation, i.e. in dimensionless form and in deviations from equilibrium growth. The system in matrix form

$$\begin{Bmatrix} v_{t+1}/v_t \\ u_{t+1}/u_t \end{Bmatrix} = \begin{bmatrix} a(1.0 - v_t) & -1.0 \\ +3.258 & 0 \end{bmatrix} \begin{Bmatrix} v_t \\ u \end{Bmatrix}$$

where a is the unique parameter open to choice and v_{t+1}/v_t and u_{t+1}/u_t each represent unity plus growth rate. The off-diagonal terms being of opposite sign, the system will oscillate. The dynamical stability is determined by the trace (sum of diagonal terms). The one diagonal term is positive for small v around equilibrium: this produces instability with a growing amplitude. For large v the term becomes negative making the system stable with a diminishing cycle of growth rates.

Consequently between the two regions there must be a barrier since no stable point can ever coincide with an unstable one. In differential equations this takes the form of one or more curves enclosing the equilibrium point. For difference equations the situation is more complicated and more interesting, as I shall try to illustrate. Enclosing the equilibrium there must exist at least one bounded, annular ring within which the cyclical limit set must lie. The limit set will cluster but not on a single curve because of the irregularity of nonlinear difference equation solutions. Since the solutions are discrete points, the approach to the limit set is not asymptotic but occurs in finite time, successive points may jump over the ring from outside to inside, but once within the band appear never to emerge.

By considering a series of values for the parameter a , one discovers first a stable cycle leading to a unique equilibrium, fixed point. Then for a higher value the solution becomes a slightly irregular limit set. For a still larger value the limit set becomes sufficiently irregular that it becomes dubious to call it a single closed band. For a small further increase in a one sees that the limit set is imploding inward towards the fixed point. The next step is a completely chaotic attracting region which fills most of the region around the fixed point. Thus one arrives in successive steps to a whole attractive region in which the phase trajectories appear to move in a seemingly random fashion.

Commencing with $a = 2.0$, one has in Fig. 4 a very stable cycle leading to a fixed point. Increasing a to 2.8 yields a bifurcation to a limit cycle set (Fig. 5). That such a solution generates a moderate but definite irregularity is shown in Fig. 6. With a increased to 3.4, Fig. 7 begins to exhibit such a marked irregularity that it is difficult to consider it a single closed band. For $a = 3.5$, it is clear that one can no longer consider the set as contained in any closed band; it is expanding in various ways towards the interior (Fig. 8). When a is increased to 3.6, there is a full-fledged chaotic attractor: the annular band has simply grown to fill much of the enclosed space (Fig. 9). The value $a = 4.0$ gives an even more complete filling of the entire region around equilibrium (Fig. 10). This means, of course, that the corresponding time series is highly erratic or chaotic (Fig. 11).

Fig. 4

Fig. 5

Fig. 6

PHASE

PORT

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

IV: GROWING IN SHORT AND LONG WAVES - SCHUMPETER

John von Neumann was a first-class mathematician with little or no interest in social or economic problems. In an inexplicable leap of the imagination he perceived that his highly original game theory could be used to illuminate economics. Significantly his theory bore at some points a resemblance to Marxian theory, presumably because both probed unconventionally and deeply into the dynamics of an economy.

This brings me to Marx. In my view a profoundly original and important social and economic thinker. In this piece it is his view that I am following: specifically that the evolutionary nature of economic society is brought about the perpetual search for profit. He was not trained as an economist, but becoming convinced of the primacy of economic attitudes, he studied the subject and, wisely, became a Ricardian. However, in a fundamental way, he departed from Ricardo by treating profit as a simple surplus which was a crucial and permanent feature of capitalism. Thus he took the best master and ignored completely the newly developing Neoclassical marginalism. To give an adequate account of his theory would mean a detour from my present purposes.

Instead, I shall discuss his most eccentric but important follower, Schumpeter. He was much influenced by Marx: whether he arrived at his position independently or whether he was influenced by the two famous Marxists, Otto Bauer and Rudolph Hilferding, who were both in the Böhm-

Bawerk seminar with him. I do not know and cannot imagine why I never asked him.

He fully believed in the Neoclassical analysis of the essential functioning of the price-market mechanism, but at the same time he departed from this essentially static analysis on the basis of a Marxist analysis. He held that technical progress demanded a thoroughly different formulation which made many of the Neoclassical conclusions faulty. Specifically he concluded that supply and demand for capital was not the clue to the determination of rate of interest/profit: rather it was innovation as a dynamic disturbance which determined the rate, and in its absence the rate would be zero. Like Marx he held that there was an enduring search for innovations which would restore eroding profit. These innovations altered the equilibrium continually, so that the economy was rarely if ever in equilibrium, though always moving towards it.

To formalize this one imagines an economy in a state of permanent evolution through technical change. In the boom high wages reduce profit and growth: this leads producers to urgent search for cost-reducing, labour-saving innovations. The source is new ideas: if we measure new ideas by the number of learned journals in the world, they have increased rather steadily at over 4% for more than two centuries.

Accepting some such fact, there remains the problem of how does such a steady flow get altered into cycles. A plausible analogy is offered by the so-called Roman fountain. Water flows into a basin; when the basin is

filled to a certain point, a syphon is primed and it emptied the basin at a greater rate than the inflow of water. If the flow in is constant, a periodic outflow of water results: if the flow varies, so does the periodicity.

New ideas arise steadily; when enough have accumulated, a group of innovations are initiated, slowly at first, then with acceleration as the methods are improved and adapted to diverse uses. This constitutes a logistic development and there is considerable agreement that this characterizes the typical course of an innovation. Such a course of events is helped by demand effects: some investment must precede the introduction of an innovation; investment stimulates demand; increased demand facilitates the spread of the innovation. Then, as all the uses of the innovations are fulfilled, the process decelerates towards zero. This explains what Schumpeter called the 'swarming' of innovations, though he rejected the demand effects, since, in the Neoclassical manner, he assumed full employment throughout.

In his final formulation he proposed a 3-cycle scheme: Kitchins, Juglars, Kondratieffs. The Kitchins were short, well observed and well understood; they relate to the stocks cycle and innovations play no role.

The Juglars were of the order of 10 years, much studied, but he related them to innovations. He was the first major economist to recognize the serious significance of Kondratieff's study of the historical statistics. He quite correctly related the evidence for long waves to the fact that the most important innovations required 30 to 100

years to become fully integrated into the economy, e.g. steam, steel, railways, steamships, electricity, the internal combustion engine. He somewhat recklessly applied his innovation theory to both the short and the long waves, and I shall follow him in this.

Lord Kahn in his path-breaking multiplier article formulated the generation of income, in terms of a lag, i.e. a dynamic difference equation. Following him, consider an economy with a propensity to spend a proportion $\bar{a}$ out of receipts of q . Reckon in deviations from whatever is happening in the economy - growth or decay, cycle or random variations. What will be the added effect on output, expenditure and income of an innovative 'swarm' of roughly 10 years? q_t is current rate of output, k_t is new innovative capacity and $\kappa \Delta k$ is the investment expenditure necessary to create the new capacity:

$$q_{t+1} = \bar{a}q_t + \kappa(\Delta k).$$

k is in the form of a logistic, i.e.

$$k_{t+1} = b k_t (1 - k_t/c), \text{ so that}$$

$$q_{t+1} = \bar{a}q_t + \kappa b k_t (1 - k_t/c) - \kappa k_t.$$

Such a roughly 9 year swarm is simulated in Fig. 1.

Fig. 1

Notice that neither the logistic nor the multiplier is cyclical but that the conjunction produces a wave, and that if each logistic innovative burst is followed by another, there will result a cycle.

Schumpeter assumes that innovational investment produces both the Short and the Long Wave, and implicitly assumes that the two are independent. This poses a complex problem, which he ignored and I shall do the same. Assume an existing cycle of about 8 years and ask what would happen if a 50 year logistic occurs in such an economy. Each has a given structure but each is affected by the other.

Given a short cycle of 8 years with constant amplitude:

$$q_{t+2} - 1.414q_{t+1} + 1.0q_t = \kappa \Delta k, \text{ with}$$

$k_{t+1} = 1.15k_t (1 - 0.0020k_t) + 0.080q_t$, where the parameter 0.080 allows for the influence of q on k . Taking account of the influence of k on q , one gets

$$q_{t+2} - 1.141q_{t+1} + 0.96q_t - 0.75k (1 - 0.0143k_t).$$

The resulting simulation is shown in Fig. 2. The short cycle is completely distorted by the logistic: first it is shortened and made milder: then it is destroyed altogether, becoming merely a slower growth rate. Finally comes the great boom at the point of most rapid innovational investment. After that comes the reverse process but always somewhat different from before. This kind of behaviour could have produced the statistical evidence first noticed by Kondratief, but largely ignored by everyone, except Schumpeter who made it a central facet of his final version of capitalist dynamics. The Big Ones like steam or steel.

would be a part of the short wave but last through several such, with great effect on the performance of each short

Fig. 2

Fig. 2a

wave. The mutual conditioning of investment and effective demand helps to explain how a disparate group of technical innovations become ordered into a common upsurge, a case of 'self-organization'.

In principle each wave returns to its initial level and shape as the logistic approaches saturation level. This, of course, is a serious defect. Schumpeter, like most Neoclassical economists, simply assumed labour-market clearing. Hence with improved productivity there must be greater output and income. Once one has understood that full employment is the exception rather than the rule, lower inputs mean lower demand and output.

To discuss this problem concisely, it is useful to rephrase it in continuous time, with differential equations. In the von Neumann equation let q_{t+1} represent a lag, not a time unit. Then $q_t = (a + u) (1 + \dot{q}/q) q_t$, so that $\dot{q}/q = 1/(a + u) - 1$, as before, but in continuous time.

$$\dot{u}/u = \dot{w}/w - g_a = \kappa(1/(1-v) - c) - g_a, \text{ and}$$

$$\dot{v}/v = \dot{q}/q - (g_a + g_N).$$

Distributive shares between wages and profits are

$$\frac{u}{1-a} + \frac{a+u}{1-a} \frac{\dot{q}}{q} = 1.$$

With $g_a = 0.015$, $g_N = 0.005$ and plausible values for the other parameters, a typical result is given in Fig. 2a. It is evident that this gives a growth cycle and thus solves Schumpeter's problem.

The model is still too simple because it assumes a constant rate of increase of productivity over a 50 year period, whereas the whole point is that productivity only

grows in consequence of new technology. Therefore one should rework the model making g_a rise only with the rise of innovative capacity, k . Therefore I take g_a as proportional to Δk , reaching a peak after 25 years and then gradually subsiding again to zero (g_N is set at zero to isolate the effects of technical progress). This is illustrated in Fig. 3. Labour's share falls as investment moves towards its maximum and then returns to its original position; the real wage, however, rises throughout. As average u falls, average v rises, only to fall back again as u rises.

Fig. 3

It is illuminating to plot v against u over a 50 year wave. Initially they move up and down in sympathy, but c cannot grow, so that v rises less and falls more than q . Then they

gradually return to their original motion. This is represented in Fig. 4.

Fig. 4

To complete this account I propose to give a simple, abstract view of the basic, essential structure of chaos, in the form proposed by Prof. Rössler of Tübingen. It is applicable to a wide variety of problems and is in dimensionless form, but I shall use the three variables, v , u , and k .

$$\dot{v} = -u - k,$$

$$\dot{u} = +v + \bar{a}u$$

$$\dot{k} = \bar{b} + k(v - \bar{c}).$$

The coefficients of v and u being $+1$ and -1 , the solutions are necessarily oscillatory and are in deviations from the fixed point equilibrium, which represents, given constant growth rates of productivity and labour force, a steady-state growth. Setting $k(0)$ equal to zero, or a negligible quantity, $\dot{v} = -u$ and $\dot{u} = v + \bar{a}u$, so that $\ddot{v} - \bar{a}\dot{v} + v = 0$. If $\bar{a} < 0$, the result is stable cycle around steady growth. However, with all parameters positive, an unstable

cycle results. As v grows and becomes greater than c , k becomes positive; $k(t)$ ceases to be negligible and the full system is operative.

The aim is to illustrate the gradual onset of chaotic solutions as one parameter is varied. Therefore one can set $a = b = 0.2$ and study the solutions as functions of parameter c alone. With $c = 2.1$ the result is a limit cycle (Fig. 5); it is also asymptotically stable but in an unusual way. From Fig. 6 one sees that the trajectory jumps from outside to inside the limit set, and that this limit set is an annular band, similar to the discrete-time case. The Figures 5 and 6 together show that there is an asymptotic approach by means of successive, slightly different cycles.

It is possible to give a rough, plausible explanation of the economic meaning of this so-called Rössler Band. It is clear that u and v can produce a cycle with somewhat realistic parameters. Then it is equally possible to choose values which give a cycle of growing amplitude. If as a result of such expansive behaviour, investment is heavily encouraged, capacity will accumulate rapidly, more rapidly than output grows, so that investment will eventually decelerate precipitating a decline in demand and the end of the boom. This introduces a new element with the consequence that the expansion will not repeat the same pattern, leading to a different period and amplitude. Two periods and two amplitudes can coexist as is demonstrated in Fig. 7 with $c = 3.1$: the aspect of band width is made even clearer. The band consists of a group of similar but not identical cycles. However, it represents not the

irregularities of a single cycle but rather similar cycles which keep changing so as to approach a final, unique limit cycle as illustrated in Fig. 8 for $140 \leq t \leq 170$ and $c = 3.1$. These evident band widths are the initial hint of the possibility of chaotic or aperiodic behaviour. For higher values of c , the number of types of cycles increases rapidly (Fig. 9). The trajectories tend to occupy the interior more fully; they implode inward in a manner reminiscent of the discrete-time behaviour. To generate chaos requires 3D in differential equations, but the 2D projections exhibit a distinct kinship with discrete-time solutions. The cycles constituting a single band vary only moderately from one another. However the multiplicity of bands indicates quite unperiodic yet bounded behaviour which is highly irregular. If c be increased to 5.7 (Fig. 10), true chaos is present with the ensemble of almost periodic bands filling most of the limit cycle region. For $c = 7.0$ only a small part of the region is left uncovered by a great variety of stable cycles (Fig. 11). This obviously produces a chaotic temporal statistic, some indication of which is given in Fig. 12 for four successive 15-year periods.

Without independent knowledge one could not extract the model from the time series, nor could one predict. Thus though the end state seems quite bizarre, we have arrived there step by step, which helps to lessen the 'strangeness' of our attractor. By varying a single parameter, one illustrates the astonishing generalization of stable equilibrium: from a fixed point to a fixed motion and

finally to a whole bounded region of non-wandering, non-repeating motions.

When Schumpeter said that the cycle is simply the form that growth takes, he implicitly destroyed a large part of the analytical structure of a whole era of economic theory. It means that the reduction of economic time series to trend, cycle and random shocks has to be rejected. One has to find a suitable, unitary model which will produce the perpetual evolutionary dynamics of the modern economy. His intuition of the essence of capitalism far out-reached the linear procedures of his time. Only now, over half a century later, are the appropriate tools available for embodying his 'vision'.

In joining with Frisch and Tinbergen to form the Econometric Society, he placed his faith in the happy combination of good theory, with good empirical foundations, the two related by the sound logic of mathematics. Alas, the theory was linear and inadequate for dynamics: the statistical techniques were linear and hence gave a misleading basis for theory: as for the mathematics, he could not even understand linear dynamics; and even the mathematicians had not yet discovered the chaotic dynamics so relevant to economics. Linear decomposition of time series may be quite useful descriptively but is false as a basis for models of actual behaviour. Basing himself on Wickseil's perturbed rocking horse, Frisch misled Schumpeter and a whole generation of cycle and econometric theorists by assuming it was necessary to have exogenous shocks to maintain the cycles and to explain the evident irregularity

of economic time series. After the discovery and analysis of chaotic attractors, we know it is not so: by means of a variety of particular types of nonlinear dynamics we can explain the persistence of cycles, their irregularity, the lack of exact periodicity, the multiplicity of cycles and the reasons why one cannot predict, even for completely endogenous, deterministic systems.

When he came to make his final statement, Schumpeter had to use the existing, linear methodology, a quite misleading tool. He spoke of 3 Kitchins to a Juglar and 6 Juglars to a Kondratief, but this is nonsense and contradicts his own 'vision' of the economic system. The Juglar cannot exist independently of the Kondratief and vice versa: one requires a single theory which will explain their mutual interdependence. The size of investments for the Kondratief will condition the Juglar and, conversely, the state of the Juglar will affect the Kondratief.

To exhibit the possibility of realizing Schumpeter's perception of the nature of the economic system, I take an abstract, discrete-time model:

$$x(t+1) = ax_t - b(yt-zt)$$

$$y(t+1) = bxt + a(yt-zt)$$

$$z(t+1) = cxt - d(yt)^3 + ezt$$

The variables are assumed to generate a positive, constant trend (as in the previous example) and are measured in deviations from that trend. Selecting suitable parameter values and taking successively three different values for c , three simulations for 100 'years' each are presented in figures 13, 14 and 15. All are recorded with the same

initial values, but after any initial effects have disappeared.

In Fig. 13 the Kondratief lasts approximately 50 years and there are 6 Juglars visible. In Fig. 14 the Kondratief is slightly under 50 years and contains only 5 Juglars. The long wave in Fig. 15 is nearer 60 years and appropriately includes 7 Juglars. From a careful inspection of these examples it is evident that both the long and the short waves are quite different each time, and this is partly due to their mutual interdependence. To me these time series look generically like economic time series, though none of the parameters derive in any way from economics. I would say this looks like the area where macrodynamics should be and perhaps always should have been. I failed to persuade Schumpeter to take linear dynamic models seriously, but I think I could have sold him this type of model.

One final point: I have not the space to treat an additional approach to chaos, i.e. an endogenous cycle subject to an exogenous cycle. Thus suppose a country has a van der Pol type indigenous cycle but which is significantly affected by a world trade cycle with a different periodicity. Such a non-autonomous model can give erratic, even chaotic, behaviour. In Figures 16 and 17 an endogenous limit cycle is subjected to three different exogenous cycles, to show the resulting irregular behaviour.

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 14

Fig. 13

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