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Massimo Morelli and In-Uck Park

EUROPEAN UNIVERSITY INSTITUTE, FLORENCE
DEPARTMENT OF ECONOMICS

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MASSIMO MORELLI

and

IN-UCK PARK

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Internal Hierarchy and Stable Coalition Structures*

Massimo Morelli
Columbia University & EUI

In-Uck Park
University of Bristol

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Abstract. *When an agent decides whether to join a coalition or not, she must consider both i) the expected strength of the coalition and ii) her position in the vertical structure within the coalition. We establish that there exists a positive relationship between the degree of inequality in remuneration across ranks within coalitions and the number of coalitions to be formed. When coalition size is unrestricted, in all stable systems the endogenous coalitions must be mixed and balanced in terms of members' abilities, with no segregation. When coalitions must have a fixed finite size, stable systems display segregation by clusters while maintaining the aforesaid feature within clusters.* (*JEL* Codes: C71, D71)

Keywords: Stable Systems, Abilities, Hierarchy, Cyclic Partition.

1 Introduction

Circumstances abound in which individual agents interact via the organizations they choose to belong to. From each agent's perspective, the consequences of joining an organization or another are determined by (i) what kind of outcome will be generated by the interaction between the organization she chooses to join and its rival organizations, and (ii) what will happen to herself within the organization under that outcome. The second aspect is likely to be determined by the internal structure of the organization and the agent's position in it.

In a political setting, for example, politicians form parties and members of each party decide on the party line and on the campaign strategy, given the perception of their strengths and the opponents' characteristics; then, the election outcome will be determined by what kind of parties have been formed and their relative strengths; and finally, the members of the winning party will be allocated a role depending

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on their relative positions within the party, which will determine their payoffs. In this and other examples (e.g., gangs and entrepreneurial organizations), the agents' *ranks* within the organization appear to be an important factor in determining their final payoffs.[1]

Understanding what determines the number and composition of coalitions (e.g., party systems, market concentration, economic and political integration) has been a recurrent focus in many strands of literature (discussed below), but, to the best of our knowledge, there is no systematic existing work on the relationship between such *horizontal* segmentation incentives and the *vertical* structure of each endogenous organization. We believe that, especially in contexts in which the relevant agents are heterogeneous in ability, studying the interplay of these two dimensions could be very important.[2] This is what we do in this paper, through a cooperative game theory analysis that yields a number of sharp results in an institution-free environment.

The first key assumption of our model is that the relevant agents[3] have heterogeneous observable abilities, and the aggregate strength of a coalition depends on the total ability of it's members. Second, we assume that each endogenous coalition must have a vertical structure, i.e., the coalition members must be ranked, or assigned to different tasks of rankable importance, and payoff shares must be non-decreasing in rank. In other words, the comparison of total abilities across coalitions is assumed to determine the coalitions' relative power (probability of winning or market share, depending on the application), whereas payoff division within each coalition is assumed to depend on the members' ranks in the vertical structure of the coalition. Then, given that abilities are observable, it is verified that competition between coalitions ensures that each coalition will endogenously assign internal ranks following the ability ordering.

For analytical tractability, we assume that there are a countable infinity of relevant agents, and that the fundamental distribution of abilities will be captured by a single parameter that determines a geometric distribution of abilities. The role of

[1]Given that our focus is on the relationship between internal structure of coalitions and the competition between the endogenous coalitions, our model and results will relate more to the formation of *competing* parties, firms or gangs, than to the formation of clubs and jurisdictions, since typically club and jurisdiction formation models are about sorting or matching preferences (for example on local public goods), and the vertical differentiation dimension is not considered.

[2]For example, the (small) literature on party formation focuses almost exclusively on the incentives that different institutional systems provide to form parties to represent different (horizontal) segments of the voters' population, whereas the impact of internal organization of parties on the stability of different party systems is not studied (as a notable exception, see Persico. et al., 2008) and could be quite important: Intuitively, the choice between becoming the leader of a new party or remaining at a lower rank of an existing party must depend on how the different ranks are treated.

[3]We use the term "relevant" because in each application there could be agents like voters (in the political application), consumers (in the industrial organization application) and victims (in the criminal organizations application) that are important in general but not relevant for the decisions to merge, split or form strategic alliances.

hierarchy for internal payoff division will also be captured by a single parameter, to be interpreted as the fractional drop in the relative payoff from one rank to the next, which will be our measure or proxy for vertical inequality. The expected utility of each member of each coalition will then depend on both parameters, one through her internal rank and the other through the expected power of the coalition.

The first part of the paper studies the conditions for the stability of different partitions of players (or coalition structures) as a function of these two parameters capturing the distribution of abilities and the distribution of payoffs within coalitions. In particular, the fixed parameter capturing vertical, intra-coalition inequality is assumed to be determined by the ratio of the marginal contributions of two consecutive ranks to the surplus achievable by the organization. In this part, the analysis follows the logic of core stability of NTU games, in the sense that payoff division or imputation rules are treated as given. Thus, the focus is on coalition formation only. Later, in the second part of the paper, we examine what happens when vertical inequality is endogenized.

The results of the first part are summarized as follows: First and foremost, we establish that the lower is the inequality of payoffs across ranks, i.e., the lower is vertical inequality, the smaller is the number of rival organizations that can be sustained in a stable partition of the relevant agents. In particular, we show that the exact range of vertical inequality that supports a stable partition consisting of K coalitions ("K-partition," for short) is higher for a larger K; Second, all stable partitions must be *cyclic* in the sense that each of K coalitions consists of every K-th player in ability ordering, and hence no two players in any K consecutive ability types may belong to the same coalition; Third, the reward of every rank in an organization belonging to a K-partition has to be between the rank's marginal contribution in a K-partition and that in a $(K-1)$-partition.

In the second part of the paper, we move on to characterize the set of "strongly stable systems," i.e., the levels of vertical inequality and coalition structures that survive coalitional deviations in an environment in which the players in each coalition can choose any level of vertical inequality they wish. Allowing for this type of endogeneity of vertical inequality, we obtain a very tight result: A system is strongly stable if and only if each coalition is cyclic and every agent receives her "marginal contribution"; given that the ability of the agent occupying each rank in the hierarchy decreases in K, the number of coalitions, it must be the case that the marginal contribution of each rank also decreases in K, implying higher vertical inequality. In other words, endogenizing the vertical inequality norm allows us to show that partitions with *any* number of coalitions of players can be stable, together with an appropriate level of vertical inequality that increases in the number of coalitions. Specifically, we first characterize the set of *symmetric* strongly stable systems, i.e., the set of strongly stable systems in which all existing coalitions use the same vertical inequality norm. We then extend the characterization to *asymmetric* strongly stable systems, i.e., systems in which different coalitions may adopt different levels of vertical inequality, with lower vertical inequality prevailing in the stronger coalitions.

While all the above analysis is conducted without imposing any restriction on coalition size, obtaining that all coalitions in a stable partition contain a countable infinity of players, in the last part of the paper we study the case in which each coalition must have a fixed finite size. In this restricted setting we will show that players endogenously form segregated clusters of coalitions, still maintaining the cyclic structure of coalitions within clusters. This result provides interesting comparative statics in terms of inequalities within and across coalitions, approximating the empirically-observed, "segregation by skill" phenomenon (elaborated later).

The paper is organized as follows: after a review of the related literature, Section 2 introduces the model and Section 3 provides a complete characterization of the necessary and sufficient conditions for existence of stable partitions, preceded by a number of general results. Section 4 fully characterizes strongly stable systems with endogenous vertical inequality. Section 5 analyzes the case in which coalitions must be of a fixed finite size, and Section 6 concludes.

Related Literature Our paper highlights at least two features of endogenous formation of rival organizations that can be contrasted with the previous literature on coalition formation: (1) the more skewed is the allocation of payoffs, the more fragmented will be the rival coalitions to be formed; and (2) coalitions consist of members from widely dispersed ability levels and have therefore similar cyclic compositions. The latter feature of the unrestricted-size version of our model is in contrast with the separation outcomes that are prevalent in the literature on some other types of group formation, such as the important literature on clubs and jurisdictions providing local public goods, preluded by Tiebout (1956).[4] In our model, a particular type of segregation appears when we assume that each coalition must be of a fixed finite size.[5]

The literatures on social classes (Akerlof, 1997), partnerships (Farrell and Scotchmer, 1988), hedonic games (Banerjee, et al., 2001; Bogomolnaia and Jackson, 2002; Le Breton, et al., 2008; Watts, 2007), social status (Milchtaich and Winter, 2002), and organisation (e.g., Demange 2004; Garicano and Rossi-Hansberg, 2006; and an earlier work on firm formation by Legros and Newman, 1996), are all related in a

[4]A sequence of formal advancements in that literature can be found in Ellickson (1973), Westhoff (1977), Wooders (1978), Guesnerie and Oddou (1981), Greenberg and Weber (1986), Demange (1994), Konishi, Le Breton and Weber (1998), Casella (2001), Ellickson et al (1999), Jehiel and Scotchmer (2001), Zame (2009). These studies differ from ours because different jurisdictions provide different local public good quantities and the endogenous coalitions do not play a constant sum game. Moreover, typically agents are not differentiated in terms of ability.

[5]Segregation by skills (see, e.g., Kremer and Maskin, 1996) and increased inequality between plants rather than within plants (see, e.g., Gavilan, 2011), are recognized phenomena in the applied literature, and our fixed finite coalition size model (in Section 5) provides a clean theoretical interpretation of such phenomena based solely on coalition formation forces.

broad sense to what we do, but our approach is distinguished from these studies by two key aspects: *rivalry* among *endogenous* coalitions and *rank-dependent* internal rewards. As a result, agents face a dilemma between teaming up with more able people for a more powerful coalition and teaming up with less able people for a higher internal rank. Damiano, *et al.* (2010) consider a similar tension but in a setting where agents decide which one to join from a *fixed* set of coalitions, motivated by different contexts from ours.[6] Watts (2007), on the other hand, analyzes two separate settings, one in which agents desire to team up with higher ability members (under the "average quality payoff"), and an opposite one in which they desire to team up with lower ability members (under the "big fish payoff").

The paper makes a conceptual contribution also in political economy, and in particular to the literature on party formation, showing that even with similar institutions and preferences, different party systems can be stable, depending on the parties' internal organization.[7] For more distantly related work on trade alliance formation, see, e.g., Yi (1996) and Casella and Feinstein (2002).

We assume that groups can coordinate deviations, using a core-like cooperative logic.[8] Hence, parties, firms, teams or gangs are more natural types of coalitions that fit our analysis than countries/jurisdictions where agents individually decide whether to move in or out, like in Jehiel and Scotchmer (2001).

In hedonic games coalitional deviations are allowed, but players' payoffs are determined by the composition of their own coalition only. In our game the players' utility depends on the rank and the degree of vertical inequality in the coalition, as well as on the aggregate strength of the coalition, so it is not a proper hedonic game.[9] Our model can also be viewed as generalizing Gamson games (see, e.g., Le Breton. et al., 2008): in this special class of hedonic games the total cake goes to whichever coalition that has more than half of the total talent, whereas our analysis includes settings where coalitions fight over market shares or power shares, with no magic value given to passing a fifty percent threshold.

[6]In Damiano. et al. (2010) agents of different abilities choose between two organisations of a fixed capacity of measure 1, when the agent's utility increases both in the average ability of the organisation (peer effect) and in her internal ranking (pecking order effect). If the value of each coalition is a function of the average ability of agents, they obtain some degree of segregation of ability types, with a larger overlap when the pecking order effect is stronger. Their results apply to very different contexts, like students' choices among a fixed set of universities, rather than endogenous formation of organisations.

[7]On the importance of party formation and pre-election coalition formation across systems, see, e.g., Levy (2004), Morelli (2004) and Bandyopadhyay, Chatterjee and Sjöström (2010). See also Dhillon (2005) for a survey.

[8]See, e.g., Aumann and Drèze (1974) for some early study on the cooperative stability of coalition structures.

[9]If we fix the degree of vertical inequality as in Section 3 and impose the ability ranking assumption, our game is hedonic. However, generally it does not satisfy known conditions for existence of a core, namely, balancedness (Scarf, 1967) and top-coalition property (Banerjee et al, 2001), neither in a finite setting nor when naturally extended to an infinite setting.

2 Model

Consider an economy with a large number N of players with heterogeneous ability. We conduct our analysis for the limit case of $N \to \infty$, i.e., with countably infinite agents. Each player $i \in \mathbb{N}$ has an observable ability a_i (which could be political ability, market ability, or criminal ability depending on the application). We order players according to their ability, with the convention that $a_1 > a_2 > \cdots$. In particular, we assume a geometric distribution of abilities:

$$a_i = a^{i-1} \quad \text{where} \quad a \in (0,1). \tag{1}$$

2.1 Hierarchical organization and coalitional strength

We consider an environment in which all players form hierarchical organizations that compete for a contestable surplus. By a "hierarchical organization" we mean a coalition of agents assigned to a collection of hierarchically ranked posts that perform complementary tasks, whose marginal contribution to the total surplus/production decreases in the post's rank. The ratio of marginal contributions of any two consecutively ranked posts is assumed to be constant and denoted by $\rho \in (0,1)$. So, the total surplus of a hierarchical organization formed by a coalition $Z \subset \mathbb{N}$ of agents is

$$S(Z) := p(Z)(1-\rho)(1+\rho+\cdots+\rho^{\#(Z)-1}) \;=\; p(Z)(1-\rho^{\#(Z)}) \tag{2}$$

where $p(Z) \in [0,1]$ is the "coalitional strength" of Z to be defined below. Multiplying by $(1-\rho)$ serves the purpose of normalizing the size of the total surplus to one when the coalition has a countable infinity of members and a coalitional strength of unity. The ratio of marginal contribution, ρ, is common to all hierarchical organizations.[10] Throughout the paper, for brevity, we will use the term "coalition" to always mean "a coalition that constitutes a hierarchical organization".

The *(coalitional) strength* of a coalition $Z \subset \mathbb{N}$ is determined by an aggregate measure $p(Z)$ of the abilities of its members, relative to the aggregate measure of the abilities of the whole population of players, $\mathbb{N}$. Letting $\theta(Z) := \sum_{i\in Z} a^{i-1}$ denote the sum of abilities of members of Z, we specifically define

$$p(Z) \;:=\; \frac{\theta(Z)}{\sum_{j\in\mathbb{N}} a^{j-1}} \;=\; (1-a)\cdot\theta(Z).^{11} \tag{3}$$

[10]In the party formation application, the assumption of a common ρ is reasonable, since $p(Z)$ can be interpreted as the probability of winning an election, and then the various tasks to which the winning party members are assigned are fixed government tasks. Hence, party members (of different parties) who expect to have the same rank conditional on winning, have the same expected marginal contribution, related to the task they will have to perform in the government structure.

[11]In future work we plan to extend the analysis to the case of structure externalities, by studying a more general contest functional form for the relative power of a coalition, say Z_i, when players

Note that the sum of $p(Z)$ across all coalitions is 1, allowing $p(Z)$ to be interpreted as the probability of winning or power share or market share depending on the application.[12] Yet, linearity of $p(Z)$ in the sum of the members' abilities ensures that the total surplus of a coalition, (2), does not depend on the composition of other coalitions. This feature exempts us from the unsettled issue of what a deviating coalition should expect regarding the responses of the remaining agents.

2.2 Imputation ratio and ability ranking

Once all coalitions are formed, they constitute a *partition* of $\mathbb{N}$, which we denote by π. Within each coalition Z the members are assigned to posts/ranks, and the generated surplus, $S(Z)$, is divided among the members. Given that a coalition has a hierarchical structure we assume that the division of surplus is also hierarchical, i.e., the surplus shares are rank-dependent. Formally, we capture the vertical inequality of payoffs across ranks with a single parameter termed *imputation ratio,* which is the ratio of the surplus share of a coalition member relative to that of the member occupying the rank immediately above. Thus, a lower imputation ratio corresponds to a greater inequality. Expressing the degree of vertical inequality with just one parameter is a simplifying assumption, which nonetheless is broadly consistent with the impression that in most societies and most contexts the different views about the relative importance of different tasks and ranks are usually summarized by simple statements or positions.[13]

In principal, the imputation ratio can be determined by a collective decision within each coalition, hence may differ across coalitions. Prior to analyzing such a case (in Section 4), we analyze in Section 3 the case in which there is an exogenously determined imputation ratio that all coalitions abide by. In particular, we present the intuitive benchmark case in which the exogenous imputation ratio is equal to ρ, the ratio of marginal contributions between consecutive ranks.[14] In this case, denoting the rank of player i in a coalition $Z \subset \mathbb{N}$ by $r_i(Z)$, the expected utility of

are partitioned in K coalitions: $p(Z_i) = \frac{(\theta(Z_i))^\beta}{\sum_{k=1}^K (\theta(Z_k))^\beta}$. Keeping $\beta = 1$ in this paper allows us to use core stability, while in the more general extension the choice of the appropriate solution concept will be an issue.

[12]When $p(\cdot)$ is interpreted as a probability of winning in a winner-take-all contest, the utility of members of losing parties is normalized to 0. Since the decision by the relevant players on whether to form one coalition or another is made ex ante, it doesn't matter whether $p(\cdot)$ is a probability of winning in a winner-take-all system or a share in a proportional power sharing system. For the difference between the two in terms of voting incentives, see Herrera and Morelli (2010).

[13]In the political economy literature it is very common to simplify the distributive views using a single parameter, like the preferred tax rate in a linear taxation system. This assumption is as brutal as ours.

[14]The case in which there is a fixed imputation ratio but different from ρ can also be analyzed, and the results deliver essentially the same main insights as the case studied in this section, although the details are more complex (available from the authors).

player i in Z is

$$u_i(Z) \;=\; \frac{S(Z)\cdot\rho^{r_i(Z)-1}}{1+\rho+\cdots+\rho^{\#(Z)-1}} \;=\; p(Z)\cdot(1-\rho)\rho^{r_i(Z)-1}, \tag{4}$$

regardless of the cardinality of Z. Thus, every agent should decide which coalition to join not only on the basis of the coalition's strength, $p(Z)$, but also on the basis of her expected rank in the coalition, r_i, and the vertical inequality, ρ.

The next issue is how the members are assigned to ranks in each coalition. To start with, we conduct the analysis assuming that ranks are assigned according to the relative ability of members (henceforth "ability ranking")[15] , i.e.,

$$r_i(Z) = \#\{n \in Z | n \leq i\}. \tag{5}$$

We then verify in Section 3.4 that ability ranking indeed endogenously obtains as a result of competition when the coalitions are free to assign ranks in any order of their choice. Therefore, even if relative payoffs are not tied to relative abilities explicitly, they are tied endogenously.

2.3 Stability

In Section 3, we study the structure of stable partitions under the assumption that every coalition will adopt ρ as its imputation ratio. This case can be interpreted as a short-run model in which the organizational structure may not be changed quickly due to some institutional reasons.[16] In a political context, for example, even though the probability of winning of a party depends on the abilities of the politicians involved, once a party grabs power and the various offices have to be filled, at that point the relative payoffs of the various party members depend on their assigned ranks (from president to secretary and so on), and these relative and absolute payoffs take the form of pre-specified rank-related remunerations. In this case, the notion of stability is akin to that of core stability in a NTU game: a partition π is *stable* if there is no "profitable" deviation/subset D of $\mathbb{N}$ that would give higher expected payoff to all members of D than in the original coalitions in the partition π.

Over time, the organizational structure may change. In Section 4, allowing for this possibility by letting coalitions determine their own imputation ratios, we characterize *strongly stable* partitions that are immune to coalitional deviations where each coalition chooses its own imputation ratio.

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[15] Ability ranking in the internal organization (first assumed and then verified) has key implications on "who to team up with", and thereby on the endogenous structure of rival coalitions.

[16] We share the view that "often, the rewards from joint effort are shared according to rather rigid rules," as conveyed, e.g., by Farrell and Scotchmer (1988) who, focusing on partnerships, analyze the implications of the equal sharing rule on the size of partnerships and welfare.

2.4 Discussion

Let us interpret and motivate here three main features of the model that have not been fully addressed yet.

Ability as a "team asset". An agent's ability contributes to the coalition she joins by enhancing the coalitional strength, but does not affect the relative productivity of the post she occupies. In this sense, a member's ability is a team (rather than personal) asset. This stresses two aspects of our environment. One is the importance of *coalitional* competition among organazations that vie for (a share of) contestable resources.[17] The other is the *complementary* nature of the organization's operation. In an industrial organization application, for example, even if an employee is recruited for a particular post, e.g., engineer, the employee's ability is valued to the extent that she contributes to extending the firm's market share, by complementing other parts of the firm's operation to enhance the overall appeal of the final product to customers.

A countable infinity of players. We consider a countable infinity of players to avoid the artificial restrictions on partition structure that finite populations may impose. For example, a partition of two equal-size coalitions is feasible with an even number of players but not with an odd number of them. In addition, a countable infinity of players places no a priori bound either on the possible number of coalitions or on the maximum coalition size. As such, it allows us to focus on the relationship between the intra-coalition structure and the endogenous formation of coalitions, with minimal interference of other factors.

A related aspect is that there is some surplus loss if a finite coalition forms (although this does not happen in stable partitions), in the sense that the total surplus, (2), is equal to $p(Z)$ for infinite coalitions but less than $p(Z)$ for finite coalitions.[18] This is certainly realistic for coalitions of small sizes: In the political competition example, if a party wins the elections in spite of having only a very small number of members, it may cover only a small number of posts, forgoing payoffs that would have been generated from lower rank offices.[19] However, it is conceivable that a finite coalition does not incur any surplus loss if it is above a certain, large threshold size. Our model is consistent with this situation, too, so long as this threshold is a random variable with an unbounded support. If this threshold is finite, on the other hand, it would impose an upper bound on the coalition size. In our setting of infinite agents, this would prevent meaningful comparisons of the

[17] In political applications, an agent's political ability contributes to the likelihood that the collective campaign efforts of a party will succeed in persuading enough voters to win the election.

[18] For example, with $\rho = 2/3$ a coalition Z of three players would have, conditional on winning (with some probability $p(Z)$), payoffs equal to $1/3$ for the first ranked player, $2/9$ for the second, and $4/27$ for the third, which sum to $19/27 < 1$.

[19] See Mattozzi and Merlo (2010) for other reasons behind the positive relationship between the size of a party and its payoff.

number of organizations that emerge under different environments, which are part of our goals.

In any case, we analyze in Section 5 what happens in a complementary case in which coalitions must have a fixed finite size (and thus, a coalition's surplus is not a function of its size).

Single parameter distributions. We specify the ability differential among players, the internal payoff inequality, and the relative contributions of ranks, each by a single parameter. This is a simplification made in order to obtain clean analytical results, since our aim in this paper is to delineate stylized, yet clear insights on endogenous formation of hierarchical organizations. Generalizations beyond our geometric structure are probably feasible using computational methods, and indeed it seems to us that the results proved in this paper could hold more generally, but further analytical generalizations are beyond the scope of this paper.

3 Stable Partitions under a Fixed Imputation Ratio ρ

For any partition $\pi = \{Z_1, Z_2, \cdots\}$ of $\mathbb{N}$, we adopt the convention of labeling the coalitions in the ability order of the most able member, or "leader' for short, i.e., we assume (without loss of generality) that $1 \in Z_1$ and $i_k < i_{k+1}$ for all $k = 1, 2, \cdots$, where $r_{i_k}(Z_k) = 1$.

In this section we assume that every coalition adopts ρ as the imputation ratio. Given the two key parameters of the model, (a, ρ), a deviation $D \subset \mathbb{N}$ is *profitable* relative to a partition π if

$$u_i(D) \;\geq\; u_i(\pi(i)) \quad \forall i \in D \neq \emptyset$$

where $\pi(i)$ is the coalition $Z_k \in \pi$ such that $i \in Z_k$, and the inequality is strict for some $i \in D$. A partition π is *stable* if there is no profitable deviation relative to it. As mentioned earlier, initially we conduct our analysis taking it for granted that every coalition adopts the ability ranking. In Section 3.4 we prove that every coalition will indeed do so in any stable partition with more than one coalition.

A specific partition structure is key to our results: A "cyclic" K-partition is $\pi^c_K = \{Z_1, \cdots, Z_K\}$ such that Z_k consists of all players $k, k+K, k+2K, k+3K, \cdots$, i.e., $Z_k = \{j \in \mathbb{N} | j = k \mod K\}$, so that, in particular,

$$\theta(Z_k) = \frac{a^{k-1}}{1 - a^K} \quad \text{for} \quad k = 1, 2, \cdots, K. \tag{6}$$

The main result of this section is that cyclic K-partitions are the only stable partitions when all coalitions adopt the imputation ratio ρ. Moreover, each cyclic K-partition is stable for a range of ρ that decreases in K as specified below, establishing a positive relationship between the vertical inequality and the number of coalitions to be formed endogeously:

Theorem 1 *The cyclic K-partition π_K^c is stable if and only if*

$$a^K \leq \rho < \frac{a^{K-1}}{1 + a^{K-1} - a^K}. \tag{7}$$

Furthermore, if a partition is stable for any (a, ρ), it is a cyclic K-partition for some $K \geq 1$ and it is the unique stable partition at (a, ρ). The fraction of area (7) within $a^K \leq \rho < a^{K-1}$ approaches 1 as $K \to \infty$.

Figure 1 illustrates the areas of the parameter values $(a, \rho) \in (0, 1) \times (0, 1)$ that satisfy (7) and thus, support stable cyclic K-partitions. Between the area for a stable K-partition and that for a stable $(K-1)$-partition, there is an area with no stable partition. The fraction of this latter area, relative to the area for a stable K-partition, converges to 0 as $K \to \infty$.

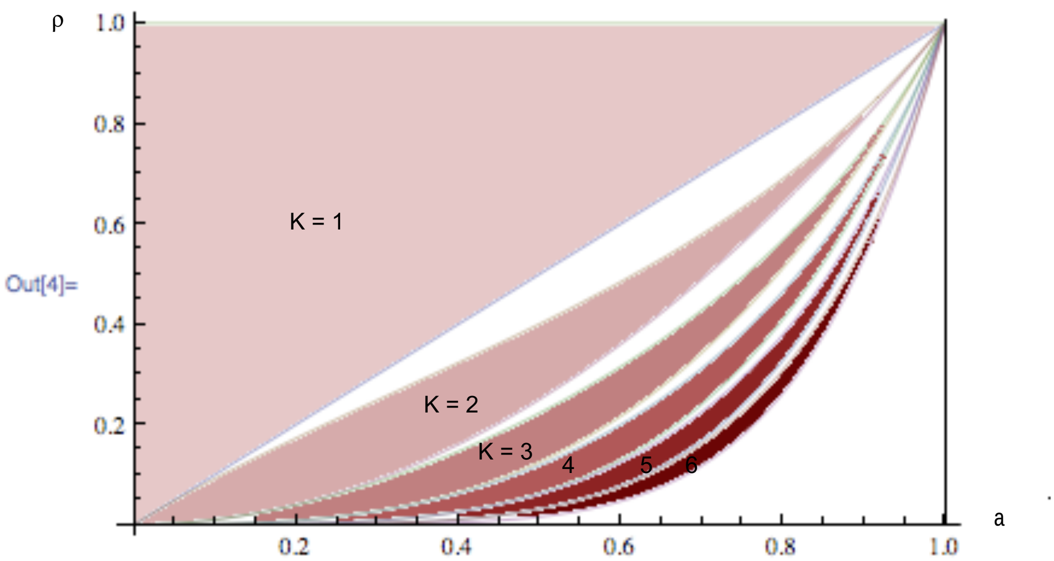

Figure 1

In the rest of this section, we gradually build up our analysis on stable partitions to prove Theorem 1 eventually.

3.1 Some general properties

We start by observing that, in a stable partition, no coalition may consist of a finite number of members because such a coalition would always be able to find an agent who would be willing to join as the lowest rank member, either because she is ranked so low in another coalition or because her own coalition's power is negligible, as formalized below.

Lemma 1 *In any stable partition, every coalition has countably infinite members.*

Proof. To reach a contradiction, suppose there is a finite coalition, say Z_f, in a stable partition π. Let ℓ be the least able member of Z_f. If there is an infinite coalition, say $Z_k \in \pi$, then there exists a large enough $j \in Z_k$ such that $j > \ell$ and $p(Z_k)(1-\rho)\rho^{r_j(Z_k)-1} < p(Z_f)(1-\rho)\rho^{\#(Z_f)}$, so that j is better off by joining Z_f. Thus, $D = Z_f \cup \{j\}$ would constitute a profitable deviation because the coalition's strength increases relative to Z_f, i.e., $\theta(D) > \theta(Z_f)$, while the ranks of members of Z_f are intact and thus, their payoffs are higher according to (4). This would contradict the supposed stability of the partition. If all coalitions of π are finite, on the other hand, there are infinitely many coalitions; consequently, there exists a large enough j such that agent j is the most able agent of a coalition, say $Z' \in \pi$, and $p(Z')(1-\rho) < p(Z_f)(1-\rho)\rho^{\#(Z_f)}$, leading to an analogous contradiction that $Z_f \cup \{j\}$ would be a profitable deviation. ■

Although this result is proved here for the case that the imputation ratio is fixed at ρ (which is assumed in this section), we emphasize that the same result continues to hold in the more general case in which the imputation ratio is endogenously determined, as analyzed in Section 4: Then, coalitions are free to choose an imputation ratio for which they would not grow above a certain finite size, yet no coalition finds it optimal to do so.

Next, we establish that in any stable partition the strength of each coalition is increasing in the leader's ability, since otherwise, the more able leader of a weaker coalition would be happy to replace the less able leader of a stronger coalition, which the latter coalition would also welcome.

Lemma 2 *If $\pi = \{Z_1, Z_2, \cdots, Z_K\}$ is a stable partition (where $K = \infty$ is allowed), then $\theta(Z_1) > \theta(Z_2) > \cdots > \theta(Z_K)$.*

Proof. If $\theta(Z_{k+1}) \geq \theta(Z_k)$ for some k, the deviation $D = \big(Z_{k+1} \setminus \{i_{k+1}\}\big) \cup \{i_k\}$ would be profitable where i_k and i_{k+1} are the most able members of Z_k and Z_{k+1}, respectively, because (i) $i_k < i_{k+1}$ so that $\theta(D) > \theta(Z_{k+1}) \geq \theta(Z_k)$, and (ii) every member in D retains the same rank as in π. ■

By the same token, if a rank of a weaker coalition is occupied by a more able agent than the same rank of a stronger coalition, then an analogous swap of the agents for that rank would constitute a profitable deviation. Thus, for any member i of any coalition in a stable partition, every stronger coalition must have more members who are more able than i than i's own coalition does. In fact, we prove a stronger result: For any two consecutively ranked members of a weaker coalition, exactly one agent exists in each stronger coalition whose ability is between them (Lemma 3). This lemma will in turn be used to show that every stable partition must consist of a finite number of coalitions.

Consider any two coalitions Z_k and $Z_{k'}$ in a stable partition π where $k < k'$. Suppose, for instance, that there is no agent in Z_k whose ability is between the two most able agents in $Z_{k'}$, say i' and j'. Let i and j be the two consecutively ranked members of Z_k such that $i < i' < j' < j$, with no agent in $Z_{k'}$ between j' and j. Then, every agent in $Z_{k'} \setminus \{i'\}$ prefers $D = (Z_{k'} \setminus \{i'\}) \cup \{i\}$ to $Z_{k'}$ because their coalition becomes stronger, i.e., $\theta(D) > \theta(Z_{k'})$, while maintaining their ranks. For the deviation D to be not profitable, therefore, the agent i should be worse off in D, i.e.,

$$\theta(Z_k)\rho^{r_i(Z_k)-1} \;>\; \theta(D) \;=\; \theta(Z_{k'}) + a^{i-1} - a^{i'-1} \;>\; \theta(Z_{k'}). \tag{8}$$

At the same time, for the deviation $D' = (Z_k \setminus \{j\}) \cup \{j'\}$ to be not profitable, since all agents in $Z_k \setminus \{j\}$ prefer D' to Z_k, the agent j' should be worse off in D', i.e.,

$$\begin{aligned}\theta(Z_{k'})\rho^{r_{j'}(Z_{k'})-1} \;>\; \theta(D')\rho^{r_j(Z_k)-1} \;=\; (\theta(Z_k) + a^{j'-1} - a^{j-1})\rho^{r_i(Z_k)} \;>\; \theta(Z_k)\rho^{r_i(Z_k)} \\ \Longrightarrow \quad \theta(Z_{k'})\rho^{r_{j'}(Z_{k'})-2} \;>\; \theta(Z_k)\rho^{r_i(Z_k)-1},\end{aligned} \tag{9}$$

where the equality follows because agent i is one rank above j in Z_k. Since $r_{j'}(Z_{k'}) = 2$, (8) and (9) lead to a contradictory conclusion that $\theta(Z_k) > \theta(Z_k)$. This establishes that there must be at least one agent in Z_k whose ability is between i' and j'. Extending the same idea, we prove that:

Lemma 3 *Let Z_k and $Z_{k'}$ be two coalitions in a stable partition π where $k < k'$. For any two adjacently ranked agents in $Z_{k'}$, exactly one agent exists in Z_k whose ability is between them.*

Proof. See Appendix. ■

This implies that agents in $Z_k \cup Z_{k'}$ who are no more able than the leader of $Z_{k'}$ alternate between the two coalitions as we go down along the ability ordering. If there were infinitely many coalitions (all of which have infinite members by Lemma 1), therefore, the number of agents whose ability is between two adjacently ranked agents in Z_1 would increase without bound as we go down the rank. This means that lower rank agents in Z_1 would be compensated increasingly better relative to their ability, which is inconsistent with stability because then there should be an agent not in Z_1 who would be willing to replace a less able agent occupying a certain rank in Z_1. Thus,

Lemma 4 *Any stable partition consists of a finite number of coalitions.*

Proof. Consider a partition $\pi = \{Z_k\}_{k \in \mathbb{N}}$ consisting of infinitely many coalitions. Let $Z_1 = \{1, \ell_2, \ell_3, \cdots\}$, where agent ℓ_r is rank r (≥ 2) in Z_1. Let i_k be the leader of Z_k. Note that $\theta(Z_k) \leq \frac{a^{i_k-1}}{1-a^k}$ by Lemma 3. For π to be stable, if $i_k < \ell_r$ then $\ell_{r+1} - \ell_r \geq k$ by Lemma 3 and consequently, $\ell_{r+1} - \ell_r \to \infty$ as $k \to \infty$, which in turn implies that $\ell_r/r \to \infty$ as $r \to \infty$.

For arbitrarily large k, find r such that $\ell_r < i_k < \ell_{r+1}$. Note that as k increases without bound so does the corresponding r. Then, $D = Z_1 \cup \{i_k\} \setminus \{\ell_{r+1}\}$ is profitable for a sufficiently large k, because $\theta(Z_1)\rho^r - \theta(Z_k) \geq \theta(Z_1)\rho^r - \frac{a^{i_k-1}}{1-a^k} = \rho^r(\theta(Z_1) - \frac{a^{i_k-1}}{\rho^r}\frac{1}{1-a^k}) > 0$ where the last inequality follows since $\ell_r/r \to \infty$ implies $\frac{a^{\ell_r-1}}{\rho^r} \to 0$ as $k \to \infty$ and thus, $\ell_r < i_k$ implies $\frac{a^{i_k-1}}{\rho^r}\frac{1}{1-a^k} \to 0$ as $k \to \infty$. ■

Thus, we may limit attention to partitions consisting of a finite number of coalitions in what follows. Consider an arbitrary stable K-partition $\pi_K = \{Z_1, Z_2, \cdots, Z_K\}$. By Lemma 3, for any two adjacently ranked agents in Z_K, there are exactly $K-1$ agents whose ability is in between them (one from every other coalition). Thus, Z_K consists of every K-th agent in ability ordering starting from its leader, i_K, and consequently, $\theta(Z_k) = \frac{a^{i_K-1}}{1-a^K}$. Note that all players of $D = Z_K \setminus \{i_K\}$ would be strictly better off in D than in Z_K if $\rho < a^K$ because then $\theta(D) = \theta(Z_K) - a^{i_K-1} = \frac{a^{i_K-1} \cdot a^K}{1-a^K} > \theta(Z_K)\rho$, i.e., because the benefit of moving up one rank dominates the reduced strength from losing the leader, i_K. This places a lower bound on ρ for stability of K-partitions as stated in Lemma 5. This bound decreases in K, which implies that the degree of vertical inequality potentially allowed in a stable partition increases with the number of coalitions.

Lemma 5 *A necessary condition for any K-partition to be stable is that $\rho \geq a^K$.*

3.2 Grand Coalition

When is the grand party $Z_G = \mathbb{N}$ stable? Suppose there is a profitable deviation D and let i be the first rank of D: She has a payoff of $(1-\rho)\rho^{i-1}$ in Z_G while $p(D)(1-\rho)$ in D. Note that the latter is largest when D consists of all players $j \geq i$, in which case $p(D) = (1-a)\sum_{n \geq i} a^{n-1} = a^{i-1}$. Thus, no deviation is profitable if

$$(1-\rho)\rho^{i-1} \geq a^{i-1}(1-\rho) \quad \forall i \quad \Longleftrightarrow \quad \rho \geq a.$$

By Lemma 5, therefore, Z_G is stable precisely when $\rho \geq a$. Since $\rho < a$ is necessary for any other coalition structure to be stable as shown in (11) below, we establish:

Proposition 1 *The grand coalition constitutes a stable partition if and only if $\rho \geq a$. Furthermore, there is no other stable partition if $\rho \geq a$.*

3.3 Multiple Coalitions

Lemma 5 provides a lower bound of ρ for a stable K-partition, $\rho \geq a^K$, based on the idea that excessive inequality would prompt lower rank members to revolt by throwing out their leader. We now derive an upper bound of ρ, based on the flip-side idea that most able players may be sought after by multiple coalitions when the inequality level is very low.

Consider an arbitrary stable K-partition. For each $k = 2, \cdots, K$, letting i_k denote the leader of Z_k, we have $\theta(Z_k) > \theta(Z_1 \setminus \{1, \cdots, i_k - 1\})$ because agents in $Z_1 \cup Z_k$ who are no more able than i_k alternate between the two coalitions as we go down along the ability ordering by Lemma 3. Let $D_k = (Z_1 \cap \{1, \cdots, i_k - 1\}) \cup Z_k$. Clearly, all members of $D_k \cap Z_1$ are better off in D_k than in Z_1. Player i_k's payoff is $(1-a)\theta(Z_k)(1-\rho)$ in Z_k; it is $(1-a)\theta(D_k)(1-\rho)\rho^{\nu_k}$ in D_k where $\nu_k = \#(Z_1 \cap \{1, \cdots, i_k - 1\})$. For D_k to be not profitable, therefore, we need

$$\theta(D_k)\rho^{\nu_k} < \theta(Z_k) \iff \rho < \Big(\frac{\theta(Z_k)}{\theta(Z_k) + \sum_{j \in Z_1 \cap \{1, \cdots, i_k - 1\}} a^{j-1}}\Big)^{1/\nu_k}. \tag{10}$$

For $k = 2$, we have $Z_1 \cap \{1, \cdots, i_2 - 1\} = \{1, \cdots, i_2 - 1\}$ so that $\nu_2 = i_2 - 1$, and $\theta(Z_2) \leq \frac{a^{i_2-1}}{1-a^2}$ by Lemma 3. Thus, (10) for $k = 2$ implies that

$$\rho < \Big(\frac{a^{i_2-1}}{a^{i_2-1} + (1-a^2)(1-a^{i_2-1})/(1-a)}\Big)^{\frac{1}{i_2-1}} = \Big(\frac{a^{i_2-1}}{1+a-a^{i_2}}\Big)^{\frac{1}{i_2-1}} < a, \tag{11}$$

identifying an upper bound of ρ for a stable K-partition, $K \geq 2$, that establishes Proposition 1 above.

However, this is not a tight upper bound as we show below. To do this, we focus on cyclic partitions defined earlier. Suppose there exists a cyclic K-partition $\pi_K^c = \{Z_1, \cdots, Z_K\}$ that is stable. Then, the condition for the deviation $D = \{1\} \cup Z_K$ to be not profitable is (10) when $k = K$, $\nu_K = 1$ and $Z_1 \cap \{1, \cdots, i_K - 1\} = \{1\}$, i.e., $\rho < \frac{a^{K-1}}{1+a^{K-1}-a^K}$. In conjunction with Lemma 5, therefore, (7) is a necessary condition for a cyclic K-partition to be stable.

Theorem 1 states that (7) is also a sufficient condition for the cyclic K-partition to be stable, and there is no other stable partitions. We prove this result formally in the Appendix. Here, we provide a sketch of the proof.

To show that (7) is also a sufficient condition for the cyclic K-partition to be stable, consider first the deviation of "i-onward break-off" from Z_k, i.e., by $Z_k^i = \{n \in Z_k | n \geq i\}$. Each player $j \in Z_k^i$ is no better off in Z_k^i than in Z_k precisely when

$$\frac{(1-a)a^{k-1}}{1-a^K}(1-\rho)\rho^{\frac{j-k}{K}} \geq \frac{(1-a)a^{i-1}}{1-a^K}(1-\rho)\rho^{\frac{j-i}{K}} \iff \rho \geq a^K. \tag{12}$$

Since this condition is independent of i and $j \in Z_k^i$, we deduce that

[A] truncated break-offs are not profitable if and only if $\rho \geq a^K$.

Next, consider a deviation by $D \subset \mathbb{N}$. If D is finite, let ℓ be the last member of D. Let $\ell \in Z_k \in \pi_K^c$. If D is profitable, i.e., all member of D is better off in

D than in π_K^c, then they are still so in the union of D and $Z_k^\ell = \{n \in Z_k | n \geq \ell\}$. Furthermore, in this union the payoff of $j \geq \ell$ is $p(D \cup Z_k^\ell)(1-\rho)\rho^{|D|-1+(j-\ell)/K}$ while that in Z_k is $p(Z_k)(1-\rho)\rho^{(j-k)/K}$, so that j is better off in $D \cup Z_k^\ell$ than in Z_k if and only if $p(D \cup Z_k^\ell)\rho^{|D|-1} \geq p(Z_k)\rho^{(\ell-k)/K}$, which is postulated to hold for $\ell \in D$. Since this inequality is independent of j, it further follows that

[B] If a finite D is profitable, there is an infinite profitable deviation.

Consequently, we only need to ensure that no deviation is profitable that consists of infinite members. Consider an infinite D. For each $n \in D$, let $e_n = r_n(D) - r_n(Z_k)$ where $n \in Z_k$. Note that if $\{e_n | n \in D\}$ is unbounded above then D is not profitable since there is n large enough such that n's payoff is lower in D because $p(D)\rho^{r_n(Z_k)+e_n-1} - p(Z_k)\rho^{r_n(Z_k)-1} = p(Z_k)\rho^{r_n(Z_k)-1}\big(p(D)\rho^{e_n}/p(Z_k) - 1\big) < 0$. Hence, let $e^* = \max_D e_n$.

Let $h = \min\{n \in D \cap Z_1 \,|\, e_n = e^*\}$ if such $h \in Z_1$ exists. If not, let $h = \min\{n \in D \cap Z_2 \,|\, e_n = e^*\}$ if such $h \in Z_2$ exists. Proceeding recursively, one can find $h = \min\{n \in D \cap Z_k \,|\, e_n = e^*\}$ for a unique k, such that $e_n < e^*$ for all $n \in Z_{k'} \cap D$ if $k' < k$. Then, define

$$D_h = D \cap \{1, 2, \cdots, h\} \quad \text{and} \quad D^* = D_h \cup Z_k^h. \tag{13}$$

As we detail in the proof of Theorem 1 (in Appendix), if an infinite deviation D is profitable relative to π_K^c, then so is D^* because $Z_k^h \setminus \{h\}$ consists of more able agents than $D \setminus D_h$ and agents in $Z_k^h \setminus \{h\}$ are happier in D^* than in Z_k so long as h is. Consequently, by delineating the conditions under which no deviation of the form $D_h \cup Z_k^h$ is profitable for any $h > 1$ and $k = 1, 2, \cdots, K$, we verify that the cyclic K-partition is stable if and only if (7) holds. The lower bound is the condition that no "break-off" deviation is profitable as explained above; The upper bound is the condition that members of Z_K would not recruit player 1 as their leader because the enhanced power (winning probability) from the recruitment is overshadowed by the payoff reduction from their ranks going down by one.

Finally, to show that there does not exist any stable partition that is not cyclic, we need the following key lemma.

Lemma 6 *Let Z_k and $Z_{k'}$ be two coalitions in a K-partition π_K where $k < k'$. Let y be the leader of $Z_{k'}$; and let $x_1, x_2, \cdots, x_\ell$ be, in that ability order, the agents in Z_k who are more able than y. If $y - x_\ell = x_{j+1} - x_j$ for all $j = 1, \cdots, \ell - 1$, and $\ell \geq 2$, then π_K is not stable.*

Proof. In Appendix. ■

Recall that Lemma 3 says that between any two coalitions their members alternate in the ability ordering except possibly for the top ability group of agents who may belong to the stronger coalition. Lemma 6 says that this top group of agents may not be superior to the leader of the weaker coalition by a certain degree. Essentially, these lemmas mean that coalitions have similar compositions in stable partitions. The proofs are relatively lengthy, so are deferred to the Appendix.

With these lemmas at hand, we are now ready to establish that no non-cyclic partition can be stable. Observe from Lemma 6 that any partition fails to be stable so long as $\{1, \cdots, \ell\} \subset Z_1$ and $\ell + 1 \in Z_2$ for some $\ell \geq 2$.

Hence, focus on partitions such that $1 \in Z_1$ and $2 \in Z_2$. To reach a contradiction, suppose that such a partition is stable but non-cyclic. Let $K \geq 3$ be the second rank agent of Z_1. Then, by Lemma 3, $j \in Z_j$ for all $j = 2, \cdots, K-1$. Find $m \geq 1$ such that $m'(K-1)+j \in Z_j$ for all $j = 1, \cdots, K-1$, for all $m' = 1, \cdots, m-1$, but $m(K-1)+j \notin Z_j$ for some $j = 1, \cdots, K-1$. Such m exists since the partition is assumed non-cyclic. Let κ be such that $m(K-1)+\kappa \notin Z_\kappa$ while $m(K-1)+j \in Z_j$ for all $j < \kappa$. The partition is not stable if $m(K-1)+\kappa \in Z_j$ for $j < \kappa$ or $\kappa + 1 \leq j \leq K-1$ by Lemma 3 (applied for $k = j$ and $k' = \kappa$ in the former case, and for $k = \kappa$ and $k' = j$ in the latter). Therefore,

$$m(K-1) + \kappa \ \in \ Z_K. \tag{14}$$

If $\kappa = 1$, then $m \geq 2$ by supposition (if $m = 1$ then $m(K-1)+1 = K \in Z_1$ as posited above). Then, by Lemma 6 (applied for $k = 1$ and $k' = K$), the partition is not stable.

Alternatively, suppose $\kappa \in \{2, \cdots, K-1\}$. First, consider the case that $m \geq 2$. Since $m'(K-1)+\kappa \in Z_\kappa$ for all $m' = 1, \cdots, m-1$, by Lemma 6 (applied for $k = \kappa$ and $k' = K$), the partition is not stable.

It remains to consider the case that $\kappa \in \{2, \cdots, K-1\}$ and $m = 1$. For the deviation $Z_1 \setminus \{1\}$ not to be profitable, we need $\theta(Z_1) - 1 \leq \theta(Z_1)\rho \ \Leftrightarrow \ \rho \geq \frac{\theta(Z_1)-1}{\theta(Z_1)}$. For the deviation $D = Z_K \cup \{\kappa\}$ not to be profitable, since the agent κ would be strictly better off in D than in Z_κ by Lemma 3, we need $\theta(Z_K) > (\theta(Z_K) + a^{\kappa-1})\rho \ \Leftrightarrow \ \rho < \frac{\theta(Z_K)}{\theta(Z_K)+a^{\kappa-1}}$. These two inequalities are inconsistent if $\theta(Z_1) \geq 1+\theta(Z_K)/a^{\kappa-1}$, because then we have $\frac{\theta(Z_1)-1}{\theta(Z_1)} \geq \frac{1+\theta(Z_K)/a^{\kappa-1}-1}{1+\theta(Z_K)/a^{\kappa-1}} = \frac{\theta(Z_K)}{a^{\kappa-1}+\theta(Z_K)}$. Indeed, $\theta(Z_1) \geq 1 + \theta(Z_K)/a^{\kappa-1}$ holds since Lemma 3 implies, for every $n \in \mathbb{N}$, that $y_n - x_{n+1} \geq \kappa - 1$ where y_n and x_{n+1} are the agents with ranks n and $n+1$ in Z_K and Z_1, respectively. This establishes that all stable partitions are cyclic and thereby, the second part of Theorem 1.

3.4 Endogenous ability ranking

We have conducted the analysis so far under the assumption that any coalition with any set of members would always rank them according to ability. We prove now that ability ranking must hold in every stable partition due to competitive pressure.

Lemma 7 *Consider any triplet of agents, i, j and ℓ, in decreasing order of ability. No stable partition can display i and ℓ in the same coalition, say Z, but in ranks of reversed order, and j in another coalition, say Z'.*

Proof. Consider $D = (Z \setminus \{\ell\}) \cup \{j\}$, where the agent j replaces the agent ℓ at the same rank. Since all members of $Z \setminus \{\ell\}$ would accept j in place of ℓ because it would increase the strength, stability would require that j be worse off in D than in Z'. This implies that j's payoff in Z' exceeds ℓ's payoff in Z, which in turn exceeds i's payoff in Z. Thus, i must be willing to replace j in Z', and the other members of Z' would accept i in place of j, upsetting stability. ■

Proposition 2 *In any stable partition with $K \geq 2$, all coalitions must satisfy "ability ranking".*

Proof. Suppose there is a stable partition π with reversed ranking within coalitions. Then, for any pair of agents i and j in the same coalition, say Z_k, such that $i < j$ yet j is ranked higher than i, switch the ranking of i and j. In the new ranking system, the partition is still stable. To verify this, suppose to the contrary that it is not stable in the new ranking system. Then, a profitable deviation D, with a deviation ranking $d : D \to \mathbb{N}$, exists that includes either i or j. If D included only i, then the same D with d would be profitable relative to the old ranking as well, because i is less happy in the old ranking. If D included only j, then D and d, with j replaced by i, would be profitable relative to the old ranking, because i in the old ranking is equally happy as j in the new ranking. If D included both i and j, then the same D with d' that switches the ranking of i and j, would be profitable relative to the old ranking, because i (j) in the old ranking is equally happy as j (i) in the new ranking.

By sequentially switching the reversed ranks in the manner described above, therefore, we can construct a new ranking of the same partition π that satisfies the ability ranking.[20] Since stability is preserved, π in the ability ranking must be cyclic by Theorem 1. This means that for any pair i and j in a coalition, there is an agent in between in another coalition. Then, the Lemma 7 dictates that no reversed ranking exists in the old ranking system. ■

[20] To be fully precise, there is a technical complication due to the possibility that one may not finish the switching process in finite steps when there are infinite instances of reversed ranking initially. However, the proofs of Lemma 3 and the relevant parts of Lemma 6 can be straightforwardly modified to apply to the agents who are ability ordered when the ability ordering prevails at least for the M most able agents, for arbitrarily large M. Consequently, the logic of the current proof works with a finite number of switchings even if there may be infinite instances of reversed ranking in the original partition under consideration. (At the end of the document we include, for referees' review, a supplementary material explaining in detail how this potential issue is resolved, which may not be included in the final manuscript.)

4 Endogenous Imputation Ratio and Strongly Stable Systems

We now extend the notion of stability to environments in which the imputation ratio, now denoted by σ to distinguish from ρ, is endogenously determined within each coalition. Since $\sigma \neq \rho$ in general, a coalition Z with an imputation ratio σ generates a total surplus $S(Z) = p(Z)(1-\rho^{\#(Z)})$ as before, but allocates according to σ so that the payoff of player $i \in Z$ is

$$\begin{aligned} u_i(Z,\sigma) &= \frac{S(Z)\cdot\sigma^{r_i(Z)-1}}{1+\sigma+\cdots+\sigma^{\#(Z)-1}} = \frac{p(Z)(1-\rho^{\#(Z)})(1-\sigma)\sigma^{r_i(Z)-1}}{1-\sigma^{\#(Z)}} \qquad (15)\\ &= p(Z)(1-\sigma)\sigma^{r_i(Z)-1} \text{ if } \#(Z)=\infty. \end{aligned}$$

Note that, for an infinite coalition, the payoffs are the same as those in the previous section, (4), with ρ replaced by σ.

Since σ can be different across coalitions now, we define a "system" to be a pair $(\pi,\vec{\sigma})$ consisting of a K-partition π and a K-vector $\vec{\sigma} = (\vec{\sigma}_1,\cdots,\vec{\sigma}_K)$ that specifies one imputation ratio $\vec{\sigma}_k \in (0,1)$ for each coalition Z_k of π. A system is *strongly stable (s-stable)* if there does not exist a profitable deviation (D,σ') in the sense that every member of D is weakly (some strictly) better off in D than in $(\pi,\vec{\sigma})$ conditional on σ' being the imputation ratio in D: That is, recalling that $\pi(i)$ denotes the coalition in π to which agent i belongs to,

$$\frac{p(D)(1-\rho^{\#(D)})(1-\sigma')(\sigma')^{r_i(D)-1}}{1-(\sigma')^{\#(D)}} \geq \frac{p(\pi(i))(1-\rho^{\#(\pi(i))})(1-\vec{\sigma}_{\pi(i)})(\vec{\sigma}_{\pi(i)})^{r_i(\pi(i))-1}}{1-(\vec{\sigma}_{\pi(i)})^{\#(\pi(i))}} \quad \forall i \in D$$

where the inequality is strict for some $i \in D$. If $\#(D) = \#(\pi(i)) = \infty$, this becomes

$$p(D)(1-\sigma')(\sigma')^{r_i(D)-1} \geq p(\pi(i))(1-\vec{\sigma}_{\pi(i)})(\vec{\sigma}_{\pi(i)})^{r_i(\pi(i))-1} \quad \forall i \in D. \qquad (16)$$

Note from $S(Z) = p(Z)(1-\rho^{\#(Z)})$ that the marginal contribution of an agent i to a coalition's total surplus is largest at $S(Z) - S(Z\setminus\{i\}) = (1-a)a^{i-1}$ when she joins a coalition of an infinite size. A key observation here is that every agent can obtain this level of payoff if coalitions are free to choose their own imputation ratios. Formally,

Lemma 8 *If an agent i's payoff in a system $(\pi,\vec{\sigma})$ is strictly less than $(1-a)a^{i-1}$, then the system is not s-stable.*

Proof. Let j_1 be the agent whose payoff in the system $(\pi,\vec{\sigma})$ is strictly lower than $(1-a)a^{j_1-1}$. Find a sufficiently low $\sigma' > 0$ such that $u' = (1-a)(1-\sigma')a^{j_1-1}$ exceeds her payoff in $(\pi,\vec{\sigma})$. For each $r = 2,3,\cdots$, one can find an agent, say j_r, whose payoff in the system $(\pi,\vec{\sigma})$ falls short of $u'\cdot(\sigma')^{r-1}$, maintaining the feature that $j_r < j_{r+1}$. This is because there exists an agent i, i arbitrarily large, whose payoff is arbitrarily low in the system either because her rank is arbitrarily low in an infinite coalition, or in case there is no infinite coalition, because she is in a coalition of arbitrarily small strength. Then, the deviation $(D' = \{j_1, j_2, \cdots\}, \sigma')$ is profitable because agent j_r would have a payoff of $(1-a)(1-\sigma')\theta(D')(\sigma')^{r-1} > u' \cdot (\sigma')^{r-1}$. This proves that $(\pi, \vec{\sigma})$ is not s-stable. ■

Lemma 8 implies that, in any s-stable system, agent i's payoff is at least $(1-a)a^{i-1}$. In fact, it is *equal* to $(1-a)a^{i-1}$ in any s-stable system, since the maximum possible surplus in the whole economy is $(1-a)\sum_{i=1}^{\infty} a^{i-1}$. This further implies that any coalition Z in an s-stable system must generate a total surplus of $(1-a)\sum_{i \in Z} a^{i-1} = p(Z)$, which is possible only if it is of an infinite size. Indeed, the symmetric K-cyclic system $(\pi_K^c, \vec{\sigma})$ where π_K^c is the K-cyclic partition and $\vec{\sigma} = (a^K, \cdots, a^K)$, delivers these payoffs and constitutes an s-stable system.

Proposition 3 *For every $a \in (0,1)$ and $K \in \mathbb{N}$, the K-cyclic system $(\pi_K^c, \vec{\sigma})$ is s-stable if $\vec{\sigma} = (a^K, \cdots, a^K)$.*

Proof. Consider a system $(\pi_K^c, \vec{\sigma})$ where $\vec{\sigma} = (a^K, \cdots, a^K)$. The equilibrium payoff of agent i in this system is routinely calculated to be $(1-a)a^{i-1}$ from (15):

$$(1-a)\frac{a^{k(i)-1}}{1-a^K}(1-a^K)(a^K)^{(i-k(i))/K} = (1-a)a^{k(i)-1}(a^K)^{(i-k(i))/K} = (1-a)a^{i-1}$$

where $k(i)$ is the first-ranked agent in the coalition that i belongs to in π_K^c (so that i's rank in the coalition is $(i-k(i))/K$).

For any deviation (D, σ'), from (16) the total surplus of this deviation is

$$\sum_{i \in D} \frac{p(D)(1-\rho^{\#(D)})(1-\sigma')(\sigma')^{r_i(D)-1}}{1-(\sigma')^{\#(D)}} = p(D)(1-\rho^{\#(D)}) \le (1-a)\sum_{i \in D} a^{i-1}.$$

Since $(1-a)\sum_{i \in D} a^{i-1}$ is the sum of payoffs of members of D in the system $(\pi_K^c, \vec{\sigma})$, we deduce that no deviation is profitable relative to $(\pi_K^c, \vec{\sigma})$. This proves that $(\pi_K^c, \vec{\sigma})$ is s-stable. ■

Observe that for every $a \in (0,1)$ there are multiple s-stable symmetric systems, namely, the symmetric K-cyclic systems $(\pi_K^c, (a^K, \cdots, a^K))$ for $K = 1, 2, \cdots$. Furthermore, every agent i is indifferent across such symmetric s-stable systems because her payoff is the same at $(1-a)a^{i-1}$ in all such systems (payoff equivalence).[21] There-

[21] This payoff coincides with the Shapley value of the TU coalitional game derived from our game by defining the value of $Z \subset \mathbb{N}$ as $v(Z) = \sum_{i \in Z} u_i(Z)$. This game is superadditive with the feature that the marginal contribution of each agent i is $(1-a)a^{i-1}$ whenever she joins an infinite coalition. Consequently, it is not hard to verify that the payoff allocation that imputes each agent this level of her marginal contribution is in the core, with any partition that consists of infinite coalitions. Given the restriction of our setting that in each coalition the imputation is governed by a single parameter, σ, the generalized-cyclic systems defined below are the only configurations in which the same imputation may be achieved.

fore, a system should be s-stable, symmetric or not, so long as every agent gets this payoff. This is possible in so far as each coalition is lifted from a cyclic K-partition for some K, along with the imputation ratio $\sigma = a^K$. We refer to such a system as a "generalized-cyclic" system, formalized as: A system $(\pi, \vec{\sigma})$ is a *generalized-cyclic system* if each coalition Z_k consists of every κ-th agent for some $\kappa \in \mathbb{N}$, starting from a first ranked agent, say i_k, with $\vec{\sigma}_k = a^\kappa$, i.e.,

$$Z_k = \{i_k + n\kappa | n \in \mathbb{N}\} \text{ for some } i_k \in \mathbb{N}, \text{ and } \vec{\sigma}_k = a^\kappa. \tag{17}$$

For example, the symmetric 4-cyclic system $(\pi_4^c, (a^4, a^4, a^4, a^4))$ constitutes a generalized-cyclic system when coalitions Z_1 and Z_3 merge and adopt a^2 as its imputation ratio. We establish that a system is s-stable if and only if it is of this form.

Theorem 2 *A system $(\pi, \vec{\sigma})$ is s-stable if and only if it is a generalized-cyclic system. Furthermore, agent i's payoff in any s-stable system is $(1-a)a^{i-1}$.*

Proof. Consider an arbitrary generalized-cyclic system $(\pi, \vec{\sigma})$. Then, it is routinely calculated that every agent i has a payoff of $(1-a)a^{i-1}$. If a deviation (D, σ') were profitable relative to this system, then D would be profitable relative to a s-stable symmetric K-cyclic system. Since this would be a contradiction to Proposition 3, any generalized-cyclic system must be s-stable.

Next, to reach a contradiction, suppose there is a s-stable system $(\pi, \vec{\sigma})$ that is not a generalized-cyclic one. We proved earlier that all coalitions in any s-stable system will consist of infinite members. Thus, there is an infinite coalition Z_k that is not of the form (17), so that for any two consecutively ranked members, their payoff ratio in the system $(\pi, \vec{\sigma})$ is constant at $\vec{\sigma}_k$, but their payoff ratio in a symmetric s-stable system, in which each agent i gets $(1-a)a^{i-1}$, is not constant. Note that the sum of the members' payoffs of Z_k is $(1-a)\theta(Z_k)$ in $(\pi, \vec{\sigma})$, and that in a symmetric s-stable system is also $\sum_{i \in Z_k}(1-a)a^{i-1} = (1-a)\theta(Z_k)$. Therefore, there must exist a member of Z_k, say j, whose payoff in the system $(\pi, \vec{\sigma})$ is strictly lower than $(1-a)a^{j-1}$. This contradicts to $(\pi, \vec{\sigma})$ being s-stable by Lemma 8. ■

5 Finite coalition size and segregation by clusters

Two questions that may independently interest the readers are: (1) what if coalitions are forced to have a fixed finite size (for example, due to technology) and (2) how can we reconcile our cyclic partition findings with the observed segregation by skills (see, for example, Kremer and Maskin, 1996)? In this section we show that the answers to these two questions are intimately connected: if we assume that each coalition has to have a finite size M, then a set of strongly stable partitions exhibits what we describe as "segregation by clusters". After proving this phenomenon in the next proposition, we will show that the empirical findings discussed in Kremer and Maskin (1996) and, to some extent, in the subsequent literature summarized, for example, in Gavilan (2011), can be compatible with our framework.

Let us first describe what a "clustered cyclic-partition" is: π_K^{cM} will denote a partition where the entire population is divided in clusters of consecutive MK players in the ability ordering, where each cluster of MK consecutively ordered players form K coalitions consisting of M members each, with the cyclic structure described in the previous sections. For example, the clustered partition π_2^{cM} has players $1, 3, ..., 2M-1$ in the first coalition, players $2, 4, ..., 2M$ in the second coalition, which completes the first cluster; then the third coalition (and first coalition of the second cluster consisting of two coalitions) has players $2M+1,\ 2M+3, ..., 4M-1$, and so on. For brevity, we use "(M,K)-partition" to refer to the clustered cyclic-partition π_K^{cM} for $M, K \geq 2$.

Proposition 4 *Suppose every coalition must consist of M members where M is a finite integer. Then, for each $K \in \mathbb{N}$, the (M,K)-partition π_K^{cM} described above, together with an imputation ratio $\sigma = a^K$ for every coalition, constitutes an s-stable system.*

Proof. Normalize to 1 the fixed operational output that any coalition Z of size M produces, so that the expected total surplus of Z is $p(Z)$. (For simplicity, we assume that a coalition of any other size produces zero output.) Then, the expected payoff of an agent, say i, of rank r in the k-th coalition of cluster $n \in \mathbb{N}$ in the (M,K)-partition, denoted by $Z_{n,k}$, is

$$\begin{aligned} u_i(Z_{n,k}) &= (1-a)\theta(Z_{n,k})\frac{(1-\sigma)\sigma^{r-1}}{1-\sigma^M} \\ &= (1-a)\frac{a^{MK(n-1)+k-1}(1-a^{MK})}{1-a^K}\frac{(1-\sigma)\sigma^{r-1}}{1-\sigma^M} \end{aligned} \tag{18}$$

where σ is the imputation ratio prevailing in the coalition $Z_{n,k}$. Here, the second equality follows from $\theta(Z_{n,k}) = a^{MK(n-1)}(a^{k-1}+a^{k-1+K}+\cdots+a^{k-1+(M-1)K})$. Observe that this agent is agent $i = MK(n-1)+k+K(r-1)$. From (18), this agent's expected payoff is $(1-a)a^{MK(n-1)+k+K(r-1)-1}$ when $\sigma = a^K$. This means that the players' expected payoffs in (M,K)-partition when $\sigma = a^K$ are identical to those in the symmetric K-cyclic system, i.e., player i's expected payoff is $(1-a)a^{i-1}$. Thus, no deviation D (of size M) with any imputation ratio σ' is profitable because the total surplus of this deviation, $(1-a)\theta(D)$, is equal to the sum of expected payoffs that members of D receive in the (M,K)-partition when $\sigma = a^K$. ■

Having shown that imposing a finite size on coalitions would generate stable partitions that display a form of segmentation/segregation, we can now make some potentially interesting remarks about income inequality within and across firms.

First of all, lower K means that there are a smaller number of firms of size M in each cluster (interpretable as an increase in concentration of each cluster). Hence, each firm is composed by agents who are closer to each other in ability, implying reduced inequality between members within a firm. At the same time, it also means that the firm-level income distribution is more widely dispersed, implying enlarged inequality across firms in the economy.[22] This is broadly consistent with the findings about within-plant and between-plant inequality summarized in Gavilan (2011).

On the other hand, payoff equivalence continues to hold and thus, no change in K can affect the skill premium. The skill premium (and hence the inequality of income between people of different ability) depends only on technology and the level of competition, which are fixed in our model. What we capture instead is the fact that even if and when the skill premium is stable, one can still observe different degrees of segmentation and a negative (positive) correlation between concentration of clusters and inequality within firms (inequality across firms in the economy).

One final caveat: our aim in this section is only to show that our model is compatible with theoretical and empirical statements made in other papers about segregation and inequality if we assume that coalitions should have a finite size. Thus, we do not offer a full characterization of all s-stable systems like in the previous section, since we believe that the connection between vertical inequality and the number and composition of parties/firms should remain our main focus, and studying this relationship required to allow for coalitions of an infinite size. We conjecture that if one attempts a full characterization of s-stable systems for coalitions of a finite size, the result will require a notion of "generalized cyclic clusters" in line with the generalized cyclic characterization result of the previous section.

6 Concluding Remarks

In this paper we have shown some important connections between vertical inequality within coalitions and the endogenous formation of coalition structures. In order to best emphasize the connection, we have first characterized how different distributions of abilities and different distributions of payoffs determine which coalition structures can be stable. Then we have shown that when the level of vertical inequality is endogenous, partitions consisting of any number of coalitions can be strongly stable, for any distribution of abilities in the considered class, as long as intra-coalition payoff inequality allows every player to receive her marginal contribution. Since in a cyclic partition the agent occupying a given rank in a coalition must have lower ability when there are more coalitions, this implies that vertical inequality must indeed increase in the number of coalitions.

We have ended the analysis by showing that if coalitions are forced to be of a finite size, then there is an endogenous determination of clusters of coalitions,

[22]Taking the range of total surplus levels of all firms in the economy as a crude measure of inequality across firms, for example, it is straightforward to see that it is higher for lower K in our model.

displaying segregation and some interesting connections between concentration of clusters and various notions of inequality.

One limitation of our model is that the value of a coalition does not depend on the partition of the rest of the players, while, for example, in plurality rule elections it makes a big difference for a coalition expecting 30 percent of the votes whether the rest is divided into 7 small parties of 10 percent each or two other parties of 35 each. This limitation is not important when coalitions are expected to be of similar strengths like in our cyclic partitions. If asymmetric coalition structures could emerge in a modified model, then the value of a coalition should reflect these asymmetries in the partition of others. An extension of the model in which the relative power of any coalition depends not only on the ability of it's members but also on some other dimension, like the exogenous distribution of voters' preferences, is in our future research agenda.

We note that our cooperative game theoretic results have the potential to be implementable and extendable in a dynamic stability setting like the one introduced by Acemoglu, Egorov and Sonin (2008), since the lack of commitment that constitutes their main tenet is conceptually or implicitly assumed even in our core-like cooperative logic. A dynamic stability analysis would therefore be a natural next step of this research, perhaps confirming that multiple steady states with different coalition structures and, accordingly, with different vertical inequality levels, can exist, in line with our Theorem 2.

A final remark or two about the potential empirical relevance of our findings are in order: In the literature on the number of parties, for example, the leading hypotheses elaborated and tested have all to do with the electoral formula (Duverger's law and Duverger's hypothesis), but it is well documented that even controlling for the electoral formula the number of parties of different countries varies enormously (think of India and the United States in the set of countries using a majoritarian system and Ireland and Italy within the set of countries using a more proportional system). Within each set of countries with homogeneous electoral institutions, one could verify whether vertical inequality across the major ranks of each party is indeed higher in countries with a larger number of stable parties. Similarly, dividing the US production of goods and services in a number of categories, we could evaluate income distribution across ranks in each industry or category and see if higher hierarchical inequality is correlated with a lower concentration in the sector (of course controlling for economies of scale and other relevant differences across sectors). However, verifying or falsifying our theoretical result is not going to be easy: first of all, Willis and Rosen (1979) rejected empirically the hypothesis that the ability or talent that matter for distribution of wages can be one dimensional; second, when evaluating in the aggregate, one could argue that increasing inequality and increasing concentration are the trends, and this connection has been discussed in the last section, but the model is too abstract to offer quantitative predictions. Perhaps one place where our results on the relationship between number of firms and inequality could be verified is by looking at hierarchical inequality of research scientists when comparing, for example, the pharmaceutical industry and the bio-tech sector, in the latter sector of which there are many more firms and vertical inequality might possibly be proved to be higher. One place where one could look for the relevance of our cyclic partition or cyclic cluster results could be the allocation of researchers across universities, or the composition of sport teams, provided that adequate data are available on the rankings of individuals based on a unidimensional measurement of ability, as well as the rankings of universities and teams.[23]

Appendix

Proof of Lemma 3: Let $o : Z_k \cup Z_{k'} \to \mathbb{N}$ be the ordering of agents in $Z_k \cup Z_{k'}$, i.e., $o(j) = \#\{i \in Z_k \cup Z_{k'} | i \leq j\}$. Let $\ell \geq 1$ be the number of agents in Z_k who are more able that the leader of $Z_{k'}$, denoted by $i_{k'}$. Using $o^{-1}(n)$ to denote the player j such that $o(j) = n$, we have $o^{-1}(\ell) \in Z_k$ and $o^{-1}(\ell + 1) = i_{k'} \in Z_{k'}$.

If Lemma 3 failed, two or more consecutively ordered agents in $\{i \in Z_k \cup Z_{k'} | i \geq i_{k'}\}$ would belong to the same coalition. To reach a contradiction, therefore, suppose that there are integers $m \geq \ell+1$ and $m' > m$ with the following property: all players $j \in Z_k \cup Z_{k'}$ such that $m \leq o(j) \leq m'$ belong to the same coalition. Without loss of generality, suppose m is the smallest such integer. We first consider the case that they all belong to Z_k, but the same argument works when they all belong to $Z_{k'}$ as well, as shown later.

By the way m is defined above, we have $o^{-1}(\ell+1) = i_{k'} \in Z_{k'}$ and $o^{-1}(\ell+2) \in Z_k$. In the deviation $(Z_k \cup \{o^{-1}(\ell+1)\}) \setminus \{o^{-1}(\ell+2)\}$, i.e., when the player $o^{-1}(\ell+1)$ replaces $o^{-1}(\ell+2)$ in Z_k, all remaining members of Z_k are strictly better off because the coalition's strength increased while their rankings remain the same. For the original partition to be stable, therefore, player $o^{-1}(\ell+1)$ should be worse off in this deviation, i.e.,

$$\big(\theta(Z_k) + a^{o^{-1}(\ell+1)-1} - a^{o^{-1}(\ell+2)-1}\big)\rho^{\ell} < \theta(Z_{k'}). \tag{19}$$

Note that, by retaking m' if necessary, we may assume $o^{-1}(m'+1) \in Z_{k'}$. In the deviation $D = (Z_{k'} \cup \{o^{-1}(m')\}) \setminus \{o^{-1}(m'+1)\}$, all members of $D \cap Z_{k'}$ are better off in D because $\theta(D) > \theta(Z_{k'})$ while their rankings remain the same. Thus, the payoff of player $o^{-1}(m')$ in D, $(\theta(Z_{k'}) + a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})(1-\rho)\rho^{(m-\ell)/2}$, should be lower than that in Z_k, $\theta(Z_k)(1-\rho)\rho^{\ell+(m-\ell)/2+m'-m-1}$, which is impossible because

[23] Even though the total ability criterion and the average ability criterion yield identical rankings for teams that have a fixed finite size, complementing our total ability criterion with some other criterion that puts some weight on other statistics could in principle be useful to capture more realistically the case of unrestricted coalition size. This study would require a separate (most likely computational) analysis, which we leave for future research.

(19) implies that the former payoff level is higher than the latter as calculated below:

$$\begin{aligned}
& (\theta(Z_{k'}) + a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m-\ell)/2} - \theta(Z_k)\rho^{(m+\ell)/2+m'-m-1} \\
> & \big(\theta(Z_k) + a^{o^{-1}(\ell+1)-1} - a^{o^{-1}(\ell+2)-1}\big)\rho^{(m+\ell)/2} + (a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m-\ell)/2} \\
& \qquad -\theta(Z_k)\rho^{(m+\ell)/2+m'-m-1} \\
= & \theta(Z_k)\rho^{(m+\ell)/2}(1-\rho^{m'-m-1}) + (a^{o^{-1}(\ell+1)-1} - a^{o^{-1}(\ell+2)-1})\rho^{(m+\ell)/2} \\
& \qquad +(a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m-\ell)/2} > 0
\end{aligned}$$

where the first inequality follows from (19).

For completeness, we now consider the alternative case that all players $j \in Z_k \cup Z_{k'}$ such that $m \leq o(j) \leq m'$ belong to $Z_{k'}$. In the deviation $(Z_{k'} \cup \{o^{-1}(\ell)\}) \setminus \{o^{-1}(\ell+1)\}$, all remaining members of $Z_{k'}$ is strictly better off because the coalition's strength increased while their rankings remain the same. For the original partition to be stable, therefore, player $o^{-1}(\ell)$ should be worse off in this deviation, i.e.,

$$\big(\theta(Z_{k'}) + a^{o^{-1}(\ell)-1} - a^{o^{-1}(\ell+1)-1}\big) < \theta(Z_k)\rho^{\ell-1}. \tag{20}$$

Note that we may assume $m'+1 \in Z_k$ without loss of generality. In the deviation $D = (Z_k \cup \{o^{-1}(m')\}) \setminus \{o^{-1}(m'+1)\}$, all members of $D \cap Z_k$ are better off in D because $\theta(D) > \theta(Z_k)$ while their rankings remain the same. Thus, the payoff of player $o^{-1}(m')$ in D, $(\theta(Z_k) + a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})(1-\rho)\rho^{\ell+(m-\ell-1)/2}$, should be lower than that in $Z_{k'}$, $\theta(Z_{k'})(1-\rho)\rho^{(m-\ell-1)/2+m'-m}$, which is impossible because (20) implies that the the former payoff level is higher than the latter as calculated below:

$$\begin{aligned}
& (\theta(Z_k) + a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m+\ell-1)/2} - \theta(Z_{k'})\rho^{(m-\ell-1)/2+m'-m} \\
> & \big(\theta(Z_{k'}) + a^{o^{-1}(\ell)-1} - a^{o^{-1}(\ell+1)-1}\big)\rho^{(m-\ell+1)/2} \\
& \qquad +(a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m+\ell-1)/2} - \theta(Z_{k'})\rho^{(m-\ell-1)/2+m'-m} \\
= & \big(\theta(Z_{k'})(1-\rho^{m'-m-1}) + a^{o^{-1}(\ell)-1} - a^{o^{-1}(\ell+1)-1}\big)\rho^{(m-\ell+1)/2} \\
& \qquad +(a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m+\ell-1)/2} > 0.
\end{aligned}$$

QED.

Proof of Theorem 1. The discussion up to (13) in the main text has been done formally. Thus, recalling the definition of D^* from (13), we now prove

[C] If an infinite deviation D is profitable relative to π^c_K, then so is D^*.

Assume $D^* \neq D$ to avoid triviality. Let $d_j > h$ denote $d_j \in D$ such that $r_{d_j}(D) = r_h(D) + j$ for $j = 1, 2, \cdots$.

To reach a contradiction, suppose $d_j < h + jK$ for some $j \geq 1$. If $d_j \in Z_k$, then $r_{d_j}(Z_k) < r_h(Z_k) + j$ so that $e_{d_j} = r_{d_j}(D) - r_{d_j}(Z_k) > r_h(D) - r_h(Z_k) = e^*$ (since

$r_{d_j}(D) = r_h(D) + j$), a contradiction. If $d_j \in Z_{k'}$ where $k' > k$, then an analogous argument establishes that $e_{d_j} = r_{d_j}(D) - r_{d_j}(Z_{k'}) > r_h(D) - r_h(Z_k) = e^*$, again a contradiction. If $d_j \in Z_{k'}$ where $k' < k$, then an analogous argument establishes that $e_{d_j} = r_{d_j}(D) - r_{d_j}(Z_{k'}) \geq r_h(D) - r_h(Z_k) = e^*$, but this still is a contradiction to selection of h described above, in particular, $e_n < e^*$ for all $n \in Z_{k'} \cap D$ if $k' < k$.

Thus, we have proved that $d_j \geq h + jK$ for all $j = 1, 2, \cdots$. Consequently, when members $d_1, d_2, \cdots$ of D are replaced by $h + K, h + 2K, \cdots$, respectively, the winning probability of the deviation increases, i.e., $p(D^*) > p(D)$. For players in D_h, therefore, the payoffs are higher in D^* than in D because the ranks do not change. Since D being profitable means $p(D)\rho^{r_h(D)-1} \geq p(Z_k)\rho^{r_h(Z_k)-1}$ and thus, $p(D^*)\rho^{r_h(D)-1} > p(Z_k)\rho^{r_h(Z_k)-1}$, it further follows that $p(D^*)\rho^{r_{h+jK}(D^*)-1} = p(D^*)\rho^{r_h(D)-1+j} > p(Z_k)\rho^{r_h(Z_k)-1+j} = p(Z_k)\rho^{r_{h+jK}(Z_k)-1}$ for all $j = 1, 2, \cdots$. This means that D^* is profitable, proving [C].

We now return to the proof of the Theorem. First, we ensure that the deviation $\{1\} \cup Z_K$ is not profitable: Since player 1 is better off in this deviation because $\theta(Z_K) + 1 > \theta(Z_1)$, we need that members of Z_K would be strictly worse off, i.e.,

$$(\theta(Z_K) + 1)\rho < \theta(Z_K) \;\Leftrightarrow\; \rho < \theta(Z_K)/(\theta(Z_K) + 1) = \frac{a^{K-1}}{1 + a^{K-1} - a^K}, \tag{21}$$

which is the second inequality of (7). Since the first inequality of (7) has already been shown to be a necessary and sufficient condition for no "break-off" deviation to be profitable, it remains to show that no other deviation is profitable if (7) holds. Due to [B] and [C] above, we only need to verify non-profitability of deviations of the form $D_f \cup Z_k^h$ for some $h \in Z_k$, where $D_f \subset N_{h-1} := \{1, 2, \cdots, h-1\}$. Hence, we focus on such deviations D below.

The following result proves useful:

[D] If D is profitable, $1 \notin D$ and (7) holds, then $\{i - \min D + 1 | i \in D\}$ is also profitable.

To see this, first note that if $\min D \notin Z_1$ then $D' = \{i - 1 | i \in D\}$ is also profitable because (i) player i's payoff in D is $(1-a)\theta(D)(1-\rho)\rho^{r_i(D)-1}$ while that of player $i-1$ in D' is $(1-a)\theta(D')(1-\rho)\rho^{r_i(D)-1} = (1-a)\theta(D)(1-\rho)\rho^{r_i(D)-1}/a$, and (ii) in the cyclic K-partition π_K^c, player i's payoff is no lower than a times that of player $i-1$ (because $\theta(Z_k) = \theta(Z_{k+1})/a$ and $a > \rho \geq a^K$). Furthermore, if D is profitable and $\min D \in Z_1 \setminus \{1\}$, then $D' = \{i - K | i \in D\}$ is also profitable. To see this, note that (i) player i's payoff in D is $(1-a)\theta(D)(1-\rho)\rho^{r_i(D)-1}$ while that of player $i-K$ in D' is $(1-a)\theta(D')(1-\rho)\rho^{r_i(D)-1} = (1-a)\theta(D)(1-\rho)\rho^{r_i(D)-1}/a^K$, and (ii) in π_K^c the payoff of player $i-K$ is $1/\rho$ times that of player i where $1/\rho \leq 1/a^K$ due to $\rho \geq a^K$. Without loss of generality, therefore, we assume that $1 \in D$ for a profitable D in the sequel.

To reach a contradiction, suppose there is a profitable deviation $D = D_f \cup Z_k^h \ni 1$. For player 1 to be no worse off in D, we must have $\theta(D) \geq \theta(Z_1)$.

First, consider the possibility that $\theta(Z_1) \leq \theta(D) < \theta(Z_K) + 1$. If player $j \in D$ is ranked strictly lower in D than in π_K^c, i.e., $r_j(D) > r_j(Z_k)$, then j's payoff in D would be no higher than $\theta(D)(1-\rho)\rho \cdot \rho^{r_j(Z_k)-1} < \theta(Z_k)(1-\rho)\rho^{r_j(Z_k)-1}$ because (21) implies that $\theta(D)\rho < (\theta(Z_K)+1)\rho < \theta(Z_K) \leq \theta(Z_k)$. Since this would mean player j is worse off in D than in π_K^c, we deduce that every player $j \in D$ is ranked at least as high in D as in π_K^c. This, in turn, means that there are at most t players in $D \cap \{1, 2, \cdots, tK\}$ for every $t \in \mathbb{N}$. The maximum possible θ-value of such D is $\theta(Z_1)$, which is obtained only when $D = Z_1$. Therefore, we conclude that no profitable deviation exists of the form $D = D_f \cup Z_k^h$ such that $\theta(Z_1) \leq \theta(D) < \theta(Z_K) + 1$.

Hence, $\theta(D) \geq \theta(Z_K) + 1$ must hold. Then, consider $D' = D \setminus \{1\}$. Any player $j \in D'$ has a payoff of $(1-a)(\theta(D)-1)(1-\rho)\rho^{r_j(D)-2}$ in D', i.e., $\frac{\theta(D)-1}{\theta(D)\rho}$ times of her payoff in D. Since $\frac{\theta(D)-1}{\theta(D)\rho}$ increases in $\theta(D)$, in conjunction with (21) we deduce that $\frac{\theta(D)-1}{\theta(D)\rho} \geq \frac{\theta(Z_K)}{(\theta(Z_K)+1)\rho} > 1$. Thus, D' is also profitable and consequently, so is $D'' = \{i - \min D' + 1 | i \in D'\}$ by [D]. Then, $\theta(D'') \geq \theta(Z_K) + 1$ must hold for the same reason that $\theta(D) \geq \theta(Z_K) + 1$ must hold as shown above. By recursively applying analogous argument to D'' and so on, we are bound to eventually come to a contradictory conclusion that $\theta(Z_1) \geq \theta(Z_K) + 1$ (because $D = D_f \cup Z_k^h$ for some finite D_f). This completes the proof that π_K^c is stable if and only if (7) holds.

The second part of the Theorem is proved in the main text. The third part follows from the observation that

$$\frac{a^{K-1} - \frac{a^{K-1}}{1+a^{K-1}-a^K}}{a^{K-1} - a^K} = \frac{a^{K-1}\left(\frac{a^{K-1}-a^K}{1+a^{K-1}-a^K}\right)}{a^{K-1} - a^K} = \frac{a^{K-1}}{1 + a^{K-1} - a^K}$$

converges uniformly to 0 as $K \to \infty$ on $[0, \bar{a}]$ for all $\bar{a} < 1$. QED.

Proof of Lemma 6: Suppose to the contrary that $\ell \geq 2$ and $y - x_\ell = x_{j+1} - x_j = b \geq 1$ is constant for all $j = 1, 2, \cdots, \ell - 1$, yet the partition π_K is stable. For notational ease, let x denote the most able agent in Z_k, i.e., $x = x_1$. Then, $x_j = x + (j-1)b$ for $j \leq \ell$; and $x_\ell = y - b$. Lemma 3 implies that each agent $i \in (x, x_2)$ belongs to a distinct coalition. This means that $k' \geq b + 1$, so that $\theta(Z_{k'}) \leq \frac{a^{y-1}}{1-a^{b+1}}$ by Lemma 3 and thus, in particular,

$$\theta(Z_{k'}) \; < \; \frac{a^{y-1}}{1-a^b}. \tag{22}$$

Furthermore, since Lemma 3 also implies, for every $n \in \mathbb{N}$, that $y_{n+1} - x_{\ell+n} \geq b$ where y_{n+1} and $x_{\ell+n}$ are the agents with ranks $n+1$ and $\ell+n$, respectively, in $Z_{k'}$ and Z_k, we deduce that

$$\theta(Z_k) \; \geq \; \frac{a^{x-1} - a^{y-b-1}}{1-a^b} + \frac{\theta(Z_{k'})}{a^b}. \tag{23}$$

For the deviation $Z_k \setminus \{x, x_2, \cdots, x_{\ell-1}\}$ not to be profitable, we need

$$\theta(Z_k)\rho^{\ell-1} \geq \theta(Z_k) - \frac{a^{x-1} - a^{y-b-1}}{1-a^b} \;\Leftrightarrow\; \rho \geq \Big(\frac{\theta(Z_k) - \frac{a^{x-1}-a^{y-b-1}}{1-a^b}}{\theta(Z_k)}\Big)^{1/(\ell-1)}. \tag{24}$$

For the deviation $Z_{k'} \cup \{x_\ell\}$ not to be profitable, on the other hand, we need either

$$\theta(Z_{k'}) \geq (\theta(Z_{k'}) + a^{y-b-1})\rho \;\Leftrightarrow\; \rho \leq \frac{\theta(Z_{k'})}{\theta(Z_{k'}) + a^{y-b-1}} \tag{25}$$

or

$$\theta(Z_k)\rho^{\ell-1} \geq (\theta(Z_{k'}) + a^{y-b-1}) \;\Leftrightarrow\; \rho \geq \Big(\frac{\theta(Z_{k'}) + a^{y-b-1}}{\theta(Z_k)}\Big)^{1/(\ell-1)}. \tag{26}$$

In addition, for the deviation $Z_{k'} \cup \{x, x+b, \cdots, x+(\ell-1)b\}$ not to be profitable, we need

$$\theta(Z_{k'}) > \Big(\theta(Z_{k'}) + \frac{a^{x-1} - a^{y-1}}{1-a^b}\Big)\rho^\ell \;\Leftrightarrow\; \rho < \Big(\frac{\theta(Z_{k'})}{\theta(Z_{k'}) + \frac{a^{x-1}-a^{y-1}}{1-a^b}}\Big)^{1/\ell}. \tag{27}$$

First, we show that (24) and (25) cannot hold simultaneously, by verifying that

$$\frac{\theta(Z_k) - \frac{a^{x-1}-a^{y-b-1}}{1-a^b}}{\theta(Z_k)} > \Big(\frac{\theta(Z_{k'})}{\theta(Z_{k'}) + a^{y-b-1}}\Big)^{\ell-1} \tag{28}$$

holds subject to (22) and (23). If $\ell = 2$, a routine calculation verifies that (28) holds unless (23) holds as an equality. In this latter case, (25) holds as equality, so that members of $Z_{k'}$ are indifferent in the deviation $Z_{k'} \cup \{x+b\}$; Agent $x+b$ is better off in this deviation since $\theta(Z_{k'}) + a^{x+b-1} - \theta(Z_k)\rho = \theta(Z_{k'}) + a^{x+b-1} - (a^{x-1} + \frac{\theta(Z_{k'})}{a^b})\frac{\theta(Z_{k'})}{\theta(Z_{k'})+a^{x+b-1}} = (\theta(Z_{k'}) + a^{x+b-1})(1 - \frac{\theta(Z_{k'})}{a^b\theta(Z_{k'})+a^{x+2b-1}}) > 0$ because $\frac{\theta(Z_{k'})}{a^b\theta(Z_{k'})+a^{x+2b-1}} < 1$ given that $\theta(Z_{k'}) < \frac{a^{x+2b-1}}{1-a^b}$ by (22).

Thus, consider $\ell > 2$. Since the RHS of both (24) and (25) increase in $\theta(Z_k)$ and $\theta(Z_{k'})$, respectively, it suffices to show that (28) holds at the lower bound of $\theta(Z_k)$ given by (23), i.e., that

$$\frac{\theta(Z_{k'})}{\frac{a^{x+b-1}-a^{y-1}}{1-a^b} + \theta(Z_{k'})} > \Big(\frac{\theta(Z_{k'})}{\theta(Z_{k'}) + a^{y-b-1}}\Big)^{\ell-1}, \tag{29}$$

or equivalently, by taking log and subtracting, that

$$\Delta := \log\Big(\frac{\theta(Z_{k'})}{\frac{a^{x+b-1}-a^{y-1}}{1-a^b} + \theta(Z_{k'})}\Big) - (\ell-1)\log\Big(\frac{\theta(Z_{k'})}{\theta(Z_{k'}) + a^{y-b-1}}\Big) > 0 \tag{30}$$

for all feasible $\theta(Z_{k'})$ as per (22). Given that $x+\ell b=y$, a routine calculation verifies that $\Delta=0$ when $\theta(Z_{k'})=a^{y-1}/(1-a^b)$, the upper bound of $\theta(Z_{k'})$ by (22). Thus, it suffices to show that the derivative of Δ wrt $\theta(Z_{k'})$,

$$\Delta' = \frac{(\frac{a^{x+b-1}-a^{y-1}}{1-a^b})B-(\ell-1)a^{y-b-1}A}{\theta(Z_{k'})AB} \tag{31}$$

is strictly negative for all $\theta(Z_{k'})<a^{y-1}/(1-a^b)$, where

$$A=\frac{a^{x+b-1}-a^{y-1}}{1-a^b}+\theta(Z_{k'}) \quad \text{and} \quad B=a^{y-b-1}+\theta(Z_{k'}). \tag{32}$$

Using $y=x+\ell b$, it is routinely calculated that the top line of Δ', evaluated at $\theta(Z_{k'})=a^{y-1}/(1-a^b)$, is $a^{x+y-2}(1-a^b)^{-1}\big(1-a^{b(\ell-1)}-(\ell-1)(1-a^b)\big)$. This is easily verified to be negative because $\big(1-a^{b(\ell-1)}-(\ell-1)(1-a^b)\big)$ increases in a to a value of 0 when $a=1$. (The derivative of it is $b(\ell-1)(a^{b-1}-a^{b(\ell-1)-1})>0$). It now therefore suffices to show that the derivative of the top line of Δ' wrt $\theta(Z_{k'})$, denoted by Δ'' by a slight abuse of notation, is positive for all $\theta(Z_{k'})<a^{y-1}/(1-a^b)$.

Observe that

$$\Delta''=\frac{a^{x-1}}{a^b-1}\Big(a^{b\ell}-a^b-a^{b(\ell-1)}(\ell-1)(a^b-1)\Big)=\frac{\alpha a^{x-1}}{\alpha-1}\Big(\alpha^{(\ell-1)}-1-\alpha^{(\ell-2)}(\ell-1)(\alpha-1)\Big) \tag{33}$$

where $\alpha=a^b$. Hence, it boils down to showing that $f(\alpha)=\alpha^{(\ell-1)}-1-\alpha^{(\ell-2)}(\ell-1)(\alpha-1)<0$ for all $\alpha\in(0,1)$, which is the case because $f'(\alpha)=-\alpha^{\ell-3}(\alpha-1)(\ell-1)(\ell-2)>0$ and $f(1)=0$. This proves that (24) and (25) cannot hold simultaneously.

This means that (26) must hold. Then, since $\theta(Z_k)<\theta(Z_{k'})+\frac{a^{x-1}-a^{y-1}}{1-a^b}$, (27) is inconsistent with (26) if

$$\left(\frac{\theta(Z_{k'})+a^{y-b-1}}{\theta(Z_{k'})+\frac{a^{x-1}-a^{y-1}}{1-a^b}}\right)^{\ell} \geq \left(\frac{\theta(Z_{k'})}{\theta(Z_{k'})+\frac{a^{x-1}-a^{y-1}}{1-a^b}}\right)^{\ell-1} \tag{34}$$

for all $\theta(Z_{k'})<a^{y-1}/(1-a^b)$, or equivalently, by taking log and subtracting, if

$$\tilde{\Delta}:=\ell\log[\theta(Z_{k'})+a^{y-b-1}]-\log[\theta(Z_{k'})+\frac{a^{x-1}-a^{y-1}}{1-a^b}]-(\ell-1)\log[\theta(Z_{k'})]\geq 0. \tag{35}$$

Since it is routinely calculated that $\tilde{\Delta}=0$ when $\theta(Z_{k'})=a^{y-1}/(1-a^b)$ as before, to establish (34) it suffices to show that the derivative of $\tilde{\Delta}$ with respect to $\theta(Z_{k'})$, denoted by $\tilde{\Delta}'$, is negative, i.e.,

$$\tilde{\Delta}' = \frac{\ell}{\theta(Z_{k'})+a^{y-b-1}}-\frac{1}{\theta(Z_{k'})+\frac{a^{x-1}-a^{y-1}}{1-a^b}}-(\ell-1)\frac{1}{\theta(Z_{k'})} \tag{36}$$

$$= C\cdot\frac{1}{\theta(Z_{k'})}\leq 0 \quad \text{for all} \quad 0<\theta(Z_{k'})<a^{y-1}/(1-a^b) \tag{37}$$

where $C = \frac{\ell\theta(Z_{k'})}{\theta(Z_{k'})+a^{y-b-1}} - \frac{\theta(Z_{k'})}{\theta(Z_{k'})+\frac{a^{x-1}-a^{y-1}}{1-a^b}} + 1 - \ell$. Observe that solving $C = 0$ for $\theta(Z_{k'})$ produces a unique solution because the term containing $\theta(Z_{k'})^2$ vanishes. Hence, it suffices to show that the value of C is negative at both boundary points of $\theta(Z_{k'})$. The value of C is $1-\ell < 0$ at $\theta(Z_{k'}) = 0$. The value at $\theta(Z_{k'}) = a^{y-1}/(1-a^b)$, calculated to be $1 - a^{y-x} - \ell(1 - a^b)$, is also negative because it is 0 when $a = 1$ and the derivative of $1 - a^{y-x} - \ell(1 - a^b)$ with respect to a is $-(y - x)a^{y-x-1} + b\ell a^{b-1} = -b\ell(a^{b\ell-1} - a^{b-1}) > 0$ for $a \in (0, 1)$ where the equality follows from $y = x + b\ell$. This establishes that (27) is inconsistent with (26) for relevant values of $\theta(Z_{k'})$, which completes the proof. QED.

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Supplementary Material: An elaboration of the proof of Proposition 2

We start with a lemma:

Lemma A. *Let $\pi_K = \{Z_1, Z_2, \cdots, Z_K\}$ be a stable K-partition with coalitions labelled according to the most able members' abilities: $i_k < i_{k'}$ for $1 \leq k < k' \leq K$ (where i_k is the most able member of Z_k). Suppose ability ordering applies to all agents $i \leq M$ for some $M > i_K$. For $Z_k, Z_{k'} \in \pi_K$ where $k < k'$, we have (i) $\theta(Z_k) > \theta(Z_{k'})$ and (ii) all agents in $\{i \in Z_k \cup Z_{k'} | i_{k'} \leq i \leq M\}$ alternate between $Z_{k'}$ and Z_k as they go down the ability order.*

Proof. (a) That $\theta(Z_k) > \theta(Z_{k'})$ is straightforward by the same reasoning as used in the proof of Lemma 2. The second part follows from an argument analogous to the proof of Lemma 3 as detailed below.

Let $o : Z_k \cup Z_{k'} \to \mathbb{N}$ be the ability ordering of agents in $Z_k \cup Z_{k'}$, i.e., $o(j) = \#\{i \in Z_k \cup Z_{k'} | i \leq j\}$. Let $\ell \geq 1$ be the number of agents in Z_k who are more able that the leader of $Z_{k'}$, denoted by $i_{k'}$. Using $o^{-1}(n)$ to denote the player j such that $o(j) = n$, we have $o^{-1}(\ell) \in Z_k$ and $o^{-1}(\ell + 1) = i_{k'} \in Z_{k'}$.

If assertion (ii) did not hold, two or more consecutively ordered agents in $\{i \in Z_k \cup Z_{k'} | i_{k'} \leq i \leq M\}$ would belong to the same coalition. To reach a contradiction, therefore, suppose that there are integers $m > \ell$ and $m' > m$ with the following property: $o^{-1}(m') \leq M$ and all players $j \in Z_k \cup Z_{k'}$ such that $m \leq o(j) \leq m'$ belong to the same coalition. Without loss of generality, suppose m is the smallest such integer. We discuss the case that they all belong to Z_k below. The same argument works when they all belong to $Z_{k'}$ as well (which we omit here).

By the way m is defined above, we have $o^{-1}(\ell+1) = i_{k'} \in Z_{k'}$ and $o^{-1}(\ell+2) \in Z_k$. In the deviation $(Z_k \cup \{o^{-1}(\ell + 1)\}) \setminus \{o^{-1}(\ell + 2)\}$, i.e., when the player $o^{-1}(\ell + 1)$ replaces $o^{-1}(\ell+2)$ in Z_k, all remaining members of Z_k are strictly better off because the coalition's strength increased while their rankings remain the same. For the original partition to be stable, therefore, player $o^{-1}(\ell + 1)$ should be worse off in this deviation, i.e.,

$$\big(\theta(Z_k) + a^{o^{-1}(\ell+1)-1} - a^{o^{-1}(\ell+2)-1}\big)\rho^{\ell} < \theta(Z_{k'}). \tag{38}$$

Without loss of generality, we may assume that either $o^{-1}(m' + 1) \in Z_{k'}$ or $o^{-1}(m' + 1) > M$. Consider the first case that $o^{-1}(m' + 1) \in Z_{k'}$. In the deviation $D = (Z_{k'} \cup \{o^{-1}(m')\}) \setminus \{o^{-1}(m' + 1)\}$, all members of $D \cap Z_{k'}$ are better off in D because $\theta(D) > \theta(Z_{k'})$ while their rankings remain the same. Thus, the payoff of player $o^{-1}(m')$ in D, $(\theta(Z_{k'}) + a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})(1 - \rho)\rho^{(m-\ell)/2}$, should be lower than that in Z_k, $\theta(Z_k)(1 - \rho)\rho^{\ell+(m-\ell)/2+m'-m-1}$, which is impossible because

(38) implies that the former payoff level is higher than the latter as calculated below:

$$\begin{aligned}
& (\theta(Z_{k'}) + a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m-\ell)/2} - \theta(Z_k)\rho^{(m+\ell)/2+m'-m-1} \\
> \; & \big(\theta(Z_k) + a^{o^{-1}(\ell+1)-1} - a^{o^{-1}(\ell+2)-1}\big)\rho^{(m+\ell)/2} + (a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m-\ell)/2} \\
& \qquad -\theta(Z_k)\rho^{(m+\ell)/2+m'-m-1} \\
= \; & \theta(Z_k)\rho^{(m+\ell)/2}(1-\rho^{m'-m-1}) + (a^{o^{-1}(\ell+1)-1} - a^{o^{-1}(\ell+2)-1})\rho^{(m+\ell)/2} \\
& \qquad +(a^{o^{-1}(m')-1} - a^{o^{-1}(m'+1)-1})\rho^{(m-\ell)/2} > 0
\end{aligned}$$

where the first inequality follows from (38).

Next, consider the other case that $o^{-1}(m'+1) > M$. Let $L > M$ be the player who is ranked highest among all $\{i \in Z_{k'} | i > M\}$, i.e., her rank in $Z_{k'}$ is $(m-\ell)/2+1$. In the deviation $D = (Z_{k'} \cup \{o^{-1}(m')\}) \setminus \{L\}$, all members of $D \cap Z_{k'}$ are better off in D because $\theta(D) > \theta(Z_{k'})$ while their rankings remain the same. Thus, the payoff of player $o^{-1}(m')$ in D, $(\theta(Z_{k'}) + a^{o^{-1}(m')-1} - a^{L-1})(1-\rho)\rho^{(m-\ell)/2}$, should be lower than that in Z_k, $\theta(Z_k)(1-\rho)\rho^{\ell+(m-\ell)/2+m'-m-1}$, which is impossible because (38) implies that the former payoff level is higher than the latter by the same calculation as above (with $o^{-1}(m'+1)$ replaced with L). ■

We now turn to the proof of Proposition 2. Suppose there is a stable K-partition, π_K, with reversed rankings within coalitions. Let M be arbitrarily large integer such that there are two agents in $\{i \leq M\}$ such that they are reversely ranked within the same coalition. For each coalition $Z_k \in \pi_K$, find the most able agent $i \in Z_k$ whose rank in Z_k is strictly lower than her ability ranking in Z_k, i.e., i's rank in Z_k is strictly lower than $\#\{j \in Z_k | j \leq i\}$, and switch the ranks of player i and the player who is currently assigned to the rank $\#\{j \in Z_k | j \leq i\}$. By the reasoning in the first paragraph of the proof of Proposition 2, the partition with the switched ranking is stable. Continue this process until all agents in $\{j \in Z_k | j \leq M\}$ are ranked at $\#\{j \in Z_k | j \leq i\}$. Within a finite number of switching this process completes and produces a new ranking of the same partition π_K that satisfies ability ranking for all agents $i \leq M$, denoted by π_K^M, which is stable. Without loss of generality, we label coalitions in π_K^M according to the best member's abilities.

In π_K^M, by Lemma A (ii), no two consecutively ranked agents in $\{i | i_2 \leq i \leq M\}$ belong to the same coalition, where i_2 is the leader of the coalition Z_2. By Lemma 7, these agents must have been ability ordered in the original ranking. Hence, the only remaining possibility of reverse ranking in the original ranking is that $i_2 > 2$ and some agents in $\{i \in Z_1 | i \leq i_2\}$ were reverse ranked.

Thus, let $\ell = i_2 - 1$ be the number of agents in Z_1 who are more able than i_2. Let $y = i_2$ for notational ease. In π_K^M, all agents $i \leq M$ are ability ordered. Consider $Z_1^\ell = \{i \in Z_1 | i \geq \ell\}$. If agents in $Z_1^\ell \cup Z_2$ do not alternate their memberships between Z_1 and Z_2 as they go down the ability order, then one can find large enough M' such that $\pi_K^{M'}$ is not stable by Lemma A (i.e., find M' such that there are two agents in adjacent ability ranking in $Z_1^\ell \cup Z_2$ that belong to the same coalition). Since this would contradict the earlier observation that stability is preserved after switching the ranks of two agents who were initially reversed ranked, we deduce that all agents in $Z_1^\ell \cup Z_2$ alternate their memberships between Z_1 and Z_2 as they go down the ability order. This means that $\theta(Z_1) \geq \frac{1-a^{\ell-1}}{1-a} + \theta(Z_2)/a$, i.e., (23) holds for the current case, namely, $x = 1,\ b = 1$, so that $y = \ell + 1$. Hence, since the proof of Lemma 6 is independent of the ranking of agents $i > i_2$, it can be established that π_K^M is not stable by the same argument. Since this would contradict the earlier assertion that π_K^M is stable, we conclude that $i_2 = 2$. Since this in turn would contradict the earlier finding that $i_2 > 2$, we reach a final conclusion that our initial supposition was wrong that a stable K-partition, π_K, had reversed rankings within coalitions, completing the proof.

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