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Stability and Cycles in a Cobweb Model

with Heterogeneous Expectations

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Abstract

We investigate the dynamics of a cobweb model with heterogeneous beliefs, generalising the example of Brock and Hommes (1997). We examine situations where the agents form expectations by using either perfect foresight, or a form of adaptive expectations with limited memory defined from the last two prices. We specify conditions that generate cycles. These conditions depend on a set of factors that includes the proportion of rational agents, the intensity of switching between beliefs, and the adaption parameter. We show that both Flip bifurcation and Neimark-Sacker bifurcation can occur as primary bifurcation when the steady state is unstable.

JEL Classification: C62, D84, E30.
Key Words: Bounded rationality, Cobweb model, Flip bifurcation, Neimark-Sacker bifurcation, Rational expectations.

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1. Introduction

As first developed in the cobweb model of Brock and Hommes (1997), expectation formation arising from rational choice between various costly forecasts may generate equilibrium instability. Brock and Hommes (1997) presented a systematic dynamical analysis based on the new concept of Adaptively Rational Equilibrium Dynamics (ARED). The present paper further develops this approach and aims at characterising such instability. We show that when the steady state is unstable, supercritical Flip bifurcation as well as Neimark-Sacker bifurcation[1] can occur under specific conditions. The resulting cycles can be attracting for a set of parameters.

Over the past decade, a growing number of papers have dealt with the role of heterogeneous expectations in generating instability (Chiarella and He, 1998, 2001; Franke and Neseman, 1999; Goeree and Hommes, 2000; Hommes, 1991). While economic implications of these studies are

[1] Both types of primary bifurcation were also present in a cobweb model with homogeneous and adaptive expectations (Hommes, 1998).

obvious for some specific markets,[2] most papers, including ours, are based on the simple cobweb model since it appears to be a useful tool.

The framework as well as the economic meaning of this paper are close to those of Brock and Hommes[3] (1997).

Let us first consider the framework. Expectation formation is modeled as an economic decision. Indeed, producers choose a predictor between two expectation functions. The predictor's performance is defined as the net realized profits in the most recent period less the cost associated with the predictor. Depending on this performance, each producer may at every period switch from a predictor to another. For producers as a whole, this switching process, which is perfectly endogenous, may occur at various levels of intensity.

Let us now turn to the economic meaning of this class of models (Brock and Hommes, 1997; Branch, 2002). Under the previous assumptions on the expectation formation and the ARED concept, the instability of the steady state is generated by a simple but powerful economic mechanism which could be intuitively described as follows.[4]

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[2] See for instance Frankel and Froot (1990) for concerns related to the Foreign Exchange Market.

[3] See also Brock and Hommes (1995).

[4] A basic but necessary assumption used in the literature on this topic is the local instability of the steady state when all agents use the less sophisticated predictor. For that purpose, the slope of the supply function must be larger than the absolute value of the slope of the demand function.

On the one hand, when the price is close to its steady-state value, very few agents use the most sophisticated predictor since its cost exceeds the benefits of its forecast. Therefore, the distance between the current price and its steady-state value grows large over time.

On the other hand, while its cost is significant, the sophisticated predictor provides a better net return when the current price is far from its steady-state value. Thus, the distance between both prices gets smaller over time.

Consequently, price oscillations are endogenously generated in the steady-state neighbourhood.

Our main contribution is to show that the mechanism described above may lead to the possibility of cycles through primary bifurcations. Their existence is directly linked to our definition of the expectation functions. While Brock and Hommes (1997) assume that costly rational expectations are competing with costless naïve expectations, we replace the latter by costless adaptive expectations. More precisely, we assume that adaptive expectations are a weighted average of the last two prices. Such an assumption[5] which is crucial for our results, seems to be more

[5] This assumption is present in Hommes (1998) who studies the homogeneous cobweb model. A similar formulation is also used in the cobweb model of Chiarella and He (1998).

appropriate than those used previously, for instance by Brock and Hommes (1997) or by Branch (2002).

Indeed, it should be noted that our costless predictor is not too unsophisticated. Our predictor is a reasonable forecasting strategy for boundedly rational agents.[6] According to proponents of Bounded Rationality Theory, such as Simon (1957) or Baumol and Quandt (1964), our assumption may be justified as follows. First, it is often believed that the agents can loose or forget information quickly. We can then imagine that beyond two periods they don't keep the information about prices. Second, one could also think that agents could also believe that the prices observed more than two periods ago will have no impact (or so little) impact on future prices that it is not necessary to take account of that information. Third, one could conjecture that the extra-cost in keeping and taking that information into account would exceed the extra benefit to be obtained. Therefore, it would be "economically rational" not to take these earlier prices into account in the prediction function. Fourth, as suggested by Simon (1957), this strategy may simply be connected to the limited capacity of individuals to store and process information.

[6] The reference to bounded rationality is quite common in the literature on heterogeneous expectations. See for instance Hommes (2000).

It should also be mentioned that it seems to be intuitive that the more sophisticated a predictor is, the more costly it is. In his theorem 8, Branch (2002, p. 77) considers a model similar to ours, i.e. a model with rational versus adaptive expectations. Although the adaptive expectations he refers to are a weighted average (with exponentially declining weights) of all past prices, he assumes that their cost is nil. However, costless adaptive expectations seem to be more realistic when they are based on a finite number of price observations, as we assume.[7]

Given the existing literature derived from Brock and Hommes (1997), our model allows us to derive two new results.

First, the model of Brock and Hommes (1997) becomes a special case of our model. Indeed, the naïve expectations they consider correspond to our adaptive expectations when all weight is put on the more recent price. As we consider an expectation function with two lags, the dimension of the dynamical system of our model increases from 2 to 3. Due to this change, we are able to demonstrate the existence of a new type of primary bifurcation, namely a primary Neimark-Sacker bifurcation.[8]

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[7] In our case, they are based on the two most recent prices.

[8] See Proposition 3. For a mathematical exposition of bifurcations, we refer to Kuznetsov (2000).

Second, Branch (2002, pp. 77-78) studies a model close to ours where agents choose between a costly predictor and a costless adaptive predictor defined as a weighted average of all past prices. One of his main conclusions (Theorem 8, p. 77) stated that the stability conditions of the steady state are broader when adaptive expectations put "enough" weight on the past. Our conclusion is more cautious. First, the stability zone is wider when the agents base their adaptive expectations on both past prices with more weight on the most recent price. Second, the instability of the steady state may lead to stable cycles. On the one hand, these cycles may appear when the agents put "enough" weight on the current price (cycles occurring through a Flip bifurcation). On the other hand, stable cycles can also occur when the agents put "reduced" weight on the most recent price (cycles occurring through the Neimark-Sacker bifurcation).

The paper is organized as follows. The cobweb model and its dynamics under rational versus adaptive expectations are presented in Section 2. The stability conditions of the steady state and of periodic equilibria are stated in Section 3. Section 4 concludes.

2. The Cobweb Model with Rational vs Adaptive Expectations

We are going to present an extension of the model of Brock and Hommes (1997) that focuses on the case of rational versus naïve expectations. The only two changes to their framework are the following. On the one hand, we consider the introduction of an adaptive expectation function with two lags rather than naïve expectations. On the other hand, the analysis is based on the relative number of agents using rational expectations compared to the number of agents using adaptive expectations, denoted by n_1. Although the second change is just a matter of presentation, the first change, through small, leads to significant differences in results. To make the results comparable with these of Brock and Hommes (1997), we follow closely their setup. More recently, Branch (2002) considers a more generalised setting. Indeed, he examines in detail the stability properties of the cobweb model when agents can choose between three predictors: the perfect-foresight predictor, the naïve predictor and adaptive beliefs. We will discuss our results with respect to his in the next section.

In the Brock and Hommes (1997) framework, supply decisions are made by choosing the output that maximises expected profits subject to the one-period production lag. That is,

$$\max_q \left(p_{t+1}^e \, q - c(q) \right)$$

where $c(q)$ is the cost function which is increasing in q.

Price expectations, p_{t+1}^{e}, are formed by choosing a predictor from a set of expectation functions. Given this heterogeneity in expectation formation, market supply is a weighted sum of the supply decisions of the heterogeneous agents. The weights are simply the proportion of agents using a specific predictor. That is, each agent chooses $H_j \in \{H_1, H_2, \ldots, H_K\}$ where each predictor depends upon a vector of past prices $\vec{P}_t = (p_t, p_{t-1}, \ldots, p_0)$. The fractions of agents using a given predictor, $n_{j,t}\left(p_t, \mathrm{H}\left(\vec{P}_{t-1}\right)\right)$ depend on the current price and on the vectors of previous predictors:

$$\mathbf{H}\left(\vec{P}_{t-1}\right) = \left(H_1\left(\vec{P}_{t-1}\right), H_2\left(\vec{P}_{t-1}\right), \ldots, H_K\left(\vec{P}_{t-1}\right)\right).$$

Therefore, market equilibrium is given by the equation:

$$D(p_{t+1}) = \sum_{j=1}^{K} n_{j,t}\left(p_t, \mathbf{H}\left(\overrightarrow{P_{t-1}}\right)\right) S\left(H_j\left(\overrightarrow{P_t}\right)\right)$$

where $D(.)$ is the demand function and $S(.)$ is the supply function.

To keep the model analytically tractable, we assume linear demand and supply. Therefore let $D(p_t) = F - B\,p_t$ be the demand and $S\left(H_j\left(\vec{P}\right)\right) = b\,H_j\left(\vec{P}\right)$, with $F, B, b \in R_+$.

Without loss of generalisation to the stability properties, we set F equal to zero. Market equilibrium when $H_j \in \{H_1, H_2\}$ is determined by the condition

$$D(p_{t+1}) = n_{1,t}\, S\left(H_1\left(\vec{P}_t\right)\right) + n_{2,t}\, S\left(H_2\left(\vec{P}_t\right)\right) \tag{1}$$

where the two predictor functions are defined as

$$H_1\left(\vec{P}_t\right) = p_{t+1} \text{ with cost } C \geq 0, \tag{2}$$

$$H_2\left(\vec{P}_t\right) = \tau\, p_t + (1-\tau)\, p_{t-1} \text{ with } 0 < \tau < 1 \text{ and no cost.} \tag{3}$$

Each period, after observing the new price and assessing the accuracy of their forecasts, producers update their prediction of next period's price. The evolution of the proportion of agents using a particular predictor is given by

$$n_{j,t+1} = Exp\left[\beta\, U_{j,t+1}\right] \Big/ \sum_{j=1}^{K} Exp\left[\beta\, U_{j,t+1}\right]. \tag{4}$$

$U_{j,t+1}$ is a measure of the welfare associated with a certain predictor. The variable β parameterises preferences over profits. The larger the β, the more likely a producer will switch to an expectation with slightly higher returns. Brock and Hommes call this the "intensity of choice" parameter. Assume that the measure of the welfare is equal to realised net profits in the last period, then we get

$$U_{j,t+1} = \pi_j\left(p_{t+1}, \mathbf{H}\left(\overrightarrow{P_t}\right)\right)$$

where $\pi_j\left(p_{t+1}, \mathbf{H}\left(\overrightarrow{P_t}\right)\right) = p_{t+1}\, S\left(H_j\left(\vec{P}_t\right)\right) - c\left(S\left(H_j\left(\vec{P}_t\right)\right)\right) - C_j$.

C_j is the fixed cost associated with H_j. The cost of production is a simple quadratic cost function $c(q) = q^2 / (2b)$. The profit functions for producers using each predictor are respectively:

$$\pi_1\left(p_{t+1}, p_{t+1}\right) = \frac{b}{2} p_{t+1}^2 - C \tag{5}$$

$$\pi_2\left(p_{t+1}, p_t, p_{t-1}\right) = \frac{b}{2}\left(\tau\, p_t + (1-\tau)p_{t-1}\right)\left(2\, p_{t+1} - \left(\tau\, p_t + (1-\tau)p_{t-1}\right)\right) \tag{6}$$

Then plugging these into (4) leads to the law of motion for the two predictors:

$$n_{1,t+1} = \mathrm{Exp}\left[\beta\left(\frac{b}{2} p_{t+1}^2 - C\right)\right] \Big/ Z_{t+1} \tag{7}$$

$$n_{2,t+1} = \mathrm{Exp}\left[\beta \frac{b}{2}\left(\tau\, p_t + (1-\tau)p_{t-1}\right)\left(2\, p_{t+1} - \left(\tau\, p_t + (1-\tau)p_{t-1}\right)\right)\right] \Big/ Z_{t+1} \tag{8}$$

where $Z_{t+1} = \sum_{\mathrm{j}=1}^{2} \mathrm{Exp}\left[\beta\, \pi_{j,t+1}\right]$ and $n_{1,t+1} + n_{2,t+1} = 1$.

The cobweb model with rational and adaptive expectations is a system (S) of non-linear difference equations that governs the law of motion of price (9) and the law of motion of the proportion of agents using the rational expectation predictor (10):

$$p_{t+1} = \phi(p_t, p_{t-1}, n_{1,t}) \tag{9}$$

$$n_{1,t+1} = \varphi(p_t, p_{t-1}, n_{1,t}) \tag{10}$$

where

$\phi(p_t, p_{t-1}, n_{1,t}) = A(n_{1,t})(\tau p_t + (1-\tau)p_{t-1})$,

$A(n_{1,t}) = b(n_{1,t} - 1)/(B + b\,n_{1,t})$, and

$$\varphi(p_t, p_{t-1}, n_{1,t}) = \frac{1}{1 + \mathrm{Exp}\left[-\beta\left\{\frac{b}{2}\left((\tau p_t + (1-\tau)p_{t-1})^2 \left(A(n_{1,t}) - 1\right)^2\right) - 2C\right\}\right]}$$

Since (9) and (10) are respectively a second-order difference equation and a first-order difference equation, the system (S) can be rewritten as a system of three first-order difference equations (S'):

$$h_{t+1} = p_t \tag{11}$$

$$p_{t+1} = \phi(h_t, p_t, n_{1,t}) \tag{12}$$

$$n_{1,t+1} = \varphi(h_t, p_t, n_{1,t}) \tag{13}$$

The stability or the instability of the steady state issued from the system (S') formed by the equations (11), (12), and (13) can be directly investigated by looking at the Jacobian matrix of (S') taken at the steady state. These stability properties will be studied in the following section.

3. Stability and Cycles

A simple computation shows that the system (S') has a unique steady state $E = \left(0,0,\bar{n}_1(\beta) = 1/\left(1 + \mathrm{Exp}[\beta C]\right)\right)$. To ease the presentation, let us assume that $C = 0$ or $C = 1$. When $C = 0$, the agents have free access to the sophisticated predictor.

Remark: $\partial A(\bar{n}_1(\beta))/\partial \beta < 0$ (The proof is left to the reader.)

Proposition 1

Assume[9] that the slopes of the supply and the demand satisfy $b/B > 1$. When the information costs are nil, the steady state is $E = \left(0, 0, \bar{n}_1(\beta) = 1/2\right)$ and is always asymptotically stable.

The proof is left to the reader.

Proposition 2 (Possibility of a primary Flip bifurcation)

Assume that the slopes of the supply and the demand satisfy $b/B > 1$. Assume that $2/3 < \tau < 1$ and the information cost is 1. In the cobweb model with rational versus adaptive expectations, there is a unique steady state defined by $E = \left(0, 0, \bar{n}_1(\beta)\right)$, where $\bar{n}_1(\beta) = 1/\left(1 + Exp\beta\right)$. Further the steady state has the following properties:

[9] This assumption is equivalent to assuming that the steady state is unstable if all agents use the less sophisticated predictor, namely adaptive expectations.

i) There exists a critical value β_1 such that for $0 \le \beta < \beta_1$, the equilibrium is asymptotically stable while for $\beta > \beta_1$, this equilibrium is unstable with eigenvalues 0, $\lambda_{1,2} = \dfrac{A(\overline{n}_1(\beta))\tau \pm \sqrt{A(\overline{n}_1(\beta))(A(\overline{n}_1(\beta))\tau^2 + 4(1-\tau))}}{2}$ with $\lambda_1 < -1$ and $-1 < \lambda_2 < 0$. At the critical value, $\overline{n}_1(\beta_1) = \dfrac{2\,b\,\tau - (b+B)}{2\,b\,\tau}$.

ii) When the steady state is unstable, the system can undergo a Flip bifurcation. This bifurcation is supercritical, i.e.a stable cycle appears.

Proposition 3 (Possibility of a primary Neimark-Sacker bifurcation)
Assume that the slopes of the supply and the demand satisfy $b/B > 1$. Assume that $0 < \tau < 1/2$ and $1/2 < \tau < 2/3$ and the information cost is 1. In the cobweb model with rational versus adaptive expectations, there is a unique steady state defined by $E = (0, 0, \overline{n}_1(\beta))$, where $\overline{n}_1(\beta) = 1/(1 + Exp\beta)$. Further the steady state has the following properties:

i) There exists a critical value β_2 such that for $0 \le \beta < \beta_2$, the equilibrium is asymptotically stable while for $\beta > \beta_2$, this equilibrium is unstable with eigenvalues 0,

$\lambda_{1,2} = \dfrac{A(\bar{n}_1(\beta))\tau \pm i\sqrt{A(\bar{n}_1(\beta))(A(\bar{n}_1(\beta))\tau^2 + 4(1-\tau))}}{2}$. At the critical value, $\bar{n}_1(\beta_2) = \dfrac{b - B - b\tau}{2b - b\tau}$.

ii) When the steady state is unstable, the system can undergo a Neimark-Sacker bifurcation. This bifurcation is supercritical when $\tau \in (0, 0.203817) \cup (0.59299, 2/3)$.

Propositions 2 and 3 are illustrated by the Figures 1 to 4.

Figure 1

Figure 1 shows how the stability of the steady state depends on the parameters values. It plots three curves in the $(\tau, A[\bar{n}_1(\beta)])$ plane. We choose $A[\bar{n}_1(\beta)]$ for the vertical axis for two reasons. On the one hand, this coefficient allows us to distinguish the two areas where the eigenvalues λ_1 and λ_2 are either real or complex. On the other hand, it is the coefficient in the law of motion of the prices. Three curves are drawn in this figure, $A[\bar{n}_1(\beta)] = -4(1-\tau)/\tau^2$ - the ‘eigen curve’, $A[\bar{n}_1(\beta)] = -1/(2\tau - 1)$ - the ‘flip curve’, and $A[\bar{n}_1(\beta)] = -1/(1-\tau)$ - the ‘NS curve’.

Above the eigen curve, the eigenvalues λ_1 and λ_2 are complex and conjugate, below that curve they are distinct and real.

On the flip curve, the real eigenvalue λ_1 is equal to –1. Above that curve, λ_1 is greater than –1, but remains negative. As λ_2 is always negative but greater –1 and the third eigenvalue is zero, the steady state is always asymptotically stable. The lighter shaded area in Figure 1 is the set of parameters values τ and $A[\bar{n}_1(\beta)]$ for which the steady state is asymptotically stable. The flip curve by itself represents the possibility of Flip bifurcation as a primary bifurcation.

On the NS curve, the complex eigenvalues have modulii equal to 1. Above that curve, the modulii of λ_1 and λ_2 are always less than 1. As the third eigenvalue is zero, the steady state is always asymptotically stable. The darker shaded area in Figure 1 is the set of parameters values τ and $A[\bar{n}_1(\beta)]$ for which the steady state is asymptotically stable. Below the curve, the modulii of λ_1 and λ_2 are greater than 1, the steady state is then always unstable. The NS curve by itself represents the possibility of Neimark-Sacker bifurcation as a primary bifurcation. The flip curve and the NS curve intersect when $\tau = 2/3$.

Thanks to this figure, we note that as β increases, the values of $A[\bar{n}_1(\beta)]$ are getting smaller, i.e. the steady state is more likely to be unstable (as already pointed out by Brock and Hommes (1997)). The possibility of bifurcation rests on specific values between τ and β. Note that the values of $A[\bar{n}_1(\beta)]$ lies between 0 and –3, i.e. for "relatively small" values of β. These results follow those of Hommes (1998) when he studies the cobweb model with linear backward-looking expectations with two lags.

Figure 2

Figure 2 illustrates Propositions 2 and 3. It plots the flip and NS curves in the (β, τ)-plane for specific values of the parameters of the demand and the supply, $B = 1$ and $b = 5$. The dotted curve represents the possibility of Flip bifurcation. The plain curve represents the possibility of Neimark-Sacker bifurcation. The two curves intersect when $\tau = 2/3$. The double-lined area shows the stability zone of the steady state. The stability zone occurs for small values of the intensity of choice. This area is "shrinking" for low values of the intensity of choice and "extreme" values of τ. The area is larger when there is a sufficiently mix of the current price and the past price for specific values of β. In other words, adaptive expectations are less destabilising for the market than naïve expectations.

Figure 3

Figure 3 illustrates Proposition 2(ii). It shows a stable cycle of period two for specific values of the parameters in the $(t, p(t))$-plane. This is an expected result. We simply extend Theorem 3.1 of Brock and Hommes (1997) to the case of adaptive expectations. As the bifurcation parameter increases, the instability of the steady state first leads to a stable cycle of period two. This result is analytically proven in the appendix.

Figure 4

Figure 4 illustrates Proposition 3(ii). We can see a limit cycle for specific values of the parameters in the $(p(t-1), p(t))$-plane. Although this result was showed by Hommes (1998) in the cobweb model with homogenous adaptive expectations, it is new in the cobweb model with heterogeneous expectations. In the appendix, we prove analytically that a primary Neimark-Sacker bifurcation can occur for specific values of the parameters. This bifurcation is supercritical when τ is around 0.6 or when τ takes values between 0 and 0.20.

4. Concluding Comments

Our paper shows how relevant the adaption parameter is in the dynamical study of the steady state. Associated with a set of parameters (that notably includes the slopes of supply and demand, the intensity of choice between predictors, the cost and the features of each predictor), we establish the conditions for stability and instability of the steady state. It demonstrates how cycles arise in the cobweb model with heterogeneous beliefs.

It shows how expectations may, by themselves and when their formation is modeled as an economic decision, be sufficient to generate endogenous fluctuations.

Future research could investigate in a more systematic way how the features of the predictors and their associated costs could generate stable periodic equilibria consistent with heterogeneous expectations.

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Appendix

<u>Proof of Propositions 2i and 3i</u>

We just need to study the stability properties of the steady state $E = \left(0,\ 0,\ \bar{n}_1(\beta) = 1/(1+\mathrm{Exp}\beta)\right)$. The steady state is asymptotically stable when all the absolute values of the real eigenvalues or all the modulii of the complex eigenvalues of the Jacobian matrix at *E* are less than 1 (Azariadis, 1993)).

The Jacobian Matrix at *E*:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ A(\bar{n}_1(\beta))(1-\tau) & A(\bar{n}_1(\beta))(\tau) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In what follows, we will denote $\bar{n}_1(\beta)$ by $\bar{n}_1$, keeping in mind that the relative weight of agents using rational expectations depends on the intensity of choice β.

If $A(\bar{n}_1)\tau^2 + 4(1-\tau) > 0$, then there are three eigenvalues: 0 and

$$\lambda_{1,2} = \frac{A(\bar{n}_1)\tau \pm i\sqrt{-A(\bar{n}_1)\left(A(\bar{n}_1)\tau^2 + 4(1-\tau)\right)}}{2}$$

Study of the modulus

$$\left|\lambda_{1,2}\right| = \sqrt{\left(\frac{A(\bar{n}_1)\tau}{2}\right)^2 + \frac{1}{4}\left(-A(\bar{n}_1)\left(A(\bar{n}_1)\tau^2 + 4(1-\tau)\right)\right)} = \sqrt{-(1-\tau)A(\bar{n}_1)}$$

$$\left|\lambda_{1,2}\right| < 1 \iff A(\bar{n}_1) > -1/(1-\tau)$$

Note that $-1/(1-\tau) > -4(1-\tau)/\tau^2$ when $0 < \tau < 2/3$.

If $A(\bar{n}_1)\tau^2 + 4(1-\tau) < 0 \iff A(\bar{n}_1) < -4(1-\tau)/\tau^2$, then there are three eigenvalues 0 and $\lambda_{1,2} = \dfrac{A(\bar{n}_1)\tau \mp \sqrt{A(\bar{n}_1)\left(A(\bar{n}_1)\tau^2 + 4(1-\tau)\right)}}{2}$.

Study of λ_1

$$\lambda_1 < -1 \iff \frac{A(\bar{n}_1)\tau - \sqrt{A(\bar{n}_1)\left(A(\bar{n}_1)\tau^2 + 4(1-\tau)\right)}}{2} < -1$$

$$\iff A(\bar{n}_1)\tau + 2 < \sqrt{A(\bar{n}_1)\left(A(\bar{n}_1)\tau^2 + 4(1-\tau)\right)}$$

If $A(\bar{n}_1)\tau + 2 < 0 \;\Leftrightarrow\; A(\bar{n}_1) < -2/\tau$, the above inequality is always true and then $\lambda_1 < -1$ whatever τ.

Let us now assume that $A(\bar{n}_1) \geq -2/\tau$ and let us find the conditions for which $-1 < \lambda_1 < 0$. We have:

$$-A(\bar{n}_1)\tau - 2 < -\sqrt{A(\bar{n}_1)\left(A(\bar{n}_1)\tau^2 + 4(1-\tau)\right)}$$

$$\Leftrightarrow \; A(\bar{n}_1) > -1/(2\tau - 1) \text{ if } \tau > 1/2$$

Note that $-1/(2\tau - 1) > -2/\tau$ when $\tau > 2/3$.

$$\Leftrightarrow \; (-3\tau + 2)/(\tau(2\tau - 1)) < 0 \text{ if } \tau > 2/3.$$

So we have shown that when $-2/\tau < -1/(2\tau - 1) < A(\bar{n}_1) < -4(1-\tau)/\tau^2$ and $\tau > 2/3$, then $-1 < \lambda_1 < 0$.

Study of λ_2

It is easy to check that $-1 < \lambda_2 < 0$.

Q.E.D.

<u>Proof of Propositions 2ii and 3ii</u> (we follow Kuznetsov (2000))

Our system (S') is three-dimensional and needs to be rewritten so that the steady state is at the origin.

$$h_{t+1} = p_t$$

$$p_{t+1} = \phi(h_t, p_t, n_{1,t})$$

$$n_{1,t+1} = \varphi(h_t, p_t, n_{1,t})$$

Let us denote $m_t = n_{1,t} - \bar{n}_1$. Then the system (S') becomes the following system (S1):

$$h_{t+1} = p_t \tag{A.1}$$

$$p_{t+1} = \phi(h_t, p_t, m_t + \bar{n}_1) \tag{A.2}$$

$$m_{t+1} = \varphi(h_t, p_t, m_t + \bar{n}_1) - \bar{n}_f = \psi(h_t, p_t, m_t) \tag{A.3}$$

The steady state is then $(0,0,0)$.

Let us denote (S1) as a discrete –time dynamical system:

$$x \to f(x) \tag{A.4}$$

We can write this system as:

$$\tilde{x} = J\,x + F(x),\; x \in R^3, \tag{A.5}$$

where J is the Jacobian matrix of (A.4) at the steady state and $F(x) = O\left(\|x\|^2\right)$ is a smooth function. Let us represent its Taylor expansion in the form

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O\left(\|x\|^4\right),$$

where $B(x,y)$ and $C(x,y,z)$ are multilinear functions.

Let us first consider the Flip case (Proposition 2ii). In that case, $A(\bar{n}_1) = -1/(2\,\tau - 1)$ and $\tau \in (2/3, 1)$.

The Jacobian matrix J of (A.4) at the steady state is:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ (\tau-1)/(2\tau-1) & -\tau/(2\tau-1) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are three eigenvalues: 0, -1 and $(\tau-1)/(2\tau-1)$. The corresponding critical eigenspace is one dimensional and spanned by an eigenvector $q \in R^3$ such that $J\,q = -q$, where $q^T = \left(1/\sqrt{2}, -1/\sqrt{2}, 0\right)$. Let $s \in R^3$ be the adjoint eigenvector, that is, $J^T s = -s$, where J^T is the transposed matrix of J. Normalise s with respect to q such that $\langle s,q \rangle = 1$, where $s^T = \dfrac{\sqrt{2}}{2-3\tau}\left(1-\tau, 2\tau-1, 0\right)$.

The bilinear function $B(x, y)$, defined for two vectors $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T \in R^3$ can be partitioned into three elements:

$$B(x, y) = \begin{pmatrix} 0 \\ x_3 B_p^{1,3}\, y_1 + x_3 B_p^{2,3}\, y_2 + x_1 B_p^{1,3}\, y_3 + x_2 B_p^{2,3}\, y_3 \\ x_1 B_m^{1,1}\, y_1 + x_2 B_m^{1,2}\, y_1 + x_1 B_m^{1,2}\, y_2 + x_2 B_m^{2,2}\, y_2 \end{pmatrix}$$

where $B_p^{1,3} = (1-\tau)\,A'(\bar{n}_1)$, $B_p^{2,3} = \tau\, A'(\bar{n}_1)$, $B_m^{1,1} = (\tau-1)^2\,\sigma$,

$B_m^{1,2} = \tau(1-\tau)\sigma$, and $B_m^{2,2} = (\tau)^2\,\sigma$,

with $\sigma = \beta\, b\, \mathrm{Exp}\,[\beta](A(\bar{n}_1 - 1))^2 \big/ (1 + \mathrm{Exp}\,[\beta])^2$ and

$$A'(\bar{n}_1) = \frac{2\tau\, b}{(2\tau-1)}\left(\frac{1}{B + b\,\bar{n}_1}\right).$$

We left to the reader to show that none of the elements of $C(x, y, z)$ is relevant for us.

Following Kuznetsov, the map (A.5) can be transformed to the normal form:

$$\tilde{\varepsilon} = -\varepsilon + \chi(0)\,\varepsilon^3 + O\left(\varepsilon^4\right),$$

where

$$\chi(0) = \frac{1}{6}\langle s, C(q,q,q)\rangle - \frac{1}{2}\left\langle s, B\left(q,(J - Id)^{-1} B(q,q)\right)\right\rangle = -\frac{1}{4}\frac{(1-2\tau)^4}{(2-3\tau)}\sigma A'\left(\bar{n}_f\right)$$

We denote by *Id* the Identity matrix.

Thus, the critical normal form coefficient $\chi(0)$, that determines the nondegeneracy of the Flip bifurcation and allows us to predict the direction of bifurcation of the two-period cycle, is always positive when $\tau > 2/3$. Therefore, the Flip bifurcation is nondegenerate and always supercritical.

Let us now consider the Neimark-Sacker case (Proposition 3ii). In that case, $A(\bar{n}_1) = -1/(1-\tau)$ and $0 < \tau < 1/2$ and $1/2 < \tau < 2/3$.

The Jacobian matrix *J* of (A.4) at the steady state is:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -\tau/(1-\tau) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are three eigenvalues: 0 and $\lambda_{1,2} = -\dfrac{\tau}{2(1-\tau)} \pm i\dfrac{\sqrt{(2-\tau)(2-3\tau)}}{2(1-\tau)} = \mathrm{Re}(\lambda) \pm i\,\mathrm{Im}(\lambda)$. J has a simple pair of eigenvalues on the unit circle $\lambda_{1,2} = e^{\pm i\theta_0}$ with $\pi/2 < \theta_0 < \pi$ and $\theta_0 \neq 2\pi/3$. Let $q \in C^3$ be a complex eigenvector corresponding to λ_1:

$$J\,q = e^{i\theta_0}\,q\,,\; J\,\overline{q} = e^{-i\theta_0}\,\overline{q}\,,$$

$q^T = (1, Re(\lambda) + i\,Im(\lambda), 0)$ and $\overline{q}^T = (1, Re(\lambda) - i\,Im(\lambda), 0)$. Introduce also the adjoint eigenvector $s \in C^3$ having the properties

$$J^T s = e^{-i\theta_0}\,s \text{ and } J^T \overline{s} = e^{i\theta_0}\,\overline{s}\,,$$

and satisfying the normalisation

$$\langle s, q \rangle = 1\,,$$

where $\langle s, q \rangle = \sum_{i=1}^{3} \overline{s}_i q_i$ is the standard product in C^3, $\overline{s}^T = \dfrac{1}{2\,\mathrm{Im}(\lambda)\,i}\left(-\mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda),\ 1,\ 0\right)$.

Following Kuznetsov, we know that in the absence of strong resonances, i.e.:

$$e^{ik\theta_0} \neq 1, \text{ for } k = 1, 2, 3, 4$$

the map (A.5) can be transformed into

$$\tilde{z} = e^{i\theta_0} z\left(1 + \kappa(0)|z|^2\right) + O\left(u^4\right),$$

with $\alpha(0) = \operatorname{Re}\kappa(0)$, that determines the direction of the bifurcation of a closed invariant curve. This real number can be computed by the following invariant formula:

$$\alpha(0) = \frac{1}{2}\operatorname{Re}\left\{e^{-i\theta_0}\left[\left\langle s, C(q,q,\bar{q})\right\rangle + 2\left\langle s, B\left(q, (Id - J)^{-1} B(q,\bar{q})\right)\right\rangle + \left\langle s, B\left(\bar{q}, \left(e^{2i\theta_0} Id - J\right)^{-1} B(q,q)\right)\right\rangle\right]\right\}$$

Therefore,

$$\alpha(0) = \frac{1}{2} Re\left\{A'(\bar{n}_1)\sigma\left(\left(Re(\lambda)\,\tau\right)\left(\left(Im(\lambda)\,\tau\right)^2 - 3L^2\right) + L\left(L^2 - 3\left(Im(\lambda)\,\tau\right)^2\right) + \frac{\left(L^2 + \left(Im(\lambda)\,\tau\right)^2\right)^2}{2}\right)\right\}$$

where $L = \tau - 1 - \tau^2 / \left(2(\tau - 1)\right)$

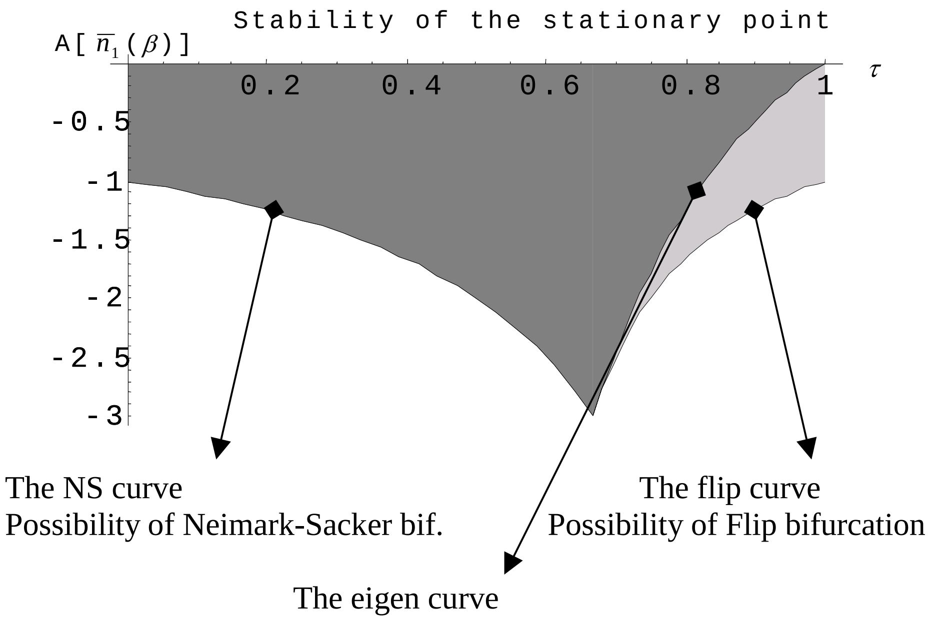
The coefficient $\alpha(0)$ is always negative when $\tau \in (0, 0.203817) \cup (0.59299, 2/3)$. Therefore, the Neimark-Sacker bifurcation is nondegenerate and supercritical on these intervals.

Let us now see what happens when $\theta_0 = 2\pi/3$. Recall the findings of the corollary. When $\tau = 0.5$ (and $A(\bar{n}_f) = -2$), the stationary equilibrium undergoes a strong resonance 1:3 as $\theta = 2\pi/3$, see Kuznetsov (2000) p. 397.

Finally, when $\tau = 2/3$, the two curves of the Neimark-Sacker bifurcation and of the Flip bifurcation intersect. The steady state has a double –1 eigenvalue, a codim-2 bifurcation occurs (See Frouzakis *et al*., 1991, p. 85).

Q.E.D.

Figure 1



The steady state is asymptotically stable in the shaded areas.

Figure 2

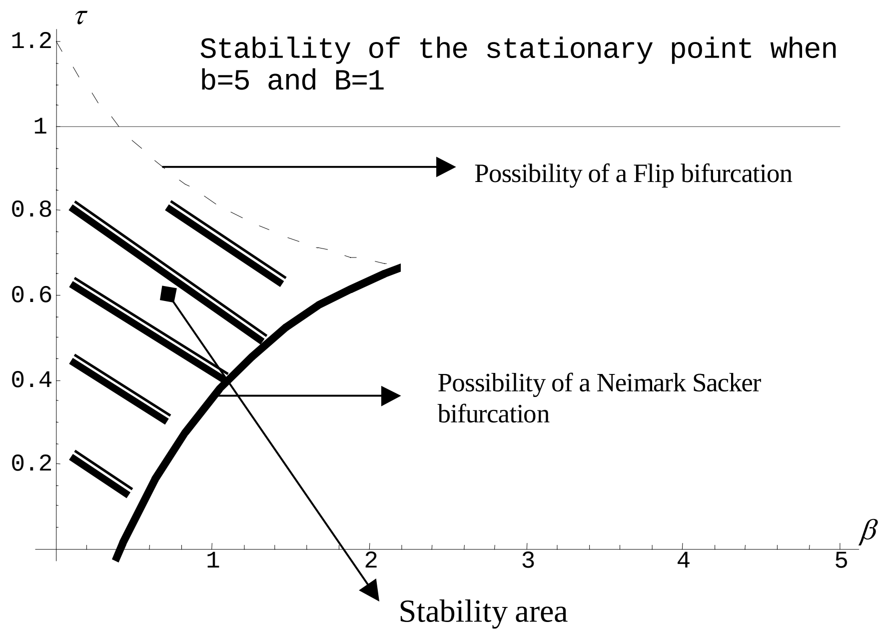
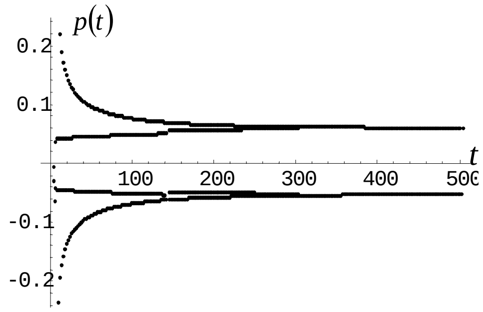


Figure 3

Stable cycle of period 2 in prices $(-0.056, 0.056)$ and $\bar{n}_1 = 0.114583$

$\tau = 0.8$, $\beta = 2.05476$, $C = 1$, $B = 0.5$, $b = 1.2$



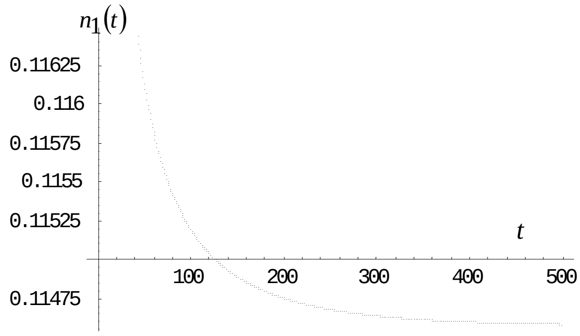


Figure 4

Stable limit cycle in prices and $\bar{n}_1 = 0.107867$

$\tau = 0.63$, $\beta = 2.11272$, $C = 1$, $B = 0.3$, $b = 1.35$