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EUI Working Papers

DYNAMIC COMPETITIVE ECONOMIES WITH COMPLETE
MARKETS AND COLLATERAL CONSTRAINTS

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May 22, 2012

Abstract

In this paper we examine the competitive equilibria of a dynamic stochastic economy with complete markets and collateral constraints. We show that, provided both the set of asset payoffs and collateral levels are sufficiently rich, the equilibrium allocations with sequential trades and collateral constraints are equivalent to those obtained in Arrow-Debreu markets subject to a series of appropriate limited pledgeability constraints.

We provide sufficient conditions for equilibria to be Pareto efficient and show that when collateral is scarce equilibria are also often constrained inefficient, in the sense that imposing tighter borrowing restrictions can make everybody in the economy better off.

We derive sufficient conditions for the existence of Markov equilibria and show that they typically have finite support when there are two agents' types. The model is then tractable and its equilibria can be computed with arbitrary accuracy. We carry out on this basis a quantitative assessment of the risk sharing and efficiency properties of equilibria.

1 Introduction

We examine the competitive equilibria of an infinite-horizon exchange economy where the only limit to risk sharing comes from the presence of a collateral constraint. Consumers face a borrowing limit, determined by the fact that all loans must be collateralized, as for example in Kiyotaki and Moore (1997) or Geanakoplos (1997), but otherwise financial markets are complete. Only part of the consumers' future endowment can be pledged as collateral, hence the borrowing constraint may be binding and limit the risk sharing possibilities in the economy. More specifically, we consider an environment where consumers are unable to commit to repay their debt obligations and the seizure of the collateral by lenders is the only loss an agent faces for his default (as in Geanakoplos and Zame (2002)). There is no additional punishment, for instance in the form of exclusion from trade in financial markets as in the model considered by Kehoe and Levine (1993), Alvarez and Jermann (2000). However, like in this model, and in contrast to Bewley (1977) and the literature which followed it[1], the level of the borrowing (collateral) constraint is endogenously determined in equilibrium by the agents' limited commitment problem.

The analysis is carried out in the set-up of a Lucas (1978) style economy with a single perishable consumption good. The part of a consumer's endowment that can be pledged as collateral can be naturally interpreted as the agent's initial share of the Lucas tree – a long-lived asset in positive supply that pays dividends at each date-event. This asset can be used, both directly and indirectly, as collateral for any short position of the consumer.

We show in this paper that this is a tractable model of dynamic economies under uncertanty, establish the existence of Markov and of finite support equilibria, analyze the welfare properties of competitive equilibria and the risk sharing pattern that is attained. More specifically, we show the equivalence between the competitive equilibria when trade occurs in a complete set of contingent commodity markets at the initial date, as in Arrow Debreu, subject to a series of appropriate limited pleadgeability constraints, and the equilibria when trade is sequential, in a sufficiently rich set of financial markets, and short positions must be backed by suitably defined collateral constraints. This allows to clearly identify market structures, and in particular the specification of asset payoffs and of the associated collateral requirements, such that the only financial friction is the collateral constraint. Second, we derive sufficient conditions for the existence of a Markov equilibrium in this model when there is a finite number of agents' types and show that Markov equilibria often 'have finite support' in the sense that individuals' consumption only takes finitely many values. Markov equilibria exist whenever all agents coefficient of relative risk aversion is bounded above by one. Under the same assumption or when all agents have identical constant relative risk aversion utility, equilibria have finite support when there are only two agents. Third, we provide some sufficient conditions for competitive equilibria to be fully Pareto efficient, that is for the amount of available collateral to be sufficiently large that the collateral constraint

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[1]See Heathcote, Storesletten and Violante (2009) for a survey.

never binds. In addition, we show that, whenever the constraint binds, competitive equilibria in this model are not only Pareto inefficient but are also often constrained suboptimal, in the sense that introducing tighter restrictions on borrowing from some date $t > 0$ (with respect to the restrictions imposed by the collateral constraints) makes all agents better off. Finally, we carry out a quantitative assessment of the efficiency properties and the risk sharing pattern of competitive equilibria for 'realistic' specifications of the economy and of the existing amount of collateral.

Several papers (from the quoted work of Kiyotaki, Moore (1997) and Geanakoplos (1997) to various others as, e.g., Aiyagari and Gertler (1999)) have formalized the idea that borrowing on collateral might give rise to cyclical fluctuations in the real activity and enhance volatility of prices. They typically assume that financial markets are incomplete, and/or that the collateral requirements are exogenously specified, so that it is not clear if the source of the inefficiency are the missing markets or the limited ability of the agents to use the existing collateral for their borrowing needs. Furthermore, dynamic models with collateral constraints and incomplete markets turn out to be very difficult to analyze (see Kubler and Schmedders (2003) for a discussion), no conditions are known that ensure existence of recursive equilibria and there are therefore few quantitative results about the welfare losses due to collateral.

We show here that considering an environment where financial markets are complete and there are no restrictions to how the existing collateral can be used to back short positions, while not immediate to formalize, allows to simplify matters considerably. In our model equilibria can often be characterized as the solution of a finite system of equation. We show that a numerical approximation of equilibria is fairly simple and a rigorous error analysis is possible. Moreover we can use the implicit function theorem to conduct local comparative statics and perform a serious quantitative analysis of the potential welfare gains from government intervention.

As mentioned above, there is also a large literature that assumes that agents can trade in complete financial markets, default is punished with the permanent exclusion from future trades and loans are not collateralized. As shown in Kehoe and Levine (1993), (2001) and Alvarez and Jermann (2000), these 'limited enforcement models' are extremely tractable since competitive equilibria can be written as the solution to a planning problem subject to appropriate constraints. Even though this is not true in the environment considered here - the limited commitment constraint has a different nature and we show that competitive equilibria may be constrained inefficient - tractability still obtains.

Chien and Lustig (2011) (also Lustig (2000) in an earlier, similar work) examine a version of the model in this paper with a continuum of agents and growth. The main focus of their analysis is on a quantitative assessment of the asset pricing implications of the model and their similarities with Alvarez and Jermann (2000). However, they also present some theoretical results that are similar to ours. Under the assumption of identical CRRA

utility they give a sufficient condition for competitive equilibria to be Pareto efficient. Here we present a necessary and sufficient condition, for the case of i.i.d. shocks, no aggregate uncertainty and finitely many consumers' types. Most importantly, their notion of recursive equilibrium also uses individuals' multipliers (Chien and Lustig call them "stochastic Pareto-Negishi weights") as an endogenous state variable and is essentially identical to ours. However, our results on the existence of such recursive equilibria and of finite support equilibria are rather different, as explained more in detail in the next sections. Also, they do not examine how the allocation can be decentralized in asset markets with collateral constraints nor they discuss the constrained inefficiency of competitive equilibria.

Lorenzoni (2008) as well as Kilenthong and Townsend (2011) obtain an analogous constrained inefficiency result to ours but in a production economy. As they point out, given the previous literature on suboptimality the result is not entirely surprising.[2] Their analysis is different as in that environment capital accumulation link different periods and the reallocation is induced by a change in the level of investment that modifies available resources. In our pure exchange set-up resources are fixed, only their distribution can vary and the reallocation is induced by tightening the borrowing constraints with respect to their level endogenously determined in equilibrium.

Geanakoplos and Zame (2002, and, in a later version, 2009) are the first to formally introduce collateral constraints and default into general equilibrium models. They consider a two period model with incomplete markets where a durable good needs to be used as collateral. They are the first to point out that, even if markets are complete and the amount of collateral in the economy is large, the Pareto efficient Arrow Debreu outcome may not be obtained unless one allows for collateralized financial securities to be used as collateral in addition to the durable good (they refer to this as pyramiding). Our equivalence result in Section 2 below makes crucial use of this insight.

The remainder of the paper is organized as follows. In Section 2 we introduce the economic model and the equivalence of equilibrium allocations in three different market environments, with complete contingent markets at the initial date and with sequential trade in financial markets. In Section 3 we present a simple example to illustrate the properties of competitive equilibria. In Section 4 we study the existence of Markov equilibria and show that they are sometimes described by a finite system of equations. In Section 5 we analyze the welfare properties of equilibria and in Section 6 we carry out a quantitative assessment of the equilibrium properties.

2 The model

We consider a standard dynamic model of an exchange economy with collateral constraints where a Lucas tree in unit net supply can be used as collateral for short positions in financial

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[2]The main contribution of Kilenthong and Townsend is to show how constrained optimality can be restored using market-based, segregated exchanges in securities.

assets. As opposed to most papers in the literature on models with collateral constraints, that assume incomplete financial markets and fixed collateral constraints (see e.g. Aiyagari and Gertler (1999), Kubler and Schmedders (2003) or Cao (2011)), we analyze here an environment with a rich asset structure.

We examine an infinite horizon stochastic model of an exchange economy with a single perishable consumption good available at each date. We represent the resolution of uncertainty by an event tree – at each period $t = 0, 1, \ldots$ one of S possible exogenous shocks $s \in \mathcal{S} = \{1, \ldots, S\}$ occurs and each node of the tree is characterized by a history of shocks $\sigma = s^t = (s_0 \cdots s_t)$. The exogenous shocks follow a Markov process with transition matrix π, where $\pi(s, s')$ denotes the probability of shock s' given s. We assume that $\pi(s, s') > 0$ for all $s, s' \in \mathcal{S}$. With a slight abuse of notation we also write $\pi(s^t)$ to denote the unconditional probability of node s^t. We collect all nodes of the infinite tree in a set Σ and we write $\sigma' \succeq \sigma$ if node σ' is either the same as node σ or a (not necessarily immediate) successor. We write $\sigma' \succ \sigma$ if σ' is a successor of (i.e. not the same as) σ.

There are H infinitely lived agents which we collect in a set $\mathcal{H}$. Agent $h \in \mathcal{H}$ maximizes a time-separable expected utility function

$$U_h(c) = E \left\{ \sum_{t=0}^{\infty} \beta^t u_h(c_t, s_t) \right\},$$

where expectations are taken with respect to the Markov transition matrix π, and the discount factor $\beta \in (0, 1)$. We assume that the possibly state dependent Bernoulli function $u_h(\cdot, s) : \mathbb{R}_{++} \to \mathbb{R}$ is strictly monotone, C^2, strictly concave, and satisfies the Inada-condition $u_h'(x, s) \to \infty$ as $x \to 0$, for all $s \in \mathcal{S}$. In the following discussion we will suppress the dependence of u^h on s whenever there is no possibility of confusion.

Each agent h's endowment over his lifetime consists of two parts. The first part is given by an amount of the consumption good which the agent receives at any date event, i.e. $e^h(s^t) = e^h(s_t)$ where $e^h : \mathcal{S} \to \mathbb{R}_{++}$ is a time-invariant function of the shock. In addition, the agent is endowed at period 0 with a share $\theta^h(s_{-1}) \geq 0$ of a Lucas tree, which pays strictly positive dividends $d : \mathcal{S} \to \mathbb{R}_{++}$ that depend solely on the current shock realization $s \in \mathcal{S}$. The tree is an infinitely lived, aggregate physical asset (can be interpreted as machines, land or houses), which exists in unit net supply, $\sum_{h \in \mathcal{H}} \theta^h(s_{-1}) = 1$, and its shares can be traded at any node σ for a unit price $q(\sigma)$. The total endowment of the consumer is therefore $\omega^h(s_t) = e^h(s_t) + \theta^h(s_{-1})d(s_t)$, where $e^h(s_t)$ can be viewed as the nonpledgeable component, which cannot be sold in advance in order to finance consumption or savings at any date before the endowment is received.

Agent h can hold any amount $\theta^h(\sigma) \geq 0$ of shares of the tree at any node σ. In addition to this physical asset, there are J financial assets (in zero net supply) which we collect in a set $\mathcal{J}$. These assets are one-period securities; asset j traded at node s^t promises a payoff $b_j(s^{t+1}) = b_j(s_{t+1}) \geq 0$ at the S successor nodes (s^{t+1}). We denote agent h's portfolio in financial assets by ϕ^h, and write $p_j(\sigma)$ for the price of asset j at node σ.

Differently from the physical asset, consumers can short any of the financial securities. They can default then at no cost on the prescribed payments. To ensure that some payments are made, each short position in a security is backed by an appropriate amount of the tree which is held (either directly or indirectly as we show below) as collateral. At each node σ, we associate with each financial security $j \in \mathcal{J}$ a collateral requirement described by the vector $k^j(\sigma) \in \mathbb{R}_+^{J+1}$. For each unit of security j sold by a consumer, she is required to hold k_{J+1}^j units of the tree as well as k_i^j units of each security $i \in \mathcal{J}$ as collateral. In the next period the agent can default on her promise to deliver $b_j(s_{t+1})$ per unit sold and will actually find it optimal to do so whenever $b_j(s_{t+1})$ is lower than the value of the collateral. In this case the buyer of the financial security gets the collateral associated with the promise. Hence the actual payoff of any security $j \in \mathcal{J}$ at any node s^{t+1} is endogenously determined by the agents' incentives to default and the collateral requirements, as in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003), and is given by the set of values $f_j(s^{t+1})$ satisfying the following system of equations, for all $j \in \mathcal{J}$:[3]

$$f_j(s^{t+1}) = \min \left\{ b_j(s_{t+1}), \sum_{i=1}^{J} k_i^j(s^t) f_i(s^{t+1}) + k_{J+1}^j(s^t)(q(s^{t+1}) + d(s_{t+1})) \right\}. \quad (1)$$

Without further restrictions on collateral requirements, the above expression might not be well defined or have several solutions for $f_j(\sigma)$, $j \in \mathcal{J}$, $\sigma \in \Sigma$. Many possible restrictions can be imposed to solve this problem: we assume here that there is a 'seniority structure' of obligations, i.e. that if a security can be used as collateral for a second security and this can in turn be used as collateral for a third security and so on, then none of these can be used as collateral for the first security. In this way, the collateral of the first security is also used to back, indirectly, the claims of the other securities along the chain. Formally we say that a security j is senior to another security j' if it can be used, directly or indirectly, as collateral for the second one: that is, if there exist a series of securities $j_1, ..., j_n$ with $j_1 = j$ and $j_n = j'$ and with $k_{j_{i-1}}^{j_i} > 0$ for all $i = 2, \ldots, n$. We assume that seniority is irreflexive, i.e. if j is senior to j' then j' cannot be senior to j.

Since agents can default on their debt obligations, at the only cost of losing the posted collateral, it is clear that the tree is ultimately backing all financial claims, directly or indirectly. Still, the assumption that not only the tree but also financial securities can be used as collateral allows to economize on the use of the tree as collateral, as we will see later. Geanakoplos and Zame (2002) refer to this assumption as 'pyramiding'.

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[3]Evidently, when the collateral requirements are set at a sufficiently high level that

$$\min_{s_{t+1}} \left\{ \sum_{i=1}^{J} k_i^j(s^t) b_i(s^{t+1}) + k_{J+1}^j(s^t)(q(s^{t+1}) + d(s_{t+1})) - b_j(s_{t+1}) \right\} \geq 0,$$

as in Kiyotaki and Moore (1997), consumers never choose to default and the payoff of all securities equal their nominal value. It will become clear in the next section that in our framework with *complete markets* it is irrelevant whether collateral requirements are set according to this rule or whether they allow for the value of collateral to fall below the promise in certain states.

A **collateral constrained financial markets equilibrium** is defined as in Kubler and Schmedders (2003) as a collection of choices $(c^h(\sigma), \theta^h(\sigma), \phi^h(\sigma))_{\sigma \in \Sigma}$ for all agents $h \in \mathcal{H}$, prices, $(p(\sigma), q(\sigma))_{\sigma \in \Sigma}$ and payoffs $(f(\sigma))_{\sigma \in \Sigma}$ satisfying (1) and the following other conditions:

(CC1) Market clearing:

$$\sum_{h \in \mathcal{H}} \theta^h(\sigma) = 1 \text{ and } \sum_{h \in \mathcal{H}} \phi^h(\sigma) = 0 \text{ for all } \sigma \in \Sigma.$$

(CC2) Individual optimization: for each agent h

$$(\theta^h(\sigma), \phi^h(\sigma), c^h(\sigma))_{\sigma \in \Sigma} \in \arg \max_{\theta \geq 0, \phi, c \geq 0} U_h(c) \text{ s.t.}$$
$$c(s^t) = e^h(s_t) + \phi(s^{t-1}) \cdot f(s^t) + \theta(s^{t-1})(q(s^t) + d(s_t)) - \theta(s^t)q(s^t) - \phi(s^t) \cdot p(s^t), \quad \forall s^t$$
$$\theta(s^t) + \sum_{j \in \mathcal{J}} k^j_{J+1}(s^t) \min[0, \phi_j(s^t)] \geq 0, \quad \forall s^t$$
$$\max\left\{\phi_j(s^t), 0\right\} + \sum_{i \in \mathcal{J}} k^i_j(s^t) \min[0, \phi_i(s^t)] \geq 0, \quad \forall s^t, \forall i \in \mathcal{J}.$$

where the second and the third constraints are the collateral constraints for the tree's and securities' holdings.

It is important to point out out that no information over the overall trades carried out by an agent is needed to enforce the collateral constraints as specified above: it suffices to post the required collateral for each short position. We can then say the contracts traded in the markets are non exclusive. This is in contrast with other limited commitment models, as Kehoe and Levine (1993, 2001), Alvarez and Jermann (2000).

Existence of a collateral constrained financial markets equilibrium is proven in Kubler and Schmedders (2003). They also show that tree prices in equilibrium are bounded.

2.1 Complete Markets

In this paper we want to analyze economies with collateral constraints where markets are *complete* in the sense that agents are able to trade securities with any payoff and any specification of the collateral requirement. Hence the only impediment to risk sharing is the limit on borrowing imposed by the available amount of collateral.

In order to formalize the notion of complete markets we consider the case where consumers can trade at $t = 0$ in a complete set of contingent commodity markets, but are subject to the constraint imposed by the non pledgeability of part of the endowment. More precisely, we define an **Arrow Debreu equilibrium with limited pledgeability** as a collection of prices $(\rho(\sigma))_{\sigma \in \Sigma}$ and a consumption allocation $(c^h(\sigma))_{\sigma \in \Sigma}^{h \in \mathcal{H}}$ such that

$$\sum_{h \in \mathcal{H}} (c^h(\sigma) - \omega^h(\sigma)) = 0, \quad \sigma \in \Sigma \tag{2}$$

and for all agents h

$$(c^h(\sigma))_{\sigma\in\Sigma} \in \arg\max_{c\geq 0} \in \arg\max U_h(c) \text{ s.t.} \tag{3}$$

$$\sum_{\sigma\in\Sigma} \rho(\sigma)c^h(\sigma) \leq \sum_{\sigma\in\Sigma} \rho(\sigma)\omega^h(\sigma) < \infty \tag{4}$$

$$\sum_{\sigma\succeq s^t} \rho(\sigma)c^h(\sigma) \geq \sum_{\sigma\succeq s^t} \rho(\sigma)e^h(\sigma) \text{ for all } s^t. \tag{5}$$

The definition is the same as that of an Arrow Debreu competitive equilibrium except for the additional constraints (5). These constraints express precisely the condition that $e^h(\sigma)$ is unalienable, i.e. this component of the endowment can only be used to finance consumption in the node σ in which it is received or in any successor node. Assuming strictly monotonic preferences the specification of the constraint follows. Note that these additional constraints are likely to be binding whenever the e^h-part of the endowments is large relative to the tree's dividends, that is when there is only a small amount of future endowments that can be traded at earlier nodes of the tree.[4]

We will show that any Arrow Debreu equilibrium allocation with limited pledgeability can also be attained at an equilibrium with sequential trading in a model with collateral constraints with a sufficiently rich asset structure. To show the result, it is convenient to introduce an alternative equilibrium notion with sequential trading, where each period intermediaries purchase the tree from consumers and issue on that basis, at no cost, a complete set of one period, shock-contingent claims (options) on the tree, which are bought by consumers and are the only assets they can trade. This specification turns out to be very useful to analyze the properties of collateral constrained equilibria when markets are complete.

More precisely, at each node s^t intermediaries purchase the tree and issue $J = S$ assets, where asset j promises the delivery of one unit of the tree the subsequent period if and only if shock $s = j$ realizes. Households in the economy can only take long positions in these assets at every node. The intermediaries' holdings of the tree ensure that all due payments can be made. At any node s^t an agent purchases a portfolio of tree-options $\theta(s^t; s) \geq 0$, $s = 1, \ldots, S$, at the prices $q(s^t; s) > 0$, $s = 1, ..., S$. The condition $q(s^t) = \sum_{s=1}^{S} q(s^t; s)$ ensures that intermediaries make zero profit in equilibrium, since the intermediation technology, with zero costs, exhibit constant returns to scale.

An **equilibrium with intermediaries** is defined as a collection of individual consumptions $(c^h(\sigma))_{\sigma\in\Sigma}^{h\in\mathcal{H}}$, portfolios $(\theta^h(\sigma))_{\sigma\in\Sigma}^{h\in\mathcal{H}}$, as well as prices $(q(\sigma))_{\sigma\in\Sigma}$, such that markets clear and agents maximize their utility, i.e.

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[4]Kehoe and Levine (1993) also consider an environment with complete contingent market where only part of the agents' endowment can be seized in the event of default, but with the additional punishment of permanent exclusion: hence the additional constraint does not have the form of a budget constraint, as (5) above, but of a constraint on the continuation utility level.

(IE1) At all nodes s^t,
$$\sum_{h \in \mathcal{H}} \theta^h(s^t; s) = 1 \text{ for all } s \in \mathcal{S}.$$

(IE2) For all agents $h \in \mathcal{H}$

$$(c^h, \theta^h) \in \arg \max_{\theta, c \geq 0} U^h(c) \text{ s.t.}$$

$$c(s^t) = e^h(s^t) + \theta(s^{t-1}; s_t) \left(\sum_{s'=1}^{S} q(s^t; s') + d(s_t) \right) - \sum_{s'=1}^{S} \theta(s^t; s') q(s^t; s')$$

$$\theta(s^t; s) \geq 0, \quad \forall s^t, s$$

It is relatively easy to show that any Arrow Debreu equilibrium allocation with limited pledgeability can also be attained as an equilibrium with intermediaries. In order to show that it can also be attained as an equilibrium with collateral constraints, one needs to construct a rich enough asset structure that ensures that the payoffs achieved with the tree options can be replicated by trading in the asset markets. We have so the following result, proved in the Appendix.

THEOREM 1 *For any Arrow Debreu equilibrium with limited pledgeability there exists an equilibrium with intermediaries with the same consumption allocation. Moreover, there exists an asset structure $\mathcal{J}$ such that there is a collateral constrained financial markets equilibrium with the same consumption allocation.*

The reverse implication can also be shown to hold for equilibria without bubbles in the tree price, thus establishing the equivalence between the three equilibrium notions presented (as long as the possibility of equilibria with bubbles is ignored).[5] Given this equivalence, in most of the paper we will focus our attention on the notion which turns out to be more convenient, the one of equilibria with intermediaries.

Remark Our equilibrium notion with intermediaries is very similar to Chien and Lustig (2010). They analyze a model with collateral requirements where, in addition to the tree, a complete set of S Arrow securities is available for trade at each node and the tree must be used as collateral for short positions in these Arrow securities. A crucial difference between their set-up and the one described above is that they assume that the tree can be used to secure short positions in several Arrow securities at the same time, i.e. the collateral constraint only has to hold ex post, for each realization of the payoff of the security. This clearly allows to economize on the use of the tree as collateral but it also requires a stronger enforcement and coordination ability among lenders, or the full observability of agents' trades, not needed as we said in the environment considered here, and seems then more difficult to justify.

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[5]Since the existence proof in Kubler and Schmedders (2003) shows that there exist collateral constrained financial markets equilibria without bubbles, Theorem 1 implies the existence also of Arrow Debreu equilibria with limited pledgeability.

9

3 An example

To illustrate the analysis, we consider the simplest possible example, with two agents, two states and no aggregate uncertainty. The shocks are i.i.d. with two possible realizations, with probabilities π_1 and π_2. For simplicity, assume that the tree has a deterministic dividend $\mathfrak{d}$ and that endowments of agent 1 are $e^1(1) = \mathfrak{h}, e^1(2) = \mathfrak{l}$, the endowments of agent 2 are $e^2(1) = \mathfrak{l}, e^2(2) = \mathfrak{h}$, where $0 < \mathfrak{l} < \mathfrak{h}$. We assume that initial conditions are $s_0 = 1$ and $\theta_1^1(s_{-1}) = 0$, i.e. the economy starts at shocks $s = 1$ with agent 1 holding no share of the tree.

Depending on the relative magnitude of $\mathfrak{d}, l$ and $\mathfrak{h}$, and the initial conditions, competitive equilibria are Pareto-efficient or inefficient, that is, the collateral constraint is slack or not. We first derive a necessary and sufficient condition for equilibria to be Pareto-efficient and then characterize their properties when they are not. Given the equivalence established in Theorem 1, we find it convenient to carry out the analysis here in terms of the notion of equilibrium with intermediaries.

3.1 Efficient equilibria

In the environment considered in this example there is no aggregate uncertainty and hence at a Pareto efficient equilibrium agents' consumption is constant, i.e. we must have $c^1(s^t) = \bar{c}^1$ for all s^t. The equilibrium price of the tree must then also be constant and given by

$$\bar{q} = \frac{\beta \mathfrak{d}}{1 - \beta}.$$

The constant value of consumption implies that the equilibrium prices of the state-contingent tree-options are $q(s; s') = \pi_{s'}\beta(\bar{q} + \mathfrak{d})$, also invariant of the current shock realization s (since shocks are i.i.d.) and such that $q(s; s') = \pi_{s'}\bar{q}$ for all s.

Given the properties established above of agents' consumption and equilibrium prices, the expressions of the budget constraint of type 1 consumers when the shock realization is, respectively, 1 and 2, are

$$\begin{aligned} \bar{c}^1 &= \mathfrak{h} + \theta_1(q + \mathfrak{d}) - \bar{q}(\pi_1\theta_1 + \pi_2\theta_2) \\ &= \mathfrak{l} + \theta_2(q + \mathfrak{d}) - \bar{q}(\pi_1\theta_1 + \pi_2\theta_2), \end{aligned} \tag{6}$$

where θ_2 is the holding in the tree-option that pays in shock 2 and θ_1 the holding of the tree-option paying in shock 1. Solving the first one for $\bar{q}$ and substituting it into the second one yields

$$\theta_2 - \theta_1 = \frac{(\mathfrak{h} - \mathfrak{l})(1 - \beta)}{\mathfrak{d}}.$$

By the market clearing conditions in the securities' market, the holdings of tree options of type 2 consumers are then $(1 - \theta_1), (1 - \theta_2)$. Feasibility, that is the non negativity of the securities' holdings for both types of consumers, requires that $\theta_1, \theta_2 \in [0, 1]$, which means

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that $\theta_1 \geq 0$ and $\theta_1 + (\mathfrak{h} - \mathfrak{l})(1 - \beta)/\mathfrak{d} \leq 1$. Thus there exists a value of θ_1 satisfying these conditions only if

$$\frac{(\mathfrak{h} - \mathfrak{l})(1 - \beta)}{\mathfrak{d}} \leq 1. \tag{7}$$

If (7) is not satisfied, a Pareto efficient competitive equilibrium does not exist. On the other hand, if (7) holds, an efficient competitive equilibrium exists for some appropriate initial endowment of the tree, equal to the equilibrium portfolio of tree options which for the type 1 consumers is given by a pair $\theta_1 \geq 0$, $\theta_2 = \frac{(\mathfrak{h}-\mathfrak{l})(1-\beta)}{\mathfrak{d}} + \theta_1 = \frac{(\mathfrak{h}-\mathfrak{l})+\theta_1(\bar{q}+\mathfrak{d})}{\bar{q}+\mathfrak{d}} \leq 1$. The equilibrium price of the tree is $\bar{q} = \frac{\beta \mathfrak{d}}{1-\beta}$ and the consumption level of type 1 consumers is obtained by substituting these values into the budget constraints (6). The consumption level of type 2 consumers obtained from the feasibility conditions is then also constant and budget feasible. Hence the one described constitutes an efficient, steady state[6] equilibrium with intermediaries.

3.2   Inefficient steady state equilibria

Consider next the case where $(\mathfrak{h} - \mathfrak{l})(1 - \beta) > \mathfrak{d}$, that is the efficiency condition (7) is violated, so that the only possible equilibrium with intermediaries is an inefficient one, where the constraints on agents' portfolios bind (at least in some state). Assume that $\mathfrak{l} > 0$. We show that in the environment of this simple example a steady state equilibrium exists even in this case, supported by the following steady state portfolios

$$\theta^1 = (0, 1), \theta^2 = (1, 0).$$

Note that these are the portfolios supporting an efficient equilibrium if $\mathfrak{h} - \mathfrak{l} = \mathfrak{d}/(1 - \beta)$.

Letting $q(s; 1)$ and $q(s; 2)$ denote, as before, the equilibrium prices in state s of the tree contingent on states 1 and 2, the consumption values supported by the above portfolios readily obtain from the budget constraints:

$$
\begin{aligned}
c_1^1 &= \mathfrak{h} - q(1; 2), \\
c_2^1 &= \mathfrak{l} + (\mathfrak{d} + q(2; 1) + q(2; 2)) - q(2; 2) = \mathfrak{l} + \mathfrak{d} + q(2; 1)
\end{aligned}
$$

The values of the equilibrium prices must satisfy the first order conditions of agent 1 for the security paying in state 2 (since agent 1 is always unconstrained in his holdings of this asset)

$$
\begin{aligned}
q(1; 2)u_1'(c_1^1) &= \beta \pi_2 (q(2; 1) + q(2; 2) + \mathfrak{d})u_1'(c_2^1) \\
q(2; 2)u_1'(c_2^1) &= \beta \pi_2 (q(2; 1) + q(2; 2) + \mathfrak{d})u_1'(c_2^1)
\end{aligned}
$$

---

[6]We use the term steady state to refer to situations where the equilibrium variables depend at most on the current realization of the shock.

and, by the same argument, the corresponding conditions of agent 2 for the security paying in state 1

$$q(1;1)u_2'(c_1^2) = \beta\pi_1(q(1;1) + q(1;2) + \mathfrak{d})u_2'(c_1^2)$$
$$q(2;1)u_2'(c_2^2) = \beta\pi_1(q(1;1) + q(1;2) + \mathfrak{d})u_2'(c_1^2)$$

From the second and the third conditions above we obtain that the following relationship must hold

$$q(1;1) = \frac{\beta\pi_1}{1 - \beta\pi_1}(q(1;2) + \mathfrak{d})$$
$$q(2;2) = \frac{\beta\pi_2}{1 - \beta\pi_2}(q(2;1) + \mathfrak{d})$$

To complete the proof that a steady state equilibrium exists with the portfolio-holdings stated above it thus remains to show that the remaining first order conditions have a solution for a positive level of the prices $q(1;2)$, $q(2;1)$ satisfying $\mathfrak{h} - q(1;2) \geq \mathfrak{l} + \mathfrak{d} + q(2;1)$,[7] or equivalently:

$$q(1;2)u_1'(\mathfrak{h} - q(1;2)) = \frac{\beta\pi_2}{1 - \beta\pi_2}(q(2;1) + \mathfrak{d})u_1'(\mathfrak{l} + \mathfrak{d} + q(2;1)) \qquad (8)$$
$$q(2;1)u_2'(\mathfrak{h} - q(2;1)) = \frac{\beta\pi_1}{1 - \beta\pi_1}(q(1;2) + \mathfrak{d})u_2'(\mathfrak{l} + \mathfrak{d} + q(1;2))$$

Note first that if there is a positive solution to system (8), it must satisfy $\mathfrak{h} - q(1;2) \geq \mathfrak{l} + \mathfrak{d} + q(2;1)$. Suppose this inequality were not satisfied; since we are considering the case where $\mathfrak{h} - \mathfrak{l} > \frac{\mathfrak{d}}{1-\beta}$, we would then have $q(1;2) + q(2;1) > \frac{\mathfrak{d}}{1-\beta} - \mathfrak{d} = \frac{\beta\mathfrak{d}}{1-\beta}$. Furthermore,

$$\frac{u_1'(\mathfrak{l} + \mathfrak{d} + q(2;1))}{u_1'(\mathfrak{h} - q(1;2))} < 1 \text{ and } \frac{u_2'(\mathfrak{l} + \mathfrak{d} + q(1;2))}{u_2'(\mathfrak{h} - q(2;1))} < 1.$$

Substituting these inequalities in (8) yields

$$q(1;2) < \frac{\beta\pi_2}{1 - \beta\pi_2}(q(2;1) + \mathfrak{d}) < \frac{\beta\pi_2}{1 - \beta\pi_2}\left(\frac{\beta\pi_1}{1 - \beta\pi_1}(q(1;2) + \mathfrak{d}) + \mathfrak{d}\right)$$

Equivalently, by collecting the terms with $q(1;2)$ on the left hand side and simplifying we obtain

$$q(1;2)(1 - \beta) < \beta\pi_2\mathfrak{d}.$$

Symmetrically, we can perform the same operation for $q(2;1)$ to obtain

$$q(2;1)(1 - \beta) < \beta\pi_1\mathfrak{d}.$$

---

[7]This condition ensures that $c_1^1 \geq c_2^1$ and hence that the no borrowing constraint for security 1 is binding for agent 1 at the specified consumption levels.

Adding up these two inequalities yields a contradiction to the inequality $q(1;2) + q(2;1) > \frac{\beta \mathfrak{d}}{1-\beta}$ above. Therefore a solution to (8) must always satisfy $\mathfrak{h} - q(1,2) \geq \mathfrak{l} + \mathfrak{d} + q(2,1)$.

To show that a positive solution to (8) exists recall the following lemma that follows directly from Brouwer's fixed point theorem (see e.g. Zeidler (1985), Proposition 2.8).

LEMMA 1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that*

$$\inf_{\|x\|=r} \sum_{i=1}^{n} x_i f_i(x) \geq 0, \text{ for some } r > 0.$$

Then f has at least one zero, i.e. there is a x with $\|x\| \leq r$ and $f(x) = 0$.

For sufficiently small $\epsilon > 0$, define $g : [-\mathfrak{d}, \mathfrak{h} - \epsilon]^2 \to \mathbb{R}^2$ by

$$g(x,y) = \begin{cases} \frac{1-\beta\pi_2}{\beta\pi_2} x u_1'(\mathfrak{h} - x) - (y + \mathfrak{d}) u_1'(\mathfrak{l} + \mathfrak{d} + y) \\ \frac{1-\beta\pi_1}{\beta\pi_1} y u_2'(\mathfrak{h} - y) - (x + \mathfrak{d}) u_2'(\mathfrak{l} + \mathfrak{d} + x) \end{cases}$$

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = g \left(\max\left[-\mathfrak{d}, \min[\mathfrak{h} - \epsilon, x]\right], \max\left[-\mathfrak{d}, \min[\mathfrak{h} - \epsilon, y]\right] \right).$$

We can apply Lemma 1 for $r = \mathfrak{h}$, using the sup-norm and obtain the existence of a zero point for f and then verify that this must also be a solution to $g(x,y) = 0$. For $x = -r$ we obtain

$$x f_1(x,y) + y f_2(x,y) \geq r(y + \mathfrak{d}) u_1'(\mathfrak{l} + \mathfrak{d} + y) \geq 0.$$

For $x = r$ we obtain that $x f_1(x,y)$ can be made arbitrarily large by choosing ϵ appropriately small while $y f_2(x,y)$ is obviously bounded below since we assume $\mathfrak{l} > 0$. By symmetry, the same is true for $y \in \{-r, r\}$ and there must be (x^*, y^*) with $f(x^*, y^*) = 0$. It is easy to see that $(x^*, y^*) \in (0, \mathfrak{h})^2$ and therefore also solve $g(x,y) = 0$.

3.3 Transition to a steady state

Note that the efficiency of competitive equilibria also depends on the initial conditions. Suppose the efficiency condition (7) holds but initial conditions are $s_0 = 1$ and $\theta_- := \theta_1^1(s_-) > 1 - \frac{\mathfrak{h} - \mathfrak{l}}{\tilde{q} + \mathfrak{d}}$, so that the initial endowment of the tree does not coincide with the portfolio holdings at an efficient steady state.

Collect all histories (nodes) which consist only of shock 1 in a set

$$\Sigma^1 = \{ s^T = (s_0, ..., s_T) : s_t = 1 \text{ for all } t = 0, ..., T \}.$$

We conjecture and then verify the following are the portfolios of type 1 consumers at an equilibrium with intermediaries:

$$\theta^1(s^t) = (\theta_-, 1), \ c^1(s^t) = \tilde{c}_1^1 \ \text{ if } s^t \in \Sigma^1$$

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and

$$\theta^1(s^t) = (1 - \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}}, 1), \ c^1(s^t) = \bar{c}^1 \text{ if } s^t \notin \Sigma^1,$$

where the consumption values $\tilde{c}_1^1$ and $\bar{c}^1$ are determined below. Hence for all $s^t \notin \Sigma^1$ we are at an efficient steady state, where prices are $q(s^t) = \bar{q} = \mathfrak{d}\frac{\beta}{1-\beta}$, $q(s^t; s') = \pi_{s'}\mathfrak{d}\frac{\beta}{1-\beta}$ and consumption levels

$$\bar{c}^1 = c^1(s_t = 1) = c^1(s_t = 2) = \mathfrak{h} + (1 - \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}})\mathfrak{d} - \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}}\pi_2\bar{q}.$$

At nodes $s^t \in \Sigma^1$ (i.e. nodes for which only shock 1 occurred up to and including date t), equilibrium values are also independent of history and given by

$$q(1; 2) = \frac{u_1'(\bar{c}^1)}{u_1'(\tilde{c}_1^1)}\beta\pi_2(\bar{q} + \mathfrak{d}) \tag{9}$$

and

$$q(1; 1) = \beta\pi_1(q(1; 1) + q(1; 2) + \mathfrak{d}). \tag{10}$$

and the associated level of consumption of type 1 consumers is obtained from his budget constraint, given the above specification of the agent's portfolio:

$$\tilde{c}_1^1 = \mathfrak{h} + (\mathfrak{d} + q(1; 1) + q(1; 2))\,\theta_- - q(1; 1)\theta_- - q(1; 2) = \mathfrak{h} + \mathfrak{d}\theta_- - (1 - \theta_-)q(1; 2).$$

Substituting this expression for $\tilde{c}_1^1$ into (9) yields one non-linear equation in the unknown $q(1; 2)$. By a standard argument (intermediate value theorem) this has a positive solution associated with positive consumption.

It remains to verify that the consumers' optimality conditions are satisfied. At the nodes $s^t \notin \Sigma^1$ the efficient steady state obtains, for which we already verified these conditions hold at the above prices and allocations. For nodes $s^t \in \Sigma^1$, for agent 1 this follows from (9) and (10) above. For agent 2, the first order condition with respect to asset 1 holds since the agent is unconstrained (the condition is in fact still given by (10)). It remains to be shown that the first order condition with respect to asset 2 holds for the type 2 consumers, who are constrained (their holdings of asset 2 equals zero)

$$q(1; 2) > \frac{u_2'(\bar{c}^2)}{u_2'(\tilde{c}_1^2)}\beta\pi_2(\bar{q} + d) \Leftrightarrow \frac{u_1'(\bar{c}^1)}{u_1'(\tilde{c}_1^1)} > \frac{u_2'(\bar{c}^2)}{u_2'(\tilde{c}_1^2)}$$

where $\tilde{c}_1^2 = \mathfrak{h} + \mathfrak{l} + \mathfrak{d} - \tilde{c}_1^1$, $\bar{c}^2 = \mathfrak{h} + \mathfrak{l} + \mathfrak{d} - \bar{c}^1$. Since there is no aggregate uncertainty the above inequality is equivalent to the condition $\tilde{c}_1^1 > \bar{c}^1$, hence the consumption of a type 1 consumer must be decreasing when going from state 1 to state 2.

We prove by contradiction that this must be the case. Suppose that $\tilde{c}_1^1 \leq \bar{c}^1$. This inequality, together with (9) and the corresponding first order condition at the efficient steady state, $\pi_2\bar{q} = \beta\pi_2(\bar{q}+\mathfrak{d})$ would imply that $q(1; 2) \leq \pi_2\bar{q}$. But by the budget constraints the inequality $\tilde{c}_1^1 \leq \bar{c}^1$ is equivalent to

$$\mathfrak{h} + (1 - \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}})\mathfrak{d} - \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}}\pi_2\bar{q} \geq \mathfrak{h} + \mathfrak{d}\theta_- - (1 - \theta_-)q(1; 2) \Leftrightarrow$$

$$(1 - \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}} - \theta_-)\,(\mathfrak{d}+q(1; 2)) \geq \frac{\mathfrak{h} - \mathfrak{l}}{\bar{q} + \mathfrak{d}}\,(\pi_2\bar{q} - q(1; 2))$$

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Since in the case under consideration $\theta_- > (1 - \frac{\mathfrak{h}-\mathfrak{l}}{\bar{q}+\mathfrak{d}})$, the left hand side of the above inequality is always negative. If $q(1; 2) \leq \pi_2 \bar{q}$ the right hand side is non-negative and we obtain a contradiction. Hence we must have $\tilde{c}_1^1 > \bar{c}^1$ and the candidate equilibrium satisfies all the consumers' optimality conditions, in addition to market clearing.

It is easy to verify that even the transition to an inefficient steady state (when (7) does not hold and initial portfolio holdings differ from $(0, 1), (1, 0)$) displays essentially the same properties.

4 Stationary equilibria

The example demonstrates that in this model there might exist steady state equilibria where consumption and prices only depend on the exogenous shock. In general one would expect current prices and consumption to also depend on an endogenous state, typically the current distribution of assets across agents. The question is then whether along the equilibrium path this endogenous state takes finitely many or infinitely many values. If it takes finitely many values, the equilibrium can be characterized by a finite system of equations, it can typically be computed easily and one can conduct local comparative statics using the implicit function theorem. In this case, we say that there exists a finite-support equilibrium (the stochastic process of the exogenous and endogenous state has finite support). While the example above obviously is one case of a finite support equilibrium, it turns out that in our model these equilibria exist for much more general specifications of preferences and endowments. In this section we give sufficient conditions for there to exist Markov equilibria and for these Markov equilibria to be finite-support equilibria.

As we show below, competitive equilibrium in our model may be constrained inefficient: the additional constraints, as (5) in the Arrow Debreu notion, depend in fact on prices. It is therefore not possible to derive equilibrium allocations as the solution to a planner's problem (as done in other limited commitment models, as Kehoe and Levine (1993, 2001)). In this respect our model is closer to models with incomplete financial markets where existence of Markov equilibria is an open problem.

4.1 Markov equilibria

While the 'natural' endogenous state space consists of beginning-of-period financial wealth across all agents, it turns out that the analysis is simplified if one does not take the distribution of wealth as the endogenous state variable but instead works with the agents' instantaneous Negishi weights, i.e. the weighted share of current consumption. Formally, we take the endogenous state at some node s^t to be $\lambda(s^t) \in \mathbb{R}_{++}^H$ where

$$(c^1(s^t), \ldots, c^H(s^t)) \in \arg\max \sum_{h \in \mathcal{H}} \lambda_h(s^t) u_h(c^h) \text{ s.t. } \sum_{h \in \mathcal{H}} (c^h - \omega^h(s_t)) = 0.$$

Strictly speaking, since these weights are endogenous they cannot be state variables, but the discussion in Kubler and Schmedders (2003) on the choice of state-variables in models with incomplete markets also applies here. Any collection of endogenous variables could serve as co-state variables as long as there is a mapping between the equilibrium values of these and the equilibrium values of all other endogenous variables at a given date-event.

Negishi's (1960) approach to proving existence of a competitive equilibrium of course shows that instead of solving for consumption values that clear markets, one can solve for weights that enforce budget balance, see also Dana (1993). Judd et al. (2003) show how to use this approach to compute equilibria in Lucas style models with complete markets (and without collateral constraints). Cuoco and He (2001) formulate recursive equilibria in models with incomplete markets with this choice of an endogenous state. Finally, as already mentioned in the introduction, Chien and Lustig (2010) (see also Chien et al. (2011)) consider a Markov equilibrium notion that features individual multipliers - interpretable as the inverse of our consumption weights - as endogenous state variable in a model with collateral constraints analogous to ours, though for a different economy. They then numerically approximate equilibria with a continuum of agents by finite histories of shocks.

Obviously we can normalize these weights to sum up to one and can take as the endogenous state space the $H-1$ dimensional simplex in $\mathbb{R}^H$ which we denote by Δ^{H-1}. Given the state space $\mathcal{S} \times \Delta^{H-1}$ a Markov equilibrium is a competitive equilibrium that can be described by a policy function, $C : \mathcal{S} \times \Delta^{H-1} \to \mathbb{R}_+^H$, that maps the current state to current consumption across all agents and a transition function, $L : \mathcal{S} \times \Delta^{H-1} \to \Delta^{H-1}$ that maps the current endogenous state and next period's shock to next period's endogenous state.

By definition of the endogenous state, the policy function is obviously given by

$$C(s, \lambda) = \arg \max_{c \in \mathbb{R}_+^H} \sum_{h \in \mathcal{H}} \lambda_h u_h(c^h) \text{ s.t. } \sum_{h \in \mathcal{H}} (c^h - \omega^h(s)) = 0. \tag{11}$$

It is useful to define (in a slight abuse of notation)

$$u_h'(s, \lambda) = u_h' \left(C^h(s, \lambda), s \right)$$

The following theorem characterizes the transition function.

THEOREM 2 *A policy function $C : \mathcal{S} \times \Delta^{H-1} \to \mathbb{R}_+^H$ together with a function $L : \mathcal{S} \times \Delta^{H-1} \to \Delta^{H-1}$ describe a Markov equilibrium if there are excess expenditure functions $V^h : \mathcal{S} \times \Delta^{H-1} \to \mathbb{R}$ for all agents $h \in \mathcal{H}$ that satisfy*

$$V^h(s, \lambda) = u_h'(s, \lambda)) \left(C^h(s, \lambda) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L(s', \lambda))$$

as well as

$$L(s', \lambda) = \frac{1}{\sum_{h \in \mathcal{H}} \lambda_h + \gamma_h} (\lambda + \gamma)$$

for some $\gamma \in \mathbb{R}_+^H$ with $\gamma_h V^h(s', L(s', \lambda)) = 0$ and $V^h(s', L(s', \lambda)) \geq 0$ for all $s' \in \mathcal{S}$.

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Proof. Given functions (C, V, L) and any $\lambda_0 \in \Delta_{++}^{H-1}$, we need to verify that there exist initial conditions and a competitive equilibrium (and here it is convenient to consider the notion of Arrow Debreu equilibrium with limited pledgeability) with $c^h(s^t) = C^h(s_t, \lambda(s^t))$ and $\lambda(s^t) = L(s_t, \lambda(s^{t-1}))$. Define $\rho(s_0) = 1$ and

$$\rho(s^t) = \rho(s^{t-1}) \beta \pi(s_{t-1}, s_t) \max_{h \in \mathcal{H}} \frac{u_h'(s_t, \lambda(s^t))}{u_h'(s_{t-1}, \lambda(s^{t-1}))}.$$

Agent h's first order conditions for optimal consumption at some node s^t can be written as follows.

$$\beta^t \pi(s^t) u_h'(c^h(s^t), s_t) - \eta^h \rho(s^t) + \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma) \rho(s^t) = 0$$

$$\mu^h(s^t) \sum_{\sigma \succeq s^t} \rho(\sigma)(c^h(\sigma) - e^h(\sigma)) = 0,$$

for multipliers $\eta^h \geq 0$ (associated with the standard budget constraint) and $\mu^h(\sigma) \geq 0$ (associated with the collateral constraint (5) at node σ). It is standard to show that for summable and positive prices these conditions, together with the budget inequalities (4) and (5) are necessary and sufficient for a maximum (see e.g. Dechert (1982)). But then at each s^t and for all agents $h = 2, ..., H$ we have

$$\frac{u_1'(c^1(s^t), s_t)}{u_h'(c^h(s^t), s_t)} = \frac{\eta^1 - \sum_{\sigma: s^t \succeq \sigma} \mu^1(\sigma)}{\eta^h - \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma)}$$

which is equivalent to the first order conditions of (11) if $1/\lambda_h(\sigma) = \eta^h - \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma)$ for all h, σ. It remains to be shown that the budget inequalities (5) as well as the market clearing conditions are satisfied. The latter is obvious, given (11). Regarding the budget inequalities we need to show that $V^h(s^t, \lambda(s^t)) = 0$ if and only if $\sum_{\sigma \succeq s^t} \rho(\sigma)(c^h(\sigma) - e^h(\sigma)) = 0$. Since for any agent $h \in \mathcal{H}$, $\frac{\rho(s^{t+1})}{\rho(s^t)} = \frac{u_h'(s_{t+1}, \lambda(s^{t+1}))}{u_h'(s_t, \lambda(s^t))}$ whenever $V^h(s_{t+1}, \lambda(s^{t+1})) \neq 0$, this follows from the definition of V^h. $\square$

Note that if there is a competitive equilibrium with $\lambda(s^t) = \lambda^*$ for all s^t, this must be an unconstrained Arrow Debreu equilibrium. The fact that $\lambda(s^t)$ does not change over time implies that the additional constraint (5) is never binding in equilibrium. The equilibrium allocation is identical to the unconstrained Arrow-Debreu equilibrium allocation and is Pareto-optimal. Therefore, if for a given Markov equilibrium there exists a vector of weights λ^* with $V^h(s, \lambda^*) \geq 0$ for all $s \in \mathcal{S}$ and all $h \in \mathcal{H}$, then there exist initial conditions (leading to the weights λ^*) for which the Markov equilibrium is identical to an unconstrained Arrow-Debreu equilibrium. In this Markov equilibrium we simply have that $L(s, \lambda^*) = \lambda^*$ for all s.

It is well known that in models where the equilibrium may be constrained inefficient Markov equilibria might not always exist. Examples of non-existence are known for models with incomplete markets (see e.g. Kubler and Schmedders (2002)). These examples

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typically consider the 'natural' endogenous state space, i.e. beginning-of-period portfolio holdings. However, from the structure of the examples it is clear that non-existence remains a problem if one considers the instantaneous Negishi weights as the state variable. For the model with collateral constraints, when financial markets are incomplete no sufficient conditions are known that ensure the existence of a Markov equilibrium (see Kubler and Schmedders (2003)). In contrast, in the environment considered here, with complete markets, the assumption that all agents' preferences satisfy the gross substitute property[8] implies that Markov equilibria always exist. As pointed out by Dana (1993), in our context the assumption of gross substitutes is equivalent to assuming that for all agents h and all shocks s, the term $c\, u'_h(c, s)$ is increasing in c; or equivalently, that the coefficient of relative risk aversion $-c\frac{u''_h(c,s)}{u'_h(c,s)}$ is always less than or equal to one. While in applied work it is often assumed that relative risk aversion is significantly above one it is also sometimes argued (see e.g. Boldrin and Levine (2001)) that a value below one might the the empirically more relevant case.

We have then the following result.

THEOREM 3 *Suppose that for all agents h and all shocks s, $c\, u'_h(c, s)$ is increasing in c for all $c > 0$. Then a Markov equilibrium exists.*

The proof is somewhat lengthy and is relegated to the Appendix. The main idea of the proof is to use Dana's (1993) insight to show that gross substitutability implies uniqueness of a zero of the excess expenditure map as a function of the Negishi consumption weights.

In the next section, we derive some sufficient conditions that ensure not only the existence of Markov equilibria but of finite support equilibria.

4.2 Markov equilibria with finite support

The main difficulty in determining if there exist Markov equilibria with finite support lies in specifying the support. We show that for the case of two agents, $H = 2$, there is a natural characterization of the support. We will then extend the analysis to the case of arbitrarily many agents.

4.2.1 Finite support Markov equilibria in economies with two agents

In this section we focus on the case where there are only two types of agents. This allows us to denote by $\lambda = \lambda^1$ the value of the consumption weight for agent 1 and take this as a state variable. In a slight abuse of notation, we write then $C^h(s, \lambda) = C^h(s, (\lambda, 1 - \lambda))$.

It turns out that for two agents the existence of a Markov equilibrium implies the existence of a finite support equilibrium. This will be made precise below. For now we conjecture that there are Markov equilibria where at most $2S$ points in the endogenous state space are visited. We denote them by $(\underline{\lambda}^*_s, \overline{\lambda}^*_s)_{s \in \mathcal{S}}$.

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[8]See Dana (1993) for an application of this assumption to an infinite horizon model.

For any $\underline{\lambda} \in [\epsilon, 1-\epsilon]^S$ and $\overline{\lambda} \in [\epsilon, 1-\epsilon]^S$ and all s we can define a function $L : \mathcal{S} \times [0, 1] \to [0, 1]$ by

$$L_{(\underline{\lambda}, \overline{\lambda})}(s, \lambda) = \begin{cases} \lambda & \text{if } \underline{\lambda}_s \leq \lambda \leq \overline{\lambda}_s \\ \underline{\lambda}_s & \text{if } \lambda < \underline{\lambda}_s \\ \overline{\lambda}_s & \text{if } \lambda > \overline{\lambda}_s. \end{cases}$$

For each $h = 1, 2$, define $2S^2$ numbers $V^h(s, \underline{\lambda}_{\tilde{s}})$ and $V^h(s, \overline{\lambda}_{\tilde{s}})$ for $s, \tilde{s} \in \mathcal{S}$ to be the solution to the following linear system of $2S^2$ equations.

$$V^h(s, \underline{\lambda}_{\tilde{s}}) = u'_h(s, \underline{\lambda}_{\tilde{s}}) \left(C^h(s, \underline{\lambda}_{\tilde{s}}) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L_{(\underline{\lambda}, \overline{\lambda})}(s', \underline{\lambda}_{\tilde{s}})), \qquad (12)$$

$$V^h(s, \overline{\lambda}_{\tilde{s}}) = u'_h(s, \overline{\lambda}_{\tilde{s}}) \left(C^h(s, \overline{\lambda}_{\tilde{s}}) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L_{(\underline{\lambda}, \overline{\lambda})}(s', \overline{\lambda}_{\tilde{s}})). \qquad (13)$$

Clearly, the solution to this system depends non-linearly on the choice of $(\underline{\lambda}_s, \overline{\lambda}_s)_{s \in \mathcal{S}}$, which determine both the transition function $L_{(\underline{\lambda}, \overline{\lambda})}$ and the value of the terms $u'_h(s, \overline{\lambda}_{\tilde{s}}) \left(C^h(s, \overline{\lambda}_{\tilde{s}}) - e^h(s) \right)$. Part of the solution consists of the $2S$ numbers $V^1(s, \underline{\lambda}_s)$, $V^2(s, \overline{\lambda}_s)$ for all $s \in \mathcal{S}$. We want to show that there exist S pairs $(\underline{\lambda}_s^*, \overline{\lambda}_s^*)$ such that

$$V^1(s, \underline{\lambda}_s^*) = V^2(s, \overline{\lambda}_s^*) = 0 \text{ for all } s \in \mathcal{S}. \qquad (14)$$

We will then show below how to construct a Markov equilibrium whose support is a subset of these values $(\underline{\lambda}_s^*, \overline{\lambda}_s^*)_{s \in \mathcal{S}}$. These pairs $\underline{\lambda}_s^*, \overline{\lambda}_s^*$ determine the boundaries of the intervals where $V^1(s, \lambda) \geq 0$ and $V^2(s, \lambda) \geq 0$.

To show the existence of a solution of (12)-(14), we can substitute out all $V^1(s, \overline{\lambda}_{\tilde{s}})$ and $V^2(s, \underline{\lambda}_{\tilde{s}})$ as well as all $V^1(s, \underline{\lambda}_{\tilde{s}})$ and $V^2(s, \overline{\lambda}_{\tilde{s}})$ for $s \neq \tilde{s}$. We obtain a function $f : [\epsilon, 1 - \epsilon]^{2S} \to \mathbb{R}^{2S}$, where each f_i, $i = 1, ..., S$ is the weighted sum of terms of the form

$$u'_1(s, \overline{\lambda}_{\tilde{s}}) \left(C^1(s, \overline{\lambda}_{\tilde{s}}) - e^1(s) \right) \text{ and } u'_1(s, \underline{\lambda}_{\tilde{s}}) \left(C^1(s, \underline{\lambda}_{\tilde{s}}) - e^1(s) \right), \qquad (15)$$

where the weights on the terms involving $\overline{\lambda}_s$ are positive (bounded away from zero) if and only if there is an s' with $\underline{\lambda}_{s'} > \overline{\lambda}_s$ (recall that $\pi(s, s') > 0$ for all s, s'). Similarly each f_i with $i = S + 1, ..., 2S$ is a weighted sum of terms

$$u'_2(s, \overline{\lambda}_{\tilde{s}}) \left(C^2(s, \overline{\lambda}_{\tilde{s}}) - e^2(s) \right) \text{ and } u'_2(s, \underline{\lambda}_{\tilde{s}}) \left(C^2(s, \underline{\lambda}_{\tilde{s}}) - e^2(s) \right), \qquad (16)$$

where the weights on the terms involving $\underline{\lambda}_s$ are positive if and only if there is an s' with $\overline{\lambda}_{s'} < \underline{\lambda}_s$. We obtain that $f(\underline{\lambda}_1, \overline{\lambda}_1, \ldots, \underline{\lambda}_S, \overline{\lambda}_S) = 0$ precisely when there exists a solution to (12) and (13) with

$$V^1(s, \underline{\lambda}_s) = V^2(s, \overline{\lambda}_s) = 0 \text{ for all } s \in \mathcal{S}.$$

We have the following lemma.

LEMMA 2 *There exist $x \in [\epsilon, 1 - \epsilon]^{2S}$ with $f(x) = 0$.*

Proof.

The Lemma follows directly by applying Lemma 1 to a slight modification of the function $f(.)$. For this let, for $x \in [\epsilon, 1 - \epsilon]^{2S}$, $g_i(x) = f_i(x)$ for $i = 1, ..., S$ and $g_i(x) = -f_i(x)$ for $i = S + 1, ..., 2S$. Extend then the function g to the whole domain $\mathbb{R}^{2S}$ by setting it continuous and constant outside of $[\epsilon, 1 - \epsilon]^{2S}$. All one needs to prove is the appropriate boundary behavior. Clearly as some $\overline{\lambda}_s$ is sufficiently large or some $\underline{\lambda}_s$ is sufficiently small, we have that $\sum_i x_i g_i(x) < 0$ since each $f_i(x)$ is bounded above. The key is to show that if $\overline{\lambda}_s$ is sufficiently small, or if $\underline{\lambda}_s$ is sufficiently large, we also have that some $|g_i(x)|$ becomes arbitrarily large. To show this note that in (16) the terms involving $\underline{\lambda}_s$ have positive (and bounded away from zero) weight whenever there is a s' with $\overline{\lambda}_{s'} < \underline{\lambda}_s$. If this is the case clearly some $f_i(x)$, $i = 1, ..., S$ can be made arbitrarily small; if it is not the case, some $\overline{\lambda}_{s'}$ becomes arbitrarily close to 1 and we are in the case above. The argument for $\overline{\lambda}_s$ is analogous. $\square$

Note that given a solution to the system (12)-(14) we can define functions $V^h : \mathcal{S} \times (0, 1) \to \mathbb{R}$ as follows.

$$V^1(s, \lambda) = u_1'(s, \lambda)(C^1(s, \lambda) - e^1(s)) + \beta \sum_{s'} \pi(s, s') V^1(s, L_{(\underline{\lambda}^*, \overline{\lambda}^*)}(s, \lambda))$$

$$V^2(s, \lambda) = u_2'(s, \lambda)(C^2(s, \lambda) - e^2(s)) + \beta \sum_{s'} \pi(s, s') V^2(s, L_{(\underline{\lambda}^*, \overline{\lambda}^*)}(s, \lambda))$$

It is easy to verify that these functions together with the transition function $L_{(\underline{\lambda}^*, \overline{\lambda}^*)}(s, .)$ satisfy the conditions of Theorem 1 above and therefore describe a Markov equilibrium if the transition function satisfies

$$L_{(\underline{\lambda}^*, \overline{\lambda}^*)}(s', \underline{\lambda}_s^*) = \underline{\lambda}_s^* \Rightarrow V^h(s', \underline{\lambda}_s^*) \geq 0 \text{ for } h = 1, 2 \tag{17}$$

$$L_{(\underline{\lambda}^*, \overline{\lambda}^*)}(s', \overline{\lambda}_s^*) = \overline{\lambda}_s^* \Rightarrow V^h(s', \overline{\lambda}_s^*) \geq 0 \text{ for } h = 1, 2 \tag{18}$$

If this is the case, $L_{(\underline{\lambda}^*, \overline{\lambda}^*)}(.)$ describes a transition function that ensures that $V^h(s, L(s, \lambda)) \geq 0$ for all λ.

In order to find conditions on the fundamentals that ensure that (17) and (18) hold, it is useful to define functions $\tilde{V}^h(s, \lambda) = \frac{1}{u_h'(s, \lambda)} V^h(s, \lambda)$ for $h = 1, 2$. It is easy to see that $\tilde{V}$ satisfies the following.

$$\tilde{V}^1(s,\lambda) = C^1(s,\lambda) - e^1(s) + \beta \sum_{s':\lambda \in [\underline{\lambda}^*_{s'},\overline{\lambda}^*_{s'}]} \pi(s,s')\frac{u_1'(s',\lambda)}{u_1'(s,\lambda)}\tilde{V}^1(s',\lambda)$$

$$+\beta \sum_{s':\lambda < \underline{\lambda}^*_{s'}} \pi(s,s')\frac{u_1'(s',\underline{\lambda}^*_{s'})}{u_1'(s,\lambda)}V^1(s',\underline{\lambda}^*_{s'})$$

$$\tilde{V}^2(s,\lambda) = C^2(s,\lambda) - e^2(s) + \beta \sum_{s':\lambda \in [\underline{\lambda}^*_{s'},\overline{\lambda}^*_{s'}]} \pi(s,s')\frac{u_2'(s',\lambda)}{u_2'(s,\lambda)}\tilde{V}^2(s',\lambda)$$

$$+\beta \sum_{s':\lambda > \overline{\lambda}^*_{s'}} \pi(s,s')\frac{u_2'(s',\overline{\lambda}^*_{s'})}{u_2'(s,\lambda)}V^2(s',\overline{\lambda}^*_{s'})$$

for all s with $\lambda \in [\underline{\lambda}^*_s,\overline{\lambda}^*_s]$. A standard argument shows that these equations always have a solution.

A sufficient condition for (17) and (18) to hold is of course that each $\tilde{V}^h(s,.)$ has a unique zero. In this case, it must be that for both $h = 1, 2$, $V^h(s,\lambda)$ is non-negative for all $\lambda \in [\underline{\lambda}^*,\overline{\lambda}^*]$.

To guarantee this we make the following strong assumption on preferences:

ASSUMPTION 1 *One of the following properties holds for preferences:*

1. *All agents coefficient of relative risk aversion is below 1.*

2. *All agents have identical constant relative risk aversion (CRRA) utility*

3. *Utility is shock-independent and there is no aggregate uncertainty*

The assumption guarantees that $\tilde{V}^h(s,.)$ is monotone for all s and both $h = 1, 2$. We prove the result for $h = 1$, the case $h = 2$ is of course analogous. If all agents' relative risk aversion is below 1, the utility satisfies the gross substitute property and the result follows from the proof of Theorem 2. Assume that agents have identical CRRA utility. Then the term $u_h'(s',\lambda)/u_h'(s,\lambda)$ is independent of λ. Therefore λ only enters through the term $C^1(s,\lambda)$, which is clearly increasing in λ and through the term $\pi(s,s')\frac{u_1'(s',\underline{\lambda}_{s'})}{u_1'(s,\lambda)}V^1(s',\underline{\lambda}_{s'})$ which is also increasing in λ since $u_1'(s,\lambda)$ is decreasing in λ. Therefore the function V^1 must be monotonically increasing. Finally, if there is no aggregate uncertainty, the term $u_h'(s',\lambda)/u_h'(s,\lambda)$ is simply equal to 1 and the same argument as for identical CRRA utility functions applies.

By the previous arguments we have then shown that Assumption 1 guarantees the existence of a Markov equilibrium with finite support. We have so the following result:[9]

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[9]In an earlier working paper version of their published paper, Chien and Lustig also characterize equilibria with finite support for the case of two shocks and two agents with identical CRRA utility. Our result holds for any number of shocks under more general conditions.

THEOREM 4 *Under any of the conditions of Assumption 1 a finite support Markov equilibrium exists in economies with two agents.*

Note that in this construction the intervals $([\underline{\lambda}_s^*, \overline{\lambda}_s^*])_{s \in \mathcal{S}}$ uniquely define the values $(V^h(s, \underline{\lambda}_{\tilde{s}}), V^h(s, \overline{\lambda}_{\tilde{s}}))_{s, \tilde{s} \in \mathcal{S}}^{h \in \mathcal{H}}$. Therefore, in the case of two agents, if Markov equilibria exist, they can be described by $2S$ numbers which determine the boundaries of the intervals in the consumption weights' space which are feasible in equilibrium. Moreover, the equilibrium dynamics of these simple equilibria is straightforward. If one starts at an initial condition which corresponds to a welfare weight on the boundary of the interval for the initial state, only finitely many different welfare weights are visited along the equilibrium. Since (17) and (18) constitute a finite number of inequalities, it is possible to verify numerically whether a Markov equilibrium exists – this is an important advantage of finite support equilibria. If equilibria have infinite support it is often extremely difficult to conduct error analysis given a computed approximate Markov equilibrium (see Kubler (2011)).

4.2.2 The example again

To illustrate the construction of Theorem 4 it is useful to reconsider the example of Section 3. In that example a Pareto-efficient steady state exists, for some parameter values, but, depending on the initial conditions, it might take arbitrarily long to reach it. This can be easily explained in this framework. Suppose for simplicity that $\mathfrak{h} - \mathfrak{l} = \frac{\eth}{1-\beta}$, $u_1(c) = u_2(c) = \log(c)$ and $\pi = 1/2$. Denote aggregate endowments by $\omega = \mathfrak{h} + \mathfrak{l} + \eth$.

For this specification we have $C^1(s, \lambda) = \lambda \omega$ for both $s = 1, 2$. Also, by the argument in Section 3.1, there exists a unique efficient steady state where each agents' consumption is given by $\frac{\omega}{2}$. At this steady state, we have therefore $\lambda = \frac{1}{2}$.

As pointed out after the proof of Theorem 2, a Pareto efficient Markov equilibrium exists (for some initial conditions) if, for some λ^*, we have $V^h(s, \lambda^*) \geq 0$ for all h and all s. In the environment considered here, since agent 1 has a high endowment in shock 1, we must have $\underline{\lambda}_1 > \underline{\lambda}_2$ and, for an efficient equilibrium to exist, we must also have $\underline{\lambda}_1 \leq \overline{\lambda}_2$. In fact, we will show that an efficient Markov equilibrium exists with $\underline{\lambda}_1 = \overline{\lambda}_2 = \frac{1}{2}$. For this, one needs to verify that, with this value of $\underline{\lambda}_1$, the solution to the following system (obtained from (12), (13), with $L(s, \underline{\lambda}_1) = \underline{\lambda}_1$ for $s = 1, 2$)

$$V^1(1, \underline{\lambda}_1) = 1 - \frac{\mathfrak{h}}{\underline{\lambda}_1 \omega} + \frac{\beta}{2} \left(V^1(1, \underline{\lambda}_1) + V^1(2, \underline{\lambda}_1) \right)$$

$$V^1(2, \underline{\lambda}_1) = 1 - \frac{\mathfrak{l}}{\underline{\lambda}_1 \omega} + \frac{\beta}{2} \left(V^1(1, \underline{\lambda}_1) + V^1(2, \underline{\lambda}_1) \right)$$

satisfies $V^1(1, \underline{\lambda}_1) = 0$ and $V^1(2, \underline{\lambda}_1) > 0$. It is easy to see that if $V^1(1, \underline{\lambda}_1) = 0$ a solution of the second equation above, when $\underline{\lambda}_1 = \frac{1}{2}$, is given by $V^1(2, \underline{\lambda}_1) = \frac{1}{1-\beta/2}(1 - \frac{\mathfrak{l}}{0.5\omega})$. The first equation is then also satisfied if

$$V^1(1, \underline{\lambda}_1) = 1 - \frac{\mathfrak{h}}{0.5\omega} + \frac{\beta}{2} V^1(2, \underline{\lambda}_1) = \frac{\mathfrak{l} + \eth - \mathfrak{h}}{\omega} + \frac{\beta(\mathfrak{h} + \eth - \mathfrak{l})}{(2 - \beta)\omega} = 0,$$

which holds whenever $(2 - \beta)(\mathfrak{l} + \mathfrak{d} - \mathfrak{h}) + \beta(\mathfrak{h} + \mathfrak{d} - \mathfrak{l}) = 0$, equivalent to our assumption that $\mathfrak{h} - \mathfrak{l} = \frac{\mathfrak{d}}{1-\beta}$.

By symmetry, $\overline{\lambda}_2 = \frac{1}{2}$ solves the corresponding system for $V^2(s, \overline{\lambda}_2)$, $s = 1, 2$. We can also solve for $\underline{\lambda}_2$ the system for $V^1(s, \underline{\lambda}_2)$, $s = 1, 2$, where $L(1, \underline{\lambda}_2) = \underline{\lambda}_1$. Using the fact that $V^1(1, \underline{\lambda}_1) = 0$ we get

$$V^1(2, \underline{\lambda}_2) = 0 = 1 - \frac{\mathfrak{l}}{\underline{\lambda}_2 \omega},$$

thus $\underline{\lambda}_2 = \frac{\mathfrak{l}}{\omega} < 1/2$. By symmetry, we have that $\overline{\lambda}_1 = 1 - \underline{\lambda}_2$.

As pointed out at the end of the previous section, the values $\underline{\lambda}_s, \overline{\lambda}_s, s = 1, 2$ completely characterize the Markov equilibrium for this example. The analysis of the transition to the steady state is then greatly simplified with respect to the one in Section 3.3. If the initial conditions are such that the initial consumption weight $\lambda_0 = 1/2$, the Markov equilibrium coincides with the efficient steady state. On the other hand if, for example, $\lambda_0 = \underline{\lambda}_2$, the state variable remains unchanged at the value λ_0 as long as only shock 2 occurs, since $V^h(2, \underline{\lambda}_2) \geq 0$ for both $h = 1, 2$; agent 1 will consume then an amount less than $1/2$. When shock 1 occurs, we have $V^1(1, \underline{\lambda}_2) < 0$ since $\underline{\lambda}_2 < \underline{\lambda}_1$ and $V^1(1, .)$ is monotone. Therefore λ must 'jump' to $\underline{\lambda}_1$ where it will stay from there on. Hence the steady state will be reached after each shock has realized at least once.[10]

Finally, note that the same argument can also be used to analyze the case where the steady state is inefficient. It is easy to see that when $\mathfrak{h} - \mathfrak{l} > \frac{\mathfrak{d}}{1-\beta}$ we have $\underline{\lambda}_1 > \overline{\lambda}_2$. There is then no efficient steady state and along the equilibrium path the instantaneous consumption weight λ oscillates between the two values $\{\overline{\lambda}_2, \underline{\lambda}_1\}$.

4.2.3 Existence and non-existence of finite support equilibria when $H > 2$

Unfortunately, for the general case with more than 2 agents' types we do not know of general conditions which ensure the existence of finite support Markov equilibria. However, it is easy to construct examples for which finite support equilibria exist. Suppose there are 3 agents and three equiprobable i.i.d. shocks. Assume again the agents have identical log-utility functions, $u_h(c) = \log(c)$ for all h, endowments are $e^1 = (0, \mathfrak{h}, \mathfrak{h})$, $e^2 = (\mathfrak{h}, 0, \mathfrak{h})$ and $e^3 = (\mathfrak{h}, \mathfrak{h}, 0)$ for some $\mathfrak{h} > 0$, while the tree pays constant dividends $\mathfrak{d} > 0$. The aggregate endowment is then deterministic and equal to $\omega = 2\mathfrak{h} + \mathfrak{d}$. Similarly to the previous example, we have $C^h(s, \lambda) = \lambda_h \omega$ for all h and s. We assume that initial conditions are $s_0 = 1$ and $\theta^1(s_{-1}) = 1$.

Using symmetry, we show in what follows that under the condition

$$\mathfrak{h} > \frac{\mathfrak{d}}{1 - \beta}$$

there exists a steady state where agents 2 and 3 are constrained in state 1, agents 1 and 3 in state 2 and agents 1 and 2 in state 3. Denoting by $\lambda(s, h)$ the value of the Negishi

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[10]Note that this argument is valid even if there is aggregate uncertainty.

weights in state s where only type h is unconstrained, we need then to find the values of the vectors $\lambda(1,1), \lambda(2,2)$ and $\lambda(3,3)$ constituting the support of the equilibrium. By symmetry, the weights of all agents when constrained are identical, across all states, i.e. $\lambda_1(2,2) = \lambda_1(3,3) = \lambda_2(1,1) = \lambda_2(3,3) = \lambda_3(1,1) = \lambda_3(2,2) = \lambda_{\mathfrak{h}}$ for some $\lambda_{\mathfrak{h}}$. Similarly, $\lambda_1(1,1) = \lambda_2(2,2) = \lambda_3(3,3) = \lambda_{\mathfrak{l}} = 1 - 2\lambda_{\mathfrak{h}}$. In this situation, the transition function must then satisfy the following property

$$L(s, \lambda) = \lambda(s,s) \text{ whenever } \lambda \in \{\lambda(1,1), \lambda(2,2), \lambda(3,3)\}.$$

Given this property of the transition function and the above specification of the states where each agent is constrained, proceeding analogously to the previous section we obtain

$$V^1(1, \lambda(1,1)) = 1 + \frac{\beta}{3} V^1(1, \lambda(1,1)) = \frac{1}{1 - \frac{\beta}{3}}$$

$$V^1(s, \lambda(s,s)) = 0 = 1 - \frac{\mathfrak{h}}{\lambda_1(s,s)\omega} + \frac{\beta}{3} V^1(1, \lambda(1,1)), \ \ s = 2, 3$$

where the equality $V^1(s, \lambda(s,s)) = 0$ holds in the states where agent 1 is constrained. Hence we must have

$$1 - \frac{\mathfrak{h}}{\lambda_{\mathfrak{h}}\omega} + \frac{\beta}{3 - \beta} = 0,$$

or

$$\lambda_{\mathfrak{h}} = \frac{(3 - \beta)\mathfrak{h}}{3(\mathfrak{d} + 2\mathfrak{h})},$$

and $1 > \lambda_{\mathfrak{h}} \geq 1/3 \geq \lambda_{\mathfrak{l}}$ given the assumption $\mathfrak{h} \geq \frac{\mathfrak{d}}{1 - \beta}$. Given the initial condition, we must have $\lambda(s_0) = \lambda(1,1)$ and we have verified the one constructed is a Markov equilibrium with finite support.

To generalize the example to any number of states and agents we again construct a finite set of possible weights on which a transition function is defined and verify that we can find associated values of the excess expenditure functions $V^h(s,.)$, also defined on this finite set. To make the construction clear, we suppose first that there exists a Markov equilibrium with excess expenditure functions $V^h(s,.)$ and then validate ex post this supposition was correct. We conjecture that in the Markov equilibrium the endogenous state variable (λ) only takes values in the finite set of values of λ for which excess expenditure is zero for all but one agent (the only one who is unconstrained)

$$\mathcal{C}^* = \{\lambda^*(1,1), \dots, \lambda^*(1, H)), \lambda^*(2,1), \dots, \lambda^*(S, H)\},$$

where each $\lambda^*(s, \tilde{h})$ is a solution of the system $V^h(s, \lambda^*(s, \tilde{h})) = 0$, $h \neq \tilde{h}$ whenever it exists. Since these are $H - 1$ equations in $H - 1$ unknowns, for any s, h there are typically finitely many solutions $\lambda^*(s, h)$ of this system. Under the assumptions of Theorem 3 the solution, if it exists, is unique for all s, h. When, as in the above example, $e^h(s) = 0$ for some h, s, a solution $\lambda^*(s, h)$ will not exist and is therefore not included in the set $\mathcal{C}^*$.

Next, we define, for $\lambda \in \mathcal{C}^*$

$$L(s,\lambda) = \begin{cases} \lambda^*(s,\bar{h}) & \text{if } \exists \gamma \geq 0, \gamma_{\bar{h}} = 0 : \lambda^*(s,\bar{h}) = \frac{\lambda+\gamma}{\sum_i (\lambda_i + \gamma_i)} \\ \lambda & \text{otherwise.} \end{cases} \tag{19}$$

assuming that $L(s,\lambda)$ is uniquely defined. The following theorem gives a sufficient condition that ensures the existence of a Markov equilibrium with finite support.

THEOREM 5 *Suppose for all $s \neq s'$ and all $h \in \mathcal{H}$ the following holds.*

$$L(s',\lambda^*(s,h)) = \lambda^*(s,h) \Rightarrow V^{\tilde{h}}(s',\lambda^*(s,h)) \geq 0 \text{ for all } \tilde{h} \in \mathcal{H}, \tag{20}$$

If the initial conditions are such that $\theta^h(s_{-1}) = 1$ for some agent h, then there exists a finite support Markov equilibrium with

$$\lambda(s^t) = L(s_t, \lambda(s^{t-1})) \text{ for all } s^t.$$

The proof of the theorem follows directly from our recursive characterization in Theorem 2 above. When $L(s',\lambda^*(s,h)) = \lambda^*(s',h')$ for some h', the definition of $\lambda^*(s',h')$ ensures that $V^h(s',\lambda^*(s',h')) \geq 0$ for all h. In contrast, when $L(s,\lambda) = \lambda$ for some $\lambda \neq \lambda^*(s,h)$ for all h, the construction of $L(s,.)$ in (19) does not ensure the same property holds. Condition (20) in the theorem ensures that for any $\lambda \in \mathcal{C}^*$ it is always the case that $V^h(s,L(s,\lambda)) \geq 0$, implying that the conditions for a Markov equilibrium are satisfied. Finally, the condition that $\theta^h(s_{-1}) = 1$ for some agent h ensures that the initial λ also lies in the finite set $\mathcal{C}^*$.

Obviously, Theorem 5 imposes a condition on the properties of a Markov equilibrium. This does not translate into a condition on fundamentals. To illustrate why it is difficult to find general conditions that ensure the existence of a finite support equilibrium, consider the following small modification of the example above. Instead of assuming that each agents' individual endowments are high in two out of the three states, suppose they are high only in one out of the three states. That is $e^1 = (\mathfrak{h}, 0, 0)$, $e^2 = (0, \mathfrak{h}, 0)$ and $e^3 = (0, 0, \mathfrak{h})$ for some $\mathfrak{h} > 0$. Under the maintained assumption of logarithmic utility, Theorem 3 ensures the existence of a Markov equilibrium. However, we will show that under the condition $\mathfrak{h} > \frac{\mathfrak{d}}{2(1-\beta)}$ there exists no finite support equilibrium of the form postulated in Theorem 5.

It is clear from the specification of the agents' endowments that in equilibrium two out of the three agents's types will always be unconstrained while the construction above postulates that two out of the three types of agents are constrained. Consider the case where the current state of the economy is (s, λ') for $s \neq 1$, and we move next period to shock 1 and weight λ, where agent 1 is constrained ($V^1(1,\lambda) = 0$). Hence agents 2 and 3 will be unconstrained and the ratio λ_2/λ_3 will be equal to λ'_2/λ'_3, and depend so on the previous period's state.

In fact, λ must satisfy the following system of equations, for some function $L(s,.)$,

$$V^1(1,\lambda) = 0 = 1 - \frac{\mathfrak{h}}{\lambda_1 \omega} + \frac{\beta}{3} \sum_{s'=2}^{3} V^1(s', L(s',\lambda))$$

$$V^1(s,\lambda) = 1 + \frac{\beta}{3} \sum_{s'=2}^{3} V^1(s', L(s',\lambda)), \quad s = 2,3.$$

Since for $s = 2,3$ we obtain that $V^1(s,\lambda) = \frac{1}{1-\frac{2\beta}{3}}$, $\lambda_1 = \lambda_{\mathfrak{h}}$ is determined as solution of the following equation.

$$1 - \frac{\mathfrak{h}}{\lambda_{\mathfrak{h}} \omega} + \frac{2\beta}{3-2\beta} = 0 \Leftrightarrow \lambda_{\mathfrak{h}} = \frac{\mathfrak{h}}{\omega} \frac{3-2\beta}{3}.$$

Under the assumption that $2\mathfrak{h} > \frac{\mathfrak{d}}{1-\beta}$ we have $1 > \lambda_{\mathfrak{h}} > \frac{1}{3}$.

By symmetry we conjecture that the same expression obtains for agent 2 in state 2 and agent 3 in state 3. We therefore postulate the following transition function that is defined on the set of all $\lambda \in \Delta^2$ with $\lambda_j = \lambda_{\mathfrak{h}}$ for some $j = 1,2,3$.

$$L(s,\lambda) = \begin{cases} (\lambda'_j)^H_{j=1} : & \begin{array}{ll} \lambda'_j = \lambda_{\mathfrak{h}} & \text{if } j = s \\ \lambda'_j = \lambda_j \frac{1-\lambda_{\mathfrak{h}}}{\sum_{h \neq s} \lambda_h} & \text{otherwise.} \end{array} \end{cases}$$

If we take as initial conditions $s_0 = 1$ and $\theta^2(s_{-1}) = \theta^3(s_{-1}) = 1/2$, a Markov equilibrium exists with transition function $L(s,.)$. The initial value of the welfare weights is given by $\lambda(s_0) = (\lambda_{\mathfrak{h}}, \frac{1-\lambda_{\mathfrak{h}}}{2}, \frac{1-\lambda_{\mathfrak{h}}}{2})$. Since $V^h(s,\lambda) > 0$ whenever $s \neq h$ and, by construction, for all $\lambda(s^t)$ along the equilibrium path $V^h(h, \lambda(s^t)) = 0$, the one specified is a Markov equilibrium. It is easy to check that this equilibrium generally does not have finite support. To see this, consider for instance a sequence of shocks for $t = 1, 2, ...$ with $s_t = 1$ if t is odd and $s_t = 2$ if t is even. It is easy to see that we must have $\lambda_3(s^{t+1}) = \lambda_3(s^t) \frac{1-\lambda_{\mathfrak{h}}}{\lambda_{\mathfrak{h}} + \lambda_3(s^t)}$ and hence

$$\frac{1}{\lambda_3(s^{t+1})} = \frac{1}{1-\lambda_{\mathfrak{h}}} + \frac{\lambda_{\mathfrak{h}}}{1-\lambda_{\mathfrak{h}}} \frac{1}{\lambda_3(s^t)},$$

which converges to $\frac{1}{1-2\lambda_{\mathfrak{h}}}$ if $\lambda_{\mathfrak{h}} < \frac{1}{2}$ or diverges otherwise. In the process it takes infinitely many values.

5   Welfare Properties

In this section we investigate the welfare properties of competitive equilibria with collateral constraints. In the simple example considered in Section 3, steady state equilibria are Pareto efficient whenever $\mathfrak{d}/(1-\beta) > \mathfrak{h} - \mathfrak{l}$, that is when the amount of available collateral is sufficiently large relative to some measure of the variability of agents' endowments. In this section we will generalize this result and derive some necessary and sufficient conditions for the existence of Pareto efficient equilibria when shocks are i.i.d. and there is no aggregate uncertainty. Next, we will turn our attention to the welfare properties of equilibria when

the collateral constraint binds so that competitive equilibria are Pareto inefficient, showing that they are also constrained inefficient. That is, even by taking the limited pledgeability constraints into account, a welfare improvement can still be obtained with respect to the competitive equilibrium.

5.1 Pareto Efficient equilibria

It is well known that in the stationary economy considered in this paper Pareto efficient allocations are always stationary, i.e. consumption only depends on the current shock (see e.g. Judd et al. (2003)). As in Section 3.1 we define a steady state to be an equilibrium where individual consumption and prices are time invariant functions of the shock alone. It follows again from Judd et al. (2003) that at such an equilibrium portfolios will be constant, i.e. will not depend on the shock. On the other hand equilibrium prices may depend on the current realization of the shock. Formally, a *steady state equilibrium* consists of prices of the tree options $(\bar{q}(s; s'))_{s\in\mathcal{S}, s'\in\mathcal{S}} \in \mathbb{R}_+^{S^2}$ (where s is the current realization of the shock and s' identifies the realization of the shock next period in which the security promises the delivery of the tree), consumption levels $(\bar{c}^h(s))_{s\in\mathcal{S}, h\in\mathcal{H}} \in \mathbb{R}_+^{HS}$ and portfolios $(\bar{\theta}_{s'}^h)_{s'\in\mathcal{S}, h\in\mathcal{H}} \in \mathbb{R}_+^{HS}$ such that for an economy with initial conditions $\theta^h(s_{-1}) = \bar{\theta}_{s_0}^h$ a competitive equilibrium obtains at the values $\left((\theta^h(s^t), c^h(s^t))_{h\in\mathcal{H}}, q(s^t)\right) : c^h(s^t) = \bar{c}^h(s_t)$, and, for all $s' \in \mathcal{S}$, $q(s^t; s') = \bar{q}(s_t; s')$, $\theta_{s'}^h(s^t) = \bar{\theta}_{s'}^h$.

Therefore a competitive equilibrium can only be Pareto efficient if it is a steady state equilibrium. As shown in the example of Section 3, even when an efficient steady state exists, for some initial conditions it may not be reached immediately. With a slight abuse of notation, we then say that *Pareto efficient equilibria exist* if there are initial conditions for which the competitive equilibrium is efficient (and therefore is a steady state equilibrium).

At a steady state equilibrium, the budget constraints of each consumer can be reduced to the following finite system of equations

$$A(q)\theta = \begin{pmatrix} c(1) - e^h(1) \\ \vdots \\ c(S) - e^h(S) \end{pmatrix}, \ \theta \geq 0. \tag{21}$$

where the matrix $A(q)$ is defined as follows:

$$A(q) = \begin{pmatrix} d(1) + \sum_{s'} q(1; s') - q(1; 1) & -q(1; 2), & \dots & -q(1; S) \\ -q(2; 1) & d(2) + \sum_{s'} q(2; s') - q(2; 2), & \dots & -q(2; S) \\ \vdots & \vdots & \ddots & \vdots \\ -q(S; 1) & -q(S; 2), & \dots & d(S) + \sum_{s'} q(S; s') - q(S; S) \end{pmatrix}.$$

Hence, from (21) it follows that a Pareto efficient allocation $(c^h(s))_{s\in\mathcal{S}, h\in\mathcal{H}}$ can be im-

plemented as a steady state equilibrium if and only if, for all agents h,[11]

$$A(q)^{-1} \begin{pmatrix} c^h(1) - e^h(1) \\ \vdots \\ c^h(S) - e^h(S) \end{pmatrix} \geq 0. \tag{22}$$

where q are the supporting prices, determined by the allocation as follows. Define, for all $s = 1, ..., S$, $\rho(s, s') = u'_h(s')/u'_h(s)$, where we use the shorthand notation $u'_h(s)$ to denote $u'_h(c^h(s))$ for each $s \in \mathcal{S}$. Note that by the Pareto efficiency of the allocation $u'_h(s')/u'_h(s)$ is h-invariant for all s, s'. From the consumers' optimality conditions we then see that the securities' prices must satisfy the following system of equations

$$q(s; s') = \beta \pi(s, s') \rho(s, s') \left(\sum_{s''} q(s'; s'') + d(s') \right), \ s, s' \in \mathcal{S}. \tag{23}$$

To solve this system it is convenient to find first the prices of the tree in every state $s \in \mathcal{S}$, $q(s) = \sum_{s'} q(s; s')$, which satisfy the following system of S equations

$$q(s) = \beta \sum_{s'} \pi(s, s') \rho(s, s') \left(q(s') + d(s') \right), \ s \in \mathcal{S}. \tag{24}$$

Having found the supporting price of the tree in each state, the price of each contingent claim readily obtains from the equations

$$q(s; s') = \beta \pi(s, s') \rho(s, s') \left(q(s') + d(s') \right), \ s, s' \in \mathcal{S} \tag{25}$$

In general, condition (22) is difficult to interpret since it involves equilibrium allocations and the supporting prices. However, when there is no aggregate uncertainty ($\sum_h e^h(s) = \sum_h e^h(s')$ and $d(s) = d(s') = d$ for all s, s'), the condition takes a simpler and tractable form. In that case in fact a Pareto efficient allocation satisfies $c^h(s) = \bar{c}^h$ for all s, so that the value of the supporting prices of the tree are, for all s, $q(s) = \bar{q} := \frac{\beta d}{1 - \beta}$ and the securities' prices are $q(s; s') = \pi(s, s')\bar{q}$. Under the additional condition that shocks are i.i.d. ($\pi(s, s')$ is s invariant), the expression of the consumer's budget constraints in (21) simplifies to

$$\begin{pmatrix} d + (1 - \pi_1)\bar{q} & -\pi_2\bar{q} & \dots & -\pi_S\bar{q} \\ -\pi_1\bar{q} & d + (1 - \pi_2)\bar{q} & \dots & -\pi_S\bar{q} \\ \vdots & \vdots & \ddots & \vdots \\ -\pi_1\bar{q} & -\pi_2\bar{q} & \dots & d + (1 - \pi_S)\bar{q} \end{pmatrix} \theta = \begin{pmatrix} \bar{c}^h - e^h(1) \\ \vdots \\ \bar{c}^h - e^h(S) \end{pmatrix}, \tag{26}$$

which implies that for, any pair of states s and s' we have

$$\theta_s - \theta_{s'} = \frac{e^h(s') - e^h(s)}{\bar{q} + d} \tag{27}$$

---

[11]It is immediate to verify that the matrix $A(q)$ defined above is always invertible.

So a necessary condition for the existence of an efficient steady state in the case of no aggregate uncertainty and i.i.d. shocks is that for all shocks s, s' and any subset of agents $\mathcal{G} \subset \mathcal{H}$ we have[12]

$$\sum_{h \in \mathcal{G}} \frac{e^h(s') - e^h(s)}{\bar{q} + d} \leq 1. \tag{28}$$

This condition says that the amount of available collateral, as measured by $\bar{q} + d = \frac{d}{1-\beta}$, is larger than the variability of the endowment of any subset of consumers across any pair of states.

To derive a sufficient condition from (27) for the existence of a Pareto efficient steady state, for some appropriate initial conditions, we proceed as follows. For each $h \in \mathcal{H}$, denote by $\hat{s}^h$ the state where agent h has the highest endowment, $e^h(\hat{s}^h) \geq e^h(s)$ for all $s \in \mathcal{S}$. Set then $\theta^h_{\hat{s}^h} = 0$, while the remaining values of θ^h_s, $s \neq \hat{s}^h$, are determined by (27). If for all $s \in \mathcal{S}$ the following condition holds

$$\sum_h \frac{e^h(\hat{s}^h) - e^h(s)}{\bar{q} + d} \leq 1 \tag{29}$$

there exists a collection of portfolios $\left(\theta^h\right)_{h \in \mathcal{H}}$ which are admissible (≥ 0) as well as feasible (less or equal than the supply, equal to one) and support a constant value of $\bar{c}^h$ for all $h \in \mathcal{H}$, that is a Pareto efficient allocation.

We can summarize the above findings in the following:

THEOREM 6 *A Pareto efficient steady state exists, for some appropriate initial conditions, if the finite set of conditions (22), (24), (25) are satisfied for some $(q(s; s'), q(s))_{s,s' \in \mathcal{S}}$ and some Pareto efficient allocation $(c^h(s))_{s \in \mathcal{S}, h \in \mathcal{H}}$. When there is no aggregate uncertainty and shocks are i.i.d., a Pareto efficient steady state exists if condition (29) holds.*

5.2 Constrained inefficiency

If the collateral in the economy is too little to support a Pareto efficient allocation, it could still be the case that the equilibrium allocation is constrained Pareto efficient in the sense that no reallocation of the resources that is feasible and satisfies the collateral constraints can make everybody better off. We show here that this is not true, by presenting a robust example for which a welfare improvement can indeed be found subject to these constraints.

We consider in particular a reallocation obtained by imposing tighter short-sale constraints on the trades of some tree options and considering the associated equilibrium where agents optimize subject to such constraints and markets clear. Such reallocation clearly respects the collateral constraints. At the same time, since the tighter constraints will change trades and hence securities' prices, the allocation obtained may not be budget feasible at the original prices and looser short-sale constraints, and hence might yield a higher welfare.

---

[12]This condition generalizes the one obtained in the example, given by (7).

We show the result in the simple environment described in Section 4.2.2, where shocks are equiprobable, $\pi_1 = \pi_2 = 1/2$, consumers have the same preferences, $u_h(c,s) = u(c)$ for $h = 1, 2$, and their endowment in the low state is zero: $\mathfrak{l} = 0$. In addition, $\mathfrak{h}(1 - \beta) > \mathfrak{d}$, so that (7) is violated and there is no Pareto efficient equilibrium, but an inefficient steady state equilibrium exists.

Suppose the economy is at this inefficient steady state, where $\theta^1 = (0, 1)$, $\theta^2 = (1, 0)$, and consider the welfare effect of tightening the portfolio restriction to $\theta_s^h(s^t) \geq \varepsilon$, for $\varepsilon > 0$ and all $s \in \mathcal{S}$. The restriction is assumed to be introduced at $t = 1$ and to hold for all $t \geq 1$. The intervention is announced at $t = 0$ after all trades have taken place. Agents' utility is then evaluated ex ante, from date 0.

We show that this intervention is Pareto improving, for an open set of the parameter values describing the economy. Thus the inefficient steady state equilibrium is also constrained inefficient: making the collateral constraint tighter in some date events improves welfare.

Given the nature of the intervention and the fact that the economy is initially in a steady state, there is a transition phase of one period before the economy settles to a new steady state[13]: prices and allocations are then going to depend now on time (whether it is $t = 1$ or $t > 1$) as well as the realization of the current shock. It is useful to use the notation $q_t(s; s')$ to indicate the price at time t and state s of the tree option that pays in state s'. As before, the price of the tree is then $q_t(s) = q_t(s; 1)) + q_t(s; 2)$. The new equilibrium portfolios are, at all dates $t \geq 1$, $\theta^1 = (\varepsilon, 1 - \varepsilon)$, $\theta^2 = (1 - \varepsilon, \varepsilon)$, that is the short-sale constraint always binds. At the date of the intervention, $t = 1$, we have then

$$
\begin{aligned}
c^1(s_1 = 1) &= \mathfrak{h} - q_1(1; 1)\varepsilon - q_1(1; 2)(1 - \varepsilon) \\
c^1(s_1 = 2) &= \mathfrak{d} + q_1(2; 1) + q_1(2; 2) - q_1(2; 1)\varepsilon - q_1(2; 2)(1 - \varepsilon) \\
&= \mathfrak{d} + q_1(2; 1)(1 - \varepsilon) + q_1(2; 2)\varepsilon
\end{aligned}
$$

At all subsequent dates, $t > 1$,

$$
\begin{aligned}
c^1(s_t = 1) &= \mathfrak{h} + \varepsilon\left(q(1; 1) + q(1; 2) + \mathfrak{d}\right) - q(1; 1)\varepsilon - q(1; 2)(1 - \varepsilon) \\
&= \mathfrak{h} + \varepsilon\mathfrak{d} - q(1; 2)(1 - 2\varepsilon) \\
c^1(s_t = 2) &= \left(\mathfrak{d} + q(2; 1) + q(2; 2)\right)(1 - \varepsilon) - q(2; 1)\varepsilon - q(2; 2)(1 - \varepsilon) \\
&= \mathfrak{d}(1 - \varepsilon) + q(2; 1)(1 - 2\varepsilon)
\end{aligned}
$$

That is, we settle at the new steady state where $q_t(s; s') = q(s; s')$ for all $t > 1$, s, s'. We have then eight new equilibrium prices to determine. By symmetry (of consumers'

---

[13]Note that this is different from what we found in Sections 3 and 4.2.2, where we showed that the transition to a steady state may take a very long time, until all shocks occurred. The reason is that the original steady state is no longer feasible when the restriction $\theta_s^h(s^t) \geq \varepsilon$ is introduced, and hence it is no longer possible for the allocation to stay the same until the shock stays the same, as before.

preferences, endowments and shocks) however these reduce to four, since $q_1(1;1) = q_1(2;2)$, $q_1(1;2) = q_1(2;1)$, as well as $q(1;1) = q(2;2)$ and $q(1;2) = q(2;1)$ for all $t = 2, \ldots.$

Using the above expressions of the budget constraints, the equilibrium prices can be obtained from the first order conditions for the consumers' optimal choices. After some substitutions, we obtain[14] the following equation that can be solved for $q(1;2) = q(2;1)$:

$$q(2;1)u'(\mathfrak{h} + \varepsilon\mathfrak{d} - q(2;1)(1 - 2\varepsilon)) - \frac{\beta(q(2;1) + \mathfrak{d})}{2 - \beta}u'(\mathfrak{d}(1 - \varepsilon) + q(2;1)(1 - 2\varepsilon)) = 0. \quad (30)$$

It is useful to denote by $q^0(2;1)$ the solution of this equation when $\varepsilon = 0$ (that is, at the initial steady state).

Differentiating (30) with respect to ε, and evaluating it at $\varepsilon = 0$ yields the following expression:

$$\left.\frac{dq(2;1)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{-\left[\beta\frac{\mathfrak{d}+q^0(2;1)}{2-\beta}u''(\mathfrak{d}) + q^0(2;1)u''(\mathfrak{h})\right](\mathfrak{d} + 2q^0(2;1))}{u'(\mathfrak{h}) - \frac{\beta}{2-\beta}u'(\mathfrak{d}) - \beta\frac{\mathfrak{d}+q^0(2;1)}{2-\beta}u''(\mathfrak{d}) - q^0(2;1)u''(\mathfrak{h})} \quad (31)$$

where $u'(\mathfrak{h}) = u'(\mathfrak{h} - q^0(2;1))$ and $u'(\mathfrak{d}) = u'(\mathfrak{d} + q^0(2;1))$ with $u''(\mathfrak{h})$ and $u''(\mathfrak{d})$ defined analogously. In the above expression the numerator is clearly positive, and so is the denominator, since equation (30) evaluated at $\varepsilon = 0$ yields $u'(\mathfrak{h}) = \frac{\mathfrak{d}+q^0(2;1)}{q^0(2;1)}\frac{\beta}{2-\beta}u'(\mathfrak{d}) > \frac{\beta}{2-\beta}u'(\mathfrak{d})$. From the above expressions of the budget constraints and the symmetry of equilibrium prices we find that the effect on equilibrium consumption in the new steady state is

$$\left.\frac{dc^1(s_t = 1)}{d\varepsilon}\right|_{\varepsilon=0} = -\left.\frac{dc^1(s_t = 2)}{d\varepsilon}\right|_{\varepsilon=0} = 2q^0(2;1) + \mathfrak{d} - \left.\frac{dq(2;1)}{d\varepsilon}\right|_{\varepsilon=0}. \quad (32)$$

From (31) we immediately see that

$$0 < \left.\frac{dq(2;1)}{d\varepsilon}\right|_{\varepsilon=0} < \mathfrak{d} + 2q^0(2;1),$$

so that $\left.\frac{dc^1(s_t=1)}{d\varepsilon}\right|_{\varepsilon=0} > 0$. Hence the new steady state equilibrium price of the tree options unambiguously increases, as a result of the intervention, since their effective supply (the amount which can be traded in the market) decreases, from 1 to $1 - 2\varepsilon$. The variability in consumption across states increases too.

We can similarly proceed to determine the effect on consumption at the transition date $t = 1$:

$$\left.\frac{dc^1(s_1 = 1)}{d\varepsilon}\right|_{\varepsilon=0} = -\left.\frac{dc^1(s_1 = 2)}{d\varepsilon}\right|_{\varepsilon=0} = q^0(2;1) - \frac{\beta(q^0(2;1) + \mathfrak{d})}{2 - \beta} - \left.\frac{dq_1(1,2)}{d\varepsilon}\right|_{\varepsilon=0},$$

where we used the fact that $q_1(11)$, evaluated at $\varepsilon = 0$, equals $q(1;1)$ and both terms are at the steady state value before the intervention, $\frac{\beta(q^0(2;1)+\mathfrak{d})}{2-\beta}$.

---

[14]The details for this as well as the similar derivation of (35) below are in the Appendix.

The effect on the discounted expected utility of consumer 1 of an infinitesimal tightening of the portfolio restriction, that is from $\varepsilon = 0$ to $d\varepsilon > 0$ is then

$$\left.\frac{dU}{d\varepsilon}\right|_{\varepsilon=0} = \frac{1}{2}\left(u'(\mathfrak{h}) - u'(\mathfrak{d})\right)\left.\frac{dc^1(s_1 = 1)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{\beta}{2(1-\beta)}\left(u'(\mathfrak{h}) - u'(\mathfrak{d})\right)\left.\frac{dc^1(s_t = 1)}{d\varepsilon}\right|_{\varepsilon=0}.$$
(33)

By symmetry, the expression for the change in consumer 2's expected utility has the same value. Hence the welfare effect of the intervention considered is determined by the sign of the expression in (33).

Since $u'(\mathfrak{h}) < u'(\mathfrak{d})$, our finding on the sign of (32) implies that the effect of the intervention considered on agents' steady state welfare, given by the second term in (33), is always negative. For the intervention to be welfare improving we need then to have a welfare improvement in the initial period that is sufficiently large to compensate for the negative effect after that period. More precisely, from (33) it follows that $\left.\frac{dU}{d\varepsilon}\right|_{\varepsilon=0} > 0$ if, and only if,

$$\left.\frac{dc^1(s_1 = 1)}{d\varepsilon}\right|_{\varepsilon=0} < -\frac{\beta}{1-\beta}\left.\frac{dc^1(s_t = 1)}{d\varepsilon}\right|_{\varepsilon=0},$$

or equivalently, substituting the expressions obtained above for the consumption changes and rearranging terms,

$$\left.\frac{dq_1(1;2)}{d\varepsilon}\right|_{\varepsilon=0} > \frac{2q^0(2;1) + \mathfrak{d}\beta}{(2-\beta)(1-\beta)} - \frac{\beta}{1-\beta}\left.\frac{dq(1;2)}{d\varepsilon}\right|_{\varepsilon=0}$$
(34)

That is, for an improvement to obtain the price change in the first period, $\left.\frac{dq_1(1;2)}{d\varepsilon}\right|_{\varepsilon=0}$, has to be sufficiently large so that $c^1(s_1 = 1)$ decreases, increasing risk sharing in this intermediate period, and by a sufficiently large amount. Again by differentiating the consumers' first order conditions with respect to ε we obtain the following expression for the price effect at the intermediate date:

$$\left.\frac{dq_1(1;2)}{d\varepsilon}\right|_{\varepsilon=0} =$$
$$\frac{q^0(2;1)\left(\frac{\beta(q^0(2;1)+\mathfrak{d})}{2-\beta} - q^0(2;1)\right)u''(\mathfrak{h}) - \frac{\beta(q^0(2;1)+\mathfrak{d})}{2-\beta}u''(\mathfrak{d})\left(\mathfrak{d}+2q^0(2;1) - \left.\frac{dq(2;1)}{d\varepsilon}\right|_{\varepsilon=0}\right) + \frac{\beta}{2-\beta}u'(\mathfrak{d})\left.\frac{dq(2;1)}{d\varepsilon}\right|_{\varepsilon=0}}{u'(\mathfrak{h}) - q^0(2;1)u''(\mathfrak{h})}$$
(35)

Substituting this expression into the sufficient condition for suboptimality we obtained above, (34), we find that this, after rearranging terms, is equivalent to the following:

$$q^0(2;1)(1-\beta)\left[\beta\mathfrak{d} - 2(1-\beta)q^0(2;1)\right]u''(\mathfrak{h}) - (1-\beta)\beta(q^0(2;1) + \mathfrak{d})u''(\mathfrak{d})\left(\mathfrak{d}+2q^0(2;1)\right)$$
$$- (2q^0(2;1) + \mathfrak{d}\beta)\left(u'(\mathfrak{h}) - q^0(2;1)u''(\mathfrak{h})\right)$$
$$+ \left[\beta(2-\beta)\left(u'(\mathfrak{h}) - q^0(2;1)u''(\mathfrak{h})\right) + \beta(1-\beta)u'(\mathfrak{d}) + (1-\beta)\beta(q^0(2;1) + \mathfrak{d})u''(\mathfrak{d})\right]\left.\frac{dq(2;1)}{d\varepsilon}\right|_{\varepsilon=0} > 0$$
(36)

This condition is stated in terms of endogenous variables which obviously raises the question if there are economies for which the equilibrium values satisfy it. We establish then the following result.

THEOREM 7 *There are specifications of economies in the environment under consideration that are robust with respect to perturbations in $(\mathfrak{h}, \mathfrak{d}, \beta)$ as well as perturbations of preferences for which Condition (36) holds and hence the competitive equilibrium is constrained suboptimal.*

To prove the theorem, we show (in the Appendix) that for sufficiently small β Condition (36) is satisfied if

$$1 + \mathfrak{d}\frac{u''(\mathfrak{d})}{u'(\mathfrak{d})} + \frac{u'(\mathfrak{h})}{u'(\mathfrak{d})} < 0. \tag{37}$$

As shown in Section 3, when $\mathfrak{h}(1 - \beta) > \mathfrak{d}$ an inefficient steady state equilibrium exists with $\frac{u'(\mathfrak{h})}{u'(\mathfrak{d})} < 1$. It then follows that the inequality $-\mathfrak{d}\frac{u''(\mathfrak{d})}{u'(\mathfrak{d})} > 1 + \frac{u'(\mathfrak{h})}{u'(\mathfrak{d})}$ is satisfied when the absolute risk-aversion is sufficiently high. Therefore Condition (36) holds and the steady state equilibrium is constrained inefficient whenever the agents' absolute risk aversion is uniformly above $2/\mathfrak{d}$ and β is sufficiently small. It is clear that this is true for an open set of parameters and utility functions.

5.2.1 Logarithmic preferences

While Theorem 7 above is all one can say in general, it is useful to illustrate for a given specification of the agents' utility function how large the set of parameter values is for which one obtains constrained inefficient equilibria. We consider here the case where $u(c) = \log(c)$. It can be verified that in this case an explicit solution of (30) for the equilibrium price can be found, given by [15]

$$q^0(2; 1) = \beta\frac{\mathfrak{h}}{2}.$$

Since utility is homothetic it is without loss of generality to normalize $\mathfrak{d} = 1$. Direct computations then show that

$$\left.\frac{dq_1(1; 2)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\beta(1 + \mathfrak{h})(1 + \beta\mathfrak{h})}{2 + \beta\mathfrak{h}}$$

and

$$\left.\frac{dq(2; 1)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\beta(-4\mathfrak{h} + \beta^2\mathfrak{h}(2 + 3\mathfrak{h}) + 2\beta(1 + \mathfrak{h} - 2\mathfrak{h}^2)}{2(\beta - 2)(2 + \beta\mathfrak{h})}.$$

According to Equation (34) an improvement is possible if

$$\beta \left.\frac{dq_t(1, 2)}{d\varepsilon}\right|_{\varepsilon=0} + (1 - \beta) \left.\frac{dq_1(1, 2)}{d\varepsilon}\right|_{\varepsilon=0} - \frac{2q(1, 2) + \mathfrak{d}\beta}{(2 - \beta)} > 0$$

Substituting these expressions into (34) we find that, in the case of logarithmic preferences the intervention considered is welfare improving if, and only if

$$2 - \beta(\mathfrak{h} - 2)\mathfrak{h} + \beta^2\mathfrak{h}^2 < 0.$$

---

[15]While it may seem surprising that the dividend level $\mathfrak{d}$ does not appear in this expression of the equilibrium price, we should bear in mind that the one considered is an inefficient equilibrium. When an efficient steady state exists we have in fact $q(1; 2) = \frac{\beta}{2}\frac{\mathfrak{d}}{(1-\beta)}$.

Figure 1 then shows, in the space $\mathfrak{h}, \beta$, the region of values of these parameters for which competitive equilibria are constrained inefficient as well as the region where equilibria are Pareto efficient. We see that the region where constrained inefficiency holds is quite large, while the region where full Pareto efficiency cannot be attained but still the intervention considered is not welfare improving is very small.

[FIGURE 1 ABOUT HERE]

6 Quantitative Assessment of the Equilibrium Properties

The findings of the previous section have important implications for the properties of equilibrium allocations as well as potentially important policy-implications. In Condition (28) we identified the minimum amount of collateral that needs to be present, in economies without aggregate uncertainty and without persistence in idiosyncratic shocks, for competitive equilibria to be fully Pareto-efficient. When available collateral is scarce and violates such condition, imposing tighter borrowing constraints than the ones imposed by the markets may lead to welfare improvements. In this section we carry out a quantitative assessment of the efficiency properties of competitive equilibria for more general and realistic economies where shocks are persistent, there may be aggregate as well as idiosyncratic uncertainty and levels of collateral are somewhat realistic.

We consider a set of economies with two types of agents, where there are both aggregate shocks and persistent idiosyncratic shocks and we allow for different possible levels of the idiosyncratic shocks and different possible degrees of their persistence. For such economies equilibria can be easily computed numerically by solving a non-linear system of equations. We provide a quantitative assessment of the values of parameters for which competitive equilibria are Pareto efficient and, when the equilibrium is inefficient, we determine the size of the welfare loss (with respect to the fist best level of welfare). We also illustrate the properties of the pattern of agents' consumption over time and across states when equilibria are inefficient.

More precisely, we examine an environment where there are four shocks, $\mathcal{S} = \{1, .., 4\}$. Aggregate endowments are $\omega(1) = \omega(2) = (1 + \zeta)$, $\omega(3) = \omega(4) = (1 - \zeta)$. The tree pays dividends equal to a fraction δ of aggregate endowments: $d(s) = \delta\omega(s)$ for each $s \in \mathcal{S}$ and individual non pledgeable endowments are

$$e^1(1) = e^1(3) = \eta(1 - \delta)\omega(s), \quad e^1(2) = e^1(4) = (1 - \eta)(1 - \delta)\omega(s)$$

The aggregate shock is i.i.d. and each of its two realizations has the same probability. In contrast, the idiosyncratic shock is persistent, with persistence measured by $\gamma = 2\pi(1, 1)$. Agents have identical CRRA utility with $\beta = 0.95$ and a coefficient of relative risk aversion of 3.

Regarding the size of the fraction of the economy's resources that are pleadgeable, given by δ, it is sometimes argued (see e.g. McGrattan and Prescott (2005)) that the share

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of output which can be attributed to tangible capital in the US economy is close to 30 percent. This is capital that in principle could be used as collateral and this number would imply for our model a value $\delta = 0.3$. In stark contrast, Chien and Lustig (2011) use NIPA (National Income and Product Accounts) to estimate tradeable or collateralizable income to be 10.2 percent of total income. In this, they include rental income, dividends, and interest payments. As they point out, this is a narrow measure, because it treats proprietary income as non collateralizable. When proprietary income is included as well, the ratio rises to 19.5 percent. In what follows we will consider these three (10, 20 and 30 percent) possible values for the share of income that is collateralizable, that is $\delta \in \{0.1, 0.2, 0.3\}$.

As far as the degree of persistence of the idiosyncratic shocks is concerned, we consider the following possible values for $\gamma \in \{0.9, 0.95, 0.99\}$. For the magnitude of these shocks, as measured by the fraction of aggregate endowment obtained in the good individual state, we consider the values $\eta \in \{0.75, 0.85, 0.95\}$.

For the magnitude of the aggregate shocks we consider the case $\zeta = 0.1$, i.e. of substantial aggregate uncertainty. It turns out that the features of the equilibrium values reported in what follows are almost identical when there is no aggregate uncertainty, i.e. when $\zeta = 0$. The presence of aggregate shocks turns out then not to play a quantitatively important role..

We report in the following table the welfare losses at a competitive equilibrium with collateral constraints with respect to the Arrow-Debreu benchmark (i.e. relative to the equilibria without collateral constraints). Welfare losses are measured in percent, in wealth equivalent terms, i.e. we report what fraction of his consumption level at an Arrow-Debreu equilibrium an agent should give up (uniformly across all nodes) in order to attain the same level of welfare as at the competitive equilibrium with collateral constraints. The initial conditions are such that at the beginning of period 0 each agent holds half of the tree and initially, coming into period 0, all 4 shocks are equally likely.[16]

δ	γ	$\eta = 0.75$	$\eta = 0.85$	$\eta = 0.95$
0.1	0.9	0	0.	0.01
0.1	0.95	0.19	0.48	0.70
0.1	0.99	4.47	7.13	8.53
0.2	0.9	0	0	0
0.2	0.95	0	0	0.07
0.2	0.99	0.57	2.54	4.25
0.3	0.95	0	0	0
0.3	0.99	0	0.10	1.16

Table 1: welfare losses at an equilibrium

The findings in the table above provide a useful complement to our theoretical analysis in the previous sections. For the value of β considered here, Condition (29) implies that,

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[16]This is slightly different from the specification adopted in the previous sections, where time runs from date 0 after s_0 has realized, and allows results here not to depend on the initial condition.

without aggregate uncertainty and with no persistence of the idiosyncratic shocks, an efficient competitive equilibrium would exist for all the values of the collateral level δ and of the size of the idiosyncratic shocks η considered in the table. We see from the results in the table that the persistence of the idiosyncratic shocks plays a crucial role for the risk sharing properties of equilibrium allocations in this economy. When idiosyncratic shocks are very persistent (and the level of existing collateral is relatively low) potential welfare losses are quite large. This is in accord with what one finds in a model with incomplete markets and exogenous borrowing constraints, where equilibria are always inefficient but welfare losses are quantitatively very small unless shocks are very persistent (see e.g. Kubler and Schmedders (2001)). In the environment considered here, welfare losses are zero when the level of collateral is sufficiently high ($\delta = 0.3$) or when idiosyncratic shocks are not very persistent ($\gamma = 0.9$).

It is then useful to examine also the degree of risk sharing at a competitive equilibrium when Pareto efficiency is not attained. Since we are considering the case where there are two types of agents with identical CRRA utility, Theorem 4 applies and the equilibrium can be described by S pairs of numbers $(\underline{\lambda}_s, \overline{\lambda}_s)_{s \in \mathcal{S}}$. For instance, take the case where $\eta = 0.75$, $\delta = 0.1$ and $\gamma = 0.99$. As we see from Table 1, the resulting equilibrium is inefficient and the pattern of consumption in the long run (at a steady state) is described by the following values:

State	1	2	3	4
$\underline{\lambda}_s$	0.2250	0.5743	0.2250	0.5889
$\overline{\lambda}_s$	0.4260	0.7752	0.4114	0.7752

Table 2: risk sharing pattern

Recall that shocks 1, 3 and 2, 4 represent different realizations of the idiosyncratic shock, while 1, 2 versus 3, 4 represent different aggregate shocks. We see then from the values in Table 2 that in the long run consumption will change, and significantly, whenever there is a change in the realization of the idiosyncratic shock, while there will be no change, or at most a much smaller change, in consumption when only the realization of the aggregate shock changes.

It is obviously beyond the scope of this paper to take a stand on which values should be considered as 'realistic' for the level of persistence and the size of the idiosyncratic shocks as well as for the amount of available collateral. It might be interesting, however, to consider an example of a calibrated economy from the applied literature. Heaton and Lucas (1996) calibrate a Lucas style economy with two types of agents to match key facts in the US economy. They take the 'dividend-share' to be earnings to stock-market capital and estimate this number to be around 15 percent of total income. They assume that aggregate growth rates follow an 8-state Markov chain and calibrate their model using the PSID (Panel Study of Income Dynamics) and NIPA (National Income and Product Accounts). Let us consider their calibration for the 'Cyclical Distribution Case' but detrend the economy to

ensure we remain in our stationary environment. We find that for their specification of the economy the competitive equilibrium is Pareto efficient in the long run. In fact, the persistence of the shocks is so small that even with only a 5 (instead of 15) percent level of collateralizable income, efficiency would still obtain. This shows that, if one considers the specification of idiosyncratic risks in Heaton and Lucas (1996) to be somewhat realistic, Pareto inefficiency does not obtain in the long run for all realistic levels of collateral.

7   Conclusion

In this paper we have considered an infinite horizon economy with complete markets and collateral constraints where the only financial friction is the limit to borrowing imposed by the existing amount of collateral. We have shown that this is a tractable dynamic stochastic model, whose equilibria can be computed fairly easily and hence the efficiency and risk sharing properties of equilibria quantitatively assessed. This is true even though they can be constrained suboptimal, in the sense that imposing tighter borrowing constraints at certain nodes of the event tree than the limits imposed by the collateral constraints can make everybody better off.

8   Appendix: Proofs

8.1   Proof of Theorem 1

We first show that each Arrow Debreu equilibrium allocation with limited pledgeability is also an equilibrium allocation in an equilibrium with intermediaries. Given the equilibrium Arrow Debreu prices $(\rho(\sigma))_{\sigma \in \Sigma}$, set the prices of the tree equal to $q(s^t) = \frac{1}{\rho(s^t)} \sum_{\sigma \succ s^t} \rho(\sigma) d(\sigma)$ and the prices of the tree-options as

$$q(s^t; s_{t+1}) = \frac{1}{\rho(s^t)} \rho(s^{t+1}) \left(q(s^{t+1}) + d(s^{t+1}) \right) \tag{38}$$

for every s^t, s_{t+1}. It is then easy to see that the set of budget feasible consumption levels are the same for the budget set in (IE2) and for the budget set defined by (4) and (5). Given a consumption sequence $(c(\sigma))_{\sigma \in \Sigma}$ that satisfies (IE2), using (38) we get

$$\rho(s^t)\theta(s^{t-1}; s_t)(q(s^t)+d(s^t)) = \rho(s^t)(c(s^t)-e^h(s^t))+\rho(s^t) \sum_{s_{t+1}\in\mathcal{S}} \theta(s^t; s_{t+1})\frac{\rho(s^{t+1})}{\rho(s^t)}(q(s^{t+1})+d(s_{t+1}))$$

for each s^t with $t \geq 1$. Substituting then recursively for the second term on the right hand side we obtain

$$\rho(s^t)\theta(s^{t-1}; s_t)(q(s^t) + d(s^t)) = \sum_{\sigma \succeq s^t} \rho(\sigma)(c(\sigma) - e^h(\sigma)) \geq 0,$$

that is (5) holds. At the root node s_0 we have

$$\theta^h(s_-)(q(s_0) + d(s_0)) = \sum_{\sigma \succeq s^t} \rho(\sigma)(c(\sigma) - e^h(\sigma))$$

equivalent to (4). The reverse implication can be similarly shown.

We show next that when the set of assets $\mathcal{J}$ is large and includes all possible kinds of securities, subject to all possible kinds of collateral constraints specified in Section 2, Arrow Debreu equilibria with limited pledgeability can be decentralized as collateral constrained financial market equilibria. For this it suffices to show that this is possible for some collection of assets $\bar{\mathcal{J}} \subset \mathcal{J}$.

One possible specification (although certainly not the only one) of the set of assets that allows to establish the decentralization result is as follows. For each security $j \in \bar{\mathcal{J}}$ we have $b_j(s) \in \{0,1\}$ for each $s \in \mathcal{S}$. The set of securities is then partitioned into the subsets $\mathcal{J}^{S-1}, \mathcal{J}^{S-2}...\mathcal{J}^1$ and we assume that for each $\bar{s} \in \{1, ..., S-1\}$ all securities in $\mathcal{J}^{\bar{s}}$ promise the payment of one unit for each realization of the shock $s = 1, ..., \bar{s}$. Within $\mathcal{J}^{\bar{s}}$ the securities only distinguish themselves by their collateral requirements. The set of the promised payoffs of the securities in $\bar{\mathcal{J}}$ plus the tree (which has a strictly positive payoff in all states) has then a triangular structure.

We specify next the set of assets which can be used as collateral for these securities. This set has to be sufficiently rich to allow us to establish the completeness of the market. More specifically, there is only one security in $\mathcal{J}^{S-1}$ and only the tree can be used as collateral for short positions in this security. There are then two securities in $\mathcal{J}^{S-2}$, one collateralized by the tree, the second one by long positions in the security in $\mathcal{J}^{S-1}$ (in turn collateralized by the tree). We have four securities in $\mathcal{J}^{S-3}$, one collateralized by the tree, the second one by the security in $\mathcal{J}^{S-1}$, the third one by the first security in $\mathcal{J}^{S-2}$ and the fourth one by the second security in $\mathcal{J}^{S-2}$. Note that in this specification a security can only serve as collateral for another security if the set of states where the first one promises a nonzero payment contains the set of states where the second one promises a payment. This and the triangular structure of the payoffs generate a natural seniority structure of the securities, as their promised payoff determines their ability to serve as collateral for other securities.

More formally, a security is identified by a pair $(C, \bar{s}) \in \{0,1\}^{S-1} \times \mathcal{S}$, specifying that the security promises the payment of one unit in the shock realizations $s = 1, ..., \bar{s}$ and is collateralized - either directly or indirectly - by the securities identified by an element of the set $\{0,1\}^{S-1}$, where the first element refers to the tree - which can be identified, with some abuse of notation, with $\mathcal{J}^S$ - and the other elements to the securities in, respectively, $\mathcal{J}^{S-1}, \mathcal{J}^{S-2}...\mathcal{J}^2$. The convention is that $C_s = 0$ if no security in $\mathcal{J}^s$ is used either directly or indirectly as collateral of the security under consideration. So $C = (1, 0, ..., 0)$ means that only the tree is used as collateral, $C = (1, 0, 1, 0, ...)$ implies that the collateral is given by the security in $\mathcal{J}^{S-2}$ which in turn is collateralized by the tree. At the other extreme we have the case $C = (1, ..., 1) \in \{0,1\}^{S-1}$ indicating that the security is collateralized by a security

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in $\mathcal{J}^2$ which is collateralized by a security in $\mathcal{J}^3$ and so on. Given the seniority structure described above, for any security $(C, \bar{s})$, C must be such that its elements $\bar{s}, \bar{s} - 1, .., 1$ are all zero.

The two securities $(C, \bar{s})$ and $(C', \bar{s})$ are then identical in terms of promised payoffs but differs for their collateral requirements. The collateral requirements induce an additional, lexicographic ordering among the securities with the same promised payoff: i.e. $C \prec_l C'$ if $C_S > C'_S$ or if $C_{S-i} = C'_{S-i}$ for all $i = 1, ..., n - 1$ and $C_{S-n} > C'_{S-n}$ (that is, the securities serving, either directly or indirectly, as collateral are more senior, in a lexicographic sense, in C than in C'). It is also convenient to denote by $j = \kappa(C, \bar{s})$ the security that needs to be used directly as collateral for $(C, \bar{s})$. We assume that, as already implicit in the above construction, exactly one security must be used as collateral, this security must belong to one of the sets $\mathcal{J}^{\bar{s}+1}, \dots, \mathcal{J}^S$. Conversely, for each $(C, \bar{s})$ and each $s < \bar{s}$, there is exactly one asset in $\mathcal{J}^s$ that uses $(C, \bar{s})$ directly as collateral, and we denote this by $\kappa^{-1}((C, \bar{s}), s)$.

We complete the description of the collateral requirements of the various securities by specifying the level of (direct) collateral requirement for each security $(C, \bar{s})$ in any given state s^t. If a security is directly collateralized by the tree, i.e. if $\kappa(C, \bar{s}) \in \mathcal{J}^S$, then

$$k^{C,\bar{s}}_{\kappa(C,\bar{s})}(s^t) = \bar{k}(s^t) \equiv \min_{s^{t+1} \succ s^t} \frac{1}{q(s^{t+1}) + d(s_{t+1})}$$

The direct collateral requirement in terms of all other financial assets is simply one. In other words

$$k^{C,\bar{s}}_{\kappa(C,\bar{s})}(s^t) = 1 \text{ if } \kappa(C, \bar{s}) \notin \mathcal{J}^S.$$

Note that this specification implies that in any given state the actual payoff of a security is either zero or an amount proportional to the value of the tree plus its dividends,

$$f_{(C,\bar{s})}(s^t) \in \{0, \bar{k}(s^{t-1}) \left(q(s^t) + d(s_t) \right)\}. \tag{39}$$

More precisely, it is zero for all $s > \bar{s}$ and nonzero (proportional to the payoff of the tree) in all other states, and is then independent of C.

Given the equivalence result shown in the first part of the proof, it suffices to show that any equilibrium allocation with intermediaries, with prices $\bar{q}(s^t, s)$ and portfolios $\bar{\theta}^h(s^t, s)$ of the tree options, for all h, s, s^t is also an equilibrium allocation with collateral constraints and asset structure $\bar{\mathcal{J}}$. We show in what follows that we can construct prices for the tree $q(s^t)$ and all the securities in $\bar{\mathcal{J}}$, $p(s^t)$, as well as portfolios $\theta^h(s^t, s), \phi^h(s^t, s)$ that support the same consumption allocation at a financial markets equilibrium with collateral constraints. Note first that, as shown in (39), the payoff of each security is proportional to the payoff of a tree option. Set then the price of the tree equal to $q(s^t) = \sum_{s \in \mathcal{S}} \bar{q}(s^t, s)$ and the security prices at

$$p_{(C,\bar{s})}(s^t) = \sum_{s=1}^{\bar{s}} \bar{k}(s^t) q(s^t, s).$$

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For each node s^t portfolio holdings are constructed as follows. Set the tree holdings for each agent h at the level $\theta^h(s^t) = \bar{\theta}^h(s^t, S)$. We denote the holdings of financial security $(C, \bar{s})$ by $\phi^h_{(C,\bar{s})}(s^t)$. For all agents $h \in \mathcal{H}$ let

$$\phi^h_{((1,0,...,0),S-1)}(s^t) = \frac{\bar{\theta}^h(s^t, S-1) - \theta^h(s^t)}{\bar{k}(s^t)}.$$

With this construction, the portfolio pays the same as the tree options in the shock realizations $s_{t+1} = S$ and $S-1$ next period. In order to guarantee the same payoffs also in the other shock realizations $s_{t+1} = 1, ..., S-2$ while satisfying at the same time the collateral requirements, define recursively for each $\bar{s} = S-2, S-3, \ldots, 1$, for each $h = 1, ..., H$

$$\gamma^h(\bar{s}) = \frac{\bar{\theta}^h(s^t, \bar{s}) - \theta^h(s^t)}{\bar{k}(s^t)} - \sum_C \sum_{i=\bar{s}+1}^{S-1} \phi^h_{(C,i)}(s^t),$$

with the convention that $\phi^h_{(C,i)}(s^t) = 0$ if the security (C, i) does not exist. With this definition $\gamma^h(\bar{s})$ denotes the total amount of securities promising a nonzero payoff in the shock realizations $s = 1, ..., \bar{s}$ which needs to be purchased to ensure that the payoffs of the portfolio in shock $\bar{s}$ replicates the payoff of the tree option contingent on $\bar{s}$.

To allocate this amount among the different securities that have the same promised payoff but different collateral requirements, we need to consider two cases. First, if $\gamma^h(\bar{s}) < 0$, that is the total position is a short one, one needs to ensure that the collateral requirements are satisfied. For each asset j that could be used as collateral, that is with higher seniority than $\bar{s}$, we define $R^j(s^t, \bar{s})$ the amount of that asset that is still 'available' in node s^t as possible collateral for j, given the collateral requirements of the holdings of assets promising to pay in shocks $s > \bar{s}$. That is, for the tree, let

$$R^S(s^t, \bar{s}) = \theta^h(s^t) + \bar{k}(s^t) \left(\sum_{i=\bar{s}+1}^{S-1} \min[0, \phi^h_{((1,0,...,0),i)}(s^t)] \right),$$

and set $\phi^h_{((1,0,...,0),\bar{s})}(s^t) = \max[\gamma^h(\bar{s}), -\frac{R^S(s^t,\bar{s})}{\bar{k}(s^t)}]$ for each $\bar{s}$. Note that this recursive specification of $\phi^h_{((1,0,...,0),i)}$ and $R^S(s^t, i)$ for all $i > \bar{s}$ ensures that $R^S(s^t, \bar{s}) \geq 0$. Similarly, for each security $(C, \hat{s})$, $\hat{s} = \bar{s}+1, .., S-1$, we have

$$R^{(C,\hat{s})}(s^t, \bar{s}) = \max\left\{ \phi^h_{(C,\hat{s})}(s^t), 0 \right\} + \sum_{i=\bar{s}+1}^{\hat{s}} \min[0, \phi^h_{\kappa^{-1}((C,\hat{s}),i)}(s^t)].$$

and we set, proceeding recursively now also for $C = (1,1,0,\ldots,0), (1,0,1,0,\ldots,0)...$

$$\phi^h_{(C,\bar{s})}(s^t) = \max[\gamma^h(\bar{s}) - \sum_{C':C \preceq_l C'} \phi^h_{(C',\bar{s})}(s^t), -R^{\kappa(C,\bar{s})}(s^t, \bar{s})].$$

The first term in the above expression is the amount of the total position $\gamma^h(\bar{s})$ that needs to be allocated to securities with collateral C or below according to the ordering $\prec_l$. The

second term indicates the (opposite of the) amount of collateral that is available for asset $(C, \bar{s})$. implied set so that the collateral. When $R^{\kappa(C,\bar{s})}(s^t, \bar{s})$ is small, $\phi^h_{(C,\bar{s})}(s^t)$ is set so that the collateral constraint holds with equality. It is clear that with this construction the collateral requirements will be satisfied and eventually $\sum_C \phi^h_{(C,\bar{s})}(s^t) = \gamma^h(\bar{s})$.

Secondly, we need to consider the case $\gamma^h(\bar{s}) > 0$. Although an agent is indifferent between a long position in security $(C, \bar{s})$ and a long position in another security $(C', \bar{s})$ the assignment cannot be arbitrary because we need to ensure market clearing, that is we need to ensure $\sum_h \phi^h_{(C,\bar{s})}(s^t) = 0$ for all $(C, \bar{s})$ in addition to $\sum_C \phi^h_{(C,\bar{s})}(s^t) = \gamma^h(\bar{s})$. But the validity of the market clearing condition in the tree options ensures that an assignment satisfying market clearing always exists.

We have thus verified that the consumption allocation at an equilibrium with intermediaries is budget feasible in the presence of collateral constraints with asset structure $\bar{\mathcal{J}}$. Moreover, the set of budget feasible consumption plans is the same and so will be the consumers' choice.

8.2 Proof of Theorem 3

We prove existence of a Markov equilibrium by showing that finite horizon truncations converge monotonically to policy and transition functions when the horizon becomes large. Throughout the proof we will crucially use the fact that our assumption on preferences guarantees the so-called gross-substitute property. The following definition and two lemmas make this precise.

DEFINITION 1 *A function $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies the strict gross substitute property if for all $y \in \mathbb{R}^n_+$ and all $x \in \mathbb{R}^n_+$ with $x_i = 0$ for some $i = 1, ..., n$ it holds $f_i(y) < f_i(y + x)$.*

The following lemma makes clear why this property is crucial for establishing existence. It is a slight variation of a result by Dana (1993).

LEMMA 3 *Suppose $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies the strict gross substitute property and is homogeneous of degree zero. Given any $x \in \mathbb{R}^n_{++}$ suppose there exist $\gamma, \gamma' \in \mathbb{R}^n_+$ such that $f(x+\gamma) \geq 0$, $f(x + \gamma') \geq 0$ and $\gamma_i f_i(x + \gamma) = 0$, $\gamma'_i f_i(x + \gamma') = 0$, for all $i = 1, ..., n$. If both γ and γ' are not strictly positive, i.e. $\gamma, \gamma' \notin \mathbb{R}^N_{++}$ then $\gamma = \gamma'$.*

Proof. Suppose to the contrary that $\gamma \neq \gamma'$. Without loss of generality we can take $\gamma > 0$. Then there must be a j with $\gamma_j > 0$ as well as a $\xi > 0$ such that $(x_j + \gamma_j) = \xi(x_j + \gamma'_j)$ and $\xi(x + \gamma') > x + \gamma$. The latter inequality holds strict because $\gamma \neq \gamma'$ and because both are not strictly positive. But since $f(.)$ is homogeneous of degree zero we must have that $f_j(\xi(x + \gamma')) = f_j(x + \gamma') \geq 0$. On the other hand, by the strict gross substitute property and since $\gamma_j > 0$ we must have

$$f_j(\xi(x + \gamma')) < f_j(x + \gamma) = 0,$$

which is a contradiction. □

We also need the following result.

LEMMA 4 *Suppose $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfies the strict gross substitute property and is homogeneous of degree zero. For any $x \in \mathbb{R}^n_{++}$ and $y > x$ suppose there exist $\gamma^x, \gamma^y \in \mathbb{R}^n_+$ such that $f(x + \gamma^x) \geq 0$, $f(y + \gamma^y) \geq 0$ and $\gamma^x_i f_i(x + \gamma^x) = 0$ and $\gamma^y_i f_i(y + \gamma^y) = 0$ for all $i = 1, ..., n$. If $x_j = y_j$ for some $j = 1, .., n$ then it must hold that*

$$f_j(x + \gamma^x) \geq f_j(y + \gamma^y).$$

Proof. If $\gamma^y_j > 0$ or if $\gamma^x = 0$ then the results holds by construction.

If $\gamma^y_j = 0$ and $\gamma^x > 0$ then we must have $x + \gamma^x \leq y + \gamma^y$. If this were not the case, there must exist an i with $\gamma^x_i > 0$ and a $\xi > 0$ such that $(x_i + \gamma^x_i) = \xi(y_i + \gamma^y_i)$ and $\xi(y + \gamma^y) > (x + \gamma^x)$.

As in the previous proof this is a contradiction since $f_i(\xi(y + \gamma^y)) = f_i(y + \gamma^y) \geq 0$ while $f_i(\xi(y + \gamma^y)) < f_i(x + \gamma^x) = 0$. □

Proof of the Theorem. Define the functions

$$V^h_0(s, \lambda) = u'_h(s, \lambda)(C^h(s, \lambda) - e^h(s)).$$

For a given V^H_n, $h \in \mathcal{H}$ define $\Lambda_n(s) = \{\lambda \in \mathbb{R}^H_{++} : V^h_n(s, \lambda) \geq 0 \text{ for all } h \in \mathcal{H}\}$ and $L_n : \mathcal{S} \times \mathbb{R}^H_{++} \to \mathbb{R}^H_{++}$ by

$$L_n(s, \lambda) = \begin{cases} \lambda & \text{if } \lambda \in \Lambda_n(s) \\ \lambda + \bar{\gamma} & \text{otherwise,} \end{cases}$$

where

$$\bar{\gamma} \in \{\gamma \geq 0 : \lambda + \gamma \in \Lambda_n(s), \quad \gamma_h V^h_n(\lambda + \gamma, s) = 0, \ \forall h\}. \tag{40}$$

Note that L_n is only well defined whenever there exists a unique $\bar{\gamma}$ that satisfies (40). By Lemma 3 and the fact that $\bar{\gamma}$ cannot be strictly positive, there exists at most one solution whenever V_n satisfies the strict gross substitute property.

If this is the case, we can define recursively

$$V^h_n(s, \lambda) = u'_h(s, \lambda)(C^1(s, \lambda) - e^1(s)) + \beta \sum_{s'} \pi(s, s') V^h_{n-1}(s', L_{n-1}(s', \lambda)). \tag{41}$$

It is easy to see that $V^0(s, .)$ satisfies the gross substitute property for all s, i.e. that $V^h_0(s, \lambda + \gamma) < V^h_0(s, \lambda)$ for all $\gamma > 0$ with $\gamma_h = 0$. It follows from Lemma 4 that each V_n then satisfies the strict gross substitute property.

It is a standard argument to show that equilibrium exist for each finitely truncated economy and that therefore $L_n(s, \lambda)$ is always well defined.

The crucial step of the proof consist of showing, by induction, that $V_n \geq V_{n-1}$ for all n. Clearly $V_1^h(s,.) \geq V_0^h(s,.)$ for all h and all s. To prove that $V_{n+1}(s,.) \geq V_n(s,.)$ whenever $V_n(s,.) \geq V_{n-1}(s,.)$ for all s, it suffices to show that

$$V_n(s, L_n(s, \lambda)) \geq V_{n-1}(s, L_{n-1}(s, \lambda)) \text{ for all } s \text{ and all } \lambda \in \mathbb{R}_{++}^H. \tag{42}$$

We can focus on the case where $L_n(s, \lambda) \neq L_{n-1}(s, \lambda)$ and we can write $L_n(s, \lambda) = \lambda + \bar{\gamma}$ for some $\bar{\gamma} \geq 0$ with $\bar{\gamma}_i = 0$ for some i. Note that if $V_n(s,.) \geq V_{n-1}(s,.)$ then $\Lambda_{n-1}(s) \subset \Lambda_n(s)$ and therefore $L_{n-1}(s, \lambda + \bar{\gamma}) > \lambda + \bar{\gamma}$. By the same argument as in the proof of Lemma 4 we must have that $L_{n-1}(s, \lambda + \bar{\gamma}) = L_{n-1}(s, \lambda)$: If $x = L_{n-1}(s, \lambda + \bar{\gamma}) \neq L_{n-1}(s, \lambda) = y$ there must be an i and a ξ with $\xi y_i = x_i$ and $\xi y > x$. It is easy to verify that $V_{n-1}^i(s, x) = 0$ and by the gross substitute property this implies that $V_{n-1}^i(s, \xi y) < 0$ which is a contradiction.

Finally observe that for each $h \in \mathcal{H}$ and each $s \in \mathcal{S}$ the function $V_n^h(s,.)$ are uniformly bounded above and therefore converge pointwise as $n \to \infty$ to some function $V_*^h(s,.)$. It is straightforward to verify that this function describes a collateral constrained Arrow-Debreu equilibrium. $\square$

8.3 Further details on the Proof of Theorem 6

Derivation of Equation (30). At $t = 1$ in state 1 the price $q_1(1; 2)$ of the tree option paying in state 2 is determined by agent 1's first order condition, since agent 2 is constrained in that state in his holdings of that asset. We have so

$$q_1(1; 2)u'(\mathfrak{h} - q_1(1; 1)\varepsilon - q_1(1; 2)(1 - \varepsilon)) = \frac{\beta}{2}(q(2; 1) + q(2; 2) + \mathfrak{d})u'(\mathfrak{d}(1 - \varepsilon) + q(2; 1)(1 - 2\varepsilon)). \tag{43}$$

For $t \geq 2$ agent 1's first order conditions with respect to the tree option paying in state 2 still determine its price in state 1 since agent 2 is constrained in that state. On the other hand, in state 2 the consumption of both agents is the same as in the subsequent date in state 2, hence both agents are not constrained in their holdings of the tree options paying in state 2 and its price is determined by the first order conditions of any of them (say again agent 1).

$$q(1; 2)u'(\mathfrak{h} + \varepsilon\mathfrak{d} - q(1; 2)(1 - 2\varepsilon)) = \frac{\beta}{2}(q(2; 1) + q(2; 2) + \mathfrak{d})u'(\mathfrak{d}(1 - \varepsilon) + q(2; 1)(1 - 2\varepsilon)) \tag{44}$$

$$q(2; 2)u'(\mathfrak{d}(1 - \varepsilon) + q(2; 1)(1 - 2\varepsilon)) = \frac{\beta}{2}(q(2; 1) + q(2; 2) + \mathfrak{d})u'(\mathfrak{d}(1 - \varepsilon) + q(2; 1)(1 - 2\varepsilon)) \tag{45}$$

From (45) we obtain for $t > 1$ that

$$q(1; 1) = q(2; 2) = \frac{\beta(q(2; 1) + \mathfrak{d})}{2 - \beta} \tag{46}$$

and therefore

$$q(2; 1) + q(2; 2) + \mathfrak{d} = \frac{2(q(2; 1) + \mathfrak{d})}{2 - \beta}.$$

Substituting this expression into equation (44) we obtain (30).

Derivation of Equation (35). The expression for the price change in (35) is obtained by differentiating (43) with respect to ε, evaluated at $\varepsilon = 0$, when $q_1(1;2)$, $q(2;1)$ and $q_1(1;1), q(2;2)$ are at their steady state values before the intervention, given respectively by $q^0(2;1)$ for the first two and by $\frac{\beta(q^0(2;1)+\mathfrak{d})}{2-\beta}$ for the last two. Noting that $\frac{dq_1(1;1)}{d\varepsilon}\varepsilon\Big|_{\varepsilon=0} = 0$, since the price $q_1(1;1)$ also changes with ϵ but the expression is evaluated at $\epsilon = 0$, we get (35).

Derivation of Condition (37). From equation (30) we find that $q^0(2,1)$ can be written in term of $u'(\mathfrak{d})$ and $u'(\mathfrak{h})$,

$$q^0(2;1) = \frac{\frac{\beta\mathfrak{d}}{2-\beta}u'(\mathfrak{d})}{u'(\mathfrak{h}) - \frac{\beta}{2-\beta}u'(\mathfrak{d})}. \tag{47}$$

Defining $\tilde{u}'(\mathfrak{h}) := \frac{u'(\mathfrak{h})}{u'(\mathfrak{d})}$, $\tilde{u}''(\mathfrak{h}) := \frac{u''(\mathfrak{h})}{u'(\mathfrak{d})}$, and $\tilde{u}''(\mathfrak{d}) := \frac{u''(\mathfrak{d})}{u'(\mathfrak{d})}$ we obtain that Condition (36) is equivalent to the condition $\frac{A}{B} > 0$ where

$$A = \begin{array}{c} 4\tilde{u}'(\mathfrak{h})\left[1 + \mathfrak{d}\tilde{u}''(\mathfrak{d}) + \tilde{u}'(\mathfrak{h})\right] - 2\beta\left[2 + (4 + 3\mathfrak{d}\tilde{u}''(\mathfrak{d}))\tilde{u}'(\mathfrak{h}) + 2\tilde{u}'(\mathfrak{h})^2 + \mathfrak{d}\tilde{u}''(\mathfrak{h})\right] + \\ \beta^2\left[\frac{1}{\tilde{u}'(\mathfrak{h})} + (3 + 2\mathfrak{d}\tilde{u}''(\mathfrak{d}))\tilde{u}'(\mathfrak{h}) + \tilde{u}'(\mathfrak{h})^2 + \mathfrak{d}\frac{\tilde{u}''(\mathfrak{h})}{\tilde{u}'(\mathfrak{h})^2} + (3 + \mathfrak{d}\tilde{u}''(\mathfrak{h}))\right] \end{array}$$

and

$$B = \begin{array}{c} \left[-2\tilde{u}'(\mathfrak{h})^2 + \beta\left(\tilde{u}'(\mathfrak{h}) + \tilde{u}'(\mathfrak{h})^2 + \mathfrak{d}\tilde{u}''(\mathfrak{h})\right)\right]\left[4\tilde{u}'(\mathfrak{h})^2 + \beta^2(1 + (2 + \mathfrak{d}\tilde{u}''(\mathfrak{d}))\tilde{u}'(\mathfrak{h}) + \tilde{u}'(\mathfrak{h})^2 + \mathfrak{d}\tilde{u}''(\mathfrak{h}))\right. \\ \left. -2\beta((2 + \mathfrak{d}\tilde{u}''(\mathfrak{d}))\tilde{u}'(\mathfrak{h}) + 2\tilde{u}'(\mathfrak{h})^2 + \mathfrak{d}\tilde{u}''(\mathfrak{h}))\right] \end{array}$$

It can then be easily seen that, since all marginal utilities are evaluated at positive numbers, that remain bounded away from zero as $\beta \to 0$, for sufficiently small β we have $\frac{A}{B} > 0$ if $1 + \mathfrak{d}\tilde{u}''(\mathfrak{d}) + \tilde{u}'(\mathfrak{h}) < 0$, or equivalently

$$1 + \mathfrak{d}\frac{u''(\mathfrak{d})}{u'(\mathfrak{d})} + \frac{u'(\mathfrak{h})}{u'(\mathfrak{d})} < 0.$$

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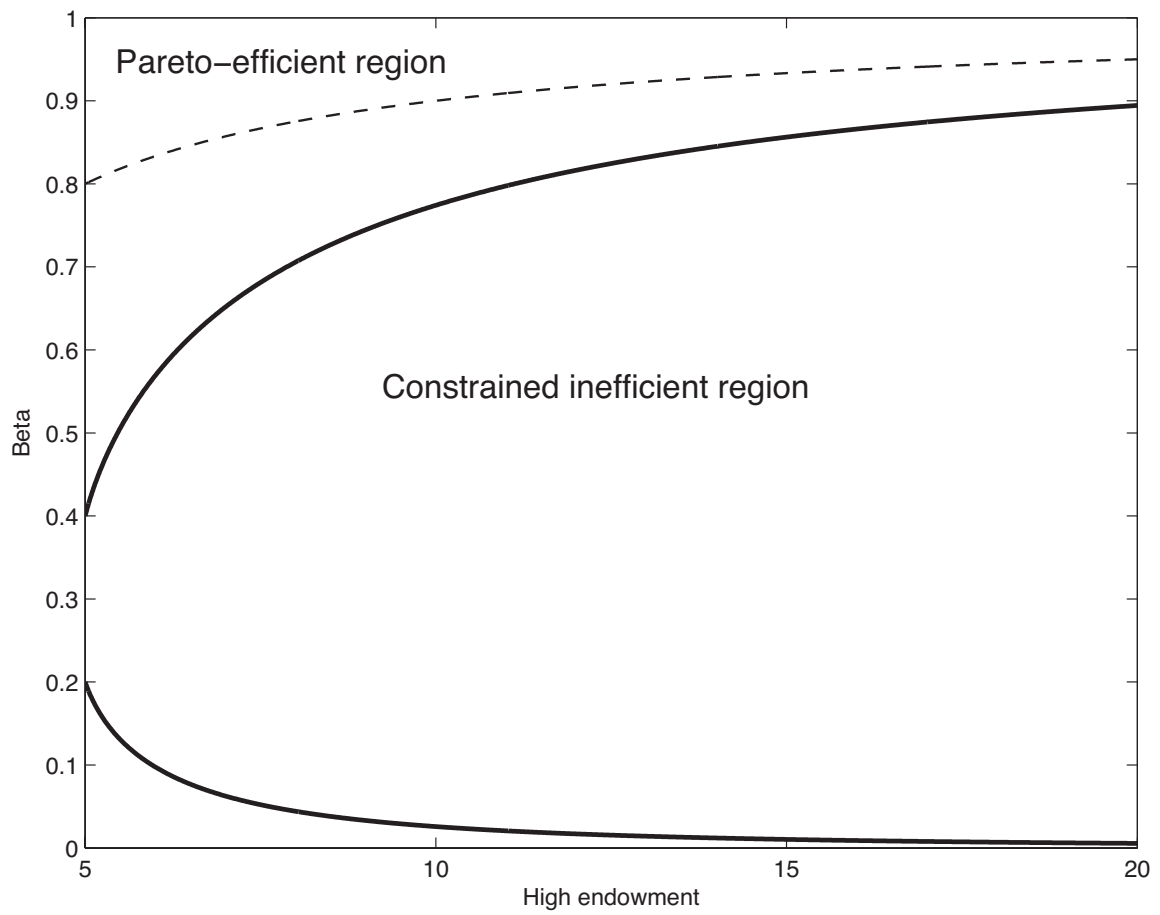[30] Zeidler, E. (1986), *Nonlinear Functional Analysis and it Applications I*, Springer Verlag, New York.

Figure 1: Constrained inefficient region

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