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THE SECOND WELFARE THEOREM IN NONCONVEX ECONOMIES

by
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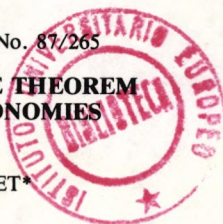
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ABSTRACT

The purpose of this paper is to report an extension of the second welfare
when both convexity and differentiable assumptions are violated.

1. INTRODUCTION

The purpose of this paper is to report an extension of the second welfare
theorem when both convexity and differentiable assumptions are violated. This
theorem, which has a well-known long history back to Allais [1], Hotelling [16],
Lange [19], Samuelson [25], has first asserted that the marginal rates of
substitution and the marginal rates of transformation must hold at a Pareto
optimal allocation. However, it was not until Arrow [2] and Debreu [9], that
this assertion was rigorously formulated and proved by means of convex analysis,
i.e., a separation theorem. In [15], in the absence of convexity and differen-
tiable assumptions, Guesnerie proved a theorem asserting that, at a Pareto
optimal allocation, it can be associated a nonzero price vector such that each
consumer satisfies the first order necessary conditions for expenditure minimi-
zation and each firm satisfies the first order necessary conditions for profit
maximization.

Our results, which extend the (finite dimensional version of the) results
of Kahn and Vohra [17], [18], Quinzii [23], Yun [28], are in the spirit of the
above assertion of Guesnerie [15], but differ from it in the way the "necessary"
conditions and the "marginal" rule are mathematically formalized. Guesnerie
uses the concept of normal cone of Dubovickii and Miljutin [12], whereas we
use, as in Cornet [8], Brown et al. [5] and the above papers, the normal cone
of Clarke [6], which allows to extend significantly the class of economies
which are considered. It allows also to use new mathematical techniques
developed during the past ten years and reported in the books [7], [26].

The next section presents the model we consider in which the preferences
of the consumers may be noncomplete and nontransitive, as in Gale and Mas-Colell
[14] (see also Fon and Otani [13]) in the convex case. The main results are
stated in Section 3; they assert that the second welfare theorem will hold
under a nonsatiation assumption of the preferences of the consumers and if
either all commodities can be freely disposed of (Theorems 3.1 and 3.3) or some
consumer has convex (or monotonic) preferences and all the production sets are
closed (Theorem 3.4). In Section 4, we give the proofs of these theorems, which

are directly deduced from a more general result (Lemma 4.1), also of interest for itself, but rather technical. The proof of it, hence of all the results of the paper, relies on a (nonsmooth) generalization of John-Kuhn-Tucker's theorem by Clarke [7, 6.1.1]. This paper will remain in the setting of a Euclidean space of commodities and we refer to [4], [18], for infinite dimensional formulations of the second welfare theorem in the spirit of Debreu [10], in the convex case.

2. THE MODEL

We consider an economy E with ℓ goods, m consumers and n firms. We let $X_i \subset \mathbb{R}^\ell$ be the consumption set of the i-th consumer and, for $x = (x_1, \ldots, x_m)$ in $X_1 x \ldots x X_m$, we let $P_i(x)$ be the set of elements in X_i which are preferred to x by the i-th consumer. In other words, the preferences of the i-th consumer are described by a correspondence P_i from $X_1 x \ldots x X_m$, to X_i. We let $Y_j \subset \mathbb{R}^\ell$ be the production set of the j-th firm and ω in $\mathbb{R}^\ell$ be the vector of total initial endowments.

DEFINITION 2.1 A $(m+n)$-tuple $((x_i^*),(y_j^*))$ of elements of $\mathbb{R}^\ell$ is said to be a __feasible allocation__ of E if (a) for all i, $x_i^* \in X_i$, (b) for all j, $y_j^* \in Y_j$ and (c) $\Sigma_i x_i^* - \Sigma_j y_j^* - \omega = 0$. A $(m+n)$-tuple $((x_i^*),(y_j^*))$ of elements of $\mathbb{R}^\ell$ is said to be a __Pareto optimum__ (resp. a __weak Pareto optimum__) if it is a feasible allocation of E and if there exists no other feasible allocation $((x_i),(y_j))$ of E such that, for all i, $x_i \in \text{cl } P_i(x_1^*, \ldots, x_m^*)$, the closure of $P_i(x_1^*, \ldots, x_m^*)$, and, for some i_0, $x_{i_0} \in P_{i_0}(x_1^*, \ldots, x_m^*)$ (resp., for all i, $x_i \in P_i(x_1^*, \ldots, x_m^*)$). In the following, we shall simply denote

In the following, we shall simply denote $P_i(x_1^*, \ldots, x_m^*)$ by P_i^*.

Our next definition of price quasi-equilibrium involves the concept of normal cone in the sense of Clarke ([6], [7]), that we now present. Firstly, for $x = (x_h)$, $y = (y_h)$ in $\mathbb{R}^\ell$, we let $x \cdot y = \Sigma_h x_h y_h$ be the scalar product of $\mathbb{R}^\ell$, $\|x\| = \sqrt{x \cdot x}$ be its Euclidean norm, $B(x,\varepsilon) = \{x' \in \mathbb{R}^\ell; \|x-x'\| < \varepsilon\}$ be the open ball of center x and radius $\varepsilon > 0$, we let $\mathbb{R}_+^\ell = \{x = (x_h); x_h \geq 0, h = 1, \ldots, \ell\}$ and the notation $x \geq 0$ means that $x \in \mathbb{R}_+^\ell$. If C is a

subset of $\mathbb{R}^\ell$, we let $\text{cl } C$ and $\text{int } C$ be, respectively, the closure and the interior of C and we say that a nonzero vector v is perpendicular to C at an element x in $\text{cl } C$, denoted $v \perp C$ at x , if there exists x' in $\mathbb{R}^\ell$ such that $v = x'-x$ and $\|x'-x\| = \inf\{\|x'-c\|; c \in \text{cl } C\}$, i.e., x is the closest element to x' in $\text{cl } C$. Then, Clarke's normal cone to C at an element x in $\text{cl } C$, denoted by $N_C(x)$, is the closed convex cone generated by the origin and the set:

$$\{v = \lim_q v_q/\|v_q\|; v_q \perp C \text{ at } x_q , \{x_q\} \subset \text{cl } C , x_q \to x \text{ and } v_q \to 0\} .$$

(We recall that a subset N of $\mathbb{R}^\ell$ is said to be a cone if $\lambda x \in N$ for all $\lambda > 0$ and $x \in N$.) We refer to Clarke's book [7, 2.5.7] for the above definition but also for properties of the normal cone $N_C(x)$ and an equivalent definition of it which takes the tangent cone as the primary concept. Incidentally, we notice that the above definition of Clarke's normal cone is unaffected if C is replaced by $\text{cl } C$, i.e., $N_C(x) = N_{\text{cl } C}(x)$, for all x in $\text{cl } C$.

DEFINITION 2.2 A $(m+n)$-tuple $((x_i^*),(y_j^*))$ of elements of $\mathbb{R}^\ell$ is said to be a quasi-equilibrium with respect to a price system, or, simply, a price quasi-equilibrium, if there exists a nonzero price vector p^* in $\mathbb{R}^\ell$ satisfying:

(α) for all i , $x_i^* \in \text{cl } P_i^*$ and $-p^* \in N_{P_i^*}(x_i^*)$ [where $P_i^* = P_i(x_1^*,\ldots,x_m^*)$] ;

(β) for all j , $y_j^* \in Y_j$ and $p^* \in N_{Y_j}(y_j^*)$;

(γ) $\Sigma_i x_i^* - \Sigma_j y_j^* - \omega = 0$.

The above definition is essentially Guesnerie's definition of P.A. equilibrium, with the only difference that we use here the concept of Clarke's normal cone, instead of Dubovickii-Miljutin's one as in Guesnerie ([15]). (See Remark 2.2 for comments on our terminology.)

REMARK 2.1 The above conditions (α) and (β) are the rigorous formalization of the assertion in the introduction that "each consumer satisfies the first order necessary condition for expenditure minimization and each firm satisfies the first order necessary condition for profit maximization." This

follows from [7, 2.4.3 and 2.2.4] since, if C is an arbitrary subset of $\mathbb{R}^{\ell}$ and p^* is a vector in $\mathbb{R}^{\ell}$, then:

x^* minimizes $p^* \cdot x$ over C implies $-p^* \in N_C(x^*)$.

REMARK 2.2 If $E = \{(X_i, P_i), (Y_j), \omega\}$ is a <u>convex economy</u>, in the sense that, for all i, P_i is convex-valued, and, for all j, Y_j is convex, then the conditions (α) and (β) are both necessary and sufficient. Indeed, by [7, 2.4.4], if C is a convex subset of $\mathbb{R}^{\ell}$ and if x is an element in cl C, then $N_C(x)$ coincides with the cone of normals in the sense of convex analysis ([24]), i.e., $N_C(x) = \{p \text{ in } \mathbb{R}^{\ell}; p \cdot x \geqq p \cdot c \text{ for all } c \text{ in } C\}$, hence, x^* minimizes $p^* \cdot x$ over C if and only if $-p^* \in N_C(x^*)$. Consequently, if, for all i, P_i^* is convex and, for all j, Y_j is convex, the conditions (α) and (β) hold if and only if:

(α') for all i, x_i^* minimizes $p^* \cdot x_i$ over the preferred set P_i^* ,

(β') for all j, y_j^* maximizes $p^* \cdot y_j$ over the production set Y_j .

Hence, for convex economies, the above notion of "price quasi-equilibrium" coincides exactly with the conclusion of Debreu [11, Thm. 6.4], the notion of "price quasi-equilibrium" in Mas-Colell [20],[21] and the one of "compensated equili-brium" in Arrow and Hahn [3]. For the (standard) way to go, from price quasi-equilibria to equilibria, i.e., from expenditure minimization to preference maximization by the consumers, we refer to the three above books. Here, we shall only consider the notion of price quasi-equilibrium.

3. STATEMENT OF THE RESULTS

Our first main result states:

THEOREM 3.1 <u>Let</u> $((x_i^*),(y_j^*))$ <u>be a weak Pareto optimum (resp. a Pareto optimum and assume</u> $m > 1$), <u>then it is a price quasi-equilibrium if, for all</u> i, $x_i^* \in$ cl P_i^* , <u>and the following condition is satisfied either, for some</u> i, <u>by</u> $C = P_i^*$ <u>(resp. cl P_i^*) at</u> $c^* = x_i^*$ <u>or, for some</u> j, <u>by</u> $C = -Y_j$ <u>at</u> $c^* = -y_j^*$:

(D) $\exists e \in \mathbb{R}^{\ell}$, $\exists \varepsilon > 0$, $\forall t \in (0, \varepsilon)$, $te + $ cl $C \cap B(c^*, \varepsilon) \subseteq$ int C .

The nonsatiation assumption that, for all i , $x_i^* \in cl\ P_i^*$, is a standard
one in convex economies with non-transitive, non-complete preferences (see Gale
and Mas-Colell [14] and Fon and Otani [13]). We interpret the vector e in (D)
as a desirable (monotonic) "direction" at x_i^* if $C = P_i^*$ (resp. cl P_i^*) and
$-e$ as an admissible "direction" at y_j^* if $C = -Y_j^*$. The vector e , in
(D), is uniformly defined on a neighborhood of c^* and this uniform property
cannot be dispensed with (see Remark 3.1, hereafter). We also point out that,
for $m = 1$, the notions of Pareto optimality and of weak Pareto optimality are
identical.

The next proposition gives several cases under which (D) will be satisfied.
The proofs of the proposition and of Theorem 3.1 are given in Section 4.

PROPOSITION 3.2 Condition (D) is satisfied by $C \subset \mathbb{R}^\ell$, $c^* \in cl\ C$, if
one of the following assertions holds:

(D.1) C is convex with a nonempty interior;

(D.2) $C + int\ \mathbb{R}_+^\ell \subset C$;

(D.2') $C + Q \subset C$, for some nonempty open cone Q of $\mathbb{R}^\ell$;

(D.3) C is epi-Lipschitzian at c^* , [25], i.e.,

  $\exists e \in \mathbb{R}^\ell$, $\exists \varepsilon > 0$, $\forall t \in (0,\varepsilon)$, $\forall e' \in B(e,\varepsilon)$, $te' + C \cap B(c^*,\varepsilon) \subset C$;

(D.3') $N_C(c^*) \cap -N_C(c^*) = \{0\}$ and C is closed;

(D.3'')   int $T_C(c^*) \ne \emptyset$ and C is closed,

  where $T_C(c^*) = \{v$ in $\mathbb{R}^\ell$; $v \cdot p \le 0$ for all p in $N_C(c^*)\}$.

Condition (D.1) needs not special comments but it is worth pointing out
that the noninteriority assumption cannot be dispensed with on the production
side (see Remark 4.1, hereafter); it will be the purpose of Theorem 3.4, however,
to weaken it (and also (D)) on the consumption side. We interpret (D.2) as a
free-disposal assumption if $C = -Y_j$; if $C = P_i^*$ (resp. cl P_i^*) then (D.2) is
satisfied if the preferences are transitive and monotonic. Condition (D.2')
is just a weakening of (D.2). Finally we point out that, by a theorem of
Rockafellar [25] (see also [7, Cor. 1 of 2.5.8]), conditions (D.3') and (D.3''),

which are equivalent, both imply the èpi-Lipschitzian condition (D.3) and the converse is true if C is closed.

The following Corollary shows that, if all commodities can be freely disposed of, the nonsatiation assumption is sufficient for the second welfare theorem to hold, i.e., the $P_i^*(i=1,\ldots,m)$ and $Y_j(j=1,\ldots,n)$ may be arbitrary nonempty subsets of $\mathbb{R}^\ell$. To make this statement precise, we modify the notion of Pareto optimality of Definition 2.1, to allow "free-disposition," as follows. Formally, the $(m+n)$-tuple $((x_i^*),(y_j^*))$ of elements of $\mathbb{R}^\ell$ is said to be a free disposal weak Pareto optimum of E if (a) for all i, $x_i^* \in X_i$, (b) for all j, $y_j^* \in Y_j$, (c') $\Sigma_i x_i^* - \Sigma_j y_j^* - \omega \leq 0$ and if there exists no other $(m+n)$-tuple $((x_i),(y_j))$ of elements of $\mathbb{R}^\ell$ satisfying (a), (b), (c') and such that, for all i, $x_i \in P_i(x_1^*,\ldots,x_m^*) = P_i^*$. The only difference with Definition 2.1 lies in the feasibility condition (c') which allows free-disposition. Clearly, $((x_i^*),(y_j^*))$ is a free disposal weak Pareto optimum of E if and only if the $(m+n+1)$-tuple $((x_i^*),(y_1^*,\ldots,y_n^*,y_{n+1}^*))$, where $y_{n+1}^* = \Sigma_{i=1}^m x_i^* - \Sigma_{j=1}^n y_j^* - \omega$ is a weak Pareto optimum of the economy $E' = \{(X_i,P_i),(Y_1,\ldots,Y_n,Y_{n+1}),\omega\}$, obtained from the original economy E, by the addition of a $(n+1)$-th firm with production set $Y_{n+1} = -\mathbb{R}_+^\ell$, which satisfies both assumptions (D.1) and (D.2). From Remark 2.2, one easily deduces that, for y_{n+1}^* in $Y_{n+1}=-\mathbb{R}_+^\ell$, $N_{Y_{n+1}}(y_{n+1}^*) = \{p^* \geq 0 ; p^* \cdot y_{n+1}^* = 0\}$. Hence, from Theorem 3.1, one deduces the:

THEOREM 3.3 Let $((x_i^*),(y_j^*))$ be a free-disposal weak Pareto optimum such that, for all i, $x_i^* \in cl\ P_i^*$. Then there exists a nonzero price vector p^* in $\mathbb{R}^\ell$ satisfying: (α) for all i, $-p^* \in N_{P_i^*}(x_i^*)$, ($\beta$) for all j, $p^* \in N_{Y_j}(y_j^*)$ and (γ') $p^* \geq 0$, $p^* \cdot (\Sigma_i x_i^* - \Sigma_j y_j^* - \omega) = 0$ and $\Sigma_i x_i^* - \Sigma_j y_j^* - \omega \leq 0$.

At this stage, we are now able to discuss the link between our results and the other ones on this subject. As already said, the main difference with Guesnerie ([15]) lies in the different concepts of normality which are used. An important justification of the choice of Clarke's normal cone (see Cornet [8]) is that it allows us to consider production sets with "inward kinks,"

such as $Y = \{(y_1, y_2)$ in $\mathbb{R}^2$; $y_1 \leq 0$ and $y_2 \leq \max\{0, -1-y_1\}\}$, of particular
economic importance, which are ruled out by Guesnerie in [15]. For a more de-
tailed discussion on this fact and the relation between Theorem 3.1 and
Guesnerie's result, we refer to Bonnisseau-Cornet [4]. The above Theorem 3.3
extends significantly the results of Kahn and Vohra [17], and [18] (in the finite
dimensional setting), Quinzii [23] and Yun [28], who all consider free disposal
Pareto optima, but assume additionally that, for all consumers and, for all firms,
the monotonicity-free disposal condition (D.2) is satisfied ([17], [23], [28])
or the epi-Lipschitzian condition (D.3) is satisfied ([18]).

At the present stage, we cannot deduce, from Theorem 3.1, the second
welfare theorem for convex economies ([2], [9]) in its more general version
in Euclidean space [11, Thm. 6.4], where no interiority assumption is made on
the sets P_i^* or Y_j . For this purpose, we have to weaken condition (D)
(which clearly implies that the set C has a nonempty interior) as in the
next theorem.

THEOREM 3.4 A (resp. weak) Pareto optimum $((x_i^*), (y_j^*))$ is a price quasi-
equilibrium if, for all i , $x_i^* \in cl\ P_i^*$, if, for all j , Y_j is closed or
convex, and, if, for some i (resp. for all i) $C = P_i^*$, $c^* = x_i^*$ satisfies:

(W.D.) $\exists e \in \mathbb{R}^\ell$, $\exists \varepsilon > 0$, $\forall t \in (0, \varepsilon)$, $te + cl\ C \cap B(c^*, \varepsilon) \subset C$.

The proofs of Theorem 3.4 and Theorem 3.1 will be given in Section 4, as a
direct consequence of a more general (but less intuitive) result (Lemma 4.1).
It is worth pointing out however that neither Theorem 2.1 implies Theorem 3.4,
nor the converse is true and that we have lost in Theorem 3.4 the symmetry
between the consumer and the producer sides of the economy that we had before
in Theorem 3.1. This latter fact will be further stressed in Remark 3.1,
hereafter.

Assumption (W.D.), which is clearly weaker than (D), needs no additional
comments to the ones made before to (D). The next proposition gives conditions
under which (W.D.) is satisfied. The proof of it is given in Section 4.

PROPOSITION 3.5 <u>Condition (W.D.) is satisfied by $C \subset \mathbb{R}^{\ell}$, $c^* \in cl\ C$, if, either (i) C is closed, or (ii) C is convex, or (iii) C and c^* satisfy condition (D).</u>

In view of Proposition 3.5 we shall prove later a slightly stronger result than Theorem 3.4, by only assuming that, for all j , $C = Y_j$, $c^* = y_j^*$ satisfy (W.D.). We are now able, however, to deduce from Theorem 3.4, the second welfare theorem for convex economies (Arrow [2], Debreu [9]) as modified by Gale and Mas-Colèll [14] (see also Fon and Otani [13] in a pure exchange economy) to the case of noncomplete nontransitive preferences.

COROLLARY 3.4 <u>Let $((x_i^*),(y_j^*))$ be a weak Pareto optimum such that, for all i , $x_i^* \in cl\ P_i^*$, P_i^* is convex and $\Sigma_j Y_j$ is convex. Then, there exists a nonzero price vector p^* in $\mathbb{R}^{\ell}$ satisfying: (α') for all i , x_i^* minimizes $p^* \cdot x_i$ over P_i^* ; (β') for all j , y_j^* maximizes $p^* \cdot y_j$ over Y_j and (γ) $\Sigma_i x_i^* - \Sigma_j y_j^* - \omega = 0$.</u>

The proof goes as follows. The (m+1)-tuple $((x_i^*), \Sigma_j y_j^*)$ of elements of $\mathbb{R}^{\ell}$ is clearly a weak Pareto optimum of the economy $\tilde{E} = \{(X_i, P_i), \Sigma_j Y_j, \omega\}$. Hence, by Theorem 3.4, Proposition 3.5 and Remark 2.2, there exists a nonzero price vector p^* such that (α') is satisfied, together with ($\tilde{\beta}$): $\Sigma_j y_j^*$ maximizes $p^* \cdot y$ over $\Sigma_j Y_j$, which clearly implies (β').

We end the section by a remark.

REMARK 3.1. The second welfare theorem does not hold, in general, if we only assume that the preference of the consumers are nonsatiated and that, for some j , Y_j is convex. In other words, the assumption that $C = Y_j$ has a nonempty interior in (D.1) of Theorem 3.1 cannot be dispensed with, in general. We consider the following economy with two goods, $\omega = (1,1)$ as the vector of initial endowments, one convex firm with production set $Y = \{(y_1, y_2) ; y_2 = -y_1\}$ and one (nonconvex) consumer with complete, transitive preferences as represented on Figure 1.
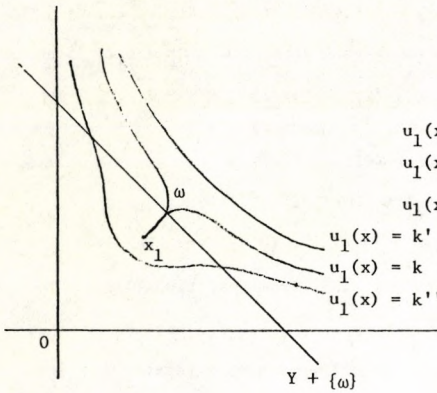
$u_1(x) = k''$
$u_1(x) = k$
$u_1(x) = k'$
$u_1(x) = k' > k$
$u_1(x) = k$
$u_1(x) = k'' < k$

$Y + \{\omega\}$

FIGURE 1

$u_1(x) = k' > k$
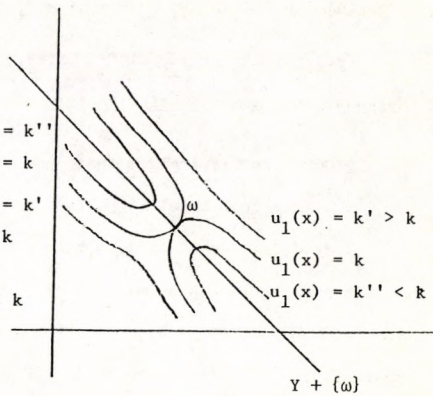$u_1(x) = k$
$u_1(x) = k'' < k$

$Y + \{\omega\}$

FIGURE 2

Then, if $x_1^* = \omega$, $y_1^* = 0$, (x_1^*, y_1^*) is a Pareto optimum but $\{0\} = -N_{P_1(\omega)}(\omega) \cap N_Y(0)$ since $N_Y(0) = Y^\perp$ and one easily sees that $N_{P_1(\omega)}(\omega) = Y$.

One may object to the above counterexample that it satisfies the nonsatiation assumption, $x_1^* \in cl\ P_1^*$, at the optimum but not on a neighborhood of x_1^* (i.e., the segment $[x_1, \omega[$ on Figure 1). However, the preferences can be modified, as in Figure 2, for the nonsatiation assumption to hold everywhere, and the above analysis remains valid.

The above example also shows that the conditions (D) (and also (W.D.)) need to be assumed on a neighborhood of x_1^* and not only at x_1^* (Figure 1) and that the vector e in (D) need to be chosen uniformly on the neighborhood $cl\ P_1^* \cap B(x_1^*, \varepsilon)$ (Figure 2) for the second welfare theorem to hold, in general.

4. PROOFS

The proofs of the Theorems 3.1 and 3.4 will be a direct consequence of the following lemma, also of interest for itself, which gives a general (but rather technical) condition under which the second welfare theorem holds.

LEMMA 4.1 The (resp. weak) Pareto optimum $((x_i^*), (y_i^*))$ is a price quasi-equilibrium if, for all i , $x_i^* \in cl\ P_i^*$ and if there exists e in $\mathbb{R}^\ell$, $\varepsilon > 0$ and $i_0 \in \{1, \ldots, m\}$ such that, for all $t \in (0, \varepsilon)$

(Δ) $te + \Sigma_i \, cl \, P^*_i \cap B(x^*_i, \epsilon) - \Sigma_j \, cl \, Y_j \cap B(y^*_j, \epsilon) \subset P^*_{i_0} + \Sigma_{i \neq i_0} cl \, P^*_i - \Sigma_j Y_j$,

(resp. $(W\Delta)$ $te + \Sigma_i \, cl \, P^*_i \cap B(x^*_i, \epsilon) - \Sigma_j \, cl \, Y_j \cap B(y^*_j, \epsilon) \subset \Sigma_i P^*_i - \Sigma_j Y_j)$.

Moreover, the nonzero price vector p^* in $(\alpha), (\beta)$ satisfies $p^* \cdot e \geq 0$.

PROOF. We first claim that, for all $t \in (0, \epsilon)$,

$$\omega - te \notin \Sigma_i \, cl \, P^*_i \cap B(x^*_i, \epsilon) - \Sigma_j \, cl \, Y_j \cap B(y^*_j, \epsilon) \ .$$

Indeed, if the claim is not true, there exists $t \in (0, \epsilon)$ such that $\omega \in te + \Sigma_i \, cl \, P^*_i \cap B(x^*_i, \epsilon) - \Sigma_j \, cl \, Y_j \cap B(y^*_j, \epsilon)$ and, from condition (Δ) (resp. $(W.\Delta)$), $\omega \in P^*_{i_0} + \Sigma_{i \neq i_0} cl \, P^*_i - \Sigma_j Y_j$ (resp., $\omega \in \Sigma_i P^*_i - \Sigma_j Y_j$), which contradicts that $((x^*_i), (y^*_j))$ is a Pareto optimum (resp. a weak Pareto optimum).

From the above claim, we now notice that $(0, (x^*_i), (y^*_j))$ is a solution of the following maximization problem:

maximize t

subject to: $\Sigma_i x_i - \Sigma_j y_j - \omega + te = 0$,

$x_i \in cl \, P^*_i$, $i = 1, \dots, m$,

$y_j \in cl \, Y_j$, $j = 1, \dots, n$,

$t \in \mathbb{R}$,

$(t, (x_i), (y_j)) \in U$,

where $U = (-\epsilon, +\epsilon) \times \Pi_i \, B(x^*_i, \epsilon) \times \Pi_j \, B(y^*_j, \epsilon)$ is an open subset of $\mathbb{R}^{1 + \ell m + \ell n}$.

From the Lagrange multiplier rule, in Clarke [7, 6.1.1 and 6.1.2 (iv)] there exist $\lambda \geq 0$ and a vector p in $\mathbb{R}^\ell$ such that $(\lambda, p) \neq (0, 0)$ and, if we let:

$$L(t, (x_i), (y_j)) = -\lambda t + p \cdot (\Sigma_i x_i - \Sigma_j y_j - \omega + te) \ ,$$

$$\nabla L(t, (x_i), (y_j)) = (-\lambda + p \cdot e, p, \dots, p, -p, \dots, -p) \in \mathbb{R} \times \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell n} \ ,$$

denote, respectively, the Lagrangian of the problem and its gradient at $(t, (x_i), (y_j))$, one must have:

$$0 \in \nabla L(0, (x^*_i), (y^*_j)) + N_{\mathbb{R} \times P^*_1 \times \dots \times P^*_m \times Y_1 \times \dots \times Y_n}(0, (x^*_i), (y^*_j)) \ .$$

But, since the normal cone to a Cartesian product sets is the Cartesian product of the normal cones [7, Cor. of 2.4.5], and since $N_{\mathbb{R}}(0) = \{0\}$, one deduces that :

$$\lambda - p \cdot e = 0 \ , \ -p \in N_{P_i^*}(x_i^*) (i = 1, \ldots, m) \ , \ p \in N_{Y_j}(y_j^*)(j = 1, \ldots, n) \ .$$

The proof will then be complete if we show that the vector p is nonzero. But, from the above equality, $p = 0$ implies $\lambda = 0$, a contradiction with $(\lambda, p) \neq (0,0)$. This ends the proof of Lemma 4.1.

We shall use later several times the following simple fact, the proof of which is immediate.

LEMMA 4.2 Let U be an open subset of $\mathbb{R}^\ell$ and let $C_1, \ldots, C_k$ be arbitrary subsets of $\mathbb{R}^\ell$. Then

$$U + \text{cl } C_1 + \ldots + \text{cl } C_k = U + C_1 + \ldots + C_k \quad \text{is an open set.}$$

PROOF OF THEOREM 3.1 follows from Lemma 4.1 by checking that the weak Pareto optimum (resp. Pareto optimum) $((x_i^*),(y_j^*))$ satisfies condition (WΔ) (resp. (Δ)). Indeed, let us assume that, for some $i_0 \in \{1, \ldots, m\}$, say $i_0 = 1$, the desirability condition (D) is satisfied by P_1^* (resp. cl P_1^*) for some e in $\mathbb{R}^\ell$ and some $\varepsilon > 0$, then, for all $t \in (0, \varepsilon)$,

$$te + \Sigma_i \text{ cl } P_i^* \cap B(x_i^*, \varepsilon) - \Sigma_j \text{ cl } Y_j \cap B(y_j^*, \varepsilon) \subset \text{int } P_1^* + \Sigma_{i \neq 1} \text{cl } P_i^* - \Sigma_j \text{cl } Y_j$$

$$(\text{resp. } \subset \text{int cl } P_1^* + \Sigma_{i \neq 1} \text{cl } P_i^* - \Sigma_j \text{cl } Y_j) \ ,$$

which, by Lemma 4.2, is a subset of $\Sigma_i P_i^* - \Sigma_j Y_j$ (resp. cl $P_1^* + \Sigma_{i \neq 1} P_i^* - \Sigma_j Y_j$) , hence (W) is satisfied (resp. hence (Δ) is satisfied, recalling that $m > 1$) . We leave the reader adapt the above proof to the case where (D) is satisfied by some firm.

PROOF OF THEOREM 3.4 We first notice that if, for all j, Y_j is closed or convex, then, by Proposition 3.5, $C = Y_j$, $c^* = y_j^*$ satisfy condition (W.D.) for some vector e_j in $\mathbb{R}^\ell$. Under this latter (and weaker) assumption, the theorem follows from Lemma 4.1 by checking that the (resp. weak) Pareto optimum $((x_i^*),(y_j^*))$ satisfies the condition (Δ) (resp. (WΔ)) for the vector $e = e_i - \Sigma_j e_j$ (resp. $e = \Sigma_i e_i - \Sigma_j e_j$) where e_i is the vector for which, for some i (resp. for all i), P_i^* satisfies (Δ) (resp. (WΔ)).

We now give the proofs of Propositions 3.2 and 3.5, for which we need the following lemma.

LEMMA 4.3 Let C be a convex subset of $\mathbb{R}^\ell$ and let c^* be in cl C, then, there exist a vector e in $\mathbb{R}^\ell$ and $\varepsilon > 0$ such that, for all $t \in (0,\varepsilon)$,

$$te + cl \ C \cap B(c^*,\varepsilon) \subset ri \ C \ , \ \text{the relative interior of} \ C \ .$$

PROOF. Let A be the affine space spanned by C and let x be in ri C. Then, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap A \subset ri \ C$ and we can further suppose that $\varepsilon < 1$. We now check that $e = x - c^*$ and ε satisfy the above property. Indeed, for all c in cl $C \cap B(c^*,\varepsilon)$ and all t in $(0,\varepsilon) \subset (0,1)$, one has $te + c = t(e+c) + (1-t)c$. But $c \in cl \ C$ and $e + c = x + c - c^* \in B(x,\varepsilon) \cap A \subset ri \ C$, hence [24, Thm. 6.1], $te + c \in ri \ C$.

PROOF OF PROPOSITION 3.2 Condition (D.1) implies (D) by Lemma 4.3 and the fact that ri C = int C, since the convex set C has a nonempty interior [24, p. 44]. Condition (D.2') implies (D) for any e in Q. It suffices to notice that, by Lemma 4.2, for all $t > 0$, $te + cl \ C \in Q + cl \ C = Q + C$, an open set, contained in C by (D.2'), hence also contained in int C. (D.3) also implies (D) by the same type of argument (with Q replaced by the open ball $B(e,\varepsilon)$). Finally, (D.3') and (D.3''), which are equivalent, both imply (D.3), hence also (D), by a theorem of Rockafellar [25] (see also [7, Cor. 1 of 2.5.8]).

PROOF OF PROPOSITION 3.5 follows from Lemma 4.3.

We end the paper by the following remarks.

REMARK 4.1 The proof of Lemma 4.1 and the Lagrange multiplier rule of [7, 6.1.1] yields, in fact, to the (slightly) stronger conclusion that:

(α'') for all i, $-p^* \in \partial d_{P_i^*}(x_i^*)$; (β'') for all j, $p^* \in \partial d_{Y_j}(y_j^*)$,

where, for $C \subset \mathbb{R}^\ell$, $x \in cl \ C$, we denote by $\partial d_C(x)$ the convex hull of the origin and the set

$$\{v = \lim_q v_q/\|v_q\| \ ; \ v_q \perp C \ \text{at} \ x_q \ , \ \{x_q\} \subset cl \ C \ , \ x_q \to x \ \text{and} \ v_q \to 0\} \ ;$$

hence, $\partial d_C(x)$ is clearly a subset of $N_C(x)$ and, by Clarke [7, 2.5.6], $\partial d_C(x)$ is exactly the generalized gradient of the distance function d_C at x. This remark allows a strengthening of the definition of price quasi-equilibria

(by replacing the conditions (α) and (β) by the above ones (α'') and (β'')) which allow us to get a property which may be of interest. Indeed, from its definition, the correspondences $\partial d_{p^*_i}(\cdot)$, $\partial d_{y_j}(\cdot)$ are of closed graph (and, in fact, upper semicontinuous, since they are bounded), a property not universally possessed by the correspondences $N_{p^*_i}(\cdot)$, $N_{y_j}(\cdot)$ (see however [25]).

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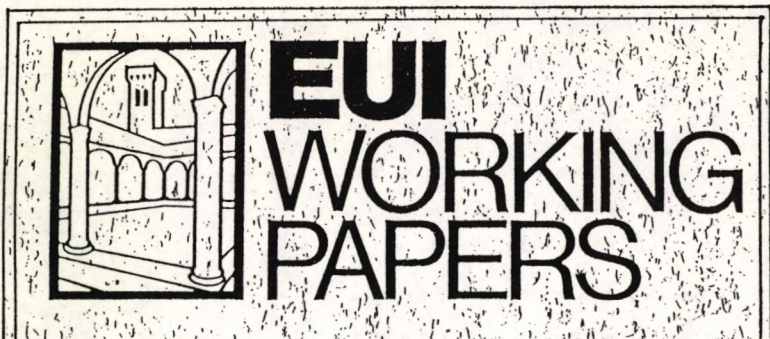
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