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Missing Observations, Additive Outliers and Inverse Autocorrelation Function

AGUSTÍN MARAVALL AND DANIEL PEÑA

* Universidad Politecnica de Madrid
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BADIA FIESOLANA, SAN DOMENICO (FI)

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European University Institute
Badia Fiesolana
– 50016 San Domenico (FI) –
Italy

SUMMARY

The paper deals with estimation of missing observations in ARIMA models. Using the inverse autocorrelation function, it is seen how estimation of a missing observation for period T is analogous to the problem of estimating the coefficient of a dummy variable associated with an additive outlier for period T. (Intervention analysis can be used, thus, to estimate missing observations.) Both problems are closely related, in turn, with the removal of noise in the time series, i.e., with signal extraction.

The results are extended to cover, first, the case of a missing observations near the two extremes of the series; then, to the case of a sequence of missing observations, and finally, to the general case of any number of sequences of any length of missing observations. In all cases the optimal estimator can be expressed, in a compact way (trivial to compute), in terms of the dual autocorrelation function. The mean squared estimation error is always equal to the inverse of the (appropriately chosen) dual autocovariance matrix.

Key words: Missing values, signal extraction, outliers, intervention analysis, ARIMA models, inverse autocorrelation function.

1. Introduction

In this paper we deal with estimation of missing observations in time series that are the outcome of Autoregressive Integrated Moving Average (ARIMA) models. We assume that the ARIMA model is known and hence concern ourselves with obtaining the conditional expectation of the missing observation given the available ones.

Let the series in question follow the general ARIMA model

$$\Phi(B)z_t = \Theta(B)a_t , \tag{1.1}$$

where $\Phi(B)$ and $\Theta(B)$ are finite polynomials in the lag operator B, and a_t is a gaussian white-noise process. Without loss of generality, we set $\sigma_a^2=1$, thus, in the following pages, all variances and mean squared errors will be implicitely expressed in units of σ_a^2. The polynomial $\Phi(B)$ may contain any number of unit roots and hence the process may be nonstationary; we shall assume however, that the model is invertible, so that the roots of $\Theta(B)$ will lie outside the unit circle. The model (1.1) can alternatively be expressed in autoregressive form as

$$\pi(B)\ z_t = a_t , \tag{1.2}$$

where $\pi(B)=\emptyset(B)\Theta(B)^{-1}=(1-\pi_1B-\pi_2B^2-\dots)$. Define the "inverse or dual model" of an ARIMA model as the one that results from interchanging the AR and MA polynomials; therefore the dual model of (1.1) is given by

$$\Theta(B)\ z_t^{*} = \emptyset(B)a_t, \tag{1.3}$$

or

$$z_t^D = \pi(B)a_t \,, \tag{1.4}$$

Since the model for z_t is invertible, model (1.3) will be stationary. Its autocorrelation function (ACF) will be given by

$$\rho^D(B) = \pi(B)\ \pi(F)/V_D \tag{1.5}$$

where $F=B^{-1}$ and V_D is the variance of the dual process, equal to

$$V_D = \Sigma_{i=0}^{\infty}\ \pi_i^2 \qquad (\pi_o=1) \tag{1.6}$$

Following Cleveland (1972), the function (1.5) will be denoted the Inverse or Dual Autocorrelation Function (DACF) of z_t. Trivially, from the ARIMA expression of the model, the DACF is immediately available.

2. Optimal Interpolation of a Missing Value and Estimation of an Additive Outlier

Consider a time series z_t with a missing value for t=T Denote by $z_{(T)}$ the vector of observed series $(\ldots,\ z_{T-2},\ z_{T-1},\ z_{T+1},\ z_{T+2},\ldots)'$. For a linear stationary series, the minimum mean-squared error (MMSE) estimator of z_T is given by

$$\hat{z}_T = E[z_T/z_{(T)}] \,,$$

that is

$$\hat{z}_T = Cov(z_T,\ z_{(T)})'\ Var^{-1}(z_{(T)})z_{(T)} \quad ,$$

where $Cov(z_T, z_{(T)})$ is a vector of components $Cov(z_T z_i)$, $(i \neq T)$ and $Var(z_{(T)})$ is the covariance matrix of $z_{(T)}$. Therefore

$$\hat{z}_T = \Sigma_{k>0} \alpha_k (z_{T-k} + z_{T+k}) \quad ,$$

that is, $\hat{z}_T$ is a linear combination of the observed values, where the α_i weights depend on the covariance structure of the process. Several authors have shown how to compute $\hat{z}_T$ recursively using the Kalman filter (Jones, 1980; Harvey, 1981; Kohn and Ansley, 1983). Others have obtained the smoothing coefficients in particular cases (Abraham and Box, 1979; Miller and Ferreiro, 1984). A general expression, however, has been available for some time (see, for example, Grenander and Rosenblatt, 1957): The optimal estimator of the missing observation can be expressed as

$$\hat{z}_T = - \sum_{k=1}^{\infty} \rho_k^D (z_{T-k} + z_{T+k}) \tag{2.1}$$

where ρ_k^D is the lag-k dual autocorrelation of z_t(i.e., the coefficient of B^k in $\rho^D(B)$.) This result has been used by Bhansali (1980), Battaglia (1983), Brubacher and Wilson (1976), and Kato (1984), among others. As we shall see, the problem of optimal interpolation of a missing value and that of estimation of an additive outlier lead to the same solution in the general, possibly nonstationary, case.

Consider the series Z_t, given by

$$\begin{aligned} Z_t &= z_t \; , \qquad t \neq T \\ Z_T &= z_T + w \end{aligned} \tag{2.2}$$

where w is an unknown constant. This is the same as saying that the series Z_t has an additive outlier at time T. In order to estimate w, define the dummy variable d_t, such that $d_t=0$ for $t \neq T$ and $d_T=1$, and write the "intervention" model (see Box and Tiao, 1975)

$$\pi(B)(Z_t - w\, d_t) = a_t \ , \tag{2.3}$$

which is obtained by combining (1.2) and (2.2). The model (2.3) can alternatively be written

$$\pi(B)\ Z_t = w\ \pi(B)\ d_t + a_t \ ,$$

and, defining the variables $y_t = \pi(B) Z_T$ and $x_t = \pi(B) d_t$, it is seen to be the simple regression model

$$y_t = w\ x_t + a_t \quad ,$$

with x_t deterministic and a_t white-noise. Assuming the observed series extends from t=1 to t=n, the optimal estimator of w is given by

$$\hat{w} = \Sigma\ y_t\ x_t / \Sigma\ x_t^2 \ , \tag{2.4}$$

where all summation signs extend from t=1 to t=n. Assume $n \to \infty$; then, after simplification, it is found that,

$$\Sigma\ y_t\ x_t = \Sigma\ \pi(B)\ Z_t\ \pi(B)\ d_t = \pi(B)\pi(F)\ Z_T$$

and

$$\Sigma\ x_t^2 = \Sigma\ \pi(B)\ d_t\ \pi(B)\ d_t = \Sigma\ \pi_i^2 = V_D \ .$$

Therefore (2.4) becomes

$$\hat{w} = [\pi(B)\ \pi(F)/V_D]\ Z_T \ , \tag{2.5}$$

in agreement with the result in Chang, Tiao and Chen (1988); and, from (1.5),

$$\hat{w} = \rho^D(B)\ Z_T \ . \tag{2.6}$$

Assume we observe Z_T and not z_T. From (2.2) the latter can then be estimated through

$$\hat{z}_T = Z_T - \hat{\omega} \tag{2.7}$$

Introducing (2.6) in expression (2.7), the estimator of the missing observation z_T can be expressed as

$$\hat{z}_T = Z_T - \rho^D(B)\, Z_T = [1-\rho^D(B)]Z_T \tag{2.8}$$

or

$$\hat{z}_T = -\sum_{k=1}^{\infty} \rho_k^D \,(z_{T+k} + z_{T-k}) \quad ,$$

identical to expression (2.1). The optimal estimator of the unobserved value is a symmetric and centered linear combination of the observed ones, where the weights are the coefficients of the dual autocorrelation function. The filter (2.1) will be finite for a pure AR model and will extend to ∞ otherwise; invertibility of the model, however, guarantees its convergence in this last case. Notice that the derivation remains unchanged when the autoregressive polynomial of the model contains nonstationary roots.

Since expression (2.8) does not depend on Z_T, the optimal estimator can be rewritten as

$$\hat{z}_T = [1 - \rho^D(B)]z_T, \tag{2.9}$$

and hence, using (1.2) and (1.5),

$$E(z_T - \hat{z}_T)^2 = E[\rho^D(B)\, z_T]^2 = \frac{1}{V_D^2}\, E[\pi(F)a_t]^2 = \frac{1}{V_D^2} \sum_{i=0}^{\infty} \pi_i^2,$$

and, considering (1.6), the Mean Squared Error (MSE) of $\hat{z}_T$ is found equal to

$$MSE(\hat{z}_T) = V_D^{-1} \quad (2.10)$$

Therefore, even for nonstationary series, the MSE of the estimator is finite; moreover, since $V_D > 1$, it will always be smaller than the one-period ahead forecast error variance, as should be expected. If the process is noninvertible, then $V_D \rightarrow \infty$ and MSE -> 0; the problem degenerates, however, because the filter (2.1) ceases to converge. Notice that (2.2) and (2.7) imply that expresion (2.10) yields also the MSE of $\hat{\omega}$, the estimator of the dummy variable coefficient in the additive outlier model.

The relationship between interpolation of a missing observation and estimation of an additive outlier can be summarized in two alternative ways: On the one hand, optimal estimation of an additive outlier is equivalent to the following procedure. First, assume the outlier is a missing observation and obtain its optimal estimate given the rest of the observations. Then, compute the coefficient of the dummy variable ($\hat{\omega}$) as the difference between the observed outlier and the interpolated value. Alternatively, estimation of a missing observation is the result of the following procedure: First, fill the "hole" in the series with an arbitrary number Z_T. Then estimate ω by intervention analysis on the "observed" series $(\ldots z_{t-1}, Z_T, z_{T+1} \ldots)$, assuming an additive outlier at T. Substracting $\hat{\omega}$ from Z_T, the estimate of the missing observation is obtained. This is of interest, since it implies that intervention analysis provides a natural way to obtain optimal estimates of missing observations (see Peña, 1987).

Notice that the procedure yields implicitely an estimated pseudo-innovation for T, equal to the difference between $\hat{z}_T$, obtained with the two-sided filter (2.1), and $\hat{z}_{T-1}(1)$, the one-period ahead forecast of z obtained at (T-1) using a one-sided filter. This pseudo-innovation is a linear combination of all innovations for periods $T+k$, $k \geq 0$.

Back to the estimator (2.1), if the process requires differencing the series (and hence is nonstationary,) $\pi(1)=0$ so that, from (1.5),

$$\rho^D(1) = 1 + 2\Sigma\, \rho_k^D = 0 \ ,$$

where the summation sign extends from 1 to ∞. Therefore $-\Sigma\, \rho_k^D=1/2$ and the sum of the weights in (2.1) is one; the estimator $\hat{z}_T$ is, in this case, a weighted mean of past and future values of the series. If the process is stationary, $\pi(1)>0$, from which it follows that

$$- \Sigma\, \rho_k^D = \frac{1}{2} \left[\frac{\Sigma\, \pi_i^2 - \pi\,(1)^2}{\Sigma\, \pi_i^2} \right] < \frac{1}{2} \ ,$$

and hence the estimator $\hat{z}_T$ represents a shrinkage towards the mean of the process.

The result (2.1) provides a compact expression for the estimator, which can be easily implemented. As an example, consider the random walk model

$$\nabla z_t = a_t \ .$$

The dual model is $z_t^D=(1-B)a_t$, and therefore $\rho_1^D=-1/2$, $\rho_k^D=0$ for k=0,1; thus

$$\hat{z}_T = (z_{T-1} + z_{T+1})/2 \ ,$$

with MSE $(\hat{z}_T)$ = .5. More generally for an AR(p) process it is easily found that

$$\rho_k^D = (-\phi_k + \Sigma_{i=1}^{p-k}\ \phi_i\ \phi_{i+k})/(1+\Sigma_{i=1}^{p}\ \phi_i^2) \ , \qquad (2.11)$$

for k=1,...,p, and $\rho_k^D=0$ for k>p. Thus the optimal estimator is

$$\hat{z}_T = (1+\Sigma_{i=1}^{p} \phi_i^2)^{-1} \Sigma_{k=1}^{p} c_k (z_{T+k}+z_{T-k}) ,$$

where

$$c_k = \phi_k - \Sigma_{i=1}^{p-k} \phi_i \phi_{i+k} ,$$

in agreement with the results of Abraham and Box (1979), Peña (1984) and Miller and Ferreiro (1984). For this general case, $MSE(\hat{z}_T) = (1+\phi_1^2+...+\phi_p^2)^{-1}$.

As another example, consider the widely used model, first popularized by Box and Jenkins (1970),

$$\nabla\nabla_{12} z_t = (1 - \Theta_1 B) (1 - \Theta_{12} B^{12})a_t \tag{2.12}$$

Table 1 displays the root mean squared error (RMSE) of the estimator $\hat{z}_T$ of a missing observation at T (expression (2.1)), for different values of Θ_1 and Θ_{12} ($\sigma_a^2 = 1$). As Θ_1 and Θ_{12} tend to 1, the RMSE of the estimator tends also to 1. This is sensible, since, in the limit, the two differences in (2.12) would cancel out, and, ignoring deterministic components, the series z_t would simply be the white-noise a_t, with variance 1. On the contrary, as the series approaches noninvertibility, the estimation error tends to zero. The RMSE of Table 1 is also that of the estimator of a dummy variable coefficient in (2.12), associated with an additive outlier. Notice that, except when the parameters are close to the noninvertibility region, Table 1 is practically symmetrical in Θ_1 and Θ_{12}.

3. Relationship with Signal Extraction

Assume we wish to decompose the series z_t into signal plus noise, as in

$$z_t = s_t + u_t , \tag{3.1}$$

Table 1

ROOT MEAN SQUARED ERROR OF A MISSING OBSERVATION ESTIMATOR

FOR MODEL (2.12)

(expressed as a fraction of the innovation variance)

THETA 1	THETA 12						
	-0,9	-0,6	-0,3	0,0	0,3	0,6	0,9
-0,9	0,068	0,130	0,165	0,189	0,205	0,216	0,222
-0,6	0,100	0,200	0,265	0,317	0,361	0,400	0,436
-0,3	0,132	0,265	0,350	0,418	0,477	0,529	0,577
0,0	0,158	0,316	0,418	0,500	0,570	0,632	0,689
0,3	0,180	0,361	0,477	0,570	0,650	0,721	0,786
0,6	0,200	0,400	0,529	0,632	0,721	0,800	0,872
0,9	0,215	0,431	0,571	0,684	0,781	0,869	0,949

where $u_t \sim niid(0, V_u)$, and s_t and u_t are mutually orthogonal. The MMSE estimator of the noise is given by (see, for example, Box, Hillmer and Tiao, 1978)

$$\hat{u}_t = V_u \pi(B) \pi(F) z_t , \qquad (3.2)$$

and comparing (3.2) with (2.5) it is seen that, except for a scale factor, the filter that provides the estimator of the noise is identical to the filter that yields the estimator of the dummy variabble coefficient.

Using (1.2) in (3.2) and (2.5), it is obtained that

$$\hat{u}_t = V_u \pi(F) a_t; \qquad \hat{\omega} = V_D^{-1} \pi(F) a_t \qquad (3.3)$$

Therefore $\hat{u}_t$ and $\hat{\omega}$ filter only contemporaneous and "future" innovations. Considering (1.5), (2.9) and (3.2), the estimator $\hat{z}_T$ can be expressed as

$$\hat{z}_T = z_T - [1/(V_D V_u)] \hat{u}_T . \qquad (3.4)$$

From the first equality in (3.3) it is obtained that

$$\hat{V}_u = (V_u)^2 V_D , \qquad (3.5)$$

where $\hat{V}_u$ denotes the variance of $\hat{u}_t$. Combining (3.4) and (3.5), the estimator $\hat{z}_T$ can then be written as

$$\hat{z}_T = z_T - k \hat{u}_T$$

where

$$k = V_u / \hat{V}_u$$

represents the ratio of the variances of the noise and of its MMSE estimator. (Since the estimator has always a smaller variance -see, for example, Maravall, 1987- the ratio k is always greater than one.) The smoothing implied by the estimation of a missing observation is therefore equivalent to extracting from the "true" series a component proportional to the noise.

In order to look at the frequency domain representation of the filters used to estimate ω and the missing observation z_T, consider, first, as an example, the Airline model of Box and Jenkins (1970, Chap. 9), given by

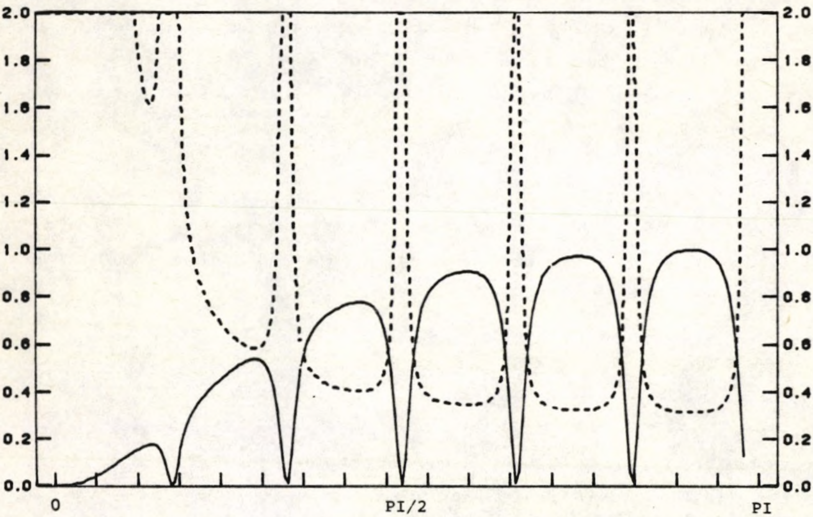
$$\nabla \nabla_{12} z_t = (1 - .4 B) (1 - .6 B^{12}) a_t . \qquad (3.6)$$

The pseudospectrum of z_t, $g_z(f)$, and the frequency domain representation of the filters used to obtain $\hat{\omega}$ and $\hat{z}_T$ are given in Figure 1 (except for a scale factor, the filter for $\hat{\omega}$ is the same as the filter that yields $\hat{u}_t$). Since the spectrum of u_t is constant, for any given frequency f, a measure of the signal to noise ratio (r) can be simply the ordinate of $g_z(f)$. For the zero frequency (associated with the trend) and for the seasonal frequencies, r goes to ∞, and the filter providing $\hat{\omega}$ has a zero. This filter transfers, however, the intraseasonal frequencies, for which the noise contribution is relatively more important. Inversely, the estimator of the missing observation filters entirely the frequencies for which $r \to \infty$.

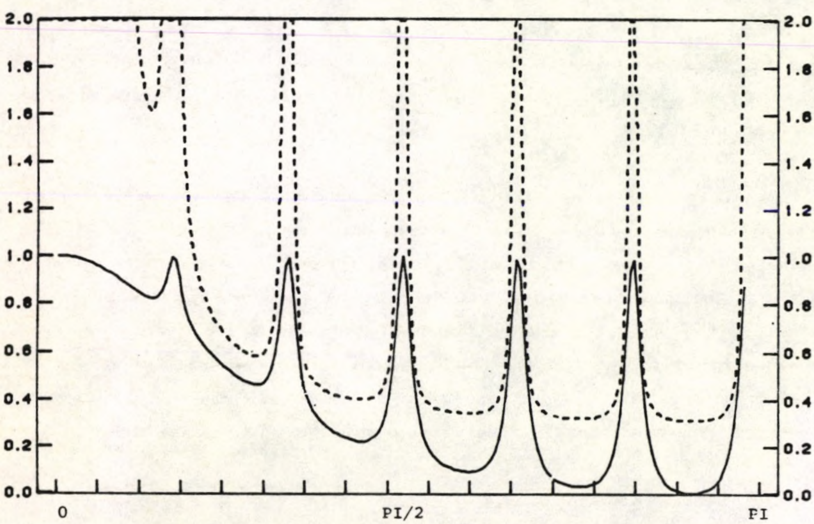
In general, the estimator of the dummy variable coefficient is obtained in the following way: for each frequency, the more the signal dominates the noise, the less will that frequency be used. Obviously, this is the same principle that should be used when estimating the noise. On the contrary, the estimator of the missing observation will be obtained by filtering the frequencies with a large signal. Estimation of missing observations or of additive outliers is, therefore, closely connected with the removal of noise, i.e., with signal extraction.

Figure 1

a) FILTER FOR THE ESTIMATOR OF THE DUMMY VARIABLE COEFFICIENT (Airline model)



b) FILTER FOR THE ESTIMATOR OF THE MISSING OBSERVATION (Airline model)



-----: pseudospectrum of the series

4. An Alternative Interpretation of the Optimal Estimator

Consider the problem of estimating a missing observation at time T for a series that follows the AR(2) model

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t \qquad (4.1)$$

An obvious estimator of z_T is the one-period-ahead forecast of the series $[\ldots, z_{T-2}, z_{T-1}]$. Denoting this estimator by z_T^o,

$$z_T^o = \phi_1 z_{T-1} + \phi_2 z_{T-2} \, , \qquad (4.2)$$

and the MSE of z_T^o, M_o, is equal to $\sigma_a^2 = 1$.

Equation (4.2) is obtained by setting $a_t = 0$ and $t = T$ in (4.1); the estimator obtained obviously ignores the information z_{T+k}, $k>0$. An alternative estimator that uses this information can be obtained by considering z_T the first element in the sequence $[z_T, z_{T+1}, z_{T+2}]$. This is equivalent to setting the innovation equal to zero and $t=T+2$ in (4.1), and the resulting equation can be solved to obtain the new estimator

$$z_T^2 = (z_{T+2} - \phi_1 z_{T+1})/\phi_2 \qquad (4.3)$$

with associated MSE $M_2 = 1/\phi_2^2$.

While z_T^o is computed as the final element of a series, z_T^2 is computed as the first element. Equation (5.1) still offers another possibility, namely, when z_T is in the middle. This will happen when $t=T+1$ in (4.1) and, solving for z_T, a third estimator is obtained:

$$z_T^1 = (z_{T+1} - \phi_2 z_{T-1})/\phi_1 \qquad (4.4)$$

with MSE $M_1 = 1/\phi_1^2$.

Finally, a pooled estimator of z_T can be obtained as a weighted average of the three previous estimators, where the weights are proportional to the precision of each estimator. If z_T^p denotes the pooled estimator,

$$z_T^p = h[z_T^o/M_o + z_T^1/M_1 + z_T^2/M_2] ,$$

where $h^{-1}=1/M_o+1/M_1+1/M_2$. Considering (4.2)-(4.4) and the values of M_o, M_1 and M_2, it is found that

$$z_T^p = [(\phi_1-\phi_1\phi_2)(z_{T-1}+z_{T+1})+\phi_2(z_{t-2}+z_{T+2})]/(1+\phi_1^2+\phi_2^2) ,$$

or, in view of (2.11),

$$z_T^p = -\rho_1^D(z_{T-1}+z_{T+1}) - \rho_2^D(z_{T-2}+z_{T+2}) .$$

Thus the pooled estimator is equal to the estimator $\hat{z}_T$, derived in Section 2 and given by (2.1). Therefore, the optimal estimator of the missing observation can be seen as a weighted average of the estimators that are obtained by assuming that the missing observation occupies all possible different positions of z in the autoregressive equation (4.1).

The previous result for the AR(2) model generalizes to any linear invertible (possibly nonstationary) model of the type (1.1). To see this, write (1.1) as (1.2), i.e.

$$z_t = \pi_1 z_{t-1} + \pi_2 z_{t-2} + \dots + a_t , \qquad (4.5)$$

or, for t=T+j, (j = 0,1,2,...) ,

$$z_{T+j} = \pi_1 z_{T+j-1} + \pi_2 z_{T+j-2} +\dots+ \pi_j z_T +\dots+ a_{T+j} . \qquad (4.6)$$

Setting $a_{T+j}=0$ and solving for z_T, the estimator z_T^j, given by

$$z_T^j = (1/\pi_j)(z_{T+j} - \pi_1 z_{T+j-1} - \dots) =$$

$$= (1/\pi_j)[\pi(B) z_{T+j} + \pi_j z_T] =$$

$$= (1/\pi_j)[\pi(B) F^j + \pi_j] z_T , \qquad (4.7)$$

is obtained (for $j=0$ we adopt the convention $\pi_o = -1$.) The MSE of z_T^j is given by $M_j = 1/\pi_j^2$. Letting $j=0,1,2,\dots$, the pooled estimator, z_T^p, is given by (all summation signs extend from $j=0$ to $j\to\infty$)

$$z_T^p = h \, \Sigma \, z_T^j / M_j , \qquad (4.8)$$

where $h^{-1} = \Sigma(1/M_j) = \Sigma \, \pi_j^2 = V_D$. Thus, using (4.7),

$$z_t^p = (1/V_D) \, \Sigma \, \pi_j \, [\pi(B) F^j + \pi_j] \, z_T =$$

$$= (1/V_D) \, [\Sigma \, \pi_j^2] \, z_T + (1/V_D) \, \Sigma \, \pi_j \, F^j \, \pi(B) \, z_T =$$

$$= z_T - (1/V_D) \, \pi(B) \, \pi(F) \, z_T = [1 - \rho^D(B)] z_T ,$$

and, considering (2.9), $z_T^p = \hat{z}_T$, as claimed.

5. Missing Observation Near the Two Extremes of the Series

5.1. Estimation

The optimal estimator of a missing observation at time T, derived in Section 2 and given by (2.1), is a symmetric filter centered at T. Although it extends theoretically from $-\infty$ to $+\infty$, invertibility of the series guarantees that the filter will converge towards zero, and hence that it can be truncated and applied to a finite length series.

For T close enough to either end of the series, however, (2.1) cannot be used since observations needed to complete the filter will not be available.

Assume that the truncated filter extends from (T−N) to (T+N); that is, for k>N $\rho_k^D \simeq 0$. Let the available series consist of the (T+n−1) observations $[z_1, z_2, \ldots, z_{T-1}, z_{T+1}, \ldots, z_{T+n}]$. Two cases can be distinguished:

(A) If N>n, the "future" values $(z_{T+n+1}, \ldots, z_{T+N})$ are needed to compute $\hat{z}_T$ with (2.1), but they have not been observed yet.

(B) If 1>T−N, the "starting" values $(z_{T-N}, \ldots, z_o)$ are needed to compute $\hat{z}_T$, yet they are not available.

To simplify the discussion, assume that T+n>2N+1 (i.e., the length of the filter is smaller than the length of the series) so that cases (A) and (B) cannot occur simultaneously. Consider first case (A), when future observations would be needed in order to apply (2.1).

Similarly to the case of unobserved components estimation (such as, for example, seasonal adjustment) one can think of extending the series beyond z_{T+n} with forecasts, and apply the filter to the extended series (see, for example, Cleveland and Tiao, 1976). This procedure, however, cannot be used in the present context because of the following consideration: Since $\rho^D(B) = \pi(B)\pi(F)/V_D$, given that n<N, the fact that $\rho_k^D \neq 0$ for $k \leq N$ implies in general that $\pi_{n+1} \neq 0$. Consider the AR representation of the one-period ahead forecast

$$z_{T+n}(1) = \pi_1 z_{T+n} + \pi_2 z_{T+n-1} + \ldots + \pi_{n+1} z_T + \ldots$$

Since $\pi_{n+1} \neq 0$, it follows that the missing observation would be needed in order to compute the forecast.

To derive the optimal estimator of z_T when T is close to the end of the series, we use the method employed in section 4 to provide an alternative derivation of $\hat{z}_T$. From expression (4.5), only (n+1) equations of the type (4.6) can be obtained, namely those corresponding to j=0,1,...,n, since z_{T+j} for j>n has not been observed yet. Therefore, expression (4.8) remains valid with the summation sign extending now from j=0 to j=n, and $h^{-1} = \Sigma_{j=0}^{n} \pi_j^2$. Denote by V_D^n the truncated variance of the dual process,

$$V_D^n = \Sigma_{j=0}^{n} \pi_j^2 ,$$

and by $\pi^n(F)$ the truncated AR polynomial

$$\pi^n(F) = (1-\pi_1 F-\ldots-\pi_n F^n) .$$

then,

$$\hat{z}_T = (1/V_D^n)\ \Sigma_{j=o}^{n}\ \pi_j\ [\pi(B)F^j+\pi_j]\ z_T =$$

$$= z_T - (1/V_D^n)\ \pi(B)(\Sigma_{j=o}^{n}\ \pi_j\ F^j)\ z_T ,$$

or

$$\hat{z}_T = z_T - (1/V_D^n)\ \pi(B)\ \pi^n(F)\ z_T . \tag{5.1}$$

Since this expression does not depend on z_T, the estimator $\hat{z}_T$ can be written as

$$\hat{z}_T = Z_T - \hat{\omega} , \tag{5.2a}$$

where

$$\hat{\omega} = (1/V_D^n)\ \pi(B)\ \pi^n(F)\ Z_T . \tag{5.2b}$$

The equations in (5.2) provide an easy way of computing the optimal estimator of the missing observation. When n=0, (5.2) yields the one-period-ahead forecast of the series, while, when $n \to \infty$, (5.2) becomes the optimal estimator (2.1) for the case of an infinite series.

If the missing observation is near the beginning of the series -case (B)- the previous derivation would remain unchanged, applied to the "reversed" series $[z_{T+n} \ldots z_1]$. In this case expression (5.2a) holds, and (5.2b) becomes

$$\hat{\omega} = (1/V_D^n)\ \pi^n(B)\ \pi(F)\ Z_T\ .$$

6.2. Mean Squared Estimation Error.

When the last observation is for period (T+n), from (5.1) the error in the estimator of the missing observation is equal to

$$z_T - \hat{z}_T = (1/V_D^n)\ \pi(B)\ \pi^n(F)\ z_T\ .$$

Since $\pi(B)\ z_T = a_T$, this expression becomes

$$z_T - \hat{z}_T = (1/V_D^n)\ \pi^n(F)\ a_T\ ,$$

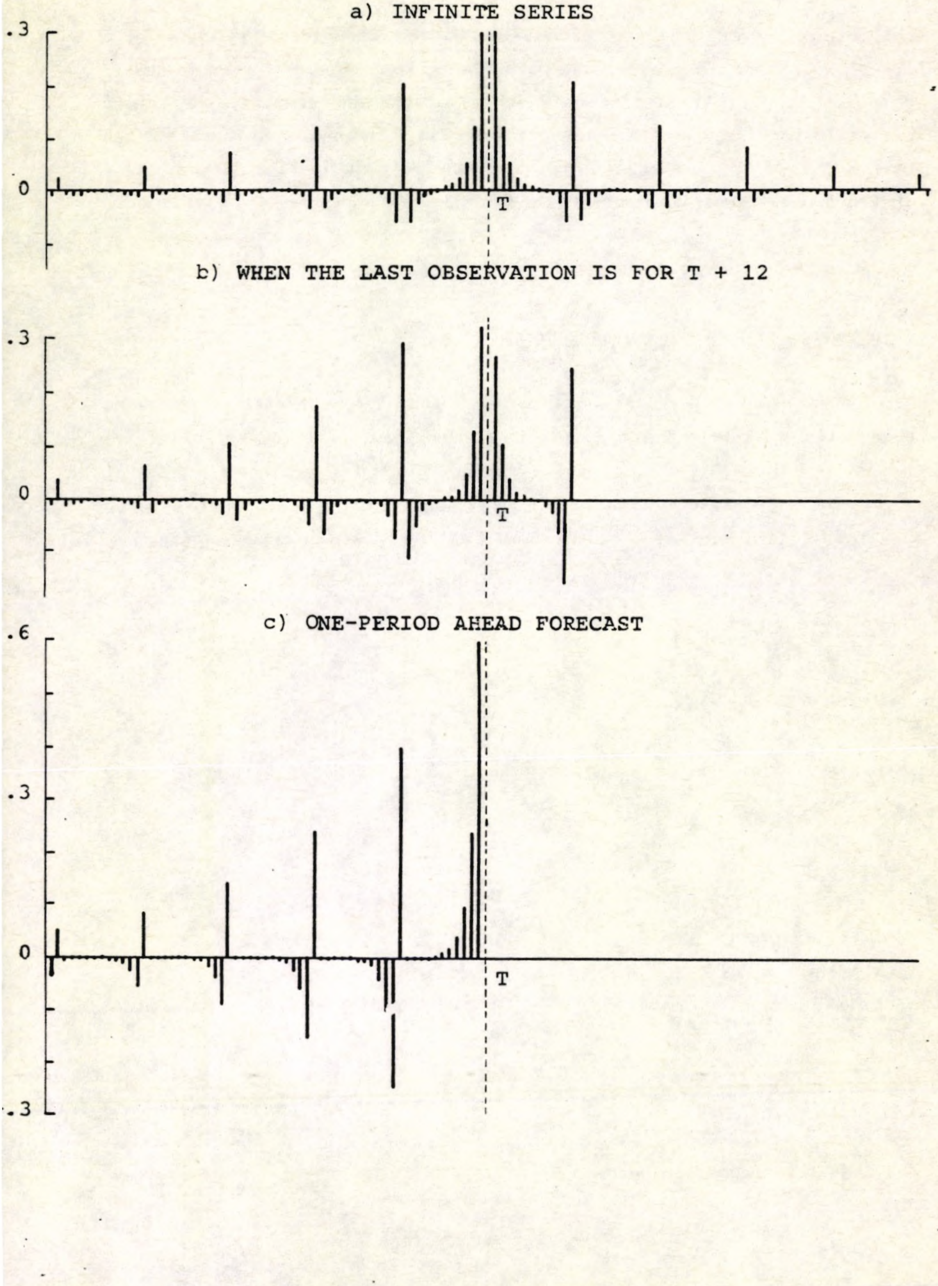
and considering that $E[\pi^n\ (F)\ a_T]^2 = V_D^n$, the MSE of the estimator is equal to

$$\text{MSE}\ (\hat{z}_T) = 1/V_D^n\ , \tag{5.3}$$

the inverse of the truncated variance of the dual process, an expression which is trivial to compute given the model (1.1). Of course, $\text{MSE}(\hat{z}_t)$ reaches a maximum for n=0 (in which case it is equal to $\sigma_a^2 = 1$), and a minimum for $n \to \infty$, when $V_D^n \to V_D$.

Expression (5.2) provides an asymmetric filter for, both $\hat{\omega}$ and $\hat{z}_T$. For the Airline model of equation (3.6), Figure 2 compares the

WEIGHTS OF THE FILTER FOR ESTIMATING A MISSING OBSERVATION Z_T (Airline Model)



complete (symmetric) filter for $\hat{z}_T$ with the one-sided filter of the one-period-ahead forecast, and with the filter when $n = 12$. Figure 3 displays the actual airline data series of Box and Jenkins, the series of estimators z_T and the associated 95% confidence intervals. The half-width of the interval in the center of the series is $1.47\ \sigma_a$; in the two extremes it widens to $1.96\ \sigma_a$.

7. A Sequence of Missing Observations

Consider a time series z_t with k consecutive missing observations at $t=T, T+1, \ldots, T+k-1$. For the rest of this section, let j take the values $0, 1, \ldots, k-1$. Proceeding as in Section 2, we first fill the k consecutive holes in the series with arbitrarily chosen numbers $Z_T, Z_{T+1}, \ldots, Z_{T+k-1}$, and define the observed series Z_t by

$$Z_t = z_t \qquad t \neq T, \ldots, T+k-1$$

$$Z_{T+j} = z_{T+j} + \omega_j \ .$$

where ω_j is an unknown constant. Then, we construct the set of dummy variables:

$$d_t^j = 0 \qquad \text{for } t \neq T+j$$

$$d_{T+j}^j = 1 \ ,$$

and the intervention model

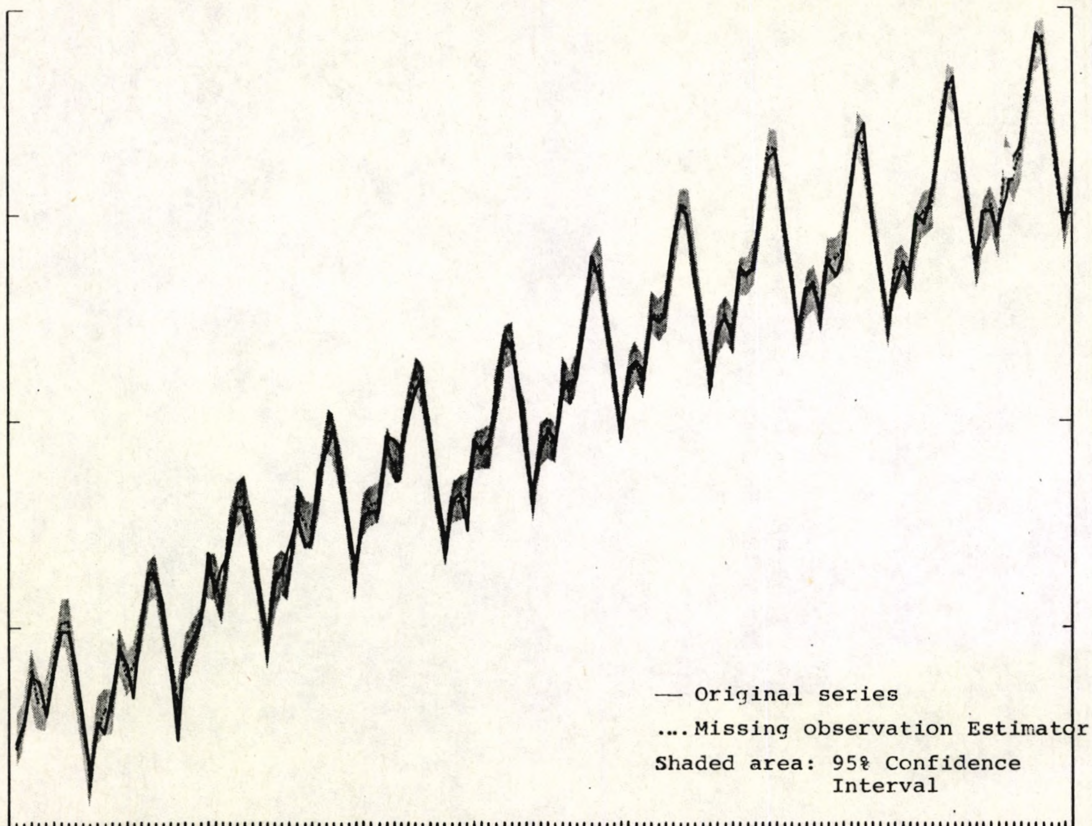
$$\pi(B)\ (Z_t - \sum_j \omega_j\ d_t^j) = a_t \ .$$

The regression equation becomes

Figure 3

MISSING OBSERVATION ESTIMATOR

(Airline Data)



$$y_t = \Sigma_j \omega_j x_{jt} + a_t , \tag{7.1}$$

where

$$y_t = \pi(B) Z_t$$

$$x_{jt} = \pi(B) d_t^j .$$

Let $\hat{\omega}$ denote the vector $(\hat{\omega}_o \ldots \hat{\omega}_{k-1})'$, X_j the column vector with element $[x_{jt}]$, and X the matrix $X=[X_o, X_1 \ldots X_{k-1}]$. Then, from (7.1),

$$\hat{\omega} = (X'X)^{-1} X'y \tag{7.2}$$

Since, summing over t and letting $n \to \infty$, it is obtained that

$$\Sigma x_{jt} y_t = \pi(B) \pi(F) Z_{t+j} \tag{7.3a}$$

$$\Sigma x_{jt}^2 = V_D \tag{7.3b}$$

$$\Sigma x_{jt} x_{j+h,t} = -\pi_h + \Sigma_{i=1}^{\infty} \pi_i \pi_{i+h} = \gamma_h^D , \tag{7.3c}$$

where γ_h^D denotes the lag-h autocovariance of the dual process, using (7.3b and c) the matrix (X'X) has all the elements of the j-th diagonal equal to γ_j^D $(V_D=\gamma_o^D)$. Therefore X'X can be written as the symmetric matrix:

$$X'X = V_D \begin{bmatrix} 1 & \rho_1^D & \cdots & \rho_{k-1}^D \\ & \cdot & & \\ & & \cdot & \rho_1^D \\ & & & 1 \end{bmatrix} = V_D R_D = \Omega_D \tag{7.4}$$

where R_D and Ω_D are the autocorrelation and autocovariance matrices of the dual process, truncated to be of order (kxk). Using (7.3a) and (7.4), (7.2) can be expressed as

$$\hat{\omega} = R_D^{-1} V_D^{-1} \pi(B)\pi(F) \begin{bmatrix} Z_T \\ \cdot \\ \cdot \\ Z_{T+K-1} \end{bmatrix} = R_D^{-1} \rho^D (B)Z \quad , \tag{7.5}$$

where Z is the vector of "invented" observations. Let $\omega_j^{(1)}$ denote the estimator of ω_j obtained by assuming that Z_{T+j} is the only missing observation (i.e., setting all other ω's equal to zero,) and using the method of Section 2. Define the vector $\omega^{(1)}=(\omega_0^{(1)},\ldots,\omega_{k-1}^{(1)})'$. Then, considering (2.6), (7.5) can be written as

$$\hat{\omega} = R_D^{-1} \omega^{(1)}, \tag{7.6}$$

and, if $\hat{z}$ denotes the vector of the missing observation estimators, $\hat{z} = (\hat{z}_T,\ldots,\hat{z}_{T+k-1})'$, it can be obtained by substracting $\hat{\omega}$ from the vector Z of arbitrarity chosen numbers. That is,

$$\hat{z} = Z - \hat{\omega} \tag{7.7}$$

To see that expression (7.7) does not depende on the vector Z, write

$$\rho^D(B)\, Z_T = \begin{bmatrix} Z_T + \Sigma\, \rho_i^D(Z_{T+i} + Z_{T-i}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Z_{T+k-1} + \Sigma\, \rho_i^D(Z_{T+k-1+i} + Z_{T+k-1-i}) \end{bmatrix} =$$

$$= Z + [B_1\ B_2\ B_3] \begin{bmatrix} Z^- \\ Z \\ Z^+ \end{bmatrix} = (I + B_2)\, Z + B_1\, Z^- + B_3\, Z^+ \quad , \tag{7.8}$$

where Z^- contains observations prior to T and Z^+ contains observations posterior to T+k-1. (Thus Z^- and Z^+ are the observations available in the series z_t.) The matrix B_2 is the symmetric matrix:

$$B_2 = \begin{bmatrix} 0 & \rho_1^D & \rho_2^D & \cdots & \rho_{k-1}^D \\ & 0 & \rho_1^D & \cdots & \rho_{k-2}^D \\ & & \ddots & & \vdots \\ & & & \ddots & \rho_1^D \\ & & & & 0 \end{bmatrix},$$

so that, from (7.4),

$$R_D = I + B_2 \,. \tag{7.9}$$

Using (7.5), (7.8) and (7.9):

$$\hat{\omega} = R_D^{-1} \, [R_D \, Z + B_1 \, Z^- + B_3 \, Z^+] = Z + R_D^{-1} \, [B_1 \, Z^- + B_3 \, Z^+] \,,$$

and, plugging this expression in (7.7), it follows that the estimator $\hat{z}$ does not depend on the invented vector Z.

7.2. Mean-Squared Estimation Error

Since the MSE of $\hat{\omega}$ in (7.2) is the matrix $(X'X)^{-1}$, using (7.4) and noticing that

$$z_{T+j} - \hat{z}_{T+j} = \hat{\omega}_j - \omega_j \,,$$

it follows that the MSE of the estimator $\hat{z}$ is given by

$$MSE(\hat{z}) = (V_D \, R_D)^{-1} = \Omega_D^{-1}$$

where Ω_D is the autocovariance matrix of the dual process.

8. The General Case

We have seen how to estimate an isolated missing observation or a sequence of consecutive missing observations. The method of Section 7 can be easily extended to cover the general case of any arbitrary mixture of missing observations, whereby some may be isolated, some may be consecutive, and their relative distances in the series may not be large enough to allow for separate estimation.

Assume that, in general, the series z_t has k missing observations at periods T, $T+m_1$, $T+m_2, \ldots, T+m_{k-1}$, where $m_1, \ldots, m_{k-1}$ are positive integers such that $m_1 < m_2 \ldots < m_{k-1}$. Let $j=0,1,\ldots,k-1$, and define the dummy variables:

$$d_t^j = 0 \quad \text{for} \quad t \neq T+m_j$$

$$= 1 \quad '' \quad t = T+m_j \quad ,$$

where $m_o=0$. The regression equation is given by (7.1), where y_t and x_{jt} are as before and the vector $\omega=(\omega_o \ldots \omega_{k-1})'$ is also given by (7.2). The expressions in (7.3) remain unchanged, except for (7.3a) which becomes now

$$\Sigma x_{jt}\, y_t = \pi(B)\, \pi(F)\, Z_{T+m_j} .$$

The matrix (X'X) is equal to $X'X=V_D\, R$, where R is the (kxk) symmetric matrix:

$$R = \begin{bmatrix} 1 & \rho^D_{m_1} & \rho^D_{m_2} \ldots & \rho^D_{m_{k-1}} \\ & 1 & \rho^D_{m_2-m_1} \ldots & \rho^D_{m_{k-1}-m_1} \\ & & 1 & \vdots \\ & & & \ddots \\ & & & 1 \end{bmatrix} \qquad (8.1)$$

and therefore

$$\hat{\omega} = R^{-1}\ \rho^{D}(B) \begin{bmatrix} Z_T \\ Z_{T+m_1} \\ \cdot \\ \cdot \\ Z_{T+m_{k-1}} \end{bmatrix} . \qquad (8.2)$$

Let, as before, Z denote the vector of arbitrarily chosen values, (Z_t) for $t = T, T+m_1, \ldots, T+m_{k-1}$, and $\omega^{(1)}$ denote the vector of estimators $(\omega_j^{(1)})$, where $\omega_j^{(1)}$ is obtained assuming that only Z_{T+m_j} is missing, i.e. $\omega^{(1)} = \rho^{D}(B)\ Z$. Then (8.2) can be rewritten

$$\hat{\omega} = R^{-1}\ \omega^{(1)} ,$$

and the missing observations estimator can be obtained through

$$\hat{z} = Z - \hat{\omega} \ . \qquad (8.3)$$

Estimation of the missing observations in the general case amounts, thus, to the following procedure: First, fill the holes in the series with arbitrary numbers. Compute then $\omega_j^{(1)}$ as in Section 2 (i.e., assuming only the observation for $T+m_j$ is missing), and hence the vector $\omega^{(1)}$. Form the matrix R given by (8.1). Then $R^{-1}\ \omega^{(1)}$ yields $\hat{\omega}$ and, substracting this vector from the vector Z, the estimators of the missing observations are obtained. Using the same derivation as in Section 7.2, the MSE matrix of the estimator $\hat{z}$ is given by the matrix $(V_D\ R)^{-1}$, or

$$MSE(\hat{z}) = \begin{bmatrix} 1 & \gamma^D_{m_1} & \gamma^D_{m_2} \cdots & \gamma^D_{m_{k-1}} \\ & 1 & \gamma^D_{m_2-m_1} \cdots & \gamma^D_{m_{k-1}-m_1} \\ & & 1 & \\ & & & \cdot \\ & & & \quad \cdot \\ & & & \qquad 1 \end{bmatrix}^{-1},$$

where γ^D_j is the j-th order autocovariance of the dual process.

By appropriately partitioning the series Z_t, the proof that (8.3) implies that the estimator z does not depend on the invented Z vector is identical to the one used in the previous section for expression (7.7).

As an example, assume the series z_t has missing observations for t=T, T+1 and T+4. The matrix R is then equal to

$$R = \begin{bmatrix} 1 & \rho^D_1 & \rho^D_4 \\ \rho^D_1 & 1 & \rho^D_3 \\ \rho^D_4 & \rho^D_3 & 1 \end{bmatrix}$$

and $\hat{\omega} = (\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_2)$ is given by

$$\hat{\omega} = R^{-1} \rho^D(B) \begin{bmatrix} Z_T \\ Z_{T+1} \\ Z_{T+4} \end{bmatrix}.$$

Dropping, for notational simplicity, the superscript "D" from the dual autocorrelations, the estimator $\hat{\omega}_o$ is found to be

$$\hat{\omega}_o = |R|^{-1}\{(1-\rho_3^2)\rho(B)Z_T-(\rho_1-\rho_3\rho_4)\rho(B)Z_{T+1}+(\rho_1\rho_3-\rho_4)\rho(B)Z_{T+4}\} , \qquad (8.4)$$

where

$$|R| = 1+2\rho_1\rho_3\rho_4 -\rho_1^2-\rho_3^2 -\rho_4^2 .$$

Since the coefficient of Z_T in $\rho(B)Z_T$, $\rho(B)Z_{T+1}$, and $\rho(B)Z_{T+4}$ is, respectively, 1, ρ_1 and ρ_4, it is easily seen that the coefficient of Z_T in (8.4) is 1. Similarly, the coefficients of Z_{T+1} and Z_{T+4} in (8.4) are seen to be zero, so that the estimator of z_T

$$\hat{z}_T = Z_T - \hat{\omega}_o$$

does not depend on the three arbitrary numbers Z_T, Z_{T+1} and Z_{T+4}.

As a final example, consider the problem of interpolating quaterly data generated from a random walk when only one observation per year is observed. The model for the series and its dual are given by

$$\nabla z_t = a_t \quad ; \qquad z_t^D = (1-B)\, a_t ,$$

and hence the dual autocorrelations are $\rho_1^D =-.5$ and $\rho_k^D = 0$, $k \neq 0,1$. Assuming that the first value is observed (so that the sequence is: 1 observation, 3 missing values, 1 observation, 3 missing values, and so on,) the matrix R of (8.1) is seem to be block diagonal, where the blocks are always the (3x3) symmetric matrix

$$R_1 = \begin{bmatrix} 1 & -.5 & 0 \\ & 1 & -.5 \\ & & 1 \end{bmatrix} \qquad (8.5)$$

Expression (8.2) consists of a set of uncoupled systems of 3 equations, corresponding to the 3 holes in each year. Let Z_0 and Z_4 denote two succesive annual observations (i.e., $Z_0=z_0$, $Z_4=z_4$), and Z_1, Z_2 and Z_3 denote the invented numbers filling the unobserved quarters. Each system of equations is of the form

$$\begin{bmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix} = R_1^{-1} \begin{bmatrix} Z_1 - 1/2\ (Z_0 + Z_2) \\ Z_2 - 1/2\ (Z_1 + Z_3) \\ Z_3 - 1/2\ (Z_2 + Z_4) \end{bmatrix}, \qquad (8.6)$$

and, from (8.5) and (8.6), it is straightforward to obtain

$$\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{bmatrix} = \begin{bmatrix} Z_1 - \hat{\omega}_1 \\ Z_2 - \hat{\omega}_2 \\ Z_3 - \hat{\omega}_3 \end{bmatrix} = \begin{bmatrix} 3/4\ z_0 + 1/4\ z_4 \\ 1/2\ z_0 + 1/2\ z_4 \\ 1/4\ z_0 + 3/4\ z_4 \end{bmatrix},$$

which is the linear interpolation formula derived by Nerlove, Grether and Carvalho (1979, pp. 101 - 102).

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