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LIPSCHITZ CONTINUOUS POLICY FUNCTIONS FOR STRONGLY CONCAVE OPTIMIZATION PROBLEMS

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Lipschitz Continuous Policy Functions for Strongly Concave Optimization Problems**

ABSTRACT

We prove that the policy function, obtained by optimizing a discounted infinite sum of stationary return functions, are Lipschitz continuous when the instantaneous function is strongly concave. Moreover, by using the notion of α-concavity, we provide an estimate of the Lipschitz constant which turns out to be a decreasing function of the discount factor.

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1. Introduction

Several dynamic economic problems can be stated in terms of optimization of a discounted sum of stationary functions subject to stationary constraints. A great deal of research has been directed to finding conditions for the dynamic stability of the optimal solutions of these models. Tipically this requires a rather large discount factor (but see Araujo and Scheinkman(1977) for a notable exception).

It is well known that in these models the optimal paths are generated by a "policy function" τ_δ which maps the current state x_t into the next state $x_{t+1} = \tau_\delta(x_t)$. This policy function depends on the discount factor δ .

Recently it has been pointed out that a high discounting of the future utilities may destroy the regular dynamic behavior of the optimal paths and that the system may even reach a chaotic regime [see Montrucchio (1986),Deneckere and Pelikan(1986),Boldrin and Montrucchio(1986)] . In particular we proved in Boldrin and Montrucchio(1986) that any C^2 dynamical system $x_{t+1} = \theta(x_t)$ can represent a policy function τ_δ when δ is small enough.

On the other side turnpike results suggest that the policy function becomes simpler as the discount factor δ increases.

We propose the Lipschitz constant of the map τ_δ as a measure of its degree of "complexity", i.e. the smaller the Lipschitz constant the simpler are the dynamic paths produced by τ_δ . A good measure of the Lipschitz constant would be obtained by computing the norm of the derivative of τ_δ with respect to x . Unfortunately it is not clear wether $\tau_\delta(x)$ is differentiable or not,even in the case in which the one-period

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return function ic C^2. To be sure,only continuity of τ_δ has been proved to be true under standard conditions.

The main achievement of this paper is to prove that strongly concave return functions produce,under some qualifications, policy functions which are Lipschitz continuous and ,furthermore,to give an estimate of the Lipschitz constant which turns out to be a decreasing function of the discount factor.

The paper is organized as follows.

In Section 2 we introduce a general discrete-time model of optimization over an infinite horizon.

In Section 3 we characterize the notion of strongly concave functions, by using the α-concavity theory of Rockafellar(1976).

In Section 4 we introduce the transform $U(x,y) \longrightarrow \Psi(t)$ which associates a real function $\Psi(t)$ to any concave function $U(x,y)$.

The last section contains the central results. We prove that if the return function is strongly concave the value function is strongly concave as well. Moreover one can evaluate its degree of concavity by means of the $\Psi(t)$ function given in Section 4. Theorem 5.2 gives the above mentioned estimate of the Lipschitz constant of the policy function τ_δ. Some remarks on the implications of this result are contained in the conclusive section.

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2. The Model

In this paper we will analyse the dynamic behavior of the solutions to the problem $P(x_0, \delta)$:

$$W_\delta(x_0) = \text{Max} \sum_{t=1}^{\infty} V(x_{t-1}, x_t)\, \delta^{t-1} \quad , \text{ subject to } \quad (x_{t-1}, x_t) \in D \quad , \tag{1}$$

$t = 1, 2, \ldots.$ and x_0 fixed in X .

Under the following assumptions :

A.1) X is a compact and convex subset of $\mathbb{R}^n$;

A.2) $V : X \times X \rightarrow \mathbb{R}$ is a continuous and strictly concave function ;

A.3) D is a closed and convex subset of $X \times X$ and $pr_1(D) = X$;

A.4) $0 < \delta < 1$ is the discount factor.

Under (A.1)-(A.4) problem $P(x_0, \delta)$ has one and only one optimal solution (x_t^*) for any given initial condition x_0 in X . Moreover the "value function" W_δ turns out to be strictly concave on X and to satisfy the Bellman equation :

$$W_\delta(x) = \underset{y}{\text{Max}} \left\{ V(x,y) + \delta W_\delta(y) \; ; \; \text{s.t.} \quad (x,y) \in D \right\} \quad . \tag{2}$$

In the theory of Dynamic Programming the optimal sequences (x_t^*) is generated by the dynamical system :

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$$x_t^* = \tau_\delta(x_{t-1}^*) \quad , \quad x_0^* = x_0 \text{ given in } X \; , \tag{3}$$

where $\tau_\delta : X \rightarrow X$ is a continuous map (the so-called optimal policy or policy function), which depends continuously on the discount parameter δ . (2) implies that τ_δ is obtained by maximizing $V(x,y)+\delta W_\delta(y)$, that is to say :

$$\underset{y}{\mathrm{Max}} \left\{ V(x,y)+\delta W_\delta(y) \; ; \text{s.t. } (x,y) \in D \right\} = V(x,\tau_\delta(x))+\delta W_\delta(\tau_\delta(x)). \tag{4}$$

It is also well known that the value function W_δ turns out to be the unique fixed point of the functional equation $U_\delta(f) = f$, where

$$U_\delta(f)(x) = \underset{y}{\mathrm{Max}} \left\{ V(x,y) + \delta f(y) \; ; \text{s.t. } (x,y) \in D \right\} \tag{5}$$

maps the space $C^0(X;\mathbb{R})$ into itself. Here $C^0(X;\mathbb{R})$ is the space of all continuous functions endowed with the uniform topology. U_δ is in fact a contraction operator : $\| U_\delta(f)-U_\delta(g) \| \leq \delta \|f - g\|$, for all $f,g \in C^0(X;\mathbb{R})$. Its unique fixed point is the value function W_δ : $U_\delta(W_\delta) = W_\delta$, see (2) .

We recall also that the successive iterates of U_δ ,starting from the zero function : $U_\delta(0)$, $U_\delta^{(2)}(0)$, $U_\delta^{(3)}(0)$, ... , yield the value functions of the problems with finite horizon . In other words :

$$U_\delta^{(T)}(0) = W_{\delta,T}(x_0) = \mathrm{Max} \sum_{t=1}^{T} V(x_{t-1},x_t)\,\delta^{t-1} \quad \text{,subject to} \tag{6}$$

the constraints $(x_{t-1},x_t) \in D$ and x_0 fixed in X .

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As it was mentioned in the introduction, there is no conclusive evidence about the differentiability of τ_δ. However a heuristic way to understand the methods we are using is that of looking at the one-dimensional case, under the assumption that W_δ is twice differentiable. If τ_δ is interior, from (4) and the implicit function theorem, we have :

$$d\,\tau_\delta\,/dx = -\,V_{12}(x,\tau_\delta(x))\,\left[V_{22}(x,\tau_\delta(x)) + \delta\,W''_\delta(\tau_\delta(x))\right]^{-1} .$$

Therefore we get the estimate :

$$|d\,\tau_\delta(x)/dx| \leq L\,(\alpha + \delta\,t_\delta)^{-1} , \text{ where } L = \mathrm{Max}\,|V_{12}(x,y)| ,$$

$\alpha = \mathrm{Min}\,|V_{22}(x,y)|$ and $t_\delta = \mathrm{Min}\,|W''_\delta(x)|$, which in turn

implies :

$$|\tau_\delta(x_1) - \tau_\delta(x_2)| \leq L\,(\alpha + \delta\,t_\delta)^{-1}\,|x_1 - x_2| . \quad (7)$$

We shall make (7) rigorous (see Theorem 5.2) by giving an appropriate meaning to L , α , t_δ . α and t_δ will be related to a measure of "curvature" of $V(x,.)$ and $W_\delta(.)$ obtained from the notion of α-concavity.

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3. α-concavity

Although our applications involve finite dimensional spaces, Hilbert spaces are a natural setting for the theory we are developping. Therefore throughout this and the next section the functions f are defined on convex sets X of a Hilbert space H . $\|\cdot\|$ denotes the Hilbert norm in H . Anagously functions U(x,y) will be defined on a set D that is a convex subset of $H \times H_1$, where H , H_1 are real Hilbert spaces.

Definition 3.1 f is said to be α-concave on X , if $f(x) + (1/2)\alpha \|x\|^2$ is concave on X , or, equivalently, if

$$f(tx_1+(1-t)x_2) \geqslant t\, f(x_1) + (1-t)\, f(x_2) + (1/2)\alpha t(1-t)\|x_1 - x_2\|^2 \tag{8}$$

holds for any $x_1, x_2 \in X$ and all $t \in [0\ ,\ 1]$.

Definition 3.2 U(x,y) is said to be α_x- concave on D if $U(x,y) + (1/2)\,\alpha\|x\|^2$ is concave on D , or, equivalently, if

$$U(tx_1+(1-t)x_2, ty_1+(1-t)y_2) \geqslant tU(x_1,y_1) + (1-t)U(x_2,y_2)+(1/2)\alpha t(1-t)\|x_1-x_2\|^2$$

for all $(x_1,y_1),(x_2,y_2) \in D$ and $t \in [0\ ,\ 1]$. (9)

Definition 3.3 We set

$$\rho(f;X) = \operatorname{Sup}\{\alpha\ ;\ f \text{ is } \alpha\text{-concave on } X\}$$
$$\rho_x(U;D) = \operatorname{Sup}\{\alpha\ ;\ U \text{ is } \alpha_x\text{-concave on } D\} \quad . \tag{10}$$

If f and U are concave : $\rho(f;X) \geqslant 0$, $\rho_x(U;D) \geqslant 0$ and they are termed the concavity parameter of f and U . The assumptions

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$\rho(f;X) > 0$ and $\rho_x(U;D) > 0$ will denote strong concavity.

We recall that in (10) : $\operatorname{Sup}\{\alpha\} = \operatorname{Max}\{\alpha\} < +\infty$ in both cases.

In the differentiable case, there are simple criteria to verify α-concavity. For example : if f is twice differentiable over an open set containing X , then f is α-concave ($\alpha \geqslant 0$) when $|w' D^2 f(x) w| \geqslant \alpha \|w\|^2$ for all $x \in X$ and $w \in H$.

We need also a notion of directional derivative which replaces differentiability. If f is a finite concave function and x_0 a point in X , then we can define the directional derivative of f at x_0 in the direction $h \in H$, as the limit :

$$f'(x_0; h) = \lim_{t \to 0+} t^{-1}\left[f(x_0 + t\,h) - f(x_0)\right] .$$

When h is feasible, i.e., $(x_0, x_0+th) \cap X \neq \emptyset$ for some $t>0$, the above limit $f'(x_0;h)$ always exists (finite or infinite). In the same way $U_2'(x,y;h)$ denotes the partial directional derivative at (x,y) along $h \in H_1$. In other words :

$$U_2'(x,y;h) = \lim_{t \to 0+} t^{-1}\left[U(x,y+th) - U(x,y)\right] .$$

Theorem 3.1 Let $U : X \times Y \to \mathbb{R}$ be a finite concave function, where $X \subset H$ and $Y \subset H_1$ are closed and convex subsets of Hilbert spaces. Assume :

i) $U(x,.)$ is upper-semicontinuous on Y for each x in X ;

ii) $U(x,.)$ is α-concave on Y for each x in X ;

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iii) $| U_2'(x_1,y;h)-U_2'(x_2,y;h)| \leq L \|x_1 - x_2\| \|h\|$ for any feasible direction $h \in H_1$ and $x_1,x_2 \in X$;

then we have :

a) there exists a unique map $\theta : X \to Y$ such that

$$\sup_{y \in Y} U(x,y) = U(x,\theta(x)) \quad ;$$

b) $\theta(x)$ is Lipschitz continuous on X and

$$\|\theta(x_1) - \theta(x_2)\| \leq (L/\alpha) \|x_1 - x_2\| , \quad x_1,x_2 \in X \quad .$$

Remark 3.1 In Theorem 3.1 the feasible set D agrees with the whole domain $X \times Y$. In the case in which D is a strict subset of $X \times Y$, we have that if the map θ such that $\text{Sup}_y [U(x,y)$; s.t. $(x,y) \in D] = U(x,\theta(x))$ is interior , then the conclusion (b) still holds. This is easily verified by looking at the proof of the theorem.

Proof : See Appendix.

We end this section with a few comments.

1) Condition (iii) implies implicitly that $U_2'(x,y;h) < +\infty$ for any $(x,y,h) \in X \times Y \times H_1$ with h feasible.

2) If U is C^2 then one can choose for L a number such that $\| U_{12}(x,y) \| \leq L$, for all $(x,y) \in X \times Y$, whereas α can be taken such that $| w'U_{22}(x,y)w | \geq \alpha \|w\|^2$, $\forall$ $w \in H_1$ and $(x,y) \in X \times Y$. With these qualifications one obtains a theorem given in Fleming and Rischel(1975; pag.170). Their assumptions are : U

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is C^2 and X , Y are finite dimensional spaces. Our extension is not fictious because it is addressed to return functions of the type $U(x,y) = V(x,y) + \delta W_\delta(y)$ (see (2)). As it was already mentioned, even when V is C^2 it does not imply that W_δ is C^2 too.

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4. The Ψ - transform

Proposition 4.1 Let $U : D \to \mathbb{R}$ be finite and bounded from above ($D \subset X \times Y \subset H \times H_1$). If U is α_x-concave on D (see Definition 3.2) ,then

$$W(x) = \sup_y \left\{ U(x,y) \ ; \text{ s.t. } (x,y) \in D \right\} \quad \text{is } \alpha\text{-concave on } X.$$

Proof : Take two points x_1 , x_2 in X and two numbers a , b such that $W(x_1) > a$ and $W(x_2) > b$. By assumptions there exist a $(x_1,y') \in D$ and $(x_2,y'') \in D$ such that $W(x_1) \geq U(x_1,y') > a$ and $W(x_2) \geq U(x_2,y'') > b$. It follows from (9) that :

$$W(tx_1+(1-t)) \geq U(tx_1+(1-t)x_2,ty'+(1-t)y'') \geq t\, U(x_1,y') +(1-t)\, U(x_2,y'')+ (1/2)\alpha t(1-t) \| x_1 - x_2 \|^2 > t\, a + (1-t)\, b + (1/2)\alpha t(1-t) \| x_1 - x_2 \|^2 .$$

As a tend to $W(x_1)$ and b to $W(x_2)$ respectively, we obtain

$$W(tx_1 + (1-t)x_2) \geq t\, W(x_1) + (1-t)\, W(x_2) +(1/2)\alpha t(1-t) \| x_1 - x_2 \|^2$$

and this completes the proof.

Definition 4.1 Let $U : D \to \mathbb{R}$ be a finite concave function. The Ψ - transform of U is the real function defined on $t \geq 0$:

$$\Psi(t ; U,D) = \sup \left\{ \alpha \geq 0 \ ; \ U(x,y) + (1/2)\alpha \| x \|^2 - (1/2)t \| y \|^2 \text{ is concave on } D \right\} \equiv \rho_x \left(U(x,y) - (1/2)t \| y \|^2 \ ; \ D \right) .$$

In the sequel the notations $\Psi(t;D)$, or even $\Psi(t)$, instead of $\Psi(t ;U,D)$ may be met.

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<u>Proposition 4.2</u> The Ψ - transform of any finite concave function $U : D \to \mathbb{R}$ is a continuous ,increasing and concave function on $t \geq 0$. Moreover it is bounded : $0 \leq \Psi(t) \leq \mathrm{Inf}_y\, \rho(U(.,y))$.

<u>Proof</u> : As one can easily verify :

$$\rho_x(c_1 U_1 + c_2 U_2) \geq c_1 \rho_x(U_1) + c_2 \rho_x(U_2) \qquad (11)$$

for $c_1, c_2 \geq 0$ and U_1, U_2 concave on D .

If $t \geq t_1$, we can write $U(x,y) - (1/2)t\|y\|^2 = \left[U(x,y) - (1/2)\, t_1 \|y\|^2\right] - (1/2)\,(t-t_1)\|y\|^2$, and ,from (11), we have $\Psi(t) \geq \Psi(t_1)$. Concavity of $\Psi(t)$ follows directly by (11) . This implies,in turn,that $\Psi(t)$ is continuous on $(0 ,+\infty)$ and that $\Psi(0) \leq \lim \Psi(t)$ as $t \to 0+$. On the other hand, it is easily verified that $\rho_x(.)$ is "upper semicontinuous". More precisely, if U_n is a sequence of concave functions which point-wise converges to U , then $\limsup_n \rho_x(U_n) \leq \rho_x(U)$. This implies that $\limsup \Psi(t) \leq \Psi(0)$,as $t \to 0+$. Consequently $\lim \Psi(t) = \Psi(0)$, as $t \to 0+$ and therefore $\Psi(t)$ is continuous at 0 too. The last part of the statement is obvious.

<u>Remark</u> : $\Psi(0) = \rho_x(U ;D)$. If U is strongly concave then $\Psi(0) > 0$.

<u>Theorem 4.1</u> Let $U(x,y)$ and $f(y)$ be finite and bounded from above on $X\times Y$ and Y respectively. If f is α-concave on Y , then $W(x) = \mathrm{Sup}_y \left\{ U(x,y) + f(y) \;;\; \text{s.t. } (x,y) \in D \subset X\times Y \right\}$ is $\Psi(\alpha \;; U,D)$- concave on X .

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<u>Proof</u> : $U(x,y) + f(y)$ can be written as $\left[U(x,y)-(1/2)\alpha \| y \|^2\right]+$ $\left[f(y) + (1/2)\alpha \| y \|^2\right]$. The first addendum is $\Psi(\alpha;U)_x$-concave, by definition, whereas the second one is 0_x-concave . Hence $U + f$ is surely $\Psi(\alpha\,;U)_x$-concave on D and the statement follows by Proposition 4.1 .

To make things more evident let us show two examples of Ψ-transforms. The computations are straightforward and, therefore, omitted.

<u>Example 1</u> : Quadratic case. Let $X = Y = \mathbb{R}^n$, $D = X\times Y$ and $U(x,y) = (1/2)\, x'Ax + x'By + (1/2)\, y'Cy$ be concave (A and C are negative definite). The Ψ-transform of U is

$\Psi(t)$ = the least eighenvalue of $\left[B\,(C - t\, I_n)^{-1} B' - A \right]$.

<u>Example 2</u> : The C^2 one-dimensional case. Let $D \subset \mathbb{R}\times\mathbb{R}$ be an open and convex set . Assume $U(x,y)$ to be a concave C^2 function on D . We have :

$$\Psi(t) = \inf_{(x,y)\in D} \left[U_{12}^2(x,y)\,(U_{22}(x,y)-t)^{-1} - U_{11}(x,y) \right] .$$

Here U_{ij} denote the second partial derivatives of U.

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5. Lipschitzian Policy Functions

In this section we apply the results of Section 3 and 4 to the problem $P(x_0,\delta)$ discussed in Section 2. The notation is the same of that section. $\rho_T(\delta) \equiv \rho(W_{\delta,T};X)$ will denote the concavity parameter of the value function of the truncated problem (see (6)). $\rho_\infty(\delta) = \rho(W_\delta;X)$ will denote the concavity parameter of the value function of problem $P(x_0,\delta)$.

Lemma 5.1 Let the return function $V(x,y)$ of $P(x_0,\delta)$ be strongly concave (i.e. $\rho_x(V;D) > 0$) and $\Psi(t) = \Psi(t;V,D)$ be the ψ-transform of V. We have :

i) The equation $\psi(\delta t) = t$ has one and only one positive solution t_δ for any δ fixed in $(0\ ,\ 1)$. Moreover t_δ increases continuously as δ increases ;

ii) the iterative system $t_n = \Psi(t_{n-1})$ is increasingly convergent to t_δ for all initial condition $t_0 \in [0\ ,\ t_\delta)$ and it is decreasingly convergent to t_δ for all $t_0 \in (t_\delta, +\infty)$.

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Proof: i) Let us write $\Psi_\delta(t) = \Psi(\delta t)$. Ψ_δ is again an increasing concave function for $t \geqslant 0$. Take $a(t) = \Psi_\delta(t) - t$. Since $a(0) > 0$ and $a(+\infty) = -\infty$,it follows that at least a t_δ exists such that $\Psi_\delta(t_\delta) = t_\delta$. It is unique. In fact,suppose not : $\Psi_\delta(t_1) = t_1$ and $\Psi_\delta(t_2) = t_2$, with $t_1 = \theta\, t_2$ and $0 < \theta < 1$.

But then $\psi_\delta(t_1) = \Psi_\delta(\theta t_2 + (1-\theta)\cdot 0\ ) \geq \theta\ \Psi_\delta(t_2) + (1-\theta)\Psi_\delta(0)$, i.e., $t_1 \geqslant \theta\ t_2 + (1-\theta)\ \Psi_\delta(0)$, that implies $\Psi(0) \leqslant 0$ which contradicts our assumption ($\rho_x(V) > 0$ is equivalent to $\Psi(0) > 0$).

In a same way it is easily seen that $\delta_1 \leqslant \delta_2$ implies $t_{\delta_1} \leqslant t_{\delta_2}$. The

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continuous dependence on δ is then easily obtained by using topological degree methods.

ii) Consider now the iterative system $t_n = \Psi_\delta(t_{n-1})$. For all $t_0 \in [0, t_\delta)$ one has $\Psi_\delta(t_0) \in [0, t_\delta)$. In fact, $t_0 < t_\delta \rightarrow \Psi_\delta(t_0) \leq \Psi_\delta(t_\delta) = t_\delta$. By uniqueness it follows $\Psi_\delta(t_0) < t_\delta$.

Moreover $\Psi_\delta(t_0) > t_0$ holds. Indeed, suppose not : $\Psi_\delta(t_0) \leq t_0$.

As $\Psi_\delta(0) > 0$, there will exist another fixed point t' of Ψ_δ belonging to $[0, t_0]$. But this contradicts part (a). Hence the sequence of iterates will be : $t_0 < t_1 < t_2 < \ldots\ldots < t_\delta$ and so $\lim t_n = t_\delta$ as $n \rightarrow \infty$.

One can deal with the case $t_0 \in (t_\delta, +\infty)$ by a similar method.

<u>Theorem 5.1</u> Assume $\rho_x(V;D) > 0$ and t_δ be defined as in Lemma 5.1 . We have :

i) $\rho_T(\delta) \geq \rho_x(V;D)$, $\rho_\infty(\delta) \geq \rho_x(V;D)$;

ii) there are two possibilities : either $\rho_T(\delta) < t_\delta$ for any T and then $\rho_T(\delta) < \rho_{T+1}(\delta)$ and $\lim \rho_T(\delta) = \rho_\infty(\delta)$ as $T \rightarrow \infty$, or $\rho_{T_0}(\delta) \geq t_\delta$ for some T_0 and then $\rho_T(\delta) \geq t_\delta$ for all $T \geq T_0$;

iii) $\rho_\infty(\delta) \geq t_\delta$.

<u>Proof</u> : Let us use the iterative system described in Section 2 (see (5) and (6)) :

$$W_{\delta,T}(x) = \underset{y}{\mathrm{Max}} \left\{ V(x,y) + \delta\, W_{\delta,T-1}(y) \; ; \; \text{s.t.} \; (x,y) \in D \right\} ,$$

$T = 1, 2, \ldots.$ and $W_{\delta,0} \equiv 0$.

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From Theorem 4.1 we have :

$\rho_T(\delta) \geq \Psi(\delta\, \rho_{T-1}(\delta))$ for $T \geq 2$ and $\rho_1(\delta) \equiv \Psi(0) \equiv \rho_x(V)$.

Denote by (t_n) the sequence generated by $t_n = \Psi(\delta\, t_{n-1})$ (see part (ii) of Lemma 5.1). One has $\rho_T(\delta) \geq t_T$, $\forall$ T .

This implies $\rho_T(\delta) \geq t_1 = \Psi(0) = \rho_x(V)$ by Lemma 5.1 .

Since $W_{\delta,T}$ converges (uniformly) to W_δ and $\rho_x(.)$ is upper semicontinuous, then $\rho_\infty(\delta) = \rho(W_\delta) \geq \limsup_n \rho_n(\delta) \geq \lim t_n = t_\delta$ and hence (iii) is true.

Part (ii) is easily proved by Lemma 5.1. In fact, suppose $\rho_n(\delta) \in (0, t_\delta)$ for all n. Then $\rho_{n+1}(\delta) \geq \Psi_\delta(\rho_n(\delta)) > \rho_n(\delta)$ and thus $\rho_1 < \rho_2(\delta) < \rho_3(\delta) < \ldots < t_\delta$. This implies that $\rho_n(\delta) \to t_\delta$.

The other case is $\rho_{T_0} \geq t_\delta$. But the set $(t_\delta, +\infty)$ is invariant under Ψ_δ by Lemma 5.1 and $\rho_T(\delta) \geq t_\delta$, $T \geq T_0$, follows.

<u>Theorem 5.2</u> Assume :

i) $| V_2'(x_1,y;h) - V_2'(x_2,y;h) | \leq L \|x_1 - x_2\| \, \|h\|$, for any $x_1, x_2, y \in X$ and any feasible $h \in \mathbb{R}^n$;

ii) either $D = X \times X$ or $(x, \tau_\delta(x))$ is interior to D for any $x \in X$;

iii) $\rho_x(V; D) > 0$;

iv) $\rho\,(V(x,.)) \geq \alpha \geq 0$ for any $x \in X$.

The policy function τ_δ of problem $P(x_0, \delta)$ is Lipschitz continuous and

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$$\| \tau_\delta(x_1) - \tau_\delta(x_2) \| \leqslant L(\alpha + \delta t_\delta)^{-1} \| x_1 - x_2 \| \qquad (12)$$

for any $x_1, x_2 \in X$ and where t_δ is defined as in Lemma 5.1.

Proof : Take $U(x,y) = V(x,y) + \delta W_\delta(y)$. Suppose at the moment that $D = X\times X$. Because $\mathrm{Max}_y U(x,y) = U(x,\tau_\delta(x))$ (see (2)) , we are in position to apply Theorem 3.1 . From assumption (iii) and by Theorem 5.1 we have $\rho(\delta W_\delta) \geqslant \delta t_\delta$. Hence $\rho(U(x,.)) \geqslant \alpha + \delta t_\delta$. Let us suppose $W'_\delta(y;h) < +\infty$. Then $U'_2(x,y;h) = V'_2(x,y;h) + \delta W'_\delta(y;h)$ and thus we have :

$|U'_2(x_1,y;h) - U'_2(x_2,y;h)| \leqslant L \|x_1 - x_2\| \|h\|$.

All the assumptions of theorem 3.1 are fullfilled and we can conclude the given estimate for τ_δ .

Remark The assumption $W'_\delta(y;h) < +\infty$ is not essential. In fact, replace the initial problem with $\mathrm{Max}_y \{U(x,y) ;\text{s.t. } y \in X^\varepsilon\}$, where $X^\varepsilon \subset X$ is an ε - contraction of X ,i.e., $X^\varepsilon = f^\varepsilon(X)$ and $f^\varepsilon : X \to X$ is defined as $f^\varepsilon(x) = (1-\varepsilon)x + \varepsilon x_0$, x_0 is a point in the relative interior of X and ε is a small positive number. Consequently we have :

$\mathrm{Max} \{U(x,y) ; \text{s.t. } y \in X^\varepsilon\} = U(x, \tau^\varepsilon(x))$ and $\tau^\varepsilon(x)$ satisfyes

$\| \tau^\varepsilon(x_1) - \tau^\varepsilon(x_2) \| \leqslant L(\alpha + t_\delta)^{-1} \| x_1 - x_2 \|$ because $W'_\delta(y;h) < \infty$ for all $x \in$ rel int X .

On the other hand $\tau^\varepsilon(x) \to \tau_\delta(x)$ as $\varepsilon \to 0$, by the Maximum Principle, and therefore we get again the desired estimate.

By Remark 3.1 we get the same estimate in the case in which D is a strict subset of $X\times X$ and τ_δ is interior.

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It is not difficult to understand the use of Theorem 5.2 to obtain stability conditions. In fact , $L(\alpha+\delta t_\delta)^{-1} < 1$ sufficies ,by Banach fixed point theorem,to assure that all the trajectories of $P(x_o,\delta)$ converge to a unique stationary trajectory $\bar{x}$ (i.e. to the unique fixed point of τ_δ).

We conclude this paper by giving an example of this fact.

Example. Take the problem $P(x_0,\delta)$ with the return function : $V : [0 , 1] \times [0 , 1] \to \mathbb{R}$ given by $V(x,y) = -4x^2y + 4xy - (1/2)y^2 - (R/2)x^2$, $R > 16$ and $D = X \times X$.

It was shown in Montrucchio(1986) that V is concave on X×X and that the policy τ_δ is chaotic for small δ . Let us find out which value of δ eliminates such a behavior. According to example 2 of Section 4 , the Ψ-transform of V is $\Psi(t) = R - 16(1-t)^{-1}$ and $t_\delta = [\delta R - 1 + (1 + \delta^2R^2 - 64\delta + 2\delta R)^{\frac{1}{2}}] / 2\delta$.
In addition we have $L = 4$ and $\alpha = 1$. By a straightforward computation we get : $L(\alpha+\delta t_\delta)^{-1} < 1$ iff $\delta > 3(R-4)^{-1}$ which is the desired "turnpike" condition . Notice that $3(R-4)^{-1} < 1/4$ for all $R > 16$.

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6. Concluding Remarks

As the example of the last section shows, the estimate (12) provides a powerful tool to compute stability conditions for the discounted Ramsey models. Note the case $L/\alpha < 1$. Because $L/\alpha \geq L/(\alpha+\delta t_\delta)$, we can deduce global stability of the model ,indipendently of the the discount factor.

A partial answer to a conjecture formulated by some authors about the role played by the discount factor is another issue of Theorem 5.2. It seems to be resonably true that the complexity of the optimal paths decreases as the rate of impatience decreases. This fact can be formulated more precisely by using the topological entropy $h(f)$ as an indicator of the dynamical complexity of a map f . From (12) it is not difficult to give the estimate

$h(\tau_\delta) \leq \dim(X) \log^+ (L/(\alpha + \delta t_\delta))$, where $\log^+(u) = \mathrm{Max}(0, \log u)$.

A last comment to be made concerns with the differentiability of the policy function. By the classical Rademacher's theorem , it follows that the policy functions are almost everywhere differentiable on X .

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Appendix

Lemma A.1 Let f be a real function defined on X. f is α-concave on X iff $f(y)-f(x) \leqslant f'(x;y-x) - (1/2)\alpha \|y-x\|^2$ (13)
for every $x,y \in X$.

Proof: (13) follows trivially by concavity of $f(x)+(1/2)\alpha\|x\|^2$.

Lemma A.2 If f is α-concave on X, then

$$\alpha \|y - x\|^2 \leqslant f'(y;x-y) + f'(x;y-x) \tag{14}$$

holds for every $x,y \in X$.

Proof: Exchanging x and y in (13), we have

$$f(x) - f(y) \leqslant f'(y;x-y) -(1/2)\alpha \| y - x\|^2 . \tag{15}$$

Summing up (13) and (15), we get (14).

Lemma A.3 Let $f : X \to \mathbb{R}$ be an upper semicontinuous and α-concave function on a closed and convex set $X \subset H$, $\alpha > 0$. f has a unique point $x^* \in X$ that maximizes f on X and, furthermore :

$$\alpha \| x - x^*\|^2 \leqslant f'(x; x^* - x) \quad , \quad \forall\, x \in X . \tag{16}$$

Proof: The sections $S_\lambda = \{ x \quad X ; \quad f(x) \geqslant \lambda \}$ are closed because f is upper semicontinuous. We prove now that S_λ are relatively compact for the weak topology. In fact, f is superdifferentiable in some point $x_0 \in X$ (actually it is superdifferentiable over a dense subset of X). Let now $u \in \partial f(x_0) \neq \emptyset$ be, where $\partial f(x_0)$ is the set of the superdifferential of f at x_0. Let $A = \{ x \in X \; ; \; f(x) \geqslant f(x_0)\}$. We have $f(x) \leqslant f(x_0) + \langle u , x-x_0 \rangle -(1/2)\alpha \|x - x_0\|^2 \leqslant f(x) + \langle u, x-x_0 \rangle - (1/2)\alpha \|x-x_0\|^2$.

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Thus $(1/2)\,\alpha \| x-x_0 \|^2 \leqslant \langle u, x-x_0 \rangle \leqslant \| u \| \, \| x-x_0 \|$,that implies

$\| x - x_0 \| \leqslant (2/\alpha) \| u \|$, $\forall\ x \in A$.

This sufficies to infer the existence of a maximal point x^* in X (see for example Ekeland and Teman (1976). The maximal point is then unique by the strong concavity of f.

Put now $y = x^*$ in (14). We have $\alpha \| x^*-x \|^2 \leqslant f'(x^*;x-x^*)+f'(x;x^*-x)$.

On the other hand $f'(x^*;x-x^*) \leqslant 0$ for any x in X ,because x^* maximizes f, and thus we get (16).

<u>Proof of Theorem 3.1</u> : (a) follows immediately from Lemma A.3 . Let now $x_1, x_2 \in X$ be . Take (16) with $f = U(x_1,.)$. We have :

$$\alpha \| y - \theta(x_1) \|^2 \leqslant U_2'(x_1, y; \theta(x_1)-y) \quad \text{for all } y \in Y \ . \tag{17}$$

Putting $y = \theta(x_2)$ in (17) ,one obtain

$$\alpha \| \theta(x_2) - \theta(x_1) \|^2 \leqslant U_2'(x_1, \theta(x_2); \theta(x_1)-\theta(x_2)).$$

On the other hand $U_2'(x_2, \theta(x_2); \theta(x_1)-\theta(x_2)) \leqslant 0$ holds because $\theta(x_2)$ maximizes $U(x_2,.)$. We can write

$$\alpha \| \theta(x_2)-\theta(x_1) \|^2 \leqslant U_2'(x_1,\theta(x_2);\theta(x_1)-\theta(x_2)) - U_2'(x_2,\theta(x_2);\theta(x_1)-\theta(x_2)) \tag{18}$$

From assumption (iii),the right side of (18) is smaller that $L \| x_1 - x_2 \| \| \theta(x_1) - \theta(x_2) \|$. Hence :

$\alpha \| \theta(x_1) - \theta(x_2) \|^2 \leqslant L \| x_1 - x_2 \| \| \theta(x_2) - \theta(x_1) \|$,i.e.,

$\| \theta(x_1) - \theta(x_2) \| \leqslant (L/\alpha) \| x_1 - x_2 \|$ and this completes the proof of (b).

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