EUROPEAN UNIVERSITY INSTITUTE, FLORENCE

DEPARTMENT OF ECONOMICS

E U I WORKING PAPER No. 67

ON THE THEORY OF EFFECTIVE DEMAND UNDER STOCHASTIC RATIONING

by

Gerd WEINRICH

October 1983

Paper presented at the European Meeting of the Econometric Society, Dublin, 1982.

BADIA FIESOLANA, SAN DOMENICO DI FIESOLE

Printed in Italy in October 1983
European University Institute
Badia Fiesolana
50016 San Domenico (Fi)
Italy.

1. INTRODUCTION

In the theory of effective demand, the most established concepts are those of Drèze-demand and Clower/Benassy-demand. Both of them, however, possess an unsatisfactory feature. They cannot be reconciled simultaneously with two basic requirements, namely, that there be a difference between effective demand and actual trade *and* that effective demand be derived from explicit maximizing behaviour with respect to the resulting trade. This inconvenience led to the development of the theory of effective demand under stochastic rationing, which promised to yield a more sound representation of demand behaviour.

Contributions have been made for instance by Svensson [9], Gale [5] and Green [7]. Svensson postulates a simple non-manipulable stochastic rationing scheme, from which he derives interesting properties of effective demand, but which he does not integrate into a model of a whole economy. In particular he does not ask the question of equilibrium.

Gale [5] on the other hand provides a general framework for the study of a large economy, where trading possibilities are uncertain. He describes the individual behaviour under stochastic rationing and introduces an adequate notion of equilibrium in this setting. His conditions for existence of an equilibrium an in particular of a non-trivial one, however, are rather technical, so that their economic meaning is not easy to illuminate and their applicability to special cases of rationing mechanisms remains questionable.

Green [7] concentrates on a special case of Gale's model, in which the stochastic rationing scheme's distribution as perceived by the individual agent depends only on his own action and on the aggregate values of demand and supply. Under the further assumption of anonymity, which means that any agent's expected trade depends only on his action and not on his name, his result states that such a rationing mechanism must be manipulable (in contradiction to Svenssons's approach) and that the expectation of such a rationing function is linear in any agent's own action, provided the individual's influence on the aggregate values is neglected and there are four or more agents in the economy.[1]

As Green points out, a consequence of the manipulability is that an agent's expected utility function is not necessarily quasiconcave, even if the underlying von Neumann-Morgenstern-utility function is. This entails that an agent's effective demand correspondence is possibly not convex-valued. Hence fixed point theorems cannot be applied to prove existence of equilibria. At this stage Green's contribution stops, leaving the question whether Gale's existence theorems apply to the type of rationing schemes Green deals with.

One possibility to solve the existence problem would be to place further restrictions on the stochastic nature of the

[1] Green claims that this hold for *three* or more agents. This is contradicted by Weinrich [10], who also reestablishes the linearity for four or more agents [11].

realization process in order to exclude non-concavities in the expected utility function. Another way is to treat the problem in the framework of a continuum economy, because then the non-concavities do no longer prevent the application of fixed point theorems.

A further reason for the continuum approach is that the economy is thought of to be a large one, where each single agent has vanishing influence on market aggregates. Likewise already Gale assumes in his article that the space of agents is non-atomic. Thus, if one wants to check if Gale's existence theorems apply to Green's Schemes, one necessarily has to use the continuum framework.

This is done in this paper. In section 2, we first discuss the foundations of the stochastic rationing approach to the concept of effective demand and its economic content. In section 3, then, Green's linearity result is extended to the case of a continuum of economic agents. Furthermore it is shown that the class of anonymous stochastic rationing schemes which depend on the individual agent's action and on the aggregate values only and which meet moreover the short-sided-rule, consists of those random functions, the mean value function of which is the uniform proportional rationing mechanism. In section 4, Green's rationing mechanism is combined with Gale's framework of an exchange economy with a continuum of agents and the existence of equilibria is investigated. It turns out that Gale's theorem concerning the existence of non-trivial equilibria cannot be applied to Green's schemes. We therefore give a different condition, and we show that it is sufficient to assert existence of non-trivial equilibria.

2. THE STOCHASTIC RATIONING APPROACH TO THE CONCEPT OF EFFECTIVE DEMAND

The concept of effective demand plays a central role in the attempt to construct a microeconomic foundation of macroeconomic equilibrium under temporarily rigid prices. Rigid prices may lead to restrictions in trade for some agents. Subjected to such constraints agents will modify their Walrasian demand, because the Walrasian demand is no longer optimal. The modified demand is called the effective demand.

Which properties should a concept of effective demand possess? There are at least two basic requirements. First, effective demand should have a sound choice-theoretic foundation, that is, it should be the explicit solution to a preference maximization problem. Second, it should reflect the fact that under trade restrictions there is dissatisfaction of some agents in the economy, for signs of dissatisfaction such as the unemployment rate are at the center of much macroeconomic analysis. Consequently, the effective demand should typically differ from actual trade. Moreover, the discrepancy between effective demand and actual trade should not be of arbitrary size but it should yield a reliable measure of dissatisfaction. This is particularly important for a theory of price changes to be based on market excess demands and supplies.

If one examines the deterministic concepts of the Drèze- and the Clower/Benassy-demand with respect to these two requirements, one recognizes that neither fulfills both requirements simultaneously. While it is true that Drèze-demand emerges as the solution of an optimization problem, it does never exceed the given constraints. Thus, an unemployed worker does not offer to work. Clower/Benassy-demand on the other hand does admit offers that exceed the trade restrictions. But apart from the fact that these excesses do not provide a reliable measure of dissatisfaction, Clower/Benassy-demand is not obtained as solution of an explicit maximization behaviour with respect to the resulting trade.

If one tries to remove these shortcomings, one realizes that with a deterministic rationing mechanism this is not possible. To see this, consider a state of equilibrium under quantity rationing, where agent a is rationed in his transaction of good h. Denote his effective demand for good h by z_{ah} and his actual transaction by x_{ah}. Then a discrepancy between z_{ah} and x_{ah} cannot provide a reliable measure of dissatisfaction, because first, the perceived rationing function $\phi_{ah} : z_{ah} \mapsto x_{ah}$ that associates demands with trades (and which is assumed to be non-decreasing), must be constant beyond z_{ah}, for otherwise the rationed agent would have expressed a higher demand:

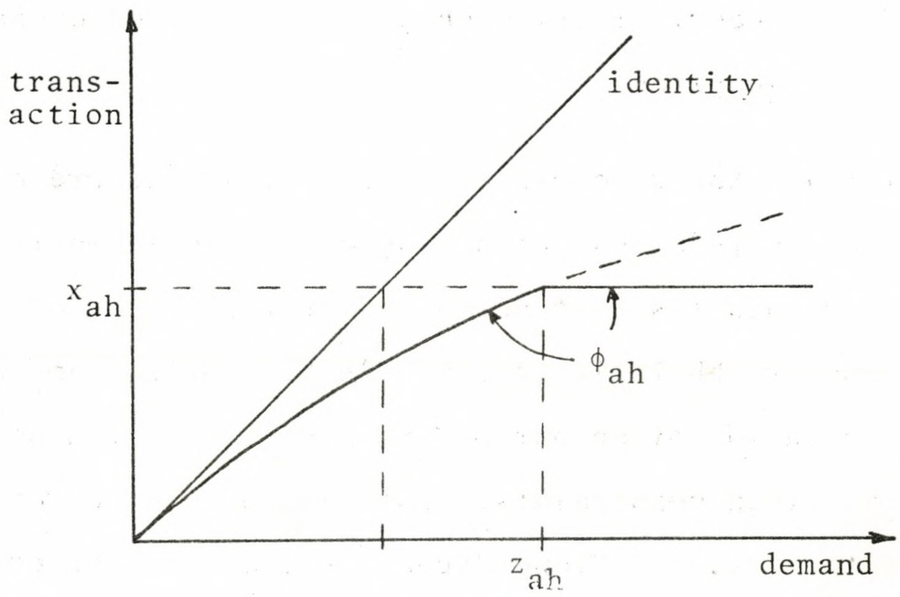


Figure 1

Hence, any demand greater than z_{ah} is optimal for the agent, too. Thus the discrepancy between z_{ah} and x_{ah} is rather arbitrary.

Second, if the agent would have wished to trade just the amount x_{ah}, also in this case it would have been optimal for him to demand the quantity z_{ah}. Therefore, if a positive difference between z_{ah} and x_{ah} would have been taken as an indication of dissatisfaction, this would have been misleading, since the agent would have actually been satisfied.

In contrast to the deterministic case, under stochastic rationing the agent is not sure which actual trade will be associated with his offer. Yet he has some information, which can be modelled by assigning a probability distribution over transactions to his offer. This distribution will depend on his own action, but also on the actions of the other agents, that is on the disequilibrium situation in the economy:

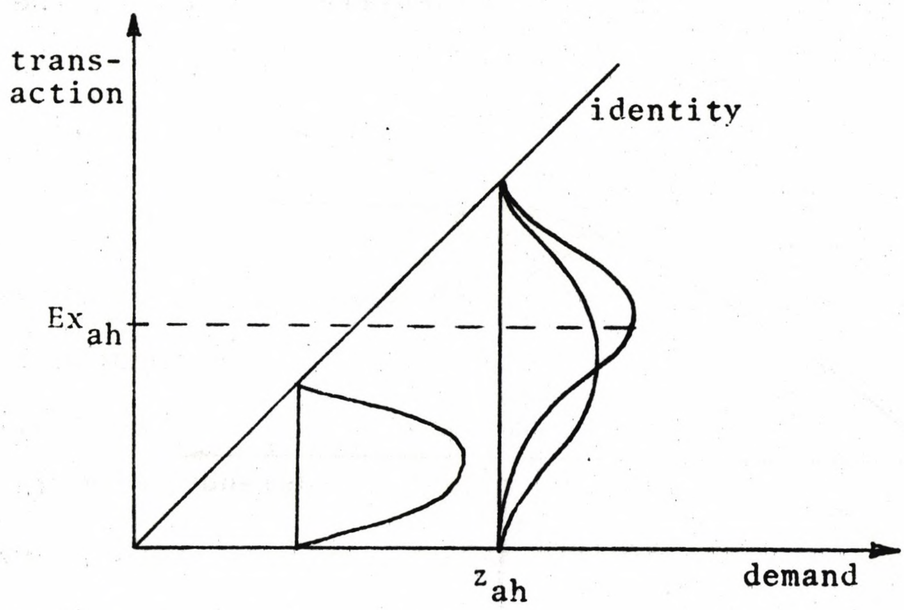


Figure 2

If one requires that the support of such a distribution extends from the demanded quantity z_{ah} to zero (or to any magnitude smaller than z_{ah}) it is clear that the expected realization will be less than z_{ah}. Therefore, if agent a desires a trade of Ex_{ah}, say, he has an incentive to overstate. On the other hand this overstatement will not become unbounded, because it is always possible that the agent realizes his whole demand. As he must meet his budget constraint under all possible states of the world, this will restrict his offer. Maximizing expected utility under the budget constraint

$$\max_{(z_{a1},\ldots,z_{a\ell})} Eu_a(z_{a1},\ldots,z_{a\ell})$$

$$\text{s.t.} \quad \sum_{h=1}^{\ell} p_h x_{ah} \leq 0 \quad \text{with prob. } 1$$

will therefore lead to effective demands $\xi_a = (\xi_{a1},\ldots,\xi_{a\ell})$ that possess the two properties required above, that is, they result from an optimization problem and they may exceed the actual transactions. Whether these excesses provide a reliable measure of dissatisfaction, remains to be seen.

So far, the justification for employing a stochastic rationing mechanism was rather a technical one. More important perhaps is an argument that explains which intrinsic motivation leads an agent to assume a stochastic rationing mechanism in calculating his effective demand. In general, the trading possibilities of agent a depend on the actions of all the other agents, that is, the transaction is a function

$$x_{ah} = \psi_{ah}(z_h) = \psi_{ah}(z_{ah}, Z_{ah}),$$

where $z_h : A \to \mathbb{R}$ is a list of effective demands on market h of all agents $a \in A$ and $z_{ah} = z_h(a)$, $Z_{ah} : A \setminus \{a\} \to \mathbb{R}$, $Z_{ah}(a') = z_h(a')$. If agent a would know the other agent's actions, he could exactly determine the outcome x_{ah} as a function of his own action z_{ah}. However, in a large economy it is not plausible that an agent has this full information. It is more likely that he receives some market information such as the unemployment rate, aggregate demand and aggregate supply or the ratio of aggregate demand and aggregate supply. However, knowing only these is not sufficient to determine accurately the transaction resulting from a certain offer. Instead suppose the agent substitutes the unknown variables by some random variable ω. Then the rationing function becomes a random variable, too:

$$x_{ah} = \phi_{ah}(z_{ah}, r_h, \omega)$$

where r_h is a vector of statistics or market signals that result from the whole vector of demands for good h,

$$r_h = \mathfrak{F}_h(z_h)$$

and whose dimension is finite and smaller than the number of agents. Thus r_h conveys less information than the whole list z_h would do. Therefore, even if the true rationing mechanism is deterministic, for the individual agent it appears stochastic. But this is all that matters, if one seeks to model an agent's behaviour, that is, his effective demand.

An equilibrium of the economy in this context is then a list of effective demand vectors $z : A \to \mathbb{R}^{\ell}$, that reproduces the signals $r = (r_1, \ldots, r_{\ell})$, that is

$$z_a \in \xi_a(r) \quad \text{for all} \quad a \in A$$

and

$$r_h = \mathfrak{F}_h(z_h) \quad \text{for all } h.$$

The special case that Green [7] has considered and which we will investigate in what follows is that r_h denotes the mean values of demand and supply on market h, that is

$$r_h = \mathcal{F}_h(z_h) = \int\left(\max\ \{z_{ah},0\},\min\ \{z_{ah},0\}\right)\nu\ (da)$$

(where ν is a probability measure on the space of agents $(A,\mathfrak{A})$). This is clearly a very simple form of the function $\mathcal{F}_h$, but it seems appropriate to begin with such a simple case before one proceeds to assume more complex functional relations.

3. THE RATIONING MECHANISM

Let $(A,\mathfrak{A},\nu)$ be a probability space of agents which is assumed to be either atomless or such that A contains a finite number of agents only, I, say, which then all have the same measure $\nu(a) = \frac{1}{I}$, $a \in A$. The effective demands and supplies of agents on a certain market are described by an element of the linear space $\mathcal{Z} = \{z : A \to \mathbb{R} \mid z \text{ is integrable}\}$[1].

Define $z^+ : A \to \mathbb{R}$ by $z^+ = \max\ \{z,0\}$ and $z^- : A \to \mathbb{R}$ by $z^- = \min\ \{z,0\}$. Then, for any $z \in \mathcal{Z}$, mean effective demand and supply are given by $Z^+ = \int z^+ d\nu$ and $Z^- = \int z^- d\nu$. A stochastic rationing mechanism is a function $\phi : \mathcal{Z} \times \Omega \to \mathcal{Z}$, where $(\Omega,\mathfrak{A},P)$ is a certain probability space. More precisely, the rationing scheme dealt with here is assumed to be a function as follows.

[1] Integrability presumes implicitly measurability. $\mathbb{R}$ is always assumed to be equipped with the Borel-σ-Algebra $\mathfrak{B}$.

A(0) For any $z \in \mathcal{Z}$, $\omega \in \Omega$, the integrable function $\phi(z,\omega) : A \to \mathbb{R}$ is of the form

$$\phi(z,\omega)(a) = \phi_a(z(a),Z^+,Z^-,\omega) \qquad a \in A.$$

For any $z \in \mathcal{Z}$, $a \in A$, $\phi_a(z(a),Z^+,Z^-,\cdot)$ is a random variable.[1]

ϕ is eventually subject to

A(i) For all $z \in \mathcal{Z}$, ν-almost all $a \in A$,

$$|\phi_a(z(a),Z^+,Z^-,\omega)| \le |z(a)| \qquad \text{P-a.e.}$$

A(ii) For all $z \in \mathcal{Z}$, ν-almost all $a \in A$,

$$z(a)\phi_a(z(a),Z^+,Z^-,\omega) \ge 0 \qquad \text{P-a.e.}$$

A(iii) For every $z \in \mathcal{Z}$,

$$\int\phi_a(z(a),Z^+,Z^-,\omega)\,\nu\,(da) = 0 \qquad \text{P-a.e.}$$

A(iv) For all $z \in \mathcal{Z}$ and ν-almost all $a_1,a_2 \in A$:

$z(a_1) = z(a_2)$ implies $E\phi_{a_1}(z_{a_1},Z^+,Z^-) = E\phi_{a_2}(z_{a_2},Z^+,Z^-)$

A(v) For ν-almost all $a \in A$, the distribution of $\phi_a(z(a),Z^+,Z^-,\cdot)$ is weakly continuous in its arguments, whenever $Z^+ > 0$ or $Z^- < 0$.[2]

---

[1] That is, a *measurable* function $(\Omega,\mathcal{A}) \to (\mathbb{R},\mathcal{B})$.

[2] Excluding the points $(z(a),0,0)$ from the continuity assumption is necessary in order to allow for rationing functions of the form $\phi_a(z(a),Z^+,Z^-,\omega) = \tilde{\phi}_a(z(a),\frac{Z^-}{Z^+},\omega)$. The distribution of such functions cannot be continuous at $Z^+ = Z^- = 0$, since they are homogeneous of degree zero in (Z^+,Z^-).

A(vi) For all $z \in \mathcal{Z}$ and ν-almost all $a \in A$:

$$z(a)(Z^- + Z^+) < 0 \text{ implies } E\phi_a(z(a), Z^+, Z^-) = z(a)$$

Conditions A(i) and A(ii) formalize the voluntariness of trade. A(iii) requires that the rationing mechanism be consistent. A(iv) states that agents who offer the same transaction, can expect the same realizations, regardless of their name. Therefore, this property is called "anonymity". Naturally it does not mean that agents expressing the same demand will also realize the same trades. While condition A(v) is a technical requirement, A(vi) expresses the "short-sided-rule", which requires that only one side of the market is rationed, namely the "long" side. An example of a rationing mechanism satisfying A(0) to A(vi) is provided by Gale [5, pp. 329-332].

THEOREM 1. Let $I \geq 4$. If ϕ is a stochastic rationing mechanism as stated in A(0), then, under conditions A(i) to A(v), ν-a.e. the functions ϕ_a can be written

$$\phi_a(z(a), Z^+, Z^-, \omega) = \begin{cases} z(a)s_a^+(z(a), Z^+, Z^-, \omega), & \text{if } z(a) \geq 0 \\ z(a)s_a^-(z(a), Z^+, Z^-, \omega), & \text{if } z(a) \leq 0 \end{cases} \quad P\text{-a.e.}$$

where, for each $z \in \mathcal{Z}$, $s_a^+(z(a), Z^+, Z^-, \cdot)$, $s_a^-(z(a), Z^+, Z^-, \cdot)$ are random variables whose mean is independent of $z(a)$.

Proof. The theorem is equivalent to the statement that ν-a.e. $E\phi_a(z(a),Z^+,Z^-)$ be linear in its first argument, over the positive and negative half-lines, but perhaps with different slopes.

Without loss of generality, we concentrate on the case $z(a) \geq 0$. Let $Z^+ > 0$,[1] $Z^- \leq 0$ be fixed and consider an integrable function z such that $\int z^+ d\nu = Z^+$, $\int z^- d\nu = Z^-$. Let $A^+ = \{a \in A | z(a) \geq 0\}$ and define X^- to be the negative of the mean expected realisation of supplies, that is,

$$X^- = -\int_{A \setminus A^+} E\phi_a(z(a),Z^+,Z^-)\nu(da).$$

Each $z(a) \geq 0$ can be written as $z(a) = \lambda(a)Z^+$, where λ is a nonnegative function on A^+ such that $\int_{A^+} \lambda(a)\nu(da) = 1$. From A(iii) one has

$\int_{A^+} E\phi_a(z(a),Z^+,Z^-)\nu(da) = X^-$. If $X^- = 0$, $E\phi_a(z(a),Z^+,Z^-) = 0$ ν-a.e. and linearity holds trivially. Thus, assume henceforth that $X^- > 0$ (which implies $Z^- < 0$). Then A(iv) and A(v) imply the existence of a continuous function $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$E\phi_a(\lambda(a)Z^+,Z^+,Z^-) = f(\lambda(a))X^- \quad \text{for } \nu\text{-almost all } a \in A^+ .$$

f has the property that for all λ such that $\int_{A^+} \lambda(a)\,\nu\,(da) = 1$, $\int_{A^+} (f \circ \lambda)(a)\,\nu\,(da) = 1$. Because of A(i), $f(0) = 0$.

It is immediate that $E\phi_a$ is linear in z(a) (in the positive half-line) if and only if f is the identity map. To see this, let $\mu = \frac{1}{\nu(A^+)}$. Then $\int_{A^+} \mu Z^+ d\nu = Z^+$. Therefore $\int_{A^+} E\phi_a(\mu Z^+,Z^+,Z^-)d\nu = X^-$ and $E\phi_a(\mu Z^+,Z^+,Z^-) = \mu X^-$ by A(iv) ν-a.e.. Consider $\lambda(a) = \alpha\mu$, $\alpha \geq 0$. Then, linearity of $E\phi_a$ implies that

$$f(\lambda(a)) = \frac{E\phi_a(\alpha\mu Z^+,Z^+,Z^-)}{X^-} = \alpha\frac{E\phi_a(\mu Z^+,Z^+,Z^-)}{X^-} = \alpha\mu = \lambda(a).$$

[1] If $Z^+ = 0$, $z^+(a) = 0$ ν-a.e., and the asserted identity holds trivially.

The problem of whether $E\phi_a$ is linear is therefore equivalent to the question whether every continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$ and $\int_{A^+} (f \circ \lambda)(a)\, \nu(da) = 1$ if $\lambda : A^+ \to \mathbb{R}, \int_{A^+} \lambda(a)\, \nu(da) = 1$ and $\nu(A^+) < 1$, has to be the identity map. (The case $\nu(A^+) = 1$ is excluded because of $Z^- < 0$). We proceed to show the second of the equivalent assertions.

First, consider the case $A^+ = A_1 + A_2$,[1] $\nu(A_1) = \nu(A_2) = \frac{1}{2\mu}$ and

$$\lambda(a) = (\mu - x)1_{A_1}(a) + (\mu + x)1_{A_2}(a),$$

where $\mu = \frac{1}{\nu(A^+)}$, 1_{A_i} is the indicator function on A_i and x any element of the interval $[-\mu,\mu]$. Then $\int_{A^+} \lambda(a)\, \nu(da) = 1$ and therefore $1 = \int_{A^+} f(\lambda(a))\, \nu(da) = \frac{1}{2\mu}[f(\mu - x) + f(\mu + x)]$ which yields

$$f(\mu - x) + f(\mu + x) = 2\mu \quad \text{for all } x \in [-\mu,\mu] \qquad (1)$$

For $x = 0$, it follows that $f(\mu) = \mu$.

Next let $A^+ = \sum_{i=1}^{n} A_i$, $n \geq 3$, $\nu(A_i) = \frac{1}{n\mu}$, $i = 1,\ldots,n$, and

$$\lambda(a) = (\mu - x_1)1_{A_1}(a) + \sum_{i=1}^{n-2} (\mu + x_i - x_{i+1})1_{A_i}(a) + (\mu + x_{n-1})1_{A_n}(a).$$

Then

$$f(\mu - x_1) + \sum_{i=1}^{n-2} f(\mu + x_i - x_{i+1}) + f(\mu + x_{n-1}) = n\mu \qquad (2)$$

for all $x_1,\ldots,x_{n-1}$ such that $\mu - x_1 \geq 0$, $\mu + x_i - x_{i+1} \geq 0$, $i = 1,\ldots,n-2$, and $\mu + x_{n-1} \geq 0$. For $n = 3$, $x_1 = x_2 = \mu$, (2) results in $f(0) + f(\mu) + f(2\mu) = 3\mu$ and hence $f(2\mu) = 2\mu$. For $n \geq 3$, $x_i = i\mu$, $i = 1,\ldots,n-1$, (2) yields $(n-1)f(0) + f(n\mu) = n\mu$, and therefore

$$f(n\mu) = n\mu \quad \text{for all} \quad n \in \mathbb{N} \qquad (3)$$

---

[1] $A^+ = A_1 + A_2$ means $A^+ = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$

Thus f is the identity on all points $n\mu$. It remains to show that f has this property for all rational multiples of μ too, from which the assertion will follow because of the continuity of f.

From (1), $f(\mu + x_1 - x_2) = 2\mu - f(\mu - x_1 + x_2)$, which together with (2) yields $f(\mu - x_1) + f(\mu + x_2) = \mu + f(\mu - x_1 + x_2)$. Set $x_2 = -x_1$ to receive $f(\mu + x_2) = \frac{1}{2}\mu + \frac{1}{2} f(\mu + 2x_2)$. Especially, for $x_2 = (n - 1 + r)\mu$, $n \geq 1$, $r \in \mathbb{R}_+$, one has

$$f((n + r)\mu) = \frac{1}{2}\mu + \frac{1}{2} f((2n - 1 + 2r)\mu). \tag{4}$$

Now consider the special case where $r = \frac{p}{2^q}$, $p,q \in \mathbb{N}$. Then, we claim that $f((n + \frac{p}{2^q})\mu) = (n + \frac{p}{2^q})\mu$ for all $n \geq 1$, $p,q \geq 0$. The proof is given by induction on q. For $q = 0$ the equality holds because of (3). For $q = 1$ one has by (4) and (3)

$$f((n + \tfrac{p}{2})\mu) = \frac{1}{2}\mu + \frac{1}{2} f((2n - 1 + p)\mu) = (n + \tfrac{p}{2})\mu.$$

Suppose the equality holds for $q - 1$. Then, using (4),

$$f((n + \frac{p}{2^q})\mu) = \frac{1}{2}\mu + \frac{1}{2} f((2n - 1 + \frac{p}{2^{q-1}})\mu)$$

$$= \frac{1}{2}\mu + \frac{1}{2}(2n - 1 + \frac{p}{2^{q-1}})\mu = (n + \frac{p}{2^q})\mu.$$

Finally, consider the case $n = 0$, $0 \leq p \leq 2^q$. Then by (1),

$$f(\frac{p}{2^q}\mu) = 2\mu - f((1 + (1 - \frac{p}{2^q}))\mu) = 2\mu - f((2 - \frac{p}{2^q})\mu) = 2\mu - (2 - \frac{p}{2^q})\mu,$$

which yields $f(\frac{p}{2^q}\mu) = \frac{p}{2^q}\mu$. Thus, $f((n + \frac{p}{2^q})\mu) = (n + \frac{p}{2^q})\mu$ for all integers $n,p,q \geq 0$ and continuity of f yields $f(\lambda) = \lambda$ for every real $\lambda \geq 0$. ||

So far, the short sided trading rule A(vi) was not used in the analysis. If it is imposed, then the theorem can be strengthened to the following

COROLLARY. If $I \geq 4$ and if the rationing mechanism ϕ satisfies A(0) to A(vi), then ν-a.e.

$$E\phi_a(z(a),Z^+,Z^-) = \begin{cases} z(a) \min\{-\frac{Z^-}{Z^+}, 1\}, & z(a) \geq 0 \\ z(a) \min\{-\frac{Z^+}{Z^-}, 1\}, & z(a) \leq 0 . \end{cases}$$

Proof. Consider $Z^+ > -Z^- \geq 0$. A(vi) implies that $\int_{A^+} E\phi_a(z(a),Z^+,Z^-)\, \nu(da) = -Z^-$. By A(iv), $E\phi_a(\mu Z^+,Z^+,Z^-) = -\mu Z^-$ ν-a.e., where $\mu = \frac{1}{\nu(A^+)}$. For $z(a) = \alpha\mu Z^+$, $\alpha \geq 0$, it follows from the theorem that ν-a.e.

$E\phi_a(z(a),Z^+,Z^-) = \alpha E\phi_a(\mu Z^+,Z^+,Z^-) = \alpha\mu(-Z^-) = z(a)(-\frac{Z^-}{Z^+})$.

If $Z^+ \leq -Z^-$, then by A(vi) $E\phi_a(z(a),Z^+,Z^-) = z(a)$ ν-a.e. and therefore $E\phi_a(z(a),Z^+,Z^-) = z(a) \min\{-\frac{Z^-}{Z^+},1\}$ ν-a.e., if $z(a) \geq 0$. The proof in the case $z(a) \leq 0$ is analogous. ||

4. EQUILIBRIUM

The linearity property of the functions ϕ_a which is asserted in Theorem 1, holds only, if a variation in the individual agent's acton does not influence the value of the means Z^+ and Z^-. This can be assumed to hold approximately in large economies, where each agent has negligible influence on market aggregates. Hence an adequate formal treatment is that of an atomless measure space of economic agents. Moreover, this framework removes non-convexities in the aggregate excess demand correspondence which can arise at the individual level and which would exclude the application of fixed point theorems in demonstrating the existence of equilibria. The latter has not yet been done for the class of rationing schemes considered in this paper. As far as Gale's existence theorems are concerned ([5], Theorem 3, p.333 and Theorem 4, p.335), they are very general in their treatment of stochastic rationing rules and for Green "it remains an open question as to whether his (Gale's) conditions can be satisfied by stochastic rationing schemes of the particular form studied in this paper" ([7], p.352). As it seems to us, Theorem 3 of Gale [5] applies (in the case of uncountably many agents), if some minor additional restrictions on Green's scheme are imposed. However, as these restrictions are not necessary in order to demonstrate the existence of equilibria for the type of rationing mechanisms considered by Green, we prefer to do without them and give a separate proof, patterned after that of Gale.

Before this can be done, some more elements of the model have to be specified. As before, $(A, \mathbf{A}, \nu)$ denotes the space of agents, where now A is assumed to be a separable metric space, $\mathbf{A}$ the Borel-σ-field and ν an atomless probability measure on $\mathbf{A}$. Each agent $a \in A$ has as endowment a non-negative vector $(e_{a1}, \ldots, e_{a\ell}) \in \mathbb{R}^\ell$ of tradeble goods and a stock of money $M_a > 0$. Then, the set of feasible net trades of agent a is

$$X_a = \{x \in \mathbb{R}^\ell \mid px \leq M_a \text{ and } x + e_a \geq 0\}$$

where $p = (p_1, \ldots, p_\ell) \in \mathbb{R}^\ell$ is a fixed strictly positive price vector. Clearly X_a is convex, compact, non-empty and contains the origin of $\mathbb{R}^{\ell}$. [1]

Agent a's preferences are represented by a von Neumann-Morgenstern utility $u_a : X_a \to \mathbb{R},\ x \mapsto u_a(x) = U_a(M_a - px, e_a + x)$. The allocation of goods to agents is thought of to be performed by a stochastic rationing mechanism $\phi = (\phi_1, \ldots, \phi_\ell)$, where each ϕ_h is a function from $Z \times \Omega$ to Z as described in A(0). Hence, for each agent a, with any proposed action $x \in \mathbb{R}^{\ell}$ and any vector of mean aggregate demands and supplies $(Z_1^+, Z_1^-, \ldots, Z_\ell^+, Z_\ell^-) = (r_1, \ldots, r_\ell) = r \in R \subset \mathbb{R}^{2\ell}$, where $R = (\mathbb{R}_+ \times \mathbb{R}_-)^{\ell}$, there is associated a distribution over final net trades $\mathcal{D}\phi_a(x,r) = \mathcal{D}(\phi_{a1}, \ldots, \phi_{a\ell})(x,r)$. For each $r \in R$, the assignment $a \to (e_a, M_a, u_a, \mathcal{D}\phi_a(\cdot, r))$ is assumed to be measurable with respect to some suitably constructed measurable space.[2]

To the rationing functions ϕ_{ah} correspond via Theorem 1 random functions s_{ah}^+, s_{ah}^-. They might be subjected to

A(vii) For ν-almost all $a \in A$, s_{ah}^+, s_{ah}^- are independent across distinct markets $h = 1, \ldots, \ell$.

Next, for each $h = 1, \ldots, \ell$, imagine a function

$$\gamma_h : \mathbb{R}_+ \times \mathbb{R}_- \to [0,1]^2 \qquad (5)$$
$$(Z_h^+, Z_h^-) \mapsto (\gamma_h^+, \gamma_h^-)$$

with $\gamma_h(0, Z_h^-) = (\gamma_h^+, 0)$, $\gamma_h(Z_h^+, 0) = (0, \gamma_h^-)$.

A(viii) For any $x_h \geq 0$, $Z_h^+ \geq 0$,

$$\text{co supp}\ \mathcal{D}\, s_{ah}^+(x_h, Z_h^+, Z_h^-) = \begin{cases} [\gamma_h^+, 1]\ , & Z_h^- < 0\ ^{3} \\ \{0\}\ , & Z_h^- = 0 \end{cases}.$$

---

[1] X_a does not necessarily contain the origin of $\mathbb{R}^{\ell}$ *in its interior*. To require this (as Gale does) would mean that the agent possesses a positive amount of each good $h = 1, \ldots \ell$. This is an economically unpleasant condition that can be dispensed with in the present context. However, this implies that Gale's locally interior-conditions ([5],Def.2.,p.325 and [6],p.363) are not fulfilled. This is one reason why Gale's theorem cannot be applied.

[2] One possible procedure is described in [5],pp.322 to 324.

[3] co supp μ denotes the convex hull of the support of a measure μ.

for any $x_h \leq 0, Z_h^- \leq 0$,

$$\text{co supp } \mathcal{D}\, s_{ah}^-(x_h, Z_h^+, Z_h^-) = \begin{cases} [\gamma_h^-, 1] \;, & Z_h^+ > 0 \\ \{0\} \;, & Z_h^+ = 0; \end{cases}$$

γ_h is continuous in (Z_h^+, Z_h^-) whenever $Z_h^+ > 0$ or $Z_h^- < 0$.

For an interpretation of these conditions, see Green [7, p.350]. In his description, $\gamma_h^+ = \gamma_h^- = 0$ for all Z_h^+, Z_h^-. Our more general version makes it possible that A(viii) can eventually be reconciled with the short sided rule A(vi). If, for example, $Z_h^+ < -Z_h^-$, then imposing A(vi) implies $\gamma_h^+ = 1$.

By A(vii), $\mathcal{D}\,\phi_a(x,r) = \bigotimes_{h=1}^{\ell} \mathcal{D}\,\phi_{ah}(x_h, r_h)$ ν-a.e. In particular, ν-a.e.

$$\text{supp } \mathcal{D}\,\phi_a(x,r) = \prod_{h=1}^{\ell} \text{supp } \mathcal{D}\,\phi_{ah}(x_h, r_h) \tag{6}$$

Let

$$\delta_h^+ = \begin{cases} 1 & \text{if } Z_h^+ > 0 \\ 0 & \text{if } Z_h^+ = 0 \end{cases} \;, \quad \delta_h^- = \begin{cases} 1 & \text{if } Z_h^- < 0 \\ 0 & \text{if } Z_h^- = 0 \end{cases} .$$

Because of A(viii) and Theorem 1, ν-a.e.

$$\text{co supp } \mathcal{D}\,\phi_{ah}(x_h, Z_h^+, Z_h^-) = \begin{cases} [\delta_h^- \gamma_h^+ x_h, \delta_h^- x_h] & \text{if } x_h \geq 0 \;^{1} \\ [\delta_h^+ x_h, \delta_h^+ \gamma_h^- x_h] & \text{if } x_h \leq 0 \end{cases} \tag{7}$$

Next define for each $a \in A$ the feasible set correspondence β_a from R into $\mathbb{R}^\ell$ by

$$\beta_a(r) = \{x \in \mathbb{R}^\ell \mid \text{supp } \mathcal{D}\,\phi_a(x,r) \subset X_a\}$$

---

[1] Although (6) and (7) imply that for all r such that $Z_h^+ \neq 0 \neq Z_h^-$ for all h = 1,...,ℓ, the convex hulls of supp $\mathcal{D}\,\phi_a(x^n, r)$ converge to the convex hull of supp $\mathcal{D}\,\phi_a(x,r)$ for all $(x^n) \to x$, the same does not necessarily hold for the supports themselves. The latter is required in Gale's treatment ([5],(6),p.323 and [6],(17),p.363).

Each agent is assumed to have rational expectations about the vector r and to know the distribution of his rationing function. So his problem is to maximize

$$v_a(x,r) = \int u_a \, d\,\mathcal{D}\phi_a(x,r)$$

on $\beta_a(r)$. This defines the optimum set relation ξ_a from R to $\mathbb{R}^\ell$,

$$\xi_a(r) = \{x \in \beta_a(r) \mid v_a(x,r) = \sup v_a(\beta_a(r),r)\}.$$

A complete list of effective demands and supplies is given by an element z of the set $\mathcal{Z}^\ell$ of all integrable functions $A \to \mathbb{R}^\ell$.[1] Each $z \in \mathcal{Z}^\ell$ results in a vector r of mean effective demands and supplies according to

$$r = \int_A (z_1^+(a), z_1^-(a), \ldots, z_\ell^+(a), z_\ell^-(a))\nu(da)$$

For notational facilitation define $F: \mathbb{R}^\ell \to R$ by $F: (x_1,\ldots,x_\ell) \mapsto (x_1^+, x_1^-, \ldots, x_\ell^+, x_\ell^-)$ and set $\mathcal{F}(z) = \int F(z(a)\nu(da)$ for each $z \in \mathcal{Z}^\ell$. One more assumption is needed:

A(ix) The function $\rho : A \to \mathbb{R}$, $\rho(a) = \sup\{\sum_{h=1}^{\ell} |x_h| \mid x \in X_a\}$ is ν-integrable.

For instance, if ν-a.e. $M_a \leq \bar{M}$ and $e_{ah} \leq \bar{e}_h$ for all h, then A(ix) holds. But A(ix) is weaker, for it allows the sets X_a to become unboundedly large. Finally, $z^* \in \mathcal{Z}^\ell$ is called an *equilibrium* if ν-a.e. $z^*(a) \in \xi_a(\mathcal{F}(z^*))$.

To show that such an equilibrium exists causes some difficulties because of the fact that the feasible set correspondence β_a is not everywhere well behaved. It is true that it is always non-empty, but if some components of the vector r are zero, $\beta_a(r)$ is neither continuous nor bounded at such an r.[2]

---

[1] Deviating from Gale, $\mathcal{Z}^\ell$ as defined here is not the set of *equivalence classes* of ν-almost everywhere identical integrable functions. Compare [5],p.326.

[2] Gale treats this problem in his more general framework by introducing the concepts of responsiveness ([5],Def.3,p.325), relative continuity and relative locally interiority ([6], p.363). In the present context, we are free not to define these notions, but the line of argument will be essentially the same as Gale's one.

Therefore, in demonstrating existence of equilibria, an approximation argument will be used.

To this end, $\beta_a(r)$ is investigated more closely. Let $R^+ \subset R$ be the set of all vectors the components of which are all non-zero and set for each $r \in R$

$$\Lambda(x,r) = \prod_{h=1}^{\ell} \begin{cases} [\gamma_h^+,1] & \text{if } x_h \geq 0 \\ [\gamma_h^-,1] & \text{if } x_h \leq 0 \end{cases}$$

Suppose $r \in R^+$. Then, by (6) and (7), ν-a.e. $x \in \beta_a(r)$ if and only if $(\alpha_1 x_1,\dots,\alpha_\ell x_\ell) \in X_a$ for all $(\alpha_1,\dots,\alpha_\ell) \in \Lambda(x,r)$. Therefore,

$$\beta_a(r) = \prod_{h=1}^{\ell} [-e_{ah}, \frac{M_a + \sum_{k \neq h} p_k \gamma_k^- e_{ak}}{p_h}] \cap X_a, \text{ for all } r \in R^+ \qquad (8)$$

If $r \notin R^+$, suppose for example that $Z_k^+ = 0$ and all other components of r are non-zero. Then, by (6) and (7), ν-a.e. $x \in \beta_a(r)$ if and only if

$(\alpha_1 x_1,\dots,\alpha_{k-1}x_{k-1},\alpha_k \max\{0,x_k\},\alpha_{k+1}x_{k+1},\dots,\alpha_\ell x_\ell)$ is an element of X_a, for all $(\alpha_1,\dots,\alpha_\ell) \in \Lambda(x,r)$. That is, x_k can become unboundedly small without implying that x does not belong to $\beta_a(r)$. To deal with this difficulty, define

$$\tilde{\beta}_a(r) = \prod_{h=1}^{\ell} [-\delta_h^+ e_{ah}, \delta_h^- \frac{M_a + \sum_{k \neq h} p_k \gamma_k^- e_{ak}}{p_h}] \cap X_a, \text{ for all } r \in R \qquad (9)$$

LEMMA. Assume A(0) to A(v) and A(vii), A(viii). Then, ν-a.e. for all $r \in R$, to any $x \in \beta_a(r)$ there exists $\tilde{x} \in \tilde{\beta}_a(r)$ such that $\mathcal{D}\phi_a(\tilde{x},r) = \mathcal{D}\phi_a(x,r)$.

For instance, if $Z_k^+ = 0$, then by A(viii) no negative value of x_k will be realized and $\tilde{x}_k = 0$ would do as well.

Proof. If $r \in R^+, \beta_a(r) = \tilde{\beta}_a(r)$. If $r \notin R^+$, assume w.l.o.g. that $Z_k^+ = 0, Z_k^-, Z_h^+, Z_h^- \neq 0$ for all $h \neq k$.

By definition, $\tilde{\beta}_a(r)$

$$= \prod_{h=1}^{k-1} [-e_{ah}, \frac{M_a + \sum_{j\neq h} p_j \bar{\gamma}_j e_{aj}}{p_h}] \times [0, \frac{M_a + \sum_{j\neq k} p_j \bar{\gamma}_j e_{aj}}{p_k}]$$

$$\times \prod_{h=k+1}^{\ell} [-e_{ah}, \frac{M_a + \sum_{j\neq h} p_j \bar{\gamma}_j e_{aj}}{p_h}] \quad \cap X_a$$

$= \{x \in \mathbb{R}^\ell \mid \alpha x \in X_a$ for all $\alpha \in \Lambda(x,r)$ and $x_k \geq 0\}$.

Now let x be an element of $\beta_a(r) \setminus \tilde{\beta}_a(r)$, that is $x_k < 0$ and $(\alpha_1 x_1, \ldots, \alpha_{k-1} x_{k-1}, 0, \alpha_{k+1} x_{k+1}, \ldots, \alpha_\ell x_\ell) \in X_a$ for all $(\alpha_1, \ldots, \alpha_\ell) \in \Lambda(x,r)$. Set $\tilde{x} = (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_\ell)$. Clearly $\tilde{x} \in \tilde{\beta}_a(r)$. Let $B_1, B_2, \ldots, B_\ell$ be elements of the σ-field $\mathcal{B}$. Then, ν-a.e.

$$\mathcal{D}\phi_a(x,r)(\prod_{h=1}^{\ell} B_h) = \prod_{h=1}^{\ell} \mathcal{D}\phi_{ah}(x_h, r_h)(B_h)$$

$$= \prod_{h=1}^{\ell} \mathcal{D}\phi_{ah}(\tilde{x}_h, r_h)(B_h) = \mathcal{D}\phi_a(\tilde{x}, r)(\prod_{h=1}^{\ell} B_h),$$

since $\tilde{x}_h = x_h$ for all $h \neq k$ and, by (6),

$\operatorname{supp} \mathcal{D}\phi_{ak}(x_k, 0, Z_k^-) = \{0\} = \operatorname{supp} \mathcal{D}\phi_{ak}(\tilde{x}_k, 0, Z_K^-)$. ||

We are now ready to prove

THEOREM 2. Under conditions A(0) to A(v) and A(vii) to A(ix), there exists an equilibrium.

Proof. There is a ν-nullset $A_0 \in \mathcal{A}$ such that the statements of the conditions A(i) to A(v), A(vii), A(viii) hold for all $a \in A \setminus A_0$. Set $(\tilde{A}, \tilde{\mathcal{A}}) = (A \setminus A_0, A \setminus A_0 \cap \mathcal{A})$ and let $A_1, A_2, \ldots$ be a sequence of subsets of $\tilde{A}$ such that $A_n \in \tilde{\mathcal{A}}$, $\nu(A_n) > 0$ for all n and $\nu(A_n) \to 0$. Define $\mathcal{Z}^+ \subset \mathcal{Z}^\ell$ by $\mathcal{Z}^+ = \mathcal{F}^{-1}(R^+)$ and fix $\bar{z} \in \mathcal{Z}^\ell$ such that $1_{A_n} \bar{z} \in \mathcal{Z}^+$ for all n. For each $(a,r) \in \tilde{A} \times R$ define

$$\tilde{\xi}_a(r) = \begin{cases} \xi_a(r) & \text{if } r \in R^+ \\ X_a & \text{if } r \notin R^+ \end{cases}$$

and set for each n

$$G_a^n(r) = \begin{cases} F(\tilde{\xi}_a(r)) & \text{if } a \notin A_n \\ \{F(\bar{z}(a))\} & \text{if } a \in A_n \end{cases}$$

and

$$\mathcal{G}^n(r) = \int_A G_a^n(r)\nu(da)$$

It has to be shown that the correspondence $\mathcal{G}^n: R \to R$ is well defined. To begin with, it makes sense to take the integral over A rather than over $\tilde{A}$, since both sets differ by a null-set only. Next, if $r \in R^+$, it is clear from (8) that, for all $a \in \tilde{A}$, $\beta_a(r)$ is compact and $\xi_a(r)$ is non-empty. As a consequence of the measurability of the assignment $a \mapsto (e_a, M_a, u_a, \mathcal{D}\phi_a(\cdot,r))$, $\xi_\cdot(r)$: $(\tilde{A}, \tilde{\mathcal{A}}) \to (|R^\ell, \mathcal{B}(|R^\ell))$ has a measurable graph ([8],Prop.1, p.59).[1] As the correspondence $a \mapsto X_a$ clearly has a measurable graph, for each $r \in R$, $\tilde{\xi}_\cdot(r)$ has a measurable graph. Hence it has a measurable selection $\tilde{g}$, say ([8], Theorem 1,p.54). As $\bar{z}$ is measurable, the function $g^n = 1_{\tilde{A} \setminus A_n}\tilde{g} + 1_{A_n}\bar{z}$ is measurable, and as F is continuous, $F \circ g^n$ is measurable, too. By A(ix), $F \circ g^n$ is integrable and hence $\mathcal{G}^n(r)$ is well defined for every n and every $r \in R$. Furthermore, A(ix) implies that $\mathcal{G}^n(r) \subset K$ for all n and all $r \in R$ for a suitably chosen compact convex set $K \subset |R^{2\ell}$.

As ν is non-atomic, $\mathcal{G}^n(r)$ is convex ([8],Theorem 3,p.62).

Next, for any $a \in \tilde{A}$ and any $n \in |N$, the correspondence $G_a^n: R \to R$ is closed. To see this, consider a sequence $(x^k, r^k) \to (x,r)$, where for each k, $x^k \in G_a^n(r^k)$. Then there exists a sequence (y^k) such that $F(y^k) = x^k$ and $y^k \in \tilde{\xi}_a(r)$ if $a \notin A_n$ and $y^k = \bar{z}(a)$ if $a \in A_n$, for all k. In the latter case $x = x^k = F(\bar{z}(a)) \in G_a^n(r)$. If $a \in \tilde{A} \setminus A_n$ and $r \notin R^+$, then $G_a^n(r) = F(X_a)$,

[1] As each agent a can be identified with his characteristics $(e_a, M_a, u_a, \mathcal{D}\phi_a)$, $\xi_a(r)$ can be written $\xi_a(r) = \xi(e_a, M_a, u_a, \mathcal{D}\phi(\cdot,r))$ $= \xi(\varepsilon(a,r))$, say, where ε: $(a,r) \mapsto (e_a, M_a, u_a, \mathcal{D}\phi_a(\cdot,r))$ is measurable, for any $r \in |R^{2\ell}$. The graph of ξ can be shown to be Borelian (see e.g. [5], Theorem 1, p.326).

and as $x^k \in F(X_a)$ for all k and $F(X_a)$ is closed, $x \in G_a^n(r)$ in this case, too. Finally, if $a \in \tilde{A} \setminus A_n$ and $r \in R^+$, assume w.l.o.g. that $r^k \in R^+$ for all k. As $y^k \in X_a$ for all k and X_a is compact, there is a subsequence $(y^{k_q}) \to y \in X_a$. Because of A(v) and (8), ξ_a is closed at $r \in R^+$. As $r^k \to r$, this implies $y \in \xi_a(r)$, and by continuity of F, $x \in G_a^n(r)$. Thus $G_a^n(r)$ is closed at r for any $r \in R$, any $a \in \tilde{A}$ and any n, and hence $\mathcal{G}^n$ is closed for any n ([8],Prop.8,p.73).

We have shown that the correspondence $\mathcal{G}^n$, restricted to the compact convex set K, satisfies the conditions of the Kakutani Fixed Point Theorem and hence has a fixed point r^n, say, for every n.

As $r^n \in \int_{A \setminus A_n} F(\tilde{\xi}_a(r^n)\nu(da) + \int_{A_n} F(\bar{z}(a)\nu(da)$, $\nu(A_n) > 0$ and $1_{A_n}\bar{z} \in \mathcal{Z}^+$, it follows that $r^n \in R^+$ for each n.

Hence $\tilde{\xi}_a(r^n) = \xi_a(r^n)$. Therefore, for each n, there exists a measurable selection z^n of $\xi_{\cdot}(r^n)$, such that

$$r^n = \int_{A \setminus A_n} F(z^n(a))\nu(da) + \int_{A_n} F(\bar{z}(a))\nu(da) \qquad (10)$$

As each r^n is an element of the compact set K, there is a subsequence, again denoted (r^n), that converges to some $r \in R$. It remains to show that $r \in \int_A F(\xi_a(r))\nu(da)$ (whether $r \in R$ or not).

Because of $\nu(A_n) \to 0$ and $\bar{z} \in \mathcal{Z}^\ell$, $\int_{A_n} F(\bar{z}(a))\nu(da) \to 0$. As $r^n \to r$, (10) and A(ix) imply $r = \lim_n \int_A F(z^n(a))\nu(da)$. Since mean convergence of a sequence implies the existence of a subsequence that converges almost everywhere, we can (r^n), r and (z^n) w.l.o.g.

assume to meet, for every $h = 1,\dots,\ell$,

$$r_h = (0,0) \text{ implies } z_h^n(a) \to 0 \quad \nu\text{-a.e.} \tag{11}$$

By A(ix),

$\lim_n \int_A F(z^n(a))\nu(da) \in \int_A Ls(\{F(z^n(a))\})\nu(da)$[1] ([8], Theorem 6, p.68).

To show that $Ls(\{F(z^n(a))\}) \subset F(\xi_a(r))$ ν-a.e., we first assert that $Ls(\{z^n(a)\}) \subset \xi_a(r)$, for ν-almost all $a \in \tilde{A}$. To see this, suppose w.l.o.g. that $z^n(a) \to x \in \mathbb{R}^\ell$. If h is such that $r_h \neq (0,0)$, then by A(viii) $\text{supp}\,\mathcal{D}\phi_{ah}(x_h,r_h) \subset Ls\,(\text{co supp}\,\mathcal{D}\phi_{ah}(z_h^n(a),r_h^n))$. If $r_h = (0,0)$, then (11) implies $\text{supp}\,\mathcal{D}\phi_{ah}(z_h^n(a),r_h^n) \to \{0\}$ $= \text{supp}\,\mathcal{D}\phi_{ah}(x_h,r_h)$. Therefore by (6) $\text{supp}\,\mathcal{D}\phi_a(x,r)$ $\subset Ls(\text{co supp}\,\mathcal{D}\phi_a(z^n(a),r^n)) \subset X_a$ and hence x lies in $\beta_a(r)$. Suppose $x \notin \xi_a(r)$. Then there is $\hat{x} \in \beta_a(r)$ such that $v_a(\hat{x},r) > v_a(x,r)$. By the Lemma there is $\tilde{x} \in \tilde{\beta}_a(r)$ with the property that $\mathcal{D}\phi_a(\tilde{x},r) = \mathcal{D}\phi_a(\hat{x},r)$, hence $v_a(\tilde{x},r) = v_a(\hat{x},r)$. By (8) and (9), there is a sequence $(\tilde{x}^n)$ such that $\tilde{x}^n \in \beta_a(r^n)$ for all n and $\tilde{x}^n \to \tilde{x}$. If r is such that Z_h^+ or Z_h^- is non-zero for all $h = 1,\dots,\ell$, then A(v) implies that $v_a(\tilde{x}^n,r^n) \to v_a(\tilde{x},r)$ as well as $v_a(z^n(a),r^n) \to v_a(x,r)$. Therefore, for large n, $v_a(z^n(a),r^n) < v_a(\tilde{x}^n,r^n)$, contradicting $z^n(a) \in \xi_a(r^n)$. If there exist some h such that $Z_h^+ = Z_h^- = 0$, then v_a is not necessarily continuous at (x,r). But, for these components h, $\tilde{x}_h = 0$ by (9) and we can set $\tilde{x}_h^n = 0$ for all n, too. Therefore, also in this case $v_a(\tilde{x}^n,r^n) \to v_a(\tilde{x},r)$. Further, by (11) $r_h = (0,0)$ implies $Ls\,\text{supp}\,\mathcal{D}\phi_{ah}(z_h^n(a),r_h^n) \to \{0\} = \text{supp}\,\mathcal{D}\phi_{ah}(x_h,r_h)$, and as $\phi_{ah}(z_h^n(a),r_h^n) \to \phi_{ah}(x_h,r_h)$ for all h such that $r_h \neq (0,0)$, again $v_a(z^n(a),r^n) \to v_a(x,r)$, by A(vii). Therefore again, for large n, $v_a(z^n(a),r^n) < v_a(\tilde{x}^n,r^n)$. This proves $Ls(\{z^n(a)\}) \subset \xi_a(r)$ ν-a.e.

---

[1] $Ls(B^n)$ denotes the topological Limes superior of the sequence (B^n). See [8], p. 15.

Finally, to see that $Ls(\{F(z^n(a))\}) \subset F(\xi_a(r))$ ν-a.e., let $a \in \tilde{\tilde{A}}$, $y \in Ls(\{F(z^n(a))\})$ and assume w.l.o.g. that $F(z^n(a)) \to y$. As $z^n(a) \in X_a$, X_a compact, there is a subsequence $(z^{n_q}(a))$ that converges to some $x \in X_a$. By construction, x is an element of $Ls(\{z^n(a)\})$, hence of $\xi_a(r)$. Since F is continuous, $y = F(x)$ and so $y \in F(\xi_a(r))$.

As $Ls(\{F(z^n(a))\}) \subset F(\xi_a(r))$ for all $a \in \tilde{\tilde{A}}$, $r \in \int_A F(\xi_a(r))\nu(da)$. Therefore there exists a selection z^* of $\xi_{\cdot}(r)$ such that $\mathfrak{F}(z^*) = r$, and z^* is the required equilibrium. $\|$

Theorem 2 guarantees that an equilibrium exists, but it does not say whether such an equilibrium is non-trivial for it is clear that $z = 0$ is always an equilibrium. On the other hand, an equilibrium z is *non-trivial* if and only if $\mathfrak{F}(z) \neq 0$. Gale deals with this problem in his Theorem 4 ([5],p.335). There, he states conditions under which a non-trivial equilibrium is asserted to exist. It is easy to see that not all of these conditions are fulfilled by the type of rationing mechanisms dealt with in this paper. It suffices to look at Gale's condition (iii). Applied to our situation, it states that there exists a measurable non-null subset E of A, such that for each $a \in E$

C) If $r \neq 0$, then $\text{supp}\, \mathcal{D}\phi_a(x,r) = \{0\}$ implies $x = 0$.

This is clearly a too strong assumption for consider the case that $Z_h^+ > 0$ and $Z_h^- = 0$ for some good h. Then $r \neq 0$, so the premise of C) is met. But for $x \in \mathbb{R}^\ell$ such that $x_h > 0$ and $x_k = 0$ for $k \neq h$, ν-a.e. $\text{supp}\, \mathcal{D}\phi_a(x,r) = \{0\}$, by A(viii), though $x \neq 0$.

Intuitively it is clear, that in order that a non-trivial equilibrium is possible, there must be some agents who are willing to trade with each other, that is, there must be at least one good, for which there are a non-null set of suppliers and a non-null set of demanders.[1] This is formalized in the following assumption, where we employ the function introduced in (5).

A(x) There is a good h, $1 \leq h \leq \ell$, such that

a) $Z_h^+ \leq -Z_h^- \Rightarrow \gamma_h^+(Z_h^+, Z_h^-) \geq \varepsilon_h^+ > 0$

$Z_h^+ \geq -Z_h^- \Rightarrow \gamma_h^-(Z_h^+, Z_h^-) \geq \varepsilon_h^- > 0$

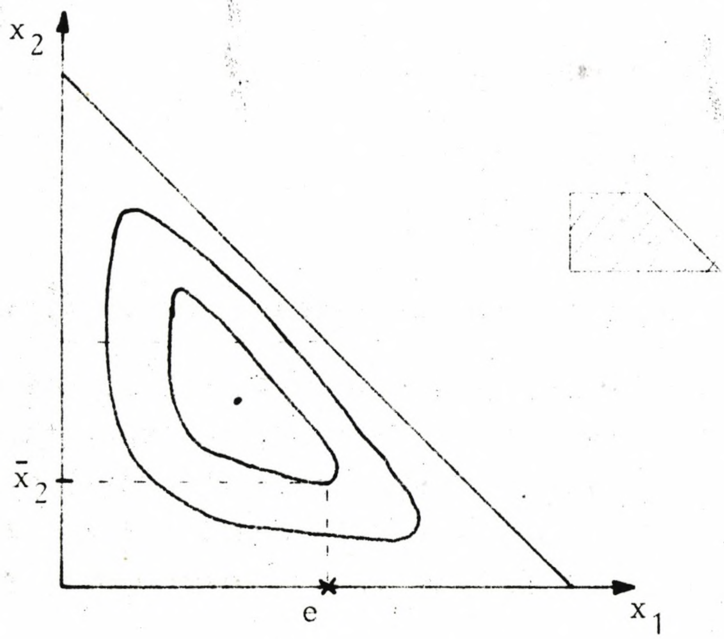
where $\varepsilon_h^+, \varepsilon_h^-$ are fixed numbers independent of Z_h^+ and Z_h^-;

b) there exist measurable non-null subsets E^+, E^- of A such that $a \in E^+$ ($a \in E^-$, resp.) implies that there is $\bar{x}_h(a) > 0$ ($x_h(\bar{a}) < 0$, resp.) such that $\bar{x}(a) = (0,\ldots,0,\bar{x}_h(a),0,\ldots,0) \in \beta_a(r)$ for all $r \in R$ and $u_a(\lambda_1\bar{x}(a)) \geq u_a(\lambda_2\bar{x}(a)) > u_a(0)$ whenever $0 < \lambda_2 \leq \lambda_1 \leq 1$.

A(x)a) states that an agent who finds himself on the short side of the market, can be sure to realize a positive fraction of his intended trade, if not all of it. This is in some sense a requirement on the efficiency of the market rationing mechanism, for if for example $\phi_h = 0$ identically, then A(x)a) can clearly not be fulfilled. On the other hand, if the short sided rule A(vi) prevails, then $\varepsilon_h^+ = \varepsilon_h^- = 1$.

[1] From this point of view, Gale's requirement of the existence of a non-null set of agents with a "positive trade point" ([5],p.334) in order to ensure existence of non-trivial equilibria seems questionable. For example, if all agents are identical, that is all have the same endowments and the same tastes, then there is no good that would be demanded as well as supplied, although all agents might have a positive trade point. But then, under rational expectations the trivial equilibrium is in fact the only one.

A(x)b) is illustrated in Fig. 3 where the agent is a de-



$= \bigcap_{r \in R} \beta(r)$

Fig. 3

mander of good 2, who meets the requirement that $u(0,\cdot)$ be monotonicly increasing between zero and $\bar{x}_2$. This situation appears to be not too artificial but fulfilled for a wide class of preference structures. For example, if the underlying utility function $U(M - p_1x_1 - p_2x_2, e_1 + x_1, e_2 + x_2)$ is Cobb-Douglas, then the indifference curves are of the shape indicated in Fig.3. The requirement that on some market there appear demanders as well as suppliers means that there are differences among agents concerning preferences or endowments.

THEOREM 3. Under conditions A(O) to A(v) and A(vii) to A(x), there exists a non-trivial equilibrium.

Proof. We refer to the proof of Theorem 2. For each $r \in R$, set $v_a^*(r) = \sup v_a(\beta_a(r), r)$ and let (r^n) be the sequence converging to r in R that was constructed in the proof of Theorem 2. Consider some $a \in E^{+} \cup E^{-}$ and suppose that $\lim_n \inf v^*_a(r^n) = u_a(0)$. Then there is a subsequence, again denoted (r^n), such that $v^*_a(r^n) \to u_a(0)$. As r^n is an element of R^{+} for each n, there is again a subsequence (r^{n_q}) such that either $0 < Z_h^{n_q +} \leq -Z_h^{n_q -}$ or $Z_h^{n_q +} \geq -Z_h^{n_q -} > 0$ for all q. W.l.o.g. assume the first case and a that $a \in E^{+}$.

Then, by A(x)

$$v^*_a(r^{n_q}) \geq v_a(\bar{x}(a), r^{n_q})$$

$$= \int_{\mathbb{R}^{\ell}} u_a(y_1,\ldots,y_{\ell}) \mathop{\times}_{k=1}^{\ell} \mathcal{D}\,\phi_{ak}(\bar{x}_k(a), r^{n_q})(d(y_1,\ldots,y_{\ell}))$$

$$= \int_{\mathbb{R}} u_a(0,\ldots,0,y_h,0,\ldots 0)\, \mathcal{D}\,\phi_{ah}(\bar{x}_h(a), r^{n_q})(dy_h)$$

$$\geq u_a(0,\ldots,0,\varepsilon_h^{+}\bar{x}_h(a), 0,\ldots 0) > u_a(0)$$

for all q. This contradicts $v^*_a(r^n) \to u_a(0)$. Thus, $\xi_a(r^n)$ is bounded away from the origin as well as $F(\xi_a(r^n))$ and therefore, as $\nu(E^{+}) > 0$, also $\mathcal{E}^n(r^n)$ is bounded away from the origin. Since $r^n \in \mathcal{E}^n(r^n)$, $r^n \not\to 0$, hence $r \neq 0$. Thus the equilibrium z referring to r, which exists by Theorem 2, is such that $\mathcal{F}(z) \neq 0$ and hence non-trivial. ||

5. CONCLUDING REMARKS

Among the assumptions A(0) to A(vi) characterizing the class of rationing mechanisms dealt with in this paper, A(i) to A(iii) and A(v), A(vi) are standard in the literature. Whereas A(iv) also appears to be natural in this kind of stochastic model, A(0) expresses a special feature of the mechanism, namely that an agent's actual transaction depends on his own action and on the aggregate values of demand and supply only. We have justified A(0) as a first step in investigating mechanisms of the form $x_{ah} = \phi_{ah}(z_{ah}, r_h, \omega)$, namely when $r_h = (\int z_h^+ d\nu, \int z_h^- d\nu)$, which implies by Theorem 1 that ϕ_{ah} must be manipulable. This suggests that the result may be generalized by weakening A(0), that is, by allowing r_h to be defined in ways differing from the one above. A logical connection between the dimension of r_h and the requirement that ϕ_{ah} be manipulable, may possibly be revealed. This is particularly of interest because practically all of the prevailing theories of equilibrium under quantity rationing rely on the use of non-manipulable rationing mechanisms. A clarification of the eventual need of manipulable schemes for reasons of consistency could therefore help to inquire into the validity of the disequilibrium literature.

As shown in this paper, stochastic manipulable schemes are compatible with the existence of non-trivial equilibria, at least in the framework of a continuum economy, the latter representing an idealization of a large finite economy. The continuum framework might be further sustained by first stating approximate equilibria for finite economies and then showing that these approximate equilibria approach an equilibrium of the continuum economy as the economy becomes large. In order to demonstrate the existence of equilibria in finite economies, one could impose as a further condition that the random functions s_{ah} introduced in Theorem 1 are independent of z_{ah}. Then the concavity of the expected utility function can be ensured by the concavity of the underlying von Neumann-Morgenstern utility function, and the proof of the existence theorem can be adapted to the finite case.

REFERENCES

1. J.P. BENASSY, Neokeynesian disequilibrium theory in a monetary economy, *Review of Economic Studies* 42 (1975), 503-523.

2. J.P. BENASSY, On quantity signals and the foundations of effective demand theory, *Scandinavian Journal of Economics* (1977), 147-168.

3. R. CLOWER, The Keynesian counterrevolution: A theoretical appraisal, *in* "The Theory of Interest Rates" (F.H. Hahn and F.R.P. Brechling, eds.), pp. 104-115, Macmillan & Co., London, 1965.

4. J. DREZE, Existence of an equilibrium under price regidity and quantity rationing, *International Economic Review* 16 (1975), 301-320.

5. D. GALE, Large economies with trading uncertainty, *Review of Economic Studies* 46 (1979), 319-338.

6. D. GALE, Large economies with trading uncertainty: A correction, *Review of Economic Studies* 48 (1981), 363-364.

7. J. GREEN, On the theory of effective demand, *Economic Journal* 90 (1980), 341-353.

8. W. HILDENBRAND, "Core and Equilibria of Large Economies", Princeton University Press, Princeton, 1974.

9. L. SVENSSON, Effective demand and stochastic rationing, *Review of Economic Studies* 47 (1980), 339-355.

10. G. WEINRICH, "On the Linearity of Stochastic Rationing Mechansisms", mimeographed, 1980.

11. G. WEINRICH, On the theory of effective demand, *Economic Journal* 92 (1982), 174-175.

EUI ECONOMICS DEPARTMENT WORKING PAPERS

No. 1 :	Jacques PELKMANS	The European Community and the Newly Industrialized Countries.
No. 3 :	Aldo RUSTICHINI	Seasonality in Eurodollar Interest Rates.
No. 9 :	Manfred E. STREIT	Information Processing in Futures Markets - An Essay on the Adequacy of an Abstraction.
No. 10 :	Kumaraswamy VELUPILLAI	When Workers Save and Invest: Some Kaldorian Dynamics.
No. 11 :	Kumaraswamy VELUPILLAI	A Neo-Cambridge Model of Income Distribution and Unemployment.
No. 12 :	Kumaraswamy VELUPILLAI Guglielmo CHIODI	On Lindahl's Theory of Distribution.
No. 22 :	Don PATINKIN	Paul A. Samuelson and Monetary Theory.
No. 23 :	Marcello DE CECCO	Inflation and Structural Change in the Euro-Dollar Market.
No. 24 :	Marcello DE CECCO	The Vicious/Virtuous Circle Debate in the '20s and the '70s.
No. 25 :	Manfred E. STREIT	Modelling, Managing and Monitoring Futures Trading: Frontiers of Analytical Inquiry.
No. 26 :	Domenico Mario NUTI	Economic Crisis in Eastern Europe - Prospects and Repercussions.
No. 34. :	Jean-Paul FITOUSSI	Modern Macroeconomic Theory An Overview.
No. 35 :	Richard M. GOODWIN Kumaraswamy VELUPILLAI	Economic Systems and their Regulation.
No. 46 :	Alessandra VENTURINI	Is the Bargaining Theory Still an Effective Framework of Analysis for Strike Patterns in Europe?.
No. 47 :	Richard A. GOODWIN	Schumpeter: the Man I Knew.
No. 48 :	J.P. FITOUSSI Daniel SZPIRO	Politique de l'Emploi et Réduction de la Durée du Travail.
No. 56 :	Berc RUSTEM Kumaraswamy VELUPILLAI	Preferences in Policy Optimization and Optimal Economic Policy.
No. 64 :	Marcello DE CECCO	Italian Monetary Policy in the 1980s.
No. 65 :	Giampaolo ROSSINI	Intraindustry Trade in Two Areas: Some Aspects of Trade Within and Outside a Custom Union.
No. 66 :	Wolfgang GEBAUER	Euromarkets and Monetary Control: The Deutschemark Case
No. 67 :	Gerd WEINRICH	On the Theory of Effective Demand under Stochastic Rationing
No. 68 :	Saul ESTRIN Derek C. JONES	The Effects of Worker Participation upon Productivity in French Producer Cooperatives
No. 69 :	Berc RUSTEM Kumaraswamy VELUPILLAI	On the Formalization of Political Preferences: A Contribution to the Frischian Scheme