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Abstract

It has long been established in the literature that the set of pure strategy Nash equilibria of any binary game of strategic complements among a set N of players can be seen as a lattice on the set of all subsets of N under the partial order defined by the set inclusion relation (subset of). If the game happens to be strict in the sense that players are never indifferent among outcomes, then the resulting lattice of equilibria satisfies a straightforward sparseness condition. In this paper, we show that, in fact, this class of games expresses all such lattices. In particular, we prove that any lattice under set inclusion on the power set of N satisfying this sparseness condition is the set of pure strategy Nash equilibria of some binary game of strategic complements with no indifference. This fact then suggests an interesting way of studying some subclasses of games of strategic complements: By attempting to characterize the subcollections of lattices that each of these classes is able to express. In the second part of the paper we study subclasses of binary games of strategic complements with no indifference, defined by restrictions that capture particular social influence structures: 1) simple games, 2) nested games, 3) hierarchical games 4) clan-like games, and 5) graphical games of thresholds.

Keywords

Peer Effects, Social Networks, Implementation

JEL Classification Codes: C70, C72, C00, D70

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1 Introduction

A game theoretic model of social influence is a game in which one of the primitives is some notion of the *social structure* defining the way in which the behaviors of the players are intertwined. These models have a number of applications in economics and other social sciences including understanding information diffusion, the adoption of new technologies, the production of locally non-excludable goods, fads and fashion, and peer effects in a variety of social environments such as schools, gangs and cults. It is also the case that, in some of these applications, the agents just decide over two alternatives: whether to adopt an innovation or not, which version of a product to buy or whether to join or not join a group. As an example, consider the task of modeling the choice of which newspaper to read in a small community. Suppose that the only data to inform the modeling exercise is the collection of the different social circles to which a person belongs and an understanding of whether each individual is a conventionalist or an agitator. While agitators tend to prefer the least common option in their social circles, conventionalists always align themselves with the majority. This example showcases three common distinguishing features of situations that are apt to be described by discrete models of social influence: 1) the modeler has very good data about the way in which behaviors among players are intertwined, but no solid grounds for making precise statements about payoffs. 2) the choice spaces of the agents are *strongly* discrete in the sense that acceptable continuous approximations are unlikely. 3) the data specifies an underlying *social structure* governing the way in which choices are entangled and many questions of interest involve understanding how behavior may change with variations in this social structure. These characteristics have three major implications over the game theoretic frameworks which are likely to be useful. First, best response correspondences are a more immediate primitive for the models than individual payoff functions. Second, mixed strategies and mixed strategy equilibria are of limited use to the extent that payoffs cannot be credibly specified. This deems many of the standard game theoretic tools for studying existence and analyzing the properties of equilibria inappropriate, as they often rely on the continuity of best response correspondences. And finally, the formulation should facilitate the analysis of the connections between the properties of the set of equilibria and social structure.

This paper studies the properties of the set of pure strategy Nash equilibria of one of the simplest classes of discrete games of social influence: complete information, simultaneous move two action games of strategic complements with no indifference, under a variety of assumptions on the underlying social structure that entangles the behaviors of the players. A nice feature of this class of games is that the best response correspondence of a player can readily be seen as an *influence structure*: a specification of the subsets of other players that induce him to take the higher action whenever they take it. Although most situations of interest involving approximately complete information and choices between just two options have a significant dynamic component, which is absent in our setting, our study is significant for the following reason: simple simultaneous move games along with the concept of pure strategy Nash equilibria capture a basic necessary condition of equilibrium behavior in dynamic settings that meet two conditions. The first is that agents can frequently and repeatedly revise their choices, and the second is that the agents disregard their own influence over the social group.[1] Under these conditions, the more refined models based upon the complicating features of a specific motivating situation would select their predictions from the set of equilibria of the simple games studied in this paper.

The paper makes two contributions. First, it characterizes the set of equilibria of strict binary games of strategic complements, and second, it provides some insights into the structure of the sets of equilibria of games in some subclasses of interest. It has long been established in the literature that the set of pure strategy Nash equilibria of any binary game of strategic complements among N players can be seen as a lattice on the set of all subsets of N under the partial order defined by the set inclusion relation ($\subseteq$) (see Topkis (1988) [8] for an in-depth treatment of strategic complementarities). Moreover, if the game happens to be strict, in the sense that players are never indifferent among outcomes, then the resulting lattice of equilibria satisfies a straightforward *sparseness condition*. In the first part of the paper, we show that, in fact, this class of games expresses all such lattices. In particular we prove that any lattice under set inclusion on 2^N satisfying this *sparseness condition* is the set of equilibria of some binary game of strategic complements with no indifference. Secondly we study the properties of the set of equilibria of the subclasses of games of strategic complements that arise by considering additional restrictions on *influence structures* motivated by features of some social environments in which binary games of strategic complements might be applicable. Among these, we focus on *influence structures* corresponding to games of strategic complements played by agents that live on graphs (graphical games of strategic complements).

More generally, the paper contributes to the growing literature on graphical games of strategic complements, which have had a wide array of applications in economics in the recent years (see for example Jackson and Yariv (2007) [5], Galeotti *et. al.* (2010) [4], Calvo-Armengol *et. al.* (2009) [1]) by contextualizing this class of games within the broader class of complete information, simultaneous move two action games of strategic complements. One potential advantage of the networks approach over the standard peer effects literature[2] lies in the fact

[1] These two requirements are the defining characteristics of many situations in small communities in which social pressures are important.

[2] Manski (1993) [7] and Lee (2004) [3] provide instances of models relying on the standard "average

that it allows us to capture the way in which global patterns of behavior are affected by the local neighborhood structure. A *most suitable* concept of equilibrium for these settings does not seem to exist. The "right" definition is often suggested by the particular features of the application, for instance the depth of the agents' rationality, the nuances of communication, or the ability of groups of players to form coalitions. At least in part however, the choice of equilibrium notion is also driven by mathematical considerations: *i.e.* existence, uniqueness, tractability. Finally, one key consideration is whether the predictions stemming from a given definition choice seem to correspond well to what is actually observed, or, at the very least whether there is a good explanation for why they fail to do so. It turns out that, in most cases, the set of pure strategy Nash equilibria is large and complex, and while finding one equilibrium is easy, navigating the whole set is in general quite difficult: there are not many straightforward structural relations between different equilibria. This multiplicity and complexity, however, is often consistent with what one intuitively expects in many motivating real world situations, and, therefore, despite its intractability and lack of parsimony, it may well be the adequate definition of equilibrium (or maybe precisely for those reasons). Moreover, we can expect the sets of equilibria implied by other definitions to be subsets of the set of Nash equilibria, since the latter, in some sense, captures the minimal requirements that one would demand from any other notion. It is therefore useful to understand the mathematical properties of the sets of pure strategy Nash equilibria.

The paper is divided into five sections. Section 2 introduces the definitions that are used throughout. Section 3 characterizes the sets of equilibria of general complete information, simultaneous move two action games of strategic complements with no indifference, to which we refer throughout as *increasing games of influence*. The main result of this section provides conditions on the structure of a lattice, which are necessary and sufficient for it to be the set of equilibria of some game in our class. Using the groundwork laid out in Section 3, Sections 4 and 5 study subclasses of increasing games of social influence that arise from imposing restrictions on the best response behavior of the agents stemming from the social structures that they reflect: 1) *simple* games in which each agent is influenced by supersets of only one core group of other agents; 2) *Nnested* games in which each agent can only be influenced by agents who are in turn influenced by other agents that influence him directly; 3) *hierarchical* games in which the agents can be embedded in a hierarchy that respects the influence structure; 4) *clan-like* games, which are *simple* games with the additional property that the influence is always mutual and therefore partitions the community; and 5) *graphical games of thresholds*, games in which the way in which agents influence each other can be represented by a network. Most of the analysis in these sections relies on the analytic

interactions" assumption whereby agents are partitioned into groups, within which every agent interacts with every other agent.

approach and insights of the literature of games of strategic complements in lattices (see Topkis (1988) [8]). We build the analysis of each of the classes of games that we consider around the question: What collections of lattices are *expressible* as the set of equilibria of some game in the class? Section 6 concludes by studying the algorithmic complexity of the problem of deciding whether a given lattice is the set of equilibria of some increasing game of social influence. The analysis of this section is based upon (Echenique 2007 [2]), which provides the most efficient algorithm available for finding all the equilibria of general finite games of strategic complements. Kempe and Tardos (2003) [6] provide algorithms and complexity bounds for the related problem of identifying targets of "infection" in order to maximize influence.

2 General Games of Social Influence

A *peer influence structure* on a set of agents $N = \{1, 2, 3, ..., n\}$ is a collection of functions $\mathcal{I}_i : 2^N \to \{0, 1\}$, one for each $i \in N$, satisfying the property that $\forall i \in N$, $\mathcal{I}_i(x) = \mathcal{I}_i(x \cup \{i\})$.[3] For a given set, $x \in 2^N$, $\mathcal{I}_i(x) = 1$ is interpreted as meaning that, when all agents $j \in x$ set $a_j = 1$, then i strictly prefers to set $a_i = 1$. Similarly, $\mathcal{I}_i(x) = 0$ is interpreted as meaning that, when all agents in $j \in x$ set $a_j = 1$, then i strictly prefers to set $a_i = 0$. The idea behind the only requirement in the definition of influence structure is to preclude self reference in action: that is, an agent is influenced by what other agents do, but not by what he himself does.[4] For the purpose of making comparisons between influence structures, we rely on the partial order $\leq$ on the collection of all influence structures on N defined by letting $\mathcal{I}' \leq \mathcal{I}$ if $\mathcal{I}'_i(x) \leq \mathcal{I}_i(x)$ $\forall x, i$. Note that, if $\mathcal{I}$ and $\mathcal{I}'$ are influence structures, then so is $\mathcal{I} \vee \mathcal{I}'$ defined by $(\mathcal{I} \vee \mathcal{I}')_i(x) = max\{\mathcal{I}_i(x), \mathcal{I}'_i(x)\}$ and $\mathcal{I} \wedge \mathcal{I}'$ defined by $(\mathcal{I} \wedge \mathcal{I}')_i(x) = min\{\mathcal{I}_i(x), \mathcal{I}'_i(x)\}$. In this way, the collection of influence structures in itself forms a lattice.

A game of social influence induced by a peer influence structure $\mathcal{I}$ is a simultaneous move game $\Gamma_{\mathcal{I}} = \langle N, \{\{0, 1\}\}_{i=1}^n, \{R_i\}\rangle$ in which each agent has strategies 0 and 1, and some strict preferences on 2^N induced[5] by the influence structure $\mathcal{I}$. This paper is only concerned with strategic behavior relying on pure strategies, and, from this perspective, it suffices to specify the ordinal properties of preferences; that is, we do not require utility functions. A pure strategy *Nash equilibrium* of a game of influence is a strategy profile $\sigma_1 \times \sigma_2 \times, ..., \times \sigma_n$, where

[3] 2^N denotes the power set of N.

[4] Formally, influence structures are best response correspondences (functions) in games for n players with two actions, in which players are never indifferent. The additional terminology that we introduce is useful as the paper studies statements about "social structure restrictions" related to how individual players are affected by other groups of players.

[5] The induced preferences for each agent are unique as orderings on 2^N.

$\sigma_i \in \{0,1\}$ has the property that $\forall i \in \{j \in N : \sigma_j = 1\}$, $\mathcal{I}_i(\{j \in N : \sigma_j = 1\}) = 1$ and $\forall i \notin \{j \in N : \sigma_j = 1\}$, $\mathcal{I}_i(\{j \in N : \sigma_j = 1\}) = 0$. Note that any strategy profile in a game of social influence can be described succinctly by the set $\{i \in N : \sigma_i = 1\}$. In this way, and for convenience, throughout the paper we will be thinking of strategy profiles, and in particular of Nash equilibria as subsets of N. We will denote the set of Nash equilibria of game $\Gamma_{\mathcal{I}}$ as $NE(\Gamma_{\mathcal{I}})$. An influence structure $\mathcal{I}$ is said to *express* a collection of subsets of N, $\mathcal{C} \subseteq 2^N$ if $\mathcal{C} = NE(\Gamma_{\mathcal{I}})$. Similarly, we say that a collection $\mathcal{C} \subseteq 2^N$ is *expressible* by a given family of influence structures $\{\mathcal{I}\}$ if some structure $\mathcal{I}^*$ in the family exists so that $NE(\Gamma_{\mathcal{I}^*}) = \mathcal{C}$. When talking about a collection of sets, we think of it as endowed with the partial order induced by the weak set containment relation ($\subseteq$).

A nonempty collection of subsets of N, $\mathcal{C} \subseteq 2N$ is expressible by some influence structure if and only if $x \in \mathcal{C}$ and $i \in x \Longrightarrow x \setminus \{i\} \notin \mathcal{L}$. *(1)*

This condition immediately follows from the definition of social influence structure. If $\mathcal{I}$ is an influence structure that expresses $\mathcal{C}$, it must be the case that $\forall x \in \mathcal{C}$, $\mathcal{I}_i(x) = 1$ for all $i \in x$ and $\mathcal{I}_i(x) = 0$ for all $i \notin x$. If for some $x \in \mathcal{C}$, it is the case that $x \setminus \{i\} \in \mathcal{C}$ for some $i \in x$ we would have $\mathcal{I}_i(x) = 1$ and $\mathcal{I}_j(x \setminus \{i\}) = 1$, which violates the definition of social influence structure. Given any collection $\mathcal{C}$ satisfying the condition in (1) we can define $\mathcal{I}$ that expresses it as follows: begin by letting $\mathcal{I}_i(x) = 0$ $\forall x$ and $\forall i$, and then for each $x \in \mathcal{C}$ and i in x set $\mathcal{I}_i(x) = 1$. If $\varnothing \in \mathcal{C}$, we are done; otherwise, let x be a minimal element of $\mathcal{C}$, and set $\mathcal{I}_i(x') = 1$ $\forall i \in x$ and $x' \subseteq x$. Given that $\mathcal{C}$ satisfies the condition in (1), this collection of functions will be a well-defined influence structure. It is clear that there are many influence structures which express a given collection of subsets of N, $\mathcal{C} \subseteq 2^N$. Note that, if $\varnothing \notin \mathcal{C}$, then the influence structure that we just constructed is *minimal* in the sense that there does not exist $\mathcal{I}'$ expressing $\mathcal{C}$, such that $\mathcal{I}' \prec \mathcal{I}$.

The following is a list of properties which are useful to characterize different kinds of social influence structures. We have chosen them because they reflect some features of various applications rather than for their performance as axioms.[6]

(P1) Increasing, if $\forall i \in N$, $x \subseteq x'$ and $\mathcal{I}_i(x) = 1 \Rightarrow \mathcal{I}_i(x') = 1$. The elements of the set $B_i = minimal\{x \subseteq N : \mathcal{I}_i(x) = 1\}$ are called the bases for action of i.

(P1-1) Simple, if for each $i \in N$, the collection B_i contains only one set, which we denote b_i.

(P1-11) Nested, if it is simple and has the property that $j \in b_i \Rightarrow b_j \subseteq b_i$.

[6] As can be seen, they are not independent. It would not be difficult to reconstruct these properties in terms of an independent set of axioms.

(P1-12) Clan-like, if it is simple and has the property that $j \in b_i \Rightarrow i \in b_j$. This is equivalent to saying that the collection $\{b_i \cup \{i\} : i \in N\}$ forms a partition of N[7].

(P1-2) Hierarchical, if the agents can be partitioned into a hierarchy $\mathcal{H} = \{H_1, H_2, H_3, ..., H_m\}$ such that if i belongs to level $H_{h(i)} \in \mathcal{H}$ then $\exists b_i^1, b_i^2, ..., b_i^r \subseteq \bigcup_{k \geq h(i)} H_k$ such that $\mathcal{I}_i(x) = 1 \Rightarrow b_i^s \subseteq x,\ s \in \{1, 2, ..., r\}$.

(P1-3) Admits a network representation, if, for each player i there exist weights $\{w_{ij}\}_{j=1}^n$, $w_{ij} \geq 0$, and a threshold $t_i \geq 0$ such that $\mathcal{I}_i(x) = 1$ if, and only if, $\sum_{j \in x} w_{ij} \geq t_i$.

3 Increasing Influence Structures

One very useful result that applies to our setting from the general theory of games of strategic complements is the following:

If $\mathcal{I}$ is an increasing influence structure (P1), then $NE(\Gamma_{\mathcal{I}})$ forms a lattice (with respect to the set containment ($\subseteq$) partial order). *(2)*

We present a proof of this statement, as it showcases an argument that frequently arises when working within this class of games of influence.
Consider $x, x' \in NE(\Gamma_{\mathcal{I}})$. Let $x' \vee x = min\{y : y \supseteq x \cup x'\} =$. To see that this minimum exists, suppose it does not. Then, it must be the case that the set $\{y : y \supseteq x$ and $y \supseteq x'\}$ has at least two minimal elements z and z'. Then some set w, $x \cup x' \subseteq w \subseteq z \cap z'$ must be an equilibrium: if $i \notin z \cap z'$ then either $i \notin z$ or $i \notin z'$. Without loss of generality, suppose that $i \in z$. Then, as z is an equilibrium, it must be the case that $\mathcal{I}_i(z) = 0$ and because $z' \cap z \subseteq z$ and $\mathcal{I}$ is increasing, we must also have $\mathcal{I}_i(z \cap z') = 0$. On the other hand, if $i \in x \cup x'$ then $\mathcal{I}_i(x \cup x') = 1$, because i belongs to either x or x', $\mathcal{I}$ is increasing and both x and x' are equilibria. So, starting at $z \cap z'$, we can iteratively remove the elements that prefer not to be active from the set. Due to the fact that $\mathcal{I}$ is increasing, it will be the case that elements not belonging to the set prefer not to be active at each iteration. And, at some point, before reaching $x \cup x'$ or at $x \cup x'$, it will also be the case that all elements in the set will prefer to be in the set. This set is the equilibrium W that we were looking for, and as $w \subset z$ and $w \subset z'$, its existence contradicts the minimality of these two elements. We conclude that the set $\{y : y \supseteq x$ and $y \supseteq x'\}$ only has one minimal element; the minimum we were looking for. With an analogous argument we can show the existence of the *meet* $x \wedge x' = max\{y : y \subseteq x \cap x'\}$. ∎

[7] Formally, b_i and $b_i \setminus \{i\}$ are always valid bases for $\{i\}$; $b_i \cup \{i\}$ is the same whichever convention is being used.

Given a lattice $\mathcal{L}$, how can we know whether it is expressible by an increasing influence structure? Providing necessary and sufficient conditions for expressibility is much harder than in *(1)*. It is not difficult to see what is needed in order to produce an influence structure which expresses some superset $\mathcal{L}'$ of $\mathcal{L}$. Making sure that $\mathcal{L}' \setminus \mathcal{L} = \varnothing$ takes some care, as by trying to rule the extra equilibria from the game, one might destroy some wanted equilibria or give rise to new ones. So, to begin with, it is not clear whether a simple characterization is possible. As the following proposition shows, thanks to the lattice structure of the set we can get rid of unwanted equilibria one by one using a simple rule.

PROPOSITION **1** *A lattice $\mathcal{L} \in 2^N$ is expressible by an increasing influence structure $\mathcal{I}$ if and only if:*

(SC) ***Sparseness Condition*** $i \in y \in \mathcal{L} \Rightarrow \forall x \in \mathcal{L}$ *such that* $y \setminus \{i\} \subseteq x$*, we have* $i \in x$*.*

Proof of Proposition 1: *Necessity*: Suppose condition (1) does not hold. That is, assume that there exist $x \in \mathcal{L}$ and $i \in N$ such that $x \in \mathcal{L}$, $i \notin x$ and $i \in y$ for some $y \in \mathcal{L}$ such that $y \setminus \{i\} \subseteq x$. But then any peer influence structure inducing the game satisfies $\mathcal{I}_i(y \setminus \{i\}) = \mathcal{I}_i(y) = 1$ and $\mathcal{I}_i(x) = 0$, so $\mathcal{I}$ is not increasing.

Sufficiency: We begin by setting $\mathcal{I}_i(x) = 0$ for all i and $x \in 2^N$ and modify them in the following steps.

Induce all required equilibria:
S1) For each $x \in \mathcal{L}$ and $i \in x$, let $\mathcal{I}_i(x') = 1$ for $x' \supseteq x \setminus \{i\}$.

Note that, in step S1), we construct an increasing peer influence structure, and, by virtue of (SC), each $x \in \mathcal{L}$ is an equilibrium of the game that it induces. Specifically note that, by construction, if $i \in x \in \mathcal{L}$, then $\mathcal{I}_i(x) = 1$. Moreover, if $i \notin x \in \mathcal{L}$, then $\mathcal{I}_i(x) = 0$. To see this, note that, to have $\mathcal{I}_i(x) = 1$, it needs to be the case that $y \setminus \{i\} \subset x$ for some $y \in \mathcal{L}$ such that $i \in y$, but by (SC) this would imply $i \in x$, which, by assumption, is false. The issue is that the game based upon the influence structure $\mathcal{I}$, that we have so far, may have other equilibria. To see this in a simple example consider the influence structure that results from the application of S1) to the lattice satisfying (SC) depicted on the left in Figure 2, and note that it actually expresses the lattice shown on the right.

As generally seems to be the case when attempting to construct games that express a given set of equilibria, it is easy to guarantee that the members of the set are all equilibria of the

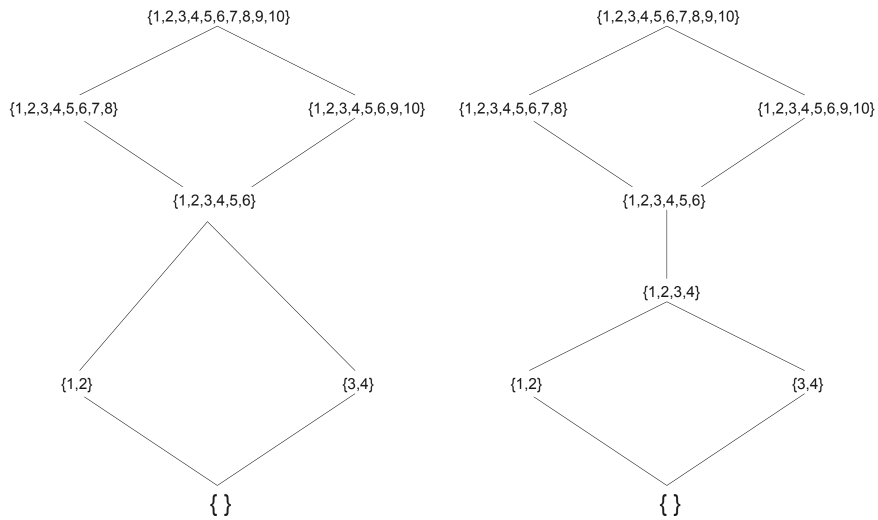

Figure 1: The figure on the right is the lattice expressed by the game induced by the influence structure constructed by applying S1) on the lattice on the left.

game, but a lot harder to guarantee that those are the only equilibria. S2) removes the unwanted equilibria, and this can be done thanks to the fact that the set in question is a lattice. So, let $\mathcal{I}^0$ denote the influence structure constructed in S1) and $\Gamma_{\mathcal{I}^0}$ the game that it induces.

Remove all unwanted equilibria:

Suppose that we have a game $\Gamma_{\mathcal{I}}$ such that every element x of $\mathcal{L}$ is an equilibrium of the game. Let $y \notin \mathcal{L}$. Then either:

2a) $\exists j \in y$ such that $\forall x \in \mathcal{L}$ such that $x \subset y$, we have that $j \notin x$; or
2b) $\exists j \notin y$ such that $j \in x$ $\forall x \in \mathcal{L}$ such that $y \subset x$.

To see that at least one of these must hold, suppose that 2a) fails, and for each $j \in y$, let $m_y \in \mathcal{L}$, $m_j \subset y$ and $j \in m_j$. As $\mathcal{L}$ is a lattice, $y \notin \mathcal{L}$ and $\bigvee_{j \in y} m_j \supseteq \bigcup_{j \in y} m_j = y$, we have that $\bigvee_{j \in y} m_j \supset y$. Consider some $w \in \bigvee_{j \in y} m_j \setminus y$, and suppose $z \in \mathcal{L}$ and $y \subset z$. Then, $z \supseteq \bigvee_{j \in y} m_j$ (by definition of $\bigvee_{j \in y} m_j$) and therefore $w \in z$ as required by 2b).

Having constructed $\mathcal{I}^0$ as above, we will show by induction that, for all $t \geq 0$, given an increasing influence structure $\mathcal{I}^t$, such that $\mathcal{L} \subseteq NE(\Gamma_{\mathcal{I}^t})$, if $NE(\Gamma_{\mathcal{I}^t}) \cap \mathcal{L}^c \neq \varnothing$, then we can produce an increasing influence structure $\mathcal{I}^{t+1}$ such that $\mathcal{L} \subseteq NE(\Gamma_{\mathcal{I}^{t+1}})$ and such that $NE(\Gamma_{\mathcal{I}^{t+1}}) \cap \mathcal{L}^c$ is a strict subset of $NE(\Gamma_{\mathcal{I}^t}) \cap \mathcal{L}^c$.

The base case:

Suppose that $\mathcal{L} \subseteq NE(\Gamma_{\mathcal{I}^0})$, $NE(\Gamma_{\mathcal{I}^0}) \cap \mathcal{L}^c \neq \varnothing$ and pick $x \in NE(\Gamma_{\mathcal{I}^0}) \cap \mathcal{L}^c$. Since it must be the case that $\mathcal{I}_i^0(x) = 1 \ \forall i \in x$, the condition in $2a$) does not hold (by virtue of the way in which $\mathcal{I}^0$ was constructed in S1)). So 2b) implies that, for each $i \in x$ there exists $y_i \in \mathcal{L}$ such that $i \in y_i \subset x$. Therefore, $x = \bigcup_{i \in x} y_i \subset \bigvee_{i \in x} y_i$, where $\bigvee_{i \in x} y_i$ is the join in $\mathcal{L}$ of the equilibria y_i (one for $i \in x$).[8] So, for each $y \in 2^N$ and $j \in N$, let:

$$I_j^1(y) = \begin{cases} 1 & : y \supseteq x \text{ and } j \in (\bigvee_{i \in x} y_i) \setminus x \\ I_j^0(y) & : otherwise \end{cases}$$

We will now show that $\mathcal{L} \subseteq NE(\Gamma_{\mathcal{I}^1})$. The definition of I^1 only makes adjustments to $\mathcal{I}^0$ on sets $y \supseteq x$. So, if $y \in \mathcal{L}$ and $y \nsupseteq x$, then $y \in NE(\Gamma_{\mathcal{I}^1})$ (as $y \in NE(\Gamma_{\mathcal{I}^0})$). So, consider some $y \in \mathcal{L}$, such that $y \supset x$. Any such y must contain $\bigvee_{i \in x} y_i$, as any set containing $x = \bigcup_{i \in x} y_i$, and the only possible difference between $\mathcal{I}^1$ and $\mathcal{I}^0$ can be on components ($\mathcal{I}_j^1$ vs. $\mathcal{I}_j^0$) involving elements in $\bigvee_{i \in x} y_i$. As these elements also belong to y which is an equilibrium of $\Gamma_{\mathcal{I}^0}$, the images $\mathcal{I}_j^0(y)$ were already 1 to begin with. So actually no change was really made to the function in these sets.

We now show that $NE(\Gamma_{\mathcal{I}^1}) \cap \mathcal{L}^c \subset NE(\Gamma_{\mathcal{I}^0}) \cap \mathcal{L}^c$. By construction of $\mathcal{I}^1$, $x \notin NE(\Gamma_{\mathcal{I}^1})$, and we will show that no other element $y \in \mathcal{L}^c$ that was not in $NE(\Gamma_{\mathcal{I}^0})$ can now be part of $NE(\Gamma_{\mathcal{I}^1})$. As above, we only need to be concerned with sets $y \supset x$. If $y \supseteq \bigvee_{i \in x} y_i$, then, as seen above, $\mathcal{I}^1$ and $\mathcal{I}^0$ are identical, so the only possible occurrence of a a new equilibrium in $\Gamma(\mathcal{I}^1)$ must involve sets y, such that $x \subset y, y \notin \bigvee_{i \in x} y_i)$. Note, however, that any such set must lack some element in $j \in (\bigvee_{i \in x} y_i) \setminus x$, and for such an element $\mathcal{I}_j^1(x) = 1$, and therefore y cannot be an equilibrium of $\Gamma_{\mathcal{I}^1}$.

The inductive step:

Now make the inductive hypothesis that this procedure can be consistently reproduced $k > 1$ times, on each occasion $m < k$ taking the set $NE(\Gamma_{\mathcal{I}^{m-1}}) \cap \mathcal{L}^c$ to a strict subset $NE(\Gamma_{\mathcal{I}^m}) \cap \mathcal{L}^c$. If $NE(\Gamma_{\mathcal{I}^k}) \cap \mathcal{L}^c = \varnothing$, then we are done. Otherwise, suppose that there exists $x \in NE(\Gamma_{\mathcal{I}^k}) \cap \mathcal{L}^c$. It must be the case that $\mathcal{I}_i^k(x) = 1 \ \forall i \in x$. However it is not immediate that condition 2a) does

[8] Note that the join in $NE(\Gamma_{\mathcal{I}^0})$ is x, which must be different from the join in $\mathcal{L}$, as by assumption $x \in \mathcal{L}^c$.

not hold (which was a key part of our argument in the base case), as it might be that for some $j \in x$ $\mathcal{I}_j^0(x) = 0$, but that, at some iteration, $m < k$, there was an update from $\mathcal{I}_j^{m-1}(x) = 0$ to $\mathcal{I}_j^m(x) = 1$. So, suppose that this is the case. Denoting by x^m the element of $NE(\Gamma_{\mathcal{I}^{m-1}}) \cap \mathcal{L}^c$ removed in iteration m, it must have been that $x^m \subset x \notin \bigvee_{i \in x^m} y_i^m$ and $j \in (\bigvee_{i \in x^m} y_i^m) \setminus x^m$. So there must have been some $l \in ((\bigvee_{i \in x^m} y_i^m) \setminus x^m) \setminus x$. But then $\mathcal{I}_l^m(x) = 1$ and $l \notin x$, which means that $x \notin NE(\Gamma_{\mathcal{I}^m}) \cap \mathcal{L}^c$, which, in turn, implies, by virtue of the inductive hypothesis that $x \notin NE(\Gamma_{\mathcal{I}^k}) \cap \mathcal{L}^c$, a contradiction. So, we can conclude that $\forall i \in x$ $\mathcal{I}_j^0(x) = 0$. But this, in turn, means that $2a)$ does not hold for $x \notin \mathcal{L}$. From this point on, we can proceed just as in the base case to construct I^{k+1} such that $\mathcal{L} \subseteq NE(\Gamma_{\mathcal{I}^{k+1}})$ and $NE(\Gamma_{\mathcal{I}^{k+1}}) \cap \mathcal{L}^c \subset NE(\Gamma_{\mathcal{I}^k}) \cap \mathcal{L}^c$.

So, now we are ready to state the last step in our construction:

S2) Let $\mathcal{I}^0$ be the increasing influence structure constructed in S1). If $NE(\Gamma_{\mathcal{I}^0}) \cap \mathcal{L}^c = \varnothing$ then we are done. Otherwise, we use the algorithm depicted above to produce a sequence of increasing influence structures $\{\mathcal{I}^m\}_{m=1}^T$, where for each $m < T$ $\mathcal{L} \subset NE(\Gamma_{\mathcal{I}^m})$ and $\varnothing \subset NE(\Gamma_{\mathcal{I}^m}) \cap \mathcal{L}^c \subset NE(\Gamma_{\mathcal{I}^{m-1}}) \cap \mathcal{L}^c$; and for $m = T$, $\mathcal{L} = NE(\Gamma_{\mathcal{I}^m})$. ▮

One nice property of increasing influence structures that also follows immediately from the theory of games of strategic complements, is that if in structure $\mathcal{I}'$ every agent is at least as willing to take action under any pattern of activity than under $\mathcal{I}$, then the equilibria of $\Gamma_{\mathcal{I}'}$ are larger than the equilibria of $\mathcal{I}$ in the following sense.

If for all i and x, $\mathcal{I}_i(x) = 1$ implies $\mathcal{I}'_i(x) = 1$, then $y \in NE(\Gamma_{\mathcal{I}})$ implies $\exists y' \in NE(\Gamma_{\mathcal{I}'})$ such that $y' \supset y$. (3)

As in the case of (2) above, the proof exhibits a way of reasoning that was extensively used in the proof of Proposition 1 and which we will continue to use throughout this paper.

Proof: Consider some $y \in NE(\Gamma_{\mathcal{I}})$, then $\mathcal{I}_i(y) = 1$ for all $i \in y$ which by assumption implies that $\mathcal{I}'_i(y) = 1$ for all $i \in y$. This means that the only reason that it may be the case that y does not belong to $NE(\Gamma_{\mathcal{I}'})$ is that, for some elements, $j \notin y$ we have that $\mathcal{I}'_i(y) = 1$. Then, let $y_0 = y$ and $y_{i+1} = y \cup \{j \notin y : \mathcal{I}'_i(y) = 1\}$. As the number of agents is finite and $\mathcal{I}'$ is increasing for some i^*, we will have $y_{i^*+1} = y^*$. By construction, $y^* \supseteq y_0 = y$ and y^* is an equilibrium of $\Gamma_{\mathcal{I}'}$. ▮

4 Simple, Nested, Clan-like and Hierarchical Influence Structures

Simple influence structures are useful to capture situations in which each agent i is influenced by a single group of agents b_i: that is, he prefers to be active if, and only if, every agent in b_i is active. The following definitions will help us to characterize these influence structures. Given a set $s \subset N$, its *completion*, which we denote by $\widehat{s}$, is the smallest superset of s containing all the bases of its elements. That is:

$$\widehat{s} = min\{a \supseteq s : a \supseteq b_j, \forall j \in a\}$$

Notice that, given any increasing influence structure, we can characterize the mapping $\mathcal{I}_i$ in terms of a finite number of bases for action $B_i = \{b_i\}_{k=1}^{m}$ for each agent $i \in N$. And, given a set s, we can therefore think of the collection of minimal supersets of s which contain *at least one base* for each of its elements. In the case of simple influence structures, however, there is always a unique minimal element, -this is the reason for which the above definition makes sense. An agent i only finds it optimal to be active if all the agents in his base for action b_i are active, and each of these, in turn, requires all the agents in his base for action to be active and so on. This means that any equilibrium to which an agent i belongs must contain a base for action for each of its elements, and therefore the set of equilibria of the game induced by an increasing influence structure $\mathcal{I}$ must be a subset of the set of equilibria of the game induced by the influence structure generated by all the possible completions of the bases of $\mathcal{I}$.

EXAMPLE **1** *Completion of bases when the influence structure is not simple.*

Consider the increasing influence structure on $\{1, 2, 3, 4, 5\}$ in which agent 1 finds it optimal to be active if either a superset of $\{2, 3\}$ or a superset of $\{4, 5\}$ are active. Assume also that each of the sets $\{2, 3, \}$ and $\{4, 5\}$ makes activity optimal for its elements. Then $\{\{1, 2, 3\}, \{1, 4, 5\}\}$ is the collection of supersets of $\{1\}$ with the property that they contain bases for all its elements. This collection does not have a minimum element. It is clear that this cannot happen when the influence structure in question is simple. $\diamond$

In the case of simple influence structures, we can take advantage of the fact that the base for action of each element has a unique completion, and make a much stronger statement.

LEMMA **1** *Let $\widehat{\mathcal{I}}$ be the influence structure, defined by the collection of completions $\{\widehat{b_i}\}$ of the bases for action of some simple influence structure $\mathcal{I}$, in which $b_i \neq \varnothing$ for each $i \in N$. Then $NE(\Gamma_{\mathcal{I}}) = NE(\Gamma_{\widehat{\mathcal{I}}})$.*

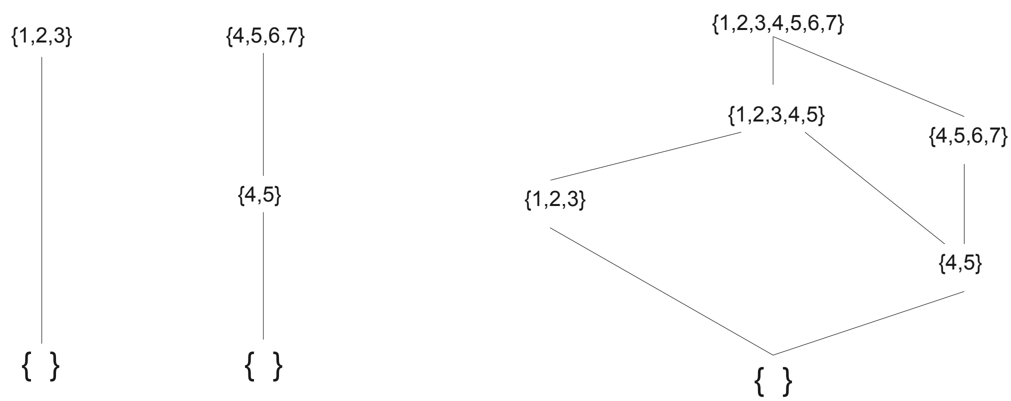

Figure 2: The figure on the right represents the product of the lattices on the left.

Proof of Lemma 1: Suppose that $x \in NE(\Gamma_{\mathcal{I}})$ and consider some $i \in x$. Then $b_i \subseteq x$ and $b_j \subseteq x$ for each $j \in b_i$. This implies that $\widehat{b_i} \subseteq x$ and therefore $\widehat{\mathcal{I}}_i(x) = 1$. Suppose that $i \notin x$, then $b_i \notin x$. As $\widehat{b_i} \supseteq b_i$ this implies that $\widehat{b_i} \notin x$ and $\widehat{\mathcal{I}}_i(x) = 0$. We therefore have that $x \in NE(\Gamma_{\widehat{\mathcal{I}}})$. Now suppose that $x \notin NE(\Gamma_{\mathcal{I}})$. This can be the case for one of two reasons, either (1) for some $i \in x$ $\mathcal{I}_i(x) = 0$ or (2) for some $i \notin x$ $\mathcal{I}_i(x) = 1$. If (1) is the case, then it must be that $b_i \notin x$, which in turn implies $\widehat{b_i} \notin x$ and $\widehat{\mathcal{I}}_i(x) = 0$. If (1) does not hold yet (2) is the case, then we have that $b_i \subseteq x$ and $b_j \subseteq x$ for all $j \in x$ (as the statement in (1) does not hold for any $j \in x$). But, then $\widehat{b_i} \subseteq x$ by definition of completion and therefore $\widehat{\mathcal{I}}_i(x) = 1$. So, in both cases, we can conclude that $x \notin NE(\Gamma_{\widehat{\mathcal{I}}})$. ▮

This simple lemma is very powerful as it allows us to characterize the set of equilibria of games induced by simple influence structures easily. Given an increasing influence structure $\mathcal{I}$ and some set s let $\uparrow_{\mathcal{I}} s$ denote the smallest set larger than s such that, conditional on everyone in s being active, no agent not in s would prefer to be active. That is, $\uparrow_{\mathcal{I}} s = min\{x \supseteq s : i \notin x \Rightarrow \mathcal{I}_i(x) = 0\}$. Note that $\uparrow_{\mathcal{I}} S$ is well-defined for any set s, given that we are just focusing on increasing influence structures. When C is a collection of sets, then $\uparrow_{\mathcal{I}} C = \{\uparrow_{\mathcal{I}} s : s \in C\}$.

Given a collection of lattices $\mathcal{S} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_k\}$, their product, denoted $\bigtimes_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$, is the collection of all possible unions of elements of $\mathcal{S}$. That is:

$$\bigtimes_{\mathcal{L} \in \mathcal{S}} \mathcal{L} = \{x : x = \bigcup_{j=i}^{k} y_j \text{ where } y_j \in \mathcal{L}_j\}$$

Proposition **2** *Let $\mathcal{I}$ be a simple influence structure, such that b_i is nonempty for each $i \in N$. Then, $NE(\Gamma_{\mathcal{I}}) = \uparrow (\bigtimes_{\mathcal{L} \in \mathcal{S}} \mathcal{L})$ where $\mathcal{S} = \{\{\varnothing, \widehat{b_i} \cup \{i\}\} : i \in N\}$. It does not matter whether* *the operator ↑ is applied with respect to $\mathcal{I}$ or $\widehat{\mathcal{I}}$, so, for simplicity, we omit the subscript.*

The following Lemma, regarding the completions of bases is very useful for the proof of Proposition 2, as it allows us exploit Lemma 1 very profitably.

LEMMA **2** *Let $\mathcal{I}$ be a simple influence structure. Then its completion $\widehat{\mathcal{I}}$ is nested. That is $j \in \widehat{b_i} \Rightarrow \widehat{b_j} \subseteq \widehat{b_i}$.*

Proof of Lemma 2: Suppose that $j \in \widehat{b_i}$, then, by definition of completion, it must be the case that $b_i \subseteq \widehat{b_j}$, which, in turn, implies that $b_k \subseteq b_j$ for all $k \in b_i$. As $\widehat{b_i}$ is the smallest set with this property, we must have that $\widehat{b_i} \subseteq \widehat{b_k}$. ▮

Proof of Proposition 2: Let $\widehat{\mathcal{I}}$ be defined as in Lemma 1. By the Lemma, we just need to show $NE(\Gamma_{\widehat{\mathcal{I}}}) = \uparrow (\bigtimes_{\mathcal{L} \in \mathcal{S}} \mathcal{L})$. Suppose that $x \in NE(\Gamma_{\widehat{\mathcal{I}}})$ and $i \in x$. Then, it must be the case that $\widehat{b_i} \cup \{i\} \subseteq x$. We therefore have that $\bigcup_{i \in x} (\widehat{b_i} \cup \{i\}) \subseteq x$ and this, in turn, implies $\uparrow (\bigcup_{i \in x} (\widehat{b_i} \cup \{i\})) \subseteq x$, given that x is a Nash equilibrium of $\Gamma_{\mathcal{I}}$[9] . We therefore have that $x = \uparrow (\bigcup_{i \in x} (\widehat{b_i} \cup \{i\}))$ and therefore $x \in \uparrow (\bigtimes_{\mathcal{L} \in \mathcal{S}} \mathcal{L})$.
Now suppose that $x \in \uparrow (\bigtimes_{\mathcal{L} \in \mathcal{S}} \mathcal{L})$. Then, $x = \uparrow (\bigcup_{i \in x} (\widehat{b_i} \cup \{i\}))$ by virtue of the fact that by Lemma 2 if $k \in \widehat{b_i}$ then $\widehat{b_k} \subseteq \widehat{b_i}$. So, suppose that $i \in x$. Then, either $i \in \bigcup_{j \in x} (\widehat{b_j} \cup \{j\})$, in which case we automatically have $\widehat{b_i} \subseteq x$ and $\widehat{\mathcal{I}}_i(x) = 1$. Otherwise, $i \in (\uparrow \bigcup_{j \in x} (\widehat{b_j} \cup \{j\}) \setminus (\bigcup_{j \in x} (\widehat{b_j} \cup \{j\}))$ and by definition of the ↑ operator, it must be the case that $\widehat{\mathcal{I}}_i(x) = 1$. If, on the other hand, $i \notin x$, then it must be the case that $\widehat{\mathcal{I}}_i(x) = 0$ by definition of ↑. We can conclude that $x \in NE(\Gamma_{\widehat{\mathcal{I}}})$. ▮

The key element in the proof of Proposition 2 is the fact that the completions of the bases are nested as shown by Lemma 2. So a slightly more general version of the Proposition holds for all nested influence structures, in which the relevant collection for the product lattice is $\mathcal{S} = \{\{\varnothing, \widehat{b_i} \cup \{i\}\} : i \in N\}$. Example 1 shows that the key property for Lemma 1 and therefore for Proposition 2 is not that $\mathcal{I}$ is simple, but rather that unique completions of bases for action of each $i \in N$ can be defined. Given some increasing influence structure $\mathcal{I}$ the *bases* for action of i can be defined as $B_i = minimal\{x : \mathcal{I}_i(x) = 1\}$, and, given some set s its set of completions by $C(s) = minimal\{a \supseteq s : a \supseteq b_j \text{ for some } b_j \in N_j, \forall j \in a\}$. We can then easily generalize Lemma 1 and Proposition 2 to the class of influence structures with the property that $C(b_i)$ is a singleton for each $b_i \in B_i$ and $C(b_i) = C(b_i')$ whenever $b_i, b_i' \in B_i$ for all elements i. These influence structures are *essentially simple* in the sense that, regarding their sets of equilibria, they can be equivalently represented by simple influence structures.

[9] Note that we could be taking ↑ with respect to $\mathcal{I}$ or $\mathcal{I}'$, as $\mathcal{I}_i'(x) \to \mathcal{I}_i(x)$ for all i.

The more general versions of the results above would be more cumbersome notationally, but would not be any more enlightening.

Among simple influence structures, clan-like influence structures, in which the bases for action partition the set of agents into equivalence classes have many applications, as they represent well the ways in which families, closely-knit groups of friends, mafias or clans act. As we see below, the set of equilibria of clan-like influence structures have a very simple structure.

PROPOSITION **3** *If $\mathcal{I}$ is clan-like, then $NE(\Gamma_{\mathcal{I}}) = \bigtimes_{\mathcal{L}\in\mathcal{B}} \mathcal{L}$ where $\mathcal{B} = \{\{\varnothing, b_i \cup \{i\}\} : i \in N\}$.*

Proof of Proposition 3: Let $x \in \bigtimes_{\mathcal{L}\in\mathcal{B}} \mathcal{L}$, then by definition of clan-like influence structure $\mathcal{I}_i(x) = 1\ \forall i \in x$. If $i \notin x$ then $b_i \notin x$, as the sets $b_j \cup \{j\}$ partition N, and therefore $\mathcal{I}_i(x) = 0$. So we have that $x \in NE(\Gamma_{\mathcal{I}})$. If $x \notin \bigtimes_{\mathcal{L}\in\mathcal{B}} \mathcal{L}$, then x must be the union of some members of the lattices in $\mathcal{B}$ and a non-empty set which has some, but not all, elements from $b_j \cup \{j\}$ for some j. Then, any such element k of $b_j \cup \{j\}$ which it does contain would rather take action 0. That is $\mathcal{I}_k(x) = 0$, so $x \notin NE(\Gamma_{\mathcal{I}})$. ∎

Besides clan-like structures, there are other simple influence structures which give rise to games with sets of equilibria that are the product of chains such that their maximum elements are disjoint. In what follows, we refer to these lattices as *simple lattices*. A simple influence structure is *sub-clan-like* if its bases $\{\varnothing\} \cup \{\widehat{b_i} \cup \{i\} : i \in N\}$ form a collection of chains[10] whose maximum elements partition N.

PROPOSITION **4** *$NE(\Gamma_{\mathcal{I}})$ is a simple lattice containing N when $\mathcal{I}$ is sub-clan-like. On the other hand if $\mathcal{L}$ is a simple lattice containing N and satisfying condition (1), then there exists a sub-clan-like influence structure $\mathcal{I}$ such that $NE(\Gamma_{\mathcal{I}}) = \mathcal{L}$.*

Proof of Proposition 4: Suppose that $\mathcal{I}$ is sub-clan-like. Let $\mathcal{C}$ be the collection of maximal chains[11] that can be formed using elements from $\{\widehat{b_i} \cup \{i\}\} : i \in N\} \cup \{\varnothing\}$. We will show that $NE(\Gamma_{\mathcal{I}}) = \bigtimes_{C\in\mathcal{C}} C$. Suppose that $x \in \bigtimes_{C\in\mathcal{C}} C$. Then, by definition, x is the union of some collection of sets $\{\widehat{b_k} \cup \{k\} : k \in I \subseteq N\}$, therefore for all $i \in x$, $\widehat{b_i} \subseteq x$, and, as a result, $\mathcal{I}_i(x) = 1$. Now suppose that $i \notin x$ and that $\mathcal{I}_i(x) = 1$. Then, it must be the case that $\widehat{b_i} \subseteq x$. By the fact that $\mathcal{I}$ is sub-cyclic, for each $k \in I$ either $\widehat{b_i} \cup \{i\} \subseteq \widehat{b_k} \cup \{k\}$, $\widehat{b_k} \cup \{k\} \subseteq \widehat{b_i} \cup \{i\}$ or $(\widehat{b_i} \cup \{i\}) \cap (\widehat{b_k} \cup \{k\}) = \varnothing$. As $i \notin x$, it cannot be the case that $\widehat{b_i} \cup \{i\} \subseteq \widehat{b_k} \cup \{k\}$ for some

[10] A chain is a lattice with the property that $x, y \in \mathcal{C} \to x \subseteq x$ or $y \subseteq x$.

[11] A chain C is maximal in a collection of chains $\mathcal{C}$ if $C \notin C'$ for any $C' \in \mathcal{C}$.

$k \in I$, so $I' \subseteq I$ must exist such that $\widehat{b}_k \cup \{k\} \subset \widehat{b}_i \cup \{i\}$ for $k \in I'$ and $\widehat{b}_i \subseteq \bigcup_{\{k \in I' \subseteq I\}} \widehat{b}_k \cup \{k\}$. Due to the fact that $\mathcal{I}$ is sub-clan-like, it must be the case that all the sets $\widehat{b}_k \cup \{k\}$ for $k \in I'$ belong to the same maximal chain (as, otherwise, the intersection of the maximum elements of at least two such chains would have to be non-empty). But this implies $\widehat{b}_i = \widehat{b}_{k^*} \cup \{k^*\}$ for some $k^* \in I'$, but then $(\widehat{b}_i \cup \{i\}) \cap (\widehat{b}_{k^*} \cup \{k^*\}) \neq \varnothing$ and therefore it must be the case that $\widehat{b}_i \cup \{i\}$ belongs to the same chain as $\widehat{b}_{k^*} \cup \{k^*\}$ and we therefore have $i \in x$, a contradiction. It must therefore be the case that $\mathcal{I}_i(x) = 0$.

Now suppose that $\mathcal{L}$ is a simple lattice containing N, then define $\mathcal{I}$ by letting $\mathcal{I}_i(x) = 1$ if and only if $x \supseteq minimal\{y \in \mathcal{L} : i \in x\}$. Consider some $x \in \mathcal{L}$, then, by construction, $\mathcal{I}_i(x) = 1$ for all $i \in x$. On the other hand, suppose that $i \notin x$. If $\mathcal{I}_i(x) = 1$, then it must be the case that $x \supseteq minimal\{y \in \mathcal{L} : i \in x\} \setminus \{i\}$. The fact that $\mathcal{L}$ is simple and that it satisfies condition (1) however, implies that all sets containing $\{y \in \mathcal{L} : i \in x\} \setminus \{i\}$ must contain (i), a contradiction, so it must be the case that $\mathcal{I}_i(x) = 0$, and we can conclude that $\mathcal{L} \subseteq NE(\Gamma_{\mathcal{I}})$ By construction, all the equilibria of $\Gamma_{\mathcal{I}}$ are unions of sets of the form $minimal\{y \in \mathcal{L} : i \in x\}$ so it is also the case that $NE(\Gamma_{\mathcal{I}}) \subseteq \mathcal{L}$. The fact that $\mathcal{I}$ is sub-clan-like follows from the fact that, as $\mathcal{L}$ is a simple lattice, for each i, the set $minimal\{y \in \mathcal{L} : i \in x\}$ is always the smallest element containing i of the only chain to which it belongs in the collection of chains whose product is $\mathcal{L}$. ∎

4.1 Hierarchical Influence Structures

The sets of equilibria of hierarchical influence structures exhibit well a theme that arises repeatedly across binary games of social influence: as the individual action spaces are so simple, all the complexity of equilibria arises from the social structure, and more specifically from indirect self reference in action.[12] Given a hierarchical influence structure $\mathcal{I}$ on a hierarchy $\mathcal{H} = \{H_1, H_2, ..., H_m\}$, an active set of agents $x \subseteq \bigcup_{k>r} H_k$ induces an increasing influence structure $\mathcal{I}^{r,x}$ on the agents in H_r given by

$$\mathcal{I}_i^{r,x}(y) = 1 \text{ if and only if } \mathcal{I}_i(y \cup x) = 1, \text{ where } y \in \mathcal{H}_r \text{ and } i \in H_r$$

Letting $\mathcal{I}^m = \mathcal{I}$, we can compute the set $NE(\Gamma_{\mathcal{I}})$ by a kind of backward induction, beginning by computing the Nash equilibria of the game restricted to the top level of the hierarchy $NE(\Gamma_{\mathcal{I}^m})$, taking advantage of the fact that the actions of agents in a given level of the

[12]This kind of indirect self reference implies a sort of irreducibility which is difficult to circumvent. The next section explores how the difficulty in computing and understanding the structure of the sets of equilibria in graphical games of strategic complements stems from the fact that they are often irreducible in the sense that they cannot be broken apart into smaller games which can be solved more easily.

hierarchy can only influence other agents at the same level or at lower levels:

$$E^m = NE(\Gamma_{\mathcal{I}^m})$$
$$E^r = \{\{x\} \times NE(\Gamma_{\mathcal{I}^r(x)}) : x \in E^{r+1}\}$$

We can proceed in this way down to the base of the hierarchy, having $NE(\Gamma_{\mathcal{I}}) = E^1$. The original problem is reduced to coping with the influence structures induced at each level of the hierarchy, by the higher levels.

5 Increasing Influence Structures that Admit a Network Representation

For the most part throughout this section we will maintain the convention of thinking of a strategy profile as the subset $x \subset N$ of agents that are active. In some parts, however, it will be convenient to rely on matrix notation to represent incentives, and binary vectors $\vec{x} \in \{0,1\}^{|N|}$ to represent strategy profiles. To avoid confusion, we consistently denote strategy profiles using the vector notation $\vec{x}$ when thinking of them as binary vectors.[13] In this notation, $\vec{x}_i = 1$ if and only if agent i chooses to be active.

An influence structure $\mathcal{I}$ admits a network representation when, for each player, i there exist weights $\{w_{ij}\}_{j=1}^n$, $w_{ij} \geq 0$, and a threshold $t_i \geq 0$ such that $\mathcal{I}_i(x) = 1$ if, and only if, $\sum_{j \in x} w_{ij} \geq t_i$. We can group all the individual weights in a matrix W and let $\overrightarrow{t} = (t_1, t_2, ..., t_i, ..., t_n)'$ denote the vector of thresholds. As W and $\vec{t}$ fully capture an influence structure, we will directly denote it by $(W, \vec{t})$, instead of using $\mathcal{I}$. The games induced by increasing structures that admit network representation are well known in the literature as *graphical games of strategic complements* or *games of thresholds*. Throughout this section, we refer to the game induced by $(W, \vec{t})$ as a *game of thresholds* and denote it by $\Gamma_{(W,\vec{t})}$. Given a lattice $\mathcal{L}$, we say that it is *expressible by a game of thresholds* if there exist some weights W and a vector of thresholds $\vec{t}$ such that $NE(\Gamma_{(W,\vec{t})}) = \mathcal{L}$.

Using this notation, we can represent the equilibrium condition using a system of inequalities. A vector $\vec{x} \in \{0,1\}^n$ is an equilibrium of game $\Gamma_{(W,\vec{t})}$ if, and only if:

$$[Diag(-t_i) + W]\overrightarrow{x} \geq \overrightarrow{0} \text{ and } [Diag(t_i - 1)) + W]\overrightarrow{x} < \overrightarrow{t}$$

The problem of finding all the equilibria of a graphical games of thresholds is therefore a subclass of the problem of finding all solutions to systems of linear inequalities, since

[13] Formally in moving back and forth between the set notation and the vector notation, we have that $i \in x \Leftrightarrow \vec{x}_i = 1$.

disregarding for an instant the strictness of the second inequality, we can write the above as[14]

$$\begin{pmatrix} Diag(-t_i) + W \\ -Diag(t_i - 1)) - W \end{pmatrix} \vec{x} \geq \begin{pmatrix} \vec{0} \\ -\vec{t} \end{pmatrix}$$

In general, the problem of finding all solutions to linear systems of inequalities is a very hard one, but we can hope to make some general statements about it by exploiting the additional structure of the matrices and vector involved stemming from the fact that they represent graphical games of strategic complements. There are two features of games of thresholds which make them very appealing: 1) their structure summarizes social interaction in a variety of settings, and 2) they can be described very succinctly: their representations are not more complex than those of *simple games of social influence*, but they have a much greater expressive power. As seen in Example 2, there are some increasing influence structures that do not admit network representations.

Example **2** *An influence structure that does not admit a network representation.*

Let $\mathcal{I}$ be an increasing influence structure on $\{1,2,3,4,5\}$ such that $\mathcal{I}_1(x) = 1$, if, and only if, $x \supseteq \{2,3\}$ or $x \supseteq \{4,5\}$. This influence structure does not admit a network representation: The reason is that $\mathcal{I}_1(\{2,3\}) = 1$ implies that either $w_{12} \geq \frac{t_1}{2}$ or $w_{13} \geq \frac{t_1}{2}$. Similarly, it must be the case that either $w_{14} \geq \frac{1}{2}$ or $w_{15} \geq \frac{1}{2}$. But this, in turn, implies that at least one of $\mathcal{I}_1(\{2,4\})$, $\mathcal{I}_1(\{2,5\})$, $\mathcal{I}_1(\{3,4\})$ or $\mathcal{I}_1(\{3,5\})$ must also be 1. $\diamond$

Example 2 shows that, in depicting agent 1's social incentives using weights and a threshold, if we want him to be triggered by groups of agents $\{2,3\}$ and $\{4,5\}$ then it must be the case that he is also triggered by at least one other group of agents that is not a superset of either of these. In general, this is the only kind of limitation that we encounter when constructing network representations of influence structures: it is straightforward to assign the weights and pick the threshold in order to have an agent prefer to be active when every element in a specified collection of subsets of N is active. What can be difficult and sometimes impossible is choosing them in order to ensure that these are the only triggers. That is, it is always possible to construct a network *approximation* of an influence structure containing all the triggers of an agent, but, in general, in any such approximation, agents will be strictly more sensitive to social influence. As shown in statement (3) in Section 3, the equilibria induced

[14] The problem involving the strict inequality can be approached by solving the weaker version involving the weak inequality and then testing one by one the individual solutions to determine the subset in which the strict inequality holds.

by these approximations will be weakly larger than those of the games induced by the original structures.

We now turn to the question of which sub-collection of lattices expressible by increasing influence structures are also expressible by games of thresholds. There are two different kinds of problems that we may face in trying to express a given lattice. The first one is that due to the approximation limitations seen in Example 2, we may gain some equilibria which we cannot destroy without compromising the equilibria that we do want to include. The other possibility is that none of the approximations expresses supersets of the lattice in question. We do not have much to say regarding the question of expressibility in the general class of influence structures admitting a network representation. In what follows, we provide some result and intuition related to a few special cases.

5.1 A Few Special Cases

Given a graph W, a *cycle* is a subset of N, $i_0, i_1, ..., i_k$, such th such that $w_{i_0 i_1} > 0, w_{i_1 i_2} > 2, ..., w_{i_{k-1} i_k} > 0$ and $w_{i_i i_0} > 0$. The first result is very simple and puts forth an important idea, which is that all the complexity of the lattice of equilibria is closely related to the existence of cycles. A graph that has no cycles is called a *tree*. In general, the issue is the same as the one highlighted in Section 4.1 in relation to hierarchical influence structures. It becomes a lot more clear in the context of networks.

Claim 1 *If W has no cycles, the game has a unique equilibrium.*

Proof of Claim 1: If W has no cycles there must exist at least an agent i, such that $w_{ij} = 0$. By virtue of $(A2)$ all such agents have a single optimal action x_i^*. In particular, $x_i^* = 0$ if $t_i > 0$ and $x_i^* = 1$ if $t_i = 0$. We can therefore create a simpler and equivalent game $\Gamma'_{W', \vec{t}'}$, by removing all such agents from the graph and adjusting the the thresholds of the remaining agents. Formall, let $H = \{i \in N : w_{ij} = 0 \ \forall j \in N\}$. Let $W' = [w_{ij}]_{i=1,2,...,n; j=1,2,...,n}$ where $w'_{ij} = w_{ij}$ if $i \notin H$ and $j \notin H$ and $w'_{ij} = 0$ otherwise. And for each $j \in N$, let $t'_j = t_j - \sum_{i \in H} 1\{t_i = 0\} w_{ji}$. As the resulting graph is also a tree, we can continue this process, each time removing a positive number of agents and assigning them their optimal actions until we are left with a single agent, and a complete assignment of the unique equilibrium actions $\vec{x}$. ∎

If we allow influence structures to be correspondences (letting agents be indifferent between playing 0 or 1) games on trees will in general have multiple equilibria. The spirit of the result nevertheless continues to hold: the set of equilibria can be found in its entirety by a form of backward induction starting in the leaves.

We now examine the problem of expressing $\mathcal{L}$ using games of thresholds on the *complete graph*, that is, the graph in which the weights $w_{ij} = w$ are the same for all pairs of agents $i \neq j$. This special case is of interest because it is equivalent to many models of peer effects in the applied literature, in which each agent chooses his activity level in response to some measure of mean activity level in the environment. In these models, the agents may have different sensitivities to the environment (in our language, different thresholds) but the environment is usually the same for everyone, which is analogous to all the weights being equal.

Claim **2** *If x and y are equilibria of a game of thresholds in the complete graph then either $x \subseteq y$ or $y \subseteq x$. That is, the set of equilibria must be a chain.*

Proof of Claim 2:

Suppose that we have two equilibria x and y. Without loss of generality, assume that $|x| \leq |y|$ and let $i \in x$. Then, $t_i \leq \sum_{j\in x} w_{ij} = |x|w$ and therefore $t_i \leq |y|w = \sum_{j\in y} w_{ij}$. So it must be the case that $i \in y$. ▮

The converse of the claim above is also true, any chain which satisfies the strictness condition (1) of Section 2 is expressible by a game of thresholds on the complete graph.

Claim **3** *If $\mathcal{L}$ is a chain with the property if $x \in \mathcal{L}$ then $x \setminus \{i\} \notin \mathcal{L}$ $\forall i \in x$*[15] *then it is expressible by a game of thresholds on the complete graph.*

Proof of Claim 3:

Let $C = \{x_1, x_2, ..., x_m\}$ be a chain satisfying the condition of the claim, and without loss of generality suppose $k' < k$ implies $x_k \subseteq x_{k'}$. Then, it must be true that $|x_{k+1} - xk| > 1$. So let, $t_i = \frac{|x_1|-1}{n}$, $\forall i \in x_1$ and in general $t_i = \frac{|x_k|-1}{k}$, $\forall i \in x_k \setminus x_{k-1}$, for each $k \leq m$. Finally, let $t_i = 1$, $\forall i \notin x_m$. Let $w_{ij} = \frac{1}{n}$ $\forall i \neq j$. By construction, the set of equilibria of this game of thresholds is precisely $C = \{x_1, x_2, ..., x_m\}$. ▮

The simple structure of the lattices that are expressible by games of thresholds on the complete network means that we are able to count the number of different (up to re-labeling of the agents) possible sets of equilibria of these games. In enunciating this counting result, we restrict attention to sets of equilibria which always include the complete set N.

Corollary **1** *There are F_{n-1} different chains, which can be expressed as equilibria of games of thresholds on the complete network on the set $N = \{1, 2, ..., n\}$ that include the set N; where F_{n-1} denotes the $n - 1^{th}$ Fibonacci number.*

[15] Condition (1) of Section 2.

Proof of Corollary 1: Since we are counting the number of different chains up to re-labeling of the agents, two chains C_1 and C_2 are different if, and only if, one of them contains a set different in size from all the sets contained in the other chain. So, in what follows, when referring to a given set, we only speak of its size. We can partition the set of expressible chains on $|N|$ players into two groups: (1) those that contain an element of size $|N-2|$ and (2) those that do not contain an element of size $|N-2|$. Each of the chains in (1) corresponds to a unique chain expressible with $|N-2|$ elements. A bijection is given by $f(c) = c \setminus N$. On the other hand, each of the chains in (2) corresponds to a unique chain expressible with $|N-1|$ elements. A bijection is given by $g(c) = c \setminus N \cup N-1$. Therefore, if we denote the number of expressible chains on $|N|$ players by c_N, we have that $c_N = c_{N-1} + c_{N-2}$. Moreover, $c_1 = 0 = F_0$, $c_2 = 1 = F_1$. ▮

Claims 2 and 3 are not very helpful in the sense that it is easy to come up with examples of threshold games with equilibria that are not nested. In general, network architectures which have many components, or have a number of highly intra-connected islands, only inter-connected by a few bridges will tend to have pairs of equilibria not comparable by set inclusion. These claims, however, do show quite succinctly that threshold games are interesting precisely due to the interplay between thresholds and network structure. No matter how much freedom we have to play with the thresholds, we will never be able to abstract away from the network structure. As seen in Example 3, allowing the network to be weighted also adds expressive power to games of thresholds.

Example **3** *The expressive power added by weights.*

The lattice $\mathcal{L}$ shown in Figure 3 cannot be expressed by a game of thresholds on a network in which all links have the same weight. To see this, suppose that there existed a a game $(W, \vec{t})$, in which $w_{ij} = 0$ or $w_{ij} = w$ and, such that $NE(\Gamma_{W,\vec{t}}) = \mathcal{L}$. We begin noting that the agents in $\{1,2,3\}$ need to be connected as otherwise one of them would need to have threshold 0, but this cannot be the case since $\varnothing \in \mathcal{L}$. Whoever among 2 and 3 is linked to 1 must have a threshold of at most $2w$, and therefore cannot be linked to $4,5,6,7$ since otherwise at least one of $\{1,4,5\}$ or $\{1,6,7\}$ would not be an equilibrium. If j among 2 and 3 is not directly linked to 1, then j must have a threshold of at most w, and just as before it cannot be linked to $4,5,6$ or 7, since, as above, at least one of $\{1,4,5\}$ or $\{1,6,7\}$, would not be an equilibrium. We can therefore conclude that neither 2 nor 3 can be linked to $4,5,6$ or 7. Therefore $\{1,4,5,6,7\}$ has to be an equilibrium as well, This contradicts the existence of an unweighted network W and a vector of weights $\vec{t}$ satisfying $NE(\Gamma_{W,\vec{t}}) = \mathcal{L}$. ⋄

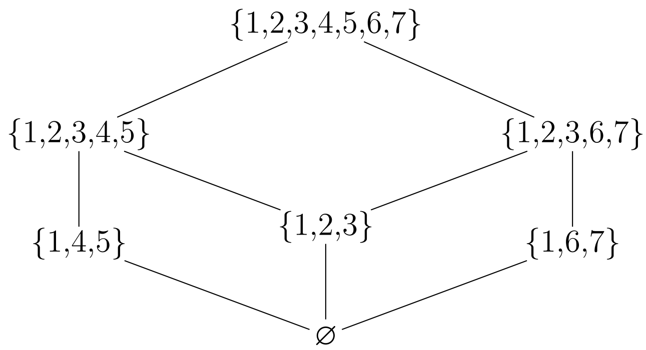

Figure 3: A lattice $\mathcal{L}$ that can only be expressed by games of thresholds on graphs with weighted links.

6 Finding All Equilibria and Deciding Expressibility

(Echenique 2007 [2]) sets forth the fastest known algorithm for computing all equilibria in general finite games of strategic complements. In what follows, we present a version of Echenique's algorithm for our games taking advantage of the notation that has been introduced in Sections 3 and 4. The main idea behind the algorithm is to traverse the space of of the 2^n subsets of N efficiently by taking advantage of the fact that the influence structure is increasing.

1) *Initialization:* Let $\mathbb{E}_0 = \varnothing$, $S_0 = \{(\varnothing, \varnothing)\}$ and $C_0 = \varnothing$.
We start with an empty set of equilibria $\mathbb{E}_0 = \varnothing$, a stack of sets (the seeds) to be inspected $S_0 = \{\varnothing\}$ just containing the empty set, and an empty stack of already-inspected elements (Checked elements) $C_0 = \varnothing$.

2) *Create $\mathbb{E}_t$, S_t and C_t from $\mathbb{E}_{t-1}$, S_{t-1} and C_{t-1}:*
Select one (any) of the elements of $(x, z) \in S_{t-1} \setminus C_{t-1}$ (If the set is empty, go to 3).
a) If $\mathcal{I}_i(\uparrow (x \cup z)) = 1$ for all $i \in z$ then:
$C_t = C_{t-1} \cup \{(y, z) : \text{ such that } y \leq x \leq \uparrow (x \cup z)\}$
$S_t = S_{t-1} \cup \{(\uparrow (x \cup z), \{j\}) : j \notin x \cup z\}$
$\mathbb{E}_t = \mathbb{E}_{t-1} \cup \{\uparrow (x \cup z)\}$.
Go back to the beginning of Step 2).

b) If $\mathcal{I}_i(\uparrow (x \cup z)) = 0$ for some $i \in z$ then:
$C_t = C_{t-1} \cup \{(y, z) : \text{ such that } y \leq x \leq \uparrow (x \cup z)\}$
$S_t = S_{t-1} \cup \{(x, z \cup \{j\}) : j \notin x \cup z\}$
$\mathbb{E}_t = \mathbb{E}_{t-1}$.
Go back to the beginning of Step 2).

3) Let $NE(\Gamma_{\mathcal{I}}) = \mathbb{E}_{t^*}$ where t^* represents the stopping time of the iterative procedure presented in 2).

The algorithm starts at the bottom of the lattice. At the beginning of each iteration set x represents agents that want to be active given that everyone else in x is active. The set z is a group of agents held artificially active by the algorithm, in order to be able to rely on the influence structure in order to navigate the power set of N. The algorithm evaluates the best response of the agents that do not belong to the set $x \cup z$, looking for the smallest superset $\uparrow(x \cup z)$ with the property that no agents not belonging to it would rather be active. Resolving whether $\uparrow(x \cup z)$ is an equilibrium just requires checking whether the agents in z are willing to be active, as, by construction, we know that all other agents in the set —the agents in x, and those that are added by the $\uparrow$ operator— are best responding. Note that regardless of whether $\uparrow(x \cup z)$ is an equilibrium or not, at that set, the algorithm cannot take advantage of the fact that the influence structure is increasing to continue navigating the power set of N. So, we must add new seeds to the stack S_t. While adding any immediate successor of $\uparrow(x \cup z)$ would suffice to re-start the process, in order to make sure that we traverse the entire power set of N all the successors ought to be added. Note that the stack C_t of already-inspected sets is only kept for efficiency, as the re-seeding process may eventually lead to considering a given pair (x, z) in 2) more than once.

Provided that the $\uparrow$ can be applied efficiently, this algorithm is a huge improvement over evaluating the best response function of each agent at each subset of N, which is the only general algorithm for finding all pure strategy Nash equilibria of arbitrary discrete games. The efficiency in the evaluation of the $\uparrow$ operator in turn, depends on the extent to which we can efficiently evaluate the best response function of the agents. If this is the case, the algorithm will terminate quickly to the extent that the gaps between the the sets $x \cup z$ and $\uparrow(x \cup z)$ are large, and therefore traversing the space does not essentially rely on the reseeding process. As the next example shows, the worst case performance can be exponential on n, and, moreover, its halting time is unrelated to the size of the set of equilibria of the problem at hand.

Example 4 *Worst case performance.*

Consider the increasing influence structure in which every agent in N finds it optimal to remain inactive regardless of what the other agents do. Then, the algorithm evaluates the best response function of every single agent at each of the 2^n subsets of N: it is forced to reseed after every single iteration.$\diamond$

Example 4 shows that the worst case performance of the algorithm is exponential on n. Formally, an instance of the problem of computing all the pure strategy Nash equilibria of one of our games is a description of the influence structure. There are many classes of increasing influence structures of interest whose representations are of polynomial size on n. Consider, for example, the class in which each agent has, at most, k bases for action (k can be any number). Then, provided that the description of each subset of N is of size $O(p(n))$, where $p(n)$ is some polynomial of n, the description of the whole problem is $O(knp(n))$, so polynomial on n. Note that the class of influence structures in which each agent has, at most, k bases includes important subclasses of the special cases studied in this paper: for example, all simple structures, and all structures admitting a network representation in which the number of neighbors of each agent is bounded. So Example 4 shows that, within the class of problems whose description are of polynomial size on n, the worst case halting time of the algorithm is exponential on n.

The problem of deciding whether a lattice $\mathcal{L}$ is expressible by an arbitrary influence structure can be decided in polynomial time on the size of the lattice; by just applying (1), we just need to verify that, for each $x \in \mathcal{L}$, we have $x \setminus \{i\} \notin x$. Similarly, the problem of deciding whether a lattice $\mathcal{L}$ is expressible by some increasing influence structure can be decided in polynomial time on the size of the lattice, by just applying (SC), provided in Proposition 1. On the other hand the problem of deciding whether a lattice is expressed by a given arbitrary influence structure[16] takes exponential time based upon the best general algorithms available. That is, the best algorithm available has worst case performance which is exponential on the size of the description of the influence structure. This follows because verifying whether a given lattice is precisely the set of equilibria of the game induced by a specific influence structure is essentially not any easier than finding all the equilibria of the game. The issue is that it requires that we test not only whether each element of the lattice is an equilibrium of the game proposed, but also whether the game proposed has no other equilibria. In order to substantially improve on this appalling performance in the more general problem of deciding expressibility, we use proposition 1, which gives us a characterization of expressibility that *only depends* on the elements of $\mathcal{L}$, just as is the case with the problem of expressibility by arbitrary influence structures.

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