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Efficient Tests of the Seasonal Unit Root Hypothesis*

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Abstract

In this paper we derive, under the assumption of Gaussian errors with known error covariance matrix, asymptotic local power bounds for seasonal unit root tests for both known and unknown deterministic scenarios and for an arbitrary seasonal aspect. We demonstrate that the optimal test of a unit root at a given spectral frequency behaves asymptotically independently of whether unit roots exist at other frequencies or not. We also develop modified versions of the optimal tests which attain the asymptotic Gaussian power bounds under much weaker conditions. We further propose near-efficient regression-based seasonal unit root tests using pseudo-GLS de-trending and show that these have limiting null distributions and asymptotic local power functions of a known form. Monte Carlo experiments indicate that the regression-based tests perform well in finite samples.

Keywords: Point optimal invariant (seasonal) unit root tests; asymptotic local power bounds; near seasonal integration.

JEL Classifications: C22.

1 Introduction

Since the introduction of formal unit root tests by Dickey and Fuller [DF] (1979,1981), a large literature, both theoretical and applied, has grown on the analysis of unit root time-series data. Much of this has been dedicated to developing and refining unit root test procedures. In particular, what has motivated researchers to search for new procedures is the characteristically low power of most unit root tests, particularly when deterministic trend variables are introduced into the test regression. A distinct aspect of several recently proposed procedures lies in the criterion used to estimate the parameters on deterministic variables; that is, the method of de-trending employed. While DF use ordinary least squares (OLS) de-trending, Elliott, Rothenberg and Stock [ERS] (1996) and Schmidt and Phillips

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(1992), use generalised least squares (GLS) and first-difference (FD) based de-trending, respectively. These approaches have been shown to yield more powerful unit root tests than those based on OLS de-trending. Most notably though, ERS have demonstrated that DF-type tests based on GLS de-trended data have asymptotic local power functions which lie arbitrarily close to the Gaussian power envelope.

Similar concerns arise in the seasonal context where DF-type tests (with OLS de-trending) for unit roots at the zero and seasonal frequencies have been developed by Hylleberg, Engle, Granger and Yoo (1990) [HEGY] for quarterly data and generalised to the case of monthly data by Beaulieu and Miron (1993) and Taylor (1998), and to an arbitrary seasonal periodicity, say S, by Smith and Taylor (1999a,b). A detailed discussion of the theoretical and empirical importance of testing for seasonal unit roots is provided in Rodrigues and Taylor (2004a). Seasonal patterns in economic time-series tend to evolve slowly over time (see, *inter alia*, Hylleberg *et al.*, 1993), a characteristic shared by seasonal unit root processes. However, as Ghysels and Osborn (2001,p.90) note, most empirical applications of the aforementioned seasonal unit root tests have lead to non-rejection of the non-seasonal unit root but to rejections of the unit root hypothesis at some, but rarely all, of the seasonal frequencies, implying the inappropriateness of either taking annual differences of the data or of commonly used seasonal adjustment procedures which assume the presence of unit roots at all of the seasonal frequencies (Ghysels and Osborn, 2001,Ch.4). The fact that unit roots cannot be rejected at all of the seasonal frequencies might be attributable to the low power of the OLS de-trended HEGY tests. This suggests that the need to develop more powerful seasonal unit root tests is arguably more compelling than for the non-seasonal case in order to better establish whether or not seasonal unit roots constitute an appropriate mechanism for modelling the seasonal patterns in economic series.

Developing FD-de-trended variants of the HEGY testing procedure is a relatively simple exercise and has already been undertaken in Breitung and Franses (1998) and Rodrigues (2002) who both find evidence of non-trivial power gains over the standard OLS de-trended HEGY tests. However, GLS de-trending in the context of the HEGY testing procedure is complicated by the fact that the power surface of the testing problem is S-dimensional and has yet to be developed. For this reason, asymptotic power bounds have also not been derived for autoregressive unit root tests in seasonal time series. Both of these deficiencies in the literature are rectified in this paper. Gregoir (2004) has developed GLS de-trended versions of the tests of Ahtola and Tiao (1987) for the null hypothesis of the presence of a pair of complex conjugate unit roots at a fixed spectral frequency $\omega_k \in (0, \pi)$. These tests assume that unit roots are not present at any other spectral frequency, and so are not appropriate to the seasonal unit roots testing scenario considered in HEGY. However, Gregoir (2004) contains a number of technical results which prove useful in the development of this paper, as does ERS which assumes that a unit root exists only at frequency zero.

The paper is organised as follows. In Section 2 we review the standard seasonal autoregressive model framework, highlighting the typology of seasonal deterministic trend functions extant in the seasonal unit root literature and outline the parameter restrictions on this model which yield (near) unit roots at the zero and seasonal frequencies. In Section 3 we develop point optimal invariant test procedures for unit roots at each of the zero and seasonal frequencies and derive expressions for their limiting power envelopes using local-to-unity asymptotics. We show that the optimal test of a unit root at a given frequency behaves asymptotically independently of whether unit roots exist at other frequencies or not. Moreover, these expressions are shown to coincide with existing representations in the literature. Feasible tests which weaken the assumptions underpinning the point optimal tests are introduced in Section 4 and are shown to have identical limiting power envelopes as the corresponding point optimal tests. In Section 5 we propose further tests, which like the modified augmented DF tests proposed in ERS, are based on a method of pseudo-GLS de-trending of the familiar HEGY autoregressive-based testing procedure for unit roots at the zero and seasonal frequencies. Where our modified HEGY tests are performed on a single unit root parameter they lie very close to the relevant power envelopes pertaining to that parameter from Section 3. Monte Carlo evidence provided in Section 6 indicates that the modified HEGY tests display vastly improved finite sample power properties over the standard HEGY tests, especially so in the case of the joint F-type tests, yet display equally good finite sample size properties in the presence of serially correlated errors. Section 7 concludes. Appendices A, B and C contain the proofs of Propositions 3.1-3.2, Theorems 3.1-3.2, and Theorem 5.1, respectively.

2 The Seasonal Unit Root Framework

2.1 The Seasonal Model

Using the set-up of Smith and Taylor (1999a,b), consider the process $\{x_{Sn+s}\}$, observed with constant seasonal periodicity S, which can be written as the sum of a purely deterministic component, μ_{Sn+s}, and a purely stochastic process; *viz.*,

$$x_{Sn+s} = y_{Sn+s} + \mu_{Sn+s},\ s = 1-S, ..., 0, n = 1, 2, \dots, N, \quad (2.1)$$
$$\alpha(L)y_{Sn+s} = v_{Sn+s} \quad (2.2)$$

where $\alpha(L) \equiv 1 - \sum_{j=1}^{S} \alpha_j^* L^j$ in (2.2) is an Sth order polynomial in the conventional lag operator, L. The disturbance process $\{v_{Sn+s}\}$ is a mean-zero covariance stationary process which admits the moving average representation $v_{Sn+s} = \psi(L)u_{Sn+s}$ where $\{u_{Sn+s}\}$ is $IID(0,\sigma^2)$ with finite fourth moments and the lag polynomial $\psi(z) \equiv 1 + \sum_{i=1}^{\infty} \psi_i z^i$ satisfies the following conditions: (i) $\psi(\exp\{\pm i2\pi k/S\}) \neq 0$, $k = 0, ..., \lfloor S/2 \rfloor$, $\lfloor \cdot \rfloor$ denoting the integer part of its argument, and (ii) $\sum_{j=1}^{\infty} j|\psi_j| < \infty$. These conditions ensure that the spectral density function of v_{Sn+s} is bounded and is strictly positive at both the zero and seasonal spectral frequencies, $\omega_k \equiv 2\pi k/S$, $k = 0, ..., \lfloor S/2 \rfloor$. For notational convenience we define $S^* \equiv (S/2) - 1$ (if S is even) or $\lfloor S/2 \rfloor$ (if S is odd).

In (2.1), μ_{Sn+s} is modelled as a linear combination of a set of deterministic regressors; that is, $\mu_{Sn+s} = \beta' z_{Sn+s}$. Following Smith and Taylor (1998,1999a,b), we consider six cases of interest:

Case 1: no deterministics.

Case 2: zero frequency intercept: $z_{Sn+s} \equiv z_{Sn+s,2} = 1$, $s = 1-S, ..., 0$, $n = 1, 2, \dots, N$, with $\beta \equiv \gamma_0$.

Case 3: zero and seasonal frequency intercepts: $z_{Sn+s} \equiv z_{Sn+s,3} = [1, \cos(2\pi\,(Sn+s)\,/S),$ $\sin(2\pi\,(Sn+s)\,/S), ..., \cos(2\pi S^*(Sn+s)/S), \sin(2\pi S^*(Sn+s)/S), (-1)^{Sn+s}]'$, $s = 1-S, ..., 0$, $n = 1, 2, \dots, N$, with $\beta \equiv (\gamma_0, \gamma_1', ..., \gamma_{S^*}', \gamma_{S/2})'$, and where $\gamma_k = (\gamma_{k,\alpha}, \gamma_{k,\beta})'$, $k = 1, ..., S^*$.

Case 4: zero-frequency intercept, zero-frequency trend: $z_{Sn+s} \equiv z_{Sn+s,4} = (1, Sn+s)'$, $s = 1-S, ..., 0$, $n = 1, 2, \ldots, N$, with $\beta \equiv (\gamma_0, \delta_0)'$.
Case 5: zero and seasonal frequency intercepts, zero-frequency trend: $z_{Sn+s} \equiv z_{Sn+s,5} = (z'_{Sn+s,3}, Sn+s)'$, $s = 1-S, ..., 0$, $n = 1, 2, \ldots, N$, with $\beta \equiv (\gamma_0, \gamma_1, ..., \gamma_{S^*}, \gamma_{S/2}, \delta_0)'$.
Case 6: zero and seasonal frequency intercepts and trends: $z_{Sn+s} \equiv z_{Sn+s,6} = (z'_{Sn+s,3}, (Sn+s)z'_{Sn+s,3})'$, $s = 1-S, ..., 0$, $n = 1, 2, \ldots, N$, with $\beta \equiv (\gamma_0, \gamma_1, ..., \gamma_{S^*}, \gamma_{S/2}, \delta_0, \delta_1, ..., \delta_{S^*}, \delta_{S/2})'$, where $\delta_k = (\delta_{k,\alpha}, \delta_{k,\beta})'$, $k = 1, ..., S^*$,

omitting $(-1)^{Sn+s}$ from $z_{Sn+s,3}$, $\gamma_{S/2}$ from β in Cases 3, 5 and 6, and $\delta_{S/2}$ from β in Case 6, when S is odd.

2.2 The Seasonal Unit Root Hypotheses

Denoting $i \equiv \sqrt{(-1)}$, we may factorise the polynomial $\alpha(L)$ at the seasonal spectral frequencies, $\omega_k \equiv 2\pi k/S$, $k = 1, ..., \lfloor S/2 \rfloor$ as:

$$\alpha(L) = \prod\nolimits_{k=0}^{\lfloor S/2 \rfloor} \omega_k(L) \tag{2.3}$$

where the lag polynomial $\omega_0(L) \equiv (1 - \alpha_0 L)$ associates the parameter α_0 with the zero frequency $\omega_0 \equiv 0$, the lag polynomial $\omega_k(L)$ corresponds to the conjugate (harmonic) seasonal frequencies $(\omega_k, 2\pi - \omega_k)$, and is defined by

$$\begin{aligned} \omega_k(L) &\equiv [1 - (\alpha_k + \beta_k i) \exp(i\omega_k) L] [1 - (\alpha_k - \beta_k i) \exp(-i\omega_k) L] \\ &= \left[1 - 2(\alpha_k \cos\omega_k - \beta_k \sin\omega_k) L + (\alpha_k^2 + \beta_k^2) L^2\right], \end{aligned}$$

with associated parameters α_k and β_k, $k = 1, ..., S^*$, together with $\omega_{S/2}(L) \equiv (1 + \alpha_{S/2} L)$, with parameter $\alpha_{S/2}$ corresponding to the Nyquist frequency $\omega_{S/2} \equiv \pi$, when S is even.

Consequently, following HEGY ($S = 4$) and Smith and Taylor (1999a) we consider testing the $(\lfloor S/2 \rfloor + 1)$ unit root null hypotheses

$$H_{0,0} : \alpha_0 = 1, \; H_{0,S/2} : \alpha_{S/2} = 1 \;\; (S \text{ even}), \tag{2.4}$$

and

$$H_{0,k} : \alpha_k = 1, \; \beta_k = 0, \; k = 1, ..., S^*. \tag{2.5}$$

The hypothesis $H_{0,0} : \alpha_0 = 1$ corresponds to a unit root at the zero-frequency while, for S even, $H_{0,S/2} : \alpha_{S/2} = 1$ yields a unit root at the Nyquist frequency. A pair of complex conjugate unit roots at the harmonic seasonal frequencies $(\omega_k, 2\pi - \omega_k)$ is obtained under $H_{0,k} : \alpha_k = 1 \cap \beta_k = 0$, $k = 1, ..., S^*$.

The alternative hypotheses of near-integration at the zero and Nyquist (S even) frequencies may be stated as,

$$H_{1,c_0} : \alpha_0 = \left(1 + \frac{c_0}{T}\right), \; H_{1,c_{S/2}} : \alpha_{S/2} = \left(1 + \frac{c_{S/2}}{T}\right) \tag{2.6}$$

where $T \equiv SN$, and at the harmonic seasonal frequencies as

$$H_{1,c_k} : \alpha_k = \left(1 + \frac{c_k}{T}\right), \; \beta_k = 0, \; k = 1, ..., S^*. \tag{2.7}$$

Cf Phillips (1987), Rodrigues (2001), Rodrigues and Taylor (2004b) and Gregoir (2004), *inter alia*. Under H_{1,c_k}, the process $\{x_{Sn+s}\}$ admits either a single root $[k = 0, S/2]$ or a pair of complex conjugate roots $[k = 1, ..., S^*]$ with modulus in the neighbourhood of unity at frequency ω_k. These roots are stable where $c_k < 0$ and explosive where $c_k > 0$. By allowing $\beta_k \neq 0$, one might also allow for the possibility of complex roots lying in the neighbourhood of the harmonic seasonal frequencies ω_k, $k = 1, ..., S^*$. However, and following Gregoir (2004), we do not permit this. This seems entirely natural given that the seasonal aspect of our data governs the spectral frequencies of interest which can therefore be reasonably assumed fixed. Finally, notice that H_{1,c_k} reduces to $H_{0,k}$ if $c_k = 0$, $k = 0, ..., \lfloor S/2 \rfloor$.

In what follows, let $\mathbf{c} \equiv (c_0, c_1, ..., c_{[S/2]})'$ be the $(\lfloor S/2 \rfloor + 1)$-vector of non-centrality parameters and denote the lag polynomial $\alpha(L)$ under $H_{1,\mathbf{c}} \equiv \cap_{k=0}^{\lfloor S/2 \rfloor} H_{1,c_k}$ as $\Delta_{\mathbf{c}} \equiv 1 - \sum_{j=1}^{S} \alpha_j^{\mathbf{c}} L^j$. Finally, notice that under $H_0 \equiv \cap_{k=0}^{\lfloor S/2 \rfloor} H_{0,k}$, $\mathbf{c} = \mathbf{0}$ and $\Delta_{\mathbf{0}} \equiv 1 - L^S$, so that $\{x_{Sn+s}\}$ evolves as a seasonal random walk process.

3 Efficient Seasonal Unit Root Tests

In what follows we will use a similar framework of analysis to that adopted by ERS and Gregoir (2004). Consequently, we will need to make the following assumption:

Assumption 3.1 *(i)* $y_s = 0$, $s = 1 - S, \ldots, 0$; *(ii) the parameters characterising the lag polynomial* $\psi(z)$ *are known; (iii)* $\{u_{Sn+s}\}$ *is* $NIID(0, \sigma^2)$ *with* σ^2 *known.*

Assumption 3.1 is necessarily unrealistic. It is made so that we may construct most powerful tests against a given point alternative, and hence develop theoretical power bounds for seasonal unit root tests in both the known and unknown deterministic scenarios. We will subsequently drop Assumption 3.1 in Sections 4 and 5 where we discuss testing procedures which can be used in practice.

3.1 Known Deterministic Component

Where μ_{Sn+s} of (2.1) is known it is observationally equivalent to Case 1 of μ_{Sn+s} outlined in Section 2.1, and so y_{Sn+s} is observable. Denoting by $L(\mathbf{c})$, minus twice the log-likelihood and ignoring the additive constant we have that

$$L(\mathbf{c}) = \Delta_{\mathbf{c}} \mathbf{y}' \Omega^{-1} \Delta_{\mathbf{c}} \mathbf{y}, \tag{3.1}$$

where Ω is the (known) non-singular covariance matrix of the vector $\mathbf{v} \equiv (v_1, ..., v_T)'$ and $\Delta_{\mathbf{c}} \mathbf{y} \equiv (\Delta_{\mathbf{c}} y_1, ..., \Delta_{\mathbf{c}} y_T)'$.

Suppose, for the present, that we wish to test the null hypothesis $H_{0,k} : c_k = 0$ of (2.4) and (2.5) against the alternative $H_{1,c_k} : c_k = \bar{c}_k$ of (2.6) and (2.7), $\bar{c}_k \neq 0$, $k \in \{0, \ldots, \lfloor S/2 \rfloor\}$, under the *maintained* hypothesis that the remaining elements of $\mathbf{c}$ are all equal to zero. That is, we wish to test $H_0 : \mathbf{c} = \mathbf{0}$ against $H_{1,\bar{\mathbf{c}}_k} : \mathbf{c} = \bar{\mathbf{c}}_k$, where $\bar{\mathbf{c}}_k$ is an $(\lfloor S/2 \rfloor + 1)$-vector whose $(k+1)$th element is equal to c_k and all other elements equal to zero. In this case, appealing to the Neyman-Pearson Lemma (see, *e.g.* Lehmann,1986,Ch. 3.2), the most powerful test of H_0 against $H_{1,\bar{\mathbf{c}}_k}$ rejects for small values of the (log-)likelihood ratio statistic $\mathcal{L}_k(\bar{\mathbf{c}}_k, \mathbf{0}) \equiv L(\bar{\mathbf{c}}_k) - L(\mathbf{0})$. Proposition 3.1 provides explicit representations for these statistics.

Proposition 3.1 *The Neyman-Pearson statistics $\mathcal{L}_k(\bar{\mathbf{c}}_k, \mathbf{0})$, $k = 0, ..., \lfloor S/2 \rfloor$, are of the form*

$$\mathcal{L}_k(\bar{\mathbf{c}}_k, \mathbf{0}) = \frac{-2\bar{c}_k}{T} (\mathbf{y}_{k,-1})' \Omega^{-1} \Delta_S \mathbf{y} + \left(\frac{\bar{c}_k}{T}\right)^2 (\mathbf{y}_{k,-1})' \Omega^{-1} \mathbf{y}_{k,-1} \quad (3.2)$$

$$= \frac{-2\bar{c}_k}{T\,(2\pi f_k)} (\mathbf{y}_{k,-1})' \Delta_S \mathbf{y} + \frac{1}{(2\pi f_k)} \left(\frac{\bar{c}_k}{T}\right)^2 (\mathbf{y}_{k,-1})' \mathbf{y}_{k,-1} + o_p(1) \quad (3.3)$$

where $f_k \equiv \frac{\sigma^2}{2\pi} [\psi(e^{-i\omega_k})] [\psi(e^{i\omega_k})]$ is the spectral density function of $\{v_{Sn+s}\}$ at frequency $2\pi k/S$, $\Delta_S \mathbf{y} \equiv (\Delta_S y_1, \Delta_S y_2, ..., \Delta_S y_T)'$, $\mathbf{y}_{k,-1} \equiv (y_{k,0}, y_{k,1}, ..., y_{k,T-1})'$, $k = 0, ..., [S/2]$ and where, corresponding to the zero and seasonal frequencies $\omega_k = 2\pi k/S$, $k = 0, ..., \lfloor S/2 \rfloor$,

$$y_{0,Sn+s} = \sum_{i=0}^{S-1} y_{Sn+s-i}, \quad y_{S/2,Sn+s} = \sum_{i=0}^{S-1} \cos[(i+1)\pi] y_{Sn+s-i}$$

$$y_{j,Sn+s} = \sum_{i=0}^{S-1} \cos[(i+1)\omega_j] y_{Sn+s-i}, \quad j = 1, ..., S^*,$$

together with $\Delta_S y_{Sn+s} \equiv y_{Sn+s} - y_{S(n-1)+s}$.

We now show in Theorem 3.1 that the most powerful test of $H_0 : \mathbf{c} = \mathbf{0}$ against $H_{1,\bar{\mathbf{c}}_k} : \mathbf{c} = \bar{\mathbf{c}}_k$ from Proposition 3.1 has precisely the same local asymptotic power function as given in Equation (4) of ERS, p.816, for $k = 0$ and $k = S/2$ (S even), and as given in the first part of Theorem 3.1 of Gregoir (2004,p.17) for $k = 1, ..., S^*$, and that these results pertain regardless of whether or not the maintained hypothesis, outlined above, holds. Consequently, the envelope power functions for these tests are available in the literature. Although the test is constructed under the maintained hypothesis that the non-centrality parameters at all frequencies not under test are equal to zero, we also demonstrate that the most powerful test which obtains on relaxing this assumption has the same local asymptotic power function.

Theorem 3.1 *Let $\{x_{Sn+s}\}$ be generated by (2.1)-(2.2) under Assumption 3.1 and $\alpha(L) = \Delta_{\mathbf{c}}$ with $\mathbf{c} = (c_0, c_1, ..., c_{\lfloor S/2 \rfloor})'$,*

$$\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0}) \Rightarrow \begin{cases} -2\bar{c}_k \int_0^1 J_{k,c_k}(r) dJ_{k,0}(r) + \bar{c}_k^2 \int_0^1 [J_{k,c_k}(r)]^2 dr, & k = 0, S/2 \ (S \text{ even}) \\ -\bar{c}_k \int_0^1 \left[J^{\alpha}_{k,c_k}(r) dJ^{\alpha}_{k,0}(r) + J^{\beta}_{k,c_k}(r) dJ^{\beta}_{k,0}(r)\right] & \\ \quad + \frac{\bar{c}_k^2}{2} \int_0^1 \left\{ \left[J^{\alpha}_{k,c_k}(r)\right]^2 + \left[J^{\beta}_{k,c_k}(r)\right]^2 \right\} dr, & k = 1, ..., S^* \end{cases}$$
$$\equiv \mathcal{F}_k(c_k, \bar{c}_k), \quad k = 0, ..., [S/2]$$

where "$\Rightarrow$" denotes weak convergence of the associated probability measures, the $J_{k,c_k}(r)$, $k = 0$, $k = S/2$ (S even), $J^{\alpha}_{k,c_k}(r)$ and $J^{\beta}_{k,c_k}(r)$, $k = 1, ..., S^*$, are independent standard

Ornstein-Uhlenbeck [OU] processes (see Proposition B.1 in Appendix B for details). Consequently, the local asymptotic power function for the α*-level test indexed by* $\bar{c}_k$ *is given by*

$$\pi_k(c_k, \bar{c}_k)^0 \equiv \Pr\left[\mathcal{F}_k(c_k, \bar{c}_k) < \ell_k(\alpha)\right] \tag{3.4}$$

where $\ell_k(\alpha)$ *satisfies* $\pi_k(0, \bar{c}_k)^0 = \alpha$*. Since the test indexed by* $\bar{c}_k$ *is optimal against the alternative* $\bar{c}_k$*, the envelope power function for frequency* ω_k *for this family of point-optimal tests is* $\Pi_k(c_k)^0 \equiv \pi_k(c_k, c_k)^0$*,* $k = 0, ..., \lfloor S/2 \rfloor$*.*

Remark 3.1: Notice that the local limiting distributions, and hence local asymptotic power functions, given for $\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0})$, $k = 0, ..., \lfloor S/2 \rfloor$, in Theorem 3.1, depend only on c_k and $\bar{c}_k$; that is, the same representations hold regardless of the elements (other than c_k) of the vector of true non-centrality parameters, $\mathbf{c}$. This powerful result is due to the asymptotic orthogonality results in (B.1) and (B.2). Of course, if the elements of $\mathbf{c}$ (other than c_k) are *not* all equal to zero then rejecting for small values of $\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0})$ will no longer yield the *exact* most powerful test of $c_k = 0$ against $c_k = \bar{c}_k$. However, as we show in Remark 3.5 below, in such cases $\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0})$ differs from the statistic upon which the exact most powerful test is based only by an asymptotically negligible term.

Remark 3.2: From Theorem 3.1, the statistics $\mathcal{L}(\bar{\mathbf{c}}_0, \mathbf{0})$ and $\mathcal{L}(\bar{\mathbf{c}}_{S/2}, \mathbf{0})$ (S even) possess identical and independent limiting representations. These are also identical to the representation given in Equation (4) of ERS p.816. Consequently, the limiting power envelopes for tests at the zero and Nyquist frequencies in a seasonally observed process, regardless of its seasonal period, both coincide with the function $\Pi(c)$ of ERS p.816, which is graphed on p.822 of ERS.

Remark 3.3: From Theorem 3.1, the $\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0})$ statistics possess independent and identical limiting representations across $k = 1, ..., S^*$, which are independent of those for $\mathcal{L}(\bar{\mathbf{c}}_0, \mathbf{0})$ and $\mathcal{L}(\bar{\mathbf{c}}_{S/2}, \mathbf{0})$ (S even). Moreover, these are identical to the representation given in the first part of Gregoir (2004, Theorem 3.1, p.17). The limiting power envelopes for tests at each harmonic seasonal frequencies therefore coincide with the function graphed in Gregoir (2004,Figure 4,p.45).

Remark 3.4: Useful computational formulae for $\Pi_k(c_k)$, $k = 0, ..., \lfloor S/2 \rfloor$, are given in Tanaka (1996,pp.344-345), which can also be used to evaluate the power functions $\pi_k(c_k, \bar{c}_k)^0$, $k = 0, ..., \lfloor S/2 \rfloor$, of (3.4); see also Tanaka (1996, Table 9.3, p.348) for some relevant tabulations. These formulae can also be used to compute the corresponding quantities for the optimal tests when the deterministic component is unknown, developed in Section 3.2.

Remark 3.5: Suppose now that we wished to test $H_{0,\mathbf{c}^*} : \mathbf{c} = \mathbf{c}^*$, where $\mathbf{c}^* \equiv (c_0^*, c_1^*, ..., c_{\lfloor S/2 \rfloor}^*)'$ has $c_k^* = 0$, versus $H_{1,\mathbf{c}^* + \bar{\mathbf{c}}_k} : \mathbf{c} = \mathbf{c}^* + \bar{\mathbf{c}}_k$, $\bar{\mathbf{c}}_k$ defined as above. Our maintained hypothesis is now that the elements (other than c_k) of $\mathbf{c}$ are known (finite) constants, not necessarily equal to zero. Then, as shown in Appendix A.1, the most powerful test rejects for small values of the statistic

$$\mathcal{L}_k(\mathbf{c}^* + \bar{\mathbf{c}}_k, \mathbf{c}^*) = \mathcal{L}_k(\bar{\mathbf{c}}_k, \mathbf{0}) + \sum_{\substack{j=0 \\ j \neq k}}^{\lfloor S/2 \rfloor} \frac{\bar{c}_k c_j^*}{T^2} \left(\mathbf{y}_{k,-1}\right)' \Omega^{-1} \mathbf{y}_{j,-1},$$

$k = 0, ..., \lfloor S/2 \rfloor$. However, the asymptotic orthogonality results (B.1)-(B.2) ensure that the additional (scaled) cross product terms introduced are each of $o_p(1)$. This then implies

that the structure of the Neyman-Pearson test of $H_{0,k} : c_k = 0$ against $H_{1,c_k} : c_k = \bar{c}_k$, is asymptotically invariant to the elements (other than c_k^*) of $\mathbf{c}^*$. Consequently, the results of Theorem 3.1 apply equally to $\mathcal{L}_k(\mathbf{c}^* + \bar{\mathbf{c}}_k, \mathbf{c}^*)$. In particular $\mathcal{L}_k(\mathbf{c}^* + \bar{\mathbf{c}}_k, \mathbf{c}^*)$ and $\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0})$ have identical local asymptotic power functions which depend only on c_k and $\bar{c}_k$.
Remark 3.6: For completeness, it is interesting to note that if the deterministic component μ_{Sn+s} were to behave as a slowly evolving trend, as defined in Condition B of ERS, p.816, then replacing y_{Sn+s} by x_{Sn+s} in the foregoing expressions will change neither the asymptotic size nor power of the Neyman-Pearson test. Indeed, there is no loss in (asymptotic) efficiency from μ_{Sn+s} being unknown in such cases. This occurs because the stochastic component of x_{Sn+s} dominates the deterministic component in such cases when n is large; see ERS for further discussion on this point.

3.2 Unknown Deterministic Component

We now consider each of Cases 2-6 of μ_{Sn+s} outlined in Section 2.1, dropping the assumption that β is known. We restrict attention to the class of β-invariant tests; that is, tests which for a given value of i, $i = 2, ..., 6$, are (exact) invariant to the group of transformations $x_{Sn+s} \mapsto x_{Sn+s} + \bar{\beta}' z_{Sn+s,i}$, for arbitrary $\bar{\beta}$. To that end, we define the T-dimensional vector $x_{\mathbf{c}}$ and the matrix $\mathbf{Z}_{i,\mathbf{c}}$, the index $i \in \{2, ..., 6\}$ indicating which of Cases 2 to 6 holds, by:

$$\begin{aligned} x_{\mathbf{c}} &= (x_1, x_2 - \alpha_1^{\mathbf{c}} x_1, x_3 - \alpha_1^{\mathbf{c}} x_2 - \alpha_2^{\mathbf{c}} x_1, ..., x_S - \alpha_1^{\mathbf{c}} x_{S-1} - \cdots - \alpha_S^{\mathbf{c}} x_1, \Delta_{\mathbf{c}} x_{S+1}, ...\Delta_{\mathbf{c}} x_T)' \\ \mathbf{Z}_{i,\mathbf{c}} &= (z_{1,i}, z_{2,i} - \alpha_1^{\mathbf{c}} z_{1,i}, z_{3,i} - \alpha_1^{\mathbf{c}} z_{2,i} - \alpha_2^{\mathbf{c}} z_{1,i}, ..., z_{S,i} - \alpha_1^{\mathbf{c}} z_{S-1,i} - \cdots \\ &\quad -\alpha_S^{\mathbf{c}} z_{1,i}, \Delta_{\mathbf{c}} z_{S+1,i}, ..., \Delta_{\mathbf{c}} z_{T,i})' \end{aligned} \tag{3.5}$$

where the $z_{Sn+s,i}$, $i = 2, ..., 6$, and $\Delta_{\mathbf{c}}$ are as defined in Sections 2.1 and Section 2.2 respectively. Using (3.5), we can re-write (3.1) as

$$L(\mathbf{c}, \beta)_i = (x_{\mathbf{c}} - \mathbf{Z}_{i,\mathbf{c}}\beta)' \Omega^{-1} (x_{\mathbf{c}} - \mathbf{Z}_{i,\mathbf{c}}\beta) \tag{3.6}$$

the subscript $i \in \{2, ..., 6\}$ again indicates which of Cases 2-6 is being considered.

In order to develop an optimal test suppose first, as in the known deterministic case, that we wish to test $H_0 : \mathbf{c} = \mathbf{0}$ against $H_{1,\bar{\mathbf{c}}_k} : \mathbf{c} = \bar{\mathbf{c}}_k$. We may then appeal directly to the results in Lehmann (1986, Ch.6.2-6.3) to obtain that, in the case of our normal likelihood, the most powerful invariant (MPI) test of H_0 against $H_{1,\bar{\mathbf{c}}_k}$ rejects for small values of the statistic,

$$\mathcal{L}_{k,T}^i = \min_\beta L(\bar{\mathbf{c}}_k, \beta)_i - \min_\beta L(\mathbf{0}, \beta)_i, \quad k = 0, 1, ..., \lfloor S/2 \rfloor,\ i = 2, ..., 6. \tag{3.7}$$

Notice that the test statistic $\mathcal{L}_{k,T}^i$ is then the difference in (weighted) sum of squared residuals from two constrained GLS regressions, both appropriate to Case i, $i \in \{2, ..., 6\}$, one imposing $H_{1,\bar{\mathbf{c}}_k}$ and the other, H_0. In Proposition 3.2 we now give a more convenient representation for the MPI statistic.

Proposition 3.2 *The MPI statistics* $\mathcal{L}_{k,T}^i$, $k = 0, ..., \lfloor S/2 \rfloor$, $i = 2, ..., 6$, *are of the form*

$$\begin{aligned} \mathcal{L}_{k,T}^i &= -2\left(\frac{\bar{c}_k}{T}\mathbf{y}_{k,-1}\right)' \Omega^{-1} \Delta_S \mathbf{y} + \left(\frac{\bar{c}_k}{T}\mathbf{y}_{k,-1}\right)' \Omega^{-1} \frac{\bar{c}_k}{T}\mathbf{y}_{k,-1} + Q^M(\mathbf{0})_i - Q^M(\bar{\mathbf{c}}_k)_i \\ &= \mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0}) + Q^M(\mathbf{0})_i - Q^M(\bar{\mathbf{c}}_k)_i \end{aligned} \tag{3.8}$$

where $\Delta_S \mathbf{y}$ *and* $\mathbf{y}_{k,-1}$, $k = 0, ..., \lfloor S/2 \rfloor$, *are as defined in Proposition 3.1, and where* $Q^M(\mathbf{a})_i = (\Delta_{\mathbf{a}}\mathbf{y})' \Omega^{-1} \mathbf{Z}_{i,\mathbf{a}} \left[\mathbf{Z}'_{i,\mathbf{a}} \Omega^{-1} \mathbf{Z}_{i,\mathbf{a}}\right]^{-1} \mathbf{Z}'_{i,\mathbf{a}} \Omega^{-1} (\Delta_{\mathbf{a}}\mathbf{y})$, *with* $\mathbf{a} = \mathbf{0}, \bar{\mathbf{c}}_k$.

Remark 3.7: Notice that under Case 1 the minimisation in (3.7) yields precisely the Neyman-Pearson statistic, $\mathcal{L}(\bar{\mathbf{c}}_k, \mathbf{0})$ of (3.2), as should be expected.

We now show in Theorem 3.2 that the MPI test of $H_0 : \mathbf{c} = \mathbf{0}$ against $H_{1,\bar{\mathbf{c}}_k} : \mathbf{c} = \bar{\mathbf{c}}_k$ has the same local asymptotic power function as given in Equations (4) and (8) of ERS (pp.416,418) for $k = 0$ and $k = S/2$ (S even), and as given in Theorem 3.1 of Gregoir (2004,p.17) for $k = 1, ..., S^*$, and that again these results hold regardless of whether or not the maintained hypothesis holds. In what follows we introduce the index ξ whose value is determined by i (the deterministic case of interest, $i = 1, ..., 6$) and the frequency under test. Precisely, for the zero frequency ω_0 tests: Case 1: $\xi = 0$; Cases 2 and 3: $\xi = 1$; Cases 4, 5 and 6: $\xi = 2$. For the seasonal frequency ω_k tests, $k = 1, ..., \lfloor S/2 \rfloor$: Cases 1, 2 and 4: $\xi = 0$; Cases 3 and 5: $\xi = 1$; Case 6: $\xi = 2$. The nomenclature ξ has this meaning throughout the paper.

Theorem 3.2 *Let* $\{x_{Sn+s}\}$ *be generated by (2.1)-(2.2) under Assumption 3.1 with* β *unknown. Then under* $H_{1,\mathbf{c}} : \mathbf{c} = (c_0, c_1, ..., c_{\lfloor S/2 \rfloor})'$,

$$\mathcal{L}^i_{k,T} \Rightarrow \mathcal{F}_k(c_k, \bar{c}_k) + Q_{\xi,k}(c_k, \bar{c}_k) \equiv \eta_{\xi,k}(c_k, \bar{c}_k) \tag{3.9}$$

where $\mathcal{F}_k(c_k, \bar{c}_k)$ *is as defined in Theorem 3.1, and where* $Q_{\xi,k}(c_k, \bar{c}_k) = 0$ *if* $\xi = 0$ *or* $\xi = 1$, $k = 0, \ldots, \lfloor S/2 \rfloor$, *and*

$$Q_{2,k}(c_k, \bar{c}_k) \equiv \begin{cases} [J_{k,c_k}(1)]^2 - \left(1 - \bar{c}_k + \frac{\bar{c}_k^2}{3}\right)^{-1} \left[(1-\bar{c}_k) J_{k,c_k}(1) + \bar{c}_k^2 \int_0^1 r J_{k,c_k}(r) dr\right]^2, & k = 0, S/2 \\ \left\{\left[J^\alpha_{k,c_k}(1)\right]^2 + \left[J^\beta_{k,c_k}(1)\right]^2\right\} & \\ - \left[\left(1 - \bar{c}_k + \frac{\bar{c}_k^2}{3}\right)\right]^{-1} \left\{(1-\bar{c}_k)\left[J^\beta_{k,c_k}(1) + J^\alpha_{k,c_k}(1)\right]\right. & \\ \qquad \left. + \bar{c}_k^2 \int_0^1 r \left[J^\beta_{k,c_k}(r) + J^\alpha_{k,c_k}(r)\right] dr\right\}^2, & k = 1, ..., S^*, \end{cases}$$

where $J_{0,c_0}(r)$, $J_{S/2,c_{S/2}}(r)$, $J^\alpha_{k,c_k}(r)$ *and* $J^\beta_{k,c_k}(r)$, $k = 1, ..., S^*$ *are as given in Theroem 3.1. Consequently, the local asymptotic power function for the* α*-level test indexed by* $\bar{c}_k$ *is given by*

$$\pi_k(c_k, \bar{c}_k)^\xi \equiv \Pr\left[\eta_{\xi,k}(c_k, \bar{c}_k) < \ell_{\xi,k}(\alpha)\right] \tag{3.10}$$

where $\ell_{\xi,k}(\alpha)$ *satisfies* $\pi_k(0, \bar{c}_k)^\xi = \alpha$. *Notice, therefore, that* $\pi_k(c_k, \bar{c}_k)^1 = \pi_k(c_k, \bar{c}_k)^0$, $k = 0, ..., \lfloor S/2 \rfloor$. *Since the test indexed by* $\bar{c}_k$ *is optimal against the alternative* $\bar{c}_k$, *for a given value of* ξ, *the envelope power function for frequency* ω_k *for this family of point-optimal tests is* $\Pi_k(c_k)^\xi \equiv \pi_k(c_k, c_k)^\xi$, $k = 0, ..., \lfloor S/2 \rfloor$.

Remark 3.8: As was observed in Remark 3.1 for the known deterministic case (Case 1), the local limiting distributions, and hence local asymptotic power functions, of the $\mathcal{L}^\xi_{k,T}$, $k = 0, ..., \lfloor S/2 \rfloor$ statistics, for given ξ, depend only on c_k and $\bar{c}_k$. That is, the stated results hold regardless of the elements (other than c_k) of the vector of true non-centrality parameters, $\mathbf{c}$. As in Remark 3.5 it is also possible, but tedious, to demonstrate that the MPI test of $H_{0,\mathbf{c}^*} : \mathbf{c} = \mathbf{c}^*$ against $H_{1,\mathbf{c}^*+\bar{\mathbf{c}}_k} : \mathbf{c} = \mathbf{c}^* + \bar{\mathbf{c}}_k$, $\bar{\mathbf{c}}_k$ (both as defined in Remark 3.5) differs from $\mathcal{L}^{\xi}_{k,T}$ by an asymptotically negligible term.

Remark 3.9: From Theorem 3.2, for a given value of ξ, the statistics $\mathcal{L}^{\xi}_{0,T}$ and $\mathcal{L}^{\xi}_{S/2,T}$ (S even) possess identical and independent limiting representations. For $\xi = 1$ these are identical to the representations given in Theorem 3.1 and hence have identical local power functions as for the known deterministic case. For $\xi = 2$ these are identical to the representation given in part (c) of Theorem 1 of ERS page 818. Consequently, the limiting power envelopes for tests at the zero and Nyquist frequencies in a seasonally observed process, regardless of its seasonal period, for $\xi = 2$ coincides with $\Pi(c)^{\tau}$ of ERS page 818, graphed on page 823 of ERS.

Remark 3.10: For a given value of ξ, it is seen from the results in Theorem 3.2 that the harmonic seasonal frequency statistics $\mathcal{L}^{\xi}_{k,T}$ possess identical and independent limiting representations across $k = 1, ..., S^*$, which are independent of those for $\mathcal{L}^{\xi}_{0,T}$ and $\mathcal{L}^{\xi}_{S/2,T}$ (S even). For $\xi = 1$ these are identical to those of Theorem 3.1 and hence have identical local power functions as for the known deterministic case. For $\xi = 2$ these are identical to the representation given in the second part of Theorem 3.1 of Gregoir (2004,p.17). Consequently, the limiting power envelopes for tests at the harmonic frequencies in a seasonally observed process, regardless of its seasonal period, for $\xi = 2$ coincides with $\pi^{\text{tr}}(c, c)$ of Gregoir (2004,pp.17,46).

4 Feasible Point-Optimal Tests

We now drop the assumption that the variance matrix Ω is known and replace the condition of zero initial values on the y_{Sn+s} process with the following weaker assumption:

Assumption 4.1 *The initial conditions $\{y_s\}_{s=1-S}^{0}$ are each of $O_p(T^{\nu})$, $\nu < 0.5$.*

Remark 4.1: Assumption 4.1 still requires that the initial conditions of the process are asymptotically negligible. In the non-seasonal case Elliott (1999) and Müller and Elliott (2003) have relaxed this to allow for a wider range of possible assumptions on the initial value. The results in this paper provide a keystone for further analysis under weaker assumptions, of the form considered in Elliott (1999) and Müller and Elliott (2003), on the intial values. Such extensions are beyond the scope of the present paper but are currently being considered by the authors.

Remark 4.2: For the purposes of this Section, the linear process structure placed on $\{v_{Sn+s}\}$ in Section 2 could be supplanted by a seasonal modification of Condition C of ERS (p.818) without altering the results in this Section. However, our ultimate focus in this paper is on the GLS-detrended HEGY tests which we introduce in Section 5 and so we have deliberately maintained a parametric structure for modelling the weak dependence in $\{v_{Sn+s}\}$. Of course, both the parameters characterising the lag polynomial $\psi(z)$ and σ^2 are no longer assumed known and the assumption of normality on $\{u_{Sn+s}\}$ has also been dropped.

Assuming that the initial conditions, $y_{1-S}, ..., y_0$ are all zero and that $v_{Sn+s} \sim NIID(0,1)$, the likelihood ratio statistics $\mathcal{L}^i_{k,T}$, $k = 0, ..., \lfloor S/2 \rfloor$, $i = 1, ..., 6$, take very simple forms. Specifically, under these conditions $\mathcal{L}^i_{k,T} = \mathcal{S}(\bar{\mathbf{c}}_k, k)_i - \mathcal{S}(\mathbf{0}, k)_i$, where

$$\mathcal{S}(\mathbf{a}, k)_i = (\Delta_{\mathbf{a}}\mathbf{y})' (\Delta_{\mathbf{a}}\mathbf{y}) - Q(\mathbf{a})_\xi$$

$\mathbf{a} = \mathbf{0}, \bar{\mathbf{c}}_k$, and where $Q(\mathbf{a})_\xi$ is as defined below (B.23) of Appendix B, are the residual sum of squares obtained from regressing $x_{\mathbf{c}}$ on $\mathbf{Z}_{i,\mathbf{c}}$ under $H_{1,\bar{\mathbf{c}}_k}$ and H_0, respectively. Under Case 1, the terms $Q(\bar{\mathbf{c}}_k)_\xi$ and $Q(\mathbf{0})_\xi$ are simply omitted from the foregoing expressions. Under the weaker conditions outlined at the start of this Section these statistics can be modified to produce asymptotically pivotal statistics. Precisely, for $k = 0$ and $k = S/2$ (S even), the statistics

$$P_{k,T}(\bar{c}_k)_i = \hat{\omega}_k^{-2} \left[\mathcal{S}(\bar{\mathbf{c}}_k, k)_i - \left(1 + \frac{\bar{c}_k}{T}\right) \mathcal{S}(\mathbf{0}, k)_i \right]$$

and for $k = 1, ..., S^*$,

$$P_{k,T}(\bar{c}_k)_i = \hat{\omega}_k^{-2} \left[\mathcal{S}(\bar{\mathbf{c}}_k, k)_i - \left(1 + \frac{\bar{c}_k}{T}\right)^2 \mathcal{S}(\mathbf{0}, k)_i \right]$$

where $\hat{\omega}_k \equiv 2\pi \hat{f}_k$ with $\hat{f}_k$ a consistent estimator[1] of f_k, $k = 0, ..., \lfloor S/2 \rfloor$, can be straightforwardly shown to attain the same asymptotic power envelope functions as the corresponding tests from Theorems 3.1 and 3.2. The results in Theorems 3.1 and 3.2 required the assumptions that Ω was known and that the errors u_{Sn+s} were normally distributed. Consequently, the family of modified tests above can attain these Gaussian limiting power envelopes under the considerably weaker conditions of this Section.

The foregoing results show that, under the conditions stated above, the tests which reject for small values of the $P_{k,T}(\bar{c}_k)_i$, $k = 0, ..., \lfloor S/2 \rfloor$, statistics have asymptotic local power functions which are tangential to the Gaussian power envelope (for the given value of ξ) at the single point $c_k = \bar{c}_k$, and, hence, no asymptotically uniformly most powerful test exists. In practice, therefore, and as is typical in this literature, we follow King (1988) and select from this family of tests that which is associated with the value of c_k whose power function is tangential to the asymptotic local power envelope in the vicinity of power one-half, when run at a given nominal level. The values of c_k appropriate for 5% level tests are readily obtainable from ERS and Gregoir (2004) [these values are replicated in Section 5], who demonstrate that the resultant tests have asymptoptic local power functions which lie arbitrarily close to the Gaussian envelope over a broad range of values of c_k.

5 Regression-based Nearly Efficient Tests

By far the most popularly applied non-seasonal ($S = 1$) unit root tests from ERS are their pseudo-GLS versions of the familiar ADF t-tests; see ERS p.824. These are computed from the OLS regression of $\Delta_1 x^i_n$ on x^i_{n-1} and lags of $\Delta_1 x^i_n$, where $x^i_n \equiv x_n - \tilde{\beta}'_{\bar{c},i} z_n$, is the pseudo-GLS de-trended x_n. ERS choose $\bar{c}_0$ to be that alternative where maximal power

[1] See, *inter alia*, Breitung and Franses (1998), Chambers and McGarry (2002) and Gregoir (2004) for detailed discussions of suitable estimators.

is approximately one-half; *viz.*, -7 for the case of a constant mean ($i = 2$) and -13.5 for the constant plus linear trend case ($i = 4$), for 5% level tests. These tests are shown to have asymptotic local power functions which are virtually indistinguishable from the power envelope in both cases. In contrast, the asymptotic local functions of the standard ADF tests, based on OLS de-trended data, lie well below the envelope in each case. Gregoir (2004) proposes pseudo-GLS versions of the ADF-type complex conjugate unit root tests of Ahtola and Tiao (1987).

Given the success of the modified ADF-based procedure suggested by ERS we now generalise that approach to the seasonal case. Precisely, our suggested approach to testing for seasonal unit roots in $\alpha(L)$ consists of two stages. First we use a (pseudo-) GLS estimator, rather than the OLS estimator of HEGY, Beaulieu and Miron (1993) and Smith and Taylor (1999a), *inter alia*, to de-trend the data, as detailed in Section 3. Provided μ_{Sn+s} is not estimated under an overly restrictive case, standard GLS regression theory tells us that the resulting unit root tests will yield exact invariant inference with respect to β.

As in ERS and Gregoir (2004), we must use a pseudo-GLS estimator of β in our de-trending regression. The problem is at first sight more involved than the situations covered by these authors since they focused only on tests of a single non-centrality parameter while in our setting we have a vector of non-centrality parameters of order $\lfloor S/2 \rfloor + 1$, the elements of which are considered simultaneously. In the foregoing analysis we have established the maximal power for tests on each element of this vector in isolation for each of Cases 2-6. Establishing an appropriate pseudo-GLS trend estimator might therefore seem complicated. However, from the asymptotic orthogonality of the power envelope functions across spectral frequencies we may use the pseudo-GLS estimator, $\widetilde{\beta}_i^\dagger$ obtained from regressing $x_{\mathbf{c}}$ on $\mathbf{Z}_{i,\mathbf{c}}$ for $\mathbf{c} = \mathbf{c}^\dagger \equiv (c_{0,\xi}^\dagger, c_{1,\xi}^\dagger, ..., c_{\lfloor S/2 \rfloor,\xi}^\dagger)'$, where $c_{j,\xi}^\dagger$ are the 50 % points off the power envelopes for a given i, $i = 2, ..., 6$, for each spectral frequency, in all cases for tests run at the 100 α % level. Consequently, we have from ERS (pp.821,823) that for 5% level tests, $c_{k,1}^\dagger = -7$ and $c_{k,2}^\dagger = -13.5$ for $k = 0, S/2$, while from Gregoir (2004,p.19) $c_{j,1}^\dagger = -3.75$ and $c_{j,2}^\dagger = -8.65$ for $j = 1, ..., S^*$. The general pseudo-GLS estimator (of which $\widetilde{\beta}_i^\dagger$ is a special case) obtained from regressing $x_{\mathbf{c}}$ on $\mathbf{Z}_{i,\mathbf{c}}$ for $\mathbf{c} = \bar{\mathbf{c}} \equiv (\bar{c}_0, \bar{c}_1, ..., \bar{c}_{\lfloor S/2 \rfloor})'$ will be denoted as $\widetilde{\beta}_i(\bar{\mathbf{c}})$. We will denote the pseudo-GLS de-trended data from this regression as $\hat{x}_{Sn+s}^i \equiv x_{Sn+s} - [\widetilde{\beta}_i(\bar{\mathbf{c}})]' z_{Sn+s,i}$. Notice that, for Case 1, $\hat{x}_{Sn+s}^1 = x_{Sn+s}$, by definition, since no de-trending is performed.

Consider again (2.1)-(2.2) under Assumption 4.1, and further assume that $\psi(z)$ is invertible with (unique) inverse $\phi(z)$, such that an autoregressive approximation is valid. Following Smith and Taylor (1999a, Equation (2.11), page 6), our second stage consists of expanding $\alpha(L)$ in (2.1) around the seasonal unit roots $\exp(\pm i 2\pi k/S)$, $k = 0, ..., \lfloor S/2 \rfloor$, to yield the auxiliary regression equation,

$$
\begin{aligned}
\Delta_S \hat{x}_{Sn+s}^i &= \pi_0 \hat{x}_{0,Sn+s-1}^i + \sum_{j=1}^{S^*} \left(\pi_k \hat{x}_{j,Sn+s-1}^i + \pi_k^\beta \hat{x}_{j,Sn+s-1}^{\beta,i} \right) \\
&+ \pi_{S/2} \hat{x}_{S/2,Sn+s-1} + \sum_{p=1}^{p^*} \phi_p \Delta_S \hat{x}_{Sn+s-p}^i + \text{error}, \qquad (5.1)
\end{aligned}
$$

omitting the term $\pi_{S/2} \hat{x}_{S/2,Sn+s-1}^i$ if S is odd, and where corresponding to the zero and seasonal frequencies $\omega_k = 2\pi k/S$, $k = 0, ..., \lfloor S/2 \rfloor$,

$$\hat{x}^i_{0,Sn+s} \equiv \sum_{j=0}^{S-1} \hat{x}^i_{Sn+s-j}, \quad \hat{x}^i_{S/2,Sn+s} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\pi]\hat{x}^i_{Sn+s-j},$$

$$\hat{x}^i_{k,Sn+s} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\omega_k]\hat{x}^i_{Sn+s-j}, \quad \hat{x}^{\beta,i}_{k,Sn+s} \equiv -\sum_{j=0}^{S-1} \sin[(j+1)\omega_k]\hat{x}^i_{Sn+s-j}, \tag{5.2}$$

$k = 1, ..., S^*$, together with $\Delta_S \hat{x}^i_{Sn+s} \equiv \hat{x}^i_{Sn+s} - \hat{x}^i_{S(n-1)+s}$. For quarterly, $S = 4$, data the relevant transformations are

$$\hat{x}^i_{0,Sn+s} \equiv (1 + L + L^2 + L^3)\hat{x}^i_{Sn+s}, \quad \hat{x}^i_{2,Sn+s} \equiv -\left(1 - L + L^2 - L^3\right)\hat{x}^i_{Sn+s},$$

$$\hat{x}^i_{1,Sn+s} \equiv -L(1 - L^2)\hat{x}^i_{Sn+s}, \quad \hat{x}^{\beta,i}_{1,Sn+s} \equiv -(1 - L^2)\hat{x}^i_{Sn+s}.$$

The existence of unit roots at the zero, Nyquist and harmonic seasonal frequencies imply that $\pi_0 = 0$, $\pi_{S/2} = 0$ (S even) and $\pi_k = \pi_k^\beta = 0$, $k = 1, ..., S^*$, in (5.1) respectively; see Smith and Taylor (1999a). Consequently, and in order to test $H_{0,k}$ against $H_{1,k}$, $k = 0, ..., \lfloor S/2 \rfloor$, we follow HEGY, Beaulieu and Miron (1993) and Smith and Taylor (1999a), *inter alia*, and suggest the following regression statistics in (5.1): $\hat{t}_0$ (left-sided) for the exclusion of $\hat{x}^i_{0,Sn+s-1}$; $\hat{t}_{S/2}$ (left-sided) for the exclusion of $\hat{x}^i_{S/2,Sn+s-1}$ (S even); $\hat{t}_k$ (left-sided) and $\hat{t}_k^\beta$ (two-sided) for the exclusion of $\hat{x}^i_{k,Sn+s-1}$ and $\hat{x}^{\beta,i}_{k,Sn+s-1}$ respectively, and $\hat{F}_k$ for the exclusion of *both* $\hat{x}^i_{k,Sn+s-1}$ and $\hat{x}^{\beta,i}_{k,Sn+s-1}$, $k = 1, ..., S^*$. Following GLN, Taylor (1998), and Smith and Taylor (1998,1999a) we also consider the joint frequency F-statistics, $\hat{F}_{1...\lfloor S/2 \rfloor}$, for the exclusion of $\{\hat{x}^i_{j,Sn+s-1}\}_{j=1}^{\lfloor S/2 \rfloor}$ and $\{\hat{x}^{\beta,i}_{j,Sn+s-1}\}_{j=1}^{S^*}$, and $\hat{F}_{0...\lfloor S/2 \rfloor}$, for H_0, the exclusion of $\{\hat{x}^i_{j,Sn+s-1}\}_{j=0}^{\lfloor S/2 \rfloor}$ and $\{\hat{x}^{\beta,i}_{j,Sn+s-1}\}_{j=1}^{S^*}$.

We now detail the limiting distribution of these tests under the general local alternative $H_{1,\mathbf{c}} : \mathbf{c} = (c_0, c_1, ..., c_{\lfloor S/2 \rfloor})'$. The parameter ξ is as defined in Section 3.2, while δ_ξ is a dummy variable such that $\delta_\xi = 0$ if $\xi = 0, 1$ and $\delta_\xi = 1$ if $\xi = 2$.

Theorem 5.1 *Let $\{x_{Sn+s}\}$ be generated by (2.1)-(2.2) under Assumption 4.1, with $\phi(z) = 1$. Then under $H_{1,\mathbf{c}} : \mathbf{c} = (c_0, c_1, ..., c_{\lfloor S/2 \rfloor})'$,*

$$\hat{t}_j \Rightarrow c_j \left\{ \int_0^1 \left[J_{j,c_j}(r, \delta_\xi \bar{c}_j)\right]^2 dr \right\}^{1/2} + \left\{ \int_0^1 J_{j,c_j}(r) dJ_{j,0}(r) - \delta_\xi \mathcal{D}_{c_j}(r, \bar{c}_j) \int_0^1 r dJ_{j,0}(r) \right\} \times \left\{ \int_0^1 \left[J_{j,c_j}(r, \delta_\xi \bar{c}_j)\right]^2 dr \right\}^{-1/2}, \quad j = 0, S/2, \tag{5.3}$$

$$\hat{t}_k \Rightarrow \frac{c_k}{2} \left\{ \int_0^1 \left[J^{\alpha,\beta}_{k,c_k}(r, \delta_\xi \bar{c}_k)\right]^2 dr \right\}^{1/2} + \left\{ \left[\int_0^1 J^\alpha_{k,c_k}(r) dJ^\alpha_{k,0}(r) + \int_0^1 J^\beta_{k,c_k}(r) dJ^\beta_{k,0}(r) \right] - 2\delta_\xi \begin{bmatrix} \mathcal{D}_{c_k}(r, \bar{c}_k) \\ \mathcal{D}^\beta_{c_k}(r, \bar{c}_k) \end{bmatrix}' \Lambda_{1,k} \begin{bmatrix} \int_0^1 r dJ^\alpha_{k,0}(r) \\ \int_0^1 r dJ^\beta_{k,0}(r) \end{bmatrix} \right\} \left\{ \int_0^1 \left[J^{\alpha,\beta}_{k,c_k}(r, \delta_\xi \bar{c}_k)\right]^2 dr \right\}^{-1/2} \tag{5.4}$$

$$\hat{t}_k^{\beta} \Rightarrow \left\{\left[\int_0^1 J_{k,c_k}^{\alpha}(r)dJ_{k,0}^{\beta}(r) - \int_0^1 J_{k,c_k}^{\beta}(r)dJ_{k,0}^{\alpha}(r)\right]\right.$$
$$\left. - 2\delta_\xi \begin{bmatrix} \mathcal{D}_{c_k}(r,\bar{c}_k) \\ \mathcal{D}_{c_k}^{\beta}(r,\bar{c}_k) \end{bmatrix}' \Lambda_{2,k} \begin{bmatrix} \int_0^1 r dJ_{k,0}^{\alpha}(r) \\ \int_0^1 r dJ_{k,0}^{\beta}(r) \end{bmatrix}\right\}\left\{\int_0^1 \left[\bar{J}_{k,c_k}^{\alpha,\beta}(r,\delta_\xi\bar{c}_k)\right]^2 dr\right\}^{-1/2} \quad (5.5)$$

$$\hat{F}_k - \frac{1}{2}\left\{[\hat{t}_k]^2 + [\hat{t}_k^{\beta}]^2\right\} = o_p(1),\ k = 1,\ldots,S^*, \quad (5.6)$$

$$\hat{F}_{1\ldots\lfloor S/2\rfloor} - \frac{1}{S-1}\left\{[\hat{t}_{S/2}]^2 + \sum_{k=1}^{S^*}\left([\hat{t}_k]^2 + [\hat{t}_k^{\beta}]^2\right)\right\} = o_p(1), \quad (5.7)$$

$$\hat{F}_{0\ldots\lfloor S/2\rfloor} - \frac{1}{S}\left\{[\hat{t}_0]^2 + [\hat{t}_{S/2}]^2 + \sum_{k=1}^{S^*}\left([\hat{t}_k]^2 + [\hat{t}_k^{\beta}]^2\right)\right\} = o_p(1) \quad (5.8)$$

omitting terms involving $\tau_{S/2,\xi}(c_{S/2},\delta_\xi)$ if S is odd. The limiting processes $J_{j,c_j}(r)$, $j = 0, S/2$, $J_{k,c_k}^{\alpha}(r)$ and $J_{k,c_k}^{\beta}(r)$, $k = 1,...,S^*$ are defined in Proposition B.1 of Appendix B, while $J_{j,c_j}(r,\delta_\xi\bar{c}_j)$ and $\mathcal{D}_{c_j}(r,\bar{c}_j)$, $j = 0, S/2$, and $J_{k,c_k}^{\alpha,\beta}(r,\delta_\xi\bar{c}_k)$, $\bar{J}_{k,c_k}^{\alpha,\beta}(r,\delta_\xi\bar{c}_k)$, $\Lambda_{1,k}$, $\Lambda_{2,k}$, $\mathcal{D}_{c_k}(r,\bar{c}_k)$ and $\mathcal{D}_{c_k}^{\beta}(r,\bar{c}_k)$, $k = 1,...,S^*$, are defined in the proof of Theorem 5.1 in Appendix C.

Remark 5.1: Setting $\bar{\mathbf{c}} \equiv \mathbf{c}^{\dagger}$ the representations given in Theorem 5.1 delineate the asymptotic local power functions (for nominal 100 α % level tests) of the pseudo-GLS de-trended seasonal unit root tests from (5.1) using the values for c_k where maximal power is approximately one-half, as discussed in the paragraph preceding Theorem 5.1. For $k = 0$ and $k = S/2$ (S even), these representations are equivalent to those given for the $DF-GLS^{\mu}$ and $DF-GLS^{\tau}$ statistics in ERS (pp.824-825) for $\xi = 0, 1$ and $\xi = 2$, respectively. Graphs of these limiting power functions for 5% significance level tests are therefore as given in Figures 2 and 3 of ERS, pp.823-824. For $k = 1,...,S^*$, the limiting power functions of the 5% level $\hat{t}_k$ and $\hat{F}_k$ tests are as graphed in Figures 2 and 3 of Gregoir (2004,p.20) for $\xi = 0, 1$ and $\xi = 2$, respectively.

Remark 5.2: Theorem 5.1 was derived under the assumption that $\{u_{Sn+s}\} \sim IID(0,\sigma^2)$. It is straightforward but incredibly tedious to show that the results for all but the $\hat{t}_k$ and $\hat{t}_k^{\beta}$, $k = 1,...,S^*$, statistics also hold under the more general condition that $\{u_{Sn+s}\}$ follows a *stationary AR(p)*, $0 \le p < \infty$, process; i.e., $\phi(L)v_{Sn+s} = u_{Sn+s} \sim IID(0,\sigma^2)$, provided $p^* \ge p$ in (5.1); *cf.* Burridge and Taylor (2001), Rodrigues (2002) and Rodrigues and Taylor (2004b). More generally, where u_{Sn+s} satisfies the linear process conditions detailed in Section 2, we conjecture that these results will also continue to hold provided p^* is of $o_p(T^{1/3})$; cf. Said and Dickey (1984). Moreover, under the linear process conditions an alternative estimation approach based on frequency domain regression might also be considered which will deliver statistics with pivotal asymptotic null distributions. This method is outlined in Chambers and McGarry (2002) for Case 1 (no deterministics) but could be readily extended to the case of pseudo-GLS de-trended data.

Remark 5.3: It can be seen from (5.3) of Theorem 5.1 that for both $\xi = 0$ and $\xi = 1$, $\hat{t}_0$ and $\hat{t}_{S/2}$ have identical standard Dickey-Fuller (1979) limiting distributions under H_0. Moreover, for $\xi = 2$ the limiting null representations of $\hat{t}_0$ for $\bar{c}_0 = c_{0,\xi}^{\dagger}$ and $\hat{t}_{S/2}$ for $\bar{c}_{S/2} = c_{S/2,\xi}^{\dagger}$ are also identical and coincide with the representation given in ERS p.824 evaluated for $c = 0$. The $\hat{t}_0$ and $\hat{t}_{S/2}$ statistics are also asymptotically independent under both H_0 and $H_{1,\mathbf{c}}$.

Remark 5.4: Representations (5.4)-(5.5) of Theorem 5.1 demonstrate that, under both H_0 and $H_{1,\mathbf{c}}$, $\hat{t}_k$ and $\hat{t}_l$, $k \neq l$, and $\hat{t}_k^\beta$ and $\hat{t}_l^\beta$, $k \neq l$, $k, l = 1, \ldots, S^*$, possess identical limiting distributions, and hence asymptotic local power functions, and are mutually asymptotically independent and are also asymptotically independent of $\hat{t}_0$ and $\hat{t}_{S/2}$.

Remark 5.5: The representations given in Theorem 5.1 allow an explanation for the similarity between critical values which occurs in different scenarios and between different statistics; cf. Table 5.1. *E.g.*, from (5.3) it is seen that $\hat{t}_{S/2}$ has an identical limiting null distribution under Cases 3 and 4, which coincides with that of $\hat{t}_0$ under Cases 2 and 3. Moreover, for any given statistic, the limiting distributions of the statistic for $\xi = 0$ and $\xi = 1$ coincide.

Remark 5.6: The F-statistics, $\hat{F}_k$, $k = 1, ..., S^*$, are asymptotically mutually independent and are asymptotically independent of $\hat{t}_0$ and $\hat{t}_{S/2}$ under both H_0 and $H_{1,\mathbf{c}}$. Moreover, the F-statistic $\hat{F}_{1\ldots\lfloor S/2 \rfloor}$ is asymptotically independent of $\hat{t}_0$ under both H_0 and $H_{1,\mathbf{c}}$.

Table 5.1 about here

Selected finite sample and asymptotic critical values for the quarterly GLS-HEGY $\hat{t}_0$, $\hat{t}_2$, $\hat{F}_1$, $\hat{F}_{0\ldots2}$ and $\hat{F}_{1\ldots2}$ tests for $\alpha = 0.05$ are provided in Table 5.1. The finite sample critical values were based on the DGP $\Delta_4 x_{4n+s} = u_{4n+s} \sim IN(0,1)$, $s = -3, \ldots, 0$, $n = 1, \ldots, N$, with $u_{4j+s} = x_{4j+s} = 0$, $j \leq 0$, with results reported for the sample sizes $T = 48, 100, 136, 200$ and 400, for the statistics computed from Cases 3 (seasonal de-meaning), 5 (seasonal de-meaning and non-seasonal de-trending) and 6 (seasonal de-meaning and seasonal de-trending) of (5.1) with $p^* = 0$.[2] The asymptpotic critical values were obtained by direct simulation of the appropriate limiting functionals from Theorem 5.1 with $c_k = 0$, $k = 0, 1, 2$, using a sample size of $T = 3,000$. The asymptotic critical values given for the $\hat{t}_0$, $\hat{t}_2$ and $\hat{F}_1$ tests are also appropriate[3] for the $\hat{t}_0$, $\hat{t}_{S/2}$ (S even) and $\hat{F}_k$, $k = 1, ..., S^*$, tests, respectively, for other values of S; cf. Remarks 5.3-5.5. Notice from Theorem 5.1 that the asymptotic critical values for Case 3 also apply to Cases 1 and 2 for all tests and also to Case 4 for the seasonal frequency tests. All reported simulations were computed using the RNDN function of Gauss 3.1 over 100,000 replications.

6 Numerical Results

In this Section we use Monte Carlo simulation methods to investigate the small sample properties (size under autocorrelated errors and power under stationary alternatives) of

[2]Cases 1, 2 and 4 do not yield tests which are *exact* similar with respect to the initial conditions and hence we recommend against their use; cf. Smith and Taylor (1998). However, finite sample critical values for these cases and for the $\hat{t}_1$ and $\hat{t}_1^\beta$ tests are available from the authors on request.

[3]The 1%, 2.5%, 5% and 10% lower-tail asymptotic critical values for the $(\hat{t}_k, \hat{t}_k^\beta)$, $k = 1, ..., S^*$, tests are respectively $(-2.57, -2.37)$, $(-2.23, -1.99)$, $(-1.93, -1.67)$, and $(-1.60, -1.30)$ for $\xi = 1$ and $(-3.92, -2.63)$, $(-3.62, -2.24)$, $(-3.37, -1.90)$, and $(-3.08, -1.48)$ for $\xi = 2$. The upper-tail asymptotic critical values for the $\hat{t}_k^\beta$, $k = 1, ..., S^*$ tests are the negative of those above due to the symmetry of the limiting null distribution.

the GLS-HEGY $\hat{t}_0$, $\hat{t}_1$, $\hat{t}_2$, $\hat{F}_1$, $\hat{F}_{1,\ldots,2}$ and $\hat{F}_{0,\ldots,2}$ tests of Section 5 ($\alpha = 0.05$) for quarterly data, $S = 4$, comparing these with the conventional quarterly OLS-HEGY tests: $\tilde{t}_0$, $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{F}_1$, $\tilde{F}_{1,\ldots,2}$ and $\tilde{F}_{0,\ldots,2}$.[4] The OLS-HEGY tests are based on the corresponding statistic from (5.1) replacing the (pseudo-)GLS-detrended data $\hat{x}^i_{Sn+s}$ by the OLS-detrended data, $\tilde{x}^i_{Sn+s} = x_{Sn+s} - [\tilde{\beta}_i]' z_{Sn+s}$, where $\tilde{\beta}_i$ is obtained from regressing x_{Sn+s} on $z_{Sn+s,i}$, $Sn + s = 1, ..., T$, $i = 2, ..., 6$. In assessing the finite-sample size and power properties of these tests we have focused on $N = 25$ and $N = 50$ (which correspond to samples of $T = 100$ and $T = 200$ observations, respectively), and on Cases 3 and 6 of (5.1), where $\xi = 1$ and $\xi = 2$ respectively for all reported tests. All tests were run at the nominal 0.05 level using finite sample critical values generated under the quarterly seasonal random walk null. The remaining cases of (5.1) and other nominal levels were also considered, as were the corresponding tests for other values of S, but in each case yielded qualitatively similar results to those reported. In all experiments, the lag truncation order p^* in (5.1) was chosen *via* a data-dependent rule. As is commonly done in practice, we followed the *general-to-specific* approach outlined in Beaulieu and Miron (1993,pp.318-19), starting with an initial four lags of the dependent variable ($p_{\max} = 4$) and progressively deleting lags found to be insignificant at the 0.10 level.

The simulation results reported in this section were programmed using the RNDN function of Gauss 3.1 and, unless otherwise stated, were based on 50,000 replications for each experiment. These programs (together with those used to compute the critical values in Table 5.1) are available from the authors on request and can replicate all experiments reported in this paper for arbitrary S and N and also allow for other methods of selecting p^*, the lag truncation order, (including deterministic) in (5.1).

6.1 Size Properties

Table 6.1 reports empirical rejection frequencies for the above tests (nominal 0.05 level) when the true DGP for $\{x_{4n+s}\}$ is:

$$\Delta_4 x_{4n+s} = u_{4n+s}, \; s = -3, \ldots, 0, n = 2, \ldots, N, \tag{6.1}$$
$$(1 - \phi L) u_{4n+s} = (1 + \theta L^2) v_{4n+s} \sim IN(0, 1), \;\; s = -3, \ldots, 0, n = -100, \ldots, N, \tag{6.2}$$

with $x_j = 0$, $j = 1, \ldots, S$. We consider the effects of $\phi = 0.9$, holding $\theta = 0$, and $\theta = \pm 0.6$, holding $\phi = 0$.[5] The first case allows for a large peak in the spectrum of $\{u_{4n+s}\}$ at the zero frequency, while the second induces a *near cancellation of roots* at both the zero and Nyquist frequencies for $\theta = -0.6$, and at the harmonic seasonal frequency for $\theta = 0.6$.

Tables 6.1 – 6.4 about here

Consider first the case where $\phi = 0.9$ and $\theta = 0$. One would hope that all of the reported tests would lie approximately on their nominal levels, given the choice of $p_{\max} = 4$. While

[4] We also considered the weighted and simple symmetric variants of the HEGY tests of Rodrigues and Taylor (2004a) together with the FD de-trended HEGY tests of Rodrigues (2002). These tests were all somewhat more powerful than the OLS-HEGY tests but less powerful than the GLS-HEGY tests reported here. Since these tests are not based on any optimality principle nor have they, to the best of our knowledge, been used in the applied literature we do not report these results. They are, however, available on request.

[5] Other parameter values were considered but qualitatively do not add to or contradict what is reported.

this is largely the case, some undersizing is seen, particularly in the case of the $\hat{t}_1$ and $\tilde{t}_1$ tests. However, it is known from results in Burridge and Taylor (2001) that these statistics have limiting distributions which depend on the serial correlation nuisance parameters so these results are not surprising. For $N = 25$ some under-sizing is seen in certain of the other OLS-HEGY and GLS-HEGY tests, especially in the case of $\xi = 2$, with these distortions largely ameliorated for $N = 50$.

Moving to the case where $\phi = 0$ and $\theta = \pm 0.6$, we see from Table 6.1 that similar patterns of size distortion are seen across the OLS-HEGY and GLS-HEGY variants of each test. However, the GLS-HEGY tests do appear to display the smallest size distortions overall, with the differences in performance between the OLS-HEGY and GLS-HEGY tests greatest for $\xi = 1$. Both variants of both the zero and π frequency t-tests distort above the nominal level for $\theta = -0.6$, while the $\pi/2$ frequency F-tests distort above the nominal level for $\theta = 0.6$, as expected from the induced near-cancellation of roots problem in each of these cases outlined above. These patterns are transmitted into the joint frequency tests in both cases, as would be expected. As with the case of pure AR shocks discussed above, size distortions tend to decrease somewhat between $N = 25$ and $N = 50$.

6.2 Empirical Power

This sub-section compares the empirical finite sample power properties of the OLS-HEGY and GLS-HEGY tests against near-seasonal unit root processes; that is, we investigate the relative finite sample local power of the tests. To that end, Tables 6.2-6.3 report the empirical power (nominal 0.05 level) of the above tests when the true DGP for $\{x_{4n+s}\}$ is the near-seasonally integrated AR model:

$$\left[1 - \left(1 + \frac{c_0}{T}\right) L\right] \left[1 + \left(1 + \frac{c_2}{T}\right) L\right] \left[1 + \left(1 + \frac{c_1}{T}\right)^2 L^2\right] x_{4n+s} = v_{4n+s} \sim IN(0,1), \quad (6.3)$$

with $s = -3, \ldots, 0, n = 1, \ldots, N$, $x_j = v_j = 0$, $j \leq 0$. We investigate the effects of varying the non-centrality parameter c_k among $c_k \in \{-1, -5, -7, -11, -15, -19\}$ in our experiments. Results for $N = 25$ are reported in Table 6.2 and for $N = 50$ in Table 6.3. Recall that the corresponding results for $N \to \infty$ are discussed in Section 5. For completeness, and to facilitate comparison, we have also simulated the limiting powers of the OLS-HEGY and GLS-HEGY tests and these are reported in Table 6.4. These were obtained by direct simulation methods, in the same manner as the results in Table 5.1, but for the appropriate value of c_k, $k = 0, 1, 2$, using $100,000$ replications and a sample size of $1,000$. Results are not reported for the tests based on either the $\hat{t}_1^\beta$ or $\tilde{t}_1^\beta$ statistics since under (6.3) both of these statistics converge in probability to zero; cf. Rodrigues (2001).

The results reported in Tables 6.2-6.3 pertain to the case where, when moving a particular non-centrality parameter c_k, $k = 0, 1, 2$, away from unity, the remaining non-centrality parameters were all held at zero.[6] For example, the entries for $\hat{t}_0$ and $\tilde{t}_0$ tests for $c_k = -5$ relate to $(c_0, c_1, c_2) = (-5, 0, 0)$. As for the joint frequency tests, for $c_k = -5$, for example, the entries for the joint seasonal frequencies tests pertain to $(c_0, c_1, c_2) = (0, -5, -5)$ while

[6] Allowing the other non-centrality parameters to deviate from zero at the same time had no discernable effect.

those for the joint test across the zero and seasonal frequencies pertain to $(c_0, c_1, c_2) = (-5, -5, -5)$.

It is quite clear from the results in Tables 6.2-6.3 that the GLS-HEGY seasonal unit root tests proposed in this paper enjoy considerable finite-sample power advantages over the corresponding conventional OLS-HEGY tests, although the power differentials between the OLS-HEGY and GLS-HEGY tests are generally smaller for $\xi = 2$, *vis-à-vis* $\xi = 1$, echoing the observations of ERS and Gregoir (2004). In fact, there is not one single example in either Table 6.2 or Table 6.3 where an OLS-HEGY test displays superior power to the corresponding GLS-HEGY test. In the mid-power range of the power functions for $\xi = 1$, as a general rule of thumb, the GLS-HEGY tests appear to have roughly double the power of the corresponding OLS-HEGY test, although in many cases the difference can be even higher. For example, for $\xi = 1$ and $c_1 = -5$ the GLS-HEGY $\hat{t}_1$ test has power of 61 %, while the corresponding OLS-HEGY $\tilde{t}_1$ test has power of only 20 %. Finally, as N is increased the reported quantities for both the OLS-HEGY and GLS-HEGY tests appear to be converging towards the asymptotic local power levels reported in Table 6.4, as should be expected.

7 Conclusions

In this paper we have derived asymptotic local power bounds for seasonal unit root tests for both known and unknown deterministic scenarios and for an arbitrary seasonal aspect. Moreover, we have shown that the optimal test of a unit root at a given spectral frequency behaves asymptotically independently of whether unit roots exist at other frequencies or not. The point optimal tests were derived under stringent assumptions (Gaussian innovations, a known error covariance matrix and zero initial conditions). We have demonstrated that these conditions can be relaxed and that modified versions of the tests can achieve the same asymptotic local power functions as the Gaussian point optimal tests. We have also proposed near-efficient regression-based (HEGY) seasonal unit root tests using pseudo-GLS de-trending and shown that these have well-known limiting null distributions and asymptotic local power functions. Our Monte Carlo results suggest that the pseudo-GLS de-trended versions, with data-dependent lag selection, display much improved finite-sample power properties over the original OLS de-trended HEGY tests, yet display very similar size properties against seasonal unit root processes driven by weakly dependent innovations.

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Appendix A

A.1 Proof of Proposition 3.1

As in section 5, expanding $\alpha(L)$ in (2.1) around the seasonal unit roots $\exp(\pm i2\pi k/S)$, $k = 0, ..., \lfloor S/2 \rfloor$, yields the auxiliary regression which is an unrestricted re-parameterisation of (2.1)-(2.2):

$$\Delta_S y_{Sn+s} = \pi_0 y_{0,Sn+s-1} + \sum_{j=1}^{S^*} \left(\pi_k y_{k,Sn+s-1} + \pi_k^\beta y_{k,Sn+s-1}^\beta \right) + \pi_{S/2} y_{S/2,Sn+s-1} + v_{Sn+s}, \quad \text{(A.1)}$$

omitting the term $\pi_{S/2} y_{S/2,Sn+s-1}$ if S is odd, and where, $y_{j,Sn+s}$, $j = 0, S/2$, $y_{k,Sn+s}$, $k = 1, ..., S^*$, and $\Delta_S y_{Sn+s}$ are as defined in Proposition 3.1, together with $y_{j,Sn+s}^\beta = -\sum_{i=0}^{S-1} \sin[(i+1)\omega_j] y_{Sn+s-i}$, $k = 1, ..., S^*$. The exact (linear) relationship between the α_j^*, $j = 1, ..., S$, parameters of (2.1) and the π_k, $k = 0, ..., \lfloor S/2 \rfloor$ and π_j^β, $j = 1, ..., S^*$, parameters of (A.1) is detailed in the Proposition given in HEGY, pp.221-222. Consequently, using (A.1), $L(\mathbf{c})$ in (3.1) can be equivalently re-written as

$$L(\mathbf{c}) = \left(\Delta_S \mathbf{y} - \sum_{j=0}^{[S/2]} \pi_j \mathbf{y}_{j,-1} - \sum_{r=1}^{S^*} \pi_r^\beta \mathbf{y}_{r,-1}^\beta \right)' \Omega^{-1} \left(\Delta_S \mathbf{y} - \sum_{j=0}^{[S/2]} \pi_j \mathbf{y}_{j,-1} - \sum_{r=1}^{S^*} \pi_r^\beta \mathbf{y}_{r,-1}^\beta \right)$$

where $\Delta_S \mathbf{y}$ and $\mathbf{y}_{j,-1}$, $j = 0, ..., \lfloor S/2 \rfloor$, are as defined in Proposition 3.1, and $\mathbf{y}_{j,-1}^\beta \equiv \left(y_{j,0}^\beta, y_{j,1}^\beta, ..., y_{j,T-1}^\beta \right)'$ $j = 1, ..., S^*$. Expanding terms, and noting from the Proposition in HEGY that in the near-integrated model the regression parameters of (A.1) reduce to $\pi_k \equiv \frac{c_k}{T}$, $k = 0, 1, ..., [S/2]$ and $\pi_j^\beta = 0$, $j = 1, ..., S^*$, we obtain that

$$L(\mathbf{c}) = (\Delta_S \mathbf{y})' \Omega^{-1} \Delta_S \mathbf{y} - 2 \sum_{j=0}^{[S/2]} \frac{c_j}{T} \mathbf{y}_{j,-1}' \Omega^{-1} \Delta_S \mathbf{y} + \sum_{i=0}^{[S/2]} \sum_{j=0}^{[S/2]} \frac{c_j c_i}{T^2} \mathbf{y}_{j,-1}' \Omega^{-1} \mathbf{y}_{i,-1}.$$

It is then straightforward to show that under the assumptions of Remark 3.5,

$$\mathcal{L}_k(\mathbf{c}^* + \bar{\mathbf{c}}_k, \mathbf{c}^*) = \mathcal{L}_k(\bar{\mathbf{c}}_k, \mathbf{0}) + \sum_{\substack{j=0 \\ j \neq k}}^{\lfloor S/2 \rfloor} \frac{\bar{c}_k c_j^*}{T^2} \mathbf{y}_{k,-1}' \Omega^{-1} \mathbf{y}_{j,-1} \tag{A.2}$$

where

$$\mathcal{L}_k(\bar{\mathbf{c}}_k, \mathbf{0}) = -2 \frac{\bar{c}_k}{T} \mathbf{y}_{k,-1}' \Omega^{-1} \Delta_S \mathbf{y} + \left(\frac{\bar{c}_k}{T} \right)^2 \mathbf{y}_{k,-1}' \Omega^{-1} \mathbf{y}_{k,-1}, \tag{A.3}$$

$k = 0, ..., [S/2]$. The decomposition in (3.3) then follows from (3.2) using ERS (Lemma A2,p.831), for $k = 0$ and $k = S/2$ (S even), and Gregoir (2004, Lemma A.5,p.37) for $k = 1, ..., S^*$.

A.2 Proof of Proposition 3.2

Notice first that under both $H_0 : \mathbf{c} = \mathbf{0}$ and $H_{1,\bar{\mathbf{c}}_k} : \mathbf{c} = \bar{\mathbf{c}}_k$, (2.1)-(2.2) can be written in matrix form as

$$\Delta_S \mathbf{x} - \bar{\pi}_k \mathbf{x}_{k,-1} = \mathbf{Z}_{i,\mathbf{c}} \beta^* + \mathbf{v}, \qquad k = 0, ..., \lfloor S/2 \rfloor, \ i = 2, ..., 6, \tag{A.4}$$

where β^* and β are linear in a vector of parameters, $\bar{\pi}_k = 0$ under H_0 and $\bar{\pi}_k = \frac{\bar{c}_k}{T}$ under $H_{1,\bar{\mathbf{c}}_k}$; and where $\Delta_S \mathbf{x} \equiv (\Delta_S x_1, \Delta_S x_2, ..., \Delta_S x_T)'$ and $\mathbf{x}_{j,-1} \equiv (x_{j,0}, x_{j,1}, ..., x_{j,T-1})'$, $j = 0, ..., [S/2]$, with $x_{j,Sn+s}$, $j = 0, ..., \lfloor S/2 \rfloor$, as defined in Proposition 3.1 replacing y_{Sn+s} by x_{Sn+s} throughout. Consequently, defining $\widetilde{\beta}_{\bar{\mathbf{c}}_k,i}$ to be the GLS estimator,

$$\widetilde{\beta}_{\bar{\mathbf{c}}_k,i} = \left[\mathbf{Z}_{i,\bar{\mathbf{c}}_k}' \Omega^{-1} \mathbf{Z}_{i,\bar{\mathbf{c}}_k} \right]^{-1} \mathbf{Z}_{i,\bar{\mathbf{c}}_k}' \Omega^{-1} \left(\Delta_S \mathbf{x} - \frac{\bar{c}_k}{T} \mathbf{x}_{k,-1} \right), \ \ i = 2, ..., 6,$$

we observe from standard GLS projection theory (cf. ERS, p.834) that,

$$\min_\beta L(\bar{\mathbf{c}}_k, \beta)_i = \left(\Delta_S \mathbf{y} - \frac{\bar{c}_k}{T} \mathbf{y}_{k,-1} \right)' \Omega^{-1} \left(\Delta_S \mathbf{y} - \frac{\bar{c}_k}{T} \mathbf{y}_{k,-1} \right) - Q^M(\bar{\mathbf{c}}_k)_i,$$

$i = 2, ..., 6$, where $Q^M(\bar{\mathbf{c}}_k)_i$ is as defined in Proposition 3.2. Consequently,

$$\begin{aligned} \mathcal{L}_{k,T}^i &= \min_\beta L(\bar{\mathbf{c}}_k, \beta)_i - \min_\beta L(\mathbf{0}, \beta)_i \\ &= \left(\Delta_S \mathbf{y} - \frac{\bar{c}_k}{T} \mathbf{y}_{k,-1} \right)' \Omega^{-1} \left(\Delta_S \mathbf{y} - \frac{\bar{c}_k}{T} \mathbf{y}_{k,-1} \right) - (\Delta_S \mathbf{y})' \Omega^{-1} \Delta_S \mathbf{y} + Q^M(\mathbf{0})_i - Q^M(\bar{\mathbf{c}}_k)_i, \end{aligned}$$

$i = 2, ..., 6$, from which the stated result follows immediately using Proposition 3.1.

Appendix B

For the purposes of Appendices B and C all references to results for the Nyquist frequency, $\omega_{S/2} = \pi$, are taken to apply only where S is even.

B.1 Proof of Theorem 3.1

In order to prove Theorem 3.1 we first reproduce the following proposition due to Rodrigues and Taylor (2004b) which details the large sample properties of the regressors from (A.1):

Proposition B.1 *Under the conditions of Theorem 3.1,*

$$\begin{aligned}
\frac{1}{\sqrt{T}} y_{0,S\lfloor rN \rfloor + s} &\Rightarrow \sigma\psi(1) J_{0,c_0}(r) \\
\frac{1}{\sqrt{T}} y_{S/2,S\lfloor rN \rfloor + s} &\Rightarrow \sigma\psi(-1)(-1)^i J_{S/2,c_{S/2}}(r),\ i = (S\lfloor rN \rfloor + s) \bmod 2 \\
\frac{1}{\sqrt{T}} y_{k,S\lfloor rN \rfloor + s} &\Rightarrow \sigma \left\{ e_s' \varrho^*_{\alpha,k} \left[a_k J^{\beta}_{k,c_k}(r) + b_k J^{\alpha}_{k,c_k}(r) \right] + e_s' \varrho^*_{\beta,k} \left[-a_k J^{\alpha}_{k,c_k}(r) + b_k J^{\beta}_{k,c_k}(r) \right] \right\} \\
\frac{1}{\sqrt{T}} y^{\beta}_{k,S\lfloor rN \rfloor + s} &\Rightarrow \sigma \left\{ e_s' \varrho^*_{\beta,k} \left[a_k J^{\beta}_{k,c_k}(r) + b_k J^{\alpha}_{k,c_k}(r) \right] - e_s' \varrho^*_{\alpha,k} \left[-a_k J^{\alpha}_{k,c_k}(r) + b_k J^{\beta}_{k,c_k}(r) \right] \right\},
\end{aligned}$$

$r \in [0, 1]$, $s = 1-S, ..., 0$, and where e_s is an $S \times 1$ seasonal selection vector with $(S-s)$th element equal to unity and all other elements equal to zero, $s = 1-S, ..., 0$, $\varrho^*_{\alpha,k} = [\cos(\omega_k), \cos(0), ..., \cos((2-S)\omega_k)]'$, $\varrho^*_{\beta,k} = [\sin(\omega_k), \sin(0), ..., \sin((2-S)\omega_k)]'$, $a_k = Im\left[\psi(\exp(-i\omega_k)\right]$, $b_k = Re\left[\psi(\exp(-i\omega_k)\right]$, $k = 1, ..., S^*$, and $J_{0,c_0}(r)$, $J^{\alpha}_{k,c_k}(r)$ and $J^{\beta}_{k,c_k}(r)$, $k = 1, ..., S^*$, and, for S even, $J_{S/2,c_{S/2}}(r)$ are mutually independent standard OU processes. Moreover, the following asymptotic orthogonality results hold between the regressors in (A.1) (see also Jeganathan, 1991):

$$T^{-2} \sum_{s=1-S}^{0} \sum_{n=1}^{N} y_{j,Sn+s-1} y_{k,Sn+s-1} \to^p \quad 0,\ \ j, k = 0, ..., \lfloor S/2 \rfloor, j \neq k \tag{B.1}$$

$$T^{-2} \sum_{s=1-S}^{0} \sum_{n=1}^{N} y_{j,Sn+s-1} y^{\beta}_{k,Sn+s-1} \to^p \quad 0,\ \ j = 0, ..., \lfloor S/2 \rfloor,\ k = 1, ..., S^*,\ j \neq k. \tag{B.2}$$

The proof of Theorem 3.1 then follows from the results in Proposition B.1 and applications of the continuous mapping theorem [CMT].

B.2 Proof of Theorem 3.2

B.2.1 Preliminary Results

In order to establish the large sample distributions of the MPI statistics from Proposition 3.2 under $H_{1,\mathbf{c}}$ we first, for notational convenience, write the scaled deterministic components for each of Cases 2-6, $\bar{\mathbf{Z}}_{i,\bar{\mathbf{c}}} = \mathbf{Z}_{i,\bar{\mathbf{c}}} N_{i,T}$, $i = 2, ..., 6$, where $\mathbf{Z}_{i,\bar{\mathbf{c}}}$ is as defined in Section 3.2, and the scaling matrices $N_{i,T}$, $i = 2, ..., 6$, are such that: $N_{2,T} = N^{1/2}$, $N_{3,T} = diag(N^{1/2}, ..., N^{1/2})$, $N_{4,T} = diag(N^{1/2}, 1)$, $N_{5,T} = diag(N^{1/2}, ..., N^{1/2}, 1)$ and $N_{6,T} = diag(N^{1/2}, ..., N^{1/2}, 1, ..., 1)$. These are now stated in Proposition B.2 which may be readily verified by direct calculation.

Proposition B.2 *The scaled deterministic matrices, $\bar{\mathbf{Z}}_{i,\bar{\mathbf{c}}}$, $i = 2, ..., 6$, take the form:*

$$\begin{aligned}
\bar{\mathbf{Z}}_{2,\bar{\mathbf{c}}} &= \bar{\mathbf{Z}}_{2,\bar{c}_0} \\
\bar{\mathbf{Z}}_{3,\bar{\mathbf{c}}} &= \left(\bar{\mathbf{Z}}_{2,\bar{c}_0}, \bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_1}, \bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_1}, ..., \bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_{S^*}}, \bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_{S^*}}, \bar{\mathbf{Z}}_{3,\bar{c}_{S/2}}\right) \\
\bar{\mathbf{Z}}_{4,\bar{\mathbf{c}}} &= \left(\bar{\mathbf{Z}}_{2,\bar{c}_0}, \bar{\mathbf{t}}_{0,\bar{c}_0}\right) \\
\bar{\mathbf{Z}}_{5,\bar{\mathbf{c}}} &= \left(\bar{\mathbf{Z}}_{3,\bar{\mathbf{c}}}, \bar{\mathbf{t}}_{0,\bar{c}_0}\right); \\
\bar{\mathbf{Z}}_{6,\bar{\mathbf{c}}} &= \left(\bar{\mathbf{Z}}_{3,\bar{\mathbf{c}}}, \bar{\mathbf{t}}_{0,\bar{c}_0}, \bar{\mathbf{t}}^{\alpha}_{1,\bar{c}_1}, \bar{\mathbf{t}}^{\beta}_{1,\bar{c}_1}, ..., \bar{\mathbf{t}}^{\alpha}_{S^*,\bar{c}_{S^*}}, \bar{\mathbf{t}}^{\beta}_{S^*,\bar{c}_{S^*}}, \bar{\mathbf{t}}_{\lfloor S/2\rfloor,\bar{c}_{[S/2]}}\right)
\end{aligned}$$

where $\bar{\mathbf{Z}}_{2,\bar{c}_0} = \sqrt{T}\mathbf{1}_0 + \frac{\bar{c}_0}{\sqrt{T}}\mathbf{e}_0 - \frac{\bar{c}_0}{\sqrt{T}}S\mathbf{1}$, $\bar{\mathbf{Z}}_{3,\bar{c}_{S/2}} = \sqrt{T}\mathbf{1}_{S/2} + \frac{\bar{c}_{S/2}}{\sqrt{T}}\mathbf{e}_{S/2} - \frac{\bar{c}_{S/2}}{\sqrt{T}}S\mathbf{1}_{\lfloor S/2\rfloor}$, $\bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_k} = \sqrt{T}\mathbf{1}_{1,k} + \frac{\bar{c}_k}{\sqrt{T}}\mathbf{e}_{1k} - \frac{\bar{c}_k}{\sqrt{T}}(S/2)\mathbf{1}^{\alpha}_k$, $\bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_1} = \sqrt{T}\mathbf{1}_{2,k} + \frac{\bar{c}_k}{\sqrt{T}}\mathbf{e}_{2k} - \frac{\bar{c}_k}{\sqrt{T}}(S/2)\mathbf{1}^{\beta}_k$, $\bar{\mathbf{t}}_{0,\bar{c}_0} = S\mathbf{1} - \mathbf{e}_0 - \frac{\bar{c}_0}{T}\left[S\mathbf{t}_0 - \left(\sum_{i=1}^{S-1} i\right)\mathbf{1}\right] + o(1)$, $\bar{\mathbf{t}}_{\lfloor S/2\rfloor,\bar{c}_{[S/2]}} = S\mathbf{1}_{\lfloor S/2\rfloor} - \mathbf{e}_{S/2} - \frac{\bar{c}_{S/2}}{T}\left[S\mathbf{t}_{\lfloor S/2\rfloor} - \left(\sum_{i=1}^{S} i\right)\mathbf{1}_{\lfloor S/2\rfloor}\right] + o(1)$, $\bar{\mathbf{t}}^{\alpha}_{k,\bar{c}_k} = \frac{S}{2}\left(\mathbf{1}^{\alpha}_k - \frac{\bar{c}_k}{T}\mathbf{t}^{\alpha}_k\right) + o(1)$, $\bar{\mathbf{t}}^{\beta}_{k,\bar{c}_k} = \frac{S}{2}\left(\mathbf{1}^{\beta}_k - \frac{\bar{c}_k}{T}\mathbf{t}^{\beta}_k\right) + o(1)$, *and where* $\mathbf{1} = (1, ..., 1)'$, $\mathbf{1}_{\lfloor S/2\rfloor} = \left[(-1)^1, ..., (-1)^T\right]'$, $\mathbf{t}_0 = (1, 2, ..., T)'$, $\mathbf{t}_{\lfloor S/2\rfloor} = \left[(-1)^1, ..., T(-1)^T\right]'$, *together with* $\mathbf{1}^{\alpha}_k = [\cos(w_k), ..., \cos(Tw_k)]'$, $\mathbf{1}^{\beta}_k = [\sin(w_k), ..., \sin(Tw_k)]'$, $\mathbf{t}^{\alpha}_k = [\cos(w_k), ..., T\cos(Tw_k)]'$ *and* $\mathbf{t}^{\beta}_k = [\sin(w_k), ..., T\sin(w_kT)]'$, $w_k = \frac{2\pi k}{S}$, $k = 1, ..., S^*$, *while* $\mathbf{1}_0 = (1, ..., 1, 0, ..., 0)'$ *is a* $T \times 1$ *vector with first* S *elements equal to one and all other elements equal to zero;* $\mathbf{e}_0$ *is a* $T \times 1$ *vector such that* $\mathbf{e}_0 = (S, ..., 1, 0, ..., 0)'$; $\mathbf{1}_{S/2} = (-1, 1, -1, ..., 1, 0, ..., 0)'$ *is a* $T \times 1$ *vector with first* S *elements equal to the first* S *elements of* $\mathbf{1}_{\lfloor S/2\rfloor}$ *and all other elements equal to zero,* $\mathbf{e}_{S/2} = \mathbf{e}_0 \odot \mathbf{1}_{S/2}$, *where* $\odot$ *denotes the Hadamard (elementwise) product;* $\mathbf{1}_{1,k}$ *and* $\mathbf{1}_{2,k}$ *are* $T \times 1$ *vectors with first* S *elements equal to the first* S *elements of* $\mathbf{1}^{\alpha}_k$ *and* $\mathbf{1}^{\beta}_k$*, respectively, and all other elements equal to zero;* $\mathbf{e}_{1k} = \mathbf{e}_0 \odot \mathbf{1}^{\alpha}_k$ *and* $\mathbf{e}_{2k} = \mathbf{e}_0 \odot \mathbf{1}^{\beta}_k$, $k = 1, ..., S^*$;

The proof of Theorem 3.2 is greatly simplified by using the well-known result that deterministic components only affect the large sample distributions of statistics at the same spectral frequency; see, *inter alia*, Chan (1989) and Smith and Taylor (1999b). Hence, of the six cases considered in Section 2, Cases 2 and 3, and Cases 4, 5 and 6 coincide asymptotically for zero frequency statistics, while Cases 2 and 4, and Cases 3 and 5 will coincide asymptotically for seasonal frequency statistics. Consequently, and making use of the results in Propositions B.1 and B.2, the following Lemma may be stated.

Lemma B.1 *Under the conditions of Theorem 3.2, defining* $\xi_{\bar{c}_k} \equiv \Delta_S\mathbf{y} - \frac{\bar{c}_k}{T}\mathbf{y}_{k,-1}$, $k = 0, ..., \lfloor S/2\rfloor$, *and* $\bar{\mathbf{Z}}^{\alpha\beta}_{3,\bar{c}_k} \equiv (\bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_k}\ \bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_k})$, $k = 1, ..., S^*$, *and as* $N \to \infty$:
Case 2:

$$T^{-1}\bar{\mathbf{Z}}'_{2,\bar{c}_0}\bar{\mathbf{Z}}_{2,\bar{c}_0} \to S \tag{B.3}$$
$$T^{-1/2}\bar{\mathbf{Z}}'_{2,\bar{c}_0}\xi_{\bar{c}_0} \to \sigma\psi(1)y_{0,S}; \tag{B.4}$$

Case 3: *The results in (B.3)-(B.4) remain valid. Additionally:*

$$T^{-1}\bar{\mathbf{Z}}'_{3,\bar{\mathbf{c}}}\bar{\mathbf{Z}}_{3,\bar{\mathbf{c}}} \to \text{diag}\{S, S/2, S/2, ..., S/2, S/2, S\} \tag{B.5}$$
$$T^{-1/2}\bar{\mathbf{Z}}'_{3,\bar{c}_{S/2}}\xi_{\bar{c}_{S/2}} \to \sigma\psi(-1)y_{S/2,S}. \tag{B.6}$$
$$T^{-1/2}\left(\bar{\mathbf{Z}}^{\alpha\beta}_{3,\bar{c}_k}\right)'\xi_{\bar{c}_k} \to \sigma\psi(e^{-i\omega_k})[y_{k,S}\ ,\quad y^{\beta}_{k,S}],\ k = 1, ..., S^*; \tag{B.7}$$

Case 4: *The results in (B.3)-(B.4) remain valid. Additionally:*

$$T^{-1}\bar{\mathbf{Z}}_{4,\bar{\mathbf{c}}}'\bar{\mathbf{Z}}_{4,\bar{\mathbf{c}}} \rightarrow \operatorname{diag}\left\{S, \left(1-\bar{c}_0+\frac{\bar{c}_0^2}{3}\right)S^2\right\} \tag{B.8}$$

$$(T)^{-1/2}\left(\bar{\mathbf{t}}_{0,\bar{c}_0}\right)'\xi_{\bar{c}_0} \rightarrow S\sigma\psi(1)\left[(1-\bar{c}_0)J_{0,c_0}(1)+\bar{c}_0^2\int_0^1 rJ_{0,c_0}(r)dr\right]; \tag{B.9}$$

Case 5: *The results in (B.3)-(B.9) remain valid. Additionally:*

$$T^{-1}\bar{\mathbf{Z}}_{5,\bar{\mathbf{c}}}'\bar{\mathbf{Z}}_{5,\bar{\mathbf{c}}} \rightarrow \operatorname{diag}\left\{S, S/2, ..., S/2, S, \left(1-\bar{c}_0+\frac{\bar{c}_0^2}{3}\right)S^2\right\}; \tag{B.10}$$

Case 6: *The results in (B.3)-(B.9) remain valid. Additionally:*

$$T^{-1}\bar{\mathbf{Z}}_{6,\bar{\mathbf{c}}}'\bar{\mathbf{Z}}_{6,\bar{\mathbf{c}}} \rightarrow \operatorname{diag}\left\{S, \frac{S}{2}, ..., \frac{S}{2}, S, \left(1-\bar{c}_0+\frac{\bar{c}_0^2}{3}\right)S^2, \left(1-\bar{c}_1+\frac{\bar{c}_1^2}{3}\right)\left(\frac{S}{2}\right)^2\frac{1}{2}, \left(1-\bar{c}_1+\frac{\bar{c}_1^2}{3}\right)\left(\frac{S}{2}\right)^2\frac{1}{2}, ...,\right.$$

$$\left.\left(1-\bar{c}_{S^*}+\frac{\bar{c}_{S^*}^2}{3}\right)\left(\frac{S}{2}\right)^2\frac{1}{2}, \left(1-\bar{c}_{S^*}+\frac{\bar{c}_{S^*}^2}{3}\right)\left(\frac{S}{2}\right)^2\frac{1}{2}, \left(1-\bar{c}_{S/2}+\frac{\bar{c}_{S/2}^2}{3}\right)S^2\right\} \tag{B.11}$$

$$(T)^{-1/2}\left(\bar{\mathbf{t}}_{S/2,\bar{c}_{S/2}}\right)'\xi_{\bar{c}_{S/2}} \rightarrow S\sigma\psi(-1)\left[(1-\bar{c}_{S/2})J_{S/2,c_{S/2}}(1)+\bar{c}_{S/2}^2\int_0^1 rJ_{S/2,c_{S/2}}(r)dr\right] \tag{B.12}$$

$$\begin{aligned} T^{-1/2}\left(\bar{\mathbf{t}}_{k,\bar{c}_k}^{\alpha}\right)'\xi_{\bar{c}_k} &\Rightarrow \frac{\sigma S}{2\sqrt{2}}\left\{(1-\bar{c}_k)\left[a_kJ_{k,c_k}^{\beta}(1)+b_kJ_{k,c_k}^{\alpha}(1)\right]\right. \\ &\quad \left.+\bar{c}_k^2\int_0^1 r\left[a_kJ_{k,c_k}^{\beta}(r)+b_kJ_{k,c_k}^{\alpha}(r)\right]dr\right\},\ k=1,...,S^* \end{aligned} \tag{B.13}$$

$$\begin{aligned} T^{-1/2}\left(\bar{\mathbf{t}}_{k,\bar{c}_k}^{\beta}\right)'\xi_{\bar{c}_k} &\Rightarrow \frac{\sigma S}{2\sqrt{2}}\left\{(1-\bar{c}_k)\left[-a_kJ_{k,c_k}^{\alpha}(1)+b_kJ_{k,c_k}^{\beta}(1)\right]\right. \\ &\quad \left.+\bar{c}_k^2\int_0^1 r\left[-a_kJ_{k,c_k}^{\alpha}(r)+b_kJ_{k,c_k}^{\beta}(r)\right]dr\right\},\ k=1,...,S^* \end{aligned} \tag{B.14}$$

where a_k *and* b_k, $k=1,...,S^*$, *and* $J_{0,c_0}(r)$, $J_{S/2,c_{S/2}}(r)$, $J_{k,c_k}^{\alpha}(r)$ *and* $J_{k,c_k}^{\beta}(r)$, $k=1,...,S^*$, *are as defined in Proposition B.1.*

Proof of Lemma B.1

Case 2:

Under Case 2, $\bar{\mathbf{Z}}_{2,\bar{\mathbf{c}}}=\bar{\mathbf{Z}}_{2,\bar{c}_0}$. Consequently from Proposition B.2 it is straightforward to establish that

$$\begin{aligned} T^{-1}\bar{\mathbf{Z}}_{2,\bar{c}_0}'\bar{\mathbf{Z}}_{2,\bar{c}_0} &= T^{-1}\left(\sqrt{T}\mathbf{1}_0+\frac{\bar{c}_0}{\sqrt{T}}\mathbf{e}_0-\frac{\bar{c}_0}{\sqrt{T}}S\mathbf{1}\right)'\left(\sqrt{T}\mathbf{1}_0+\frac{\bar{c}_0}{\sqrt{T}}\mathbf{e}_0-\frac{\bar{c}_0}{\sqrt{T}}S\mathbf{1}\right) \\ &= \mathbf{1}_0'\mathbf{1}_0+o(1)\rightarrow S. \end{aligned}$$

Recalling that $y_s=0$, $s=1-S,...,0$, the result in (B.4) follows from the fact that

$$\begin{aligned} T^{-1/2}\bar{\mathbf{Z}}_{2,\bar{c}_0}'\xi_{\bar{c}_0} &= T^{-1/2}\left(\sqrt{T}\mathbf{1}_0+\frac{\bar{c}_0}{\sqrt{T}}\mathbf{e}_0-\frac{\bar{c}_0}{\sqrt{T}}S\mathbf{1}\right)'\left(\Delta_S\mathbf{y}-\frac{\bar{c}_0}{T}\mathbf{y}_{0,-1}\right) \\ &= \mathbf{1}_0'\Delta_S\mathbf{y}+o_p(1)=\sigma\psi(1)y_{0,S}+o_p(1). \end{aligned}$$

Case 3:
The diagonality of the matrix in the left member of (B.5) follows immediately from the asymptotic orthogonality of the columns of $T^{-1/2}\bar{\mathbf{Z}}_{3,\bar{\mathbf{c}}}$. The result for the zero frequency component is as given in the proof of Case 3, while the result for the Nyquist frequency follows in exactly the same way. For the remaining diagonal elements of the matrix, we have from Proposition B.2 that

$$\begin{aligned} T^{-1}\left(\bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_k}\right)'\bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_k} &= T^{-1}\left(\sqrt{T}\mathbf{1}_{1,k}+\frac{\bar{c}_k}{\sqrt{T}}\mathbf{e}_{1k}-\frac{\bar{c}_k}{\sqrt{T}}(S/2)\mathbf{1}^{\alpha}_k\right)'\left(\sqrt{T}\mathbf{1}_{1,k}+\frac{\bar{c}_k}{\sqrt{T}}\mathbf{e}_{1k}-\frac{\bar{c}_k}{\sqrt{T}}(S/2)\mathbf{1}^{\alpha}_k\right) \\ &= \mathbf{1}'_{1,k}\mathbf{1}_{1,k}+o(1)\to S/2 \end{aligned}$$

and, similarly,

$$\begin{aligned} T^{-1}\left(\bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_k}\right)'\bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_k} &= T^{-1}\left(\sqrt{T}\mathbf{1}_{2,k}+\frac{\bar{c}_k}{\sqrt{T}}\mathbf{e}_{2k}-\frac{\bar{c}_k}{\sqrt{T}}(S/2)\,\mathbf{1}^{\beta}_k\right)'\left(\sqrt{T}\mathbf{1}_{2,k}+\frac{\bar{c}_k}{\sqrt{T}}\mathbf{e}_{2k}-\frac{\bar{c}_k}{\sqrt{T}}(S/2)\,\mathbf{1}^{\beta}_k\right) \\ &= T^{-1}T\mathbf{1}'_{2,k}\mathbf{1}_{2,k}+o(1)\to S/2. \end{aligned}$$

Turning to the results in (B.6) and (B.7), we have that

$$T^{-1/2}\bar{\mathbf{Z}}'_{3,\bar{c}_{S/2}}\xi_{\bar{c}_{S/2}}=\sigma\psi(-1)\sum_{i=1}^{S}(-1)^i y_i+o_p(1)\equiv\sigma\psi(-1)y_{S/2,S}+o_p(1)$$

and, for $k=1,...,S^*$,

$$\begin{aligned} T^{-1/2}\left(\bar{\mathbf{Z}}^{\alpha\beta}_{3,\bar{c}_k}\right)'\xi_{\bar{c}_k} &= T^{-1/2}\left[\left(\bar{\mathbf{Z}}^{\alpha}_{3,\bar{c}_k}\right)'\xi_{\bar{c}_k}\ ,\quad \left(\bar{\mathbf{Z}}^{\beta}_{3,\bar{c}_k}\right)'\xi_{\bar{c}_k}\right] \\ &= \left[\mathbf{1}'_{1,k}\Delta_S\mathbf{y}+o_p(1)\ ,\quad \mathbf{1}'_{2,k}\Delta_S\mathbf{y}+o_p(1)\right] \\ &= \sigma\psi(e^{-i\omega_k})[y_{k,S}\ ,\quad y^{\beta}_{k,S}]+o_p(1). \end{aligned}$$

Finally, the result for the zero frequency was established in the proof of Case 2.
Case 4:
In addition to the results proved in Case 2, we have that

$$T^{-1}\bar{\mathbf{Z}}'_{2,\bar{c}_0}\bar{\mathbf{t}}_{0,\bar{c}_0}=T^{-1}\left(\sqrt{T}\mathbf{1}_0+\frac{\bar{c}_0}{\sqrt{T}}\mathbf{e}_0-\frac{\bar{c}_0}{\sqrt{T}}S\mathbf{1}\right)'\left(S\mathbf{1}-\mathbf{e}_0-\frac{\bar{c}_0}{T}\left[S\mathbf{t}_0-\left(\sum_{i=1}^{S-1}i\right)\mathbf{1}\right]\right)+o(1)\to 0$$

and

$$\begin{aligned} T^{-1}\left(\bar{\mathbf{t}}_{0,\bar{c}_0}\right)'\bar{\mathbf{t}}_{0,\bar{c}_0} &= T^{-1}\left(S\mathbf{1}-\mathbf{e}_0-\frac{\bar{c}_0}{T}\left[S\mathbf{t}_0-\left(\sum_{i=1}^{S-1}i\right)\mathbf{1}\right]\right)' \\ &\quad\times\left(S\mathbf{1}-\mathbf{e}_0-\frac{\bar{c}_0}{T}\left[S\mathbf{t}_0-\left(\sum_{i=1}^{S-1}i\right)\mathbf{1}\right]\right)+o(1) \\ &= T^{-1}S^2\mathbf{1}'\mathbf{1}-\frac{2\bar{c}_0}{T^2}S^2\mathbf{t}'_0\mathbf{1}+T^{-1}\left(\frac{\bar{c}_0}{T}\right)^2S^2\mathbf{t}'_0\mathbf{t}_0+o(1) \\ &\to \left(1-\bar{c}_0+\frac{\bar{c}_0^2}{3}\right)S^2. \end{aligned}$$

To prove the result in (B.9), observe that

$$\begin{aligned} T^{-1/2}\left(\bar{\mathbf{t}}_{0,\bar{c}_0}\right)'\xi_{\bar{c}_0} &= S\{T^{-1/2}\mathbf{1}'\Delta_S\mathbf{y} - T^{-3/2}(\bar{c}_0\mathbf{t}_0)'\Delta_S\mathbf{y} \\ &- T^{-3/2}\bar{c}_0\mathbf{1}'\mathbf{y}_{0,-1} + T^{-5/2}\bar{c}_0^2\mathbf{t}_0'\mathbf{y}_{0,-1}\} + o_p(1). \end{aligned}$$

Using the following results which follow directly from Phillips (1987):

$$\begin{aligned} T^{-1/2}\mathbf{1}'\Delta_S\mathbf{y} &= T^{-1/2}\sum_{Sn+s=1}^{T} e^{(T-(Sn+s))c_0/T}v_{Sn+s} + o_p(1) \Rightarrow \sigma\psi(1)J_{0,c_0}(1) \\ T^{-3/2}\mathbf{t}_0'\Delta_S\mathbf{y} &= T^{-3/2}\sum_{Sn+s=1}^{T}(Sn+s)\,e^{(T-(Sn+s))c_0/T}v_{Sn+s} + o_p(1) \Rightarrow \sigma\psi(1)\int_0^1 rdJ_{0,c_0}(r) \\ T^{-3/2}\mathbf{1}'\mathbf{y}_{0,-1} &= \sum_{Sn+s=1}^{T}\sum_{j=1}^{Sn+s} e^{(Sn+s-j)c_0/T}v_{Sn+s} + o_p(1) \Rightarrow \sigma\psi(1)\int_0^1 J_{0,c_0}(r)dr \\ T^{-5/2}\mathbf{t}_0'\mathbf{y}_{0,-1} &= T^{-5/2}\sum_{Sn+s=1}^{T}(Sn+s)\sum_{j=1}^{Sn+s} e^{(Sn+s-j)c_0/T}v_{Sn+s} + o_p(1) \Rightarrow \sigma\psi(1)\int_0^1 rJ_{0,c_0}(r)dr \end{aligned}$$

we then obtain immediately from applications of the CMT that

$$T^{-1/2}\left(\bar{\mathbf{t}}_{0,\bar{c}_0}\right)'\xi_{\bar{c}_0} \Rightarrow S\sigma\psi(1)\{J_{0,c_0}(1) - \bar{c}_0\left(\int_0^1 rdJ_{0,c_0}(r) + \int_0^1 J_{0,c_0}(r)dr\right) + \bar{c}_0^2\int_0^1 rJ_{0,c_0}dr\}$$

from which the stated result follows using the well-known identity $\int_0^1 rdJ_{0,0}(r) + \int_0^1 J_{0,c_0}(r)dr \equiv J_{0,c_0}(1)$.

Case 5:

The proof of the results stated in Case 5 follows from the relevant proofs in Cases 2-4, again appealing to the asymptotic orthogonality result.

Case 6:

The diagonality of the matrix in the left member of (B.11) again follows immediately from the asymptotic orthogonality of the columns of $T^{-1/2}\bar{\mathbf{Z}}_{6,\bar{\mathbf{c}}}$. The result for the zero frequency diagonal element is as given in the proof of Case 4. The result for the Nyquist frequency follows in an entirely analogous fashion and is therefore omitted. For the remaining diagonal elements of the matrix, we have from Proposition B.2 that for $k = 1, ..., S^*$,

$$\begin{aligned} \frac{1}{T}\left[\left(\frac{S}{2}\right)^2(\mathbf{1}_k^\alpha)'\,\mathbf{1}_k^\alpha\right] &= \left(\frac{S}{2}\right)^2\frac{1}{T}\sum_{j=1}^{T}[\cos(j\omega_k)]\,[\cos(j\omega_k)] \\ &= \left(\frac{S}{2}\right)^2\frac{1}{T}\sum_{j=1}^{T}\left(\frac{1}{2}e^{-i\omega_k j} + \frac{1}{2}e^{i\omega_k j}\right)^2 \\ &= \left(\frac{S}{2}\right)^2\frac{1}{T}\sum_{j=1}^{T}e^{2i\omega_k j}\left(\frac{1}{4}e^{-4i\omega_k j} + \frac{1}{2}e^{-2i\omega_k j} + \frac{1}{4}\right) \\ &= \left(\frac{S}{2}\right)^2\frac{1}{T}\sum_{j=1}^{T}\frac{1}{2} + o(1) \to \left(\frac{S}{2}\right)^2\frac{1}{2}, \end{aligned} \quad \text{(B.15)}$$

$$\begin{aligned}
\frac{1}{T}\left[-2\frac{\bar{c}_k}{T}\left(\frac{S}{2}\right)^2(\mathbf{1}_k^\alpha)'\,\mathbf{t}_k^\alpha\right] &= -2\bar{c}_k\left(\frac{S}{2}\right)^2\frac{1}{T^2}\sum_{j=1}^{T}(\cos(j\omega_k))(j\cos(j\omega_k))\\
&= -2\bar{c}_k\left(\frac{S}{2}\right)^2\frac{1}{T^2}\sum_{j=1}^{T}j\left(\frac{1}{2}e^{-i\omega_k j}+\frac{1}{2}e^{i\omega_k j}\right)^2\\
&= -2\bar{c}_k\left(\frac{S}{2}\right)^2\frac{1}{T^2}\sum_{j=1}^{T}e^{2i\omega_k j}\left(\frac{1}{4}j+\frac{1}{4}je^{-4i\omega_k j}+\frac{1}{2}je^{-2i\omega_k j}\right)\\
&= -2\bar{c}_k\left(\frac{S}{2}\right)^2\frac{1}{T^2}\sum_{j=1}^{T}\frac{1}{2}j+o(1)\to-\bar{c}_k\left(\frac{S}{2}\right)^2\frac{1}{2} \qquad \text{(B.16)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{T}\left[\left(\frac{\bar{c}_k}{T}\right)^2\left(\frac{S}{2}\right)^2(\mathbf{t}_k^\alpha)'\,\mathbf{t}_k^\alpha\right] &= \bar{c}_k^2\left(\frac{S}{2}\right)^2\frac{1}{T^3}\sum_{j=1}^{T}(j\cos(j\omega_k))(j\cos(j\omega_k))\\
&= \bar{c}_k^2\left(\frac{S}{2}\right)^2\frac{1}{T^3}\sum_{j=1}^{T}j^2\left(\frac{1}{2}e^{-i\omega_k j}+\frac{1}{2}e^{i\omega_k j}\right)^2\\
&= \bar{c}_k^2\left(\frac{S}{2}\right)^2\frac{1}{T^3}\sum_{j=1}^{T}e^{2i\omega_k j}\left(\frac{1}{4}j^2+\frac{1}{4}j^2e^{-4i\omega_k j}+\frac{1}{2}j^2e^{-2i\omega_k j}\right)\\
&= \bar{c}_k^2\left(\frac{S}{2}\right)^2\frac{1}{T^3}\sum_{j=1}^{T}\frac{1}{2}j^2+o(1)\to\bar{c}_k^2\left(\frac{S}{2}\right)^2\frac{1}{6}. \qquad \text{(B.17)}
\end{aligned}$$

Combining (B.15) - (B.17) yields the required results, $k=1,...,S^*$. Identical asymptotic limits to those given in (B.15), (B.16) and (B.17) are obtained for $\frac{1}{T}\left[\left(\frac{S}{2}\right)^2\left(\mathbf{1}_k^\beta\right)'\mathbf{1}_k^\beta\right]$, $\frac{1}{T}\left[-2\frac{\bar{c}_k}{T}\left(\frac{S}{2}\right)^2\left(\mathbf{1}_k^\beta\right)'\mathbf{t}_k^\beta\right]$ and $\frac{1}{T}\left[\left(\frac{\bar{c}_k}{T}\right)^2\left(\frac{S}{2}\right)^2\left(\mathbf{t}_k^\beta\right)'\mathbf{t}_k^\beta\right]$, respectively. Since the proof follows along similar lines, it is omitted.

The proof of (B.9) is as given under Case 3, while the proof of the result in (B.12) is very similar and, hence, is omitted. Turning to the proof of (B.13), observe from Proposition B.2 that

$$\begin{aligned}
T^{-1/2}\left(\bar{\mathbf{t}}_{k,\bar{c}_k}^\alpha\right)'\xi_{\bar{c}_k} &= T^{-1/2}\left[\frac{S}{2}\left(\mathbf{1}_k^\alpha-\frac{\bar{c}_k}{T}\mathbf{t}_k^\alpha\right)+o(1)\right]'\xi_{\bar{c}_k}\\
&= T^{-1/2}(\mathbf{1}_k^\alpha)'\,\Delta_S\mathbf{y}-T^{-3/2}\bar{c}_k(\mathbf{1}_k^\alpha)'\,\mathbf{y}_{k,-1}-T^{-3/2}\bar{c}_k(\mathbf{t}_k^\alpha)'\,\Delta_S\mathbf{y}\\
&\quad+T^{-5/2}\bar{c}_k^2(\mathbf{t}_k^\alpha)'\,\mathbf{y}_{k,-1}+o_p(1). \qquad \text{(B.18)}
\end{aligned}$$

For the following four results, the weak convergence result in each case following from Proposition B.1 and applications of the CMT, establish the large sample behaviour of the four non-vanishing elements constituting the right member of (B.18):

$$\begin{aligned} T^{-1/2}\left(\mathbf{1}_k^{\alpha}\right)'\Delta_S\mathbf{y} &= T^{-1/2}\sum_{s=1-S}^{0}\sum_{n=1}^{N}\cos\left[(Sn+s)\omega_k\right]\Delta_S y_{Sn+s} \\ &= T^{-1/2}\sum_{s=1-S}^{0}\cos(s\omega_k)\sum_{n=1}^{N}\Delta_S y_{Sn+s} \\ &= T^{-1/2}\sum_{s=1-S}^{0}\cos(s\omega_k)\sum_{n=1}^{N}\left[-\left(1-2\cos(\omega_k)L+L^2\right)\right]y_{k,Sn+s+1} \\ &\Rightarrow \frac{\sigma}{\sqrt{2}}\left[a_k J_{k,c_k}^{\beta}(1)+b_k J_{k,c_k}^{\alpha}(1)\right] \end{aligned} \quad \text{(B.19)}$$

$$\begin{aligned} T^{-3/2}\bar{c}_k\left(\mathbf{1}_k^{\alpha}\right)'\mathbf{y}_{k,-1} &= T^{-3/2}\bar{c}_k\sum_{s=1-S}^{0}\sum_{n=1}^{N}\cos\left[(Sn+s)\omega_k\right]y_{k,Sn+s-1} \\ &= T^{-3/2}\bar{c}_k\sum_{s=1-S}^{0}\cos(s\omega_k)\sum_{n=1}^{N}y_{k,Sn+s-1} \\ &\Rightarrow \frac{\sigma\bar{c}_k}{\sqrt{2}}\int_0^1\left[a_k J_{k,c_k}^{\beta}(r)+b_k J_{k,c_k}^{\alpha}(r)\right]dr \end{aligned} \quad \text{(B.20)}$$

and

$$\begin{aligned} T^{-3/2}\bar{c}_k\left(\mathbf{t}_k^{\alpha}\right)'\Delta_S\mathbf{y} &= T^{-3/2}\bar{c}_k\sum_{s=1-S}^{0}\sum_{n=1}^{N}(Sn+s)\cos\left[(Sn+s)\omega_k\right]\Delta_S y_{Sn+s} \\ &\Rightarrow \frac{\sigma\bar{c}_k}{\sqrt{2}}\int_0^1 r d\left[a_k J_{k,c_k}^{\beta}(r)+b_k J_{k,c_k}^{\alpha}(r)\right] \end{aligned} \quad \text{(B.21)}$$

and, along similar lines, that

$$T^{-5/2}\bar{c}_k^2\left(\mathbf{t}_k^{\alpha}\right)'\mathbf{y}_{k,-1}\Rightarrow\frac{\sigma\bar{c}_k^2}{\sqrt{2}}\int_0^1 r\left[a_k J_{k,c_k}^{\beta}(r)+b_k J_{k,c_k}^{\alpha}(r)\right]dr. \quad \text{(B.22)}$$

We then obtain the result in (B.13) immediately from (B.19)-(B.22) and further applications of the CMT. Correspondingly, for the result in (B.14), we obtain from Proposition B.2 that

$$\begin{aligned} T^{-1/2}\left(\bar{\mathbf{t}}_{k,\bar{c}_k}^{\beta}\right)'\xi_{\bar{c}_k} &= T^{-1/2}\left\{\frac{S}{2}\left(\mathbf{1}_k^{\beta}-\frac{\bar{c}_k}{T}\mathbf{t}_k^{\beta}\right)+o_p(1)\right\}'\xi_{\bar{c}_k} \\ &= T^{-1/2}\left(\mathbf{1}_k^{\beta}\right)'\Delta_S\mathbf{y}-T^{-3/2}\bar{c}_k\left(\mathbf{1}_k^{\beta}\right)'\mathbf{y}_{k,-1} \\ &\quad -T^{-3/2}\bar{c}_k\left(\mathbf{t}_k^{\beta}\right)'\Delta_S\mathbf{y}+T^{-5/2}\bar{c}_k^2\left(\mathbf{t}_k^{\beta}\right)'\mathbf{y}_{k,-1} \end{aligned}$$

then using the following results, which are proved in a similar manner to (B.19)-(B.22),

$$\begin{aligned}
T^{-1/2}\left(\mathbf{1}_k^{\beta}\right)'\Delta_S\mathbf{y} &\Rightarrow \frac{\sigma}{\sqrt{2}}\left[-a_kJ^{\alpha}_{k,c_k}(1)+b_kJ^{\beta}_{k,c_k}(1)\right]\\
T^{-3/2}\bar{c}_k\left(\mathbf{1}_k^{\beta}\right)'\mathbf{y}_{k,-1} &\Rightarrow \frac{\sigma\bar{c}_k}{\sqrt{2}}\int_0^1\left[-a_kJ^{\alpha}_{k,c_k}(r)+b_kJ^{\beta}_{k,c_k}(r)\right]dr\\
T^{-3/2}\bar{c}_k\left(\mathbf{t}_k^{\beta}\right)'\Delta_S\mathbf{y} &\Rightarrow \frac{\sigma\bar{c}_k}{\sqrt{2}}\int_0^1 rd\left[-a_kJ^{\alpha}_{k,c_k}(1)+b_kJ^{\beta}_{k,c_k}(1)\right]\\
T^{-5/2}\bar{c}_k^2\left(\mathbf{t}_k^{\beta}\right)'\mathbf{y}_{k,-1} &\Rightarrow \frac{\sigma\bar{c}_k^2}{\sqrt{2}}\int_0^1 r\left[-a_kJ^{\alpha}_{k,c_k}(r)+b_kJ^{\beta}_{k,c_k}(r)\right]dr
\end{aligned}$$

the stated result, (B.14), obtains immediately from the CMT.

B.2.2 Proof of Theorem 3.2

Using ERS (p.833, Eq. A8) and Gregoir (2004, Lemma A.5,p.37) we have that for $i = 2, ..., 6$,

$$\left[Q^M(\bar{\mathbf{c}}_k)_i - Q^M(\mathbf{0})_i\right] - g_k^{-1}\left[Q(\bar{\mathbf{c}}_k)_\xi - Q(\mathbf{0})_\xi\right] = o_p(1) \quad \text{(B.23)}$$

where

$$\begin{aligned}
Q(\bar{\mathbf{c}}_k)_\xi &= \left(\Delta_S\mathbf{y} - \frac{\bar{\mathbf{c}}_k}{T}\mathbf{y}_{k,-1}\right)'\mathbf{Z}_{i,\bar{\mathbf{c}}_k}\left[\mathbf{Z}'_{i,\bar{\mathbf{c}}_k}\mathbf{Z}_{i,\bar{\mathbf{c}}_k}\right]^{-1}\mathbf{Z}'_{i,\bar{\mathbf{c}}_k}\left(\Delta_S\mathbf{y} - \frac{\bar{c}_k}{T}\mathbf{y}_{k,-1}\right)\\
Q(\mathbf{0})_\xi &= (\Delta_S\mathbf{y})'\mathbf{Z}_{i,\mathbf{0}}\left[\mathbf{Z}'_{i,\mathbf{0}}\mathbf{Z}_{i,\mathbf{0}}\right]^{-1}\mathbf{Z}'_{i,\mathbf{0}}(\Delta_S\mathbf{y})
\end{aligned}$$

and $g_k \equiv 2\pi f_k$, f_k the spectral density of $\{v_{Sn+s}\}$ evaluated at frequency $2\pi k/S$, $k = 0, ..., \lfloor S/2\rfloor$.

In order to establish the limit distributions of Theorem 3.2, we must therefore establish convergence results for $g_k^{-1}\left[Q(\bar{\mathbf{c}}_k)_\xi - Q(\mathbf{0})_\xi\right]$, $k = 0, ..., \lfloor S/2\rfloor$. This is straightforward given the results derived in Lemma B.1. Precisely, under Cases 1, 2 and 3, where $\xi = 0$ or 1 for all frequencies, we observe that $g_k^{-1}\left[Q(\bar{\mathbf{c}}_k)_\xi - Q(\mathbf{0})_\xi\right]$ are all of $o_p(1)$, $k = 0, ..., \lfloor S/2\rfloor$, and hence the limiting distributions reduce to the corresponding results in Theorem 3.1. For the zero frequency tests when a zero frequency time trend is included (Cases 4, 5 and 6) and for the Nyquist frequency test when a Nyquist frequency trend is included (Case 6) we have from Lemma B.1 and the CMT that,

$$Q(\bar{\mathbf{c}}_j)_2 - \sigma^2[\psi(j)]^2 y_{j,S}^2 \Rightarrow \sigma^2[\psi(j)]^2\left\{\frac{\left[(1-\bar{c}_j)J_{j,c_j}(1) + \bar{c}_j^2\int_0^1 rJ_{j,c_j}(r)dr\right]^2}{\left(1-\bar{c}_j+\frac{\bar{c}_j^2}{3}\right)}\right\}$$

and, setting $\bar{c}_j = 0$, $Q(\mathbf{0})_2 - \sigma^2[\psi(j)]^2 y_{j,S}^2 \Rightarrow \sigma^2[\psi(j)]^2 J_{j,c_j}(1)$, for $j = 0, \lfloor S/2\rfloor$. Consequently, again by the CMT,

$$g_j^{-1}\left[Q(\bar{\mathbf{c}}_j)_2 - Q(\mathbf{0})_2\right] \Rightarrow \frac{\left[(1-\bar{c}_j)J_{j,c_j}(1) + \bar{c}_j^2\int_0^1 rJ_{j,c_j}(r)dr\right]^2}{\left(1-\bar{c}_j+\frac{\bar{c}_j^2}{3}\right)} - \left[J_{j,c_j}(1)\right]^2,$$

which, coupled with Theorem 3.1, and applications of the CMT completes the proof for the zero ($j = 0$) and Nyquist ($j = S/2$) frequency statistics.

Consider now the statistic at the kth harmonic frequency pair when a time trend at the harmonic frequencies is included (Case 6), and define $Q(\mathbf{a}_k)_2^* \equiv Q(\mathbf{a}_k)_2 - \sigma^2[\psi(e^{-i\omega_k})]^2[(y_{k,S})^2 +$

$(y^{\beta}_{k,S})^2]$, where either $\mathbf{a}_k = \bar{\mathbf{c}}_k$ or $\mathbf{a}_k = \mathbf{0}$. We then obtain from Lemma B.1 and the CMT that, for $k = 1, ..., S^*$,

$$\begin{aligned} Q(\bar{\mathbf{c}}_k)^*_2 \;&\Rightarrow\; \frac{\sigma^2}{\left(1-\bar{c}_k+\frac{\bar{c}_k^2}{3}\right)}\left\{\left[(1-\bar{c}_k)\left[a_kJ^{\beta}_{k,c_k}(1)+b_kJ^{\alpha}_{k,c_k}(1)\right]+\bar{c}_k^2\int_0^1 r\left[a_kJ^{\beta}_{k,c_k}(r)+b_kJ^{\alpha}_{k,c_k}(r)\right]dr\right]^2\right. \\ &+\left.\left[(1-\bar{c}_k)\left[-a_kJ^{\alpha}_{k,c_k}(1)+b_kJ^{\beta}_{k,c_k}(1)\right]+\bar{c}_k^2\int_0^1 r\left[-a_kJ^{\alpha}_{k,c_k}(r)+b_kJ^{\beta}_{k,c_k}(r)\right]dr\right]^2\right\}. \end{aligned}$$

After simple but tedious manipulations we obtain that

$$Q(\bar{\mathbf{c}}_k)^*_2 \Rightarrow \frac{\sigma^2\left(a_k^2+b_k^2\right)}{\left(1-\bar{c}_k+\frac{\bar{c}_k^2}{3}\right)}\left\{(1-\bar{c}_k)\left[J^{\beta}_{k,c_k}(1)+J^{\alpha}_{k,c_k}(1)\right]+\bar{c}_k^2\int_0^1 r\left[J^{\beta}_{k,c_k}(r)+J^{\alpha}_{k,c_k}(r)\right]dr\right\}^2,$$

$k = 1, ..., S^*$. Moreover,

$$\begin{aligned} Q(\mathbf{0})^*_2 \;&\Rightarrow\; \sigma^2\left[a_kJ^{\beta}_{k,c_k}(1)+b_kJ^{\alpha}_{k,c_k}(1)\right]^2+\sigma^2\left[-a_kJ^{\alpha}_{k,c_k}(1)+b_kJ^{\beta}_{k,c_k}(1)\right]^2 \\ &\equiv\; \sigma^2\left(a_k^2+b_k^2\right)\left\{\left[J^{\alpha}_{k,c_k}(1)\right]^2+\left[J^{\beta}_{k,c_k}(1)\right]^2\right\},\; k=1,...,S^*. \end{aligned}$$

Consequently, appealing to the CMT, we obtain that

$$\begin{aligned} g_k^{-1}\left[Q(\bar{\mathbf{c}}_k)_2-Q(\mathbf{0})_2\right] \;&\Rightarrow\; \frac{\sigma^2\left(a_k^2+b_k^2\right)}{g_k\left(1-\bar{c}_k+\frac{\bar{c}_k^2}{3}\right)}\left\{(1-\bar{c}_k)\left[J^{\beta}_{k,c_k}(1)+J^{\alpha}_{k,c_k}(1)\right]\right. \\ &\left.+\bar{c}_k^2\int_0^1 r\left[J^{\beta}_{k,c_k}(r)+J^{\alpha}_{k,c_k}(r)\right]dr\right\}^2 \\ &-\frac{\sigma^2\left(a_k^2+b_k^2\right)}{g_k}\left\{\left[J^{\alpha}_{k,c_k}(1)\right]^2+\left[J^{\beta}_{k,c_k}(1)\right]^2\right\},\; k=1,...,S^* \end{aligned}$$

and, noting that $g_k = \sigma^2\left[\psi(e^{-i\omega_k})\right]\left[\psi(e^{i\omega_k})\right] = \sigma^2\left(a_k^2+b_k^2\right)$, $k = 1, ..., S^*$, this simplifies to

$$\begin{aligned} g_k^{-1}\left[Q(\bar{\mathbf{c}}_k)_2-Q(\mathbf{0})_2\right] \;&\Rightarrow\; \frac{1}{\left(1-\bar{c}_k+\frac{\bar{c}_k^2}{3}\right)}\left\{(1-\bar{c}_k)\left[J^{\beta}_{k,c_k}(1)+J^{\alpha}_{k,c_k}(1)\right]\right. \\ &\left.+\bar{c}_k^2\int_0^1 r\left[J^{\beta}_{k,c_k}(r)+J^{\alpha}_{k,c_k}(r)\right]dr\right\}^2 \\ &-\left\{\left[J^{\alpha}_{k,c_k}(1)\right]^2+\left[J^{\beta}_{k,c_k}(1)\right]^2\right\},\; k=1,...,S^* \end{aligned}$$

which, coupled with Theorem 3.1, and applications of the CMT completes the proof.

Appendix C

Proof of Theorem 5.1

Proofs for Case 1 (where no de-trending occurs) are already established in the literature, see Rodrigues and Taylor (2004b), and are not reproduced here. For each of Cases 2-6, the pseudo-GLS estimator $\widetilde{\beta}_i(\bar{\mathbf{c}})$ is obtained from regressing $x_{\mathbf{c}}$ on $\mathbf{Z}_{i,\mathbf{c}}$ for $\mathbf{c} = \bar{\mathbf{c}} \equiv (\bar{c}_0, \bar{c}_1, ..., \bar{c}_{\lfloor S/2\rfloor})'$, to obtain the pseudo-GLS de-trended data,

$$\hat{x}^i_{Sn+s} \equiv x_{Sn+s} - \left[\widetilde{\beta}_i(\bar{\mathbf{c}})\right]' z_{Sn+s,i},$$

$i = 2, ..., 6$. After considerable algebraic manipulation (using standard trigonometric identities) it can be shown that the transformed level variables entering into (5.1) take the form,

$$\hat{x}^i_{0,Sn+s} \equiv \begin{cases} y_{0,Sn+s} - S\left(\widetilde{\gamma}_0(\bar{c}_0) - \gamma_0\right), \ i = 2,3 \\ y_{0,Sn+s} - \left[S + \left(\sum_{i=1}^{S-1} i\right)\right]\left(\widetilde{\gamma}_0(\bar{c}_0) - \gamma_0\right) - S\left(\widetilde{\delta}_0(\bar{c}_0) - \delta_0\right)(Sn+s), \ i = 4,5,6 \end{cases}$$

$$\hat{x}^i_{S/2,Sn+s} \equiv \begin{cases} y_{S/2,Sn+s}, \ i = 2,4 \\ y_{S/2,Sn+s} - S\left(\widetilde{\gamma}_{S/2}(\bar{c}_{S/2}) - \gamma_{S/2}\right)(-1)^{Sn+s}, \ i = 3,5 \\ y_{S/2,Sn+s} - \left[S + \left(\sum_{i=1}^{S-1} i\right)\right]\left(\widetilde{\gamma}_{S/2}(\bar{c}_{S/2}) - \gamma_{S/2}\right)(-1)^{Sn+s} \\ -S\left(\widetilde{\delta}_{S/2}(\bar{c}_{S/2}) - \delta_{S/2}\right)(Sn+s)(-1)^{Sn+s}, \ i = 6 \end{cases}$$

$$\hat{x}^i_{k,Sn+s} \equiv \begin{cases} y_{k,Sn+s}, \ i = 2,4 \\ y_{k,Sn+s} - \frac{S}{2}\left(\widetilde{\gamma}_{k,\alpha}(\bar{c}_k) - \gamma_{k,\alpha}\right)' z^{*\mu_{\omega_k}}_{Sn+s}, \ i = 3,5 \\ y_{k,Sn+s} - [S/2 + k^*]\left(\widetilde{\gamma}_{k,\alpha}(\bar{c}_k) - \gamma_{k,\alpha}\right)' z^{*\mu_{\omega_k}}_{Sn+s} - \frac{S}{2}\left(\widetilde{\delta}_{k,\alpha}(\bar{c}_k) - \delta_{k,\alpha}\right)' \Lambda_{1,k} z^{\tau_{\omega_k}}_{Sn+s}, \ i = 6 \end{cases}$$

$$\hat{x}^{\beta,i}_{k,Sn+s} \equiv \begin{cases} y^\beta_{k,Sn+s}, \ i = 2,4 \\ y^\beta_{k,Sn+s} - \frac{S}{2}\left(\widetilde{\gamma}_{k,\beta}(\bar{c}_k) - \gamma_{k,\beta}\right)' \bar{z}^{*\mu_{\omega_k}}_{Sn+s}, \ i = 3,5 \\ y^\beta_{k,Sn+s} - [S/2 + k^*]\left(\widetilde{\gamma}_{k,\beta}(\bar{c}_k) - \gamma_{k,\beta}\right)' \bar{z}^{*\mu_{\omega_k}}_{Sn+s} - \frac{S}{2}\left(\widetilde{\delta}_{k,\beta}(\bar{c}_k) - \delta_{k,\beta}\right)' \Lambda_{2,k} z^{\tau_{\omega_k}}_{Sn+s}, \ i = 6 \end{cases}$$

where the spectral intercept and time-trend parameters, γ_j and δ_j, $j = 0, ..., S^*, S/2$, are as defined in Section 2.1 with $\widetilde{\gamma}_j(\bar{c}_j)$ and $\widetilde{\delta}_j(\bar{c}_j)$, $j = 0, ..., S^*, S/2$, the pseudo-GLS estimators of these parameters for each of $i = 2, ..., 6$. The notation k^* denotes an asymptotically irrelevant term which results from the filtering of the seasonal trends, while

$$\Lambda_{1,k} = \begin{bmatrix} \cos(\omega_k) & -\sin(\omega_k) \\ \sin(\omega_k) & \cos(\omega_k) \end{bmatrix}, \quad \Lambda_{2,k} = -\begin{bmatrix} \sin(\omega_k) & \cos(\omega_k) \\ -\cos(\omega_k) & \sin(\omega_k) \end{bmatrix}$$

and $z^{*\mu_{\omega_k}}_{Sn+s} = (\cos[(s+1)\omega_k], \ \sin[(s+1)\omega_k])'$, $\bar{z}^{*\mu_{\omega_k}}_{Sn+s} = (-\sin[(s+1)\omega_k], \ \cos[(s+1)\omega_k])'$, and $z^{\tau_{\omega_k}}_{Sn+s} = ((Sn+s)\cos(s\omega_k), \ (Sn+s)\sin(s\omega_k))'$.

Before continuing, it will prove useful to state the following proposition whose proof follows directly from results presented in Appendix B and applications of the CMT. The results stated in the proposition are understood to apply only in those of Cases 2-6 where the parameter being studied features in the parameter vector β.

Proposition C.1 *Under the conditions of Theorem 5.1, as $N \to \infty$,*

$$\begin{aligned}
\sqrt{T}\left(\widetilde{\gamma}_k(\bar{c}_k) - \gamma_k\right) &= o_p(1), \;\; k = 0, ..., \lfloor S/2 \rfloor \\
\sqrt{T}\left(\widetilde{\delta}_k(\bar{c}_k) - \delta_k\right) &\Rightarrow \sigma \frac{\left[(1-\bar{c}_k)J_{k,c_k}(1) + \bar{c}_k^2 \int_0^1 rJ_{k,c_k}(r)dr\right]}{S\left(1 - \bar{c}_k + \frac{\bar{c}_k^2}{3}\right)} \equiv \frac{\sigma}{S}\mathcal{D}_{c_k}(r, \bar{c}_k), \;\; k = 0, S/2 \\
\sqrt{T}\left(\widetilde{\delta}_{k,\alpha}(\bar{c}_k) - \delta_{k,\alpha}\right) &\Rightarrow \frac{2\sqrt{2}\sigma}{S} \frac{\left[(1-\bar{c}_k)\, J^{\alpha}_{k,c_k}(1) + \bar{c}_k^2 \int_0^1 rJ^{\alpha}_{k,c_k}(r)dr\right]}{\left(1 - \bar{c}_k + \frac{\bar{c}_k^2}{3}\right)} \\
&\equiv \frac{2\sqrt{2}\sigma}{S}\mathcal{D}_{k,c_k}(r, \bar{c}_k), \;\; k = 1, ..., S^* \\
\sqrt{T}\left(\widetilde{\delta}_{k,\beta}(\bar{c}_k) - \delta_{k,\beta}\right) &\Rightarrow \frac{2\sqrt{2}\sigma}{S} \frac{\left[(1-\bar{c}_k)\, J^{\beta}_{k,c_k}(1) + \bar{c}_k^2 \int_0^1 rJ^{\beta}_{k,c_k}(r)dr\right]}{\left(1 - \bar{c}_k + \frac{\bar{c}_k^2}{3}\right)} \\
&\equiv \frac{2\sqrt{2}\sigma}{S}\mathcal{D}^{\beta}_{k,c_k}(r, \bar{c}_k), \;\; k = 1, ..., S^*
\end{aligned}$$

where $J_{0,c_0}(r)$, $J_{S/2,c_{S/2}}(r)$, $J^{\alpha}_{k,c_k}(r)$ and $J^{\beta}_{k,c_k}(r)$ are as defined in Proposition B.1.

Remark C.1: Representations for the limiting distributions of the elements of the scaled pseudo-GLS estimator, $\widetilde{\beta}_i^{\dagger}$, obtain setting $\bar{c}_k = c_k^{\dagger}$, $k = 0, ..., \lfloor S/2 \rfloor$, in the foregoing expressions.

Using Proposition C.1, we may state the following lemma which details the large sample behaviour of the scaled regressors from (5.1).

Lemma C.1 *Under the conditions of Theorem 5.1, and defining δ_ξ such that $\delta_\xi = 0$ if $\xi = 0, 1$ and $\delta_\xi = 1$ if $\xi = 2$, then as $N \to \infty$,*

$$\begin{aligned}
\frac{1}{\sqrt{T}}\hat{x}^i_{0,S\lfloor rN \rfloor + s} &\Rightarrow \sigma \begin{cases} J_{0,c_0}(r), & i = 2, 3 \\ J_{0,c_0}(r) - r\mathcal{D}_{c_0}(r, \bar{c}_0), & i = 4, 5, 6 \end{cases} \\
&= \sigma\left[J_{0,c_0}(r) - r\delta_\xi \mathcal{D}_{c_0}(r, \bar{c}_0)\right] \equiv \sigma J_{0,c_0}(r, \delta_\xi \bar{c}_0)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{T}}\hat{x}^i_{S/2,S\lfloor rN \rfloor + s} &\Rightarrow \sigma \begin{cases} (-1)^j J_{S/2,c_{S/2}}(r), & i = 2, 3, 4, 5 \\ (-1)^j J_{S/2,c_{S/2}}(r) - r(-1)^j \mathcal{D}_{c_{S/2}}(r, \bar{c}_{S/2}) & i = 6 \end{cases} \\
&= \sigma\left[(-1)^j J_{S/2,c_{S/2}}(r) - r\delta_\xi(-1)^j \mathcal{D}_{c_{S/2}}(r, \bar{c}_{S/2})\right] \equiv \sigma(-1)^j J_{S/2,c_{S/2}}(r, \delta_\xi \bar{c}_{S/2})
\end{aligned}$$

where $j = (S\lfloor rN \rfloor + s) \bmod 2$, and

$$\begin{aligned}
\frac{1}{\sqrt{T}}\hat{x}^i_{k,S\lfloor rN \rfloor + s} &\Rightarrow \frac{\sigma}{\sqrt{2}} \begin{cases} \cos\left[(s+1)\omega_k\right] J^{\alpha}_{k,c_k}(r) + \sin\left[(s+1)\omega_k\right] J^{\beta}_{k,c_k}(r), & i = 2, 3, 4, 5 \\[2ex] \cos\left[(s+1)\omega_k\right] J^{\alpha}_{k,c_k}(r) + \sin\left[(s+1)\omega_k\right] J^{\beta}_{k,c_k}(r) & \\ -\frac{\sqrt{2}S}{2}\begin{bmatrix} \frac{2\sqrt{2}}{S}\mathcal{D}_{k,c_k}(r, \bar{c}_k) \\ \frac{2\sqrt{2}}{S}\mathcal{D}^{\beta}_{k,c_k}(r, \bar{c}_k) \end{bmatrix}' \Lambda_{1,k} \begin{bmatrix} r\cos(s\omega_k) \\ r\sin(s\omega_k) \end{bmatrix}, & i = 6 \end{cases} \\
&= \frac{\sigma}{\sqrt{2}}\Bigg\{\cos\left[(s+1)\omega_k\right] J^{\alpha}_{k,c_k}(r) + \sin\left[(s+1)\omega_k\right] J^{\beta}_{k,c_k}(r) \\
&\quad -2\delta_\xi \begin{bmatrix} \mathcal{D}_{k,c_k}(r, \bar{c}_k) \\ \mathcal{D}^{\beta}_{k,c_k}(r, \bar{c}_k) \end{bmatrix}' \Lambda_{1,k} \begin{bmatrix} r\cos(s\omega_k) \\ r\sin(s\omega_k) \end{bmatrix}\Bigg\} \equiv \frac{\sigma}{\sqrt{2}} J^{\alpha,\beta}_{k,c_k}(r, \delta_\xi \bar{c}_k)
\end{aligned}$$

$$\frac{1}{\sqrt{T}}\hat{x}^{\beta,i}_{k,S\lfloor rN\rfloor+s} \Rightarrow \frac{\sigma}{\sqrt{2}}\begin{cases} \cos\left[(s+1)\omega_k\right] J^{\beta}_{k,c_k}(r) - \sin\left[(s+1)\omega_k\right] J^{\alpha}_{k,c_k}(r), \ i=2,3,4,5 \\ \\ \cos\left[(s+1)\omega_k\right] J^{\beta}_{k,c_k}(r) - \sin\left[(s+1)\omega_k\right] J^{\alpha}_{k,c_k}(r) \\ -\frac{\sqrt{2}S}{2}\begin{bmatrix} \frac{2\sqrt{2}}{S}\mathcal{D}_{k,c_k}(r,\bar{c}_k) \\ \frac{2\sqrt{2}}{S}\mathcal{D}^{\beta}_{k,c_k}(r,\bar{c}_k)\end{bmatrix}' \Lambda_{2,k}\begin{bmatrix} r\cos(s\omega_k) \\ r\sin(s\omega_k)\end{bmatrix}, \qquad i=6 \end{cases}$$

$$= \frac{\sigma}{\sqrt{2}}\Bigg\{\cos\left[(s+1)\omega_k\right] J^{\beta}_{k,c_k}(r) - \sin\left[(s+1)\omega_k\right] J^{\alpha}_{k,c_k}(r)$$

$$-2\delta_\xi\begin{bmatrix} \mathcal{D}_{k,c_k}(r,\bar{c}_k) \\ \mathcal{D}^{\beta}_{k,c_k}(r,\bar{c}_k)\end{bmatrix}' \Lambda_{2,k}\begin{bmatrix} r\cos(s\omega_k) \\ r\sin(s\omega_k)\end{bmatrix}\Bigg\} \equiv \frac{\sigma}{\sqrt{2}}\bar{J}^{\alpha,\beta}_{k,c_k}(r,\delta_\xi\bar{c}_k).$$

where: $J_{0,c_0}(r)$, $J_{S/2,c_{S/2}}(r)$, $J^{\alpha}_{k,c_k}(r)$ *and* $J^{\beta}_{k,c_k}(r)$, $k=1,...,S^*$ *are as defined in Proposition B.1;* $\mathcal{D}_{c_0}(r,\bar{c}_0)$, $\mathcal{D}_{c_{S/2}}(r,\bar{c}_{0S/2})$, $\mathcal{D}_{k,c_k}(r,\bar{c}_k)$ *and* $\mathcal{D}^{\beta}_{k,c_k}(r,\bar{c}_k)$ *are as defined in Proposition C.1.*

Proof: The results for Cases 2 and 3 for the zero frequency, and Cases 2,3,4 and 5 for the seasonal frequencies are well established in the literature and are not reproduced here; see Rodrigues and Taylor (2004b) for full details. For the zero frequency under Cases 4,5 and 6, observe that

$$\frac{1}{\sqrt{T}}\hat{x}^{i}_{0,S\lfloor rN\rfloor+s} = \frac{1}{\sqrt{T}}\left\{y_{0,S\lfloor rN\rfloor+s} - \left[S+\left(\sum_{j=1}^{S-1} j\right)\right](\widetilde{\gamma}_0(\bar{c}_0)-\gamma_0) - S\left(S\lfloor rN\rfloor+s\right)\left(\widetilde{\delta}_0(\bar{c}_0)-\delta_0\right)\right\}$$

$$= \frac{1}{\sqrt{T}}\left[y_{0,S\lfloor rN\rfloor+s} - S\left(S\lfloor rN\rfloor+s\right)\left(\widetilde{\delta}_0(\bar{c}_0)-\delta_0\right)\right] + o_p(1)$$

$$\Rightarrow \sigma\left\{J_{0,c_0}(r) - r\mathcal{D}_{c_0}(r,\bar{c}_0)\right\}, \ i=4,5,6.$$

Turning to the Nyquist frequency for Case 6, observe that

$$\frac{1}{\sqrt{T}}\hat{x}^{6}_{S/2,S\lfloor rN\rfloor+s} = \frac{1}{\sqrt{T}}\Bigg\{y_{S/2,S\lfloor rN\rfloor+s} - \left[S+\left(\sum_{j=1}^{S-1} j\right)\right](-1)^{S\lfloor rN\rfloor+s}\left(\widetilde{\gamma}_{S/2}(\bar{c}_{S/2})-\gamma_{S/2}\right)$$

$$-S\left(S\lfloor rN\rfloor+s\right)(-1)^{S\lfloor rN\rfloor+s}\left(\widetilde{\delta}_{S/2}(\bar{c}_{S/2})-\delta_{S/2}\right)\Bigg\}$$

$$= \frac{1}{\sqrt{T}}\left[y_{S/2,S\lfloor rN\rfloor+s} - S\left(S\lfloor rN\rfloor+s\right)(-1)^{S\lfloor rN\rfloor+s}\left(\widetilde{\delta}_{S/2}(\bar{c}_{S/2})-\delta_{S/2}\right)\right] + o_p(1)$$

$$\Rightarrow \sigma\left\{(-1)^j J_{S/2,c}(r) - r(-1)^j\mathcal{D}_{c_{S/2}}(r,\bar{c}_{S/2})\right\}$$

where $j = (S\lfloor rN \rfloor + s) \bmod 2$. Finally, for the seasonal harmonic frequencies in Case 6, for $k = 1, ..., S^*$, observe that

$$\begin{aligned}
\frac{1}{\sqrt{T}}\hat{x}^{6}_{k,S\lfloor rN\rfloor+s} &= \frac{1}{\sqrt{T}}\Big\{y_{k,S\lfloor rN\rfloor+s} - [S/2+k^*]\left(\widetilde{\gamma}_{k,\alpha}(\bar{c}_k)-\gamma_{k,\alpha}\right)' z^{*\mu_{\omega_k}}_{Sn+s} \\
&- \frac{S}{2}\left(\widetilde{\delta}_{k,\alpha}(\bar{c}_k)-\delta_{k,\alpha}\right)' \Lambda_{1,k} z^{\tau_{\omega_k}}_{S\lfloor rN\rfloor+s}\Big\} \\
&= \frac{1}{\sqrt{T}}\left\{y_{k,S\lfloor rN\rfloor+s} - \frac{S}{2}\left(\widetilde{\delta}_{k,\alpha}(\bar{c}_k)-\delta_{k,\alpha}\right)' \Lambda_{1,k} z^{\tau_{\omega_k}}_{S\lfloor rN\rfloor+s}\right\} + o_p(1) \\
&= \frac{1}{\sqrt{T}}\left\{y_{k,S\lfloor rN\rfloor+s} - \frac{S}{2}\left(\widetilde{\delta}_{k,\alpha}(\bar{c}_k)-\delta_{k,\alpha}\right)' \Lambda_{1,k} \left[\begin{array}{c}(S\lfloor rN\rfloor+s)\cos(s\omega_k)\\(S\lfloor rN\rfloor+s)\sin(s\omega_k)\end{array}\right]\right\} + o_p(1) \\
&\Rightarrow \frac{\sigma}{\sqrt{2}}\Big\{\cos[(s+1)\omega_k]\, J^{\alpha}_{k,c_k}(r) + \sin[(s+1)\omega_k]\, J^{\beta}_{k,c_k}(r) \\
&-2\left[\begin{array}{c}\mathcal{D}_{k,c_k}(r,\bar{c}_k)\\ \mathcal{D}^{\beta}_{k,c_k}(r,\bar{c}_k)\end{array}\right]' \Lambda_{1,k}\left[\begin{array}{c} r\cos(s\omega_k)\\ r\sin(s\omega_k)\end{array}\right]\Big\},\ k=1,...,S^*,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{T}}\hat{x}^{\beta,6}_{k,S\lfloor rN\rfloor+s} &= \frac{1}{\sqrt{T}}\Big\{y^{\beta}_{k,S\lfloor rN\rfloor+s} - [S/2+k^*]\left(\widetilde{\gamma}_{k,\beta}(\bar{c}_k)-\gamma_{k,\beta}\right)' \bar{z}^{*\mu_{\omega_k}}_{S\lfloor rN\rfloor+s} \\
&- \frac{S}{2}\left(\widetilde{\delta}_{k,\beta}(\bar{c}_k)-\delta_{k,\beta}\right)' \Lambda_{2,k} z^{\tau_{\omega_k}}_{S\lfloor rN\rfloor+s}\Big\} \\
&= \frac{1}{\sqrt{T}}\left\{y^{\beta}_{k,S\lfloor rN\rfloor+s} - \frac{S}{2}\left(\widetilde{\delta}_{k,\beta}(\bar{c}_k)-\delta_{k,\beta}\right)' \Lambda_{2,k} z^{\tau_{\omega_k}}_{S\lfloor rN\rfloor+s}\right\} + o_p(1) \\
&= \frac{1}{\sqrt{T}}\left\{y_{k,S\lfloor rN\rfloor+s} - \frac{S}{2}\left(\widetilde{\delta}_{k,\beta}(\bar{c}_k)-\delta_{k,\beta}\right)' \Lambda_{2,k} \left[\begin{array}{c}(S\lfloor rN\rfloor+s)\cos(s\omega_k)\\(S\lfloor rN\rfloor+s)\sin(s\omega_k)\end{array}\right]\right\} + o_p(1) \\
&\Rightarrow \frac{\sigma}{\sqrt{2}}\Big\{\cos[(s+1)\omega_k]\, J^{\beta}_{k,c_k}(r) - \sin[(s+1)\omega_k]\, J^{\alpha}_{k,c_k}(r) \\
&-2\left[\begin{array}{c}\mathcal{D}_{k,c_k}(r,\bar{c}_k)\\ \mathcal{D}^{\beta}_{k,c_k}(r,\bar{c}_k)\end{array}\right]' \Lambda_{2,k}\left[\begin{array}{c} r\cos(s\omega_k)\\ r\sin(s\omega_k)\end{array}\right]\Big\},\ k=1,...,S^*.
\end{aligned}$$

In Lemma C.2 we now establish the large sample behaviour of various objects involving the transformed regressors from (5.1), which will feature in the definitions of our statistics.

Lemma C.2 *Under the conditions of Lemma C.1, as $N \to \infty$,*

$$\begin{aligned}
&i)\ T^{-2}\sum_{Sn+s=1}^{T}\left(\hat{x}^{i}_{j,Sn+s-1}\right)^2 \Rightarrow \sigma^2\int_0^1 \left[J_{j,c_j}(r,\delta_\xi\bar{c}_j)\right]^2 dr,\ j=0,S/2 \\
&ii)\ T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^{i}_{k,Sn+s-1}\right)^2 \Rightarrow \frac{\sigma^2}{4}\int_0^1 \left[J^{\alpha,\beta}_{k,c_k}(r,\delta_\xi\bar{c}_k)\right]^2 dr,\ k=1,...,S^* \\
&iii)\ T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^{\beta,i}_{k,Sn+s-1}\right)^2 \Rightarrow \frac{\sigma^2}{4}\int_0^1 \left[\bar{J}^{\alpha,\beta}_{k,c_k}(r,\delta_\xi\bar{c}_k)\right]^2 dr,\ k=1,...,S^*
\end{aligned}$$

$$iv)\ T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{i}_{j,Sn+s-1}v_{Sn+s} \Rightarrow \sigma^2\left\{\int_0^1 J_{j,c_j}(r)dJ_{j,0}(r) - \delta_\xi \mathcal{D}_{c_j}(r,\bar{c}_j)\int_0^1 r dJ_{j,0}(r)\right\},\ j=0,S/2$$

$$v)\ T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{i}_{k,Sn+s-1}v_{Sn+s} \Rightarrow \frac{\sigma^2}{2}\left\{\left[\int_0^1 J^{\alpha}_{k,c_k}(r)dJ^{\alpha}_{k,0}(r) + \int_0^1 J^{\beta}_{k,c_k}(r)dJ^{\beta}_{k,0}(r)\right]\right.$$
$$\left. - 2\delta_\xi \left[\begin{array}{c}\mathcal{D}_{c_k}(r,\bar{c}_k)\\ \mathcal{D}^{\beta}_{c_k}(r,\bar{c}_k)\end{array}\right]'\Lambda_{1,k}\left[\begin{array}{c}\int_0^1 r dJ^{\alpha}_{k,0}(r)\\ \int_0^1 r dJ^{\beta}_{k,0}(r)\end{array}\right]\right\},\ k=1,...,S^*$$

$$vi)\ T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{\beta,i}_{k,Sn+s-1}v_{Sn+s} \Rightarrow \frac{\sigma^2}{2}\left\{\left[\int_0^1 J^{\alpha}_{k,c_k}(r)dJ^{\beta}_{k,0}(r) - \int_0^1 J^{\beta}_{k,c_k}(r)dJ^{\alpha}_{k,0}(r)\right]\right.$$
$$\left. - 2\delta_\xi \left[\begin{array}{c}\mathcal{D}_{c_k}(r,\bar{c}_k)\\ \mathcal{D}^{\beta}_{c_k}(r,\bar{c}_k)\end{array}\right]'\Lambda_{2,k}\left[\begin{array}{c}\int_0^1 r dJ^{\alpha}_{k,0}(r)\\ \int_0^1 r dJ^{\beta}_{k,0}(r)\end{array}\right]\right\}, k=1,...,S^*.$$

in each case for $i = 2,...,6$, where δ_ξ is as defined in Lemma C.1.

Proof: The proof of parts (i)-(iii) follows straightforwardly from Lemma C.1 and applications of the CMT. To prove part (iv), let $z^{\tau_j}_{Sn+s-1}$ denote the $(Sn+s-1)$th element of $\mathbf{t}_j$, then since

$$T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{i}_{j,Sn+s-1}v_{Sn+s} = T^{-1}\sum_{Sn+s=1}^{T}\left[x_{j,Sn+s-1} - S\delta_\xi z^{\tau_j}_{Sn+s-1}\left(\widetilde{\delta}_j(\bar{c}_j)-\delta_j\right)\right]v_{Sn+s} + o_p(1)$$
$$= T^{-1}\sum_{Sn+s=1}^{T}x_{j,Sn+s-1}v_{Sn+s} - \frac{S}{T}\delta_\xi\left(\widetilde{\delta}_j(\bar{c}_j)-\delta_j\right)\sum_{Sn+s=1}^{T}z^{\tau_j}_{Sn+s-1}v_{Sn+s} + o_p(1)$$

the stated result follows immediately. Turning to parts (v) and (vi), observing that

$$T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{i}_{k,Sn+s-1}v_{Sn+s} = T^{-1}\sum_{Sn+s=1}^{T}x_{k,Sn+s-1}v_{Sn+s}$$
$$- \frac{\delta_\xi S}{2T}\left(\widetilde{\delta}_{k,\alpha}(\bar{c}_k)-\delta_{k,\alpha}\right)'\Lambda_{1,k}\sum_{Sn+s=1}^{T}\left[\begin{array}{c}(Sn+s)\cos(s\omega_k)\\ (Sn+s)\sin(s\omega_k)\end{array}\right]v_{Sn+s} + o_p(1)$$

and that

$$T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{\beta,i}_{k,Sn+s-1}v_{Sn+s} = T^{-1}\sum_{Sn+s=1}^{T}x^{\beta}_{k,Sn+s}v_{Sn+s}$$
$$- \frac{\delta_\xi S}{2T}\left(\widetilde{\delta}_{k,\beta}(\bar{c}_k)-\delta_{k,\beta}\right)'\Lambda_{2,k}\sum_{Sn+s=1}^{T}\left[\begin{array}{c}(Sn+s)\cos(s\omega_k)\\ (Sn+s)\sin(s\omega_k)\end{array}\right]v_{Sn+s} + o_p(1)$$

the stated results follow immediately.

Making use of the asymptotic orthogonality of the regressors, which follows directly from (B.1)-(B.2), the following proposition, whose proof is given in Rodrigues and Taylor (2004b), provides a convenient form for the $\hat{t}_0$, $\hat{t}_{S/2}$, $\hat{t}_k$ and $\hat{t}^{\beta}_k$, $k=1,...,S^*$, statistics from (5.1).

Proposition C.2 *The t-statistics from (5.1) are given by*

$$\hat{t}_j = \frac{T}{\widehat{\sigma}}\pi_j \left[T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^i_{j,Sn+s-1}\right)^2\right]^{1/2} + \frac{T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^i_{j,Sn+s-1}v_{Sn+s}}{\widehat{\sigma}\left[T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^i_{j,Sn+s-1}\right)^2\right]^{1/2}} + o_p(1),\ j = 0, S/2$$

$$\hat{t}_k = \frac{T}{\widehat{\sigma}}\pi_{\alpha,k} \left[T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^i_{k,Sn+s-1}\right)^2\right]^{1/2} + \frac{T^{-1}\sum_{Sn+s=1}^{T}\widehat{x}^{\alpha,i}_{k,Sn+s-1}v_{Sn+s}}{\widehat{\sigma}\left[T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^i_{k,Sn+s-1}\right)^2\right]^{1/2}} + o_p(1)$$

$$\hat{t}^\beta_k = \frac{T}{\widehat{\sigma}}\pi_{\beta,k} \left[T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^{\beta,i}_{k,Sn+s-1}\right)^2\right]^{1/2} + \frac{T^{-1}\sum_{Sn+s=1}^{T}x^{\beta}_{k,Sn+s-1}v_{Sn+s}}{\widehat{\sigma}\left[T^{-2}\sum_{Sn+s=1}^{T}\left(\widehat{x}^{\beta,i}_{k,Sn+s-1}\right)^2\right]^{1/2}} + o_p(1),$$

$k = 1, \ldots S^*$, $i = 2, ..., 6$, *where* $\hat{\sigma}^2$ *denotes the usual OLS estimator of* σ^2 *from (5.1).*

The results stated for the t-statistics in (5.3), (5.4) and (5.5) then follow directly from Proposition C.2, Lemma C.2 and applications of the CMT. The representations for the $\hat{F}_k$, $k = 1, ..., S^*$, $\hat{F}_{1\ldots\lfloor S/2\rfloor}$ and $\hat{F}_{0\ldots\lfloor S/2\rfloor}$ statistics then follow directly from the asymptotic orthogonality result and applications of the CMT.

Table 5.1: Critical Values for GLS-Detrended Seasonal Unit Root Tests.

	$\hat{t}_0$				$\hat{t}_2$				$\hat{F}_1$				$\hat{F}_{1\ldots 2}$				$\hat{F}_{0\ldots 2}$			
T	.010	.025	.050	.100	.010	.025	.050	.100	.900	.950	.975	.990	.900	.950	.975	.990	.900	.950	.975	.990
(a) Case 3																				
48	-3.22	-2.90	-2.63	-2.34	-3.22	-2.89	-2.63	-2.34	3.81	4.65	5.47	6.60	3.89	4.60	5.32	6.23	3.88	4.52	5.14	5.95
100	-2.99	-2.66	-2.40	-2.11	-2.97	-2.66	-2.40	-2.11	3.14	3.91	4.66	5.71	3.09	3.71	4.32	5.11	3.04	3.56	4.08	4.76
136	-2.90	-2.58	-2.32	-2.03	-2.92	-2.59	-2.32	-2.03	2.97	3.72	4.46	5.50	2.89	3.50	4.07	4.84	2.81	3.31	3.81	4.48
200	-2.83	-2.51	-2.23	-1.93	-2.83	-2.51	-2.23	-1.94	2.80	3.54	4.28	5.26	2.66	3.24	3.81	4.54	2.57	3.05	3.51	4.13
400	-2.71	-2.38	-2.10	-1.80	-2.71	-2.38	-2.11	-1.80	2.59	3.32	4.04	4.98	2.43	2.97	3.53	4.22	2.31	2.77	3.21	3.76
∞	-2.59	-2.25	-1.97	-1.64	-2.59	-2.25	-1.97	-1.64	2.44	3.15	3.84	4.78	2.24	2.78	3.31	3.98	2.11	2.55	2.99	3.53
(b) Case 5																				
48	-4.02	-3.66	-3.39	-3.09	-3.29	-2.94	-2.67	-2.37	3.85	4.71	5.57	6.79	3.97	4.71	5.45	6.44	4.89	5.66	6.40	7.34
100	-3.78	-3.48	-3.21	-2.92	-3.01	-2.69	-2.42	-2.13	3.15	3.93	4.72	5.76	3.14	3.79	4.40	5.17	3.98	4.59	5.18	5.93
136	-3.70	-3.39	-3.14	-2.86	-2.92	-2.61	-2.34	-2.04	2.97	3.71	4.48	5.49	2.90	3.51	4.09	4.84	3.71	4.29	4.82	5.53
200	-3.62	-3.31	-3.06	-2.78	-2.83	-2.51	-2.24	-1.95	2.80	3.54	4.29	5.24	2.68	3.25	3.81	4.57	3.44	3.98	4.51	5.18
400	-3.52	-3.22	-2.97	-2.70	-2.73	-2.38	-2.11	-1.81	2.59	3.27	3.99	4.97	2.41	2.97	3.50	4.21	3.13	3.64	4.14	4.79
∞	-3.42	-3.12	-2.87	-2.58	-2.59	-2.25	-1.97	-1.64	2.44	3.15	3.84	4.78	2.24	2.78	3.31	3.98	2.86	3.36	3.82	4.39
(c) Case 6																				
48	-4.25	-3.87	-3.57	-3.24	-4.23	-3.86	-3.57	-3.24	8.51	9.86	11.23	13.00	8.56	9.78	10.96	12.54	8.50	9.61	10.67	12.04
100	-3.86	-3.54	-3.28	-2.99	-3.85	-3.54	-3.28	-2.99	7.18	8.31	9.40	10.77	7.04	7.95	8.83	9.90	6.88	7.69	8.45	9.39
136	-3.74	-3.45	-3.20	-2.91	-3.76	-3.44	-3.18	-2.91	6.82	7.86	8.88	10.20	6.59	7.43	8.25	9.31	6.42	7.16	7.86	8.75
200	-3.67	-3.36	-3.11	-2.82	-3.68	-3.36	-3.11	-2.83	6.47	7.45	8.43	9.69	6.19	7.02	7.78	8.77	6.01	6.70	7.35	8.18
400	-3.55	-3.25	-2.98	-2.71	-3.55	-3.24	-2.99	-2.70	6.04	7.06	8.01	9.33	5.70	6.50	7.25	8.23	5.48	6.15	6.78	7.56
∞	-3.42	-3.12	-2.87	-2.58	-3.42	-3.12	-2.87	-2.58	5.68	6.66	7.57	8.76	5.26	6.01	6.72	7.57	5.00	5.64	6.21	6.89

Table 6.1: Empirical Size of Seasonal Unit Root Tests (Nominal 0.05 level) DGP (6.1)-(6.2). Auxiliary Regression (5.1) with $p_{\max} = 4$
$\xi = 1$ indicates Case 3 of (5.1), $\xi = 2$ indicates Case 6 of (5.1)

N	ϕ	θ		t_0		t_2		t_1		F_1		$F_{[1...2]}$		$F_{[0...2]}$	
				$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$
25	.9	.0	GLS	.05	.04	.04	.02	.05	.02	.05	.02	.04	.02	.05	.03
			OLS	.06	.04	.04	.03	.04	.03	.03	.02	.04	.04	.05	.04
	.0	-.6	GLS	.28	.44	.28	.43	.05	.04	.06	.04	.22	.38	.29	.51
			OLS	.28	.49	.28	.48	.05	.04	.06	.04	.22	.39	.29	.55
	.0	.6	GLS	.06	.05	.06	.05	.26	.34	.19	.31	.18	.36	.18	.39
			OLS	.05	.05	.05	.05	.24	.43	.27	.44	.24	.47	.24	.48
50	.9	.0	GLS	.05	.05	.05	.03	.05	.02	.05	.03	.05	.03	.05	.04
			OLS	.06	.05	.05	.04	.03	.01	.05	.04	.04	.04	.05	.04
	.0	-.6	GLS	.22	.37	.22	.36	.04	.04	.05	.04	.16	.26	.22	.35
			OLS	.22	.38	.22	.38	.03	.04	.05	.04	.15	.27	.20	.37
	.0	.6	GLS	.04	.05	.04	.04	.22	.38	.14	.33	.11	.28	.09	.26
			OLS	.04	.05	.04	.04	.23	.40	.19	.37	.16	.34	.15	.33

Note: The notation GLS and OLS in the fourth column indicates the method of de-trending used.

Table 6.2: Empirical Power of Seasonal Unit Root Tests (Nominal 0.05 level). DGP (6.3), $N = 25$. Auxiliary Regression (5.1) with $p_{\max} = 4$.
$\xi = 1$ indicates Case 3 of (5.1), $\xi = 2$ indicates Case 6 of (5.1).

		$c_k = -3$		$c_k = -5$		$c_k = -7$		$c_k = -11$		$c_k = -15$		$c_k = -19$	
Test		$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$
t_0	GLS	.16	.07	.25	.11	.38	.15	.63	.29	.82	.45	.91	.65
	OLS	.09	.06	.13	.08	.18	.11	.35	.20	.54	.34	.73	.48
t_2	GLS	.15	.07	.25	.11	.37	.15	.62	.29	.82	.46	.92	.65
	OLS	.08	.06	.12	.08	.17	.11	.34	.20	.54	.33	.73	.48
t_1	GLS	.33	.09	.61	.17	.82	.28	.97	.56	.99	.80	1.00	.91
	OLS	.12	.07	.20	.10	.33	.16	.64	.34	.87	.56	.95	.77
F_1	GLS	.24	.09	.46	.15	.67	.25	.92	.52	.98	.76	.99	.89
	OLS	.11	.07	.17	.09	.28	.15	.56	.31	.81	.52	.93	.73
$F_{1...2}$	GLS	.31	.10	.57	.18	.79	.31	.97	.62	1.00	.86	1.00	.96
	OLS	.13	.08	.23	.12	.38	.19	.71	.38	.92	.63	.98	.83
$F_{0...2}$	GLS	.37	.11	.67	.21	.88	.37	.99	.72	1.00	.93	1.00	.99
	OLS	.15	.08	.27	.13	.45	.20	.81	.45	.96	.72	.99	.90

Note: The notation GLS and OLS in the second column indicates the method of de-trending used.

Table 6.3: Empirical Power of Seasonal Unit Root Tests (Nominal 0.05 level). DGP (6.3), $N = 50$. Auxiliary Regression (5.1) with $p_{\max} = 4$. $\xi = 1$ indicates Case 3 of (5.1), $\xi = 2$ indicates Case 6 of (5.1).

		$c_k = -3$		$c_k = -5$		$c_k = -7$		$c_k = -11$		$c_k = -15$		$c_k = -19$	
Test		$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$
t_0	GLS	.17	.07	.29	.11	.44	.17	.73	.33	.90	.53	.97	.72
	OLS	.09	.06	.13	.08	.19	.12	.36	.21	.58	.35	.77	.52
t_2	GLS	.17	.08	.28	.11	.44	.17	.73	.33	.90	.54	.97	.73
	OLS	.09	.06	.12	.08	.19	.11	.36	.21	.58	.35	.77	.52
t_1	GLS	.37	.09	.68	.18	.89	.30	.99	.64	1.00	.88	1.00	.96
	OLS	.12	.07	.21	.12	.34	.18	.68	.38	.91	.64	.98	.84
F_1	GLS	.23	.09	.47	.16	.71	.27	.96	.57	1.00	.83	1.00	.95
	OLS	.11	.07	.17	.11	.28	.16	.58	.34	.84	.58	.96	.79
$F_{1...2}$	GLS	.32	.10	.63	.19	.86	.34	.99	.71	1.00	.93	1.00	.99
	OLS	.13	.08	.23	.12	.39	.20	.76	.44	.95	.72	.99	.91
$F_{0...2}$	GLS	.40	.11	.74	.23	.93	.42	1.00	.82	1.00	.97	1.00	1.00
	OLS	.15	.08	.27	.13	.45	.20	.81	.45	.96	.72	.99	.90

Note: See Note for Table 6.2.

Table 6.4: Asymptotic Local Power of Seasonal Unit Root Tests (Nominal 0.05 level).

		$c_k = -3$		$c_k = -5$		$c_k = -7$		$c_k = -11$		$c_k = -15$		$c_k = -19$	
Test		$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$	$\xi = 1$	$\xi = 2$
t_k	GLS	.17	.08	.32	.12	.50	.18	.82	.37	.96	.61	1.00	.81
$k = 0, S/2$	OLS	.08	.06	.12	.08	.18	.11	.36	.22	.60	.37	.80	.57
t_k	GLS	.38	.10	.72	.19	.92	.34	1.00	.72	1.00	.95	1.00	1.00
$k = 1, ..., S^*$	OLS	.11	.07	.19	.11	.34	.18	.71	.41	.94	.69	1.00	.90
F_k	GLS	.23	.09	.48	.16	.74	.28	.98	.62	1.00	.89	1.00	.99
$k = 1, ..., S^*$	OLS	.10	.07	.17	.10	.28	.16	.62	.35	.88	.62	1.00	.85
$F_{1...2}$	GLS	.31	.12	.64	.22	.89	.40	1.00	.81	1.00	.98	1.00	1.00
	OLS	.13	.08	.23	.13	.40	.22	.80	.50	.98	.81	1.00	.96
$F_{0...2}$	GLS	.37	.12	.76	.27	.96	.50	1.00	.91	1.00	1.00	1.00	1.00
	OLS	.14	.09	.29	.15	.51	.26	.91	.61	.99	.91	1.00	.99

Note: See Note for Table 6.2.