EUROPEAN UNIVERSITY INSTITUTE
DEPARTMENT OF ECONOMICS

EUI Working Paper **ECO** No. 2004 / 31

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Published in Italy in December 2004
European University Institute
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November 2004

Abstract

In this paper, we analyse the properties of recursive trend adjusted unit root tests. We show that OLS based recursive trend adjustment can produce unit root tests which are not invariant when the data is generated from a random walk with drift and investigate whether the power performance previously observed in the literature is maintained under invariant versions of the tests. A finite sample evaluation of the size and power of the invariant procedures is presented.

Keywords: Recursive Trend Adjustment, Unit root tests, Invariance.
JEL: C12, C22.

*I thank Helmut Lütkepohl and Robert Taylor for their useful comments and suggestions. I am also grateful to the European University Institute for their kind hospitality during my stay as a Jean Monnet Fellow and to the Portuguese Science Foundation for their financial support through the POCTI program and FEDER (grant ref. POCTI/ECO/49266/2002).

1 Introduction

Recently, an interesting and simple approach for bias reduction in autoregressive model estimation was proposed by So and Shin (1999), Shin and So (2001) and Taylor (2002), based on recursively adjusting the deterministic components. The motivation behind this alternative way of accounting for the deterministic component of a process is due, as indicated by So and Shin (1999), to the lack of efficiency of conventional mean and trend adjustment procedures. Conventional mean and trend adjustment based on the full sample of data, induces downward small sample bias, as a result of the imposed correlation between the demeaned/detrended regressors and the error term.

Recursively adjusting for the deterministic terms, however, reduces this bias since at time t only sample observations, y_k, up to time $t-1$ ($k \leq t-1$) are used, thus avoiding correlation of the mean with residuals, ε_j, occurring in time periods $j \geq t$. Also, according to recent literature, this approach leads to considerable power improvements of unit root tests (see So and Shin, 2001 and Taylor, 2002).

Recursive demeaning has been shown to avoid the spurious result observed by Leybourne *et al.* (1998) for the DF test (see Cook, 2002), and recursively adjusting the deterministics has also been used in panel data models (see Choi, Mark and Sul, 2004, Chang, 2002, and Chang and Park, 2004) as well as in nonlinear instrumental variable estimation of autoregressions (see Phillips, Park and Chang, 2004). Shin and So (2002) demonstrate how well recursive demeaning performs in different nonstationary contexts and Taylor (2002) provides powerful recursive demeaned and recursive detrended unit root tests for nonseasonal and seasonal processes.

However, one problem which we highlight in this paper (see also Sul, Phillips and Choi, 2003) is that recursive trend adjustment based on recursive ordinary least squares (OLS) estimates may produce tests which are not invariant to trend parameters that may exist in the data generation process (DGP).

This problem is here addressed as well as the size and power performance of invariant versions of these tests under recursive detrending, by adopting similar transformations considered in Chang (2002), Chang and Park (2004), Phillips, Park and Chang (2004) and Taylor (2002).

This paper is organised as follows. The next section briefly introduces the recursive trend-adjusted unit root test and the potential invariance results. Section 3 presents invariant versions of the procedures and looks at efficiency gains and finite sample performance. Finally, section 4 concludes the paper and an appendix provides details of the results obtained.

2 Recursive Trend Adjustment

Consider the following data generating process (DGP),

$$y_t = \gamma_0^0 + \gamma_1^0 t + x_t, \tag{2.1}$$

$$(1-\alpha L)x_t = \varepsilon_t, \qquad t = 1, ..., T \tag{2.2}$$

where x_0 is an arbitrary $O_p(1)$ random variable, $\varepsilon_t \sim iid(0, \sigma^2)$ and the null hypothesis of interest is,

$$H_0 : \alpha = 1. \tag{2.3}$$

The model of interest includes a constant and time trend, so that the vector of deterministic variables considered is $Z_t = (1, t)'$, with corresponding vector of parameters to be estimated, $(\gamma_0, \gamma_1)'$.

In order to consider the recursive trend adjustment, we take an OLS-based approach whereby the vector of estimators of the deterministic component at time t is given by,

$$\widetilde{\gamma}_t^r = \left(\sum_{k=1}^{t} Z_k Z_k' \right)^{-1} \sum_{k=1}^{t} Z_k y_k. \tag{2.4}$$

Thus, once the $T \times 2$ vector of parameters of the deterministic component is estimated as in (2.4), following Shin and So (2001) the test regression is set up using the following recusively adjusted variables,

$$\widetilde{y}_t = y_t - Z_{t-1}' \widetilde{\gamma}_{t-1}^r, \tag{2.5}$$
$$\widetilde{y}_{t-1} = y_{t-1} - Z_{t-1}' \widetilde{\gamma}_{t-1}^r. \tag{2.6}$$

As mentioned earlier in (2.5) and (2.6), only the sample mean of the observations up to time $t-1$ is considered.

Hence, according to (2.5) and (2.6), we can obtain the test regression of interest,

$$\Delta \widetilde{y}_t = (\alpha - 1)\, \widetilde{y}_{t-1} + \varepsilon_t \tag{2.7}$$

with the relevant test statistic given as, $\tau = \left(\widehat{\alpha - 1} \right) / se(\widehat{\alpha - 1})$, where $se(\widehat{\alpha - 1})$ denotes the standard error of $(\widehat{\alpha - 1})$.

Remark 2.1: In order to account for potential autocorrelation, model (2.7) can be augmented with lags of the dependent variable as in the conventional augmented DF (ADF) test; see, *inter alia*, Shin and So (2001) and Taylor (2002).

2.1 Non-Invariance of Recursively-Detrended Tests

Contrary to OLS-based recursive demeaning, some caution is required when recursive detrending is necessary. In particular, if a constant and a time trend are considered as necessary deterministic variables in a modelling exercise, from (2.5) and (2.6) we observe that, $\widetilde{y}_t$ and $\widetilde{y}_{t-1}$ will have the following expressions:

Proposition 1 *Under the DGP considered in (2.1) and (2.2), with* $\alpha = 1$ *and* $\gamma_1^0 \neq 0$ *we can observed that,*

$$\widetilde{y}_t^{\tau} = S_t - Z_{t-1}' \overline{S}_{t-1}^{\tau} + \gamma_1^0 \tag{2.8}$$

and

$$\widetilde{y}_{t-1}^{\tau} = S_{t-1} - Z_{t-1}' \overline{S}_{t-1}^{\tau} \tag{2.9}$$

where $\overline{S}^{\tau}_{t-1} = \left(\sum_{k=1}^{t-1} Z_k Z_k' \right)^{-1} \sum_{k=1}^{t-1} Z_k S_k$ *and* $Z_{t-1} = (1, t-1)'$.

From this proposition we can establish that $\widetilde{y}^{\tau}_t$ is, among other things, a function of the nuisance parameter γ_1^0, and consequently $\Delta \widetilde{y}_t = \gamma_1^0 + \varepsilon_t$. Hence, a test based on the variables in (2.8) and (2.9) will therefore be affected by this nuisance parameter. For closer observation, consider Table 2.1 where DF^{τ}_r represents a t-statistics computed from an auxiliary regression such as (2.7) based on (2.8) and (2.9).

Table 2.1: Rejection probability of the DF^{τ}_r when the DGP is (2.1) and (2.2), with $\alpha = 1$, $\gamma_1^0 \neq 0$, $\varepsilon_t \sim nid(0,1)$ and a 5% significance level is considered.

	T=100	T=200
γ_1^0	DF^{τ}_r	DF^{τ}_r
0	.050	.050
0.25	.114	.182
0.50	.242	.328
0.75	.312	.373
1	.341	.391
5	.352	.404

Note: The results are based on 50000 Monte Carlo simulations and the critical values are taken from Table 3.1 below.

Table 2.1 shows the sensitivity of the distribution of DF^{τ}_r to γ_1^0, suggesting, in line with proposition 1, that simply using (2.5) and (2.6) is not an adequate approach for recursively detrending.

3 Invariant Versions

In this section we consider invariant versions of recursively detrended procedures using the following transformations:
Transformation 1

$$\widetilde{y}^{\tau}_{1,t} = y_t - y_0 - \frac{1}{T} \sum_{t=1}^{T} \Delta y_t - \sum_{k=1}^{t-1} \frac{1}{k} (y_k - y_0) \tag{3.10}$$

$$\widetilde{y}^{\tau}_{1,t-1} = y_{t-1} - y_0 - \sum_{k=1}^{t-1} \frac{1}{k} (y_k - y_0); \tag{3.11}$$

Transformation 2

$$\widetilde{y}^{\tau}_{2,t} = y_t + \frac{2}{t-1}\sum_{k=1}^{t-1} y_k - \frac{1}{T}\sum_{t=1}^{T}\Delta y_t - \frac{6}{t(t-1)}\sum_{k=1}^{t-1} k y_k \qquad (3.12)$$

$$\widetilde{y}^{\tau}_{2,t-1} = y_{t-1} + \frac{2}{t-1}\sum_{k=1}^{t-1} y_k - \frac{6}{t(t-1)}\sum_{k=1}^{t-1} k y_k; \qquad (3.13)$$

see also Chang (2002), Chang and Park (2004), Phillips and Chang (2004);

A slightly different version of the invariant *transformation 2* has been suggested by Taylor (2002), who instead of subtracting the mean of Δy_t from (3.12) recursively detrends y_t with data up to time t instead of $t-1$ as used in (3.12). We will refer to this approach as *transformation 3.*

Proposition 2 *Under the DGP considered in (2.1) and (2.2), with $\alpha = 1$ and $\gamma_1^0 \neq 0$ we observe that for transformation 1,*

$$\widetilde{y}^{\tau}_{1,t} = S_t - \sum_{k=1}^{t-1} \frac{1}{k} S_k$$

$$\widetilde{y}^{\tau}_{1,t-1} = S_{t-1} - \sum_{k=1}^{t-1} \frac{1}{k} S_k;$$

for transformation 2,

$$\widetilde{y}^{\tau}_{2,t} = S_t + 2\overline{S}_{t-1} - \frac{6}{t(t-1)}\sum_{k=1}^{t-1} k S_k$$

$$\widetilde{y}^{\tau}_{2,t-1} = S_{t-1} + 2\overline{S}_{t-1} - \frac{6}{t(t-1)}\sum_{k=1}^{t-1} k S_k;$$

and for transformation 3,

$$\widetilde{y}^{\tau}_{3,t} = S_t + 2\overline{S}_t - \frac{6}{t(t+1)}\sum_{k=1}^{t} k S_k$$

$$\widetilde{y}^{\tau}_{3,t-1} = \widetilde{y}^{\tau}_{2,t-1}$$

where $S_k = \sum_{i=1}^{k} \varepsilon_i$ *and* $\overline{S}_{t-1} = \frac{1}{t-1}\sum_{k=j}^{t-1}\sum_{i=1}^{k} \varepsilon_i.$

3.1 Efficiency Gains

The advantage of recursively adjusting for deterministics is in the resulting bias reduction and the consequent efficiency gains that prove useful for various of different applications. Hence, in this section we analyse the magnitude of efficiency gains obtained from the proposed invariant procedures. Table 3.1 shows the averages of the parameter estimates computed from 50000 Monte Carlo simulations where the DGP is an AR(1) model such as, $y_t = \rho y_t + \varepsilon_t$, $\varepsilon_t \sim nid(0,1)$, and the estimated model an AR(1) but adjusted for a constant and time trend. The four approaches adopted in this context draw on the procedures described earlier, *i.e.*, $\widehat{\rho}_{OLS}$ - represents the estimator based on conventional OLS; $\widehat{\rho}_r$ is obtained by recursive detrending employing the variables in (2.8) and (2.9); and, $\widehat{\rho}_{1r}$, $\widehat{\rho}_{2r}$ and $\widehat{\rho}_{3r}$, are obtained based on transformations 1,2 and 3, respectively.

Table 3.1: Average parameter estimates obtained under conventional and recursive trend adjustment

	T=50					T=100					T=200				
ρ	$\widehat{\rho}_{OLS}$	$\widehat{\rho}_r$	$\widehat{\rho}_{1r}$	$\widehat{\rho}_{2r}$	$\widehat{\rho}_{3r}$	$\widehat{\rho}_{OLS}$	$\widehat{\rho}_r$	$\widehat{\rho}_{1r}$	$\widehat{\rho}_{2r}$	$\widehat{\rho}_{3r}$	$\widehat{\rho}_{OLS}$	$\widehat{\rho}_r$	$\widehat{\rho}_{1r}$	$\widehat{\rho}_{2r}$	$\widehat{\rho}_{3r}$
1	.806	.960	.973	.960	.851	.901	.980	.987	.980	.919	.950	.990	.994	.990	.957
.97	.799	.954	.969	.952	.843	.888	.970	.982	.969	.905	.932	.975	.986	.974	.938
.95	.789	.945	.964	.942	.832	.874	.957	.975	.956	.891	.915	.960	.979	.959	.921
.90	.756	.912	.947	.910	.798	.833	.918	.956	.917	.849	.869	.917	.959	.917	.876
.80	.679	.831	.907	.831	.720	.743	.830	.913	.830	.761	.773	.823	.915	.823	.781
.70	.593	.739	.861	.741	.636	.648	.734	.866	.734	.669	.675	.725	.867	.725	.685

From Table 3.1 we observe that recursive detrending based on transformation 2 provides in essence the same efficiency gains as with recursive detrending based on the estimates from (2.4). Moreover, although transformation 1 is proving useful in terms of making the procedure invariant to drift terms in the DGP, it proves to be inefficient, particularly when $\rho \leq 0.97$ and as T increases, thus determining the procedure as inadequate. Note also that the procedure suggested by Taylor (2002), in comparison to other recursive trend adjustment methods (see the results for $\widehat{\rho}_r$ and $\widehat{\rho}_{2r}$) produces some efficiency loss when $\rho \geq 0.9$. Finally, accounting for the deterministics using the full sample as is conventionally done also produces inefficient estimates (as is known in the literature); see the results for $\widehat{\rho}_{OLS}$.

3.2 Finite Sample Performance

In this section we look at the finite sample performance of the procedures. The data generation process considered in the simulations is a conventional random walk, such as, $y_t = y_{t-1} + \varepsilon_t$, with $\varepsilon_t \sim$ niid(0,1) and the test regression used was an AR(1) adjusted for a constant and a time trend using the different approaches described in Section 2.

3.2.1 Finite Sample Critical Values

Tables 3.2 - 3.5 provide the critical values for the three recursive-based trend adjustment procedures. All critical values are based on 50000 Monte Carlo replications.

Table 3.2: Finite Sample Critical Values for the Unit Root Test Based on recursive detrending using (2.8) and (2.9) (DF_r^τ) and $\gamma_1^0 = 0$

T	0.010	0.025	0.050	0.100	0.500	0.900	0.950	0.975	0.990
100	-2.544	-2.149	-1.833	-1.473	-0.228	1.028	1.401	1.709	2.074
200	-2.504	-2.137	-1.810	-1.456	0.236	1.025	1.393	1.724	2.097
500	-2.504	-2.140	-1.827	-1.477	-0.235	1.014	1.367	1.688	2.047
1000	-2.507	-2.146	-1.843	-1.489	-0.246	1.010	1.368	1.690	2.070

Table 3.3: Finite sample critical values for the unit root test based on transformation 1 (DF_{1r}^τ)

T	0.010	0.025	0.050	0.100	0.500	0.900	0.950	0.975	0.990
100	-2.176	-1.804	-1.512	-1.187	-0.271	0.541	0.821	1.097	1.446
200	-2.135	-1.788	-1.493	-1.167	-0.247	0.561	0.846	1.102	1.423
500	-2.126	-1.751	-1.471	-1.146	-0.241	0.549	0.831	1.080	1.413
1000	-2.119	-1.751	-1.452	-1.138	-0.229	0.567	0.840	1.102	1.432

Table 3.4: Finite sample critical values for the unit root test based on transformation 2 (DF_{2r}^τ)

T	0.010	0.025	0.050	0.100	0.500	0.900	0.950	0.975	0.990
100	-2.411	-2.039	-1.729	-1.371	-0.202	0.897	1.215	1.492	1.829
200	-2.387	-2.034	-1.699	-1.365	-0.215	0.895	1.205	1.499	1.833
500	-2.388	-2.016	-1.717	-1.373	-0.215	0.888	1.188	1.466	1.778
1000	-2.363	-2.026	-1.726	-1.383	-0.225	0.881	1.188	1.475	1.794

Table 3.5: Finite sample critical values for the unit root test based on transformation 3 (DF_{3r}^τ)

T	0.010	0.025	0.050	0.100	0.500	0.900	0.950	0.975	0.990
100	-3.684	-3.358	-3.071	-2.771	-1.795	-0.870	-0.560	-0.268	0.066
200	-3.671	-3.350	-3.084	-2.790	-1.847	-0.934	-0.633	-0.364	-0.036
500	-3.677	-3.375	-3.119	-2.829	-1.882	-0.991	-0.703	-0.434	-0.113
1000	-3.698	-3.398	-3.135	-2.849	-1.901	-1.010	-0.719	-0.463	-0.133

3.2.2 Finite Sample Size and Power Analysis

Table 3.6: Size and power analysis under $\gamma_1^0 = 0$

		T=100				T=200			
γ_1^0	ρ	DF^τ	$DF^\tau_{1,r}$	$DF^\tau_{2,r}$	$DF^\tau_{3,r}$	DF^τ	$DF^\tau_{1,r}$	$DF^\tau_{2,r}$	$DF^\tau_{3,r}$
0	1	.049	.050	.049	.050	.051	.050	.051	.050
	0.95	.087	.109	.097	.105	.195	.239	.247	.268
	0.90	.197	.249	.237	.267	.640	.540	.731	.774
	0.80	.657	.572	.734	.783	.999	.817	.999	1.00

From Table 3.5 we can observe that the loss of efficiency is translated into a loss of power of the $DF^\tau_{1,r}$ statistic. Note that this procedure performs well when $\rho \geq 0.95$, however, it's performance deteriorates considerably when $\rho < 0.95$. On the other hand, $DF^\tau_{2,r}$ and particularly $DF^\tau_{3,r}$ have, as expected when compared with other results available in the literature, a good power performance; see So and Shin (2001) and Taylor (2002). The results show that the invariant transformation suggested by Taylor (2002), which sacrificed some of the efficiency gains, produces tests which have the best power performance (corresponding to $DF^\tau_{3,r}$).

4 Conclusion

In this paper we show how OLS-based recursive trend adjustment can pose some problems. Moreover, invariant versions of recursively-detrended unit root tests are examined and shown to provide a useful advantage in improving the power of unit root tests. Critical values for the invariant procedures are presented as well as finite sample performance of size and power which show the superior behaviour of these tests.

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Appendix A

Throughout this Appendix we consider that data is generated from (2.1)-(2.2) under the null hypothesis that $\alpha = 1$.

A.1 Non-Invariance of OLS-based Recursive Detrending

Assuming $\gamma_0^0 \neq 0$ and $\gamma_1^0 \neq 0$, (2.1) is,

$$y_t = \gamma_0^0 + \gamma_1^0 t + x_t \tag{A.1}$$

or put more succinctly,

$$y_t = Z_t^{0\prime}\gamma^0 + x_t, \tag{A.2}$$

where $Z_t^0 = (1,t)'$ and $\gamma^0 = (\gamma_0^0, \gamma_1^0)'$.

Moreover, under the null hypothesis, $\alpha = 1$, consider $z_t^0 = (0,t)'$ so that the DGP can be written as,

$$y_t = z_t^{0\prime}\gamma^0 + y_0 + \sum_{j=1}^{t} \varepsilon_j. \tag{A.3}$$

Now, considering $Z_t = (1,t)'$ as the vector of deterministic variables which enters the test regression, we can write the OLS recursively detrended variable based on (2.4) as,

$$\widetilde{y}_t = y_t - Z_{t-1}'\widetilde{\gamma}_{t-1}^r \tag{A.4}$$

$$\widetilde{y}_{t-1} = y_{t-1} - Z_{t-1}'\widetilde{\gamma}_{t-1}^r \tag{A.5}$$

where the elements of the $T \times 2$ vector of recursive estimators of γ^0 are computed as

$$\widetilde{\gamma}_t^r = \left(\sum_{k=1}^{t} Z_k Z_k'\right)^{-1} \sum_{k=1}^{t} Z_k' y_k. \tag{A.6}$$

Hence, since $Z_k = (1,k)'$, assuming $\gamma_1^0 \neq 0$, it follows that

$$\begin{aligned}
\widetilde{\gamma}_{t-1}^r &= \left(\sum_{k=1}^{t-1} Z_k Z_k'\right)^{-1} \sum_{k=1}^{t-1} Z_k y_k \\
&= \begin{pmatrix} \frac{4t-2}{(t-1)(t-2)} & -\frac{6}{(t-1)(t-2)} \\ -\frac{6}{(t-1)(t-2)} & \frac{12}{t(t-1)(t-2)} \end{pmatrix} \begin{matrix} \sum_{k=1}^{t-1} y_k \\ \sum_{k=1}^{t-1} k y_k \end{matrix} \\
&= \begin{pmatrix} \frac{4t-2}{(t-1)(t-2)} \sum_{k=1}^{t-1} y_k - \frac{6}{(t-1)(t-2)} \sum_{k=1}^{t-1} k y_k \\ -\frac{6}{(t-1)(t-2)} \sum_{k=1}^{t-1} y_k + \frac{12}{t(t-1)(t-2)} \sum_{k=1}^{t-1} k y_k \end{pmatrix}
\end{aligned} \tag{A.7}$$

and therefore

$$
\begin{aligned}
\widetilde{y}_t^{\tau} &= y_t - Z_{t-1}'\widetilde{\gamma}_{t-1}^{r} \\
&= S_t + z_t^{0\prime}\gamma^0 + y_0 - Z_{t-1}'\widetilde{\gamma}_{t-1}^{r} \\
&= S_t + \gamma_1^0 t + y_0 + 2\overline{y}_{t-1} - \frac{6}{t(t-1)}\sum_{k=1}^{t-1} k y_k \\
&= S_t + \frac{2}{t-1}\sum_{k=1}^{t-1} S_k + \frac{6}{t(t-1)}\sum_{k=1}^{t-1} k S_k + \gamma_1^0. \qquad \text{(A.8)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\overline{y}_{t-1} &= \frac{1}{t-1}\sum_{k=1}^{t-1} y_k \\
&= \frac{1}{t-1}\sum_{k=1}^{t-1}\left(z_k^{0\prime}\gamma^0 + y_0 + \sum_{j=1}^{k}\varepsilon_j\right) \\
&= \frac{1}{t-1}\sum_{k=1}^{t-1}\left(k\gamma_1^0 + y_0 + S_k\right) \\
&= \gamma_1^0\frac{t}{2} + y_0 + \frac{1}{t-1}\sum_{k=1}^{t-1} S_k \qquad \text{(A.9)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=1}^{t-1} k y_k &= \sum_{k=1}^{t-1} k\left(z_k^{0\prime}\gamma^0 + y_0 + \sum_{j=1}^{k}\varepsilon_j\right) \\
&= \sum_{k=1}^{t-1} k\left(k\gamma_1^0 + y_0 + S_k\right) \\
&= \gamma_1^0\frac{(t-1)\,t(2t-1)}{6} + y_0\frac{(t-1)\,t}{2} + \sum_{k=1}^{t-1} k S_k \qquad \text{(A.10)}
\end{aligned}
$$

As can be observed from (A.8), when $\gamma_1^0 \neq 0$, $\widetilde{y}_t^{\tau}$ is a function γ_1^0.

A.2 Invariant Tests

Consider the transformations used in Chang and Park (2004) and Chang (2002) in a panel data context, *i.e*, *Transformation 1,*

$$\widetilde{y}_{1,t}^{\tau} = y_t - y_0 - \frac{1}{T}\sum_{t=1}^{T}\Delta y_t - \sum_{k=1}^{t-1}\frac{1}{k}\left(y_k - y_0\right).$$

Considering again the DGP under the null with $\gamma_1^0 \neq 0$ as,

$$y_t = z_t^{0\prime}\gamma^0 + y_0 + S_t, \tag{A.11}$$

we observe that,

$$\begin{aligned}\sum_{k=1}^{t}\frac{1}{k}(y_k - y_0) &= \sum_{k=1}^{t}\frac{1}{k}\left(z_t^{0\prime}\gamma^0 + y_0 + S_k - y_0\right)\\ &= \gamma_1^0 t + \sum_{k=1}^{t}\frac{1}{k}S_k.\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{y}_{1,t}^{\tau} &= y_t - y_0 - \frac{1}{T}\sum_{t=1}^{T}\Delta y_t - \sum_{k=1}^{t-1}\frac{1}{k}(y_k - y_0)\\ &= \gamma_1^0 t + S_k - \gamma_1^0 - \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t - \gamma_1^0(t-1) - \sum_{k=1}^{t-1}\frac{1}{k}S_k\\ &= S_k - \sum_{k=1}^{t-1}\frac{1}{k}S_k\end{aligned}$$

and the result for $\widetilde{y}_{1,t-1}^{\tau}$ follows a similar approach.

In terms of *Transformation 2,*

$$\widetilde{y}_{2,t}^{\tau} = y_t + \frac{2}{t-1}\sum_{k=1}^{t-1}y_k - \frac{1}{T}\sum_{t=1}^{T}\Delta y_t - \frac{6}{t(t-1)}\sum_{k=1}^{t-1}ky_k$$

we observe from (A.9) and (A.10) that,

$$\begin{aligned}\widetilde{y}_{2,t}^{\tau} &= y_t + \gamma_1 t + 2y_0 + 2\overline{S}_{t-1} - \gamma_1^0 - \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t - \gamma_1(2t-1) - 3y_0 - \frac{6}{t(t-1)}\sum_{k=1}^{t-1}kS_k\\ &= S_t + \gamma_1(2t-1) + 3y_0 + 2\overline{S}_{t-1} - \gamma_1(2t-1) - 3y_0 - \frac{6}{t(t-1)}\sum_{k=1}^{t-1}kS_k\\ &= S_t + 2\overline{S}_{t-1} - \frac{6}{t(t-1)}\sum_{k=1}^{t-1}kS_k\end{aligned}$$

where $\overline{S}_{t-1} = \frac{1}{t-1}\sum_{k=1}^{t-1}S_k$. The results for $\widetilde{y}_{2,t-1}^{\tau}$ and for the transformation proposed by Taylor (2002) follow a similar approach.