**EUROPEAN UNIVERSITY INSTITUTE**
DEPARTMENT OF ECONOMICS

EUI Working Paper **ECO** No. 2004 / 34

# A Consumption and Debt Dynamics with (Rarely Binding) Borrowing Constraints

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Published in Italy in December 2004
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**Consumption and Debt Dynamics with (Rarely Binding) Borrowing Constraints***

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7 November 2004

Abstract. This paper examines consumption and savings dynamics in a standard model of incomplete markets. Existence of equilibrium requires the imposition of exogenous debt limits but these are often ignored because of the computational difficulties that arise in models with occasionally binding constraints. I claim that borrowing constraints have a significant qualitative and quantitative effect on equilibrium allocations even if they rarely bind. Contrary to standard results in the literature, debt exhibits mean reversion, consumption responds strongly to idiosyncratic income shocks and interest rates respond to both aggregate *and* idiosyncratic innovations in income. The implication is that market incompleteness can generate much lower consumption correlations than was previously thought.

JEL classification: C63, E21, E32

Keywords: Incomplete markets, consumption/savings dynamics, non-linear dynamic methods.

*I have benefited from suggestions and comments by Wouter DenHaan, Albert Marcet, Morten Ravn, Tom Sargent and Andrew Scott. All errors and omissions are mine.

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‡I wish to acknowledge financial support from the European Commission, MAPMU network under contract HPRN-CT-2002-00237.

## 1. Introduction

In the literature on incomplete markets it is often claimed (see for example Baxter (1995)) that the optimal debt policy implies debt dynamics that exhibit a unit root. This paper challenges this view and shows that debt, and consequently consumption, actually follow stationary stochastic processes. The difference in the predictions here arise from a consistent treatment of borrowing constraints. These are found to have a significant qualitative and quantitative effect on consumption and debt policy functions even when they rarely bind in equilibrium. It is shown that the unit root behaviour, described in other studies, arises from the local linear approximation methods used for computation of equilibrium and disappears when one uses a non-linear numerical method. This could have important implications concerning standard findings on international risk sharing. Whereas it is usually found that international consumption correlations are not significantly reduced by the assumption of market incompleteness, I show that those correlations are overestimated by the linear approximation methods.

Stochastic dynamic general equilibrium models characterize equilibria through a collection of non-linear stochastic difference (or differential) equations that are necessary for equilibrium to be obtained. Most often, that system of difference equations for optimal policy functions presents a non-trivial problem that does not admit an analytical solution and these models are therefore commonly analyzed using numerical solution methods. A common method consists in obtaining a local linear approximation of the system in a neighbourhood of the deterministic steady state, a procedure described, for example, in King, Plosser and Rebelo (1988). Equilibrium laws of motion for all variables can then be computed using the methods presented by Blanchard and Kahn (1980), Uhlig (1999) or Christiano (2001) amongst others. In a special issue of the Journal of Business Economics and Statistics in 1990, a number of researchers combined in comparing a variety of numerical methods and their accuracy using the stochastic growth model. Christiano (1990) and McGrattan (1990), in particular, provided linear methods and found that the accuracy was acceptable. Being local approximations, these methods rely on the assumption that the economy under uncertainty fluctuates in a region around the corresponding deterministic economy. In one sense, finding accuracy to be acceptable validates the above assumption, in the context of the particular model used.

The same linear methods have also been used by Baxter (1995), Baxter and Crucini (1995) and Kollman (1996) amongst others, to study international business cycles with incomplete markets. The model considered in these studies consists of two countries maximizing their welfare in the face of idiosyncratic as well as aggregate uncertainty but with limited ability to insure against idiosyncratic risk through financial markets. It is found that these ingredients imply unit root dynamics in bonds when a local linear approximation is used. This has strong implications for the accuracy of the local linear approximation. First, the assumption that the economy fluctuates around the deterministic steady state is no longer valid since, even if the economy begins at this point, it will eventually drift away  Second, unit root dynamics are incompatible with debt limits that are necessarily imposed for the existence of equilibrium.

In the interest of clarity of exposition, I abstract from endogenous production and assume income is exogenous. The model I use is, thus, similar to those used by Telmer (1993) and Marcet and Singleton (1999) to study asset prices under incomplete markets.[1] First, I show that using a linear approximation leads to a unit root in the equilibrium law of motion for bonds even after the simplification of the production side.[2] Subsequently, I compute equilibrium policy functions using a non-linear Euler based method which approximates the desired functions over the entire state space (i.e. it is not a *local* approximation method). The method is a version of the Parameterized Expectations Approach (PEA) discussed in Den-Haan and Marcet (1990), specifically adapted to deal with occasionally binding constraints. I find that borrowing constraints have a significant impact on equilibrium policy functions even when debt is zero. Indebted countries find it optimal to repay some of their debt even in relatively bad times. This is because they anticipate the possibility of future binding constraints and attempt to avoid such a situation. This is markedly different from the prediction of the linear equilibrium, namely that bad times always imply an increase in debt because of consumption smoothing. Bond price behaviour is also different from the linear case. Whereas under linearization bond prices only respond to aggregate income changes, in the non-linear equilibrium the level of debt also affects the bond price. The reason has to do with the concavity of the consumption function (see Caroll and Kimball (1996)) and has been eloquently explained in Den Haan (1996). However the effect on prices is quantitatively small, which is consistent with the finding of Telmer (1993).

I also investigate the size of the difference in the policy functions between the two methods. This difference is found to be increasing in the variance of the income process and in the patience parameter and decreasing in income persistence. Even though the dependence on the parameters provides some insights, the absolute size of the difference is difficult to interpret economically. For this reason, I proceed to compute cross-country correlations of consumption based on simulations using the two policy functions. This experiment yields a striking result. Consumption correlations are significantly lower in the non-linear case, an observation that provides some explanation for the difficulty in obtaining lower (empirically relevant) consumption correlations when market incompleteness is introduced. It would be interesting to repeat the experiment in a calibrated full business cycles model to see whether incomplete markets can actually provide a resolution of the consumption correlation puzzle.

In section 2, I briefly present the model considered and define its equilibrium. In section 3, I provide a linear approximation and discuss the problems associated with it. In section 4, I present an alternative method that overcomes those prob-

[1] Telmer has a different exogenous income process and Marcet and Singleton also introduce equities as well as risk free bonds. Many other researchers have used variations of this model, including Heaton and Lucas (1996) and Den Haan (1996).

[2] Indeed, it is straightforward to show that this result carries over to many incomplete markets settings including models with two goods, models with equities as well as bonds as in Marcet and Singleton (1999) but also Ramsey models like the ones analyzed in Marcet and Scott (2001).

lems and in section 5, I analyse the differences in predictions arising from the two solution methods. Finally, section 6 concludes.

## 2. THE MODEL

The model analysed here has been presented before in several studies, notably Telmer (1993). It is a two country general equilibrium model. Markets are complete within each country, so that we can assume a representative agent in each country whose aim is to maximize the expected sum of future discounted utilities of consumption $c_{it}$

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}) \tag{1}$$

where $\beta$ is the rate of time preference. The utility function $u(.)$ is assumed to be increasing and concave, $u'(c) > 0$ and $u''(c) < 0$.[3] The agent in country $i$ receives an exogenous stochastic endowment $y_{it}$ in every period $t$. I assume that the exogenous random processes $y_{it}$, $i = 1, 2$, form a bivariate process that is autoregressive of order 1

$$\begin{pmatrix} y_{1t} - \bar{y} \\ y_{2t} - \bar{y} \end{pmatrix} = A \begin{pmatrix} y_{1,t-1} - \bar{y} \\ y_{2,t-1} - \bar{y} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \tag{2}$$

where $\bar{y}$ is mean income, $A$ is a $2 \times 2$ matrix and $\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$ is a bivariate normal random variable with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and covariance matrix $\sigma_e^2 \begin{pmatrix} 1 & \psi \\ \psi & 1 \end{pmatrix}$. Asset markets are restricted by assuming that countries can only trade one-period risk free bonds $b_{it}$ which are bought at price $p_t$ and promise one unit of consumption next period. We can now write the budget constraint of each country as

$$\begin{aligned} c_{it} + p_t b_{it} &= y_{it} + b_{i,t-1} \\ b_{i,-1} &= b^* \text{ given} \end{aligned} \tag{3}$$

It is customary in the literature to impose an additional restriction on the countries' budget sets, namely that debt cannot exceed a certain limit $K$

$$b_{it} \leq -K \tag{4}$$

The aim of this restriction is to exclude the possibility of a Ponzi scheme being followed by an agent.[4] Ideally, if one wanted to avoid introducing additional

[3] The Inada conditions are also assumed to hold

$$\begin{aligned} \lim_{c \to 0} u'(c) &= \infty \\ \lim_{c \to \infty} u'(c) &= 0 \end{aligned}$$

[4] This is a condition that is substantially different from the transversality condition. Eschewing the issues related to the ambiguity of the transversality condition under incomplete markets, it should be pointed out that the transversality condition is a necessary condition for optimization whereas the no Ponzi scheme condition is imposed a priori to make the problem economically interesting. One can construct examples where Ponzi schemes do not violate the transversality condition, so that the former is a stricter condition than the latter.

frictions to the model on top of the missing asset markets, one would want to make sure these limits never bind in equilibrium. Anticipating the results of the next section, it should be noted here that this is impossible under a linearized solution. It is also important to clarify that, even if limits never bind in equilibrium, they may affect the properties of the equilibrium. Under rational expectations, agents anticipate that there is a point above which their debt cannot grow any further and take this into account in their decisions, more so the closer they are to the limit. Indeed that is the whole purpose of the no Ponzi scheme condition - otherwise it would be redundant.[5]

An equilibrium for this economy is a set of prices $\boldsymbol{p} = \{p_t\}_{t=0}^{\infty}$ together with consumption plans $\boldsymbol{c} = \{\boldsymbol{c}_i\}_{i=1}^{2} = \{\{c_{it}\}_{t=0}^{\infty}\}_{i=1}^{2}$ and debt plans $\boldsymbol{b} = \{\boldsymbol{b}_i\}_{i=1}^{2} = \{\{b_{it}\}_{t=0}^{\infty}\}_{i=1}^{2}$ such that

i) $\boldsymbol{c}_1$ and $\boldsymbol{b}_1$ maximize (1) subject to (2)-(4) given prices $\boldsymbol{p}$.

ii) $\boldsymbol{c}_2$ and $\boldsymbol{b}_2$ maximize (1) subject to (2)-(4) given prices $\boldsymbol{p}$.

iii) Prices $\boldsymbol{p}$ adjust to clear the bond market, $b_{1t} + b_{2t} = 0$, for all $t$.

iv) The good market also clears (due to Walras' Law), $c_{1t} + c_{2t} = y_{1t} + y_{2t}$, for all $t$.

Existence of such an equilibrium is proved in Magill and Quinzii (1994) and in Levine and Zame (1996). Necessary conditions for equilibrium are given by

$$p_t u_c(c_{it}) \geq \beta E_t u_c(c_{i,t+1}) \qquad i = 1, 2 \tag{5}$$

$$\gamma_{it}(b_{it} + K) = 0 \qquad i = 1, 2 \tag{6}$$

together with budget constraints (3) and the bond market clearing condition. Goods market clearing will be satisfied automatically by Walras' law. Equation 5 is a combination of the first order conditions for $c_{it}$ and $b_{it}$. It states that the marginal benefit from increasing consumption at $t$ by increasing debt must be greater or equal to the expected marginal loss at $t+1$ arising from the additional debt. Equation 6 is the complementary slackness condition with $\gamma_{it}$ being the multipliers on the inequality constraints. The Euler equation for agent $i$ holds as an equality whenever the debt limit does not bind, i.e. when $\gamma_{it} = 0$. If the debt limit binds in period $t$, the marginal benefit from increasing debt at $t$ will be larger than the expected marginal loss but the agent is debt constrained and cannot take advantage of this benefit.

## 3. SOLUTION BY LOG-LINEARIZATION

In this section I provide a solution that is based on the log-linearization method described in King, Plosser and Rebelo (1988). The method consists in transforming the model to one that is linear in relative deviations from steady state. The linear model is then solved exactly, using the method of Blanchard and Kahn (1980) for linear stochastic difference equations. Alternatively, one could interpret this as a local approximation of the non-linear model around the steady state. The approximation is (almost) exact at the steady state and, in theory, deteriorates

[5]For a thorough analysis of debt limits and transversality conditions under complete markets see Ljungvist and Sargent(2003). Levine and Zame (1996) and Magill and Quinzii(1994) discuss those in the case of market incompleteness.

as the economy moves away from that point.[6] The quality of the approximation depends on the interaction between the degree of non-linearity of the true policy function and the width of the stationary distribution in the stochastic model. If it happens that the true policy function of the non-linear model is actually linear, then the approximation is perfect regardless of the width of the stationary distribution. Conversely, if the width of the stationary distribution is almost zero, then the approximation is almost perfect regardless of the degree of non-linearity. The approximation will become inaccurate when there is a combination of significant non-linearity with significant variation in the state variables.

The first step in the procedure consists in finding the steady state of the deterministic version of the economy. Obviously, in a setting where no uncertainty is present and incomes are constant there is no role for asset trade. Therefore, steady state bond holdings equal initial bond holdings. I assume that debt inherited in the initial period is zero, $b^* = 0$. This maintains the ex ante symmetry of the two countries, which significantly reduces the computations in the non-linear algorithm employed in section 4. Once steady state bond holdings have been pinned down, the budget constraints give steady state consumption $c^{ss} = \bar{y}$ and the Euler equation gives steady state prices $p^{ss} = \beta$.

We are now in a position to linearize the equilibrium conditions. First, I use the bond market clearing condition to substitute for $b_{2t}$. Once $b_{1t}$ is known we can recover $b_{2t}$, thus we can summarize the state of the economy at each point in time by $(b_{1,t-1}, y_{1t}, y_{2t})$.[7] I take logarithms on both sides of each equilibrium condition and then obtain a first order Taylor approximation around the non-stochastic steady state. Hats on variables denote relative deviations from steady state so that, for any variable $x$, $\hat{x} = \frac{x-x^{ss}}{x^{ss}}$. For bond holdings I define $\hat{b} = \frac{b}{y^{ss}}$ since $b^{ss} = 0$. The resulting system of stochastic linear difference equations is

[6] The qualifier 'almost' refers to the fact that the non-stochastic steady state is actually slightly different than the centre of the stationary distribution in the stochastic model. This is due to Jensen' inequality which states that

$$f(E(x)) > E(f(x))$$

for a strictly concave function $f$. In the log linearisation method, one repeatedly assumes the above holds as equality, an assumption that yields an average interest rate equal to the time preference rate $\frac{1-\beta}{\beta}$, the non-stochastic steady state. Under uncertainty, equilibrium interest rates are necessarily less than $\frac{1-\beta}{\beta}$ (see Aiyagari (1994)). In most cases, the magnitude of this error is small.

[7] Compared to the complete markets version of this economy, the state vector now includes an additional variable, namely $b_{1,t-1}$. In the language of Ljungvist and Sargent (2003), the incomplete markets model is history dependent. This history dependence can be conveniently summarised in the inherited asset position $b_{1,t-1}$, so that recursivity of the equilibrium is restored.

given by

$$
\begin{aligned}
\hat{p}_t + \xi \hat{c}_{1t} &= \xi E_t \hat{c}_{1,t+1} \\
\hat{p}_t + \xi \hat{c}_{2t} &= \xi E_t \hat{c}_{2,t+1} \\
\hat{c}_{1t} + \beta \hat{b}_{1t} - \hat{b}_{1,t-1} &= \hat{y}_{1,t} \\
\hat{c}_{2t} - \beta \hat{b}_{1t} + \hat{b}_{1,t-1} &= \hat{y}_{2,t} \\
\begin{pmatrix} \hat{y}_{1t} \\ \hat{y}_{2t} \end{pmatrix} &= A \begin{pmatrix} \hat{y}_{1,t-1} \\ \hat{y}_{2,t-1} \end{pmatrix} + \begin{pmatrix} \hat{\varepsilon}_{1t} \\ \hat{\varepsilon}_{2t} \end{pmatrix}
\end{aligned}
$$

where $\xi = \frac{u_{cc}(c^{ss})c^{ss}}{u_c(c^{ss})}$ is the coefficient of relative risk aversion that corresponds to the choice of utility function $u(c)$ and $\hat{\varepsilon}_{it} = \frac{\varepsilon_{it}}{y^{ss}}$. After simple algebraic manipulations this reduces to

$$
\begin{aligned}
E_t \hat{b}_{1,t+1} &= \frac{\beta+1}{\beta} \hat{b}_{1,t} - \frac{1}{\beta} \hat{b}_{1,t-1} - \frac{E_t \hat{y}_{2,t+1} - \hat{y}_{2,t}}{2\beta} + \frac{E_t \hat{y}_{1,t+1} - \hat{y}_{1,t}}{2\beta} \\
\hat{p}_t &= \frac{\xi}{2} (\hat{y}_{1,t} - E_t \hat{y}_{1,t+1} + \hat{y}_{2,t} - E_t \hat{y}_{2,t+1}) \\
\hat{c}_{1t} &= \hat{y}_{1,t} - \beta \hat{b}_{1t} + \hat{b}_{1,t-1} \\
\hat{c}_{2t} &= \hat{y}_{2,t} + \beta \hat{b}_{1t} - \hat{b}_{1,t-1}
\end{aligned}
$$

so that we need to solve a second order difference equation in $\hat{b}_{1t}$ and then implied consumptions can be read from the linearized budget constraints. Note that the bond price depends only on the exogenous endowments and, in particular, is independent of the level of debt. This provides some intuition for the result of Schmitt-Grohe and Uribe (2003) who obtain a stationary law of motion in a partial equilibrium version of this model, by introducing debt-elastic interest rates. The difference equation for $\hat{b}_{1,t}$ has an associated homogeneous equation that has roots 1 and $\frac{1}{\beta}$. The second root is explosive and is ruled out by the transversality condition. We are thus left with a unit root on the equilibrium law of motion for bonds. The policy functions that will be used for simulations are as follows[8]

$$
\begin{aligned}
\hat{b}_{1,t} &= \hat{b}_{1,t-1} + \frac{1-\rho}{2(1-\beta\rho)} (\hat{y}_{1,t} - \hat{y}_{2,t}) \\
\hat{p}_t &= \frac{\xi(1-\rho)}{2} (\hat{y}_{1,t} + \hat{y}_{2,t}) \\
\hat{c}_{1t} &= \hat{y}_{1,t} - \beta \hat{b}_{1t} + \hat{b}_{1,t-1} \\
\hat{c}_{2t} &= \hat{y}_{2,t} + \beta \hat{b}_{1t} - \hat{b}_{1,t-1}
\end{aligned}
$$

which show that borrowing is decreasing in relative income so that the agent with the higher (lower) relative income will be lending (borrowing). Bond prices are only dependent on aggregate income and in a positive manner. Finally, simple

[8] In these I have assumed that there are no spillovers in the exogenous endowment process and that the persistence of $y_{1t}$ and $y_{2t}$ is the same and equal to $\rho$. Thus $A = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$ in equation 2. Obviously, the unit root in $b_{1t}$ is not affected by the process for the exogenous shock.

substitution of the bond equation in the consumption equations shows that consumption is increasing in both home and foreign incomes and increasing in assets inherited from last period. The elasticity of consumption with respect to one's own income is higher than with respect to the other's income which implies that risk sharing is imperfect.

In the procedure followed above, the debt limit conditions were ignored since the method cannot deal with inequality constraints. This is not simply a technical detail, it is at the heart of the problems associated with a linear solution. One can only linearly approximate functions that are differentiable and the debt limits introduce non-differentiabilities in the problem. Thus the procedure can only make sense locally, that is as long as the limits do not bind. But the unit root in the bond law of motion implies that the limits will actually bind with strictly positive probability. This is a contradiction that will plague any linear solution. Furthermore, in ignoring the constraints in the derivation of the policy function, we are not allowing rational forward looking agents to realize their existence and thus we are, in some sense, forcing them to act myopically. In the following section I present a method that can handle this type of non-linearity.

## 4. Solution by non-linear PEA algorithm

The method employed in this section is a version of the stochastic parameterized expectations approach (PEA) that is discussed in DenHaan and Marcet (1990).[9] It is an Euler equation method since it focuses on approximating the Euler equations as opposed to using a Bellman equation. In particular, the strategy is to find an accurate approximation of the expectation function appearing in the first order conditions. The infinite dimensional problem of approximating a function is reduced to a finite dimension one, by assuming a particular parametric form for the function and optimizing on the choice of the parameters. I briefly explain the stochastic PEA method as it applies to the present model and give some intuition for why it is well suited to this case. Thereafter I discuss a couple of modifications that improve the efficiency and accuracy of the method.

In the present model, the relevant expectation function that has to be approximated is the conditional expectation at $t$ of next period's marginal utility of consumption. All information needed by an agent to make optimal decisions at any time $t$ can be summarized in that agent's state vector $s_{it}$, where $s_{1t} = (y_{1t}, y_{2t}, b_{1,t-1})^T$ and $s_{2t} = (y_{2t}, y_{1t}, b_{2,t-1})^T = (y_{2t}, y_{1t}, -b_{1,t-1})^T$. Thus the expectation conditional on information available at time $t$ is a function of $s_{it}$. The parametric form chosen is the family of exponentiated polynomial functions. This is because exponentiated polynomials span the space of positive valued functions, meaning that by increasing the order of the polynomial one can approximate arbitrarily well any function that takes strictly positive values. The conditional expectation of next period's marginal utility of consumption is such a function. Thus a first choice is to replace conditional expectations by exponentiated linear

[9]A similar method was used by Den Haan (1996). A more detailed description is given in an appendix.

functions

$$\begin{aligned}\phi_{1t} &= \phi(b_{1,t-1}, y_{1t}, y_{2t}; \delta) = \exp[\delta(1, y_{1t}, y_{2t}, b_{1,t-1})^T] \\ \phi_{2t} &= \phi(-b_{1,t-1}, y_{2t}, y_{1t}; \delta) = \exp[\delta(1, y_{2t}, y_{1t}, -b_{1,t-1})^T]\end{aligned}$$

where $\delta$ is a row vector of unknown parameters to be estimated. One can then check if higher order terms are orthogonal to the prediction error implied by the above choice. If some higher order term is not orthogonal, then it is added to the exponentiated polynomial and an additional parameter needs to be estimated. This method for the choice of the order of the approximating polynomial is discussed in DenHaan and Marcet (1994). Note that I have exploited the ex ante symmetry between the two agents in specifying the two functions $\phi_{1t}$ and $\phi_{2t}$ to be the symmetric. This simplifies the computations by reducing the number of parameters to be estimated by a factor of two.

The virtue of replacing expectations with the $\phi_{it}$'s lies in the fact that, for a given $\delta$, the $\phi_{it}$'s are known at time $t$. Thus it is possible to produce a series of data for the variables of interest, in particular for $u_c(c_{i,t+1})$, in the following fashion. First, produce a series of exogenous innovations $\{\varepsilon_{1t}, \varepsilon_{2t}\}_{t=1}^{M}$ and choose an initial vector of parameters $\delta_{init}$. Second, use the equilibrium conditions to produce a simulation of length $M$. The following strategy is used at every step of the simulation. Calculate variables ignoring the debt limits. If debt for either agent exceeds the limit, set it equal to the limit and recalculate consumptions and price using only the other agent's Euler equation and the budget constraints. In this way one produces a series $\{u_c(c_{1t}), u_c(c_{2t})\}_{t=1}^{M}$. Third, use this series as the dependent variable in a non linear least squares algorithm to compute the implied new vector of parameters, $\delta_{impl}$. Finally, set $\delta_{init} = \alpha\delta_{init} + (1-\alpha)\delta_{impl}$, $\alpha \in [0, 1)$, and start again. This process is repeated until $\delta_{init}$ and $\delta_{impl}$ are close enough. Convergence of this algorithm depends on the initial choice for the parameters. I choose as an initial value, the vector implied by the log-linear model of section 3 and this always gives convergence.[10]

At this point one has found a rational expectations equilibrium since the vector used to produce the simulation is equal to the one estimated ex post. Another way to state this is that the perceived expectations of the agents actually coincide with the statistical conditional expectations. Seen from this point of view, the PEA algorithm is essentially an adaptive learning algorithm like the ones presented in Marcet and Sargent (1989). That is a particularly important observation for our purposes for the following reason. The simulations used in the learning algorithm will often include situations where the debt limits bind. Thus agents will learn that there is the possibility of constraints binding and will gradually include this in their perceived expectations. When equilibrium is reached, agents make optimal decisions in a rational forward looking manner, incorporating in their considerations all possible future states. This feature was absent from the log-linear solution because debt limits were necessarily ignored in the derivation of optimal decision functions.

[10] This is far from being a detailed explanation of the PEA algorithm. For such an analysis see DenHaan and Marcet(1990) and Marcet and Lorenzoni(1999).

I introduce two modifications, based on the ideas discussed in Judd (1999), to the standard method described above. First, instead of using stochastic simulations to obtain the necessary data, I discretise the state space and choose the points at which I evaluate the conditional expectation according to the zeros of Chebyshev polynomials. Second, I evaluate the conditional expectation at each point using Hermite quadrature. Christiano and Fisher (2000) introduce similar modifications to the PEA method in a model with occasionally binding constraints. They find that the modified PEA outperforms conventional PEA in both accuracy and efficiency. A number of choices for algorithm parameters need to be made. The number of points used for the quadrature step is set equal to 3. Increasing this number to 9 makes a negligible difference in the expectation values computed. In discretising I have used both 11 and 21 points in each state variable implying a total of $11^3$ and $21^3$ points respectively. The improvement when the finer partition is used is insignificant, indicating that there is little scope in increasing the number of points considered. I present results obtained using the finer partition. In terms of the order of the polynomial approximating the expectation, I use a second order complete polynomial basis and add a linear spline described below[11].

One final modification is applied to the approximating polynomials. An additional term is introduced that consists of a linear spline. That term is the piecewise linear function of $b_{1,t-1}$

$$\begin{aligned} S(b_{1,t-1}) &= 0, & if\ b_{1,t-1} > -\lambda K \\ &= (b_{1,t-1} + \lambda K), & if\ b_{1,t-1} \leq -\lambda K \end{aligned}$$

where $\lambda$ is a parameter that specifies at what point the non-linearity kicks in. I use $\lambda = 0.8$ meaning that consumers are particularly unwilling to accumulate further debt once they have already a debt exceeding 80% of their limit.[12] The reason for this is as follows. Optimal asset holdings are increasing in relative income for most values of $b_{1,t-1}$. But for $b_{1,t-1} = -K$, a decrease in relative income leaves asset holdings unchanged. The policy function for assets $B(b_{1,t-1}, y_{1t}, y_{2t})$ would seem to have a kink at $b_{1,t-1} = -K$. One of the points stressed by this paper is that, under rational expectations, consumers become progressively less willing to keep accumulating debt, the closer they are to the limit. Therefore we expect the function $B(.)$ to be smoother close to the limit so that the limits bind less often compared to the linear approximation. Nevertheless, there will always be states where the limit will bind so the expectations will exhibit a high degree of non-linearity close to the bounds. That non-linearity is captured by the linear spline $S$.

## 5. NUMERICAL RESULTS

Having established that the log linear approximation method can potentially be highly inaccurate, I proceed in this section to analyse consumption and savings behaviour when equilibrium is computed using non-linear methods. First, I present consumption and bond policy functions and identify the differences between the

[11] Similar results were obtained when third order complete polynomials are used.

[12] I have experimented with alternative values for $\lambda$ and the results are very similar.

linear and non-linear equilibria. Subsequently, I look at the implied impulse response functions to show that behaviour can be markedly different even in a region close to the steady state. I then go on to construct a measure of the differences between the two solutions based on the policy functions and study the effect of the model parameters on that measure. Finally, I present consumption correlations implied by the two solution methods and show that these can be overestimated by the linear equilibrium.

**Parameter values.** I choose the following values for the benchmark parameters. The discount factor $\beta$ is 0.95, implying an average interest rate just above 5%. Utility is of the constant relative risk aversion (CRRA) form

$$\begin{aligned} u(c) &= \frac{c^{1-\gamma}-1}{1-\gamma} \quad \text{for } \gamma \neq 1 \\ &= \log(c) \qquad \text{for } \gamma = 1 \end{aligned}$$

and $\gamma = 1$. The exogenous income processes are independent ($\psi = 0$) with persistence $\rho = 0.9$. The innovations are normally distributed with mean 0 and standard deviation $\sigma_\varepsilon = 0.1$. I choose an exogenous debt limit $K = y^{ss}$ so that consumers are allowed to borrow up to 100% of their mean income. Aiyagari (1994) and Ljungvist and Sargent (2003), amongst others, discuss the natural debt limit. This is the limit up to which debt can be repaid almost surely, that is, even if income realizations were at their minimum from then on. These are the 'loosest' possible limits that are consistent with equilibrium. If limits are less strict than that, consumers are allowed to accumulate debt that is impossible to repay under some circumstances and equilibrium collapses due to the non-negativity of consumption. In the present setup, the natural debt limit is not well defined. First, income is normally distributed which means that, in theory, there is no lower bound. In practice, this is the lesser of the difficulties since, in the numerical method employed, a grid for income is defined and therefore a minimum income does exist and it is three standard deviations below mean income $\bar{y}$. Second, to calculate natural debt limits one computes the amount of saving an agent can accumulate in the future, given minimum income and assuming zero consumption. Given the utility specification here, zero consumption is not an option for any agent. Thus, a natural debt limit calculated in this way will overestimate the true ability to repay under the worst circumstances. Third and most important is the issue of bond prices. Natural debt limits are usually calculated in environments where interest rates are exogenous and fixed. Here interest rates, or equivalently bond prices, are endogenous and therefore not constant. Even worse, marginal rates of substitution can differ across agents when limits bind, so it is not obvious what is the correct price to be used. These issues are discussed in Levine and Zame (1996) and are beyond the scope of the present study. For illustrative purposes, I compute the maximum debt that can be repaid almost surely, assuming prices are fixed at their non-stochastic steady state, using the budget constraint and the transversality condition

$$\begin{aligned} b_{1,t-1} &= c_{1t} - y_{1t} + \beta b_{1t} \Rightarrow \\ b_{1,t-1} &= \sum_{s=0}^{\infty} \beta^s (c_{1,t+s} - w_{1,t+s}) \Rightarrow \\ b_{\min} &= -\frac{w_{\min}}{1-\beta} \end{aligned}$$

Under the benchmark parametrization this implies a debt limit of 13 times mean income. Noting that prices would be less than $\beta$ when income is less than its mean, one can expect the true natural debt limit to be stricter. Nevertheless, one can be confident that the debt limit imposed in the benchmark parametrization, $K = y^{ss}$, is stricter than the natural debt limit. I experiment with alternative values for the limit as well as the rest of the parameters at the end of this section.

**Policy Functions.** In this section I present optimal asset and consumption policies as functions of the inherited asset position[13]. The exogenous income shocks are fixed to particular values and then optimal allocations are plotted against the level of inherited assets. To be more precise, let $B(b, y_1, y_2)$ and $C(b, y_1, y_2)$ be the policy functions for bonds and consumption respectively. It is obviously unfeasible to provide full plots of these functions (that would be a plot in 4 dimensions). Therefore I choose to fix $y_1$ and $y_2$ to particular values and plot $B(.)$ and $C(.)$ as functions of the inherited level of bonds. There are a large number of such functions corresponding to all combinations of exogenous income shocks. I find it convenient to distinguish between purely aggregate and purely idiosyncratic shocks. The first corresponds to a situation where income shocks are exactly equal for the two agents, $y_1 = y_2 = \bar{y} + \varepsilon^{agg}$. The second is the situation where innovations to income are of equal magnitude but opposite sign, so that aggregate income remains unchanged. In this second case, $y_1 = \bar{y} + \varepsilon^{id}$, $y_2 = \bar{y} - \varepsilon^{id}$ so $y_1 + y_2 = 2\bar{y}$.[14]

In Figure (1), optimal responses for assets ($B(b, \bar{y} + \varepsilon^{agg}, \bar{y} + \varepsilon^{agg})$) and consumption ($C(b, \bar{y} + \varepsilon^{agg}, \bar{y} + \varepsilon^{agg})$) respectively are plotted for the case of purely aggregate shocks. Solid lines correspond to the true solution and dotted lines correspond to the log linear solution. For each solution method, I plot the response to minimum aggregate shocks ($\varepsilon^{agg} = -3\sigma_y$) and the response to maximum aggregate shocks ($\varepsilon^{agg} = 3\sigma_y$). All intermediate cases (not shown) lie between these two. One such intermediate case, where incomes are at their mean ($\varepsilon^{agg} = 0$), is also plotted. These three cases are labelled 'low', 'high' and 'mean' respectively.

When shocks are purely aggregate, that is when relative income does not change, no exchange of assets takes place under the log linear rule. The log linear asset policy function is thus a straight line of slope one going through 0. This is

[13] Only policy functions for agent 1 are presented. Due to symmetry, the corresponding ones for agent 2 are mirror images of those and add no further insights.

[14] The terminology used here might be slightly misleading. As noted by Den Haan (1996), there is no such thing as a purely idiosyncratic shock in an economy with two agents, since one agent comprises half the population. I use this terminology from here onwards for want of a better one, the meaning should be clear from the definitions given.

always true regardless of the level of the aggregate endowment. By contrast, in the non-linear solution the optimal asset holdings *do* depend on the level of aggregate income. As expected, no asset trade takes place if initial debt is 0, so that all bond policy functions in Figure (1) go through the origin $(0, 0)$. But when an agent is hit by a positive income shock while already being in debt, she has a tendency to repay part of her debt even if her relative income position has not changed. This is indicated by the line of smallest slope (marked 'high') that plots the case where income is at its highest for both agents. The opposite happens when income is at its lowest for both agents, as illustrated by the line of largest slope (marked 'low'). In that case, the agent actually increases her debt slightly to cover part of her interest payments. There is however an asymmetry between the two cases: the repayment when income is higher than average is greater than the increase in debt when income is below average. Indeed, the case where incomes are at their mean for both agents almost coincides with the one at lowest income. Therefore debt repayment takes place when income is at its mean and even in some cases (not shown) where income is lower than average. The conclusion is that debt is mean reverting on average and will exhibit unit root or even explosive behaviour only when income shocks are at their lowest levels consistently. Despite these important considerations that the log linear solution fails to capture, it appears that the magnitude of the difference is not very high. It is virtually nil close to steady state (origin). It is growing as we move further away from steady state but appears to be moderate even close to the limits. In the consumption policy functions of Figure (1), we see that these small discrepancies in asset allocations are somewhat exacerbated in consumption allocations. The consumption function in the non-linear solution always has a higher slope than under linearization. There are two reasons why this is so. First, the linearized equilibrium imposes certainty equivalence. The slope of consumption as a function of bonds is everywhere constant and equal to $1 - \beta$. The non-linear equilibrium, on the other hand, takes into account the effects of uncertainty. Zeldes (1989) and Kimball (1990) have shown that uncertainty implies a marginal propensity to consume that is everywhere higher than under certainty. In addition to uncertainty, there is also an effect from the borrowing constraints which increases the slope of consumption, especially close to the debt limits.

In Figure (2), I plot the bond price as a functions of bond holdings for agent 1. For clarity, I only show the case $\varepsilon^{agg} = 0$ so income of both agents is fixed at the mean. Under linearization, the bond price is independent of bond holdings and equal to the deterministic steady state value $p^{ss} = \beta = 0.95$. However, taking non-linearities in to account reveals that the price does depend on the level of borrowing. The higher the amount of bonds held (by either agent) the higher is the price. This can be understood by recalling the result of Caroll and Kimball (1996) that consumption is a concave function of wealth.[15] Given equality of exogenous incomes, when agent 1 is a lender ($b > 0$) she is wealthier than agent 2. Holding prices fixed, concavity implies that a marginal increase in $b$ will induce an increase in her consumption that is smaller than the decrease in consumption

[15] This is proved to hold under very general conditions which are satisfied here.

of agent 2. As a result, agent 1 will want to save more than agent 2 wants to borrow. In equilibrium, this excess supply of bonds will cause an increase in the bond price. Thus the price is increasing in $b$ for $b > 0$.[16] The opposite happens when $b < 0$. Close to the debt limits, the excess supply is exacerbated by the unwillingness of the borrower to increase his debt and the bond price jumps to much higher levels. At $b = 0$, agents are equally wealthy and marginal changes in $b$ have no effect on price. That is why linear approximation close to the steady state $b = 0$, yields a bond price that is independent of bond holdings. In addition to the slope, also the level of the price differs between solution methods. The price is everywhere higher in the non-linear equilibrium, an observation first made by Aiyagari (1994).

In Figure (3), policy functions for fixed purely idiosyncratic shocks are plotted ($B(b, \bar{y} + \varepsilon^{id}, \bar{y} - \varepsilon^{id})$ and $C(b, \bar{y} + \varepsilon^{id}, \bar{y} - \varepsilon^{id})$). Three cases are considered corresponding to $\varepsilon^{id} = -3\sigma_y$, $\varepsilon^{id} = 3\sigma_y$ and $\varepsilon^{id} = \sigma_y$. These three cases are labelled 'low', 'high' and 'moderate' respectively, labels referring to the income of agent 1 relative to the one of agent 2. In all three cases, $y_1 + y_2 = 2\bar{y}$ - aggregate income is constant and equal to its mean.

As above, it is apparent that the PEA solution yields mean reverting debt behaviour with all bond policy functions less steep than the linearized ones, which are always at slope equal to one. The distance of the linearized policy functions from the true ones is now more pronounced. Asset policies and, especially, consumption policies are now clearly different than the true ones. Consumption differences can be as high as 10% of mean consumption when relative income is at its highest (or lowest).[17] Even when debt is 0 and relative income shocks are moderate, linearization underestimates the response of consumption to income by approximately 2.5%. To put it differently, the linearization method predicts significantly different consumption allocations even in the neighbourhood of the steady state. Bond prices, as before, depend on initial bond holdings. These are plotted in Figure (4) only for the middle case, $y_1 = \bar{y} + \sigma_y$ and $y_2 = \bar{y} - \sigma_y$. Whether the price is increasing or decreasing in initial bond holdings depends on relative wealth. It is helpful to distinguish between financial wealth and human wealth, as defined in Zeldes (1989). Financial wealth refers to the amount of the financial asset held. Human wealth refers to the expected present value of future endowments. When $b < 0$, agent 2 is has higher financial wealth than agent 1. However, remember that in the current experiment we have fixed exogenous endowments so that human wealth is higher for agent 1 than agent 2. Therefore, the price function can be read as follows. When agent 1 is highly indebted, he is less wealthy than 2 and the price is decreasing. But if agent 1's debt is not very large (and when she is a lender), her large endowment renders her more wealthy and prices are increasing in $b$.

To summarize, a linear approximation obscures rich dynamics of the model close to the borrowing constraints. More disturbingly, it mistakes both the level

[16] The intuition for this result was first discussed in Den Haan (1996).

[17] In fact, at the extreme ends of the state space where the debt limits bind, consumption can be underestimated (or overestimated) by more than 30%.

and slope of the equilibrium policy functions even at the steady state. In a neighbourhood of the steady state, it was found that the slope of the bond policy function is lower and the slope of the consumption policy function is higher than the linear case. Bond prices were found to be higher in level and to have a clear dependence on bond holdings. At least with respect to consumption, the differences appear quantitatively significant. Although the shape of the policy functions is invariant to parameter changes, the distance from the linear equilibrium inevitably depends on the particular parameter choices made. The following section considers a range of alternative parametrization and analyzes the importance of the different parameters of the model in determining the size of that distance.

**Quantitative differences under alternative parametrizations.** To make the quantitative discrepancy resulting from linearization more concrete and comparable across parametrizations, I compute the following statistic. For every point in the state space, define $\Delta x = \left|\frac{x^{PEA} - x^{LIN}}{x^{PEA}}\right|$, where $x$ can be consumption or asset price. In the case of consumption for example, this is the percentage difference of the approximate linear consumption choice from the consumption choice under PEA. For assets I simply compute the difference $\Delta b = b^{PEA} - b^{LIN}$. In Table (1), I report the maximum of those values over the state space and also the mean over the state space.[18] These estimates are biased downwards (towards a small difference) for three reasons. First, I choose to ignore the extreme points of the state space where the limits bind. At those points, the difference can be huge but it seems unfair to take those into account since one would generally try to avoid periods of binding limits when using the linearization method. When these extreme points are taken into account, errors are some order of magnitudes larger. Second, the range of values for debt considered is limited by the debt constraints. In the log-linear solution, debt and consumption will eventually drift away from this range. Finally, one should keep in mind that these errors will accumulate over time in a simulation. Given these considerations, the values reported in Table (1) are surprisingly large.

The above statistics are calculated for a range of deviations in risk aversion, income standard deviation, income persistence, discount factor $\beta$ and debt limits from the benchmark parametrization. For income persistence, I follow Aiyagari(1994) in keeping overall income variance constant as income persistence is changed. For a higher persistence, I decrease the innovation variance enough so that the resulting income variance remains constant at its benchmark value. This is intended to separate the effect of persistence from the effect of income variance. Reading each row from left to right one observes the following dependencies. The difference is strictly increasing in the standard deviation of income and in the discount factor. Surprisingly, the difference is actually decreasing in

[18] It would seem more natural to weight each point in the state space by its unconditional probability of occurring. The probability density can be approximated by the empirical distribution in the stationary PEA solution but the same cannot be done for the log-linear solution since bonds have a unit root. One could use the density calculated from the PEA solution in both cases. However, this would severely underestimate the average error incurred when the log-linearisation method is used.

income persistence for a given income variance. The relationship between $\Delta c$ and risk aversion is not monotonic. In fact, this difference is higher for values of risk aversion both higher and lower than 1. This seems to indicate that there is a minimum at 1, implying that, with logarithmic utility, the model is closest to being linear in logs[19]. The value of the exogenous limit on debt also appears to have a non-monotonic effect on $\Delta c$. Importantly, as the limit is relaxed further above some value, the difference between approximations is increasing, which suggests that choosing very loose debt limits does not make the linear approximation better.

In terms of magnitude, the difference is especially sensitive to changes in the exogenous income process. When income standard deviation is doubled to 0.2, $\Delta c$ can be as high as 63%. In this case, even the mean of $\Delta c$ over the state space is quite high at more than 11% of consumption. On the other hand, when persistence is raised to 0.99, differences are significantly lower. Consumption is on average only about 1% away from the non-linear case. Across parametrizations, the pricing functional is closer to the one implied by certainty equivalence than the consumption and asset holdings functions. This is reversed when one considers points in the state space where the limits bind.

To summarize, there are important qualitative differences between a log-linear solution, which imposes certainty equivalence, and the non-linear solution that takes into account uncertainty and the effects of borrowing limits. These are quantitatively significant for a wide range of parametrizations. They are exacerbated when income is variable but not persistent and when agents are patient. In the following sections, I consider the effect of these differences on statistics widely studied in the international business cycle literature, namely impulse response functions and consumption correlations.

**Impulse Responses.** In Figure (5), I present IRFs (impulse response functions) for consumption, assets, asset price, net exports and interest payments for the two solutions. Innovations are set to zero for 500 periods so that all variables settle to their 'stochastic steady state'. The steady state values are equal for the two cases, except for bond prices. As explained in the previous sections, bond prices in the non-linear solution are slightly higher than the linear case where they are simply equal to $\beta$. This confirms the approximation error due to Jensen' inequality under the linear solution[20]. In period 501, agent 1 experiences a positive one standard deviation innovation in her income. After that innovations are set to 0 again for another 500 periods and the effect on income dies out exponentially. Innovations in agent 2's income are set to 0 throughout the experiment. I plot the difference of each variable from its 'stochastic steady state' from period 501 onwards.

On the left panel are the responses under linearization. A temporary but persistent income shock hits agent 1. She responds by increasing her consumption,

[19] This adds support to the findings in the previous sections where the benchmark parameterization ($\gamma = 1$) was used. It suggests that the deviations from certainty equivalence observed there would be even higher if a different value for $\gamma$ was assumed.

[20] The error is quantitatively small.

but by less than the increase in income. The remaining income is lent to agent 2. As the income innovation gradually dies out, agent 1 gradually reduces her consumption, always making sure that some part of the surplus income is channelled towards acquiring assets. When income has fully returned to its mean, agent 1 has accumulated a significant amount of assets from agent 2. As long as no other innovations are introduced, these assets remain fixed and give agent 1 an additional source of income through interest payments. Thus her wealth has been permanently increased and as a result her consumption also has permanently increased. The other side of the coin is that agent 2 now has a permanently smaller consumption than what she started out with. She is willing to accept this because overall her utility has also increased by virtue of her high consumption in the initial periods. That is, she is only willing to give up a portion of her consumption forever because she discounts the future and this loss is counterbalanced by increased consumption in the present which she values more than the future. The bond price is the only variable that does not inherit the unit root behaviour from assets. As one can see from the price equation under linearization, the price is a simple linear function of aggregate income. It therefore increases initially and then reverts to its mean exponentially mirroring the aggregate income behaviour.

The right panel of Figure (5) shows the responses in the non-linear solution. There is a striking difference in those responses. All variables show strongly mean reverting behaviour. It is important to stress here that the impulse response exercise is a local analysis. That is, limits never even come close to be binding.[21] Stationarity obtains because of the possibility of constraints binding in the distant future. Assets respond by slightly less than in the linear case and, more importantly, accumulate for a shorter time and then begin to fall until they return to 0. The reason for this is that agent 2 is not willing to maintain debt and endeavours to eventually pay it off. Maintaining a substantial amount of debt as in the linear case would mean that her probability of being constrained in the future has increased. she does take advantage of the temporary increase in aggregate income by borrowing and increasing her consumption. However, after some periods she lowers her consumption below steady state and, in particular, below income so that she can gradually repay her debt. Bond prices show very similar behaviour across solution methods, in the non-linear case the price increases by slightly less due to the fact that assets respond by less. Agent 1 raises her consumption initially by more and agent 2 by less. Thus, at least in the initial period after the innovation, risk sharing is overestimated by the linear approximation.[22]

The qualitative difference observed in the IRFs is present under any choice of parameters. On the other hand, whether the differences are quantitatively significant will depend on the parameterization. Looking at the linear laws of motion for assets and consumption leads to the following observation. A change in the standard deviation of innovations, $\sigma_\varepsilon$, does not affect the form of these functions. This is quite different from the PEA solution where the volatility of the

[21] Actually, assets never come close to even 80% of the limit, which means that the splines imposed in the equilibrium expectation do not come into play.

[22] Judging overall risk sharing is complicated by the permanence of the responses under linearisation.

exogenous innovations does matter for the estimated coefficients in the equilibrium laws of motion. The same observation can be made with respect to the value of risk aversion, $\gamma$, which only affects prices but has no impact on assets and consumptions under log-linearization. These observations lead one to the conclusion that those two parameters are crucial in evaluating how close the linear approximation can be to a global non linear one like the PEA. Changing those parameters will affect the IRF's under PEA but not the linear ones.

**The consumption correlation puzzle revisited.** As a final experiment, I compute consumption correlations implied by the two solution methods under alternative parametrizations. These are to be compared to the consumption correlation implied by a complete markets version of this model. Under market completeness, given the absence of leisure in the utility function, consumption correlation is exactly 1 regardless of the parametrization. Using a log-linearization method, Baxter (1995) finds that when exogenous shocks are permanent ($\rho = 1$), consumption correlations move closer to the complete markets case. Results in the previous sections indicate that the log-linear solution can be highly inaccurate when the variability of the exogenous forcing process is high. When $\rho = 1$ this variability is infinite, which casts some doubt on that result. On the other hand, the effect of persistence for fixed overall income variance, is to increase the approximation accuracy. In this section, I will follow Baxter (1995) in changing persistence $\rho$, while at the same time, allowing for the effects on variability.

Implied consumption correlations are computed as follows. For each solution, the economy is simulated for 100 periods. The HP filter is then applied to the series of consumptions and the correlation is computed. This is repeated 200 times and the average of the correlation over these 200 replications is computed and reported in Table (2).[23] Persistence is varied between 0 and 0.97.[24] The first column shows cross country consumption correlations under complete markets. These are all equal to 1 regardless of the parametrization of the shocks. The second and third columns show the same statistic under incomplete markets for the two solution methods, namely log-linearization and PEA. Income correlation is exogenous in this paper and equal to 0. All correlations reported in Table (2) are positive and therefore higher than income correlation. Simulations not reported here, show that changes in the other parameters also fail to produce negative consumption correlations. In this setting, the consumption correlation puzzle cannot be fully resolved by a simple change in parameter choices. However, Baxter (1995)'s result can be detected in Table (2). As income persistence is increased, consumption correlations fall significantly below 1. Remember that there are two effects of increased persistence. Income is more persistent *and* more variable. Both of these effects reduce consumption correlations. The log linear approximation shows similar tendencies but underestimates the magnitude of the effect. It would be interesting to explore whether the above conclusion carries over to a model with

[23] The estimated correlation shows small variation across replications and one can be confident in the accuracy of the estimate.

[24] It is not feasible to compute solutions for the case of persistence $\rho = 1$ since the methods used rely on the assumption of stationary shocks.

endogenous production.

## 6. Conclusion

In this paper I have analysed consumption/savings behaviour in a two-agent model with aggregate and idiosyncratic risk but with restricted financial markets and borrowing constraints. I employed numerical techniques that can deal with non-linearities and found that the model's implications are very different than what is predicted by commonly used linear approximation methods. In particular, it was found that debt exhibits mean reverting dynamics because agents prefer to avoid hitting their borrowing limits. Bond policy functions are thus found to be S-shaped as opposed to the linear shape imposed by the linear methods. Consumption inherits this property and takes an (inverted) S-shape. The implication is that, at least close to the limits, agents engage in bond smoothing as opposed to consumption smoothing. More importantly, consumption dynamics were found to be significantly different from those implied by certainty equivalence imposed by linearization even far from the limits. In addition to that, it was shown that important interest rate (bond price) dynamics arise when the certainty equivalence assumption is dropped. The size of these differences was calculated for a wide range of parameters and was found to be quite large for consumption but less so for interest rates. Risk aversion, patience and income variability were identified as parameters that are crucial for the magnitude of the differences. Importantly, it was shown that loosening the debt constraints does not improve the picture for the linear methods.

In the international business cycles literature, similar models have been used to assess the importance of missing asset markets in explaining the discrepancies between the empirical evidence and the predictions of a model with complete asset markets. Especially in this strand of the literature linear methods are typically used and the models are analysed by looking at impulse responses and second moments of allocations. It was shown here that impulse responses can yield very different predictions when a non-linear methodology is employed. This difference is both qualitatively and quantitatively significant. As a final experiment, the consumption correlation puzzle was revisited here with striking results. Consumption correlations under incomplete markets were found to be overestimated by the methods commonly used in the international business cycle literature. This suggests a (partial) resolution to the consumption correlations puzzle, simply arising from a more accurate approximation of the rational expectations equilibrium. More specific conclusions can only be drawn by repeating the current experiment for a full blown international business cycles model.

## APPENDIX: SOLUTION METHOD

This appendix describes in detail the computational algorithm used to obtain the rational expectations equilibrium. The equilibrium conditions that need to be satisfied at every possible state of the economy $s_t$ are

$$\begin{aligned} p(s_t)c_1(s_t)^{-\gamma} - \gamma_1(s_t) &= \beta E_t[c_1(s_{t+1})^{-\gamma}] \\ p(s_t)c_2(s_t)^{-\gamma} - \gamma_2(s_t) &= \beta E_t[c_2(s_{t+1})^{-\gamma}] \\ c_1(s_t) + p(s_t)b_1(s_t) &= y_1(s_t) + b_1(s_{t-1}) \\ c_2(s_t) - p(s_t)b_1(s_t) &= y_2(s_t) - b_1(s_{t-1}) \\ \gamma_1(s_t)(b_1(s_t) + K) &= 0 \\ \gamma_2(s_t)(-b_1(s_t) + K) &= 0 \end{aligned}$$

Note that the bond market clearing condition has already been used to substitute out $b_2(s_t) = -b_1(s_t)$. All variables are expressed as functions of the state vector $s_t$. There is a large literature discussing the existence of a recursive equilibrium in this model. Ljungvist and Sargent (2003) make clearly the point that the equilibrium is not recursive in the natural state variables $(y_{1t}, y_{2t})$. This is because, contrary to the complete markets case, there is wealth redistribution taking place. Therefore one needs to keep track of the distribution of wealth at every point in time in order to have all information needed for optimal decisions to be taken. As a consequence, the state vector is usually expanded to include information on the wealth distribution - this is true for all models of incomplete markets whether incompleteness is exogenous as here or endogenous (see Marcet and Marimon (1999)). Because the current model consists of only two agents, adding the variable $b_1$ to the state vector is enough to keep track of all wealth redistribution. Thus the state vector will be $s_t = (y_{1t}, y_{2t}, b_{1t-1})$. Note that exogenous income is trivially dependent on the state vector as $y_1(s_t) = y_{1t}$ and $y_2(s_t) = y_{2t}$.

The difficulty in computing rational expectations equilibria lies in the conditional expectations appearing in the Euler equations. To compute policy functions for consumption, bonds and prices one needs to know the conditional expectations. But to know the conditional expectations one needs knowledge of the consumption policy function. The PEA method breaks this circle by noticing that the conditional expectation at $t$ is itself a function of the state at $t$ and assuming a particular parametric form for this expectation. So let

$$\begin{aligned} \phi_1(s_t) &= \phi(b_{1,t-1}, y_{1t}, y_{2t}; \delta) = \exp[P_n(b_{1,t-1}, y_{1t}, y_{2t}; \delta)] \\ \phi_2(s_t) &= \phi(-b_{1,t-1}, y_{2t}, y_{1t}; \delta) = \exp[P_n(-b_{1,t-1}, y_{2t}, y_{1t}; \delta)] \end{aligned}$$

be such parametric functions that will replace the above expectations. Here $P_n(-b_{1,t-1}, y_{2t}, y_{1t}; \delta)$ is a polynomial of order $n$ in $b_{1,t-1}, y_{1t}$ and $y_{2t}$, with $\delta$ being a vector of the polynomial coefficients. Thus we assume that the expectations of future marginal utilities are exponentiated polynomials in the state vectors. This choice of parametric function is guided by the fact that marginal utility is always positive and by the fact that exponentiated polynomials span the space of positive valued functions. The ex ante symmetry between the two agents has been exploited by noticing that expectation of marginal utility will be the same parametric function with inverted arguments. So we only need to find one function and this will be achieved when we find an appropriate parameter vector $\delta$. This is achieved by taking an initial guess $\delta^{init}$ and using the equilibrium conditions to compute implied allocations, prices and expectations of next period's marginal utilities. At this point we have a new parameter vector $\delta^{impl}$. We then update our initial guess in the direction of $\delta^{impl}$ and repeat the procedure until a fixed point is found, i.e. until the initial guess is equal to the implied expectations. The PEA is only locally convergent which means that, unless one is careful in the choice of $\delta^{init}$, the iterations might not converge to a fixed point. In what follows, I describe the procedure for determining $\delta$ in detail.

The first step is to discretize the state space. There are three state variables so we build a three-dimensional grid with $N$ number of points in each direction of the grid. The width of the grid for bonds is dictated by the bond limit $K$, that is the grid for bonds will lie in $(-K, K)$. The support of the exogenous income processes is, in theory, infinite. Given that these shocks are normally distributed, we know that with a probability of 99% they will be less than 3 standard deviations away from mean. The grid for income is thus specified to be $(\bar{y} - 3\sigma_y, \bar{y} + 3\sigma_y)$, where $\bar{y}$ and $\sigma_y$ are the mean and standard deviation of $y$ respectively. The actual points within each grid are chosen according to the zeros of the Chebyshev polynomials. This choice implies that more points are chosen close to the boundaries of the grid, which improves the accuracy of the function approximation compared to an equidistant grid (see Christiano and Fisher (2000) and also in Judd (1999)). This is especially relevant in the current case because the function approximated is expected to be highly non-linear close to the debt limits $K$. To summarize up to now, we have obtained a collection of $N^3$ points in the state space $s_i = (b_{1j}, y_{1k}, y_{2r})$, $i \in \{(j, k, r) : j = 1, ...N,\ k = 1, ...N,\ r = 1, ...N\}$.

The second step consists in computing the conditional expectation of marginal utility at each point in the grid. Given a choice for $P_n$ and an initial choice $\delta^{init}$ for the parameter vector we can compute $\phi_1(s_i)$ and $\phi_2(s_i)$. Using those to replace the expectations in the equilibrium conditions, we compute allocations and price as follows:

1. Assume limits do not bind so that $\gamma_1(s_i) = \gamma_2(s_i) = 0$. Straightforward algebra gives

$$\begin{aligned}
c_1(s_i) &= \frac{\phi_2(s_i)^{\frac{1}{\gamma}}}{\phi_2(s_i)^{\frac{1}{\gamma}} + \phi_1(s_i)^{\frac{1}{\gamma}}}(y_{1k} + y_{2r}) \\
c_2(s_i)) &= \frac{\phi_1(s_i)^{\frac{1}{\gamma}}}{\phi_2(s_i)^{\frac{1}{\gamma}} + \phi_1(s_i)^{\frac{1}{\gamma}}}(y_{1k} + y_{2r}) \\
p(s_i) &= \beta\frac{\phi_1(s_i)}{c_1(s_i)^{-\gamma}} = \beta\frac{\phi_2(s_i)}{c_2(s_i)^{-\gamma}} \\
b_1(s_i) &= \frac{y_{1k} + b_{1j} - c_1(s_i)}{p(s_i)}
\end{aligned}$$

2. If $b_1(s_i) < -K$, then redo the calculations setting $b_1(s_i) = -K$, $\gamma_2(s_i) = 0$

but $\gamma_1(s_i) > 0$.

3. If $b_1(s_i) > K$, then redo the calculations setting $b_1(s_i) = K$, $\gamma_1(s_i) = 0$ but $\gamma_2(s_i) > 0$.

It is not possible to obtain closed form expressions in cases 2 and 3, so the non-linear equations have to be solved numerically. At this point, we have prices and allocations for this point in the state space $(s_i)$ but we still need to compute the conditional expectation of next period's marginal utility $\psi(s_i)$. Note that given $b_1(s_i)$, the expectation is only over the two exogenous shocks whose distribution is known to be normal. Thus the expectation can be computed, albeit numerically using Gauss-Hermite quadrature. This computation involves choosing a number $H$ of Hermite nodes (possible next period shocks) for $y_1$ and $y_2$. For each pair of nodes $(y_1(h), y_2(h))$, steps 1 to 3 are repeated and next period's marginal utility $u_c(h)$ is computed. The conditional expectation is then given by the Gauss-Hermite quadrature formula (see Judd (1999))

$$\psi(s_i) = \frac{1}{\pi} \sum_{h=1}^{H} w(h) u_c(h)$$

where $w(h)$ are weights that depend on the number of quadrature points $H$.

When this is done for all points $s_i$ in the grid, we have values for the conditional expectation of next period's marginal utility $\psi(s_i)$ for each $s_i$. This allows us to proceed to the third step in which we choose new values for $\delta$ $(\delta^{impl})$ so as to fit the $P_n$ function to these data points. Given that we have more data points than parameters the approximation method used is regression. In particular, we regress the logarithm of $\psi(s_i)$ on $s_i$, $i = 1, ..., N^3$ and obtain $\delta^{impl}$.

Finally, the parameter vector is updated in the direction of the newly estimated vector

$$\delta^{new} = \alpha\delta^{init} + (1-\alpha)\delta^{impl} \qquad \alpha \in [0,1)$$

and the procedure is repeated starting at $\delta^{init} = \delta^{new}$. The iterations stop when the absolute difference between all elements of $\delta^{impl}$ and $\delta^{init}$ is less than a tolerance level $\varepsilon$, i.e. when

$$\max \left| \delta^{impl} - \delta^{init} \right| < \varepsilon$$

A number of choices for the algorithm's parameters need to be made. Both $N = 11$ and $N = 21$ have been used and the increased accuracy given by a higher number of grid points ($N = 21$) seems to have a negligible effect on the equilibrium values for $\delta$ (and for consumption). Results reported are based on the finer grid $N = 21$. It is well known that Gauss-Hermite quadrature can perform extremely well even for a low number of nodes used ($x$ nodes accomplish *exact* integration of a polynomial of degree $2x - 1$). I use 3 points for $y_1$ and 3 for $y_2$ which implies a total number of quadrature points $H = 9$. The difference in the conditional expectations when 9 points for each shock are used ($H = 81$) is miniscule. The tolerance level is set to $\varepsilon = 10^{-6}$. I have experimented with polynomials of order $n = 1, 2, 3$. In all cases I add a linear spline to the polynomial $P_n$. That term is the piecewise linear function of $b_{1,t-1}$

$$\begin{aligned} S(b_{1j}) &= 0, \qquad\qquad if\ b_{1j} > -\lambda K \\ &= (b_{1j} + \lambda K),\ if\ b_{1j} \leq -\lambda K \end{aligned}$$

where $\lambda$ is a parameter that specifies at what point the non-linearity kicks in. I use $\lambda = 0.8$ meaning that consumers are particularly unwilling to accumulate further debt once they have already a debt exceeding 80% of their limit. This captures the high degree of non-linearity close to the constraints. Once this term is introduced, the expectation functions and the policy functions are almost linear away from the boundaries. This might suggest using $n = 1$ and the spline. However, I tested the orthogonality of higher order terms when this specification is used and it appears that second order terms need to be introduced. When $n = 2$ and the spline is also present, higher order terms seem to be orthogonal (based on simulations of 20000 periods). Thus, the results reported are for $n = 2$. The updating parameter $\alpha \in [0, 1)$ ensures that $\delta$ does not jump to much and remains in a convergent path. Obviously, for $\alpha$ very close to 1 the algorithm is more stable but takes longer. Given the fact that I have good initial conditions $\delta^{init}$, $a = 0.1$ is enough to ensure convergence without slowing down the algorithm. Good initial conditions $\delta^{init}$ are provided by the linearized solution.

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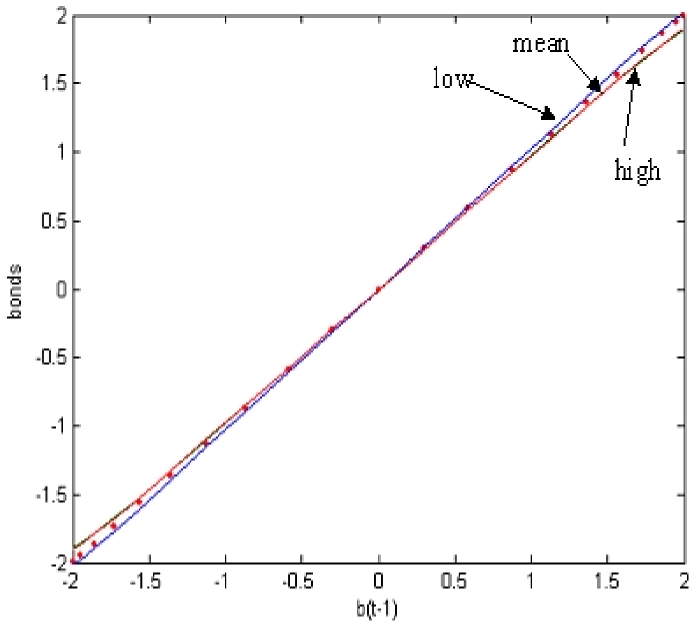


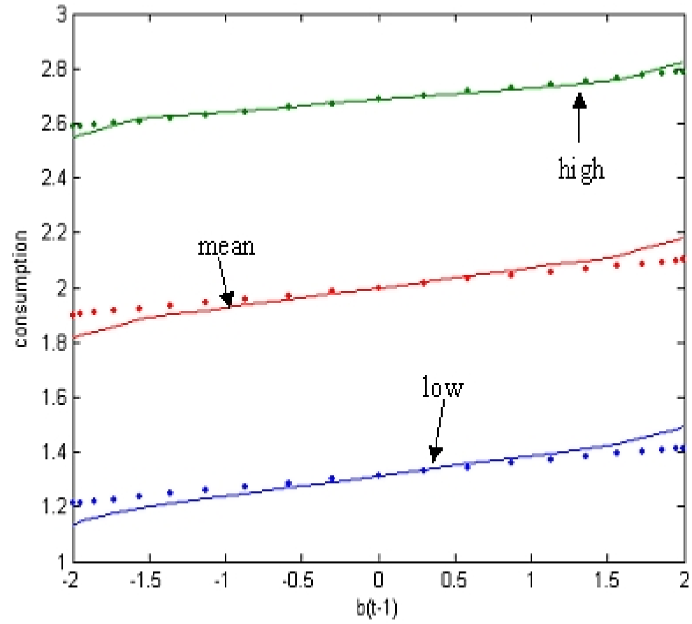


Figure 1: Policy functions for purely aggregate shocks. Dotted line is for linearised solution and solid line for PEA solution. Low is for $y_1 = y_2 = \bar{y} - 3\sigma_y$, high is for $y_1 = y_2 = \bar{y} + 3\sigma_y$ and mean is for $y_1 = y_2 = \bar{y}$. For the linearised solution all cases give the same policy function for bonds.

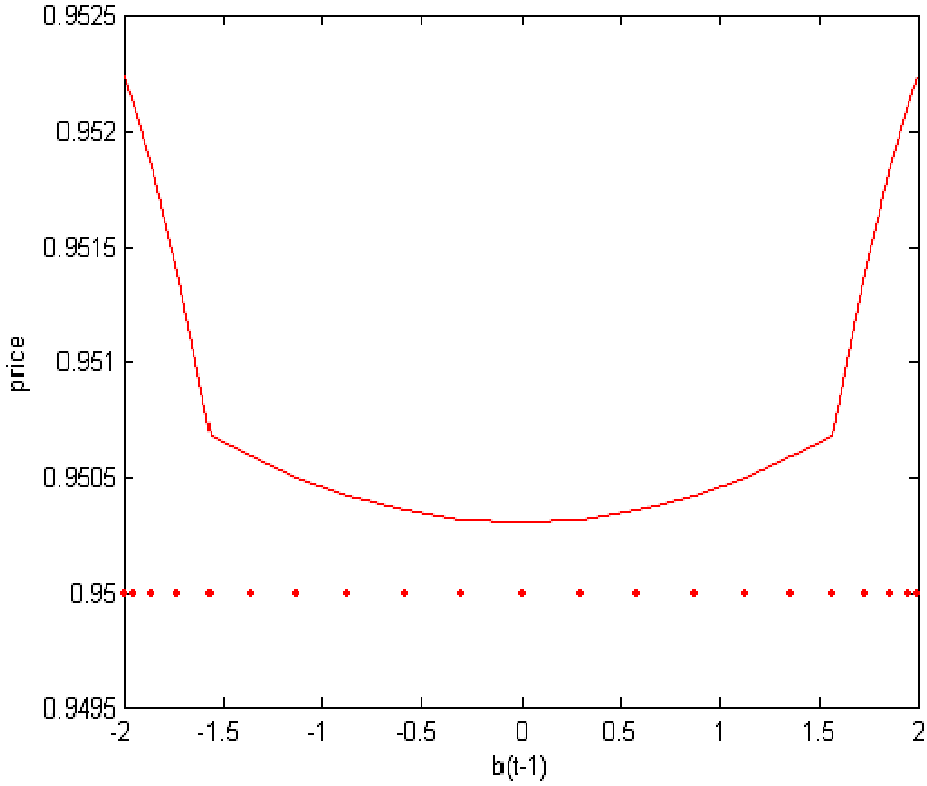


Figure 2 : Price as function of initial bond holdings for purely aggregate shocks. Dotted line is for the linear solution and solid line for the PEA solution. Exogenous endowments are fixed to $y_1 = y_2 = \bar{y}$ .

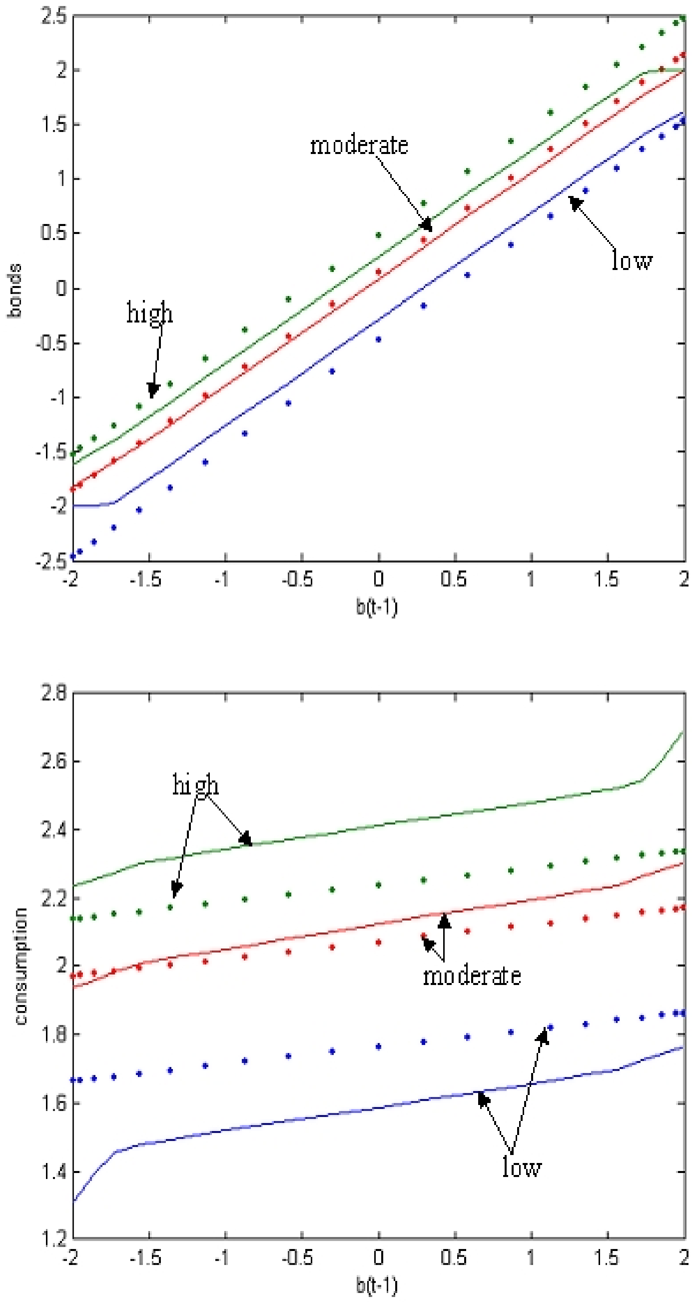


Figure 3: Policy functions for purely idiosyncratic shocks. Dotted lines are for linearised solution and solid lines for PEA solution. Low is for $y_1 = \bar{y} - 3\sigma_y$ and $y_2 = \bar{y} + 3\sigma_y$, high is for $y_1 = \bar{y} + 3\sigma_y$ and $y_2 = \bar{y} - 3\sigma_y$, moderate is for $y_1 = \bar{y} + \sigma_y$ and $y_2 = \bar{y} - \sigma_y$.

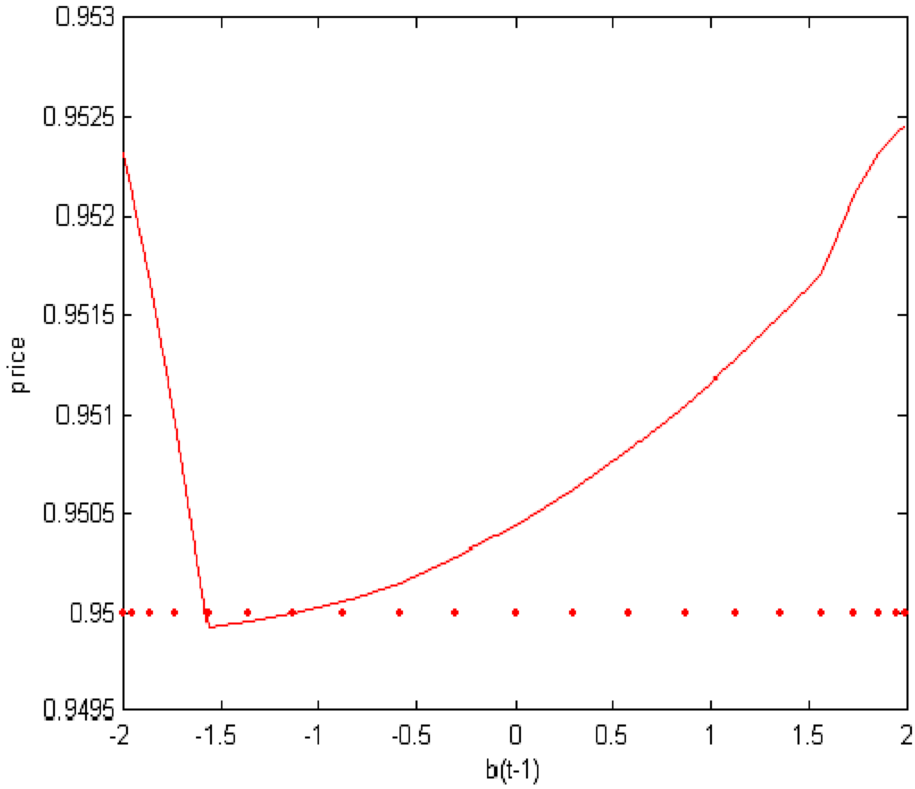


Figure 4 : Price as function of initial bond holdings for purely idiosyncratic shocks. Dotted line is for the linear solution and solid line for the PEA solution. Exogenous endowments are fixed to $y_1 = \bar{y} + \sigma_y$ and $y_2 = \bar{y} - \sigma_y$ .

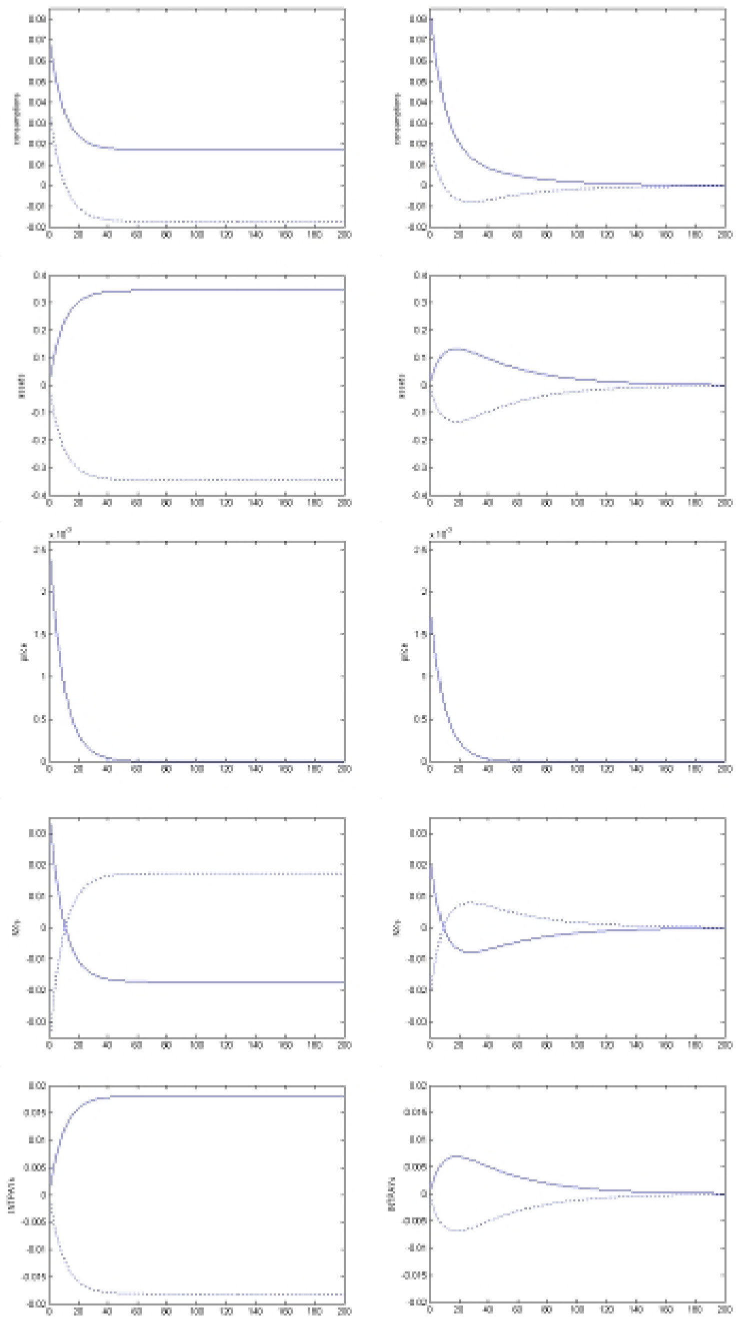

Figure 5: Impulse Responses to an income innovation for linearisation (left) and PEA (right) methods. Solid lines are for agent 1 and dotted lines for agent 2.

| | | | | | | |
|---|---|---|---|---|---|---|
| risk aversion | | 0.5 | 1 | 3 | 5 | |
| | | | | | | |
| | max | 0.226 | 0.221 | 0.241 | 0.317 | |
| $\Delta b$ | mean | 0.074 | 0.070 | 0.070 | 0.087 | |
| | max | 0.009 | 0.018 | 0.039 | 0.068 | |
| $\Delta p$ | mean | 0.002 | 0.003 | 0.010 | 0.024 | |
| | max | 0.142 | 0.135 | 0.151 | 0.207 | |
| $\Delta c$ | mean | 0.037 | 0.035 | 0.036 | 0.042 | |
| | | | | | | |
| income st. deviation | | 0.05 | 0.1 | 0.15 | 0.2 | |
| | | | | | | |
| | max | 0.095 | 0.221 | 0.386 | 0.605 | |
| $\Delta b$ | mean | 0.030 | 0.070 | 0.116 | 0.177 | |
| | max | 0.004 | 0.018 | 0.056 | 0.135 | |
| $\Delta p$ | mean | 0.001 | 0.003 | 0.008 | 0.019 | |
| | max | 0.052 | 0.135 | 0.284 | 0.635 | |
| $\Delta c$ | mean | 0.014 | 0.035 | 0.062 | 0.114 | |
| | | | | | | |
| persistence | | 0 | 0.5 | 0.9 | 0.95 | 0.99 |
| | | | | | | |
| | max | 0.623 | 0.339 | 0.221 | 0.177 | 0.065 |
| $\Delta b$ | mean | 0.154 | 0.098 | 0.070 | 0.060 | 0.021 |
| | max | 0.181 | 0.045 | 0.018 | 0.012 | 0.007 |
| $\Delta p$ | mean | 0.012 | 0.007 | 0.003 | 0.002 | 0.002 |
| | max | 0.441 | 0.254 | 0.135 | 0.114 | 0.042 |
| $\Delta c$ | mean | 0.031 | 0.035 | 0.035 | 0.030 | 0.011 |
| | | | | | | |
| debt limit/income | | 0.5 | 1 | 1.5 | 2.5 | 4 |
| | | | | | | |
| | max | 0.256 | 0.221 | 0.212 | 0.237 | 0.379 |
| $\Delta b$ | mean | 0.081 | 0.070 | 0.065 | 0.065 | 0.087 |
| | max | 0.017 | 0.018 | 0.019 | 0.023 | 0.034 |
| $\Delta p$ | mean | 0.003 | 0.003 | 0.003 | 0.004 | 0.008 |
| | max | 0.160 | 0.135 | 0.129 | 0.164 | 0.391 |
| $\Delta c$ | mean | 0.040 | 0.035 | 0.032 | 0.031 | 0.047 |
| | | | | | | |
| discount factor | | 0.9 | 0.95 | 0.97 | 0.99 | |
| | | | | | | |
| | max | 0.145 | 0.221 | 0.276 | 0.355 | |
| $\Delta b$ | mean | 0.046 | 0.070 | 0.086 | 0.110 | |
| | max | 0.019 | 0.018 | 0.018 | 0.018 | |
| $\Delta p$ | mean | 0.003 | 0.003 | 0.003 | 0.003 | |
| | max | 0.085 | 0.135 | 0.169 | 0.217 | |
| $\Delta c$ | mean | 0.022 | 0.035 | 0.044 | 0.057 | |

Table 1: Absolute relative deviation of assets, asset price and consumption implied by the log-linear solution from the corresponding variable implied by the PEA solution. For every point $s$ in the state space, $\Delta b(s) = \left|b^{PEA}(s) - b^{LL}(s)\right|$, $\Delta c(s) = \left|\frac{c^{PEA}(s)-c^{LL}(s)}{c^{PEA}(s)}\right|$ and $\Delta p(s) = \left|\frac{p^{PEA}(s)-p^{LL}(s)}{p^{PEA}(s)}\right|$. For each variable, the top row corresponds to the maximum over all $s \in S$ and the bottom line corresponds to the average over all $s \in S$.

| persistence | Cross country consumption correlations | | |
|---|---|---|---|
| | Complete markets | Incomplete markets | |
| | | Log-Linear | PEA |
| 0.97 | 1 | 0.43 | 0.10 |
| 0.95 | 1 | 0.60 | 0.26 |
| 0.9 | 1 | 0.80 | 0.48 |
| 0.5 | 1 | 0.99 | 0.95 |
| 0 | 1 | 1.00 | 0.99 |

Table 2: Consumption correlations under alternative assumptions for income persistence. The first column shows correlations under complete markets. The second and third show correlations under incomplete markets for the log-linear and the PEA method respectively.