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Optimal Life Cycle Saving With Borrowing Constraints: A Graphical Solution

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EUROPEAN UNIVERSITY INSTITUTE, FLORENCE

ECONOMICS DEPARTMENT

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OPTIMAL LIFE CYCLE SAVING WITH BORROWING CONSTRAINTS: A GRAPHICAL SOLUTION

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Abstract

As in the orthodox life-cycle theory of consumption and saving behaviour, it is assumed that there is no uncertainty and an additively separable intertemporal utility function. In addition to the usual life-time budget constraint, the consumer is assumed to face upper bounds at each date on the amount that can be borrowed. For special families of utility functions — notably those in the HARA class — it is possible to adapt a simple graphical technique due to Tobin (1975) in order to determine which borrowing constraints bind. As in Mariger (1987), the main effect of binding borrowing constraints turns out to be like a shortening of the planning horizon. It follows that current responses to all wealth, income, and interest rate changes before the time at which there is next a binding borrowing constraint are limited to those that would occur if the planning horizon really were truncated at the time of the constraint. For consumers who are either credit constrained already, or else expect that they might be so in the near future, this helps to explain why the marginal propensity to consume out of current wealth, temporary changes in income, or windfalls can be much larger than the standard life-cycle theory would suggest. But for consumers who are far away from any binding borrowing constraint, the standard life-cycle model can be expected to be a good approximation.

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1. Introduction

1.1. Theoretical Motivations for Studying Borrowing Constraints

In recent years the assumption of perfect capital markets has been questioned with increasing frequency. Everybody who is not a theoretically minded economist understands that there are limits to what they can borrow. Among such economists, Stigler (1967) entered an objection to the practice of postulating imperfect capital markets without any theory of how the imperfections arose. This gap has recently been filled to a large extent by important work such as that of Jaffee and Russell (1976), Stiglitz and Weiss (1981, 1983), Allen (1981, 1983), etc. These authors all consider what happens when a lender does not know how likely it is that a borrower can afford to repay, and is forced to judge this probability in part on how much interest the borrower is willing to agree to. The higher the rate of interest, *ceteris paribus*, the more likely it is that the lender thinks that the borrower may default. So eventually there will be a threshold rate of interest at which the lender becomes unwilling to make the loan at all. Individuals willing to bid interest rates for loans higher than this threshold inevitably become confronted by credit rationing. This prevents the usual kind of market clearing in which individuals' demands are subject only to a single linear budget constraint.

The work mentioned in the previous paragraph mostly considered "adverse selection." This occurs when a borrower's true probability of being able to repay is unknown, so that the lender is forced to infer what he can from what he knows, which is the interest rate which the borrower seems willing to accept, the collateral he has to offer, his existing wealth and income, etc. It presumes, however, that individuals who can afford to repay will do so, although when they can borrow they may choose an *ex ante* riskier plan than they would if they were using their own finances exclusively. The latter is a form of "moral hazard." Hammond (1989), however, following Keynes (1936, p. 144), has emphasized a somewhat different form of moral hazard, concerning whether the borrower will choose to repay a loan at all, or simply consume excessively intending never to repay, but rather to take the consequences of default. Of course, such behaviour is often punishable as a

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kind of fraud, and wise lenders will seek to deter it, and yet it seems to occur too often in practice.

The point of this moral hazard is that the lifetime budget constraints of orthodox theory are obviously unenforceable literally. In principle they allow consumers to acquire enormous debts during their lifetimes, which are supposed to be repaid out of the assets which survive when the debtor dies. But no enforcement mechanism can possibly recover the obligations of somebody who chooses to be heavily in debt at death. At best borrowers can be confronted with a sequence of borrowing constraints which either restrict their credit to a level below what both borrower and lender know can be repaid, or, if not, may encourage the borrower to default in some circumstances. As Foley and Hellwig (1975) point out, with perfect and complete information these credit limits could be set at levels which would just permit the individual to carry out the fully optimal lifetime consumption plan which would be chosen if capital markets were perfect and all loans were repaid promptly and in full. But with asymmetric information, a borrower may not be able to convince any lender that all the debt which this fully optimal lifetime consumption plan requires the borrower to have will be repaid. Then the consumer will face binding borrowing constraints forcing the choice of only a constrained optimum.

So, to summarize, if there were perfect and complete information, the assumption of a perfect capital market might not be so objectionable. But information is neither complete nor perfect, which implies that capital markets cannot be perfect either. Indeed, there is a sense in which the lifetime budget constraint of orthodox theory can only be enforced by facing the consumer with an appropriate sequence of borrowing constraints. Then it becomes very likely that the information to fix these borrowing constraints appropriately will be lacking. The implication is that individuals may well face several different binding borrowing constraints during the course of their lifetimes. Intertemporal wealth is not all liquid or fungible. Yet, with a few notable exceptions, rather little attention has been given to analysing how intertemporal consumption theory changes when there

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are borrowing constraints. This paper is part of an attempt to set out a much more systematic microeconomic theory with borrowing constraints. One can argue that these are no more *ad hoc* than the clearly unenforceable life-time budget constraints of orthodox theory. An important by-product may be a much closer link between microeconomic theory and the actual relationships that have been established empirically.

1.2. Review of Some Existing Literature

Of course, much has already been written on the effect of borrowing constraints on consumer demand over the course of many time periods. Some of the most important work will be briefly reviewed here.

As explained in the introduction to Section 3, Tobin (1975) is the most useful work to date on which this paper builds. Tobin reformulated the borrowing constraint at each date t by showing that it was equivalent to having the present discounted value of accumulated consumption up to date t be bounded above by the total present discounted value of initial wealth and accumulated income up to date t. For the special case of "smoothed consumption" — when the optimal consumption path in the absence of binding intermediate borrowing constraints would involve a constant level of consumption — he gave a graphical description of the solution. It was general in that it could allow alternating time intervals of binding and slack borrowing constraints. This should be contrasted with much of the later literature to be described below, which often allows just one or at most two transitions between binding and slack borrowing constraints during the consumer's lifetime. But Tobin's formulation had no explicit intertemporal utility maximization of the kind to be considered here.

Also very useful is the article by Heller and Starr (1979), who formulate in discrete time a particular version of the general problem to be considered in Section 2. Unlike Tobin, they do set up an explicit intertemporal utility maximization problem whose solution involves smoothed consumption. For this special case, they also show that consumption never falls, and only rises just after a borrowing constraint has ceased to bind. They also have results very similar to those in

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Section 4.2.

Helpman (1981) presents a rather different model, with an infinite time horizon, constant income each period, and a constant rate of utility discount. Since the only asset is regarded as "money," its nominal rate of interest is taken to be zero. In addition, the "money price of goods" is taken to be constant for all time, so the real rate of interest is also zero. Rather than a simple borrowing constraint, there is a liquidity constraint preventing income from being spent until the period after it is received; this, however, can be reformulated as a borrowing constraint, as shown in Patel (1990). In Helpman's model, the optimal time path of consumption typically involves an initial interval in which each period's borrowing constraint is slack and spare cash balances are run down to zero. From then on the consumer just consumes his constant income as fast as he receives it. This optimal time path is described by a system of simultaneous equations for which it is impossible in general to find an analytic solution. For the special examples of a logarithmic or negative exponential utility function, however, in Patel (1990) it is shown how the analytic solution which Helpman is able to derive can also be found by means of graphical techniques similar to those presented in Sections 3.2 and 3.3. Also, in this paper there will be no presumption that the consumer is facing an infinite horizon which implies making an optimal transition to a steady state path of consumption. Nor will the possibility of alternating slack and tight borrowing constraints be excluded.

The most general model to date of additively separable intertemporal utility maximization subject to borrowing constraints is probably the brief theoretical part of Mariger's (1987) paper. He shows that the first phase of an optimal consumption plan consists of a path up to some date v which is optimal subject only to the borrowing constraint at that date, and no others. Call this the "locally optimal" path up to date v. Then the date v must be chosen optimally, subject to all the borrowing constraints which the consumer faces. As argued in Section 4, and as Heller and Starr (1979) had already shown for the special case of smoothed consumption, v must be chosen to *minimize* consumption on the locally optimal

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path up to date v, in order that consumption should not be so high that some other borrowing constraint binds before that at date v. Thereafter, for his empirical work, Mariger considers a particular isoelastic utility function, so his analytical consumption function turns out to be a special case of that derived in Section 4.3.

Some other articles raise issues not treated here, but worth considering in later work. Pissarides (1978) has a model which is in many ways more general than those considered in this paper, since it includes several assets, transactions costs, and differential borrowing and lending rates. The latter feature is also to be found in Flemming's (1973) earlier analysis. Some comparative static results are derived, with a marginal propensity to consume out of current income which usually exceeds that for anticipated future income of the same present discounted value, etc. But naturally no attempt is made to derive explicit consumption functions in such a complicated setting.

Another article of great interest is by Artle and Varaiya (1978). Not only do they realize the value of Tobin's (1975) work in this connection, but they go on to consider some interesting continuous time models of optimization subject to borrowing constraints, solved by means of optimal control techniques. In their first model (Section 3 of their paper), for a special case they anticipate the results to be derived in Section 3.2. Utility is logarithmic. Income and both the rates of interest and utility discount are constant. No borrowing is allowed. When the rate of interest is less than the rate of utility discount, this implies that consumption falls steadily over time, except possibly during an intermediate time interval when the borrowing constraint binds and so consumption is held equal to the constant income level. While their first model is therefore a special case of those to be considered below, the more important later model (in Section 4 of their paper) introduces features that go beyond what is done here. They assume that the consumer can borrow only in order to switch from renting to buying a house, and will want to do this as soon as possible, given the consumption profile and assuming that the down payment can be afforded, because renting is more expensive than servicing a mortgage. With these assumptions, for the typical consumer who

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is better off buying a house at some stage rather than renting throughout his lifetime, an optimal consumption path involves saving initially in order to meet the down payment, then having liquid assets drop to zero as he buys the house. Thereafter he enjoys higher constant consumption while liquid assets remain at zero. Eventually, however, near the end of his life, he sells his house and then runs down the assets derived from the proceeds while living once again in a rented house. What is particularly interesting about this example is how it includes borrowing constraints which can be relaxed by purchasing suitable durable goods which can serve as collateral. It is also rather special, however, because of the indivisibility of the house. Another model of the consumer with some similar features appears in Jackman and Sutton (1982).

All the above work ignored the important problems posed by uncertainty. It will also be neglected in this paper, whose main aim is to derive analytical solution to intertemporal utility maximization problems with borrowing constraints. Uncertainty appears to make this impossible. As Zeldes (1989, p. 315) writes, "a closed-form solution to the unconstrained problem has been derived only under very restrictive assumptions, and no one has derived a closed-form solution to the problem with constraints imposed." For this reason, the (more) interesting case with uncertainty has had to be left for future work.

Last but not least, Section II of Blundell's (1988) survey article discusses many of the important issues which relate to models with borrowing constraints.

1.3. Outline of Paper

Following this general introduction, Section 2 begins by formulating in continuous time the problem of intertemporal utility maximization subject to a sequence of borrowing constraints. The key to all the results in this paper is a simple transformation of these constraints. For the borrowing limit at time t will not be exceeded if and only if the total accumulated present discounted value of consumption expenditure up to time t does not exceed the sum of: (i) current liquid assets; (ii) the present discounted value of future income up to time t; (iii) the present discounted value of the borrowing limit at time t itself. This simple transformation

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leads to an optimal control problem with integral constraints. While this is not a standard optimal control problem, there are simple and intuitively appealing techniques available for treating it, and for deriving first-order conditions analogous to those of Kuhn-Tucker for solving optimization problems subject to inequality constraints.

The first order conditions of Section 2 take the expected form when it is not known which borrowing constraints bind — namely, there are pairs of complementarily slack inequalities. To derive further results, much more needs to be said about which borrowing constraints do bind on the (unique) optimal path and which do not. In fact, the general effects of borrowing constraints upon consumption behaviour have been graphically analysed by Tobin (1975, Figure 33.5, p. 164, and Figure 34.3, p. 186) for a special case in which an optimal path of consumption in the absence of borrowing constraints would be perfectly smooth. But there has been no systematic analysis of the problem — in particular, no general discussion of how to determine which borrowing constraints bind and which do not. The main task of Section 3 is to show how Tobin's technique can be modified and generalized in order to treat successfully a much wider range of special utility functions than just those which make smoothed consumption desirable. The graphical technique, however, does vary slightly between the different special classes of utility function for which explicit analytical solutions can be found.

The first of these special cases occurs when the consumer maximizes the discounted present value of total utility, with an increasing strictly concave utility function which is the same each period, and when the interest rate each period equals the rate at which the consumer discounts future utility. Then consumption would be perfectly smooth in the absence of any borrowing constraints. For this special "smoothed consumption" case — which matches the case for which an optimal path subject to borrowing constraints seems easiest to work out — Section 3.1 shows how Tobin's technique is appropriate. The other special utility functions are then rather more familiar — isoelastic utility each period (Section 3.2), negative exponential (Section 3.3). For both of these cases, a modification

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of Tobin's technique will be used. Another special case which can be treeated in a similar manner is the "hyperbolic" utility function — i.e., the HARA ("hyperbolic absolute risk aversion") class of utility functions which feature prominently in intertemporal portfolio theory. Space did not permit this case to be considered, however.

After the graphical techniques of Section 3 have suggested a workable procedure for solving the intertemporal utility maximization problem — at least in some special cases — Section 4 presents an analytical form of consumption function. This requires one to solve a minimization problem of choosing when a borrowing constraint should bind for the first time. It is a minimization rather than a maximization problem because consumption has to be held down low enough at first so that a borrowing constraint does not bind too soon later on. This result is similar to Mariger's (1987) discussion of the discrete time version of the problem.

Finally, Section 5 discusses some general implications of the results reported here, and also includes some suggestions for future work.

2. General Formulation

2.1. Budget and Borrowing Constraints

Consider a continuous time model in which the consumer starts with initial wealth $a(0)$ at time 0 and then plans consumption ahead up to the horizon at time T. At that time it is required that teminal wealth $a(T)$ satisfy the budget constraint

$$a(T) \geq 0 \tag{1}$$

which is supposed to prevent the consumer from ending in debt at time T. As in the case of perfect capital markets, at each time t in the interval $[0, T]$ the consumer is assumed also to face the budget constraint

$$\dot{a}(t) \leq r(t)\, a(t) + y(t) - c(t) \tag{2}$$

in which:

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(i) $a(t)$ denotes the value of the consumer's financial assets, as a differentiable function of time almost everywhere, with time derivative $\dot{a}(t)$;

(ii) $r(t)$ denotes the rate of interest which the consumer both earns from his financial assets, and has to pay on any amount which he borrows;

(iii) $y(t)$ denotes the consumer's income from all other sources;

(iv) $c(t)$ denotes consumption.

For obvious technical reasons it will be assumed that the exogenous functions $r(t)$ and $y(t)$ are both integrable on the interval $t \in [0, T]$.

Suppose now that the consumer also faces capital market imperfections in the form of the additional borrowing constraints

$$a(t) \geq -b(t) \tag{3}$$

at all times t before the terminal date T. Here $b(t)$ denotes a limit on the consumer's credit, which would be zero if no borrowing at all were allowed. Actually, if there were a delay between the receipt of income $y(t)$ and the earliest time at which the consumer had access to it in order to spend it, then $b(t)$ could even be negative. One might have $b(t) = -\int_{t-\tau}^{t} y(s)\,ds$ for example, because income has to be held for a spell of at least τ time periods before being spent or used to offset the borrowing constraint. For technical reasons it will be assumed that $b(t)$ is also an integrable function of t on the interval $t \in [0, T]$.

In effect, (3) implies that the consumer faces a sequence of budget constraints requiring the present discounted value of consumption up to each time t in the interval $[0, T]$ not to exceed the wealth available to the consumer up to time t. The latter includes the present discounted value of both the future income which will accrue up to time t, and the present discounted value of the credit limit $b(t)$ available to the consumer at time t. This can be seen as follows. First, multiply each inequality (2) by the same discount or integrating factor

$$D(t) := \exp - \left[\int_0^t r(s)\,ds \right]. \tag{4}$$

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Then rearrange slightly to obtain

$$D(t)\,[\dot{a}(t) - r(t)\,a(t)] \leq D(t)\,[y(t) - c(t)]. \tag{5}$$

The left hand side of this inequality is the exact time derivative of the function $D(t)\,a(t)$. So integrate (5) with respect to s from 0 up to t to get

$$D(t)\,a(t) - D(0)\,a(0) \leq \int_0^t D(s)\,[y(s) - c(s)]\,ds. \tag{6}$$

Because $D(0) = 1$ and also the borrowing constraint (3) must be satisfied, this can be expressed as

$$C(t) := \int_0^t D(s)\,c(s)\,ds \leq W(t) := a(0) + \int_0^t D(s)\,y(s)\,ds + D(t)\,b(t). \tag{7}$$

Here the left hand side $C(t)$ measures the level of the individual's *accumulated consumption* up to time t, defined as the cumulative present discounted value of consumption up to that time. The right hand side $W(t)$ of (7) measures the exogenous level of the individual's *consumable wealth* up to time t, defined as the sum of the following three items:

(i) initial assets $a(0)$;

(ii) the cumulative present discounted value $\int_0^t D(s)\,y(s)\,ds$ of the future income stream up to time t;

(iii) the present discounted value $D(t)\,b(t)$ of the credit line $b(t)$ available to the consumer at time t.

So the consumer faces the continuum of budget constraints (7) for all $t \in [0, T]$, instead of the single budget constraint which arises when there is a perfect capital market.

In order that the constraints (7) can be satisfied without the consumer being forced to have negative consumption at any time, it will naturally be assumed that

$$W(t) := a(0) + \int_0^t D(s)\,y(s)\,ds + D(t)\,b(t) \geq 0 \tag{8}$$

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for all $t \in [0, T]$. In the case when the consumer is originally in debt, so $a(0) < 0$, (8) will be true if and only if the borrowing constraint becomes tighter only at a rate which allows the consumer to continue servicing his debt, if it becomes tighter at all. When $a(0) \geq 0$, then (8) will be true provided that $b(t)$ is not too negative — i.e, provided that forced saving is not too high.

2.2. Intertemporal Maximization

Suppose now that the consumer maximizes the intertemporally additively separable utility function

$$\int_0^T u(t, c(t))\, dt \tag{9}$$

of the consumption stream $c(t)$ ($t \in [0, T]$). Suppose too that this integral exists whenever the consumption stream itself is integrable — e.g., because the function $u(t, c)$ is jointly continuous, or at least pointwise continuous, in the two variables t and c together. Assume too that for each t in the interval $[0, T]$, the utility function $u(t, c)$ is twice continuously differentiable with respect to consumption c, and that its first and second partial derivatives with respect to consumption satisfy the standard assumptions

$$u'_c(t, \cdot) > 0; \quad u''_{cc}(t, \cdot) < 0; \quad \text{and} \quad u'_c(t, 0) = +\infty. \tag{10}$$

When (9) is to be maximized subject to the sequence of budget constraints (7), a corresponding Lagrangean is

$$\begin{aligned} \mathcal{L} := \int_0^T \left\{ u(t, c(t)) - \lambda(t) \left[\int_0^t D(s)\, c(s)\, ds - W(t) \right] \right\} dt \\ -\mu(T) \left[\int_0^T D(s)\, c(s)\, ds - W(T) \right]. \end{aligned} \tag{11}$$

Here $\lambda(t)$ is the Lagrange multiplier associated with the borrowing constraint at time t, whereas $\mu(T)$ is the Lagrange multiplier associated with the terminal budget constraint at time T. This terminal budget constraint is treated differently from the others, since it could be the only binding budget constraint — it will be, for instance, if no intermediate borrowing constraint binds. The interpretation

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of the Lagrange multiplier $\mu(T)$ associated with the terminal budget constraint is standard; it represents the marginal benefit of being able to relax that constraint by ending in debt at time T. The interpretation of the Lagrange multiplier $\lambda(t)$ associated with the intermediate borrowing constraint at time t is rather different, however, since in continuous time it is virtually impossible to relax just the borrowing constraint at time t without also relaxing the constraints at neighbouring times. Instead, if the consumer has small additional wealth $\Delta W(t)$ available at all times t, as an integrable function of t — possibly zero at many or even most times — then the integral $\int_0^T \lambda(t)\,\Delta W(t)\,dt$ is, to first order, the utility benefit which the consumer derives.

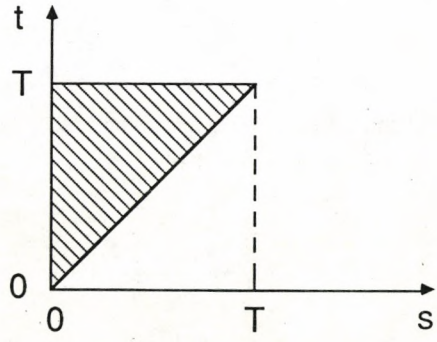

FIGURE 1

The double integral term in (11) can be rewritten as

$$\int_0^T \lambda(t)\left[\int_0^t D(s)\,c(s)\,ds\right]dt = \iint_S \lambda(t)\,D(s)\,c(s)\,ds\,dt \tag{12}$$

where

$$S := \{\,(s,t) \in \Re^2 | 0 \le t \le T, 0 \le s \le t\,\} = \{\,(s,t) \in \Re^2 | 0 \le s \le T, s \le t \le T\,\} \tag{13}$$

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is a triangular set, as illustrated in Figure 1. Therefore

$$\begin{aligned}\int_0^T \lambda(t)\left[\int_0^t D(s)\,c(s)\,ds\right]\,dt &= \int_0^T \left[\int_s^T \lambda(t)\,dt\right]\,D(s)\,c(s)\,ds \\ &= \int_0^T \left[\int_t^T \lambda(s)\,ds\right]\,D(t)\,c(t)\,dt\end{aligned} \tag{14}$$

where the last line follows simply from interchanging the two variables of integration s and t. This implies that the Lagrangean (11) can be rewritten as[1]

$$\int_0^T \left\{u(t,c(t)) - \left[\mu(T) + \int_t^T \lambda(s)\,ds\right]\,D(t)\,c(t) + \lambda(t)\,W(t)\right\}\,dt + \mu(T)\,W(T)$$

or as

$$\int_0^T \left[u(t,c(t)) - \mu(t)\,D(t)\,c(t) + \lambda(t)\,W(t)\right]\,dt \;+\; \mu(T)\,W(T) \tag{15}$$

where

$$\mu(t) := \mu(T) + \int_t^T \lambda(s)\,ds. \tag{16}$$

Thus $\mu(t)$ denotes the total marginal utility of a windfall which gives one extra unit of credit or wealth which first becomes available at time t, and then lasts for all future times up to the horizon T, but which cannot be used to finance any extra consumption before time t.

If an optimum does exist, it must be unique (up to a null set of times) because a strictly concave objective function is being maximized over a convex feasible set. And a standard argument establishes that sufficient first order conditions for optimality are that

$$\begin{aligned}&\mu(t) = \mu(T) + \int_t^T \lambda(s)\,ds; \qquad u_c'(t,c(t)) = \mu(t)\,D(t); \\ &\lambda(t) \geq 0; \quad \int_0^t D(s)\,c(s)\,ds \leq W(t) \quad \text{(comp)}\end{aligned} \tag{17}$$

[1] This technique of reversing the order of integration in order to solve an "integral" optimal control problem is apparently due to Vinokurov (1969) and Bakke (1974). In mathematical economics it has been used by Arthur and McNicoll (1977), Kamien and Muller (1976), and Nickell (1975), amongst others.

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for almost all t in the interval $[0, T]$. Each pair of inequalities in the last line of (17) is complementarily slack in the usual sense that at least one of the pair must be an equality.

After using the equation $\lambda(t) = -\dot{\mu}(t)$ to eliminate $\lambda(t)$, (17) becomes

$$\begin{aligned} u_c'(t, c(t)) &= \mu(t)\, D(t) \qquad (t \in [0, T]) \\ \mu(T) &\geq 0; \quad C(T) \leq W(T) \quad \text{(comp)} \\ \dot{\mu}(t) &\leq 0; \quad C(t) \leq W(t) \quad \text{(comp)} \qquad (t \in [0, T)\ ). \end{aligned} \tag{18}$$

This is the form which will often be used later on. Note that if $\dot{\mu}(t) < 0$, so that the marginal benefit $\mu(t)$ of a windfall at time t is a decreasing function of time, this must be because the borrowing constraint at time t is binding.

So far I have merely set out the essential preliminary analysis of the first order conditions, analogous to those of Kuhn-Tucker, for the problem of maximizing an additively separable intertemporal utility function subject to borrowing constraints. Yet even these preliminaries already bring out some important differences from the usual model with a single lifetime budget constraint. For example, in the standard life-cycle model, the discount factor which is applied to both consumption and income at any future time t is just a function of the instantaneous interest rates at all future times up to and including t. But with borrowing constraints, the discount factor needs augmenting by terms which reflect the positive shadow price of liquidity at those times when the consumer faces a binding borrowing constraint. These shadow prices are partly determined by the consumer's own preferences, including of course his pattern of time preference, but also his aversion to fluctuations in consumption. They also depend on the timing of income, of course.

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2.3. General Properties

First some notation will be developed for describing when borrowing constraints bind. Indeed, let $c^*(t)$ denote the optimal path of consumption, and $C^*(t)$ that of the accumulated present discounted value of consumption. On the optimal path the time intervals during which the borrowing constraint $C^*(t) \leq W(t)$ binds will alternate with those when it is slack. In somewhat rare and possibly rather contrived examples, there could be an infinite number of alternating time intervals with tight and slack borrowing constraints. For simplicity, however, I shall consider only the case when there is just a finite set of times when there is switch from $C^*(t) \leq W(t)$ being a binding borrowing constraint to it being slack, or *vice versa*. From Figure 2 of Section 3 below, the reader will be able to get some impression of how much generality is lost by this assumption.

In fact it will be true that the borrowing constraint is slack on the very first time interval which begins at time 0, and is tight for the last time interval which ends at time T. For, since $C^*(0) = 0$ and since $W(0) = a(0)+b(0)$ which cannot be negative and is typically positive, the first (possibly empty) time interval will be one during which the budget constraint is slack. By convention, however, it will be convenient to redefine $b(0)$ as being equal to $-a(0)$. Then at all the end points of time intervals during which the borrowing constraint is slack, including the initial interval which starts at time $t = 0$, it will be as if the borrowing constraint were tight. Also, on an optimal path, the last interval ending at time T must have a tight borrowing constraint. Otherwise, if the borrowing constraint were slack, more consumption would be possible during this last time interval, contradicting optimality.

Accordingly, let the time intervals during which the constraint $C^*(t) \leq W(t)$ is tight be denoted by $T_j := [s_j, t_j]$, and the complementary time intervals during which the same constraint is slack by $S_j := (t_{j-1}, s_j)$ ($j = 1$ to m). Note that the budget constraint is assumed to hold with equality in closed time intervals, and to be slack in open time intervals. Also, the above assumptions imply that

$$0 = t_0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < t_{j-1} < s_j < t_j < \ldots < s_m < t_m = T. \quad (19)$$

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The first order complementary slackness conditions (18) now reduce to

$$u_c'(t, c(t)) = \mu(t)\, D(t) \quad (t \in [0, T]); \qquad C(T) = W(T); \tag{20}$$

and

$$\begin{aligned} C(t) &= W(t); \quad \dot{\mu}(t) \leq 0 \quad (t \in T_j) \\ C(t) &< W(t); \quad \dot{\mu}(t) = 0 \quad (t \in S_j) \end{aligned} \tag{21}$$

for $j = 1, 2, \ldots, m$. Thus the Lagrange multiplier $\mu(t)$ generally falls, and certainly does not increase, during the time intervals T_j when the borrowing constraint (3) is tight. During these same time intervals it follows that $\dot{a}(t) = -\dot{b}(t)$. Therefore the budget constraint (2) implies that $c(t) = r(t)\, a(t) + y(t) + \dot{b}(t)$. So it is only during the time intervals S_j when the budget constraint is slack that the level of consumption still needs to be determined. During those intervals it follows from (21) that the Lagrange multiplier $\mu(t)$ is constant.

The above equations can be summarized further by defining

$$\nu_j := \mu(t) \quad (t \in S_j; \quad j = 1, 2, \ldots, m) \tag{22}$$

as the successive different values of $\mu(t)$ during slack intervals during which $\mu(t)$ is constant. Also, for $j = 1$ to m, let

$$\Omega_j := W(s_j) - W(t_{j-1}) = D(s_j)\, b(s_j) - D(t_{j-1})\, b(t_{j-1}) + \int_{t_{j-1}}^{s_j} D(\tau)\, y(\tau)\, d\tau \tag{23}$$

indicate by how much accumulated consumption is allowed to increase between the two ends t_{j-1} and $s_j$of the open time interval $S_j = (t_{j-1}, s_j)$. Then the budget equations $C(t) = W(t)$ (all $t \in T_j$) in (20) imply that $C(t) = W(t)$ for $t = t_{j-1}$ and for $t = s_j$ in particular. Therefore

$$\int_{t_{j-1}}^{s_j} D(\tau)\, c(\tau)\, d\tau = C(s_j) - C(t_{j-1}) = W(s_j) - W(t_{j-1}) = \Omega_j. \tag{24}$$

So (20) and (21) can be rearranged so that, for $j = 1, 2, \ldots, m$ one has

$$\int_{S_j} D(\tau)\, c(\tau)\, d\tau = \Omega_j; \quad \nu_{j-1} \geq \nu_j \tag{25}$$

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and also

$$C(t) < W(t); \quad u_c'(t, c(t)) = \nu_j \, D(t) \tag{26}$$

for all $t \in S_j$. In fact, the complications introduced by the borrowing constraints concern the decomposition of the single lifetime budget constraint into a whole sequence of constraints concerning what can be spent during the time intervals when the borrowing constraint is slack. They also concern the decreases in the marginal utility of consumption during the time intervals when the borrowing constraint is tight.

3. Graphical Analysis of Binding Borrowing Constraints

3.1. First Special Case: Smoothed Consumption

The first special case for which a graphical solution is possible occurs when consumption would be perfectly smoothed if capital markets were perfect — i.e., in the absence of borrowing constraints. The discrete time version of this case is discussed, for instance, in Heller and Starr (1979, p. 459). In the continuous time framework used throughout this paper, the first special case occurs when the consumer's intertemporally additively separable utility function takes the special form

$$\int_0^T u(t, c(t))\, dt \equiv \int_0^T D(t)\, u(c(t))\, dt, \tag{27}$$

with $u(c(t))$ assumed to be independent of t. Here, of course, $D(t)$ denotes the discount factor to be applied to the utility of consumption in period t. Also, the utility function $u(\cdot)$ is assumed to be differentiably increasing and strictly concave, and to satisfy the condition $u'(0) = +\infty$ which excludes corner solutions.

In this special case, (20) becomes just $u'(c(t)) = \mu(t)$. It follows that $\dot{\mu}(t) = \frac{d}{dt} u'(c(t))$. Because $u'(c)$ is a strictly decreasing function of c, it follows that $\dot{\mu}(t) \leq 0$ if and only if $\dot{c}(t) \geq 0$. Therefore (18) or (21) implies that

$$\dot{c}(t) \geq 0; \quad C(t) \leq W(t) \quad \text{(comp)}. \tag{28}$$

Thus consumption never decreases, and only increases during the time intervals T_j when a tight borrowing constraint is holding consumption down.

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When optimal consumption would be perfectly smooth in the absence of borrowing constraints, the graphical technique is relatively simple and obvious, as well as being closely related to Tobin's original approach. The first step is to transform the time axis from t to a strictly increasing function $x(t)$ of t so that, whenever $c(t)$ is equal to a constant $\bar{c}$ throughout a time interval $t \in [s, s']$, the accumulated discounted consumption function $C(t)$ defined by (7) is linear in x. To ensure this, define

$$x(t) := \int_0^t D(\tau)\, d\tau \tag{29}$$

so that, if $c(t) = \bar{c}$ on the interval $[s, s']$, then

$$C(t) - C(s) = \int_s^t D(\tau)\, c(\tau)\, d\tau = \bar{c} \int_s^t D(\tau)\, d\tau = \bar{c}\,[x(t) - x(s)] \tag{30}$$

for all such t.

Now define $t^{-1}(x)$ to satisfy

$$x(t^{-1}(x)) \equiv x \qquad \text{and} \qquad t^{-1}(x(t))) \equiv t \tag{31}$$

for all $t \in [0, T]$ and all $x \in [x(0), x(T)]$. In other words, $t^{-1}(x)$ is the well defined inverse of the strictly increasing function $x(t)$ of t. Then define the two functions $\Gamma(x)$ and $\Omega(x)$ on the interval $[0, x(T)]$ by

$$\Gamma(x) := C[t^{-1}(x)]; \quad \Omega(x) := W[t^{-1}(x)]. \tag{32}$$

Notice then that the two equations

$$C(t) = \Gamma[x(t)]; \quad W(t) = \Omega[x(t)] \tag{33}$$

are satisfied at all times t in the interval $[0, T]$. Also, because of (30), Γ is a linear function of x on any interval $[x(s), x(s')]$ whose end points correspond to those of a time interval $[s, s']$ on which consumption is constant.

After these preliminaries, it becomes clear that on any interval of values of x in $[0, x(T)]$ on which $\Gamma(x) < \Omega(x)$, then $C[t^{-1}(x)] < W[t^{-1}(x)]$ at the corresponding times $t^{-1}(x)$ in $[0, T]$. Then (28) implies that $\dot{c}(t) = 0$. Therefore $c(t)$ is constant

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and so, by (30), the function $\Gamma(x)$ is linear, with constant slope equal to $c[t^{-1}(x)]$. But on any interval when $\Gamma(x) = \Omega(x)$, and so $C[t^{-1}(x)] = W[t^{-1}(x)]$ at the corresponding times $t^{-1}(x)$. This implies that $c(t)$ could be increasing and so the function $\Gamma(x)$ may well have an increasing slope — although this can only happen throughout a non-trivial interval on which the function $\Omega(x)$ itself has an increasing slope. This is all illustrated in Figure 2.

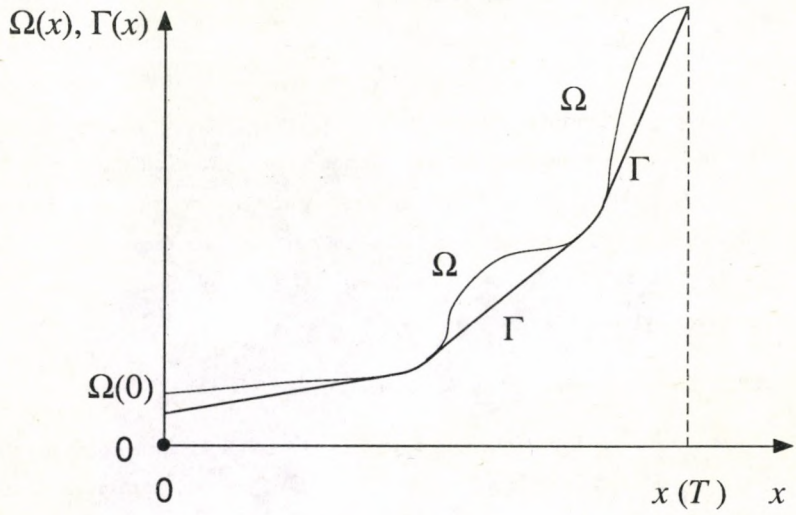

FIGURE 2

In fact Figure 2 clearly suggests a graphical technique for determining which borrowing constraints bind and so finds the optimal path. First construct the graph of the function $\Omega(x)$ which is composed from the exogenous available wealth function $W(t)$ and the inverse of the exogenous function $x(t)$, as in (32) above. In the absence of any intermediate borrowing constraints, the unconstrained optimal

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path would have constant consumption, and so would be a straight line joining the initial point $(0,0)$ to the final point $(x(T), \Omega(x(T)))$ that corresponds to the final time T and final wealth $W(T)$. When borrowing constraints bind, it is because relatively low early values of the function $\Omega(x)$ force the graph of the function $\Gamma(x)$ down below this optimum for the case when there are no binding borrowing constraints. Instead, the consumer is forced to choose from among all those paths from $(0,0)$ to $(x(T), \Omega(x(T)))$ which always stay on or below the graph of $\Omega(x)$ and so satisfy all the borrowing constraints. The best path among these is the shortest. For on the shortest path $\Gamma(x)$ is linear wherever it is below the graph of $\Omega(x)$, convex only when the graph of $\Gamma(x)$ runs along that of $\Omega(x)$, and is never concave. Since the slope of the function $\Gamma(x)$ is equal to consumption at the time $t^{-1}(x)$, it follows that consumption is constant at all times when the borrowing constraint is slack, that consumption may be increasing at times when the borrowing constraint binds, and that consumption never decreases. These are precisely the sufficient first order conditions (28), so the optimal path has indeed been found graphically.

3.2. Second Special Case: Isoelastic Utility

A second special case allowing a graphical solution is when the consumer's intertemporally additively separable utility function $\int_0^T u(t, c(t))\, dt$ takes the special isoelastic form with

$$u(t, c(t)) \equiv \begin{cases} \dfrac{[\beta(t)]^\epsilon\, [c(t) - \bar{c}(t)]^{1-\epsilon}}{1-\epsilon} & \text{if } \epsilon > 0,\ \epsilon \neq 1 \\ \\ \beta(t)\, \ln[c(t) - \bar{c}(t)] & \text{if } \epsilon = 1 \end{cases} \tag{34}$$

where $[\beta(t)]^\epsilon$ denotes the discount factor to be applied to the utility of consumption in period t. It is also assumed that $\beta(t)$, $c(t)$ and $\bar{c}(t)$ are all integrable functions of time on the interval $[0, T]$.

In this special isoelastic case, the first equation of (20) becomes just

$$[\beta(t)]^\epsilon\, [c(t) - \bar{c}(t)]^{-\epsilon} = \mu(t)\, D(t). \tag{35}$$

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Using η to denote $1/\epsilon$, this equation obviously has the solution

$$c(t) = \bar{c}(t) + \beta(t)\,[\mu(t)\,D(t)]^{-\eta} \tag{36}$$

for all $t \in [0, T]$.

In this second special case, it is again possible to determine which budget constraints bind by graphical techniques similar to those used in Section 3.1. As before, the trick is to draw a graph in which optimal paths in the absence of borrowing constraints appear as straight lines. The first step, therefore, is to construct the strictly increasing function $x(t)$ of t in order to ensure that, when no borrowing constraints bind and so $\mu(t)$ is constant throughout a time interval $t \in [s, s']$, then the accumulated discounted excess consumption function defined by $E(t) := \int_0^t D(\tau)\,[c(\tau) - \bar{c}(\tau)]\,d\tau$ is linear in x. To this end, note that the first order conditions (36) imply that

$$E(t) = \int_0^t \beta(s)\,[\mu(s)]^{-\eta}\,[D(s)]^{1-\eta}\,ds. \tag{37}$$

Then define

$$x(t) := \int_0^t \beta(\tau)\,[D(\tau)]^{1-\eta}\,d\tau, \quad \bar{C}(t) := \bar{W}(t) := \int_0^t D(\tau)\,\bar{c}(\tau)\,d\tau. \tag{38}$$

The latter indicates the present discounted value of subsistence consumption or wealth over the period $[0, t]$. Now, if $\mu(t) = \bar{\mu}$ on the interval $[s, s']$, then

$$\begin{aligned} E(t) - E(s) &= \int_s^t \beta(\tau)\,[\mu(\tau)]^{-\eta}\,[D(\tau)]^{1-\eta}\,d\tau = \bar{\mu}^{-\eta} \int_s^t \beta(\tau)\,[D(\tau)]^{1-\eta}\,d\tau \\ &= \bar{\mu}^{-\eta}[x(t) - x(s)] \end{aligned} \tag{39}$$

for all such t.

Now define the function $t^{-1}(x)$ exactly as in (31) of Section 3. And let

$$\Gamma(x) := E(t^{-1}(x)), \quad \Omega(x) := W(t^{-1}(x)) - \bar{W}(t^{-1}(x)). \tag{40}$$

Then $\Gamma(x)$ is a linear fu tion of x on any interval $[x(s), x(s')]$ whose end points correspond to those of i ne interval $[s, s']$ on which the Lagrange multiplier

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function $\mu(t)$ is constant. The only difference now is that the Lagrange multiplier has replaced consumption as the variable being held constant on linear paths in the graph being constructed, as is entirely appopriate given the different form of the first order conditions which are sufficient for an optimum. Exactly as in Section 3.1, on any interval of values of x in $[0, x(T)]$ on which $\Gamma(x) < \Omega(x)$, then $C[t^{-1}(x)] < W[t^{-1}(x)]$ at the corresponding times $t^{-1}(x)$ in $[0, T]$. Then it is clear from (36) that $\mu(t)$ is constant at all such times, and that the function Γ must therefore be linear, with constant slope equal to $\{\mu[t^{-1}(x)]\}^{-\eta}$ in the isoelastic case currently being considered. But on any interval when $\Gamma(x) = \Omega(x)$, and so $C[t^{-1}(x)] = W[t^{-1}(x)]$ at the corresponding times $t^{-1}(x)$, the function Γ may well have an increasing slope.

Exactly as in Section 3.1, therefore, Figure 2, which is based on the graphs of the exogenous functions $\Omega(x)$ and $x(t)$ as explained above, indicates a graphical technique to determine which borrowing constraints bind and so find the solution. As in Section 3.1, the optimal path is the shortest among all those from $(0, 0)$ to $(x(T), \Omega(x(T)))$ which always stay on or below the graph of $\Omega(x)$ and so satisfy all the borrowing constraints.

3.3. Third Special Case: Negative Exponential Utility

The third special case allowing an analytical solution is when the consumer's intertemporally additively separable utility function $\int_0^T u(t, c(t))\, dt$ takes the special negative exponential form, with

$$u(t, c(t)) \equiv -\frac{\alpha(t)}{\beta(t)} \exp - \left[\frac{c(t)}{\alpha(t)}\right]. \tag{41}$$

This negative exponential utility function has the disadvantage that corner solutions with $c(t) = 0$ at certain times really need to be considered. The following calculations are only relatively simple because, as in Merton (1971), negative consumption is allowed, at least in principle. Conditions for optimal paths to have non-negative consumption at all dates will be presented, however.

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In this special isoelastic case, the first equation of (20) becomes just

$$\exp - \left[\frac{c(t)}{\alpha(t)}\right] = \mu(t)\,\beta(t)\,D(t). \tag{42}$$

This equation obviously has the solution

$$c(t) = -\alpha(t)\ln[\mu(t)\,\beta(t)\,D(t)] \tag{43}$$

for all $t =\in [0, T]$. Of course, this solution will be negative at any time t satisfying $\mu(t)\,\beta(t)\,D(t) > 1$. This is a case which should either be excluded by assumption, or else the appropriate corner solution investigated.

A fairly obvious adaptation of the techniques used in Sections 3.1 and 3.2 works for this third special case of negative exponential utility as well. First, transform the time axis from t to a strictly increasing function $X(t)$ of t, with inverse $t^{-1}(X)$. Then transform the accumulated consumption and wealth variables from $C(t)$ and $W(t)$ to $Y(t)$ and $Z(t)$ respectively. Define the functions $\Gamma(X)$ and $\Omega(X)$ on the interval $[0, X(T)]$ by

$$\Gamma(X) := Y[t^{-1}(X)]; \quad \Omega(X) := Z[t^{-1}(X)]. \tag{44}$$

Perform these transformations so that, whenever no borrowing constraints bind throughout a time interval $t \in [s, s']$, and so $\mu(t)$ is constant, then $\Gamma(X)$ is linear on the associated interval $[X(s), X(s')]$ of values of X. Indeed, to ensure that this is true, define

$$\begin{aligned} X(t) &:= \int_0^t \alpha(\tau)\,D(\tau)\,d\tau; \\ Y(t) &:= C(t) + \int_0^t \alpha(\tau)\,D(\tau)\,\ln[\beta(\tau)\,D(\tau)]\,d\tau; \\ Z(t) &:= W(t) + \int_0^t \alpha(\tau)\,D(\tau)\,\ln[\beta(\tau)\,D(\tau)]\,d\tau \end{aligned} \tag{45}$$

for all t in the interval $[0, T]$. Then

$$Y(t) = \quad {}^{t}\,\mathcal{D}(\tau)\,\{c(\tau) + \alpha(\tau)\,\ln[\beta(\tau)\,D(\tau)]\}\,d\tau \tag{46}$$

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and so, after using (43), the above definitions imply that

$$\dot{Y}(t) = D(t)\{c(t) + \alpha(t)\ \ln[\beta(t)\, D(t)]\} = -\alpha(t)\, D(t)\ \ln \mu(t) = -\dot{X}(t) \ln \mu(t) \quad (47)$$

for almost all $t \in [0, T]$. Thus, as desired, $Y(t)$ is indeed linear in $X(t)$ and so $\Gamma(X)$ is linear in X over any time interval in which the Lagrange multiplier $\mu(t)$ is constant. The constant slope of the graph of $\Gamma(X)$ on any such time interval is equal to the constant value of $-\ln \mu(t)$.

With this construction it is clear that, exactly as in Sections 3.1 and 3.2, on any interval of values of X in $[0, X(T)]$ for which $\Gamma(X) < \Omega(X)$, and so for which $C[t^{-1}(X)] < W[t^{-1}(X)]$ at the corresponding times $t^{-1}(X)$ in $[0, T]$, the Lagrange multiplier $\mu[t^{-1}(X)]$ is constant. Therefore (47) implies that the function Γ is linear, with constant slope equal to $-\ln \mu[t^{-1}(X)]$. But on any interval when $\Gamma(X) = \Omega(X)$, and so $C[t^{-1}(X)] = W[t^{-1}(X)]$ at the corresponding times $t^{-1}(X)$, the function Γ may well have an increasing slope. Thus Figure 2, based now on the graphs of the exogenous functions $\Omega(X)$ and $X(t)$, suggests an identical graphical technique to that of Sections 3.1 and 3.2 for determining which borrowing constraints bind and so finding the solution. All that has changed is the method of constructing these two functions, since that is now given by (44) and (45) instead of the earlier (29) and (32).

4. Consumption Functions with Borrowing Constraints

4.1. Characterizing the Optimum

The necessary and sufficient first order conditions (25) for an optimum have solutions whose general form can be found as follows. First, for each interval S_j ($j = 1, 2, \ldots, m$) in which the borrowing constraint is slack, and for each time $t \in S_j$, the equation $u'_c(t, c(t)) = \nu_j\, D(t)$ has a solution of the form

$$c^*(t) = \phi(t, \nu_j D(t)\,), \quad (48)$$

where $c = \phi(t, \nu)$ is the well defined inverse of the decreasing function $\nu = u'_c(t, c)$.

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Substituting this solution into (24) gives

$$\int_{S_j} D(\tau)\,\phi(\tau, \nu_j D(\tau))\,d\tau = \Omega_j \qquad (j = 1, 2, \ldots, m). \tag{49}$$

Because the left hand side is a strictly decreasing function of the variable ν_j, (49) has a unique solution. This can then be used in order to construct the marginal utility function

$$\nu_j \equiv \nu_j(\langle D(t)\rangle_{t\in S_j}; \Omega_j) \tag{50}$$

of the exogenous discount factors $D(t)$ $(t \in S_j)$ and the permissible increase Ω_j in accumulated consumption during the j-th time interval during which the borrowing constraint is slack. Substituting (50) into (48) then determines the consumption function

$$c^*(t; \langle D(s)\rangle_{s\in S_j}; \Omega_j) \equiv \phi(t; \nu_j(\langle D(s)\rangle_{s\in S_j}; \Omega_j)) \tag{51}$$

at all times $t \in S_j$. Here, of course, each segment $\langle c^*(t; \langle D(s)\rangle_{s\in S_j}; \Omega_j)\rangle_{t\in S_j}$ of the optimal path $c^*(t)$ must maximize the partial utility integral $\int_{t\in S_j} u(t, c(t))\,dt$ subject to the single ordinary budget constraint $\int_{t\in S_j} D(t)\,c(t)\,dt \leq \Omega_j$, because it satisfies all the first order conditions for solving this constrained optimization problem over the interval S_j. This is an important simplification which results from the assumption of intertemporal additive separability.

Needless to say, explicit solutions cannot always be found as they will be for the special utility functions introduced in Section 3. Nevertheless, for each $s \in [0, T]$, let $\hat{c}^s(t)$ denote the level of consumption which the consumer would choose at time t if faced with a single budget constraint with the credit limit $b(s)$ imposed at the terminal date s. Then since the first binding budget constraint is that for period s_1, the above equations imply that

$$c^*(t) = \hat{c}^{s_1}(t) \tag{52}$$

for all $t \in [0, s_1]$. The reason is that the above equations for the time interval $[0, s_1]$ are precisely the sufficient first order conditions for maximizing the partial utility integral $\int_0^{s_1} u(t, c(t))\,dt$ subject to this single budget constraint.

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For each $s \in [0, T]$, let $\mu^s(t)$ denote the shadow price which would be appropriate at time t if the consumer were indeed faced with a single budget constraint with the credit limit $b(s)$ imposed at date s. And let $\mu^*(t)$ denote the Lagrange multiplier at time t for the optimal path. Then, of course, it must also be true that $\mu^*(t) = \mu^{s_1}(t)$ for all $t \in [0, s_1]$.

Note that $\mu^s(0)$ must be the minimum shadow price with the property that, if the shadow price path satisfies $\mu^s(t) = \mu^s(0)$ for all $t \in [0, s]$, then the corresponding path of consumption $\langle \hat{c}^s(t) \rangle_0^s$ satisfies the borrowing constraint at time s. Since the optimal path $\langle c^*(t) \rangle_0^T$ must satisfy the borrowing constraint at all times, it follows that $\mu^*(0) = \mu^{s_1}(0) \geq \mu^s(0)$ for all $s \in [0, T]$. Because of diminishing marginal utility, this implies that the corresponding levels of consumption satisfy $c^*(0) = \hat{c}^{s_1}(0) \leq \hat{c}^s(0)$ for all $s \in [0, T]$. Thus s_1, the first time at which a borrowing constraint binds, must actually be chosen in order to *minimize* $\hat{c}^s(0)$ with respect to s; that way, consumption is just low enough to ensure that no future borrowing constraint ever forces the Lagrange multiplier $\mu(t)$ to increase. In other words,

$$c^*(0) = \min_s \{\hat{c}^s(0) \mid s \in [0, T]\} \tag{53}$$

and s_1 must be the value at which the right hand side of (53) reaches a minimum. In later periods as well, it must always be true that

$$c^*(t) = \min_s \{\hat{c}^s(t) \mid s \in [t, T]\} \tag{54}$$

where now $\hat{c}^s(t)$ is the level of consumption which would be chosen in period t if the consumer had inherited the asset level $a^*(t)$ appropriate at time t for an optimal path of consumption, and then faced a single budget constraint with the credit limit $b(s)$ imposed at time s. These results accord with Mariger's (1987).

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4.2. Smoothed Consumption

The graphical analysis in Figure 2 of which borrowing constraints bind at the optimum can now be supplemented with an algebraic analysis for each of the three special cases considered in Section 3. This will yield an analytical form for the consumption function, in terms of current assets, future incomes, and all future credit limits which could lead to binding borrowing constraints in the future.

Consider the first special case of smoothed consumption, which occurs when $u(t, c(t)) \equiv D(t)\, u(c(t))$, with $u(c)$ independent of t, as in (27) of Section 3. Then, for $j = 1$ to m and for all $t \in S_j$, the equation $u'_c(t, c(t)) = \nu_j\, D(t)$ in (26) of Section 2 reduces to just $u'(c(t)) = \nu_j$. Therefore the optimal consumption level must satisfy $c^*(t) = \gamma_j$ where γ_j is defined as the unique constant for which $u'(\gamma_j) = \nu_j$. But then (24) implies that

$$\Omega_j = \int_{t_{j-1}}^{s_j} D(\tau)\, \gamma_j\, d\tau \quad (j = 1, 2, \ldots, m). \tag{55}$$

From this and the definition (23) of Ω_j it follows that, for $j = 1$ to m and for all $t \in S_j$, one has

$$c^*(t) = \gamma_j = \frac{D(s_j)b(s_j) - D(t_{j-1})b(t_{j-1}) + \int_{S_j} D(s)\, y(s)\, ds}{\int_{S_j} D(s)\, ds}. \tag{56}$$

Recall the convention that the credit ceiling at time 0 is taken as equal to initial wealth — i.e., $b(0) := -a(0)$. The formula (56) is similar to what would apply in the case of a single lifetime budget constraint. The denominator is the appropriate total discount factor for a constant consumption flow over the interval S_j between the two times t_{j-1} and s_j; the numerator is the present discounted value of the difference between the available credit limits at the two ends of the time interval S_j at which the borrowing constraint binds, plus the present discounted value of the income which accrues during the times in between when the borrowing constraint is slack.

Indeed, because the last binding budget constraint is that for period s_1, by

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definition, equation (56) above yields

$$c^*(0) = \hat{c}^{s_1}(0) := \frac{a(0) + \int_0^{s_1} D(\tau)\, y(\tau)\, d\tau + D(s_1)\, b(s_1)}{\int_0^{s_1} D(\tau)\, d\tau} \tag{57}$$

as the expression for optimal consumption in period 0. The right hand side $\hat{c}^{s_1}(0)$ of this equation is the constant level of consumption which the consumer would choose in period 0 if faced with a single budget constraint with the credit limit $b(s_1)$ imposed at the terminal date s_1. In the special case when the consumer has no initial assets to spare and so is immediately faced with a binding borrowing constraint, then $s_1 = 0$ and

$$c^*(0) = \hat{c}^0(0) := r(0)\, a(0) + y(0) + \dot{b}(0). \tag{58}$$

Now (53) implies that

$$\begin{aligned} c^*(0) &= \min_s \{\hat{c}^s(0) \mid s \in [0,T]\} \\ &= \min_s \left\{ \frac{a(0) + \int_0^s D(\tau)\, y(\tau)\, d\tau + D(s)\, b(s)}{\int_0^s D(\tau)\, d\tau} \mid s \in [0,T] \right\} \end{aligned} \tag{59}$$

and s_1 must be the value at which the right hand side of (59) reaches a minimum. This is the new form of the consumption function when the consumer faces a series of borrowing constraints. In later periods as well, let $a^*(t)$ denote the asset level on the optimal path which the consumer should have at time t. Then (54) implies that

$$\begin{aligned} c^*(t) &= \min_s \{\hat{c}^s(t) \mid s \in [t,T]\} \\ &= \min_s \left\{ \frac{D(t)\, a^*(t) + \int_0^s D(\tau)\, y(\tau)\, d\tau + D(s)\, b(s)}{\int_0^s D(\tau)\, d\tau} \mid s \in [t,T] \right\}, \end{aligned} \tag{60}$$

where now $\hat{c}^s(t)$ is the constant level of consumption which would be chosen in period t if the consumer was left with assets $a^*(t)$ at the start of that period and then faced a single budget constraint with the credit limit $b(s)$ imposed at time s. This is in accord with Heller and Starr's (1979, p. 459) discussion. Thus optimal consumption $c^*(t)$ in any period t is indeed a function of assets $a^*(t)$ left over from the past and, in principle, of all future discount factors $D(\tau)$, incomes $y(\tau)$, and credit ceilings $b(\tau)$ ($\tau \in [t,T]$).

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4.3. Isoelastic Utility

Consider next the second special case of an isoelastic utility function given by (34). Then the second equation of (26) in Section 2 implies that the optimal consumption level must satisfy $c^*(t) = \bar{c}(t) + \beta(t)[\nu_j\, D(t)]^{-\eta}$, for $j = 1$ to m and for all $t \in S_j$. But then (24) implies that

$$\Omega_j - \bar{\Omega}_j = (\nu_j)^{-\eta} \int_{t_{j-1}}^{s_j} \beta(\tau)\,[D(\tau)]^{1-\eta}\, d\tau \quad (j = 1, 2, \ldots, m) \tag{61}$$

where

$$\bar{\Omega}_j := \int_{S_j} D(\tau)\,\bar{c}(\tau)\, d\tau \tag{62}$$

is the present discounted value of the wealth needed to sustain subsistence consumption throughout the time interval S_j. Then (61) can be solved for ν_j to give

$$\nu_j = \left[\int_{S_j} \beta(\tau)\,[D(\tau)]^{1-\eta}\, d\tau\right]^{1/\eta} (\Omega_j - \bar{\Omega}_j)^{-1/\eta}. \tag{63}$$

Thus, for all $t \in S_j$ one has

$$c^*(t) = \bar{c}(t) + \beta(t)\,[\nu_j\, D(t)]^{-\eta} = \bar{c}(t) + \frac{\beta(t)[D(t)]^{-\eta}\,(\Omega_j - \bar{\Omega}_j)}{\int_{S_j} \beta(\tau)\,[D(\tau)]^{1-\eta}\, d\tau}. \tag{64}$$

From this and the definitions (23) and (62) of Ω_j and $\bar{\Omega}_j$ it follows that the expression (64) for $c^*(t)$ can be written as

$$\bar{c}(t) + \frac{\beta(t)\,[D(t)]^{-\eta}\left\{D(s_j)b(s_j) - D(t_{j-1})b(t_{j-1}) + \int_{S_j} D(s)\,[y(s) - \bar{c}(s)]\, ds\right\}}{\int_{S_j} \beta(s)\,[D(s)]^{1-\eta}\, ds}, \tag{65}$$

for $t \in S_j$ and for $j = 1, 2, \ldots, m$. As before, it is understood that $b(0) := -a(0)$. Of course, this formula is very similar to what would apply in the case of a single lifetime budget constraint.

When $\epsilon = \eta = 1$, so that we have the linear expenditure system, then (64)

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and (65) become

$$c^*(t) = \bar{c}(t) + \frac{\beta(t)}{\nu_j\, D(t)} = \bar{c}(t) + \frac{\beta(t)\,(\Omega_j - \bar{\Omega}_j)}{D(t)\int_{S_j} \beta(s)\, ds}$$
$$= \bar{c}(t) + \frac{\beta(t)\left\{D(s_j)\, b(s_j) - D(t_{j-1})\, b(t_{j-1}) + \int_{S_j} D(s)\,[y(s) - \bar{c}(s)]\, ds\right\}}{D(t)\int_{S_j} \beta(s)\, ds}. \tag{66}$$

Next, define

$$\hat{c}^s(0) := \bar{c}(0) + \frac{\beta(0)\,\left\{a(0) + D(s)b(s) + \int_0^s D(\tau)\,[y(\tau) - \bar{c}(\tau)]\, d\tau\right\}}{\int_0^s \beta(\tau)\,[D(\tau)]^{1-\eta}\, d\tau}. \tag{67}$$

This is the level of consumption which the consumer would choose in period 0 if faced with a single budget constraint with the credit limit $b(s)$ imposed at the terminal date s. Because the first binding budget constraint is that for period s_1, by definition, (65) implies that optimal consumption in period 0 satisfies $c^*(0) = \hat{c}^{s_1}(0)$. In the special case when $s_1 = 0$ because $b(0)$ is so low that the consumer is immediately faced with a binding borrowing constraint, then (58) must be true as before.

Now (53) and (67) together imply that $c^*(0)$ must be given by

$$c^*(0) = \bar{c}(0) + \beta(0) \min_s \left\{ \frac{a(0) + D(s)b(s) + \int_0^s D(\tau)\,[y(\tau) - \bar{c}(\tau)]\, d\tau}{\int_0^s \beta(\tau)\,[D(\tau)]^{1-\eta}\, d\tau} \mid s \in [0, T] \right\} \tag{68}$$

and s_1 must be the value at which the right hand side of (68) reaches a minimum. This is the new form of the consumption function when the consumer faces a series of borrowing constraints. In later periods as well, let $a^*(t)$ denote the asset level on the optimal path which the consumer should have at time t. Then, as in (54), $c^*(t)$ in any period t must be given by

$$\bar{c}(t) + \beta(t) \min_s \left\{ \frac{a^*(t) + \int_t^s D(\tau)\,[y(\tau) - \bar{c}(\tau)]\, d\tau \;+\; +D(s)b(s)}{\int_t^s \beta(\tau)\,[D(\tau)]^{1-\eta}\, d\tau} \mid s \in [t, T] \right\}. \tag{69}$$

Once again, therefore, this is a function of assets $a^*(t)$ left over from the past and, in principle, of all future discount factors $D(\tau)$, incomes $y(\tau)$, and credit ceilings

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$b(\tau)$ ($\tau \in [t,T]$). Of course, it also depends now on the taste parameters $\beta(\tau)$ ($\tau \in [t,T]$) and on the elasticity parameter η.

4.4. Negative Exponential Utility

For the third special case $u(t,c(t))$ is a negative exponential utility function given by (41). Then the second equation in (26) of Section 2 above implies that the optimal consumption level must satisfy $c^*(t) = -\alpha(t)\,\ln[\nu_j\,\beta(t)\,D(t)]$, for $j = 1$ to m and for all $t \in S_j$. It follows from (24) that

$$\Omega_j = \int_{t_{j-1}}^{s_j} D(t)\,c(t)\,dt = -\int_{S_j} \alpha(t)\,D(t)\ln[\nu_j\,\beta(t)\,D(t)]\,dt \tag{70}$$

for $j = 1$ to m. This can be solved for ν_j to give

$$\nu_j = \exp - \left[\frac{\Omega_j + \int_{S_j} \alpha(t)\,D(t)\ln[\beta(t)\,D(t)]\,dt}{\int_{S_j} \alpha(t)\,D(t)\,dt}\right]. \tag{71}$$

Thus

$$c^*(t) = -\alpha(t)\,\ln[\beta(t)\,D(t)] + \alpha(t)\left[\frac{\Omega_j + \int_{S_j} \alpha(\tau)\,D(\tau)\ln[\beta(\tau)\,D(\tau)]\,d\tau}{\int_{S_j} \alpha(\tau)\,D(\tau)\,d\tau}\right]. \tag{72}$$

This is non-negative at all dates $t \in S_j$ as long as

$$\Omega_j \geq \int_{S_j} \alpha(\tau)\,D(\tau)\Big(\max\{\ln[\beta(t)\,D(t)] \mid t \in S_j\} - \ln[\beta(\tau)\,D(\tau)]\Big)\,d\tau. \tag{73}$$

In this third special case as well, combining (72) with the definition (23) of Ω_j yields

$$\begin{aligned} &-\alpha(0)\,\ln[\beta(0)\,D(0)] \\ &+\alpha(0)\left[\frac{a(0) + D(s)\,b(s) + \int_0^s [D(\tau)\,y(\tau) + \alpha(\tau)\,D(\tau)\ln[\beta(\tau)\,D(\tau)]\,d\tau}{\int_0^s \alpha(\tau)\,D(\tau)\,d\tau}\right] \end{aligned} \tag{74}$$

as the expression for optimal consumption $\hat{c}^s(0)$ in period 0 when the consumer faces a single budget constraint at time s, and cannot have debt exceeding $b(s)$ at that date. Then (53) and (74) together imply that $c^*(0)$ must be equal to the minimum of $\hat{c}^s(0)$ as s ranges over the interval $[0,T]$, and the time s_1 at which

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the interval S_1 ends must be the value at which this minimum is achieved. In the special case when $s_1 = 0$ because $b(0)$ is so low that the consumer is immediately faced with a binding borrowing constraint, then (58) must be true as before. This gives the new form of the consumption function when the consumer faces a series of borrowing constraints.

In later periods as well, let $a^*(t)$ denote the asset level on the optimal path which the consumer should have at time t, and let $\hat{c}^s(t)$ is the level of consumption which would be chosen in period t if the consumer started that period with assets $a^*(t)$ left over from last period and then faced a single budget constraint with the credit limit $b(s)$ imposed at time s. Then (54) and (74) together imply that $c^*(t)$ must be equal to the minimum of $\hat{c}^s(t)$ as s ranges over the interval $[t, T]$. Thus optimal consumption $c^*(t)$ in any period t is indeed a function of assets $a^*(t)$ left over from the previous period and, in principle, of all future discount factors $D(\tau)$, incomes $y(\tau)$, and credit ceilings $b(\tau)$ ($\tau \in [t, T]$). Of course, it also depends now on the taste parameters $\alpha(\tau)$ and $\beta(\tau)$ ($\tau \in [t, T]$).

5. Conclusions

For the three special families of intertemporal utility function introduced in Section 3 — smoothed consumption, isoelastic, and negative exponential respectively — an intuitively appealing graphical technique has been presented which completely determines the set of times at which borrowing constraints bind. In fact, all utility functions of the HARA class can also be treated this way. Moreover, in Patel (1990) the same technique has also been adapted to deal with discrete time, with many consumption goods in a linear expenditure system, and even with a single durable good which the consumer enjoys holding.

For these three special families of utility function, and in the absence of uncertainty, this chapter has derived analytic forms for the consumption function in the presence of borrowing constraints. Compared with the recent work of Mariger (1987), these results are much more specific for utility functions in the three special families being considered here. Section 6 also provided similar results for a

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general additively separable intertemporal utility function, even though these are less clear cut because we lack a general procedure for finding which borrowing constraints bind.

It would clearly be desirable to allow uncertainty about future incomes, interest rates, borrowing limits, and even about the consumer's own future tastes. This has to be left for future work, however. Even so, many of the factors which would be important even in a model allowing uncertainty can already be seen from the simple model with certainty presented here. In particular, current consumption can be affected not just by expectations concerning future lifetime wealth, but also about borrowing limits at all future dates. In addition, current consumption is also crucially dependent on the timing of income, as one would expect when capital markets are imperfect. This in marked contrast to what happens when there are no (binding) borrowing constraints. By borrowing, if necessary, in a perfect capital market, consumption is then spread relatively evenly over the whole of an individual's lifetime.

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