YO2. 5-

THE EUROPEAN UNIVERSITY INSTITUTE

Department of Economics

An Essay on Some Aspects of the Economic Theory of Public Goods

Luciano PASI

June 1993

Thesis submitted for assessment with
a view to obtaining the Degree of Doctor
of the European University Institute.

 DOI: 10.2870/7482

 DOI: 10.2870/7482

Florence, June 1993

An Essay on Some Aspects of the Economic Theory of Public Goods

Luciano PASI

Thesis Committee:

Pierre Dehez, Université Catholique de Louvain, supervisor
Peter Hammond, Stanford University, supervisor
Alan Kirman, EUI
Michel Le Breton, GREQE, Marseille
Ignazio Musu, Università di Venezia

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute
DOI: 10.2870/7482

TABLE OF CONTENT

INTRODUCTION AND MOTIVATION

PART 1. ON CORES AND PUBLIC GOOD ECONOMIES

Chapter 1.1. Introduction

Chapter 1.2. Some remarks on the theory of the core

1.2.1 Basic definition and results

1.2.2 Examples

1.2.2.1 The core in a replica economy

1.2.2.2 Problems in defining the core with an external diseconomy

1.2.3 The non-emptyness of the core

1.2.4 The core and market failures

1.2.5 The core of a coalition production economy

Chapter 1.3. Public goods

1.3.1 Nash Cournot behaviour with public goods

1.3.2 Nash Cournot equilibria

1.3.3 Nash equilibria versus Pareto allocations

Chapter 1.4. A survey of the literature on core, externalities and public goods

1.4.1 Lindahl's solution and the core of an economy with public goods, 1970

1.4.2 External economies and cores, 1971

1.4.3 The core and the Lindahl equilibrium of an economy with a public good: an example, 1972

1.4.4 Theory of value with public goods: a survey article, 1972

1.4.5 A generalization of the pure theory of public goods, 1973

1.4.6 The core of a public good economy, 1974

1.4.7 The Lindahl solution for economies with public goods, 1974

1.4.8 Public goods with exclusion, 1980

1.4.9 Continuum economies with finite coalitions: core, equilibrium, and widespread externalities, 1989

1.4.10 Cost share equilibria: a Lindahlian approach, 1989

1.4.11 The equivalence of core and cost share equilibria in an economy with a public good, 1991

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

TABLE OF CONTENT

PART 2. ON THE GAME-THEORETIC APPROACH TO PUBLIC GOOD ECONOMIES

Chapter 2.1. Introduction

Chapter 2.2. Game theory and public goods

Chapter 2.3. Consideration on games in characteristic form

2.3.1 On characteristic functions

2.3.1.1 The Harsanyi modification

2.3.1.2 Games in partition-function form and limits to coalition formation

2.3.2 On S-equivalence and normalization

2.3.2.1 Characteristic functions and probability measure

2.3.3 On payoff vectors

Chapter 2.4. On payoff imputations in the core

2.4.1 On restrictions to the set of the outcomes available to coalitions

2.4.2 On a characteristic function that assumes that outsiders do contribute

Chapter 2.5. The cooperative approach to externalities and public goods

Chapter 2.6. The Vickrey auction

PART 3. ON TAXES AND PUBLIC GOODS

Chapter 3.1. Taxes and public goods

3.1.1 Some basic notions on tax systems

3.1.2 How much of public goods with different taxes?

Chapter 3.2. Tax allocation systems

Chapter 3.3. Voting

Chapter 3.4. Voting over tax allocation systems

PART 4. AGGREGATION OF PREFERENCES AND RESTRICTIONS IN VOTING MODELS

Chapter 4.1. Introduction

Chapter 4.2. Aggregation of preferences

4.2.1 Social functions of individual data

4.2.2 On convenient restrictions on sets of orderings

4.2.3 A "respect the minority" restriction on permissible sets of alternatives

 DOI: 10.2870/7482

TABLE OF CONTENT

4.2.4 Appropriate restrictions associated to public goods, and their effects on Arrow's impossibility results

Chapter 4.3. On usual restrictions in voting models

PART 5. THE MEDIAN VOTER THEOREM

Chapter 5.1. Introduction

Chapter 5.2. The model

PART 6. HOW MUCH ROOM IS THERE REALLY FOR STRATEGIC VOTING?

Chapter 6.1. Introduction

Chapter 6.2. An example of strategic voting

Chapter 6.3. A combinatorial model

6.3.1 Computing the number of orderings and profiles

Chapter 6.4. The Borda Rule as an aggregation rule

6.4.1 Some notions and definitions

6.4.2 How the Borda Rule works

6.4.3 The algorithm used in the analysis

Chapter 6.5. Two examples and some results

Chapter 6.6. Non manipulability from the set of total scores

Chapter 6.7. A modified Borda Rule

PART 7. CONCLUSIONS AND FUTURE RESEARCH

PART 8. BIBLIOGRAPHY

 DOI: 10.2870/7482

 DOI: 10.2870/7482

INTRODUCTION AND MOTIVATIONS

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

Introduction and motivations

Since my first course on microeconomic theory -which dates back to 1979- I was impressed by the relatively scarce attention dedicated by the economic thought to two subjects that I consider of primary importance for the understanding of that part of human activities which is the object of economic and social sciences. These are public goods and coalitional behaviour.

There are excellent reasons that support the fundamental role of the competitive model based on the analysis of individual actions: the mere fact that most economists agree on these reasons explains why the majority of textbooks, most of the literature, and in general the standard training in economics are not concerned with public economics and group actions.

This description of the situation is both closer to the reality and more unjustifiable in a country like Italy: not only Italian scholars provided in the past important contributions to the field of public economics, but in Italy the value of public goods and of publicly produced private goods accounts for the largest share of the annual gross national product. Moreover, in this country many economic decisions, both at the private and at the governmental level, are taken in a process that is far away from what we consider the typical framework of competitive systems: what we could describe as "feudal behaviours in a pre-capitalistic system" is fairly well summarized by "a general tendency toward agreements and collusion".

When I started working on this thesis I found that there is indeed an impressive amount of literature in the field of public economics: however, it must be said that most of the work on theoretical matters dates back to the seventies and that we still lack a general model of a public good economy that has gained the same importance of the Arrow-Debreu model for the private good economy. Quite immodestly I had decided to provide such a standard representation: the task turned out to be well beyond my capacity. To understand earnestly this simple fact took me a lot of time during which I wrote the background for such a task and much more: what remains after eliminating the more trivial parts constitutes this thesis, which is made of 6 main parts.

The idea that the Lindhal equilibrium could represent in public good economies, what the competitive equilibrium means in private good economies, has been around for quite a long time. The study of this subject

 DOI: 10.2870/7482

is linked to the development of core-related concepts. The first part of this thesis is a discussion about the core and public goods, as well as a review of the relevant literature, which is mainly aimed to show the progression of the field.

Part 2 deals with the game theoretic approach to economic behaviour: it provides -beside other things- a discussion of the notion of domination that offers a clear explanation of the relation between the core and stable sets. I elaborate the idea that the usual characteristic function is not adequate to describe the process of coalition formation: the presence of public goods makes this task impossible. Informally, it is based on excessively pessimistic assumptions about the behaviour of the complementary coalition. Moreover, it seems to pay too much attention to the simple dialectic between one coalition and its complement: a more complex structure of coalitions is typically stable in the real world and I argue that the outcome described by a game in characteristic form -or some alternative description- should be able to explain it fully.

Part 3 contains some basic notions on taxes and analyse the properties of tax allocation systems that have been proposed in the literature. Moreover, it studies the problem of voting as a way to provide legitimated choices of tax systems.

In Part 4 I am concerned with a particular allocation mechanism, that finds its theoretical environment in the literature on collective choices: the determination of the level and of the composition of public good bundles by means of voting systems. I consider existing voting models, and in particular, the relevance of the needed restrictions on the choice domain and on the distribution of preferences to assure the existence of an equilibrium outcome. I also prove that there is a simple way to avoid the non-egalitarian characteristics of the majority rule without altering its "good" properties.

Since its presentation by Bowen (1943), the median voter theorem influenced many researches on spatial models of the electoral process. In Part 5 I give a slight generalization of the original theorem.

Part 6 is devoted to the study of strategic voting: its main outcome is that the act of voting strategically is very difficult and information-demanding in all relevant situations. The properties of the Borda rule are studied and a modification is proposed in order to further its stability against strategic actions.

From my point of view the quite massive adjoined bibliography deserves the right to be presented as Part 8: it lists most of the relevant literature in the field of public economics appeared before 1993.

Many scholars and fellow students of the EUI helped me by giving helpful comments and by offering criticisms. Prof. Pierre Dehez provided a constant support and many helpful remarks from the beginning of my staying at the EUI. Prof. Peter J. Hammond patiently answered my too many questions and waited for my few conclusions. The usual caveat applies.

PART 1:

ON CORES AND PUBLIC GOOD ECONOMIES

Table of Contents

1 Introduction 1

2 Some remarks on the theory of the core 4
2.1 Basic definition and results 4
2.2 Examples 6
2.2.1 The core in a replica economy 6
2.2.2 Problems in defining the core with an external diseconomy 8
2.3 The non-emptiness of the core 10
2.4 The core and market failures 11
2.5 The core of a coalition production economy 12

3 Public goods 14
3.1 Nash Cournot behaviour with public goods. 17
3.2 Nash Cournot equilibria 20
3.3 Nash equilibria versus Pareto allocations. 21

4 A survey of the literature on core, externalities and public goods. 23
4.1 Lindahl's solution and the core of an economy with public goods, 1970 24
4.2 External economies and cores, 1971 33
4.3 The core and the Lindahl equilibrium of an economy with a public good: an example, 1972 34
4.4 Theory of value with public goods: a survey article, 1972 34
4.5 A generalization of the pure theory of public goods, 1973 39
4.6 The core of a public good economy, 1974 40
4.7 The Lindahl solution for economies with public goods, 1974 42
4.8 Public goods with exclusion, 1980 43
4.9 Continuum economies with finite coalitions: core, equilibrium, and widespread externalities, 1989 43
4.10 Cost share equilibria: a Lindahlian approach, 1989 44
4.11 The equivalence of core and cost share equilibria in an economy with a public good, 1991 46

1,092

1 Introduction

As is well known, the core of an economy is the set of all allocations which cannot be improved upon by any coalition of members of the economy.

The core solution provides an alternative approach to the price-guided competitive solution: prices emerge from the analysis without being introduced a priori as in the traditional Walrasian approach. Beside that, it represents an important characterization of competitive equilibrium since it has been shown by Debreu and Scarf (1963) that under some assumptions the core shrinks to the set of competitive equilibria as the number of individuals in the economy increases.

Moreover, the notion of Pareto optimum acquires a fresh interpretation in the light of the concept of improving coalition: a Pareto optimal allocation is one that cannot be improved upon by the coalition of all agents.

Finally, the core approach allows for freedom of choice of each agent; in the case of large economies it derives the price taking behaviour as a necessary consequence instead of merely assuming it.

1) A short history of the subject

Edgeworth (1881) conjectured that the set of possible *final settlements* in recontracting should converge to the competitive equilibrium.

Lindahl (1919) proposed a notion of equilibrium for economies with public goods: at an equilibrium a set of personalized prices exists such that every one demands (and consumes) the same quantity of each public good. The fact that these prices have to be different across individuals causes computational difficulties and discloses the problem of individual incentive compatibility.

It was Shapley (1953) who associated the Edgeworth (1881) notion of *final settlement* to Gillies' (1953) concept of the core in the theory of n-person games with transferable utility. As a solution to these games the core is certainly an appealing one; however, even in the class of games restricted by the transferable utility condition it may be the case that some games have no outcome in the core and others have, annoyingly, many outcomes in the core.

The appropriateness of this condition of money-like transferable utils was questioned and debated by economists. As a result, a theory of cooperative games without side payments has been developed, mainly through the reformulation of the characteristic function: for, giving up transferable utility, it is no longer possible to describe the payoffs available to a particular coalition as a sum of utility that the group can guarantee. In this way, starting from the cardinal and linear-in-money utils, an ordinal framework evolved. A brief taxonomy is given by saying that noncooperative games include cooperative games as a special case, just as cooperative games without side payments generalize cooperative games with side payments.

Debreu and Scarf (1963) established the equivalence of the core and the competitive equilibria in large private economies -with countably many agents- formally proving Edgeworth's conjecture.

This result led Aumann (1964) to the proof of the equivalence theorem for the core and the set of competitive equilibria of an economy with a continuum of agents.

The case for the relevance of Lindahl equilibria would be stronger if one could prove some analogue of the Debreu-Scarf theorem on the core of a competitive economy. This could support the Lindahl solution as the most politically relevant allocation.

Foley (1967 and 1970) began the search for an equivalent-type result between core allocations and Lindahl equilibria in economies with public goods. Any allocation mechanism for public goods should achieve an outcome in the core: its stability property makes sense per se. In fact, under the assumption of constant returns to scale in the production of public goods and if individual prices are non negative[1], then Lindahl equilibria belong to the core of the public goods economy (see both Foley (1970) and Milleron (1972). However, as shown in Muench (1972), the core of a public goods economy can be much larger than the set of Lindahl allocations even with a continuum of agents.

Starting with the seminal work of Foley, many authors noted that the task of providing an equivalent-type theorem for economies with public goods does not seem feasible. In the process of trying to accomplish the task most of scholars pointed out the following problems:

1 Negative individual prices are possible when there is a public "bad" or when there are redistributive transfers.

-with the usual definition of "improving coalition", since the cost of producing a given level of public good is independent of the number of agents who consume it, improving upon is very difficult;

-in every paper dealing with the core in a public good economy the basic problem of an ad hoc definition of coalition is pointed out. In the presence of public goods the restrictions induced on the set of feasible allocations by the not blocking condition are weaker than in a private economy. As Muench (1972) and Roberts (1974) wrote, the core of a public goods economy is "too large";

-the core is too large or empty when public goods are present: when decisions are taken by simple-majority and the winning coalition has unlimited taxation power there is no core. The non-emptiness of the core when we introduce minority protection depends on the form of the protection (Shubik (1984) chapters 19 and 20, and Kaneko (1977b));

-the decision about the level of production of public goods is usually modelled assuming unanimity: in this case all imputations are in the core;

-Champsaur (1976) focuses on the lack of symmetry of Lindahl equilibria, which contrasts with the *equal treatment* property of allocations in the core: two agents of the same type must receive the same bundle[2];

-Ellickson (1973) and Kaneko (1977a) refer to examples in which the Lindahl equilibrium is never in the core. Shubik (1982) says that Lindahl equilibria lack any strong relationship with the core: the latter's existence depends on the characteristic function and the Lindahl equilibrium does not.

In general, the literature focused on problems in the definition of the core inasmuch it is difficult to take public goods into account.

Then, somebody started working on the definition of equilibrium, instead[3]. Mas-Colell (1980) and Mas-Colell and Silvestre (1989) respectively generalized a Lindahl equilibrium by allowing for non linear personalized prices, and introduced the concept of a *cost sharing equilibrium* (CSE). The set of CSE is in a one-to-one correspondence with the set of Lindahl-Foley equilibria. The correspondence is established by varying the profit shares parameters which characterize the Lindahl-Foley equilibria. Also, any CSE is in the core. In general, these are special models, e.g. with one public good and/or one private good.

2 This property is due to Theorem 2 in Debreu and Scarf (1963)

3 It must be said that also this line of research began with the definition of a "public competitive equilibrium" in Foley (1967).

Weber and Wiesmeth (1990b) go further proving that any core allocation can be obtained as a CSE and be supported by a monotonic cost share system. Moreover, they give necessary and sufficient conditions for a subset of core allocations to be supported as a Lindahl-Foley equilibrium (LCSE).

2 Some remarks on the theory of the core

2.1 Basic definition and results

Let a be an agent of the set A and C be a non-empty coalition of agents; let X_a be the consumption set of agent a and assume that there are l commodities; define a function $f:A \rightarrow X$; then, $f = (f_1, f_2, \ldots) \in X \subseteq \Re^l$ is an allocation. It is feasible if $\sum_A f_a \leq \sum_A \omega_a$, where $\omega_a \in \Re^l$ is the individual endowment.

Definition. We say that C can improve f if there exists an allocation g such that:

* it is feasible, that is $\sum_C g_a \leq \sum_C \omega_a$

* $g_a >_a f_a \quad \forall a \in C$

The core of an exchange economy $E(\{X_a, \geq_a, \omega_a\})$, denoted by $C(E)$, is the set of allocations which cannot be improved by any coalition at once. If f is in the core, then it must be Pareto efficient: otherwise the grand coalition could improve it. To be clear, no coalition can redistribute its total endowments among its members so as to improve the position of all them without worsening the position of any other agent in the economy.

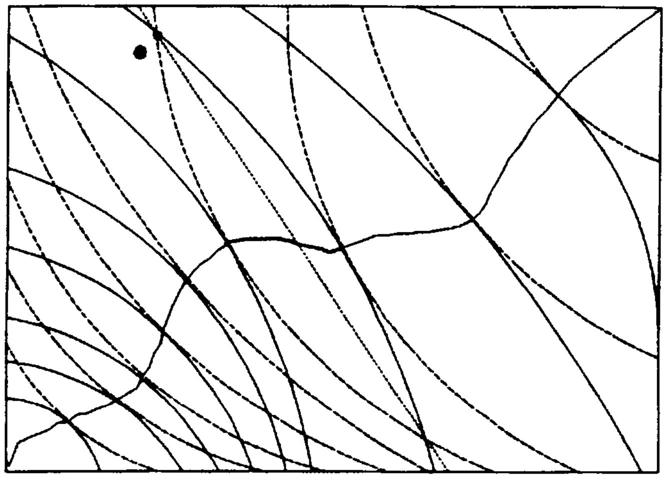


Figure 1: the Edgeworth box

In the figure above we represent a two-person two-good model embedded in a perfectly competitive price system, so that whatever the prices of the two goods they are exogeneous to decision made by the traders to offer exchanges. Any pair of prices for goods *1* and *2* determines their relative prices. A straight line through the endowment point *e* reflects a price ratio, and both agents have the option to refuse trading. Thus, only lines through the core can represent possible relative prices, since traders will refuse any final allocation which is not individually rational. If the unit prices of good *1* and *2* are respectively denoted by p_1 and p_2, then p_1/p_2 is the slope of the line passing through *e*: any allocation along that line reflects the fact that p_1 units of good *1* have equivalent value to p_2 units of good *2*. In this example, since the core is large, there are many lines through *e* and the core that define possible price ratios. The equivalence theorem tells us that when the core shrinks to a unique allocation also the price ratio is unique: the price system is then an outcome of the model.

Recall now the definition of a ***Walrasian equilibrium*** (WE) in an exchange economy: $g_a(p, p \cdot \omega_a)$ is the consumer demand function, and a WE is a pair (p^*, g^*) such that:

$$g_a^* = g_a(p^*, p^* \cdot \omega_a), \qquad \sum_A g_a^* = \sum_A g_a(p^*, p^* \cdot \omega_a) \leq \sum_A \omega_a, \quad \text{and} \qquad p^* \sum_A g_a = \sum_A p^* \omega_a$$

that is, p^* is a WE price vector if there is no good for which there is positive excess demand. We have the following standard result:

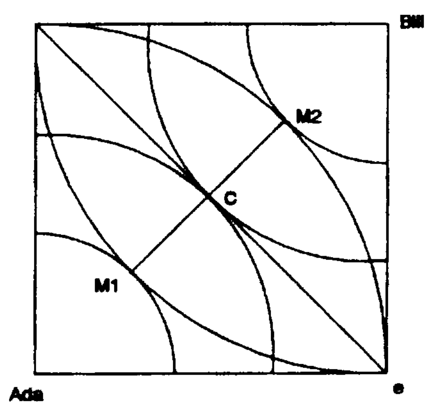
Proposition. If (p^*, g^*) is a WE with endowments ω, then g^* is in the core. To prove this proposition, assume it is not true: then there exists an allocation f and a coalition C such that $f >_a g^* \quad \forall a \in C$ and $\sum_C f_a = \sum_C \omega_a$. But if g^* is a market equilibrium, then for all $f > g^*$ we have $p \cdot f_a > p \cdot \omega_a \quad \forall a \in C$ so that $p \cdot \sum_C f_a > p \cdot \sum_C \omega_a$ which is a contradiction.

Thus, the same assumptions that ensure the existence of a WE guarantee that the core is not empty. However, it is clear from the figure that we will typically have other allocations in the core than just the competitive equilibrium. With more coalitions there are more opportunities for improvements and this suggests that with many more consumers the core shrinks. The device of the replica economy is used because the core is a subset of the allocation space and its dimension changes when the number of agents varies.

2.2 Examples

2.2.1 The core in a replica economy

The first and second players have respectively the following endowments: $e_1 = (4,0)$, $e_2 = (0,4)$ (the endowment point is the right low corner of the square, the familiar Edgeworth box).



Both the players are monopolists. Their common utility function is $u(x,y) = x^{.5} + y^{.5}$. Hence we have indifference curves that are strictly convex and symmetric with respect to the main diagonal of the square. This diagonal is also the set of Pareto optima.

If players decide to exchange their goods using prices whose ratio is 1 they will settle at the point $C = (2,2)$. For each player this point is the tangency between his budget line and the highest indifference curve compatible with it. The allocation C is on the contract curve and it is the only competitive allocation: there does not exist another price system that can equalize total demand and total supply given that both players act competitively.

The set M_1M_2, that is the part of the contract curve between the two indifference curves passing through the endowment point, contains many allocations and so far there is no reason to believe that, between them, the competitive allocation should play a particular role, at least if we drop the hypothesis that both players behave competitively.

The point is that there exists an inverse relation between the cardinality of the set of the core allocations and the number of players. A large number of agents in an economy is one of the hypotheses of perfect competition: basically this allows for the influence of each agent on the price system to be negligible.

The core is defined as the set of allocation which no coalition can improve. Intuitively it is clear that the number of possible coalitions increases with the number of agents. Since there are more coalitions it will be more difficult for an allocation not to be improved: then we can see that there will be fewer core allocations.

In this example we want to show how the core shrinks as the number of agents increases.

The allocation M_1, which assigns the vector (1,1) to Ada and the vector (3,3) to Bill, belongs to the core. Let us consider an economy with 4 agents, adding the third and the fourth agent respectively with the characteristics of Ada and Bill. Then we have:

$$e_{a1} = e_{a2} = (4,0)$$

$$e_{b1} = e_{b2} = (0,4)$$

$$u_1 = x^{.5} + y^{.5} \quad , \qquad i = 1,2,3,4.$$

We call Ada and the other agent with equal characteristics "consumers of type-I", and Bill and the last agent "consumers of type-II".

This economy is called a replica economy: the competitive allocation is the same as before, since whiten each type, each agent goes to the same point of equilibrium. At the equilibrium, reached following a price ratio equal to one, each agent demand the vector of consumption $C = (2,2)$. This allocation is feasible since the total supply is now (8,8).

On the other side, the core has changed. The allocation that gives the consumption vector (1,1) to the agents of type I, and the consumption vector (3,3) to the type-II agents is not in the core anymore. In fact the coalition of the two type-I agents and of one agent of type-II can improve this allocation. The type-II agent has initially 4 units of the second good and each type-I agent has 4 units of the first good: together they have 8 units of the first good and 4 units of the second good. Let's define the following allocations:

$$y_{11} = y_{12} = (5/2, 1/2) \qquad \text{and} \qquad y_{21} = (3,3).$$

Then we have:

$$y_{11} + y_{12} + y_{21} = (5/2 + 5/2 + 3,\ 1/2 + 1/2 + 3) = (8,4),$$

that is a feasible allocation, strictly preferred by type-I agents and indifferent to type-II to the allocation (1,1) and (3,3). In fact, $u(5/2,1/2) = (5/2)^{.5} + (1/2)^{.5} = 2.29$ is larger than $2 = (1)^{.5} + (1)^{.5} = u(1,1)$. The increase of the number of agents makes the core shrink.

2.2.2 Problems in defining the core with an external diseconomy

There are two firms, labelled by 1 and 2; let their output be denoted respectively by y_1 and y_2 and the price of both their products be $p=5$. Assume that the product of the first firm causes an external diseconomy to the second firm. Then, their profit functions are the following:

$$\pi_1 = 5y_1 - y_1^2$$

$$\pi_2 = \begin{cases} 5y_2 - y_1 y_2^2 & \text{for } y_1 > 0 \\ 5y_2 - y_2^2 & \text{for } y_1 = 0 \end{cases}$$

The maximum of π_1 is 6.25 which corresponds to the output level $y_1 = 2.5$. Given this output level of firm 1, the maximum of π_2, reached when $y_2 = 1$, is 2.5. These two payoffs coincides with the values of the characteristic function (namely the values of the two one-member coalitions) when the firm 1 is legally entitled to pollute without amends: notice that in $\max_{y_2}\min_{y_1}(5y_2 - y_1y_2^2)$ firm 2 cannot set y_1 below 2.5.

If we assume that the law forbids firm 1 to pollute and that there is no available technology to abate pollution enterely, then firm 1 has to close: thus in this case the input level $y_2 = 2.5$ and the profit of firm 2 is 6.25. As an example of an intermediate case, assume that firm 1 has a technology to prevent its own pollution at the cost $C(y_1) = y_1$: its profit function becomes $\pi_1 = 5y_1 - y_1^2 - y_1$. Thus, the maximum of π_1 becomes 4 (obtained when $y_1 = 2$), while the maximum profit of firm 2 remains 6.25.

To compute the joint profits of the two firms we maximise the sum of their profit functions. However, the definition of $\pi = \pi_1 + \pi_2$ is not straightforward, for we can have three cases:

a) if no law prohibits pollution, then $\pi_a = 5y_1 + 5y_2 - y_1^2 - y_1y_2^2$;

b) if pollution is forbidden, then $\pi_b = 5y_2 - y_2^2$;

c) if pollution from firm 1 can be abated (and it must be) at the cost $C(y_1) = y_1$, then $\pi_c = 4y_1 + 5y_2 - y_1^2 - y_2^2$.

Assuming for the sake of simplicity that $y_1 \geq 1$ -notice that π_a is unbounded if $y_1 = 0$- we have the following maxima: $\pi_a^* = 10.25$ at $y_1 = 1$ and $y_2 = 2.5$; $\pi_b^* = 6.25$ at $y_2 = 2.5$; $\pi_c^* = 10.25$ at $y_1 = 2$ and $y_2 = 2.5$. The following table summarize our findings:

Cases	π_1	π_2	$\pi_1 + \pi_2$
Pollution allowed	6.25	2.25	10.25
Pollution forbidden and no abatement technology	0	6.25	6.25
Pollution forbidden and abatement technology	4	6.25	10.25

From the table the core can easely be computed and it can be reckoned that its size depends crucially on the assumptions: in the first case the core is the set of pairs of payoffs which are respectively equal or larger than (6.25,2.25) and whose sum adds to 10.25. In the second and third case the core contains only the imputation which assigns to each firm the individually rational payoffs. Notice that these results depend crucially on the assumption regarding the cost of internalizing the external diseconomy. This example shows that the size of the core is related to the underlying economic and social conditions: different settings may imply an empty core and influence distributional issues.

2.3 The non-emptiness of the core

In an exchange economy, the set of core allocations is such that at each of them all individuals are at least as well off as at their initial endowment point. In a simple two-person exchange economy, points outside the core are not interesting because at least one of the traders, acting as a coalition of one, can do better by refusing to trade and staying at her/his endowment point. The set of core allocations may be large, reflecting significant scope for bargaining. It has been proved by Debreu and Scarf (1963) that if each different agent is replaced by n economically identical agents, the set of core allocations converges to the set of competitive equilibria as n increases. This is hardly surprising: the scope for bargaining becomes small when there are many identical individuals that exchange standardized articles. In Section 4 I summarize the most important papers that deal with this convergence when public goods are introduced.

A basic point to be considered when externalities and public goods are introduced is the following. Recall the definition of the core and the idea of "improving upon" a redistribution. A coalition of agents can improve upon a redistribution if that coalition, by using the endowments available to it, can make each of its members better off, regardless of the actions of the agents not belonging to the coalition. This last condition is fulfilled if one assumes that the welfare of every agent depends only on the commodity bundle allocated to him. This means that there are no externalities in consumption and no public goods. This is the reason for which the core theory has to be modified when considering a public good economy. Starting with the article by Foley (1970), there have been many attempts to find definitions of the core that fit better the public good case.

That the core is non-empty is obviously important, for if this is not the case, any discussion on its properties becomes inessential with regard to competitive allocations. Since it has been proved that every competitive equilibrium is in the core, it is clear that if the core is empty there exists no competitive equilibrium. On the other hand, it might be the case that the core is non empty while no competitive equilibrium exists.

There are many examples of an economy with an empty core. One of them is shown by Debreu and Scarf (1963). In view of that, it is plausible to argue that the convexity of preferences plays a substantial role in claiming the nonemptiness of the core. However, Shapley and Shubik (1969) suggested that this property is not as crucial as it seems if the number of individuals is large. In this case the core may be empty but there is a set of allocations which can be improved upon only with very small preference on the part of the blocking coalition. This consideration leads to the concept of the quasi-core: assume that a coalition can improve upon an allocation only if the increase in the utility function of each member of the improving coalition is at least as great as some $\varepsilon > 0$. Then, the so called ε-core, defined in term of such an "ε-improvement", is always non empty when the number of participants is large enough. Nonetheless, this result is obtained under the "transferable utility" assumption.

Another result by Aumann (1964) points out that if there is a non atomic measure space of agents (a continuum of traders), then the core in the strict sense is nonempty even with nonconvex preferences and nontransferable utility. He defines the core using concepts from the measure theory. In his proof of the equivalence between the core and the competitive equilibria there is no need to suppose various type of agents with the same number of individuals in each type as in the Debreu-Scarf approach. Following the same method, Hildenbrand (1968) introduces production and showed that the consumption set does not need to be restricted to $\mathfrak{R}^n_+$. As a further development, measure spaces of economic agents with atoms have been considered in the literature. This approach can be interpreted as a model of an exchange economy where there are "big" and "small" agents with regard to endowments.

2.4 The core and market failures

The strong relation between the core and competitive equilibrium yields an important insight in the case of competitive mechanism failures. If the core is empty, then the competitive mechanism will fail and conversely if the competitive mechanism fails, then it is likely that the core may be empty. This suggests a close connection between the theory of the core and the theory of market failures.

A famous case of market failure due to externalities is that provided by Shapley and Shubik (1969). They argue that in certain cases of diseconomies, the core may be empty. Thus the competitive equilibrium does not exist. They also show that the core will exist in the case of external economies if they are internalized by being listed as explicit commodities. Arrow, Rader, and Mack (1970) suspect that this difference may be really due to a lack of convexity in the production set, instead of the presence of diseconomies. Heller and Starrett (1976) support this view (see also in the same volume the succeeding comment to their paper by J. Ledyard). Ellickson (1973) analyzes this problem, among other things.

Another example of market failure concerns the commodity called "information". Apart from that information usually traded for royalties, there is much information that is not traded at all, or for which a competitive market does not exist. For example, assuming that patents exist and are effectively enforced, the owner of technical know-how cannot be imitated by others and can form a blocking coalition giving rise to a monopoly.

This last problem points out that the relative difference in the cost of forming some coalitions among the possible coalitions in the economy may be very important in explaining monopolies. They can arise only as a result of the ease of coalition formation and without any regard to the scarcity of elements like know-how and skills. All that indicates a need for introducing the cost of forming coalitions into the theory of the core.

2.5 The core of a coalition production economy

Since most of the early literature is concerned with simple exchange economies, it seems worth to consider productive capabilities of the agents. Here I simply summarize some definitions and results by Hildenbrand (1982), Böhm (1974a), and others.

In his model there are no given firms: it describes instead the productive capabilities of each coalition C by the production possibility set Y_C, a set of input-output combinations, that is, of net output vectors. The finite set of the agents is A. The following convention holds: $y \in Y_C$ means that C, using the input vector $\max(0,-y)$, produce the output vector $\min(0,y)$. Then, in $Y_C \subsetneq \Re^l$, the commodity space, input quantities are negative numbers and output quantities are positive numbers. Technological and institutional conditions as well as factors which do not belong to $\Re^l$ determine Y_C. An economic agent is thus described by $\{X_a, \succsim_a, \omega_a\}$, respectively her consumption set, preference relation and initial endowment.

A *coalition production economy* (E, Y) is a mapping which associates $\{X_a, \succsim_a, \omega_a\}$ with every $a \in A$ and a production set correspondence Y which associates Y_C with every coalition.

An *allocation* is a mapping $f: A \rightarrow \Re^l$ such that $f_a \in X_a \quad \forall a \in A$. An allocation is said to be *attainable* if:

$$\textstyle\sum_A (f_a - \omega_a) \in Y_A$$

where Y_A is the superadditive cover.

Definition. Let f be an attainable allocation of (E, Y). The coalition C can *improve* f if there exists a state g such that:

* $g_a >_a f_a \quad \forall a \in C$ and
* $\sum_{a \in C} (g_a - \omega_a) \in Y_C$

The *core* of the coalition production economy, denoted by $C(E, Y)$, is the set of states which no coalition can improve.

Let us finally recall the assumptions on Y that are required for the core to be nonempty. Most of them have a clear meaning, but condition 3 below requires a definition and a comment. A family $\aleph$ of coalitions is called *balanced* if, $\forall C \in \aleph$,

$$\exists \delta_C \quad , \quad \delta_C \geq 0 \quad , \quad \text{such that} \qquad \sum_{C \in \aleph, a \in C} \delta_C = 1 \quad \forall a \in A$$

The correspondence Y is called *balanced* if for every balanced family $\aleph$ of coalitions and associated weights (δ_C) it follows that:

$$\sum_{C \in \aleph} \delta_C Y_C \subset Y_A$$

The intuition behind balancedeness goes as follows. $\aleph$ is a family of coalitions which does not necessarily constitute a partition of A, which means that an agent can appear in different coalitions. He can do some business with a coalition and some with another. Each of the coalitions in $\aleph$ demands the fraction δ_C of the resources of each of its members. The above condition simply requires that these fractions can be chosen so that they sum up to *1* for each agent in A, therefore accounting for his whole resources. Notice that superadditivity for a characteristic function is usually expressed by $\sum_a^n v(C_a) \leq v(A)$ when the family $\{C_a\}$ is a partition of A. Thus, by writing

$$\delta_{C_1} v(C_1) + \delta_{C_2} v(C_2) + \ldots + \delta_{C_n} v(C_n) \leq v(A)$$

we generalize superadditivity to the case in which an agent is allowed to participate in more than one coalition.

Eventually, it can be shown[4] that C(E,Y) is nonempty if the following conditions hold:

1. $\forall a$ X_a is closed, convex and bounded from below, and $\omega_a \in X_a$
2. $\forall a$ the weak preference relation $\succeq_a$ is convex
3. Y is balanced, Y_C is closed, and contains the null element, and the asymptotic cone of $Y_A \cap \Re^l_+ = \{0\}$

3 Public goods

At a Pareto optimum the marginal rates of transformation for public goods are equal to the sum of marginal rates of substitution : these are called the Samuelson conditions, after his seminal paper (Samuelson (1954)). The public good model formalized by Samuelson certainly has a simple structure: if any given agent increases her contribution to the total amount X of the provided public good by some Δx_i, then each and every agent's consumption of X increases by the same amount. The only difference between individuals is that the contributor has had to reduce his private consumption. Considering the single individual's utility function $U^i(y^i,X)$, it is often said that a given increment of X affects individual i in the same way despite of its source. Indeed, this is not the case since even the sign of $\partial U^i(y^i,X)/\partial X$ may differ across individuals: take as an example the case of a good that is a public good for one person and a public bad to another.

This model has been criticized because it fits only the pure public good case instead of dealing with many real world situations that lie between this case and the pure private good one. Cornes and Sandler (1986) claim that this criticism is misleading: what accounts for the simplicity of the model is rather the presence of only one public good. It is also argued that few services, among those carried out by modern governments, are equally consumed by all members of society: the environment variables in Milleron's (1972) paper are the right example. Others complain that most public services are subject to congestion in contrast to the zero marginal user cost of the pure public good (see Ellickson (1973)). However, this objection is easely circumvented by defining each person's utility net of congestion costs.

It may be useful to stress the role that exclusion plays with respect to public goods. We can show that the optimal conditions are identical whether or not exclusion can be exercised. Since the marginal utility of the public good is non-negative it is never efficient to exclude any individual from consuming any part of public good output. The significance of exclusion rests with the characteristic of private market provision of

4 See for example, Böhm (1974b).

the public good and with the financing options open to government when it decides to provide the public good, but not with the fundamental properties of public goods themselves. As a general framework for what follows, we introduce the model of Cornes and Sandler.

Each individual consumes a private good y and contributes toward units of the public good whose total amount is X; x_i denotes units of the public good acquired by the subscriber i that derives utility from her own contribution and from the total amount X available to all. The individual utility function is

$$U_i = U_i(y_i, x_i, X) \qquad i = 1,2,\ldots,n \tag{1}$$

We can also think of this model as one with n public goods: x_i is both a public good available to all and a public good of which agent i is the sole supplier. Then, we can express the utility function as

$$U_i = U_i(y_i, x_1, x_2, \ldots, x_n) \qquad i = 1,2,\ldots,n \tag{2}$$

Now, with n agents in the economy, the aggregate resource constraint is $\sum_i y_i + p\sum_i x_i = \sum_i M_i$ where p is taken as the given money price of a unit of acquisition of the public good. It may also be thought of as reflecting any constant cost technology that produces either two final goods from the given quantity of primary resource $M = \sum_i M_i$: in this case p is the marginal rate of transformation between the public and private good.

If (2) represents the tastes of consumers, in order to get the first order conditions for a Pareto optimum, we can maximize the following strictly concave social welfare function:

$$W = W(U_1, U_2, \ldots, U_n)$$

$$\text{s.t.} \qquad T(Y, pX, M) = 0$$

Where we simplify the aggregate resource constraint as $Y + pX = M$. The associated Lagrangian is:

$$L(y_i, x_1, x_2, \ldots, x_n, \lambda) = W(U_1, U_2, \ldots, U_n) + \lambda(M - Y - pX)$$

Equating all the first order partial derivatives to zero we get the *(2n+1)* conditions for a global maximum:

$$\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial y_1} - \lambda = 0$$
$$\frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial y_2} - \lambda = 0$$
$$\vdots$$
$$\frac{\partial W}{\partial U_n}\frac{\partial U_n}{\partial y_n} - \lambda = 0$$

$$\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1} + \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial x_1} + \dots + \frac{\partial W}{\partial U_n}\frac{\partial U_n}{\partial x_1} - \lambda\, p = 0$$
$$\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_2} + \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial x_2} + \dots + \frac{\partial W}{\partial U_n}\frac{\partial U_n}{\partial x_2} - \lambda\, p = 0$$
$$\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_n} + \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial x_n} + \dots + \frac{\partial W}{\partial U_n}\frac{\partial U_n}{\partial x_n} - \lambda\, p = 0$$

$$M - Y - pX = 0$$

Where this last condition simply requires that the constraint be met at the maximum.

From the first n conditions we obtain:

$$\frac{\partial U_i}{\partial y_i} = \lambda \qquad i = 1,2,\dots,n$$

while from the second n conditions we get:

$$\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_i} + \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial x_i} + \dots + \frac{\partial W}{\partial U_n}\frac{\partial U_n}{\partial x_i} = \frac{\partial W}{\partial U_i}\frac{\partial U_i}{\partial y_i}p \qquad \forall i$$

from which by eliminating $\frac{\partial W}{\partial U_i}$ we get:

$$\sum_j^n \frac{\partial U_j}{\partial x_i} = \lambda p \qquad i = 1,2,\dots,n$$

In the usual pure public goods model $\partial U_j/\partial x_i = \partial U_j/\partial x_k$, $\forall k$, so that the one is left with the single Samuelson condition that equates the sum of marginal rates of substitution to the marginal rate of trasformation.

As an example of the generalization reached with (1), consider the following: it may be the case that X is a public bad, that is $\partial U_i(y_i,x_i,X)/\partial X < 0$. At the same time it may be $\partial U_i(y_i,x_i,X)/\partial x_i > 0$, that is, agents will intentionally consume the individual quantities x_i and they may prefer to make a positive contribution. Clearly, when both the partial derivatives with regard to X and to x_i are positive, we are dealing with the impure public good case.

Most of the theoretical and experimental work in the literature concerns models in which a single pure public good is supplied. In contrast with the model that has received the most attention so far, the model above considers that a consumer's utility depends not only on the aggregate amount of contribution, but also on his own contribution. Many theoretical papers deal with Nash equilibrium in which each consumer assumes that the contribution of others will be independent of his own, while experimental papers have been trying to verify whether behaviour is consistent with the Nash hypothesis.

In the following 2 subsections I will review some basic concepts of the theory of public goods. Namely, Nash and Pareto equilibria and the game theory approach. Needless to say, this material is well established and I do not pretend to be original here.

3.1 Nash Cournot behaviour with public goods.

This is a simple example of the model introduced previously. Consider a consumer whose preferences are defined over two commodities. Let y indicate an ordinary private good, used as a numeraire, and let X be the total available quantity of a public good completely consumed by the agent. The consumer's utility function, $U(y,X)$, is continuous, strictly increasing in y, strictly quasi concave and let it be $C^{(2)}$ everywhere. The agent receives an exogenous money income M_i which can be used either to purchase quantities of the private good or to acquire additional units of the public good denoted by x_i.

In the case of n agents we have x_i, $i = 1,2,\dots,n$ and: $\sum_{i=1}^{n} x_i = X$. Let's denote by $\tilde{X}_i = X - x_i$ the contribution of the rest of the community.

From the point of view of each agent, x_i and $\tilde{X}_i$ are perfect substitutes; apart from the fact that acquiring x_i instead of consuming X involves an opportunity cost in terms of the private good foregone: that is, the agent faces the following constraint with regard to which she maximizes his utility function: $y + px_i = M_i$, with p being the given money price of a unit of the public good.

Since this budget constraint holds with equality, it is possible to eliminate the private good from the utility function and to define an equivalent utility function as follows (let $x_i = x_1$ and $\hat{X}_i = x_2$):

$$U_1(y,X) = U_1(M_1 - px_1, x_1 + x_2) = V_1(x_1, x_2 | p, M_1)$$

Clearly, the variables p and M, still affecting utility, can be regarded as exogenously fixed. In this way we are left with a function that generates a family of indifference curves in the plane (x_1, x_2): $V: X_1 \times X_2 \to \Re$; it is increasing in x_2. The set of all allocations preferred to any reference point is convex as it is the set of feasible consumption vectors defined by the budget constraint. Being an intersection of those two sets, the set of points weakly preferred to any reference allocation is also convex.

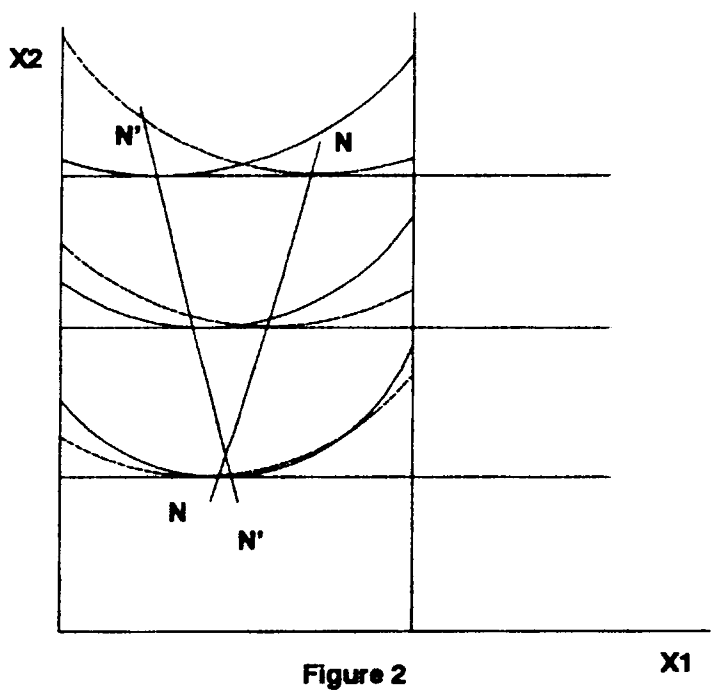


Figure 2

With regard to $x_1 \in X_1$, the set of units of the public good acquired by the consumer, the domain of V is bounded above by $\bar{x}_1 = M_1/p$ which is the value that would exhaust the consumer budget.

The individual choice of x_1 depends on the value of x_2. Following a Nash-Cournot assumption, the agent will choose the most preferred x_1 consistent with his budget constraint given the value of the rest of the community' s contribution. The agent thinks that any adjustment in his contribution will not affect that of the community: then, for any x_2, the individual perceives that alternative feasible allocations lie along an horizontal line (when x_2 is represented on the vertical axis as in the figure above).

The point of tangency between such a line and an indifference curve is the individual optimal choice. Letting x_2 vary, a locus of tangencies is generated: this is called the consumer's Nash-Cournot reaction curve. Points on NN can be determined introducing the following notation:

$$V_{x_i} = \left(\frac{\partial V(x_i^*, x_j^*)}{\partial x_i^*}\right) \quad \text{and} \quad U_{x_i} = \left(\frac{\partial U(M - px_i^*, x_i^* + x_j^*)}{\partial x_i^*}\right), \quad i,j = 1,2 \quad i \neq j, \quad \text{with}$$

$$\mathbf{V}_x = \left(\frac{\partial V(x_1^*, x_2^*)}{\partial x_1^*}, \frac{\partial V(x_1^*, x_2^*)}{\partial x_2^*}\right) \quad \text{and} \quad \mathbf{U}_x$$

analogously. Consider a perturbation vector $\mathbf{dx} = (dx_1, dx_2)$, being $\mathbf{x} = (x_1^*, x_2^*)$ an optimal choice. Along any indifference curve we must have: $\mathbf{V}_x\mathbf{dx} = \mathbf{U}_x\mathbf{dx} = \mathbf{0}$ that is:

$$V_{x_1^*} dx_1^* + V_{x_2^*} dx_2^* = -U_{M - px_1^*} dx_1 p + U_{x_1 + x_2} dx_2 = 0$$

substituting:

$$\frac{\partial V(x_i^*, x_2^*)}{\partial x_1^*} dx_1 + \frac{\partial V(x_i^*, x_2^*)}{\partial x_2^*} dx_2 =$$

$$-\frac{\partial U(M - px_1^*, x_1^* + x_2^*)}{\partial (M - px_1^*)} dx_1^* p + \frac{\partial U(M - px_1^*, x_1^* + x_2^*)}{\partial (x_1^* + x_2^*)} dx_1^* + \frac{\partial U(M - px_1^*, x_1^* + x_2^*)}{\partial (x_1^* + x_2^*)} dx_2^* = \mathbf{0}$$

then, setting $y = M - px_1^*$, the slope of the indifference curve is:

$$\left.\frac{dx_2}{dx_1}\right|_{x = x^*} = -\frac{V_{x_1}}{V_{x_2}} = p\frac{U_y}{U_{x_1 + x_2}} - 1$$

This expression is 0 on the Nash Cournot reaction curve where $pU_y = U_X$. An increment of x_1 yields a marginal benefit of $U_{x_1+x_2}dx_1$, while the associated marginal cost is related to the quantity of the private good that has to be sacrificed; its utility cost is $pU_y dx_1$. The optimum implies equality of marginal cost and benefit.

The idea that the higher the expected contribution by the rest of the community, the lower will be the individual's own contribution, is commonly summarized by the term "free rider" behaviour. That simply means that individual Nash Cournot reaction curves are downward sloping. Actually, this is not the case as long as the public good is a normal good and the real income effect due to a change in the contribution of the rest of the community is sufficiently strong. It is possible to show that the slope of the individual Nash Cournot reaction curve lies between -1 and $-\infty$ if both the private good and the public good are assumed to be normal. The validity of this assumption is an empirical matter. At the theoretical level it ensures the uniqueness of the Nash equilibrium. From the perspective of the existence of such an equilibrium, the Nash Cournot reaction curve is required to be continuous: this property is ensured by the quasi concavity of the utility function and the resulting convexity of preferences in (X_1, X_2). It is possible to say more about easy riding by comparing the outcome of a subscription equilibrium and of an equilibrium that is also a Pareto optimum: this will be done in the next sections.

3.2 Nash Cournot equilibria

Consider an economy with two consumers; their income and preferences may differ but they face the same price. Their maximization problem is the following:

$$\max_{y,X} \quad U^i(y^i, x_i + x_j) \qquad \text{or} \qquad \max_{\substack{x_i \in X_i \\ x_j \in X_j}} \quad V^i(x_i, x_j \mid p, M)$$

$$\text{both s.t.} \qquad y_i + px_i = M_i, \quad i,j = 1,2 \quad i \neq j$$

Clearly $x_i + x_j$ is equal to the total provision of the public good and the rest of the community's contribution is simply the other individual's provision. As before, we get two indifference maps and two reaction curves in the (X_1, X_2) plane. The intersection E of these reaction curves is called a Nash equilibrium (see figure 3).

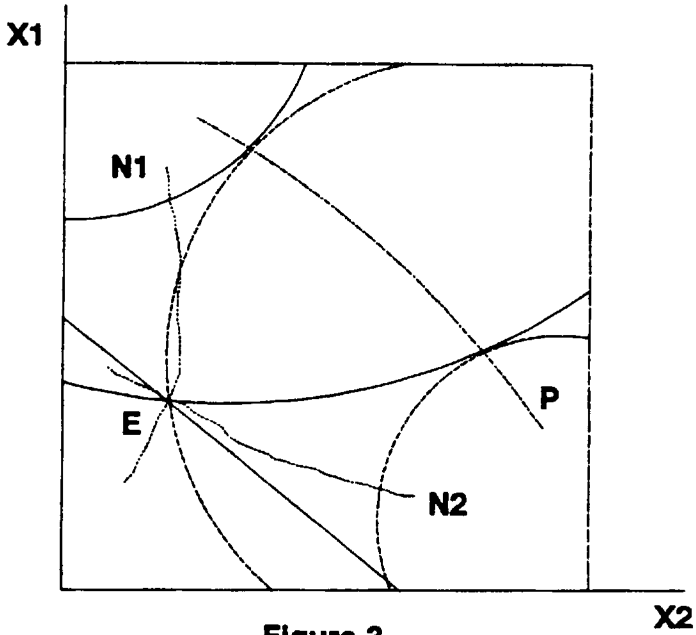


Figure 3

Bergstrom et al. (1986), work with a model in which people are concerned only about their private consumption and the total supply of public goods. They recall that since the work by Warr (1983) a noticeable result concerning the provision of a public good in a voluntary Nash equilibrium is known: the provision of a public good that is provided at positive levels by individuals is independent of a redistribution of income. Bergstrom et al. (1986) (page 29, theorem 1) generalize this result to the case of multiple public goods. Moreover, they point out that income redistribution among contributors will not change the supply of a public good if it does not change the set of contributing consumers[5]. Needless to say, many applications involve income redistributions that do change this set. Following a change in wealth distribution, the decision whether or not to became a contributor are at least as important as the decision of how much to contribute.

3.3 Nash equilibria versus Pareto allocations.

Firstly, a Nash equilibrium is typically not optimal. As an elementary proof look at figure *3* where the shaded area is the set of allocations that dominate *E*.

5 A simple argument in the next section shows that this result is at least misleading.

Secondly, even if individuals are identical there is no guarantee that E is unique. This feature allows for a stable equilibrium to involve identical consumers choosing different allocations. Indeed, the asymmetric behaviour regarding public good contributions underlined by Olson (1971) does not require income differences. It contrasts with the "equal treatment of equals" property of equilibrium concepts in private good economies.

To see this last point in more detail consider a Lindahl equilibrium for an economy with public goods and compare it with a competitive equilibrium. Agents who have the same preferences and the same initial resources are treated identically at every competitive equilibrium: they are indifferent between their own consumption vector and other agents' consumption vector. In this sense the competitive equilibrium is symmetric. On the other hand, Champsaur (1976) shows examples of non-symmetric Lindahl equilibria. Moreover, in Lindahl equilibria the quantities of public goods consumed by agents are equal while individual prices (taxes) differ.

It is also clear from figure *3* that a public good has a tendency to be provided at suboptimal levels. While the bias of an individual to contribute less in the face of higher perceived contributions by the rest of the community may be called "microlevel easy riding", this suboptimality should be termed "systemic easy riding".

In order to get an index of this phenomenon, we need to compare a Nash equilibrium with one of the multiple optimal levels of provision. Let's pick out, between them, the optimum that is consistent with the shares of individual contributions implied at the Nash equilibrium.

Graphically, that means to find the intersection between the ray passing through the origin, the Nash equilibrium point and the locus of Pareto optimal allocations. Let' s call P^* this intersection. Then, the scalar $0E/0P^*$ that lies between *0* and *1* by construction, expresses the Nash equilibrium production of (x_1+x_2) as a proportion of the associated optimal level, and can be regarded as an index of easy riding.

By looking at figure 3 it is also possible to clarify the important point concerning the assumed independence of the optimal level of public good provision from the distribution of private goods. In fact, such a separation between efficiency and distribution is not justified unless the curve PP had a slope of *-1*. In figure 3 the absolute value of the slope of PP is always greater than *1*, and the optimum level of X becomes higher when the first individual has a larger quota of the private good. A simple geometric argument makes clear that moving from below along *PP*, that is choosing points that imply higher utility levels for the first agent, the sum $x_1 + x_2$ increases.

Bergstrom and Cornes (1983) formally derived the necessary and sufficient conditions for the independence between the optimal provision of public goods and the distribution of income sketched above. However, the whole subject seems to be a typical empirical matter. On this ground, Piggott and Whalley (1991) address the same question using an applied general equilibrium model of the Australian economy. They are able to conclude that the traditional separation of allocation and distribution in determining the level of public good supply may be a justifiable empirical simplification, except where very dramatic redistributions are involved.

There has been much discussion of how the relationship between Nash equilibria and Pareto optima is affected by the size of the community. Many analyses are based on the presumption that the tendency to easy riding increases with the size of the set of agents, that is with the expected contribution by the rest of the community. Olson himself (1971, page 35) suggested that *"the larger the group, the farther it will fall short of providing an optimal amount of a collective good"*. More precisely, if a quasi linear form of utility function, say $U(y, x_1 + x_2) = y + f(x_1 + x_2)$, is used, such a result is confirmed. This form implies that income elasticity of demand for the public good is zero. For given values of $(x_1 + x_2)$ the marginal rate of substitution is constant; hence, the slope of the Nash Cournot reaction curev is *-1*. This means that the aggregate equilibrium provision is independent of the size of the community. On the other hand, optimum provision rises with it, hence the consequence of free riding becames greater in larger communities.

4 A survey of the literature on core, externalities and public goods.

The main aim of this survey is to get acquainted with the relevant literature, the traditional tools of analysis, and the acknowledged major concepts in the field. Articles are presented chronologically, in order to focus on the progression of the ideas. It seems that no pertinent contributions are left out. As we will see, there have been periods in which the theoretical discussion on public goods in the general equilibrium framework was particularly fashionable. Some of the seminal articles on the subject are not explicitly considered: those by Samuelson (1954), Debreu and Scarf (1963), and Shapley and Shubik (1969) are among them. They are well known and their results are continuously referred to by subsequent literature.

4.1 Lindahl's solution and the core of an economy with public goods, 1970

Summary

The main purpose of this paper by Foley is to define the theory of resource allocation with public goods. Moreover, it concerns the relationship between the Lindahl's solution and the core. Among others, the definition of a "public competitive equilibrium" is given. The existence of a Lindahl equilibrium is shown, as well as the proof that any Lindahl equilibrium is in the core as long as the production of the public commodity exhibits constant returns to scale and the personalized prices are non negative.

The economic system assumed by Foley includes a production sector. One of the assumptions for the production (namely B.5) is that no public good is necessary as a production input. It is indeed a strong assumption that is avoided in later papers.

The presence of public goods makes a vector of taxes take place in the definition of the public competitive equilibrium, together with the usual concepts of feasible allocation and price system. The equilibrium implies that there is not another public sector proposal with taxes to pay for it, that leaves every individual better off.

The discussion about prices follows the idea of the Samuelson condition: the sum of the individual price vectors is a social price for the public goods and it establishes the indication to be kept by profit maximizing producers.

For what follows we need some notation, assumptions, and results that are often used in the analysis of a private good economy. An economy consists of n individuals, indexed by superscripts. A bundle of commodities is a vector $(y_1,..,y_r; x_1,..,x_m)$ where the first r goods are public and the other m are private goods. Each individual has a convex weak preference ordering R_i over the space of commodity bundles; notation P means strictly preferred. Each individual has an endowment of private goods e^i, with $\sum^n e^i = e$; no public goods are initially owned. Production is denoted by a vector with inputs negative and outputs positive. Y, the set of all technically possible production plans, is a convex cone. Public goods cannot be used as inputs.

In order to define a Lindahl equilibrium the concept of lump-sum transfer is needed. Let y, x and e indicate respectively the vector of public goods, private goods and endowments of an agent i matching each Pareto optimum. Let also p_y^i and p_x be the corresponding price vectors. Then,

$$L^i = p_y^i \cdot y + p_x \cdot x^i - p_x \cdot e^i$$

is a *lump-sum transfer*: the difference between the total consumption value and the endowment income. Now, a competitive equilibrium requires that all the lump-sum transfers must be zero: the value of public good received by each individual must be equal to the total tax she pays.

Definition. A *Lindahl equilibrium* with respect to e is a *feasible allocation* (see below) $(y;x^1,\dots,x^n)$ and a price system $(p_y^1,\dots,p_y^n;p_x) \geq 0$ such that:

$$(a) \qquad \left(\sum_{i=1}^{n} p_y^i;p_x\right)\left[y;\sum_{i=1}^{n}(x^i-e^i)\right] \geq \left(\sum_{i=1}^{n} p_y^i;p_x\right)(\bar{y};\bar{z}) \quad \forall \quad (\bar{y};\bar{z}) \in Y$$

$$(b) \qquad (\bar{y}^i;\bar{x}^i)\mathrm{P}_i(y;x^i) \quad \Rightarrow \quad p_y^i \cdot \bar{y}^i + p_x \cdot \bar{x}^i > p_y^i \cdot y + p_x \cdot x^i = p_x \cdot e^i$$

The proof of the existence of the Lindahl equilibrium is based on the idea of constructing a private economy to which an existence lemma applies and of showing that its quasi equilibrium is actually a Lindahl equilibrium. In the subsequent definition of the core the concept of improving allocation encompasses two significant features: firstly let $\bar{x}^i$ be agent i's vector of consumption of private goods after the improving coalition S has been established, with $i \notin S$. The definition allows for $\bar{x}^i = 0$, which presumably cannot be caused by a reallocation within the coalition. Secondly, the vectors of consumption of public goods in the two allocations have to be different. Now, why should we introduce exclusion for the public good ? It would be interesting to allow for a reallocation between $i \in S$ concerning only private goods, considering fixed the overall individual contribution to public good provision.

The introduction of exclusion for a pure public good seems contradictory. In fact this is due to the interpretation of the concept of domination in the presence of a public good. The feasibility condition is usually understood to mean that members of a coalition cannot enjoy public goods produced by nonmembers. This is also the reason behind the intuition that improving an allocation is much harder when public goods are present. For a coalition has to rely entirely on its own members' contributions to the public good, and forego the benefits from the quantity provided by others. In the following Section it is proved that a Lindahl equilibrium with regard to the endowment is in the core with regard to the same endowment.

In his conclusion, Foley points out the problems related to the collection of information for a Lindahl equilibrium to be the target of any pragmatic political process. Moreover, he stresses the difficulty mentioned above: the definition of domination makes improving an allocation difficult because a would be coalition has to produce its own public goods and lose the benefits of the externalities generated by the rest of the economy.

The notion of public competitive equilibrium

The model built around the concept of competitive equilibrium leads to the analysis of the relations between the same competitive equilibrium and Pareto optimal states. Among the many assumptions of this model there is one particularly inadequate to the case of the presence of public goods: when public goods are present it is not generally true that the sum of the commodities consumed by the individuals is less than or equal to the total produced. Indeed, consider the production level of a pure public good as determined by the quantity of private goods used as inputs: it does not matter how many agents enjoy the consumption of the public good, that is, the maximum consumed "quantity" is not determined by the production level.

In a competitive economy preferences determine consumers' behaviour and producers seek to maximize profits. Since a government's main task is to provide public goods, there must be some similar rule of governmental behaviour. In the most general sense, the government's choice about financing and producing public goods derives from consumers' preferences expressed through some political mechanism.

We can conceive a government making different *public sector proposals*. A proposal for the public sector is a pair composed of a bundle of public goods and a vector of personal taxes. The consumer makes his decision by considering a proposal and the vector of private goods he can buy with his after tax income. It is worthwhile stressing that the demand for private goods does in general depend on the available bundle of public goods. For example, the demand for privately provided transportation services depends on the publicly provided ones (however, public transportation services are usually impure public goods, since exclusion is possible). If a new proposal is made, which may have smaller amounts of public goods and less taxes, the consumer will consider the private good bundles that, when combined with the proposed bundle of public goods, make him better off than he is already. If his after tax income associated with the new proposal allows him to buy one of these bundles, the consumer will favour the new proposal. Clearly, a bundle of private goods that would leave the consumer better off may be smaller than the old one, reflecting the fact that public goods may satisfy some of his needs better than private goods.

Definition. A feasible allocation is a set of vectors $(y^1,x^1;\ldots;y^n,x^n)$ with $(y^i,x^i)\geq 0 \quad \forall i$, one for each consumer, such that:

$$(a) \qquad y^1 = y^2 = \ldots = y^n = y$$

$$(b) \qquad \left(y, \sum_{i=1}^{n} x^i - e\right) \in Y$$

To any feasible allocation there corresponds a *net trade* $z^i = x^i - e^i$ for each consumer. Statement (b) above can also be written $(y, \sum^n z^i) \in Y$.

Statement (a) says that everyone consumes all the production of public goods; since it holds, we can write a feasible allocation as $(y;x^1,\ldots,x^n)$.

A public competitive equilibrium is a situation in which producers are maximizing profits, consumers are maximizing their utilities given their after tax income (they are buying the bundle they most prefer among those that they can afford), and there is no proposal that is favoured by everyone. Formally:

Definition. A public competitive equilibrium (PCE) is a feasible allocation $(y;x^1,\ldots,x^n)$, a price system $p = (p_y;p_x)$, and a vector of taxes $(t^1,\ldots,t^n)$ with $p_y \cdot y = \sum_{i=1}^{n} t^i$, such that:

$$(a) \quad p \cdot \left(y; \sum_{i=1}^{n}(x^i - e^i)\right) \geq p \cdot (\bar{y};\bar{z}) \quad \forall \quad (\bar{y};\bar{z}) \in Y$$

$$(b) \quad p_x \cdot x^i = p_x \cdot e^i - t^i \quad \text{and} \quad (y;\bar{x}^i)\mathrm{P}_i(y;x^i) \Rightarrow p_x \cdot \bar{x}^i > p_x \cdot x^i$$

$$(c) \quad \neg\, \exists \quad (\bar{y};\bar{t}^1,\ldots,\bar{t}^n) \quad \text{with} \quad p_y \cdot \bar{y} = \sum_{i=1}^{n} \bar{t}^i \quad \text{such that} \quad \forall i$$

$$\exists \quad \bar{x}^i \quad \text{with} \quad (\bar{y};\bar{x}^i)\mathrm{P}_i(y;x^i) \quad \text{and} \quad p_x \cdot \bar{x}^i \leq p_x \cdot (e^i - \bar{t}^i).$$

Definition. A Pareto optimum is a feasible allocation $(y;x^1,\ldots,x^n)$ such that there is no other feasible allocation $(\bar{y};\bar{x}^1,\ldots,\bar{x}^n)$ with $(\bar{y};\bar{x}^i)\mathrm{R}_i(y;x^i)$ for all i.

Theorem: a public competitive equilibrium is a Pareto optimum.

Proof: (Refer to parts (a), (b), and (c) of the definition of PCE) Let $(p;y;x^1,..,x^n;t^1,..,t^n)$ be the public competitive equilibrium. Suppose it is not a Pareto optimum: then there is a feasible allocation $(\bar{y};\bar{x}^1,..,\bar{x}^n)$ such that:

$$(\bar{y};\bar{x}^i)P_i(y;x^i) \quad \forall i \tag{1}$$

Either $\bar{y}=y$ or $\bar{y}\neq y$. First take the case $\bar{y}=y$. Part (b), saying that $p_x \cdot \bar{x}^i > p_x \cdot x^i$ for all i, implies that:

$$p_x \cdot \left(\sum_{i=1}^{n} \bar{x}^i\right) > p_x \cdot \left(\sum_{i=1}^{n} x^i\right), \quad \text{or,} \quad p \cdot \left(\bar{y}, \sum_{i=1}^{n} \bar{x}^i\right) > p \cdot \left(y, \sum_{i=1}^{n} x^i\right)$$

But this contradicts part (a) if $(\bar{y};\bar{x}^1,..,\bar{x}^n)$ is a feasible allocation as a consequence of the assumption. Therefore we must have $\bar{y}\neq y$. By part (a), since $(\bar{y},\sum_{i=1}^{n}\bar{z}_i)\in Y$, it follows that

$$p \cdot \left(\bar{y}, \sum_{i=1}^{n} \bar{x}^i\right) \leq p \cdot \left(y, \sum_{i=1}^{n} x^i\right) \tag{2}$$

That is, the total expenditure on the preferred bundle is smaller, which leaves room for a unanimously favoured public sector proposal. It can be constructed as follows. Since (1), let

$$\bar{t}^i = t^i + p_x x^i - p_x \bar{x}^i \tag{3}$$

Then, by (2):

$$\sum_{i=n}^{n} \bar{t}^i = p \cdot \left(y, \sum_{i=1}^{n} x^i\right) - p_x \cdot \left(\sum_{i=1}^{n} \bar{x}^i\right) \geq p_y \cdot \bar{y} \tag{4}$$

Or, by rearranging (3):

$$p_x \cdot \bar{x}^i \leq p_x \cdot x^i + t^i - \bar{t}^i \tag{5}$$

The proposal $(\bar{y};t^1,..,t^n)$ with $t^i \leq \bar{t}^i$ and $p_y \cdot \bar{y} = \sum_{i=n}^{n} t^i$ allows each individual to get to $(\bar{y},\bar{x}^i)P_i(y,x^i)$ which contradicts proposition (c).

Therefore, no feasible allocation satisfying (1) and proposition (b) exists, and the PCE must be a Pareto optimum. Q.E.D..

Neither the convexity of preferences nor that of the production set Y are used to prove this theorem. Nevertheless, assumptions such that of the independence of the individual preference ordering of the consumption and behaviour of other consumers, are needed. Further, the fact that Y is a convex cone and that consumer preferences are convex, is needed for the following theorem. (Foley, 1967).

Theorem: To any Pareto optimum $(y;x^1,..,x^n)$ and any set of endowments $(e^1,..,e^n)$, with $e > 0$, there correspond a price system p and a vector of taxes $(t^1,..,t^n)$ such that $(p;y;x^1,..,x^n;t^1,..,t^n)$ is a public competitive equilibrium.

If there were no public goods, there would be a no-tax competitive equilibrium, still associated with a Pareto optimum, that could be used as a reference point for income redistributions. In the case of public goods there is not such a base point, since all Pareto optima involve taxes -unless there is one that involves no public goods. In a private good economy taxes only transfer wealth from one consumer to another; on the other hand, those taxes are used to finance the production of the public goods whose effect on each consumer's real income depends on his own preference for the particular good: a national park that is run with tax revenues benefits a bird watcher much more than an urban professional that loves watching soccer on television during his weekends. The redistribution effect of taxes in a public good economy is not only due to the intrinsic properties of the tax system but also to the quantity and the quality of public goods supplied in relation to consumer preferences. In this case some individuals are forced to give up part of their wealth in the form of a pecuniary asset, while some others receive it as (partially-free) goods and services.

In general, we can say that there are two sources of conflict in the public sector. Firstly, given the amounts and the types of public goods to be produced, each individual wants to contribute as little to the total cost as he can, shifting as much as possible to other consumers. Any shift of taxes from one individual to another is clearly a gain for the first and a loss to the second; then, there is a monetary redistribution involved. Lastly, given the distribution of taxes, individuals with different tastes and after tax income will like the government to spend on different public goods; then, it is clear that a non-monetary redistribution is also associated with any provision of public goods. Moreover, individual preferences on the space of proposals may be not "nicely behaved" in a sense to be specified (see the example on cars and public transport).

In other words, the introduction of public goods and associated taxes in a private good economy causes a redistribution of income which is due to two factors: the first, which I call *pecuniary effect*, depends directly on the cost sharing scheme, that is, on the chosen tax vector. The second, the *non-pecuniary effect*, is due essentially to the heterogeneity of preferences: since the utility derived from the consumption of a certain level of a public good varies across individuals[6] according to their preferences, the introduction of a public good benefits some individuals more than others.

As a consequence of this duality, the analysis should have two stages: given the vector of public goods, we can determine the tax vector and its effects; given the tax vector, we can study the proposed public good bundle. This also means that, if we want to analyse possible outcomes of a voting process over tax schemes -when also the public good vector is under discussion- it is simpler to compare public good bundles which have the same non-pecuniary redistribution effect: otherwise, we should assume not only that individuals have complete information about pecuniary redistribution effects of different tax systems, but also that they can compute non-pecuniary redistribution effects of different public goods bundle.

However, we can also use the following assumption: public good bundles only differ in the relative quantities of each public good and not in their type. This could means, for example, that a higher tax revenue only implies a proportionately higher level of provision of each public good, and not a larger number of provided public goods.

Another problem rises from the fact that after-tax income is a parameter of each consumer when he decides upon the optimal amount of public goods: it is clear that, when individual after-tax incomes change as a consequence of the vote, so will individual preferences for the public goods.

With regard to the vector of taxes which guarantees the correspondence between any Pareto optimum and a PCE, we only know that it exists. So far the analysis could imply an irregular incidence of taxation over the members of a society, where "irregular" means a tax structure that is completely different from those used

6 Recall that neither quantity adjustment nor personal (efficient) taxes are allowed: the first is impossible because of the very nature of a pure public good, while the Lindahl solution is too much demanding in terms of information.

in the real world. Foley (1967) proved that a constant proportional tax always exists that satisfies the last theorem. The existence of a public competitive equilibrium for any arbitrary tax structure (and then for a progressive income tax as well) has been proved in a different setting by Mantel (1975).

Now, let us go back to the problem of the plausibility of the idea of a public competitive equilibrium. As we have seen before, a consumer favours a public sector proposal if it appears to increase his real income at present prices.

The problem with the concept of PCE is that it only applies when all individuals agree about the effect of the new proposal on their real income: it rules out only a specific type of incompatibility of the allocation with the political mechanism, that is, the existence of a unanimously favoured alternative proposal. Clearly, the political mechanism in the real world is able to mediate between different preferences of individuals and to guarantee stability on the basis of a different kind of criterion than that of unanimity: typically, non-unanimously favoured proposals are the ones actually adopted.

Then, it appears of great interest to study those proposals that cannot be overthrown by any other according to some political decision rule: they have an intrinsic stability whose reasons we fail to understand using our model of behaviour. Let us try to characterize such proposals: given a class A of all alternative proposals and a political mechanism as a decision rule, the set of stable proposals A^* should be a proper subset of A (it may be a singleton) whose elements cannot be upset by any other element of that class through the political decision rule.

Using this definition it is evident that unanimity is itself an undesirable political rule since the class of all alternatives is stable against it. Every Pareto optimum would be stable under that rule if technology and tastes do not change, and it would not help us in finding any unique final distribution of welfare. If the society starts from a PCE the political mechanism would produce no decision at all.

If the rule of majority is applied to the class of all alternatives, this class will be unstable. To understand this it suffices to consider a given proposal with its associated production of public goods and its tax system. Then form a new tax system by diminishing the taxes of the 51% of individuals that are paying the highest taxes, and by increasing the taxes of the others to cover the loss: the new proposal will pass, as any other proposal of this kind will do with regard to every possible initial situation.

Even restricting the class A to the class of all progressive income taxes, we have no stability against majority rule (Foley 1967, p. 86). As a consequence of these problems, and without trying to model the political process, we cannot go further than the following definition, which is an modification of the idea of the PCE.

Definition. A PCE is *realizable* under some specified political mechanism if the public sector proposal cannot be defeated by any other proposal within that mechanism, assuming that consumers support or oppose proposals as described above.

Condition (c) in the definition of PCE catches a particular kind of stability within a political mechanism: the stability with regard to unanimity. Then, a PCE is an element of the general class of realizable public competitive equilibria.

Summing it up, the equilibrium must satisfy the usual conditions of profit maximization by firms, of consumer maximization over private goods, and of equality of demand and supply. Moreover, the public sector proposal has to be stable under the particular political mechanism. This means that, at equilibrium prices, no other tax and public goods vector can be chosen by the political mechanism. The only necessary condition for a stable proposal is that no other proposals exist which leave *all* individuals better off: this means that the political mechanism is quite unspecified and that it reduces to the generally unfeasible rule of unanimity [7].

It is well known that the concept of core, applied to a private good economy, gives a strong characterization of the idea of stability for some allocations. It is also well known that the core does not serve the analysis in the same way when public goods are present, since the difficulty found in generalizing it: we are not able to assess what a coalition can do by itself when economic externalities due to public goods are present. The core of an economy in which coalitions are allowed to improve allocations by producing both private and public goods with their own resources is certainly large and does not shrink to some single point or even to a small set of allocations. The reason is that the public sector is a source of surplus that cannot be imputed to single individuals or coalitions: the gain each individual makes by joining a coalition is very large and this makes improving very difficult.

7 Also the Lindahl model uses the unanimity rule; here tax rates differ across individuals matching everybody's preferences for public goods and then the equilibrium is a Pareto optimum. However, it is well known that the quantity of information needed to implement the allocation mechanism and the incentive to misreveal preferences make the Lindahl model unrealistic.

It is exactly this problem that would prompt a different approach: is there any connection between the core of a public good economy and the tax system? As I said before, the main question is the following: if there are externalities, then one coalition cannot make a decision without affecting another and the idea of what a coalition can accomplish for itself becomes subtle. It is actually the natural formalization of this idea that allows for the appropriateness of the theory of the core for private good economies. Since there is not a plain formalization of the same concept in a public good economy, we could study the core of an economy in which consumers consider their after-tax income as given, after a vote is performed. Allocations only have private goods as elements and the core would be (presumably) a normal private good economy core. Then, we could ask what is the influence of the political decision rule and of the income structure of the society on this core. The influence is conveyed through the tax system, chosen by, say, a voting mechanism: it determines individual after-tax income which in turn is the constraint on maximization of utility from private goods. All this implies a different equilibrium concept: we will see how this task has been accomplished more than twenty years after Foley's paper.

4.2 External economies and cores, 1971

Essentially, Rosenthal presents an example which deals with external economies and cores. He defines firstly the usual concepts in the theory of the core. Then he allows an individual's utility to be increased through the production of goods by coalitions of which he is not a member. This matches the case of a public good without the restrictions made by Foley imposing the exclusion property. Rosenthal denies what Shapley and Shubik claim about the fact that, under certain assumptions, a non-empty core also exists in the presence of external economies.

Rosenthal, working with utility functions and characteristic functions in a very simple economy, shows that the use of the domination relation can lead to cores that are in sharp contradiction to reasonable behavioural assumptions about the people involved.

Finally, he gives some alternative definitions of domination-like relations, based on the strategies available to each coalition. They include concerns about reasonable actions by nonmembers of the coalition. Varying the idea of "reasonable" from individual to individual and including considerations of group rationality, different cores are obtained as different sets of undominated outcomes: it is shown that these cores do not all coincide for economies with beneficial externalities.

The differences between the various cores are due to discordant views of the significance of threats made by individuals. Rosenthal claims that his concepts at least clarify the analysis of problems involving threats.

4.3 The core and the Lindahl equilibrium of an economy with a public good: an example, 1972

This paper by Muench starts with an accurate analysis of the Edgeworth conjecture about the shrinking of the core, both in competitive and public goods economies. The process whereby the core of a public goods economy shrinks to the set of Lindahl equilibria, is called the "modified Edgeworth conjecture" by Muench. A part from the usual criticism on incentive grounds of the Lindahl process (see Section 4.5 , first block), he argues that, in general, a strong objection to it would be to prove that the modified Edgeworth conjecture is not true.

The main aim of Muench is to provide an example which shows that the set of core allocations is not identical with the Lindahl equilibrium. This means that there is more room for bargaining that the set of Lindhal equilibria would indicate. Firstly Muench proves that a Lorenz curve (where income is replaced by utility) can be associated with every allocation belonging to the core and to the set of Lindahl equilibria. Then he uses it to compare allocations. One interesting result is that all core allocations have the property that the consumption of the public good is the same as that for the Lindahl equilibrium. Thus, the only question left with regard to allocations, is the distribution of the cost of the provision of the public good among the agents. It is worth considering that the model treats each consumer's allocation of the private good as an infinitesimal, and his allocation of the public good as a macro quantity: this feature seems to correspond to the idea lately explored by Hammond et al. (1987) (see also Section 4.9). Finally, let's recall that in Muench's opinion, different ways of getting a "limit economy", or different definitions of the core, might lead to different conclusions, that is, to show that the Lindahl equilibrium has the same position in the theory of public goods economies as the competitive concept occupies in the theory of private economies.

4.4 Theory of value with public goods: a survey article, 1972

This article by Milleron is divided into five parts: - in the first Section there is a clarifying discussion of the concept of public good - the second Section is devoted to the study of pareto optimality in economies with public goods - the third Section analyzes various concepts of equilibrium for the same economies: the Lindahl equilibrium, and the Nash Cournot equilibrium, among others - the fourth part is concerned with the problem of defining a core in presence of public goods and with the possible equivalence between the core and the set of Lindahl equilibria - finally and less interestingly for our subject, some procedures for the execution of an optimal and just program are reconsidered.

There are three key concepts in the theory of public goods: the exclusion of use, the individual or collective concern, and free disposal. A combination of these three criteria yields a taxonomy of private and public goods. Public goods for which there is no free disposal are called "environment variables": they have an increasing or decreasing effect on the utility of agents but they are not consumed in the usual sense.

With regard to excludability, let's write the scarcity constraints for the two cases of pure private goods and of pure public goods. Suppose the following: there are n agents (either consumers or firms) in the economy; r_h is the sum of initial resource and production of a private commodity h; $z_{hj} \in \Re_+$ is the consumption of the commodity h by agent j. Then we can represent the scarcity constraint, respectively considering free and not free disposal, as follows:

$$\sum_{j=1}^{n} z_{hj} = r_h$$

$$\sum_{j=1}^{n} z_{hj} \leq r_h$$

On the other hand, in the case of a pure public good there will be as many constraints as there are agents:

$$z_{kj} = r_k \qquad j = 1,2,\ldots,n$$

$$z_{kj} \leq r_k \qquad j = 1,2,\ldots,n$$

An intermediate case can be represented with the following formulation:

$$r_q = (1-\mu)z'_{qj} + \mu \sum_{j}^{n} z''_{qj} \qquad j = 1,2,\ldots,n \qquad \mu \in [0,1]$$

Where z' accounts for the consumption by each agent due to the non escludability property and z'' is the direct consumption.

In the second part Milleron shows how the Samuelson condition concerning the allocation of public goods may be derived from a pareto optimal program. The problem is to find a price vector compatible with optimality. Briefly, he proceeds as follows:

The basic model with public goods consists of an economy E defined as follows: as many consumption sets as there are agents, that is m; m preference relations; a vector of initial resources; n production sets. It is worth noting that here public goods are allowed to be inputs for production. Then, definitions of attainable state for E and pareto optimality are straightforward.

Following an idea by Foley, the next step is the definition of an economy E' with only private commodities. It is such that there is a one-to-one correspondence between feasible states in it and feasible states in E. This is proved by lemma 2.1. As a consequence, any pareto optimal state in E may be associated with a Pareto optimal state in E'. The existence of a price vector compatible with a given pareto optimal state for E, is then proved by means of the equivalence with E' using a theorem of Debreu.

Milleron emphasizes that it is possible to characterize Pareto optimal solutions without introducing the personalized prices of the previous approach. This is done along the idea of public competitive equilibrium introduced by Foley and generalized here. The problem is to find a vector of taxes *tsub1*,*tsub2*,...,t_m such that the budgetary equation for the production of public goods is satisfied: $p_Q y_Q = \sum_{i=1}^{m} t_i$. The definition of public competitive equilibrium is modified and the equivalence with pareto optimality is proved in two stages. For the proof that in a Pareto optimal state, for any distribution of endowments and any system of shares, there exists a vector of prices and a vector of taxes such that it is a public competitive equilibrium, Milleron does not use the properties of the associated private economy; instead, the theorem is proved directly. At the end of this part a question is raised concerning the fact that since there is no constraint on the taxes, the set of public competitive equilibria can be very large: is it possible to define a stronger kind of equilibrium?

The third part gives an affirmative answer through a detailed analysis of the Lindahl equilibrium, with definitions and existence proofs. In order to cope with externalities in consumption of private goods, the duality approach is also followed, using the properties of the indirect utility function. It leads to an alternative proof of the existence of the Lindahl equilibrium.

Since it is easy to find examples in which public goods are provided on a free subscription basis, the properties of a Nash Cournot equilibrium are sketched. On this basis, the usual assumption is that each agent optimizes his utility considering the decisions of the others as given. The existence of the Nash Cournot equilibrium is proved following Castellan (1970). Milleron underlines that a general proof of the property according to which the utility of agents increases in passing from Nash Cournot to Lindahl equilibrium as the number of agents increases, is not available.

The fourth part is concerned with the problem of the core of an economy with public goods. It has to be noted that in order to study the allocative decisions concerning public goods, it is not sufficient to analyze an exchange economy. On the other hand, at the time of the article, the theory of the core did not include an explicit representation of production. Then, Milleron assumes a simple simulation of the production sector. There is just one technology and it is available to any coalition. Regarding the initial resources of the public good, the assumption is that any coalition can use the totality of them. This does not mean that produced public goods can also be consumed by any coalition. With regard to this point I make the same observation I did earlier: in particular when dealing with pure public goods it seems unnatural to allow exclusion. However, let's clarify the model.

As before, the indexes L and Q refer respectively to private and public goods; the set Q is partitioned into E, the subset of public goods for which there is no free disposal, and F, for which free disposal property holds. Superscripts "+" and "-" denote respectively output and input vectors. The initial distribution of resources is denoted by $\{\omega_i\}$.

Let $C \subset M$ be a coalition of the set of all possible coalitions. Define a consumption and production program as:

$$\{(x_{Li}, x_{Qi}),\quad i \in C \quad ; \quad (y_L, y_Q^+, y_Q^-)\}$$

Now, with X_i and Y respectively the consumption set of agent i and the overall production set, Milleron says that C can enforce a program if $\forall \quad i \in C$ the following conditions hold:

$$(x_{Li}, x_{Qi}) \in X_i$$

$$(y_L, y_Q^+, y_Q^-) \in Y$$

$$\sum_{i \in C} x_{Li} = y_L + \sum_{i \in C} \omega_{Li}$$

$$\forall i \in C \quad : \quad x_{Ei} = \omega_E + y_E^+$$

$$x_{Fi} \leq \omega_F + y_F^+$$

$$y_E^- = \omega_E + y_E^+$$

$$y_F^- \leq \omega_E + y_F^+$$

Then, a consumption program $\{x_{Li}^*, x_{Qi}^*\}$, $\forall i$ is improved upon by a coalition C if C itself can enforce the following program:

$$\{(\hat{x}_{Li}, \hat{x}_{Qi}), \quad i \in C \quad ; \quad (\hat{y}_L, \hat{y}_Q^+, \hat{y}_Q^-)\} \quad \text{such that :}$$

$$\forall i \in C, \quad (\hat{x}_{Li}, \hat{x}_{Qi}) \geq_i (x_{Li}^*, x_{Qi}^*)$$

with at least one strict preference relation. Clearly, a program is in the core if its consumption program cannot be improved upon by any coalition.

The main question is the nonemptiness of the core; since it is proved that any Lindahl equilibrium is in the core, the key for the nonemptiness is the assumption of the existence of the Lindahl equilibrium. In order to prove that the Lindahl equilibrium is in the core, a Lemma saying that the total profit of the single enterprise is nonpositive at the Lindahl equilibrium is needed. Moreover, the free disposal assumption must hold for all public goods and all consumers.

The following question is whether or not the core and the set of Lindahl equilibria are equivalent. A first method suggests taking the asymptotic point of view, that is, to let the economy replicate. Milleron says that, a priori, one might doubt that such a technique could work, since the production of public goods tends to infinity as long as the initial resources go to infinity. It seems that this difficulty can be defeated allowing the preferences of the consumers to change when the number of agents changes: for example this can be expressed by saying that the appreciation of the consumers about the amount of the public good is dependent on the size of the economy. Otherwise, it is possible to think of the agent becoming increasingly small when the economy is replicated: so then, do her resources, avoiding the problem of increasing production of the public good.

The second method is to work with a measure space of economic agents. But it is true that the construction of an associated private economy is in this case more difficult, since its commodity space is not a finite dimensional space anymore.

Then Milleron recalls the problem of the definition of coalition in the presence of a public good that Foley also points out. In the case of a public good economy the restrictions induced on the set of feasible allocations by the nonblocking condition are weaker than in a private economy. To elucidate this point, he gives an example of an n-replica economy which leads to the statement that with the usual definition of the core it is not possible that an equivalence theorem holds in an economy with public goods. Once again, the focus was on the problem of the notion of "improving upon".

4.5 A generalization of the pure theory of public goods, 1973

This paper is about the concept of public good when exclusion or crowding occurs. Introducing the problem in Section 2, I already remarked that many goods are public, but few are purely public. Ellickson reviews the theory of pure public goods in terms of a three person economy and then introduces crowding to show cases where the Lindhal allocation is outside the core and where the core is empty. This seems to support Samuelson's criticism concerning the relevance of Lindahl equilibrium to the political economic decision making process. (See the chapter by Samuelson in Margolis and Guitton (1969)). The author examines the nature of the nonconvexity introduced by crowding and exclusion in the aggregate technology set. He studies also the problem of sharing the cost of public goods and finally relates his concept of exclusion and jurisdictions to Buchanan's theory of clubs (See, among others, Buchanan 1968).

Since the blocking ability of a coalition is reduced by the presence of a crowded public good that has to be produced by its own resources, it seems natural to consider the possibility that the coalition can introduce exclusion: because exclusion reduces crowding, the blocking ability of any coalition is enhanced. By excluding nonmembers it might be possible to produce a given level of public good at lower cost. Following this idea, Ellickson introduces crowding in his three person economy and proves that if public goods are crowded, then the Lindahl equilibria may not be in the core. Moreover, he proves that if public goods are crowded, the core may be empty.

The discussion of exclusion is interesting: as is well known, Musgrave and Musgrave (1984) provide a taxonomy of goods using the two concepts of excludability and rivalness; they get four different types of public good. Ellickson argues that from the point of view of the analysis of the allocations in the core, the four cases reduce to just two: the first is the pure public good case, which involves no exclusion ability and no crowding or both. The second is the crowded public good with exclusion. The two cases of crowded and uncrowded public goods where exclusion is impossible and the uncrowded case with exclusion, have precisely the same core.

4.6 The core of a public good economy, 1974

Richter affirms that the crucial issue in the theory of the core is how to formulate what a coalition can achieve for itself. This is exactly the gray area when dealing with public good economies, or with economies involving externalities. The problem is that unlike the situation in a private good economy, it is not possible to identify a set of utility levels achievable by a coalition independently of the choices of agents not in the coalition.

For the author it is clear that by the definition of a pure public good, the agents belonging to a coalition are able to consume the public good produced by others. This is precisely the idea that I have been developing while reading other papers that simply denied it by assuming exclusion. It is evident that a coalition can attain for itself all utility vectors associated to private and public goods produced using its own initial endowments and to the consumption of the public goods produced by other coalitions.

In order not to allow the forming of the coalition, the complementary one has to threaten to produce no public goods since by doing so it will abate any free consumption of its public goods by the members of the new coalition. This seems to me quite illogical: first of all, the member of the complementary coalition may suffer remarkably if such a threat is carried out. Secondly, as long as the good is a pure public one (no congestion appears, for example), there is no loss for the coalition if other agents "consume" it (recall Milleron's discussion on consuming environment variables). But let us follow Richter.

 DOI: 10.2870/7482

Foley (1970) and Muench (1972) found that, under this assumption on the behaviour of the complementary coalition, the core is quite large and does not shrink as the number of agents becomes larger. The reason is that any agent makes a large gain joining with others in producing the public good: costs are shared but everyone consumes the total amount produced. Following an idea of Rosenthal (1971), Richter defines some rationality constraints that reduce the threatening power of the complementary coalition. They enlarge the set of utility vectors which a coalition can attain for itself, hence there are more utility vectors for the economy which each coalition can improve upon: the core is narrowed.

It is interesting to summarize some basic definitions of Richter's work. Let N be the set of the agents in the economy; $C \subset N$ is a coalition and $\overline{C}$ is the complementary coalition; E^C is the euclidean space of dimension equal to the number of agents in C; $V_C \subset E^C$ is the set of possible utility levels achievable by C; $v = (v_1, \dots, v_n)$ is the vector of utility levels achieved by agents in N; for $v \in V_N$, v^C is its projection onto E^C. Clearly C can improve upon v if $\exists \quad \hat{v} \in V_C$ with $\hat{v} \gg v^C$. Besides that, the notation needed is the following:

Y, a nonempty convex cone, is the production technology available to any C; ω^i is the endowment vector of private good for agent i; $x_C \geq 0$ is the m-vector of public goods produced by C; y_C^i is the k-vector of private goods consumed by $i \in C$; y_C is the vector of k-tuples y_C^i arranged in ascending order (from the left) of i; (x_C, y_C) is an allocation for C if agent $i \in C$ consumes the bundle (x_C, y_C^i); $u_i(x_C, y_C)$ is the utility received by agent i from the allocation (x_C, y_C); $u^C(x_C, y_C)$ is the vector of utilities $u_i(x_C, y_C)$ for $i \in C$ arranged in ascending order of i.

Now, (x_C, y_C) is a *feasible distribution* for C if $(x_C, \sum_{i \in C}(y_C^i - \omega^i)) \in Y$.

An allocation $(x_{\overline{C}} + x_C, y_{\overline{C}})$ for $\overline{C}$ is *group rational relative to* x_C if $(x_{\overline{C}}, y_{\overline{C}})$ is a feasible distribution for $\overline{C}$ and there does not exist a feasible distribution $(\hat{x}_T, \hat{y}_T)$ for any $T \subset \overline{C}$ such that $u^T(\hat{x}_T + x_C, \hat{y}_T) \gg u^T(x_{\overline{C}} + x_C, y_{\overline{C}})$

If $T = \overline{C}$, then $(x_{\overline{C}} + x_C, y_{\overline{C}})$ for $\overline{C}$ is *pareto efficient relative to* x_C.

If $T = \{i\}, \quad i \in C$, then $(x_{\overline{C}} + x_C, y_{\overline{C}})$ for $\overline{C}$ is *individually rational relative to* x_C.

These are three different rationality constraints imposed upon $\overline{C}$; if C produces x_C, then $\overline{C}$ has to consume either an individually rational, a Pareto efficient, or a group rational allocation relative to x_C.

The interesting point is that by imposing any of these rationality constraints, a pure public goods economy that still satisfies the usual competitive assumptions has an empty core. Shapley and Shubik (1969) showed that when the conventional assumptions are made, the core of such an economy is nonempty. Richter gives an examples of an empty core and then derives a necessary condition for an empty core in a public goods economy.

To sum up, it is possible to say that this necessary condition is more likely to be satisfied the larger the easy riding (for the use of the term "easy" instead of "free", see Cornes and Sandler (1986)) coalition C receives on the public goods production of $\overline{C}$. The usual definition of what a coalition can attain on its own and Richter's definition with the several rationality restrictions on $\overline{C}$ are two extremes of the set of possible definitions: the first one gives a very large core and the second may give no core at all. It would be interesting, and still pretty hard in Richter's opinion, to find a definition which would narrow the core without letting it vanish.

4.7 The Lindahl solution for economies with public goods, 1974

This is a neat expository paper that surveys the theory of Lindahl equilibrium as well as various related topics. We are mainly concerned with its Section III, where the possibility of achieving and maintaining a Lindahl equilibrium is discussed in terms of its relationship with the core. Roberts argues that the intuitive notion that the core is intended to formalize is one of social stability: presumably any allocation not in the core would be overthrown. Since any Lindahl equilibrium belongs to the core under some assumption (see Foley (1970), Milleron (1972), it is stable in the appropriate sense. However, we cannot go further to say that the Lindahl allocations are the only core allocations; as Muench (1972) has shown, a Debreu-Scarf equivalence theorem does not hold when public goods are introduced.

Roberts recalls that Lindahl himself thought that Edgeworth's conjecture cannot be valid for public goods economies: as the number of consumers increases, so new prices have to be introduced and further room for bargaining is added. Also Roberts emphasises that the same definition of the core makes it "too large" in the presence of public goods: core allocations exist under which many agents, even deprived of any of the benefits coming from production and trade, cannot improve upon the allocation. On this matter, Roberts' conclusion is that the core offers limited support for the Lindahl mechanism to be a solution concept.

4.8 Public goods with exclusion, 1980

This paper by Drèze is quite massive: it extends from theoretical definitions to practical considerations, from existence theorems to efficiency concerns. For what concerns us, it is interesting to recall its starting point. Drèze argues that in practice price discrimination is either nonexistent or restricted to a few categories of consumers, whereas equilibrium concepts for public goods presented in the literature count on "individualized prices" as in the case of Lindahl equilibria and subscription equilibria. These are examples of unlimited price discrimination between consumers. Clearly, there are many good reasons for this practice: the consumer tendency not to reveal the true willingness to pay; the cost of gathering and processing individual information; the existence of political and ethical constraint on public services.

A second important point coincides with the very subject of the paper: public goods with exclusion (see also Section 3). These are goods the consumption of which can be controlled and restricted. For them, physical feasibility only demands that no single agent should consume more than the whole output. They are typically sold at fixed prices up to the total quantity produced. Each individual may buy at given price any amount she chooses, not exceeding the total output. Other times some price discrimination is allowed, either between consumers or considering the quantities purchased. This is in contrast with the idea underlying the Lindahl equilibrium, where each consumer buy the same amount at different price. Both price regulation and quantity regulation by a public regulatory agency are considered in the paper. In any case the choice of the optimum level for the dependent variable comes from the solution of a maximization problem.

The interest in this two points is due to the fact that they lead to analyze more realistic aspects of the general pure public good model (see Section 3).

4.9 Continuum economies with finite coalitions: core, equilibrium, and widespread externalities, 1989

This paper concerns a perfectly competitive exchange economy with recontracting. When there is a very large number of individuals, each of them is usually supposed to have negligible influence on economic phenomena. On the other hand, the activity of recontracting suggests that individuals are influential in pursuing their own interest. In Aumann's model (1964), where the set of agents is a nonatomic measure space, each agent is also negligible relative to any recontracting group. Here the model consists both of a nonatomic measure space and of finite coalitions, i.e., ones with a finite number of agents: this last feature allows for the agent to be effective in recontracting. The solution concept used in this framework is the *f*-core, for which a definition is provided. Another important notion is that of ***widespread externalities***: it means that individuals' preferences depend on their own consumption of goods as well as on the entire allocation. When widespread externalities do appear - i.e., there is no self-evident definition of the Aumann-core - the equivalence of the *f*-core and the Walrasian allocations is proved to hold.

4.10 Cost share equilibria: a Lindahlian approach, 1989

In Mas-Colell (1980) the concept of a Lindahl equilibrium[8] is generalized by allowing for nonlinear personalized prices: these are called *valuation functions*, and the equivalence between the core and the set of valuation equilibria is shown. The problem with valuation functions is that they are not cost shares, since the sum of supporting prices does not cover the cost of production for all levels of the public good.

More recently Mas-Colell and Silvestre (1989) introduced the notion of a *cost share equilibrium* in a Lindahlian framework: this is a special case of valuation equilibrium, for the individual monotonic share function give rise to cost shares which are not necessarily linear. Given unanimity about the desired level of the public good, the decentralized allocation process yields efficient outcomes and assures that individual contributions are proportional to individual benefits. The idea of unanimity behind the notion of equilibrium is customary in normative economics: needless to say, this assumption hides the most important feature of the real world's decision processes that is, people do not agree in general on what is the best allocation. In order to give an idea of the framework used by Mas-Colell and Silvestre, I summarize their basic definitions.

There is one private good x, which is taken as a numéraire, and M public goods $y = (y_1, y_2, \ldots, y_M) \geq 0$ produced under a technology given by the single cost function $C(y)$. It is assumed that C is continuous, strictly increasing and unbounded above, and that $C(0) = 0$. There are N consumers, each endowed with a strictly positive amount of the private good, $e_i > 0$, that have continuous and increasing utility functions: $u_i(y, x_i)$ also satisfies the assumption of convex preferences.

A *state* of the economy is a vector $(y,x) \in \Re^M_+ \times \Re^N_+$ which is *feasible* if $C(y) \leq \sum_i (e_i - x_i)$, and *optimal* if there is no other feasible state (y',x') such that $u_i(y',x'_i) \geq u_i(y,x_i) \quad \forall i \in N$ with strict inequality for at least one i.

A *cost share system* (CSS) is a set of N functions $g_i : \Re^M_+ \rightarrow \Re$ such that $g_i(0) = 0$ and $\sum_i g_i(y) = C(y) \quad \forall y$. Notice that with $g_i(0) = 0$ the possibility of lump-sum transfers unrelated to the production of public goods is

8 Recall that at a Lindahl equilibrium individual prices are such that everyone demands the same quantity of each public good.

ruled out. Furthermore, a CSS is *linear* when it is of the form $g_i(y) = a_i \cdot y + b_i C(y)$, where $b = (b_1, \dots, b_N) \in \Re^N$ and $a = (a_1, \dots, a_N) \in \Re^{NM}$; this coefficients must be restricted: we put $b_i \geq 0$ and $\sum_i b_i = 1$, while $a_i \in \Re^M$ are such that $\sum_i a_i = 0$.

A *linear cost share equilibrium* (LCSE) is a pair formed by a feasible state and by a linear CSS, $((y^*, x^*), (g_1, \dots, g_N))$, such that for all i we have both $x_i^* = e_i - g_i(y^*)$ and $u_i(y^*, x_i^*) \geq u_i(y, e_i - g_i(y))$.

In this setting, is easy to prove that any cost share equilibrium is associated with an optimal state: this is done comparing an equilibrium state of the economy with any other feasible state in terms of the associated utility levels, following the definition.

The interpretation of the parameter of a linear CSS is clear: the b_i are individual shares of the unit cost of production, while the a_{ij} are a sort of side compensation based on consumption. These last coefficients are not the personalized prices of the Lindahl-Foley type, since they specify the economic environment without the need for profit shares. This is due to the fact that the framework is different. In fact, for the general variable returns case Foley's model has two problems: first, since production take place at a price-taking and profit maximizing point, there will not be an equilibrium in the case of increasing returns; second, in the case of decreasing returns the resulting positive profits have to be shared between agents. In the same way, if profit maximization is replaced by marginal cost pricing, as in Guesnerie (1975), losses resulting from increasing returns in the production of the public good have to be shared exogenously. On the contrary, here the shares are in equilibrium if, given them, there is unanimity on the desired level of public good production: profit maximization is replaced by the stronger informational requirement that agents have to know the cost function and agree on the share system.

The authors show the relation between a linear cost share equilibrium and a Lindahl-Foley equilibrium: the strength of this relation depends upon assumptions of convexity and of differentiability of the cost function. The study of this analytical correspondence makes clear that the problem of sharing a possible profit is still there: it appears in the form of *N-1* degrees of freedom of the LCSE that correspond to the profit shares in Foley's model. To remove this degree of freedom, that is, to design an endogenous scheme of profit distribution, it is natural to require that the net individual transfer be zero. This is in accordance with the basic idea that individual contributions and benefits must coincide. Thus, we have the following definition: A LCSE is *balanced* if $a_i \cdot y^* = 0$ for all i.

It is easily shown that a balanced LCSE corresponds to a Lindahl-Foley equilibrium in which share profits are proportional to individual gross expenditures, $p_i \cdot y^*$ ($p_i \in \Re^{MN}$ are the Lindahl personalized prices). Notice that the benefit theory of taxation is reflected here in the fact that individual contributions are in line with personal valuations: this is only partially true in Foley's model.

Eventually, the existence of a balanced LCSE is proved under some restrictions: the basic one is that utility functions must be weakly increasing in every argument, that is, no public bads are present.

4.11 The equivalence of core and cost share equilibria in an economy with a public good, 1991

Both Mas-Colell (1980) and Mas-Colell and Silvestre (1989) constitute the reference of this important paper by Weber and Wiesmeth (1991). It represents the ending point of the research begun with Foley's work, for it proves the equivalence between the core and a set of allocations that satisfy an ad hoc concept of equilibrium for an economy with a public good. More precisely, an allocation belongs to the core if and only if it is a cost share equilibrium: the concept of cost share equilibrium can be considered as a generalization of the Lindahl equilibrium. This result is related to the work presented in Mas-Colell (1980) which regards the equivalence of the core and the set of valuation equilibria. Weber and Wiesmeth use a model with only one private good and one public good: it is not clear if the generalization to any number of goods is straightforward.

Muench (1972) has shown that the core can be much larger than the set of Lindahl allocations, which implies that Edgeworth's conjecture cannot be extended to public good economies in terms of the core and the Lindahl equilibrium. However, Muench himself wrote that the key to overcome this problem could be found by adapting the definition of the core; this is what many authors have tried to do. Actually, Weber and Wiesmeth adjust the equilibrium concept instead: in fact, as shown in Mas-Colell and Silvestre (1989), the cost share equilibrium notion embodies the Lindahl equilibrium as a particular case (that of constant returns to scale and of proportional sharing schemes).

In an economy with n agents there is one private good y which also serves as input for the production of the public good x under a technology represented by the cost function $C(x)$, which is continuous, strictly increasing, unbounded above, and such that $C(0) = 0$. Agents are endowed with the quantity e_i of the private good and $e = \sum_{i \in N} e_i$ is the total endowment. Agents have continuous and strictly monotonic preference orderings over consumption bundles $(x, y) \in \Re^2_{++}$ which are represented by continuous utility functions

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

$u_i : \Re^2_+ \to \Re$. Let t_i be the individual contribution and S be a coalition of agents; then, the vector $X = (x;(t_i)_{i \in S})$ is called an *S-allocation* if $\sum_{i \in S} t_i = C(x)$ with $t_i \le e_i \quad \forall i \in S$. A_S, and A denote respectively the set of all *S-allocations* and the set of all *N-allocations*, which are simply called allocations. The definition of dominance and of the core are as follows: a pair (S,Y), where S is a coalition and $Y = (y;(\tau_i)_{i \in S})$ is an S-allocation, blocks an allocation X if for all agents in S we have $u_i(y, e_i - \tau_i) > u_i(x, e_i - t_y)$. The core of the economy is the set of allocations X for which there is no coalition S and S-allocation Y such that (S,Y) blocks X.

Now, let X be an allocation and consider its associated utility levels $u_i(X)$: the assumptions assure that for each $z > 0$ and $\forall i \in N$ there exists a unique $\rho_i^X(z)$ such that $u_i(z, e_i - \rho_i^X(z)) = u_i(X)$. If z is smaller than x, then $\rho_i^X(z) < t_i$ so that agent i is compensated for the loss of some units of the public good by a larger consumption of the private good. Define $r_i^X(z) = \max\{0, \rho_i^X(z)\}$ and $R_S^X(z) = \sum_{i \in S} r_i^X(z)$. It is clear that the value R_N^X is the minimal amount of the total contributions needed to produce a quantity z of the public good and to assure to each agent the utility level $u_i(X)$. These concepts were already used in Mas-Colell (1980) to provide the following characterization of core allocations.

The fact that an allocation X belongs to the core is equivalent to $R_N^X(z) \le C(z) \quad \forall z \ge 0$. When the minimal amount of the total contribution is no larger than the relative production cost it is possible to share the cost of any production level z in such a way that all agents agree upon the optimal amount of the public good.

To reach their main result Weber and Wiesmeth introduce a new restriction to the cost share system of Mas-Colell and Silvestre: monotonicity of the CSS requires that whenever a higher level of public good is produced all individual cost shares increase accordingly: this is in line with the view that individual contributions should correspond to individual benefits.

A *monotonic cost sharing method* (MCSS) ϕ is a system of functions ϕ_i, one for each agent, that determine individual shares of the production cost of the public good. These are continuous and increasing functions of the production cost of the public good in terms of the private one. Thus, their domain is $I := [0, C^{-1}(e)]$ -where e is the total endowment of private good- while their range is $[0, e_i]$ -where e_i is the individual endowment.

When ϕ is given, an allocation is called a *ϕ-allocation* if all individual contributions are determined by ϕ_i.

A *cost share equilibrium* (CSE) is an allocation $X = (x;(t_i)_{i \in N})$ for which there exists a ϕ such that X is a ϕ-allocation and for all $i \in N$ we have $u_i(x, e_i - t_i) \geq u_i(z, e_i - \phi_i(z)) \quad \forall z \geq 0$

The first main result is obtained under an additional assumption which states the quasi-concavity of the utility functions, their twice continuous differentiability, the convexity of the cost function, and its continuous differentiability.

Proposition 1. An allocation $X^* = \left(x^*;(t_i^*)_{i \in N}\right)$ is a cost sharing equilibrium with respect to a monotonic cost sharing system ϕ if and only if it belongs to the core.

Since the concept of a CSE does not refer to profit maximization a natural question arises about its relation with equilibria characterized by profit maximization itself. Kaneko (1977a) recalls that he had given examples showing that if the cost function is not convex -that is, the economy does not exhibit constant returns to scale and profits or losses are generated- it is not necessarely true that a Lindahl equilibrium belongs to the core: this depends on the distribution of profits. Thus, not any Lindahl equilibrium is also a CSE. However, Mas-Colell and Silvestre (1989), showed that "linear CSE" are in one-to-one correspondence to Lindahl-Foley equilibria for some profit share scheme and some personalized prices in the more general case of convex cost function and nonincreasing returns to scale.

Given this result, the characterization of those core allocation which can be supported by a linear CSS amounts to the characterization of the set of Lindahl-Foley equilibria in the core of the public good economy.

A CSS $\phi = ((\phi_i)_{i \in N})$ is linear if for each i there exist real numbers a_i, b_i, $b_i \geq 0$, $\sum_{i \in N} a_i = 0$, and $\sum_{i \in N} b_i = 1$ such that $\phi_i(z) = a_i \cdot z + b_i \cdot C(z)$, $\forall z$ with $\phi_i(z) < \min\{e_i\}$, $\forall i \in N$. An allocation X is a linear CSE if there exists a linear CSS ϕ supporting X as CSE.

Consider now the relative marginal rate of substitution for an allocation X and for an agent i: $\varepsilon_i(X) = (dx/dt_i)\cdot(t_i/x) = t_i/(x \cdot MRS_i(X_i))$ where t_i is the individual contribution in terms of the private good. To prove that only core allocations with the values of $\varepsilon_i(X)$ within some well defined bounds can be supported by a linear CSS, the authors define the following function for $x > 0$:

$$\lambda^*(x) = \frac{C(x)/x - C'(0)}{C'(x) - C'(0)}$$

Notice that under the assumptions regarding the utility and cost functions, we have $0 < \lambda^*(x) < 1$, and that if $C'(0) = 0$, then $\lambda^*(x)$ represents the inverse of the cost elasticity. The second important result now follows:

Proposition 2. An allocation $X^* = \left(x^*;(t_i)_{i\in N}\right)$ is a linear CSE if and only if it belongs to the core and $\lambda^*(x^*) \leq \varepsilon_i(X^*) \leq 1 \quad \forall i \in N$ with $\varepsilon_i(X^*) < 1$ for at least one $i \in N$.

Thus, we have upper and lower bounds on the relative marginal rate of substitution for a public good which are a necessary and sufficient condition for a core allocation to belong to the set of linear CSE. It is interesting to remark that this condition on the bounds of $\varepsilon_i(X^*)$ implies a certain uniformity of individual preferences.

PART 2:

ON THE GAME THEORETIC APPROACH TO PUBLIC GOOD ECONOMIES

Table of Contents

1 Introduction 1

2 Game theory and public goods 1

3 Considerations on games in characteristic form 4
3.1 On characteristic functions. 5
3.1.1 The Harsanyi modification 11
3.1.2 Games in partition-function form and limits to coalition formation 12
3.2 On S-equivalence and normalization 14
3.2.1 Characteristic functions and probability measure 15
3.3 On payoff vectors. 16

4 On payoff imputations in the core 20
4.1 On restrictions to the set of the outcomes available to coalitions 24
4.2 On a characteristic function that assumes that outsiders do contribute 24

5 The cooperative approach to externalities and public goods 26

6 The Vickrey auction 30

1 Introduction

This paper elaborates the idea that the usual characteristic function is not adequate to describe the process of coalition formation: the presence of public goods makes this task impossible. Informally, it is based on excessively pessimistic assumptions about the behaviour of the complementary coalition. Moreover, it seems to pay too much attention to the simple dialectic between one coalition and its complement: a more complex structure of coalitions is typically stable in the real world and I feel that the outcome described by a game in characteristic form -or some alternative description- should be able to explain it fully.

In the last section I comment the idea of second price auction developed by Vickrey (1961).

2 Game theory and public goods

It is clear that externalities and public goods introduce strong interdependences between agents that cannot be mediated through prices: the benefits to an individual resulting from a given set of actions depend heavily on the consumption or production activities by others. This gives the rationale for using the game theory approach in analyzing the field.

The standard public good model encompasses agents that do not coordinate their actions; they simply form expectations about their economic environment and then act without making binding agreements with other agents. Those situations are modelled by non-cooperative games.

A special class of them is that in which agents choose a binary variable instead of a continuous one. It may be an option between cooperating with others or not, and contributing nothing to the provision of a public good or just contributing one unit. I think that, these models being very simple representations of the game structures, the best way to use them is as a framework within which to extend the models to intertemporal settings. I find it interesting to study whether the threat of punishment by others in successive stages of a repeated game can deter would-be free riders. In a more general sense, repeated games give a chance to learn about the more likely behaviour of other players and may lead to reduce the bias toward suboptimal equilibrium levels of public good provision; this result could be important, since it would be obtained without the need for binding agreements.

It must be said that current results in the literature do not leave much hope for such an event to occur. One example is given by the *Chainstore Paradox* presented in Selten (1978). Using backward induction the author shows that the strategy combination *{Enter-Collude}* is a subgame perfect equilibrium, and that there is no difference between the one shot game and the finitely repeated one. The same argument applies to the game which consists of finite repetitions of *Prisoner's dilemma*: it is common knowledge to both players that they will fink in the last repetition, no matter what happens before, so that again by backward induction the unique perfect equilibrium outcome is *{Fink-Fink}* in every period. Moreover, it is also the only Nash outcome. It is usually claimed that building a reputation is pointless because in the last period is not going to matter.

The argument of backward induction is not valid in the case of the infinitely repeated[1] Prisoner's Dilemma. Now a simple perfect equilibrium is reached if both players cooperate. Unfortunately, the Folk Theorem tells us that any other outcome is a perfect equilibrium, including that given by always finking.

However, following the hint that always finking is not a dominant strategy, I argue the following: a situation is conceivable such that for at least one player the sum over the repeated games of the relative gains of cooperating is larger than the loss he suffers when the other will fink in the last period. This argument reintroduces the need for analysing reputation: when a game is repeated (both finitely and infinitely) a player may be willing to cooperate in early periods in order to establish a reputation for honesty which will be valuable to himself. Rasmusen (1989), page 120n, refers to a paper in which by a suitable expansion of the game strategy, it is shown that cooperation can be enforced in a multi-person finitely repeated Prisoner's Dilemma.

A different way to model the evolution of cooperation is developed in a recent paper by Kramarz and Ponssard (1991): they prove that the common knowledge of standards of behaviour in long term relationships gives rise to tacit cooperation provided that intermediary observations are feasible. They claim that their result is more robust with regard to players' incentives than the standard one based on the reputation effect.

The two-person two-strategy noncooperative game involving binary choices can be studied with the help of a matrix representation. The pattern of payoffs when the game involves public goods is often consistent with the *Prisoner's dilemma*-type of situation. In this case three out of the four possible outcomes are Pareto optimal, whereas the fourth is dominated. The main point is that the fourth outcome is typically the equilibrium of the noncooperative game.

---

1 In passing, it seems worthwhile to note the following: the relevance of using infinite horizon games is clearly not given by the fact that human relationships have infinite duration. What justifies their use is that in some situations players simply expect that there will be more plays after the present one, without considering the end. On the other side, the finitely repeated game is suitable to model situations in which the last period is definitively considered by the players. In this sense an infinitely repeated game can actually describe short games.

It is true that, even in a simple setting there are problems in identifying rational individual behaviour and the related social outcome. The public goods problem may not conform to the Prisoner's dilemma and this means that the Nash equilibrium concept becomes meaningless. However, Johansen (1982) claims that in a game with complete information the Nash equilibrium strategy is the only one that is consistent with four plausible rationality postulates and therefore should be considered the natural noncooperative equilibrium concept.

The analysis can be extended in three directions: first, we should be able to work with more than two players. Second, we can investigate whether repetition may favour cooperative behaviour. Third, one should consider alternative assumptions of rationality and see if they lead to non Nash behaviour still being a part of a rational strategy.

Situations in which the problem for the agents is to decide the aggregate level of a continuously variable public good, in the same noncooperative framework, are widely examined in the literature. As said before, Johansen (1963) proposes four axioms to characterize rational behaviour. A fifth postulate is implied by the fulfilment of the others and this set of rules leads to the Nash noncooperative equilibrium. I recall briefly the five postulates.

Let a_j be the action of agent j out of n agents and A_h be the set of feasible actions for h. Let the utility of agent h be represented by $U_h(a_1, a_2, \ldots, a_n)$.

1. a player chooses a_h from A_h only on the basis of information concerning the set $\{A_1, \ldots, A_n\}$ and of the preference functions of all players $U_1(a_1, \ldots, a_n), \ldots, U_n(a_1, \ldots, a_n)$.
2. each player assumes that others are rational in the same sense as she is rational.
3. if some decision of a player is rational, then it can be correctly predicted by others.
4. predicting the actions to be taken by other players, a player's own decision maximizes his preference function given the predicted actions of others.
5. a decision is rational if the player does not regret it after having observed the other players' decisions and the outcome of the game.

Then, the chosen actions $\hat{a}_h$ satisfy the condition:

$$U_h(\hat{a}_1,\ldots,\hat{a}_h,\ldots,\hat{a}_n) \geq U_h(\hat{a}_1,\ldots,a_h,\ldots,\hat{a}_n) \qquad h = 1,2,\ldots,n \quad \forall a_h \neq \hat{a}_h$$

Finally, I would like to recall the result of a concise paper by Guesnerie and Oddou (1979) that considers a cooperative setting with transferable utility[2]. Let N be the set of players, S the strategy space of the game, and P the payoff vector. Consider K and L, two disjoint subsets of N, and $v(K)$, $v(L)$ their characteristic function. Then we say that, for a game (N,S,P), $v(K)$ is superadditive if $v(K \cup L) \geq v(K) + v(L)$. Most of the time economic situations are modelled considering superadditive games but the authors present a simple economy with a public good and few agents which is not necessarily superadditive; they argue that this might be often the case. A wealth tax can finance the public good and this leads to a second best situation. It turns out that an adaptation of the core concept is needed in order to ensure its existence in this case. The generalization when the game is not superadditive is called C-stable solution; the core is a set of particular C-stable solutions.

3 Considerations on games in characteristic form

The set of players is $N = \{1,..,i,..,n\}$. Nonempty subsets of players, denoted K,M,N and so on, are called *coalitions* if they can make binding agreements.

The set of all possible coalitions is κ , and $\#\kappa = (2^N - 1)$. The condition that a coalition can make binding agreement is usually not formalized, but is a strong one. Not all the possible coalitions take place, both because of information problems and because binding agreements are not enforceable.

Moreover, in deciding which coalition to join, each player can choose among $(2^N - 2^{N-1})$ of them. Consider, for example, that if $N = 5$ he can neither join the four 3-player coalitions allowed to the other four players, nor the one 4-player coalition formed by them. In general, the number of coalitions available to agent i is $(2^N/2)$, while the number of coalition not available to him is $(2^N/2 - 1)$. I denote by κ_i the proper subset of κ that contains all coalitions available to player i.

2 The transferable utility assumption (which gives rise to *side payment* games) is equivalent to postulating the existence of a private good with linear utility that enters additively into the individual utility functions.

I argue that in the real world coalition formation is a function of the incentive to cooperate, and of other important factors: habits, cultural biases[3], and available information. The level of trust among players depends on all these factors: a binding agreement concerns mainly the way side-payments take place after the game has been played (in a transferable utility setting).

Thus we can say that to enter a coalition one player looks at the probability of receiving his final payoff, whatever it be: he has an expectation. The individual incentive to cooperate may well need to be strictly positive and larger than a certain threshold, in order for the coalition to form.

On the other hand, we can think of which coalition is more likely to form and compare the outcomes that coalitions can guarantee to their members. A coalition is more likely to form if it guarantees to its members a higher payoff, that is, if their incentive to collude is larger. In this sense an index of the incentive to form a coalition may be conceived as in the following definition, where $v(\cdot)$ is the standard characteristic function.

Definition. The incentive to cooperate associated to each coalition is given by $v(K)-\sum_{i\in K} v(\{i\})$.

This is different from the *Shapley value* (see below) which is the individual payoff defined as the average marginal worth of a player to all coalitions. This index does not deal with the distribution of individual payoffs, but it conveys the likelihood of the formation of a coalition. Dividing the index by one of the addenda we get a relative measure which is useful to compare different coalitions.

In the following subsections I will use this setting to elaborate two issues: the characteristic function and payoff vectors.

3.1 On characteristic functions.

Before considering the characteristic function, the main tool of analysis in cooperative game theory, let us recall three basic concepts of equilibrium in terms of domination.

When coalitions do not consider deviating from a proposed imputation, they do not anticipate any retaliatory move. This hypothesis, in which the agents in a coalition take the strategy of the complementary coalition as fixed, leads to a cooperative solution concept called *Strong Cournot Nash equilibrium*.

3 For example, consider this old saying of southern Italy: "Cooperatives are very good, provided that the number of members is odd and strictly smaller than three".

On the other hand, we can consider the case in which coalitions do contemplate deviating if better levels of utilities can be reached for their members. Now we can think as if time played a role, inasmuch the outcome depends on "who moves first", in the sense of who sets his strategy first.

If the members of a coalition K move first, we define what they can assure themselves. "Assure" or "guarantee" are used in the sense that the joint strategy of the coalition is a necessary condition for the considered outcome. Sufficiency is added later, when the complementary coalition reveals its joint strategy, and only then the first coalition strategy gives rise to a payoff function that is a well defined one-to-one mapping to the final outcome.

If K moves after the complementary coalition has chosen its strategy, we define what its members cannot be prevented from attaining.

The first situation leads to the set of payoffs (and to the set of corresponding cooperative equilibria) known as α-*core*. Scarf (1971) proved for it a general existence theorem[4].

The second situation leads to the definition of β-*core*: it lacks a general existence theorem. As we will see, underlying both the α- and the β-core there is a pessimistic valuation of what a coalition can do if its complement deviates. The β characteristic sets always include the α characteristic sets.

In order to be more precise, I introduce some standard notation. Denote by x_K and $x_{N\setminus K}$ the vectors of individual payoffs of members of a coalition K and of members of its complement. Denote by s^K (respec. $s^{N\setminus K}$) a generic element of $S^K = \times_{i\in K} S_i$ ($S^{N\setminus K} = \times_{i\in N\setminus K} S_i$), the product of the strategy set of each agent in K (in $N\setminus K$). A cooperative equilibrium is a strategy profile s such that there is no coalition that dominates the payoff vector $x(s) = (x_1(s),\ldots,x_N(s))$. This means that there exists no coalition K that can give to its members individual payoffs x_i such that $x_i > x_i(s) \quad \forall i \in K$. Denote by $X(K)$ the set of vectors of individual payoffs that a coalition K can ensure for its members. Now we can express formally the three different sets of payoff vectors that give rise to the three cooperative solution concepts recalled above.

$$X_{SCV}(K) = \{x_K : x_K \le x_K(s^K, s^{N\setminus K}) \quad \text{for some} \quad s^K \in S^K\} \tag{1a}$$

4 For a generalization of Scarf's theorem see a paper by Atsushi Kajii, forthcoming in the Journal of Economic Theory.

$$X_\alpha(K) = \{x_K \mid \exists s^K : x_K \le x_K(s^K, s^{N\setminus K}) \quad \forall \quad s^{N\setminus K}\} \tag{1b}$$

$$X_\beta(K) = \{x_K \mid \quad \forall \quad s^{N\setminus K} \quad \exists s^K : x^K \le x_K(s^K, s^{N\setminus K})\} \tag{1c}$$

Having said all that, I go directly to the characteristic function. The first step is to consider the payoff that a single player can get. Let $S^{-i} = \times_{j\in N}^{j\neq i} S^j$ be the set of all strategy combinations that all the players except i can use against him. An element of S^{-i} is a strategy combination $s^{-i} = (s_1, .., s_{i-1}, s_{i+1}, .., s_N)$. If player i uses one of his strategies, s^i, then the lowest payoff that he can get is the following:

$$\overline{x}_i(s^i) = \min_{s^{-i} \in S^{-i}} x_i(s^i, s^{-i}) \tag{2a}$$

This is called player i's *security level* for his strategy s^i. The highest security level that player i can obtain by using any of his strategies, is called his *maximin payoff* from the game, and is given by:

$$x_i^* = \max_{s^i \in S^i} \min_{s^{-i} \in S^{-i}} x_i(s^i, s^{-i}) \tag{2b}$$

Any strategy s^{*i} having the same maximin payoff x_i^* as the security level is called a *maximin strategy*.

By definition, the characteristic function gives to each player a value that is equal to his maximin payoff. Moreover, assuming transferable utility and free disposal, the characteristic function assigns a maximin value to each coalition: for a given game each coalition has associated with it the largest total payoff that it is capable of attaining.

Notice that in *symmetric games*, where the power of a coalition is a function only of its size, the set of possible values of $v(K)$ has cardinality N. In general it is supposed to take (2^N) values, since it is usually defined also on $\{\varnothing\}$; in contrast I assume that a coalition is a nonempty set of agents.

Clearly, $v(K)$ is not continuous and it is not necessarily a one-to-one function. Let's state the usual definition in terms of the α-characteristic function.

$$v_\alpha(K) = \max_{t^K \in S^K} \quad \min_{s^{N\setminus K} \in S^{N\setminus K}} \quad \sum_{i \in K} P_i(t^K, s^{N\setminus K}) \tag{2c}$$

Where $s^K \in S^K$, and $S^K = \times_{i \in K} S_i$ is the joint strategy space. $(t^K, s^{N\setminus K})$ is the strategy combination in which player i is using either $s^{N\setminus K}$ if $i \notin K$ or t^K if $i \in K$. Two considerations follow:

1) The domain and the range of v. Nothing can be said about its form in terms of the nice behaviour usually requested of functions in economics.

The characteristic function is defined on the set of all possible coalitions. We denoted it by K and we know that $\#\mathrm{K} = (2^N)$ if the empty set is admitted. Now, using data from a real situation, suppose we order the elements of this set by two criteria: first the cardinality of each element in ascending order and then the alphabetical order applied to the names of members. Thus we have a list of coalitions ordered as follows: first, all single player coalitions, second all two player coalitions, and so on until the grand coalition; moreover, in each sublist of coalitions with the same number of agents coalitions are ordered lexicographically by agents' names.

This simple structure of the domain of v is not enough to say something on its behaviour, even if we can expect that $v(K) \in \Re_+$ is increasing with the cardinality of K.

Suppose we know more about each player: his power, for example, measured by his marginal contribution to each coalition. Still, this is not enough to characterize easily the behaviour of v.

In fact, individual marginal contributions vary across different coalitions. Moreover, and this is just another way to put it, even information on $v(\{i\})$, each member's maximin value, cannot help, since superadditivity holds and the value of a coalition might be bigger than the sum of individual values. However, it is true that in constant-sum games, that are a particular case of superadditivity, this problem disappears. Recall that a game is *constant-sum* if:

$$v(N) = v(K) + v(N \setminus K) \tag{3a}$$

The total payoff is a fixed quantity, regardless of the strategies chosen. Since it is true for all K it is also true for $K = \{i\}$, that is, for each agent. However, this does not imply

$$v(\{i\}) = v(\{j\}) \quad for \quad i \neq j \tag{3b}$$

To get something more we have to assume that (3a) takes the following form:

$$v(K) = v(M) + v(K \setminus M) \quad \forall \quad K \subseteq N, \quad \forall M \subseteq K \tag{4}$$

This means that each game, restricted to some of the players of the original game, is constant-sum. In this case it is clear that $v(\{i\})$ is the same for all K which he can belong to. However, this is still not a sufficient condition for (3b).

What I am saying is that the worth of each agent in a coalition depends both on his original characteristics and on the particular coalition that he joins: this is also reflected in the definition of the Shapley value. Clearly I do not mind here about the way the total payoff is distributed among members: I am considering what is the reasonable individual payoff that a player would like to get. Formally I would write $w_i(K), \forall i \in N$, $\forall K \in \mathrm{K}_i$, meaning also that it is possible that $w_i(K) \neq w_i(L), K, L \in \mathrm{K}_i$. I further argue that player i's payoff in a certain coalition might well depend on the way $N \setminus K$ players arrange themselves in coalitions. This problem is better analyzed considering the coalition as a whole: I do that in point 2) and in sec. 3.1.2.

Summing up, the characteristic function embodies much information that varies with the particular situation involved: thus, it is not possible to characterize its form in general.

2) **The coalition structure.** It should be emphasized that $v(K)$ is the maximin value of K in the game played between K and $N \setminus K$. The concept was born for zero-sum games and its extension to non-zero-sum games is considered straightforward, even though it is not. For any K we let $v(K)$ be the optimal security level of the coalition K in the non-cooperative two-person non-zero-sum game K versus $N \setminus K$.

So, the partition (the pattern of the coalitions) considered is very simple: it consists of members of K and of all the others players acting as a coalition in $N \setminus K$. Moreover, it is clear that if a subset K of players form a coalition, the fact that the remaining players also form a coalition is the former's worst strategic prospect.

I believe that the value of $v(K)$ would be bigger if it faces smaller coalitions or even single individuals. Formally, $v(K)$ should also be a function of the coalition structure.

The usual assumption is restrictive because it demands that conflicts of interest always reduce to two strictly opposing coalitions. A more adequate descriptive theory has to consider the expectations of each possible coalition in relation to the actions of the remaining players arranged in any conceivable coalition: this is what I try to do below, after some other comments on related subjects.

In constant-sum games the worst threats against the coalition are precisely the strategies that maximize the payoff to the players outside the coalition. Only in this case does the pessimistic assumption of the characteristic function seem reasonable. In any case, the worth of one-person coalitions (and of bigger coalitions, as a consequence) depends critically on the threat structure, and the threat structure in turn depends on the basic legal concepts of property rights: this is even more delicate to consider in the case of externalities, where the structure of property rights is not well defined.

The presence of threats leads to the following question: why should either coalition expect its complement to choose the most punitive strategy irrespective of the costs to itself of implementing it ? In a variable sum game one would rather expect that each coalition would compromise between the need to minimize the joint payoff of the complementary coalition and the exigency of maximizing its own joint payoff. In general, the von Neumann and Morgenstern characteristic function is suitable for all games in which a joint strategy is available that concurrently maximizes one coalition's payoff and minimizes its complementary coalition's payoff. This must be true for each pair of complementary coalitions in the game. Both constant-sum games and orthogonal games (that do not allow positive damaging actions, see below) have this property.

Alternatives to the maximin characteristic function may be found both by modifying it and by modifying the coalitional description of a game. I deal with these alternatives in sec.3.1.1 and 3.1.2 below.

As I have already remarked, superadditivity is usually assumed (see Guesnerie and Oddou (1979) for a counter-example with economic implications):

$$\forall \quad K,M, \quad K \cap M = \varnothing, \quad v(K \cup M) \geq v(K) + v(M) \tag{5}$$

Note that strict superadditivity implies that the game is not constant-sum (see equation (3a) above).

Games in characteristic function form can be divided in two classes. It is conceivable that there are games in which no coalition of players is more effective than the several players of the coalition operating alone: such games are called *inessential*. Formally, for all disjoint coalitions K,M, we have:

$$v(K \cup M) = v(K) + v(M) \tag{5a}$$

It is clear that the strict superadditivity assumption rules out inessential games. It is not difficult to see that (5a) is equivalent to the following, which shows better that nothing is gained by forming coalitions in inessential games:

$$v(N) = \sum_{i \in N} v(\{i\}) \tag{5b}$$

On the other side, a game is said to be *essential* if and only if it is not true that the total payoff to the grand coalition is the same as the sum of payoffs of all the individual players. Formally:

$$v(N) > \sum_{i \in N} v(\{i\}) \tag{5c}$$

Another interesting class of games is that of *orthogonal coalitions*: the underlying idea is that no coalition can alter the payoffs of players outside of it. On the other hand, the only threat by outsiders against a coalition is to boycott it. While the situation in constant-sum games is "you are either with us or against us", in orthogonal games "you are either with us or we do not care of what you do". In both cases, the characteristic function assumes the worst for the coalition.

3.1.1 The Harsanyi modification

In order to understand this matter, I have to recall that it is linked to the value solution theory developed both from the Zeuthen-Nash solution to two-person bargaining and from the Shapley value formula proposed as a solution for n-person games representable by maximin characteristic functions.

The value solution determines a single payoff vector that expresses the value of the game to each of the players. It is not necessarily a core imputation. In the following formula for the Shapley value, $c(k) = \{(n-1)!/(n-k)!(k-1)!\}$ is the number of coalitions of size k containing the player i:

$$\phi_i = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{c(k)} \sum_{i \in K, \#K = k} [v(K) - v(K - \{i\})] \tag{6}$$

The individual payoff to player i is his average marginal worth to all the coalitions available to him; note that the weights are different across coalitions.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

If we allow games to have complex strategies which reflect the existence of variable threats, that is, if we consider typically non-c-games[5], we can extend the value concept using a modification proposed by Harsanyi (1959) and Harsanyi (1977). However, it seems to me that no theory based on ordinal utility can deal directly with threats in a satisfactory manner, since the statement "*this will hurt you more than me*" makes no sense in a purely ordinal framework.

The basic idea goes as follows: let U^K (respectively $U^{N\setminus K}$) be the joint payoff of coalition K $(N\setminus K)$. Consider the difference $D = U^K - U^{N\setminus K}$. Then define the modified value of K, in terms of the modified characteristic function $h(K)$, as that value of U^K which corresponds to the maximin value of D. Let us see that in more detail.

In the two-person case let Γ denote a non-constant-sum game in strategic form, with payoffs (x_1,x_2). We can consider this game as if it were formed by two parts: the strictly cooperative game Γ_s with payoffs (x_1+x_2,x_1+x_2) and the strictly competitive game Γ_d with payoffs (x_1-x_2,x_2-x_1) . Thus $\Gamma = 1/2\Gamma_s + 1/2\Gamma_d$, and the Nash-Selten value of Γ , $\phi = (\phi_1,\phi_2)$, is given by the following two equations:

$$\phi_1+\phi_2 = f(\Gamma_s) \qquad \phi_1-\phi_2 = f(\Gamma_d) \tag{7}$$

where $f(\Gamma_s)$ is the joint maximum solution to Γ_s, and $f(\Gamma_d)$ is the minimax solution of the zero-sum game Γ_d.

In the n-person case Harsanyi considered each coalition K in turn, faced it with its complement and used the values ϕ_K , $\phi_{N\setminus K}$ to evaluate the worth of coalitions instead of $v(K)$ and $v(N\setminus K)$ as in (2c). This new function, denoted by h, depends on asymmetries in threat potential and is not superadditive. Replacing v by h in (6) we can calculate the Harsanyi-Selten value for games with transferable utility.

3.1.2 Games in partition-function form and limits to coalition formation

The ideas on the significance of the partition structure, introduced above, seems to be similar to those of Thrall and Lucas (1963) as reported in Shubik (1982).

---

5 A c-game is such that nothing essential to the purpose of the model is lost in passing from the extensive or strategic description to the characteristic form. The fact that the characteristic form is often an inadequate description of constant sum games is well known: see for example Luce and Raiffa (1957), page 190.

In a game with N players, denote by κ the set of all possible coalitions; we know that $\#\kappa = 2^N - 1$. The set of all coalitions which a player i can belong to is denoted by κ_i; we know that $\#\kappa_i = 2^N/2 = 2^{N-1}$. On the other hand, there is the set of $2^{N-1} - 1$ coalitions which a player cannot belong to, denoted by $\kappa_{N\setminus\{i\}}$.

Assuming that a coalition K forms, I am concerned here with the description of the behaviour of players outside K in terms of the pattern of coalitions which they can give rise to. It seems useful to present an example first:

Consider a game where there are four players *(a, b, c, d)*. If a choses to stay alone, the set of coalitions available to the other players is $\kappa_{N\setminus\{a\}} = (\{b\},\{c\},\{d\},\{bc\},\{bd\},\{cd\},\{bcd\})$. It is clear that, with regard to the participation of each player, only some of these coalitions can actually form: for example, we may see the set $(\{b\},\{cd\})$ form, but not the set $(\{b\},\{bc\},\{cd\})$. In both cases many of the possible coalitions do not form, and in the second case player b appears simultaneously in two coalitions. We need a rule to characterize plausible set of coalition of members of $N\setminus K$ that can arise in reaction to the forming of a certain coalition K. First, denote by $\kappa_{N\setminus K}$ the set of all coalitions available to members of $N\setminus K$.

Definition. A *coalition partition* of $\kappa_{N\setminus K}$ is any set of coalitions of players not belonging to K such that:

i) any player $i \in N\setminus K$ appears at least in one coalition.

ii) no player appears in more than one coalition.

Denoting by $P_{N\setminus K}$ a generic element of $\mathrm{P}_{N\setminus K}$, the collection of all possible coalition partitions of $\kappa_{N\setminus K}$, we can think of a function that attributes a payoff to a coalition K for all possible coalition partitions that may face K.

Definition. Let a real valued set function $w_K: \; P_{N\setminus K} \to \Re^+$ assign a payoff to the coalition K for all possible $P_{N\setminus K}$. The *worth* w_K^* of a coalition K is the maximum of w_K

Presumably the set $w_K(P_{N\setminus K}) \subset \Re^+$ has a minimum for $P_{N\setminus K} = (\{N\setminus K\})$, and the worth w_K^* corresponds to $P_{N\setminus K} = (\{i\},..,\{j\}) \quad i,j \in N\setminus K$, reflecting the old motto "*divide et impera*". The above definition leads to two

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

related questions: firstly, does an allocation that assigns at least its worth to every coalition exist? Secondly, starting from such an allocation, if it exists, is there room for Pareto improvements? Both questions are unanswered for the moment.

In the usual approach we pessimistically define the value of a coalition, $v(K)$, in a way that reflects its power when it faces its complementary coalition. A link with the partition function approach is given by the fact that $v(K)$ equals the minimum of w_K.

$$v(K) = \min_{P_{NK}} \{w_K(P_{NK})\} \tag{8}$$

The forming of the complementary coalition might be costly or even impossible and this is in general not considered. On the other hand, the study of the worth of a coalition motivates us to consider explicitly the limits to coalition formation. This measure of the value of a coalition seems to be more realistic, also because the function w_K might well not be defined for certain implausible coalition partitions.

3.2 On S-equivalence and normalization

The idea underlying this paragraph is that two games that have different characteristic functions may be the same in their essentials.

It is common in mathematics to define a large class of objects, all of them satisfying certain conditions, and then to realize that this class can be partitioned into subclasses that are homogeneous with regard to some criterion or theory. We always need to be sure that the theory is invariant under the equivalence concept that allows the partition.

If this is the case, one element of each class can be selected and considered as a representative. This idea applies also to the class of characteristic functions: if two games reduce to the same *zero-one normalized form* (see definition below), then they are considered to belong to the same equivalence class. They are said to be S-equivalent. The relation of S-equivalence can be easily shown to satisfy the three conditions of an equivalence relation: it is reflexive, symmetric, and transitive.

Definition. Two n-person games with characteristic function v and v^*, are said to be *S-equivalent* if there exists a vector of constants $(c_1, \dots, c_n)$ and a constant k such that:

$$v^{*}(K) = kv(K) + \sum_{i \in K} c_i \qquad \forall K \subset N \tag{9}$$

A useful transformation of characteristic functions is their normalization. A type of normalization is the so called zero-one normalization. Let's recall the definition of a zero-one normalized game:

Definition. Let $\Gamma = (N, v)$ be a game. Then its *zero-one normalization*, (N, v^{*}), is given by:

$$v^{*}(K) = \frac{v(K) - \sum_{i \in K} v(\{i\})}{v(N) - \sum_{i \in N} v(\{i\})} \qquad K \subset N \tag{10}$$

We note that for non-essential games this transformation has no meaning since the dominator equals zero, and that $v^{*}(\{i\}) = 0$, since the numerator vanishes; moreover, $v^{*}(N) = 1$.

It is possible to prove that two S-equivalent games have cores, bargaining sets, and Shapley Values that are related by the same transformation that defines their zero-one normalization.

A useful transformation of $v(K \cup M)$, where K and M are disjoint coalitions, consists in subtracting from it the sum of the values of K and M: $t(K,M) = v(K \cup M) - [v(K) + v(M)]$. This transformation gives a measure of the incentive for K and M to coordinate their strategies and it is useful to check whether a game is symmetric or not (in symmetric games the power of a coalition is an increasing function of its size). In fact, it is easy to see that if the game is symmetric then $t(K,M) > 0$ for all non-empty K and M; the converse does not hold.

3.2.1 Characteristic functions and probability measure

Luce and Raiffa (1957) underline the relation between 0,1-normalized characteristic functions and the definition of a probability measure over the subsets of a finite set. However, superadditivity does not hold for a probability measure and we cannot have $p(\{i\}) = 0 \quad \forall i$.

This comparison illustrates the fact that a characteristic function can be considered as a special case of the class of all finite, normalized, real-valued set functions. If the measure is additive we have the theory of discrete probabilities. The theory of characteristic functions can be viewed as the study of superadditive measures.

An example of the close relation mentioned above is given by substituting the condition of 0,1-normalization into the definition of an imputation. By doing this we find that an imputation x is an N-tuple which satisfies the following conditions:

(11*a*) $x_i \geq 0 \quad \forall i \in N$ and (11*b*) $\sum_{i \in N} x_i = 1$

From (11a) and (11b) it is clear the analogy between the set of imputations and the set of all probability distributions over the elements of N: an imputation is a distribution of the total payoff to the individual players. If $x = (x_1,..,x_n)$ is a probability distribution over N, we can consider the set function $x(K) = \sum_{i \in K} x_i$ as a probability measure over N.

Note, at this point, that an imputation is said to be in the core if $v(K)$ is at least equal to the sum of the payoffs of members of K. Clearly, successful imputations will depend on the interplay between coalitions and on the threats that can be enforced by coalitions. A theory that tries to deal with all this has to consider that if $v(K)$ is much larger than $x(K)$ there will be a strong reason for the coalition K to form and to impose an imputation x' that is "closer" to $v(K)$. This means that the equilibrium problem consists in finding a probability measure x which approximates in some sense to be defined the normalized superadditive measure v.

3.3 On payoff vectors.

Definition. A payoff vector $x \in \Re_n$ is an *imputation* for $\Gamma = (N, v)$ if it satisfies the following three conditions:

$x_i \geq v(\{i\}) \quad \forall i \in N \quad (IR)$ $\sum_{i \in N} x_i \leq v(N) \quad (F)$ $\sum_{i \in N} x_i \geq v(N) \quad (PO)$

In the familiar Edgeworth box all allocations in the lens shaped area bordered by the two indifference curves passing through the endowment point satisfy both individual rationality (IR) and feasibility (F). All allocations on the part of the contract curve inside the same area also satisfy Pareto optimality (PO).

They form the imputation set, denoted by $I(N, v)$. It is worth noting that in any two-person game the core coincides with the imputation space, since there are no intermediate coalitions between the singletons and the grand coalition.

The set of feasible vectors such that $\sum_{i\in K} x_i \geq v(K) \quad \forall K \in \mathrm{K}$ is called the *core*. This is a condition stronger than PO: indeed, it is so strong that there may be no imputation which satisfies it. It is easy to see that if the core of a constant-sum game is non-empty, the game has to be inessential; in other words, all essential constant-sum games have empty cores and lack equilibrium imputations (see Owen, 1982). The condition for an imputation to belong to the core can be seen as a logical extension of PO to all possible coalitions, and therefore be named *group rationality*. We can also define the core starting from the concept of "domination" or "improving". This definition was originally given by von Neumann and Morgenstern (1944), where it led to the concept of a solution for *n*-person cooperative games. Their solution is currently known under the name of "stable set".

Definition. For x,y in $I(N,v)$, y *dominates* x (or *improves* it) via K (written yD_Kx) if:

$$\left.\begin{array}{ll} y_i \geq x_i \forall i \in K & \\ y_i \neq x_i \quad \text{for at least one} & i \in K \end{array}\right\} \quad (12) \qquad \text{and} \qquad \sum_{i\in K} y_i \leq v(K) \quad (F_K) \quad (13)$$

Where F_K means the "feasibility condition for K". We write D_K for the relation of domination via K and simply D for the case in which domination holds via at least one coalition; $\overline{D}$ will be used when domination holds for all possible coalitions as a short form for $yDx \quad \forall K \subseteq N$. It is easy to see the three possible cases that can occur with respect to disjoint coalitions: i) yDx and $x \neg Dy$; ii) yDx and xDy ; iii) $y \neg Dx$ and $x \neg Dy$. Note also that in general D is not a transitive relation, since the coalitions may be different; however, $\overline{D}$ is transitive.

Luce and Raiffa (1957) change condition (12) to $y_i > x_i \quad \forall i \in K$ which is in general a stronger condition that is not necessary for $y^K > x^K$ (where $y^K = \sum_{i\in K} y_i$). So do Owen (1982) and Shubik (1982). However, in a transferable utility setting, it is assumed that if condition (12) holds, then it is possible to re-allocate the total gain in such a way that $y_i > x_i \quad \forall i \in K$.

Note also that there do not exist $x,y \in I(N,v)$ such that $y_i > x_i$ for all $i \in N$, since for y and x both (F) and (PO) hold.

Condition (12) states that at least one member of K prefers y to x and no one prefers x to y, while condition (13), the feasibility for K, says that the members of K are capable of attaining y. There might be some agent in N that prefers x to y: the second condition says that those that prefer y to x are strong enough to enforce the choice of y.

Let us say something also about the first condition in the form $x_i > y_i \quad \forall i \in K$. It means that all members of K have to be better off in order for an imputation x to dominate y. Clearly, $(x_i - y_i)$ may be different across members. This is also true in the case we write $x^K > y^K$, which is implied by condition (12): but now even the sign of $(x_i - y_i)$ might differ since it is possible that $(x_i - y_i) < 0$. Actually, $y^K > X^K$and $0 < (x_i - y_i) \quad \forall i, \quad i, j \in K - \{h\}$ together are equivalent to $(x_h - y_h) < 0$.

Then, we are left with the problem of explaining why should h join the coalition. Even the fact that $0 < (x_i - y_i) \neq (x_j - y_j) > 0$ requires awareness: it implicitly means that there is an agreement on the way to divide $\sum_{i \in K}(x_i - y_i)$.

Having said that, let us go back to the core.

Definition. The *core* is the set of all imputations that are not $\overline{D}$ dominated.

The following result is well known (see for example Owen (1982), who has a slightly different version):

Theorem. For y in $I(N,v)$, y is said to be in the core $C(N,v)$ if:

(14a) $\sum_{i \in K} y_i \geq v(K) \quad \forall K \in \mathrm{K}$ or if (14b) $\neg \exists K \in \mathrm{K} \quad | \quad \sum_{i \in K} y_i < v(K)$

The interpretation of the preceding definition and theorem can be misleading: they say that the core is the set of all imputations that are not dominated by others through all possible coalition. This means that if x is in the core and y is not in it, then we cannot have $y\overline{D}x$; however, we can have yD_Kx for some K. Condition 14 makes the core contain only the imputations that cannot be improved by all coalitions: if x is in the core we cannot find another imputation y which is preferred to x by all coalitions. Condition 14 works with all possible coalitions as much Pareto optimality does with regard to the grand coalition. To be clear, the fact that x is in the core does not preclude the existence of an imputation y -that is not in the core because it does not exists another imputation h dominated by y through all coalitions- such that yD_Kx for some coalition K: the core usually contains many D-dominated imputations. For example, each coalition of one individual prefers an imputation that assigns the total payoff to itself. This also reflects the fact that in disputing an undesirable imputation in the core a coalition K may propose nothing better than an imputation which dominates the first one through K and that would be dominated by a third imputation through another coalition. Eventually, we could ask why should the players be satisfied with a core outcome that can be improved by some coalition? One answer is that even with this definition there are many ordinary games that have empty cores in which every feasible imputation can be improved by all coalitions (e.g. the three players voting game).

Let us define *external stability* as the capacity of a set of imputations to dominate all imputations outside the set through some coalitions: the core lacks it. Some useful insights on this matter are provided by the comparison between the core and stable sets.

In order to clarify the differences between the core and the stable set, introduced below, let's restate the definition as follows:

Definition. The core $C(N,v) \subset I(N,v)$ is the set of all y in $I(N,v)$ such that:

(*) if $x, y \in C(N,v)$, then $x \neg \overline{D} y$

(**) if $x \notin C(N,v)$, then $\exists \quad y \in I(N,v)$, such that $x \neg \overline{D} y$

We shall now give the definition of stable set.

Definition. A stable set, $C^*(N,v) \subset I(N,v)$ is the set of all y in $I(N,v)$ such that:

(*) if $x, y \in C^*(N,v)$, then $x \neg Dy$

(**) if $x \notin C^*(N,v)$, then $\exists \quad y \in C^*(N,v)$ and $K \in \kappa$ such that $yD_K x$

The stable set has maximal internal stability given by (*), since $x \neg Dy \Rightarrow x \neg \overline{D} y$. At the same time its external stability property is minimal: an imputation x in a stable set can be dominated by an imputation y outside it. Clearly, it is always possible to find an $x' \in C^*$ that dominates y. But there may exist a y' outside the stable set which dominates x'; again, it is always possible to find an $\dot{x} \in C^*$ that improves y', and so on.

If a nonempty core exists, then any stable set must include it, since points in the core are not dominated by all coalition. Moreover, if the core is externally stable, which is not true in general, then it is the unique stable set.

Notice that $xDy \neg\Rightarrow x\overline{D}y$ but $x \neg Dy \Rightarrow x \neg \overline{D}y$. Moreover, $x \neg \overline{D}y$ implies yD_Kx for some coalition K but not $y\overline{D}x$.

4 On payoff imputations in the core

The concept of the core is defined in terms of sums of individual payoffs. We can find the minimal value Z of $\sum_{i\in N} x_i$ that allows a nonempty core by solving the following problem:

$$Z = \min \sum_{i\in N} x_i \qquad s.t. \quad \sum_{i\in K} x_i \geq v(K) \quad \forall K \subset N\,, \quad K \neq N \tag{15}$$

where any solution is bounded below by $\sum_{i\in N} v(\{i\})$, that is, by the sum of the individual payoffs determined by IR through the definition of the α-characteristic function.

Nothing is said, apart from the assumption of transferable utility, about the way each coalition distributes the total payoff between members via side payments. Each agent will join the coalition K if $x_i^K \geq v(\{i\})$, that is, if each individual final payoff satisfies IR. Recall that in order for x_i^K to be an imputation it has also to satisfy (F) and (PO). For what follows I use superscripts to denote that an individual or total payoff is associated to a certain coalition.

I argue that there is room for a more stringent criterion in choosing whether to join a coalition or not than that of individual rationality condition as expressed in the preceding section. As with any other criterion it also should help to explain the emerging of the actual partition.

For what follows keep also in mind that the worth of a coalition K might depend on the pattern of coalitions between nonmembers of K that actually occurs, as shown above. However, I do not deal directly with this hypothesis that clearly complicates the matter.

A player should analyze each agreement that he faces and join a coalition only if he is better off with regard to all other coalitions available. In the case of symmetric games, under strict superadditivity, the formation of the grand coalition is highly probable: recall that in symmetric games the power of a coalition is a function only of its size. The reference point for player i is the payoff that he gets staying alone, that is, his individual payoff when the $(N-\{i\})$ coalition actually forms against him. Then, before deciding whether to join the grand coalition, he will compare $v(\{i\})$ with x_i^N.

I argue that in non symmetric games, when the actual choice of the patterns of coalitions is less determined, the player should look at all the payoffs conceivable under the many possible pattern of coalitions.

Let's see this in more detail. Suppose that y improves x via K and that $x_i = v(\{i\})$. I mean that with the imputation x agent i gets the minimum payoff he can expect from the game, since $v(\{i\})$ is computed with the minimax criterion: he stays alone against the coalition of all other players. Define the "*K-extra value*" as follows:

$$ev(K) = \sum_{i \in K} (y_i - x_i) \tag{16}$$

Under transferable utility this quantity can be computed: it gives a measure of how much the coalition K can guarantee to its members more than the imputation whose elements are the minimax payoffs to each agent playing alone.

Suppose an agreement is reached to divide $ev(K)$ between members of K in such a way that $\frac{ev(K)}{\#K} = ev_i(K) > 0$ for all $i \in K$. This last quantity is the quota of extra value of each agent in the coalition. As another example of a sharing system consider that $ev_i(K)$ is agreed to be proportional to x_i for all i or to some y_i^M, $M \neq K$: in the first case the distribution of extra value consider the "original" payoff, giving more of it to players that have a higher x_i ; in the second case the distribution consider how much each player would get if he join another coalition.

Then each player i faces a set of $ev_i(K) = (y_i^K - x_i)$, $\forall K \in \mathcal{K}$; clearly, $\mathcal{K}$ can be restricted according to the need of not considering implausible coalitions. Choose the maximum of this set and denotes it by $ev_i^*(K^*)$; it depends on the particular K^* and it may not be unique.

I claim the following:

1) in the process of deciding to join some coalition, each agent looks at the minimum $ev_i(K)$ and not only at x_i as a reference point.

2) an agent may find $ev_i^*(K^*)$ and join consequently K^*. Otherwise, facing the difficulty of finding and forming such a favourable coalition, he can accept a smaller extra value.

The minimum positive $ev_i(K)$ is associated to the minimum y_i^K for all $K \neq \{i\}$. All this just means that each agent, chosing to join a coalition, will compare his payoff with the payoffs he would have got by joining other coalitions.

Recall that the set of coalitions that the player can belong to, is strictly smaller than K , the set of all possible coalitions. Then, it is probably useful to consider the set of individual extra-values as row vectors in which $ev_i(K) = 0$ if $i \notin K$ and it is presumably larger than zero if $i \in K$. In this case $ev_i(K)$ is a function from K to $\Re^+$.

The choice of the domain of $ev_i(\cdot)$ is delicate: recall that I said that the extra-values of each agent for any coalition are "presumably" larger than zero if $i \in K$. These are row vectors in which the first string of $2^N/2$ elements are agent i's extra-values associated to his available coalitions. The second string of $(2^N/2 - 1)$ elements is formed by agent i' s extra-values that result when coalitions form without him and he stays alone. The problem is what constitutes x_i in the elements of this second part of the row. Let us see 3 cases, considering that $\mathrm{K}_{N\setminus\{i\}}$ denotes the set of all coalitions which player i does not belong to:

1) If $x_i = v(\{i\})$ then $ev_i(M) \quad \forall M \in \mathrm{K}_{N\setminus\{i\}}$ is at the worst equal to zero (that is $y_i^M = x_i$), and not less than zero since agent i can always stay with his endowment.

2) If $x_i = x_i^K$, K being one of his available coalitions, we can partially see the effects of the coalition partition of K on the imputation proposed by K. Partially because $ev_i(M) \quad \forall M \in \mathrm{K}_{N\setminus\{i\}}$ will refer only to one of the coalitions available to agent i.

3) If we consider $x_i = x_i^K, \quad \forall K \in \mathrm{K}_i$ we have a complete knowledge but the row vector is very long!

Moreover, if we consider the effects of any coalition partition for the second string of the vector, we should do the same also for the first part. Thus I should have written $ev_i(K,P_{N\backslash K})$, meaning that it depends both on the coalition chosen by i and on the partition resulting between agents not in K. Under assumption 3), $ev_i(\cdot)$ is a function from $\mathrm{K}_i \times \mathrm{P}_{N\backslash K}$ to $\Re$.

The cardinality of K_i is $2^N/2$; we cannot calculate the cardinality of $\mathrm{P}_{N\backslash K}$, also because it clearly varies with the particular K in K_i . The resulting cardinality of this domain of $ev_i(\cdot)$ looks quite a big number anyway. Moreover, consider that the row vector $ev_i(\cdot)$ is different from the row vector $ev_j(\cdot)$, and we have N of them: this gives rise to a intimidating matrix of real numbers quite difficult to evaluate.

However, I feel that in real situations the partition of K typically consists of a few coalitions, even if N is "large".

In a simpler way, we can see $ev_i(\cdot)$ as a function from K_i, the set of all coalitions available to agent i, to $\Re^+$. Clearly in doing so, it does not gather information on the influence of the coalitions which i does not belong to.

I started all this looking for a different lowest bound from $x_i = v(\{i\})$ for individual rationality: I think that introducing a different criterion for (IR) will change significantly the size of the resulting core. Moreover, it seems that the size and the existence of a nonempty core will be better seen to depend on the successful partition of coalition.

That the size of the core depends on the relative value of the grand coalition with regard to the values of the other coalitions is clear from (15): the larger the difference Z-$v(N)$ the more numerous are the imputations that satisfy the linear programming problem.

The second point regards the partition structure associated to core imputations: are all imputations in the core available to each single player regardless of the existence of particular effective coalitions? In other words, is the set of core imputations free from the influence of a particular partition of K? It seems that this problem concerns more the choice of the final payoff than the selection of core imputations: as long as the core is large, the study of the partition structure can provide a way to characterize the final outcome.

4.1 On restrictions to the set of the outcomes available to coalitions

Rosenthal (1972) examines one way to restrict the set of available outcomes. He starts giving two reasons for the inadequacy of the characteristic function form (*cf*-form in his notation), and he supports them by two examples. The analysis is developed in a non transferable utility setting, while all my previous discussions are on transferable utility games.

The first reason is that any asymmetry in the role of players, clearly described by the normal form, is not evident from the *cf*-form description. The second reason for which the *cf*-form appears inadequate is its restricted view of threat possibilities.

The *effectiveness* form of a game or *e*-form, proposed by Rosenthal in a non-side payment setting, consists essentially in a function (the effectiveness function) which for each coalition, maps every outcome x in the set of all possible outcomes X, into a collection of subsets of X. Clearly, there is then an ordinal, vector valued utility function, that maps each outcome into the set of n-tuples of utility payoffs. The effectiveness function identifies, for any proposed outcome x, the collection of alternative sets of outcomes which the coalition can enforce against x. This is equivalent to saying that the coalition can restrict the negotiation process to some of the outcome in X.

It is evident that the *e*-form allows for the evaluation of any proposed outcome in terms of what alternatives might be available as counter-proposals. With regard to the restriction of the set of outcomes, the *e*-form goes in the same direction as the partition function form recalled above. With regard to the way this restriction is actually performed, the *e*-form is close to the ideas which the bargaining set and all related concepts of objections and counter-objections are based on. Rosenthal claims that his framework, still being general enough to model cooperative games in normal form, is not so general as to preclude a meaningful theory of stability.

4.2 On a characteristic function that assumes that outsiders do contribute

The title of this paragraph might be misleading. I just want to describe a situation in which the members of a coalition, before setting their strategy, have some information on the joint strategies used by the complementary coalition. An information may well be a belief, or an objective probability, about the particular event "use strategy A". As a particular case, the use of strategy A can imply the contribution to the provision of a public good: from this, the title.

Let's define by $x^K = \sum_{i \in K} x_i$ (respec. $x^{N\setminus K}$) the joint payoff of coalition K. $S^{N\setminus K}$ is the joint strategy space of coalition $N\setminus K$. For members of K the complementary coalition can randomly use any strategy in $S^{N\setminus K}$. Then, we consider $s_r^{N\setminus K}$ as one of the R values that can be assumed by the discrete random variable $s^{N\setminus K}$.

In this setting x^K is a function of a random variable and itself a random variable, $x^K(s^K, s^{N\setminus K})$ for any strategy of the coalition K. Then we can compute the following:

$$\mathrm{E}\,(x^K(s^K, s^{N\setminus K})) = \sum_{r=1}^{R} x_r^K(s^K, s_r^{N\setminus K}) p_r(s_r^{N\setminus K}) \quad \forall s^K \tag{17}$$

where $p_r(s_r^{N\setminus K})$ is the probability that the coalition $N\setminus K$ uses the strategy $s_r^{N\setminus K}$. If members of K know the discrete density function of $s^{N\setminus K}$ they can calculate $\mathrm{E}\,(x^K)$ for any s^K and then they can use the s^{*K} that maximize x^K.

$$x^{*K} = \max_{s^K \in S^K} \mathrm{E}\,(x^K(s^K, s^{N\setminus K})) \tag{18}$$

This model is barely meaningful without an intertemporal setting. For members of K, to know the probability distribution of the other coalition's strategy, means that they observed in the past its behaviour: they have objective probability and they take decisions in a risky environment. Alternatively, we consider them as having subjective probabilities or beliefs about $s^{N\setminus K}$, and as acting under uncertainty. In this case it would be interesting to model also the updating of this beliefs after some realizations of $s^{N\setminus K}$ have been observed.

It seems to me that any recognition of this sort leads to models that restrict the set of available strategies to coalitions. For example, the application of a social norm might be regarded as such a restriction. If this is the case, it is not consistent to consider that the norm applies to one coalition while the other simply maximize thereafter. Thus, if we recognize that every coalition follows a social norm and continuously updates its beliefs on other people's behaviour, we have to consider also their interaction in terms of chosen strategies.

It should be clear how this discussion is related to the problem of providing a public good: improving is made more difficult for coalitions since it is assumed both that the per capita cost of the public good is a decreasing function of the size, and that the members of other coalitions will not contribute to it, enjoying free riding. What I am saying here is that a social norm might exist that provides a rationale for a tax system which is based on every agent regardless of his membership. In other words, I am considering a social norm as a private and widespread incentive to a certain behaviour: in such an environment each agent faces the probability that the others will use a certain strategy instead of another.

Consider further that the recognition of such a probability is a characteristic of individuals and suppose that it concerns a cooperative strategy as opposed to a Nash strategy. We can study its distribution over individuals and say that a society enjoys a confident environment when this distribution satisfies some criteria of the statistics of relevant data. It seems to me that a confident environment is a major condition for any economic activity, while homo economicus always behaves in a fiction of the type "homo homini lupus".

5 The cooperative approach to externalities and public goods

Having decided to use the coalitional form to model a situation, the first step is to settle whether or not to assume side-payments. The use of money as a transferable commodity with constant marginal utility in partial equilibrium analysis seems to be reasonable.

We can assume the following utility function for each player: $U(x,m) = u(x) + m$, where x is a vector of goods and m is u-money, clearly with constant marginal utility. With this assumption a comparison of utilities is indeed implied, but only by way of the money mechanism: fundamentally the individual utilities are still independent and money has the role of a decoupler. Notwithstanding this consideration, it seems fair to say that when the economy is being studied as a whole the side-payment assumption is unreasonable.

The second important step is to verify that the game is actually a c-game: this means that the coalitional form represents all the vital informations contained in the strategic form.

This is not the case, in general: it must be recognized that the acceptability of the characteristic function is strongly related to the absence of externalities to and from members of a coalition.

For example, when the threat structure is not symmetric the characteristic function hides the important fact that the cost for some player to hold another player in a certain position (a payoff associated with a particular coalition) might be much higher than the cost for the latter to hold the former.

This is the case for many situation involving public goods. In the literature models are presented in such a way that they are fit by a characterization function based on the maximin criterion. I argue that if

1,051 Rossi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

models were more realistic (in the sense of being more descriptive) they would typically constitute non c-games under the usual characteristic function. Is it possible to re-define the last one in order for such models of public goods to be c-games?

For example, Rosenthal (1971) suggests that economies with externalities are not adequately modelled by characteristic function describing games with cores as in the famous example of Shapley and Shubik (1969). Basically, his criticism concerns what constitutes "reasonable actions" by agents in the complementary coalition to K when one is trying to evaluate $v(K)$. Actually, the characteristic function gives a pessimistic lower bound on what a coalition can obtain: it rules out the possibility that the same coalition might get more if nonmembers acted reasonably.

A different definition of the characteristic function does not imply a change in the concept of domination. Clearly, if what a coalition can ensure itself changes, the conditions for dominations will be affected as a consequence. Still, the condition of preference and that of feasibility might be expressed in the same way.

When public goods and externalities are present the characteristic function used as a basis for the analysis of the core must be at least justified in an ad hoc manner in order to establish if it is a good enough representation of reality. Rosenthal (1972) suggested a modification of games in coalitional form called "cooperative games in effectiveness form".

In general it can be said that, by the very nature of externalities and public goods, until the political and social constraints on feasible actions have been specified, the game is not sufficiently well defined for analysis. Public goods are supplied by processes that aggregate individual decisions politically; external economies and diseconomies are under strategic control of individuals or groups that can act directly. A direct implication is that a game that models such situations should at least have a description of the strategic power of groups at the voting level or at the level of any other social decision mechanism.

Thus, the main distinction between externalities and public goods for what concerns modelling, is that the latter imply decisions of production that involve public institution and political processes.

Even in the case of a private provision of public goods, it is clear that the relevant decisions are not straightforward, being linked to the delicate process of knowing the individual willingness to pay. On the other side, externalities can be caused by any number of individuals engaged in economic activities. Thus, a distinction is useful only if we are concerned by mechanisms. It is clear that for what concerns the coalitional form, externalities are caused by the production of public goods because, by definition, we cannot exclude any agent from the consumption of the latter.

In economies with pure public goods the concept of Lindahl equilibrium has been thought to play, mutatis mutandi, the same role as the competitive equilibrium. At the Lindahl equilibrium there exists a vector of personalized prices such that at those prices every agent demands the same level of the public good.

The first-best condition for an optimal provision of a pure public good states the equality of the sum of the marginal rates of substitution (between the public good and some private good) and of the marginal rate of transformation. The intuitive interpretation of this condition is clear: the total marginal benefit of an extra unit of the public good is equal to the sum of the benefits that each agent gets.

This condition applies in a first-best situation: either a fully controlled economy or a competitive economy in which the government is able to levy first-best lump-sum taxes, both to finance the expenditure and to redistribute income.

The analysis of the equilibrium supply of a public good in absence of the government (subscription equilibrium) is usually derived from the assumption that each agent takes the others' supply of the public good as given. Under the rather strong assumption of identical individuals, this leads to a Nash equilibrium in which, typically, the total supply is smaller than that requested by the optimality condition. Each agent determines his expenditure in such a way that his own marginal rate of substitution is equal to the marginal rate of transformation. The inefficiency arises because each consumer faces a prices equal to that of the public good, whereas some of the benefits accrues to others. To be clear, each agent is only concerned with the benefit he personally can get from additional units of public good, without regard to other agents' benefits. Thus, in the simple case of one private and one public good, the sum of individual MRS is larger a number of times equal the number of agents, than the MRT. With regard to the reference point, that obtained by maximizing social welfare allowing for lump-sum taxation, the Samuelson condition is met at a lower level of public good output.

Dropping the restrictive assumption of identical individuals, the comparison between the subscription equilibrium and the optimum is not straightforward, even if the governments is assumed to be able to impose lump-sum taxes. In this case the optimum choice depends on individual tax shares: if shares happen to be relatively large for individuals who do not like the public good with respect to those of others, then the subscription equilibrium may be such that it coincides with a level of public good output which is higher than the optimum.

On the other side, also the determination of public spending by means of voting has been deeply investigated in the literature. No decisive result emerged, anyway: it is not possible to affirm that majority voting leads to a level of public spending that is below or above the social optimum. It is clear that, in order to reach more positive conclusions, it is necessary to take account of the role played by the political procedures, by the legislator and bureaucrats, (in general of the machinery involved by a representative democracy) and by the lobbies that are likely to appear.

More recently, the conditions for an optimum private provision of a public good were studied by Bergstrom, Blume and Varian (1986), Güth and Hellwig (1986), and Varian (1990) among others.

Any Lindahl equilibrium satisfies the necessary conditions for a Pareto-efficient supply of a public good in a full optimum. A closer inspection of the Lindahl equilibria shows that they are not as attractive as the competitive equilibria. Apart from the problem of implementation, Champsaur (1976) has shown that Lindahl equilibria are not even necessarily symmetric.

Kaneko (1975, unpublished paper, see Kaneko (1977)) proved that the Lindahl equilibria never coincide with the core in an economy with one private good and freely transferable utility if the public good is produced at a Lindahl equilibrium.

A more general criticism[6] to the appropriateness of the concept of Lindahl equilibrium goes as follows: Foley (1970) showed that the Lindahl equilibria belong to the core when the production set is a convex cone. When this is not the case the public good producer obtains pure profits, and it depends on their distribution whether or not the Lindahl equilibria belong to the core. To select an appropriate distribution is a tricky question: for example, if we say that profits should be equally distributed then we should do the same for production costs, and this make Lindahl prices useless. Furthermore, the set of distributions such that the Lindahl equilibrium is in the core is quite large, as suggested by the fact that the core itself is large.

6 I am not concerned here with the main obstacle to the Lindahl solution: this is its implementation, due to information problems.

a) if $w_i > w_j^*$ he gets the good, he pays w_j^* , and loses $(w_j^* - \hat{w}_i) > 0$.

b) if $w_i = w_j^*$ there is a tie.

c) if $w_i < w_j^*$ he does not get the good.

B) If $w_i = \hat{w}_i$ he does not get the good.

C) If $w_i < \hat{w}_i$ he does not get the good.

Notice that I assume that agents are not indifferent between acquiring the good at the price they are willing to pay and not getting the good; moreover, the second highest valuation is regarded as the reference price of the good. Then, a statement such as "...he does not get the good and loses $(\hat{w}_i - w_j^*) > 0$ " makes sense.

It is often said that to tell the truth is a dominant strategy in this context: this means that a player has no incentive to move away from it whatever action is taken by the others. Rationality in this situation means trying to get the good as long as the price paid is smaller or equal to the willingness to pay.

Consider now, that under Case 1) the path A) gives less risk of not having the good, at zero cost; by a corresponding reasoning the path C) is not rational. Under Case 2), the path A) is the only way to get the good. In both cases it is rational to offer a price higher than his own willingness to pay, since it is the only way to get the good. Under Case 3), it is not rational to get the good and agent i should offer a price lower than his willingness to pay in order to avoid the path 3)A)a). Then, telling the truth is just the only way to avoid 3)A)a): in this sense, and considering the limit of the model, it seems to me that the term "dominant strategy" is misleading.

From a descriptive task we can go to a normative one: the problem is that no agent knows in advance what w_j^* will be. Nobody knows in which case he is playing, but has to estimate the valuations of other players. I argue that a less risk averse player (and more expert) would make a hypothesis about the case he faces. Suppose that he thinks that Case 1) is highly probable: he will not tell the truth but will announce a price higher than his willingness to pay.

PART 3:

ON TAXES AND PUBLIC GOODS

The first sign of taxes is found in the Sumerian civilization: among many clay cones inscribed in cuneiform which clearly deal with tax rules, there is one that reads as follows: "You may have a Lord, you may have a King, but the man to fear is the tax collector". Actually, the birth of democracy came about for tax motives: the demands of King John of England had been getting more and more exacting, and in 1215 the barons forced him to grant the Magna Charta, which ruled that "no scrutage or aid, save the customary feudal one, shall be levied except by the common consent of the realm". Eventually, this led to the principle of "no taxation without representation" which is now contained in most constitutions.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

 DOI: 10.2870/7482

LP

Table of Contents

1 Taxes and public goods 1
1.1 Some basic notions on tax systems 4
1.2 How much of public goods with different taxes ? 5

2 Tax allocation systems 8

3 Voting 21

4 Voting over tax allocation systems. 23

1,013

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

1 Taxes and public goods

If there were no governments, taxes would not exist. A government does provide some sort of public good, even at a minimal level: defense against enemies for all the people of a tribe is a public good supplied by the government through its army. The cost of sustaining the army is faced by each citizen according to some share system. There is no substantial difference between a tax on personal income [1], which allows for the consumption of a publicly provided good, and the monetary value paid for a private good (its price times the purchased quantity), apart from the fact that taxes are typically compulsory.

However, even from the simple example of the military defense, two peculiarities of the tax with respect to competitive prices do emerge: -firstly, individuals cannot directly choose the quantity of public good that they want to consume: in general, for the very nature of the goods they are uniformly provided; -secondly, given the provided level of the defense, and given any cost sharing scheme which does not take into account individual preferences, then some groups of people get more benefits than some others: consider for example, the case in which all individual taxes whose sum covers military costs, are equal. In general, due to information problems, individual taxes do not reflect individual marginal benefits.

Summing it up, individuals cannot adjust their consumption to their preferences by either acquiring different quantities (at fixed "prices") or paying different prices. This basically gives rise to important distributional issues.

In any economic system taxes directly transfer wealth from one consumer to another when subsidies are allowed; furthermore, taxes are used to produce (public) goods whose effect on each consumer's real income depends on his own preference for the particular good: a national park that is run with tax revenues benefits much more a bird watcher than a urban professional that loves watching television during his weekends. The redistribution effect of taxes in a public good economy is not only due to the intrinsic properties of the tax system but also to the quantity and the quality of public goods supplied in relation to consumer preferences. Typically, some individuals are forced to give up part of their wealth in the form of a pecuniary asset, while some others receive it as (partially-free) goods and services.

1 In general, a fundamental difference between a tax and a price that an agent has to pay to consume some good or service arises from the fact that the person who effectively pays a tax is not necessarily the person upon whom the tax is levied. The true incidence of a tax has been studied with regard to different groups in the economy and from other points of view. An example of the first approach is given by considering the effect of a tax on the production of a commodity on producers' profits, on consumers' income, and on the incomes of those who supply factors. The study of regional incidence and that of the effect of taxes or government expenditures on the position of individuals at different points on the income scale, are examples of the other approach.

 DOI: 10.2870/7482

In general, we can say that there are two sources of conflict in the public sector. Firstly, given the amounts and the types of public goods to be produced, each individual wants to contribute as little to the total cost as he can, shifting it as much as possible to other consumers. Any shift of taxes from one individual to another is clearly a gain for the first and a loss to the second; thus, there is a monetary redistribution involved. Lastly, given the distribution of taxes, individuals with different tastes and after tax income will like the government to spend on different public goods; then, it is clear that a non-monetary redistribution is also associated to any provision of public goods.

In other words, the introduction of public goods and associated taxes in a private good economy causes a redistribution of income which is due to two factors: the first, which I call *pecuniary effect*, depends directly on the cost sharing scheme, that is, on the chosen tax vector. The second, the *non-pecuniary effect*, is due essentially to the heterogeneity of preferences, and to the fact that everyone consumes the same level of the public good output: since the utility given by the consumption of a certain level of a public good varies across individuals[2] according to their preferences, the introduction of a public good benefits some individuals more than others[3].

Assume that individual utilities depend on the chosen vector of public goods q and on individual after-tax income, given by private endowment e_i minus the individual tax share: $u_i(q, e_i - t_i)$. The second argument depends directly upon the tax allocation method which splits the total tax $T = \sum_N t_i$ among individuals according to a rule used by public decision makers. The first argument is a vector of levels of public goods whose components can be varied according with public decisions, from zero to any maximum level within the resource and technology constraints.

It depends on the particular political decision making process whether is T or q that is decided first. In the real world the distinction between these two stages is usually blurred. For the sake of simplicity, let us assume that q is defined first. Then, the definition of tax allocation method implies directly a choice between pecuniary effects, since it defines after-tax income. On the other side, assuming T fixed it is easy to understand

2 Recall that neither quantity adjustment nor personal (efficient) taxes are allowed: the first is impossible for the same nature of a pure public good, while the Lindahl solution is too much demanding in terms of information.

3 In some cases this is true simply because only some groups are entitled to the use of the provided public goods: for example, food stamps are allocated only to low income individuals.

how variations of q give rise to non-pecuniary effects: given heterogeneous preferences, a variation in the level of one public good causes different variations of individual utilities. The marginal utility of a public good varies across individuals and this leads to non-pecuniary distributive effects when a variation in output occurs.

To clarify this point, consider two public good bundles $q^1 q^2$, and a partition of the set of agents into two complementary coalitions A and B. Then, assume that $q^1 \neq q^2$ implies $(q^2, e_i - t_i)P_i(q^1, e_i - t_i)$ for all $i \in A$ and $(q^1, e_i - t_i)P_i(q^2, e_i - t_i)$ for all $i \in B$. It is apparent that the choice of one of these public good bundles is not neutral with regard to the overall distribution of utility levels.

As a consequence of this duality, a partial equilibrium analysis of a public good economy should have two stages: given the vector of public goods, we can determine the tax vector and its effects; given the tax vector, we can study the proposed public good bundle. This also means that, if we want to analyse possible outcomes of a voting process over tax schemes -when also the public good vector is under discussion- it is simpler to compare public good bundles which have the same non-pecuniary redistribution effect: otherwise, we should assume not only that individuals have complete information about pecuniary redistribution effects of different tax systems, but also that they can compute non-pecuniary redistribution effects of different public goods bundle.

Alternatively, we can also use the following assumptions: -public good bundles only differ for the relative quantities of each public good and not for their type. This means that a higher tax revenue only implies a proportionally higher level of provision of each public good, and not a higher number of provided public goods; -moreover, these different public good bundles cause the same qualitative redistribution effect, in the sense that only its quantitative character changes. Formally, we would require that, following the preceding example with two coalitions, we only deal with those $q^j \neq q^1$ such that the choice of any q^j implies $(q^j, e_i - t_i)P_i(q^1 e_i - t_i)$ for all $i \in A$ and $(q^1, e_i - t_i)P_i(q^j, e_i - t_i)$ for all $i \in B$.

In general, concerning the question of voting over cost sharing schemes we face another problem: it rises from the fact that after-tax income is a parameter of each consumer when he decides upon the optimum amount of public goods: it is clear that, when individual after-tax incomes change as a consequence of the vote, so will do individual preferences for the public goods.

1.1 Some basic notions on tax systems

A tax is nondistortionary if and only if there is nothing an individual can do to alter his tax liability: such taxes are called lump-sum taxes. Any tax on income and on commodities is distortions. The magnitude of the inefficiencies is measured by the so called *deadweight loss*, the difference in revenues that could be obtained from a lump-sum tax as compared to a distortions tax, with the same effect on the level of welfare of consumers.

The consequence of any tax can be decomposed into an income effect and a substitution effect. Since only the income effect is associated with a lump-sum tax, the magnitude of the deadweight loss is related to the magnitude of the substitution effect. The deadweight loss increases in general more than proportionally with increases in the tax. It is also proportional to the compensated elasticity of demand and to the elasticity of supply. In particular, in the case of taxation on labour supply, the income effect increases the amount of labour while the substitution effect leads to a decrease: thus, the total effect is ambiguous.

A tax system is said to be *horizontal equitable* if individuals who are the same in all relevant respects are treated equally. Many taxes treat differently individuals who differ in tastes.

A tax system is said to be *vertically equitable* if some individuals who are in the position of paying higher taxes than others, actually do so. This definition reveals three problems: how to determine this individual position; which has to be the difference of the tax between people in different position; which is the mechanism that implements the rule.

Any tax system cannot attempt to measure overall well-being and has to be based on a narrower definition of welfare: then, it must be inherently unfair. Typical measures of welfare which represent individual ability to pay are real income, potential income, taxable income[4] and consumption. Alternatively, the basis of taxation may be the benefits that individuals receive from public services: however, they are very difficult to quantify.

Let t_i be the burden of a tax on individual i, and let M_i be his total income. The ratio t_i/M_i is called the *effective tax rate* on that individual. When this ratio is the same for all income levels the tax is called *proportional*; when it is higher for the rich (poor) than for the poor (rich), is called *progressive* (*regressive*).

4 Taxable income is traditionally defined as income over and above the subsistence level.

A comprehensive uniform sales tax is a uniform tax on output and is thus equivalent to a uniform income tax. A comprehensive uniform sales tax is also equivalent to a uniform value-added tax.

In a competitive market the more (less) elastic the demand curve and the less (more) elastic the supply curve, the more the burden of a tax on production is borne by producers (consumers). Moreover, it makes no difference whether the tax is imposed on the producers or on the consumers. This result analogously holds in the case of a tax on labour. However, the effect of the imposition of a tax depends critically on the nature of the market; with monopolists the result also depend on the marginal cost curve. While in a competitive market there is no difference between a *specific tax* and an *ad valorem tax*, the latter is superior to the former in the case of monopolistic industries: the monopolistic' s output is higher with an ad valorem tax. It is not possible to make any general and definite predictions about the incidence of taxation in the case of oligopoly.

If everyone were identical there would be no reason to impose a distorting tax structure instead of a uniform lump-sum tax. The use of distortions taxes is due to the desire of redistribute income. A government could impose higher lump-sum taxes on individuals who are in a better position, provided it was able to ascertain personal abilities directly. However, a government can only base the tax on observed variables such as income and expenditures, and the resulting tax structure is inevitably distortions. It is then apparent that there is a trade-off between equality and efficiency, between a distortions tax structure and a lump-sum tax. The more progressive the tax structure and the greater the deadweight loss.

The optimum tax structure is defined as the tax structure that maximizes society's welfare, where the balance between deadweight loss and inequality reflects social attitudes toward equality and efficiency.

Taxes that minimize the deadweight loss associated with raising a given amount of government revenue are called *Ramsey taxes*: under some simplifying assumption they are proportional to the sum of the reciprocals of the elasticities of demand and supply.

1.2 How much of public goods with different taxes ?

Since the work of Samuelson (1954) the basic conditions for the optimum supply of a public good are well known: the sum of the marginal rates of substitution (MRS) between the public good and some private good must equal the marginal rate of transformation (MRT). The marginal benefit of an extra unit of a public good is indeed the sum of the individual benefits, while an extra unit of a private good is only given to one person and then consumed by him. This basic rule is obtained by maximizing a SWF subject to a constraint given by an aggregate production relationship between one public good and a vector of private goods. Clearly this model is not concerned by the financing problem: it describes the situation in which the government is able to levy first-best lump-sum taxes.

The Samuelson efficiency conditions need to be amended if the public good must be financed by distortional taxes. Pigou (1947), Stiglitz and Dasgupta (1971), Atkinson and Stern (1974). That distortions caused by taxes will affect the optimum supply of public goods has been known by economists as far back as Pigou.

Basically, policy-makers have to insure that the Samuelson condition hold. A part from the question of their actual behaviour, which can be influenced by different goals, the main problem is how to obtain the information necessary to achieve efficiency. The relevant information are as follows: tastes and costs associated to public goods; individual contributions; reactions of the private sector to policy parameters.

One of the main information problems is suggested by the fact that efficient conditions depend upon the distribution of welfare: this means that the efficient output is not unique. Aaron and McGuire (1969) show that this is true even with a given distribution of income. To deal with these problems, economists proposed several approaches: individual benefit taxation, incentive compatible mechanisms, voting, cost benefit analysis and the so called *Tiebout mechanism* when the public goods have a spatial dimension. In the benefit approach the relation between the tax-payer and the government is one of *do ut des*, with the emphasis on exchange; this leads naturally to ideas such as the Lindahl equilibrium. The ability to pay approach separates contributions from benefits received; again, this drives attention to distributional issues.

The Lindahl approach requires that individuals be taxed an amount equal to their marginal valuation times the level of the provided public good. It implies that the determination of efficient public good levels simultaneously carries with it a method for financing the public good production. This approach breaks down since the demand curve has to be supplied by individuals who will act as to minimize their tax payments.

Groves and Ledyard (1977) and Green and Laffont (1979), between others, suggested that honest revelation can be induced by making a person's tax almost independent of his stated preferences. The main problem is again that, since the chosen level of public good provision is approximately independent of individual's response, individuals will rationally refuse to participate to the mechanism. However, it is apparent that the availability of low cost technology to monitor some individual actions, would surmount this type of problems.

The determination of an endogenous tax allocation method has been examined also by considering voting processes. An equilibrium outcome may well not exist; to some extent this can be solved by fixing the tax shares and thus (partially[5]) eliminating distributional considerations from the space of the alternatives. If the choice of public good supply is made by majority rule, then the outcome will be the personal optimum of the individual with the median preferences: it is very unlikely that such an outcome will be efficient. Moreover, since intensity of preferences are difficult to report, the process cannot take into account distributional considerations.

Under the strong assumption of identical individuals, Atkinson and Stiglitz (1980) are able to extend the Samuelson-type analysis to the case of distortionary taxation on the consumption of private goods, formally proving some long standing intuitions.

The use of indirect taxes which generate an excess burden clearly modifies the basic formula on the optimum provision. To see this, consider that the idea behind this efficiency result is that the public good is produced up to the point in which the sum of the marginal rates of substitutions equals the marginal economic rate of transformation. However, with non lump-sum taxes the marginal physical rate of transformation needs not to coincide with the economic one since, both the social cost of the marginal revenue and the marginal utility of income vary with the tax structure. The problem is the same of that of finding the correct social rate of discount for project appraisals: the private rate of return -which in a competitive model equals the physical MRT between outputs in different periods- does not equal the rate at which society and the government can transfer resources between periods. Then, in a world with distortions, the social desirability of taxes is not implied by the private profitability of the project.

More precisely, an increase in the production of the public good may well increase or decrease the consumption of taxed goods. In turn, this respectively reduces or enlarge the amount to be raised. Secondly,

5 Recall the argument on non-pecuniary distribution effects.

one of the assumptions underlying the Samuelson' s formula is that the social cost of raising the revenue of one dollar equals the individual marginal utility of income: this is not true in the presence of non lump-sum taxes.

This analysis, carried out by the finding of first order conditions, does not establish the optimum quantity of the public good: this question can only be answered studying particular cases: for example by assuming that the utility function is additively separable between the utility from the private good and the quantity of the public good. We can plot the transformation curve between these two variables and find the optimum level of provision by maximizing the welfare of a representative individual subject to individual and government budget constraints. The slope of the transformation curve varies with the tax structure, while social indifference curves do not: as usual, optimum requires that they have a common tangent plane at a point. In general we cannot expect that the transformation curve assumes a particular form. However, given that it is concave and that it is steeper when considering a distortions taxation than when only lump-sum taxes are levied, the optimum level of the public good is indeed smaller when indirect taxation is emploied.

Apart from these considerations on the efficiency of the public good provision when different tax schemes are used, we can also discuss how the optimum supply is influenced by distributional considerations. Again, the results of this analysis are not general and depend on the assumptions about the utility functions of individuals, about the type of public goods, and about the way a SWF is obtained. When such variables are chosen and a framework is settled, results depend on the tax structure: considering lump-sum taxes, it is clear that the government faces a set of feasible taxes typically restricted to a uniform poll tax. The introduction of a distortions indirect tax leads to the study of the combined implication of both deadweight loss and distributional objectives. In general, the distributional effect of public goods modifies the value of the sum of MRS, while the distributional effect of a tax structure different from the first best lump-sum taxation influences the value of the MRT.

2 Tax allocation systems

There are two approaches to the problem of cost sharing. The work of Young (1988) is an example of the axiomatic approach. Kaneko (1977a), Kaneko (1977b), Mas-Colell (1980), Bergstrom, Blume, and Varian (1986), Mas-Colell and Silvestre (1989) are examples of the equilibrium approach: it deals with the problem of finding cost allocations which are endogenously generated in equilibrium. In general, these cost allocations are characterized on the grounds of equity and efficiency. Usually, the first step is to prove the existence and the uniqueness of efficient and fair allocations, associated with a tax allocation method. The second step explores the possibility of implementing such allocations, mainly in a decentralized way.

In paying taxes, people think about how much they are giving up rather than what they are allowed to keep: the focus is on costs rather than gains. Tax-payers clearly benefit from public expenditure, and the magnitude of individual benefits depend on preferences; but the emphasis is on the fact that any single individual enjoys essentially the same benefits irrespective of how much tax he alone pays: from the individual point of view the tax paid is a loss. Then, the problem of distributive justice in taxation can be seen like that of the fair allocations of costs among individuals. The ability to pay approach disregards the fact that benefits from public goods differs across individuals and studies the problem from the point of view of cost allocation. It is apparent that the condition for the efficient provision of public goods can hardly be met following this approach, since the individual marginal rate of substitutions are not directly involved in the solution of the cost allocation problem. In general the device used to consider individual benefits, along with a tax system based on ability to pay, is the rule of unanimity applied to utility functions relative to different allocations. Firstly, a tax structure associated to a range of distributions of after tax income is assumed. Secondly, the choice of the appropriated pair of good vector and tax vector is made subject to certain conditions on individual and group rationality.

The problem of fair allocation of costs among individuals has been given much attention in the literature[6]. The usual solution consists of allocating costs proportionally to the ability to pay[7]: this leads in general to a regressive tax structure. However, utilitarianism derives from the concept of equal sacrifice its argument in support of nonregressive taxation. The principle of equal sacrifice may be interpreted in different ways: we can distribute taxes so as to equalize the marginal rate of utility of all individuals, or the absolute loss in utility, or the rate of loss in utility. This principle suffers from several difficulties: the utility functions must be specified and measured; a specific criterion must be chosen and interpersonal comparisons have to be made. With regard to this problem, Young (1988) shows that, assuming that a tax structure satisfies a set of reasonable and desirable properties, there exists a utility function relative to which individual sacrifices are equal. Since this is a fundamental issue, I review his contribution.

6 See for example, Champsaur (1975) Young (1985), Young (1988), Mas-Colell (1989), and Weber-Wiesmeth (1990).

7 The proportional method equalizes absolute individual sacrifices relative to the utility function $U(x) = \ln x$

Let x_i, a measure of the ability to pay of individual i, be a component of the vector $x \in \Re_+^n$[8], and let N be the set of n individuals. Let T, $0 \le T \le \sum_i x_i$, be the amount of taxes to be collected. Let t_i with $0 \le t_i \le x_i$ and $\sum_i t_i = T$, denote the individual tax, a component of the vector t. Then , define a *tax problem* as the pair (x, T). Also define a *tax allocation method* (TAM, for short) by a function F from the set of all possible tax problems to the set of all possible individual tax vectors t. Thus, $F:\{(x,T)\} \to \{t\}$ and $t_i = F_i(x,T)$. Assume that $x_i = x_j \Rightarrow t_i = t_j,\ i,j \in N,\ i \neq j$ (symmetry for F), and that F is continuous on $\{(x,T)\}$.

A *tax scheme*, $t = f(x)$, is a function that gives a vector of individual taxes for each given distribution of taxable income: it is not a TAM since it does not depend on the total amount to be raised T. Introducing a parameter whose value depend on T, we can define a *parametric tax scheme* (PTS) $t = f(x, \gamma)$ for all $x > 0$ and all $\gamma \in [a,b] \subset \Re \mid f(x,a) = 0 \wedge f(x,b) = x$. A PTS must be continuous, and weakly monotone increasing in γ. To ensure that the total amount of tax is collected, chose a γ such that $\forall i,\ t_i = f(x_i, \gamma)$ and $\sum t_i = T$.

Definition 2.1 A TAM is consistent if for every $I \subset N$, x, and T:

$$t = F(x,T) \Rightarrow t_J = F\Big(x_J, \sum_J t_J\Big) \forall J \subseteq I \tag{1}$$

Consistency means that the way a coalition shares a given tax depends only on its own ability to pay.

Definition 2.2 A TAM is monotonic if everyone' s tax increases when the total tax increases. (It is strict monotonic if strict inequality holds in the right hand side of the following statement)

$$\{0 \le T \le T' \le \sum x_i\} \Rightarrow \{F(x,T) \le F(x,T')\} \tag{2}$$

Monotonicity does not answer the question of how a tax increase should be shared. If a TAM satisfies the composition principle, then every increment of tax is assessed equitably relative to tax-payers' current after-tax income.

Definition 2.3 A TAM satisfies the compensation principle if for all tax problems where $T < T'$ and $t < x$,

8 An assumption such as $x_i > 0$ means, for example, that x_i is individual i's taxable income.

$$\{t = F(x,T) < x\} \Rightarrow \{F(x,T') - F(x,T) = F(x - t, T' - T)\} \tag{3}$$

We would also like that a person who has a bigger ability to pay than another, still has a bigger income after the tax is paid.

Definition 2.4 A TAM is said to be *order-preserving* if, for all tax problems (x,T) and all i, j,

$$\{t = F(x,T) \wedge x_i > x_j\} \Rightarrow \{x_i - t_i \geq x_j - t_j\} \tag{4a}$$

A TAM F is said to be *strictly order-preserving* if

$$\{0 \leq T < \sum x_i \wedge t = F(x,T) \wedge x_i > x_j\} \Rightarrow \{x_i - t_i > x_j - t_j\} \tag{4b}$$

Definition 2.5 A TAM is *homogeneous* if for all $\theta > 0$ and all tax problems

$$F(\theta x, \theta T) = \theta F(x,T) \tag{5}$$

Given this desiderata to be satisfied by a tax allocation method, Young shows that they characterize what is called an equal sacrifice method. Two theorems follows the next definition.

Definition 2.6 Let $U(x)$, increasing in its strictly positive arguments, be a generic utility function for income. A TAM equalizes absolute sacrifice relative to $U(x)$, and then is called an *equal sacrifice method* [9], if for all tax problems and income distributions

$$t = F(x,T) \Leftrightarrow \exists c \geq 0 \mid [U(x_i) - U(x_i - t_i) = c] \tag{6}$$

Theorem 2.1 A tax allocation method satisfies consistency, strict monotonicity and order-preservation, and composition iff it is an equal sacrifice method.

9 Notice that F equalizes absolute sacrifice relative to $U(x)$ iff it equalizes the rate of loss in utility relative to $e^{U(x)}$. The criterion of sacrifice is not related to the form of the tax method.

Theorem 2.2 A tax allocation method satisfies consistency, strict monotonicity and order-preservation, composition, and homogeneity iff it equalizes absolute sacrifice relative to the utility function of the form $U(x) = \ln x$ or of the form $U(x) = -x^{\alpha}, \alpha < 0$. In the former case the TAM induces the flat tax scheme; in the latter case it is a parametric tax scheme of the form:

$$t = x - [x^{\alpha} + \gamma^{\alpha}]^{1/\alpha} \qquad \alpha < 0, \gamma \in [0,\infty]$$

Finally, it is worth noting that every homogeneous and equal sacrifice tax method is progressive in the usual sense: whenever $t = F(x,T)$, $x_i \leq x_j$ implies $t_i/x_i \leq t_j/x_j$.

Summing it up, we have seen how some reasonable principle of distributive justice imply the use of utility functions relative to which all individual sacrifices are equal. In other words, they imply the use of the class of equal sacrifice tax allocation methods; furthermore, these methods are progressive.

The main elements in all tax systems are tax bases, rate structures, and special provisions, such as exemptions and deductions. A theoretical analysis of a tax structure must explain these elements in terms of private and public choices: while private behaviour is usually modelled as self-interested, public decision makers are assumed to choose policies according to some social criteria that balance the trade off between efficiency and equity. I do not deal here with the reasons for the emergence of a particular tax system from a given set of political institutions. I will rather examine simple models that consider both the private and the public sector; moreover, some performances of linear income taxes are sketched.

The basic issues of fiscal policy are the following: first, is to be decided how much revenue is needed to finance public production; this question is equivalent, up to a given technology and for given prices, to the problem of deciding which and how much of public goods and publicly provided private goods are to be supplied by the public sector. Second, given a certain amount of revenue to be raised by taxing the public, how should the tax burden be distributed among economic agents?[10] More specifically, under how general a set of conditions can we establish the existence of a tax structure and competitive price system that is optimum from a social point of view when a government supplies the public good ? Given public goods, private goods,

10 On the assumption that considers fixed the required revenue see Greenberg (1975) and below.

and government, the question becomes: Utilizing a general array of taxes, can a government arrange its policy so that there will be the convenient generalized equilibrium together with a production of public goods such that a social optimum is achieved? In what follows, I will review the most relevant contributions in the literature.

Using a game theoretical approach, Guesnerie and Oddou (1979, 1981) introduced tax-rate restrictions directly into the definition of the characteristic function. In their model the production of the public good is financed through a proportional tax and each individual pays the same percentage of the private good with which he is endowed. Their task was to prove that the core of the associated game is non empty: this would guarantee the existence of a tax rate so that no coalition can benefit all of its members by proportionally taxing only their own resources. Since a necessary and sufficient condition for the non-emptiness of the core is that the game is Scarf-balanced, they proved that with a weak notion of superadditivity the proportional tax game is balanced if the number of individuals is no greater than 4. With weaker assumptions and using the notion of binary superadditivity Greenberg and Weber (1982) were able to extend the same result to any number of individuals.

It has been noted that minority protection and the existence of a nonempty core seem to be related. When unanimity is required, all imputations are in the core. With simple-majority voting and unlimited taxation there is no core. If minorities are forbidden to destroy resources to avoid taxation, the simple model in which a majority takes everything and a minority is given nothing, is a c-game [11]. If the minority can destroy resources, then we need to take this threat into account.

Mantel (1975) and Greenberg (1975) approached the same problem in a general equilibrium setting. In both their works the government, clearly a larger player than the others, appears as a *deus ex machina*. If we are to develop a coherent theory that includes the production and the distribution of public goods, we must also specify how the government forms its social welfare function[12].

Mantel (1975) states two results for an economy with private and public goods, interdependent preferences, and a tax structure[13] to finance the government's production. Firstly, there exists a competitive equilibrium for a given tax structure; secondly, there exists a socially optimum tax structure. The definition

11 Shubik (1982) defines a c-game as a game in which nothing essential to the purpose of the model is lost in the process of condensing the extensive or strategic description into a characteristic function.

12 At a more general level a fundamental question could be addressed by a theory of how consumers acquire preferences.

13 "Tax structure" is synonymous of TAM.

of social optimum is as follows: a tax structure $\bar{t}$ is socially optimum if there exists an equilibrium allocation $\bar{x}$ relative to it such that there is no tax structure t with associated equilibrium allocation x which is preferred by society to $\bar{x}$.

Greenberg (1975) compares the efficiency of lump-sum and sales taxes in a general equilibrium analysis. A lump-sum tax is such that it cannot be influenced by the consumer, it is independent of any economic variable and it is efficient, since it does not vary the (marginal) Pareto conditions. The equality of the rate of production transformation and of the rate of commodity substitution does not hold in the case of a direct tax (a sales tax) and then it is usually not efficient: its associated burden is bigger than that of an indirect tax.

However, these propositions answer the question of whether, given a fixed amount of taxes to be raised to finance public production, it is a direct or an indirect tax that imposes the greater burden. Greenberg argues that the restriction of fixed revenues makes no sense in general equilibrium analysis since prices vary with the tax system and then the expenditure for the same amount of public goods is not fixed. The main results of his work are the following: first, using a model similar to that of Foley (1967) and defining a Pareto optimum for it, any Pareto optimum can be achieved by lump-sum taxes; second, having defined a competitive equilibrium in an economy with public goods financed by sales taxes, it is shown that such an equilibrium exists; third, in general sales taxes are not efficient. Eventually, an example of efficient sales taxes is provided.

It is worth noting that two definitions of competitive equilibrium are given: the difference being that one of them allows for public goods to be used as production factors in the technology of the private goods. The equivalence of the two notions is proved without the need of an assumption of irreversibility (as in Milleron, 1972 [14]) since the private sector does not actually buy nor sell any public good.

Closer to the idea of voting over a TAM is the work of Romer (1975). He describes a basic model of a tax function that is linear in before-tax income: even if this is a very simple example of the set of all possible tax systems, it is apparent that the study of the distributive effects of its parameters provides useful insights on the alternatives available to the government. The basic data of the problem are as follows: -each individual is characterized by a non-negative index of ability n; -working a fraction L_n of a unit time period, each

14 The irreversibility in production has generally two meanings in the literature that deals with the provision of public goods: first, it guarantees that a sales tax is not transformed into a subsidy on purchases; second, with Lindahl-prices, when firms are allowed to buy from the government any level of public good to be used in production, it may be the case that the selling price to consumers is higher than that paid by the firm: then, there will be no equilibrium, since profits are not bounded.

individual of ability n earns a before income tax given by $y_n = nL_n$; -there is no saving, so individual consumption, dropping the index, is given by $C = y - T(nL)$, where this last term is the amount paid as income tax; -each individual maximizes the same Cobb-Douglas type utility function $U(C,L)$ subject to the budget constraint; -the tax function is assumed to be linear in before tax income: $T(y) = k + ty$, where parameters k and t are respectively the lump-sum parameter and the marginal tax rate.

From these assumptions and individual maximization, it follows that the before tax income depends on the type n of each individual, and for given n, depends on the pair of tax parameters. Conversely, the before tax income is an increasing function of the ability level (for n larger than a certain skill level m) for a fixed pair of tax parameters. An important restriction to the set of available tax parameters is given by the condition that after-tax income has to be non negative for all individuals

At the aggregate level it is natural to add other restrictions to the tax system available to government: -total taxes must not be less than the required revenue; -the revenue requirement must not exceed national income.

Values of (k,t) satisfying these restrictions and, for a given required revenue G yielding at least the required revenue, define the *tax possibility frontier* (TPF). It uniquely determines k for given values of G and t; also, given G, there is a unique value of k for each permissible value of t. Mapping the TPF on the (k,t) plane clearly shows the trade-off between the two parameters. As the tax parameters vary, individuals will adjust their work-leisure choice so that per-capita income before tax will be changing along the TPF.

The simple case in which for a given G no one would stop working is fully examined (the set of tax parameters is restricted in such a way that m is smaller than the lowest ability index n_0). This case is simpler to study since with everyone working, the shape of the TPF and then the behaviour of individual preferences as the tax parameters are varied are independent of the distribution of ability: they depend only on the relative values of the mean skill level of all individuals, $\bar{n}$, and of n_0. The first point to be noted is that a negative value of the lump sum parameter can be feasible only for a sufficiently large marginal tax rate and for sufficiently small per capita revenue requirements. Secondly, given the constraint that no one has negative consumption, if per capita revenue requirements exceeds the earning capacity of the least skilled individual, then negative marginal tax rates are not allowed. Thirdly, the ranking of after-tax income is the same as that of before tax income; however, the imposition of the tax will affect the distribution of income: an increase in the marginal tax rate increases the burden of those with higher than average ability.

The more general case in which $m > n_0$, that is, individuals with relatively low skill levels do not work, is analytically more complex. The qualitative results of the previous case hold only if preferences are single-peaked over the permissible set of tax parameters. The analytical complexity is given by the evidence that, if there are people idle, then the size of the work force varies as the tax parameters change: so, all the interesting economic variables now depend on the entire skill distribution. As a result, there is no guarantee that single-peakedness holds; however, Romer presents sufficient conditions for it to hold in this general case.

Mas-Colell (1980)[15] presents a model with k public goods denoted by $x \in K$ (which he calls *projects*), and with a unique private good called m, (which stands for money). n agents are endowed with nonnegative amounts of money w_i. Each agent is supposed to have preferences defined over $k+1$-tuples (x,m) belonging to $K \times (0,\infty]$: they satisfy the usual assumptions of being complete, transitive, and continuous. Moreover, if the amount of money is positive, every tuple (x,m) is strictly preferred to $(x,0)$: this means that every consumption vector with any public good bundle and positive amount of money is strictly preferred to a consumption vector where there is no money at all. Additionally, a continuous cost function $c:K \rightarrow (-\infty,\infty]$, is introduced.

I am only concerned here with the device implemented to ensure the covering of the costs of projects. For this purpose an upper semicontinuous function $v_i:K \rightarrow [-\infty,\infty)$ is defined for each agent. A n-vector v of such individual functions is called a *valuation system*: there is an agency which computes, announces, and enforce these valuation systems. Necessary conditions for the state of the economy, $(\overline{x},\overline{m})$, to be supported by a valuation system are the following: for some vector $\Pi \in \Re^n$, such that $\sum_i \Pi_i = \sum_i v_i(\overline{x}) - c_i(\overline{x})$, the state maximizes individual utilities subject to budget constraints, and the project $\overline{x}$ maximizes the total profits on K. Mas-Colell is able to say that a state of the economy is in equilibrium if it can be supported by a valuation system. The notion of equilibrium is related to unanimity: the author proves that a state is a valuation equilibrium if and only if it is Pareto optimum.

15 Not only the basic model, but also some of the ensuing definitions are quite similar to those of Kaneko (1977a). However, Mas-Colell seems not to be aware of this.

Furthermore, the usual definition of the core is introduced: a state, feasible for the whole society, is in the core if there is no coalition that can find another state, feasible for its members, which is preferred to the first state by all of its members. Then, Mas-Colell also proves that another necessary and sufficient condition for a state to be a valuation equilibrium is that it belongs to the core. However, in this case the valuation system which supports the equilibrium has both to be nonnegative and to yield zero profits.

While it is easy to prove the existence of Pareto optimum states under the assumptions made, and then to derive the existence of a valuation equilibrium for the state, the situation is different for the core: in particular it may well be empty. Recall that the equivalence between the existence of a valuation equilibrium and the fact that a state belongs to the core holds under assumptions that are stronger than those needed for the equivalence between Pareto optimum and the existence of a valuation equilibrium.

Summing it up, it is proved that a valuation equilibrium exists and that it is Pareto optimum: clearly, the set of such equilibrium states may be very large: it is apparent that if individual taxes are announced by a deus ex machina like agency, there will be many states which cannot be improved by unanimous proposals. In searching for a clearer characterization through the concept of the core, Mas-Colell finds the usual problem of its emptiness. The question of how to choose among the Pareto optimum states remains unanswered.

A classical paper in the field of public economics is that of Milleron (1972). His discussion on tax structures is mainly based on Foley (1967) and Foley (1970). In Foley's model the government finances public goods with taxes levied on each individual. The total value of public goods (the sum of each price time the quantity of the related public good) must equal the total revenue $\sum_i t_i$. The government cover the expenses corresponding to the value of the public good, given the primary distribution of income. Furthermore, since t_i may be positive, negative and zero, the government can also transfer resources from one agent to another. It must be noted that these taxes are quite arbitrary nonetheless Foley can characterize Pareto optimum solutions: however, the absence of constraints on individual taxes leads to a very large set of public competitive equilibria. The vector of individual taxes may be thought as a vector of lump-sum taxes, even if nothing restrains t_i from being a function of some individual characteristic.

In fact Foley did provide a less generic tax structure together with an equilibrium existence theorem for this case. If constant returns to scale prevail, the ratio of total taxes to total wealth equals the ratio of the value of public goods to total wealth s, when the government budget is satisfied. Then, individual taxes equal

s times individual wealth: each consumer pays a percentage of public expenditure equal to his share in the total wealth. This model is just one of the many ways of dealing with the distribution problem. It must be said that many restrictive assumptions on the production sets and on the individual utility functions are needed to prove the equilibrium existence in this setting.

From the point of view of the procedures implementing optimum public good provision programs, the central problem is that of the correct revelation of preferences by consumers. Thus, is apparent that for what concerns the tax structure, the benefit approach is followed.

In Kaneko (1977a) there are t agencies, each producing one public good. The cost of producing each public good $C^t(x^t)$, expressed in terms of the only private good of the model, is monotonically increasing with the level of production. Each agency attach a ratio r_i^t to each individual i, so that $\sum_t r_i^t C^t(x^t)$ is the individual contribution to the provision of the public goods. Obvious conditions ensures feasibility and the absence of profits.

The conclusions of this paper concern the fact that the voting game is stable, that is, it has a nonempty core, when the bureaus do provide an equilibrium vector of ratios to share the production cost between agents. The notion of equilibrium is twofold, since it regards both a vector of ratios and an allocation of public goods (r,x): this is called a ratio equilibrium if it maximizes individual utility subject to an individual budget constraint. However, as far as the equilibrium ratio is concerned, no insight is provided about the way in which it is determined. This problem is clearly linked to the revelation of individuals demand for public goods: it may not be possible to avoid the usual free rider problem.

However, this crucial question is addressed by a second paper, Kaneko (1977b), in a very different way. Essentially, a new voting game is constructed in which not only the amount of public goods but also a precise ratio is decided. Thus, the chosen ratio crucially depends upon the definition of the game: namely what a winning coalition is allowed to do. In this framework a winning coalition that propose a new configuration (an allocation and the associated cost sharing scheme) must bear the cost at greater or equal proportion to the old configuration. This model is interesting since it stresses the importance of the stability of the adopted configuration. Notwithstanding, it does not provide an explicit treatment of the tax structure, apart from the fact that the concept of ratio equilibrium is clearly related to minority rate protection.

The tax allocation method presented by Greenberg (1977) is not distorting: the individual tax is a function from the product space of public and private goods and of the price simplex to an Euclidean space which dimension equal the number of agents. Essentially, the tax scheme is a function of observable variables such as individual endowments and the value of public goods. The only conditions regard the possibility of individual survival with after tax income, and the usual assumption of balanced budget. The description of the tax allows then for many different structures which give rise to any type of income distribution. It is apparent that the nondistortionary assumption is quite demanding in terms of information. The interesting result is that there exists a public competitive quasi equilibrium for any tax structure: adding ad hoc assumptions, such an equilibrium becomes a public competitive equilibrium. Clearly, a p.c.e. Matches efficiency criteria. Then, since we can impose that a tax structure be socially accepted (in a sense defined by Greenberg), a p.c.e. Can achieve both equity and efficiency.

A recent paper by Weber and Wiesmeth (1990) addresses both these questions in an equilibrium framework, using the concept of egalitarian-equivalence, introduced by Pazner and Schmeidler (1978).

Their model is quite simple: there are n agents in the set N, one public good x and one private good y. Each agent is endowed with a quantity e_i of the private good and the total endowment is e. One unit of the private good is needed to produce one unit of the public good. There is a cost sharing scheme, (a tax allocation method in particular) which assigns to each agent a personal tax t_i such that $0 \leq t_i \leq e_i$. The utility level associated to the generic consumption bundle is $u_i(t, e_i - t_i)$, since t is the total tax (equal to the level of public good) and the second argument is the after-tax consumption of the private good. The tax allocation scheme is represented by the system $\phi = (\phi_1, \phi_2, .., \phi_n)$ where ϕ_i:$[o, e] \rightarrow [0, e_i]$; one of the assumptions requires that $\sum_N \phi_i(t) = t$ for $0 \leq t \leq e$ [16]. A ϕ-allocation is a vector $(t, t_i, \ldots t_n)$ if for all agents $t_i = \phi_i(t)$. The set of all ϕ-allocations is denoted by Φ.

The first task of the paper is to prove the existence and uniqueness of an efficient allocation in Φ.

The second stage is to adopt a concept of equity and to prove that there exists an allocation in Φ which is both equitable and efficient.

16 Notice that the proportional cost sharing method is a particular case, where $\phi_i(t) = \gamma e_i$, with the parameter γ chosen so as to ensure a balanced budget.

The tax allocation method ϕ is given. Consider, for all agents, those levels t of total tax such that:

$$u_i(t, e_i - \phi_i(t)) \geq u_i(0, e_i) \qquad (1)$$

Then, for these t, define the level of public good $x_i^{\phi}(t)$ as the level which satisfies

$$u_i(x_i^{\phi}(t), e_i) = u_i(t, e_i - \phi_i(t)) \qquad (2)$$

with $x_i^{\phi}(t) = 0$ for all other t. Furthermore, define $x^{\phi}(t) = \min\{x_i^{\phi}(t) \mid i \in N\}$. Lastly, define a ϕ-allocation $(t^*, t_1^*, ..., t_n^*)$ as a ϕ-egalitarian equivalent allocation (ϕ-EEA) if t^* maximizes $x^{\phi}(t)$ for t.

By (1) we characterize all levels t of total tax such that the consumption bundle composed by the quantity t of the public good and by the quantity of the private good left by the given tax allocation ϕ yields an individual utility level equal or larger than the utility level associated to the consumption of the private good endowment alone. Notice that $x_i^{\phi}(t)$ is the quantity of the public commodity relative to t which, together with the private consumption of the whole endowment, yields to individual i the same utility level as the utility given by the following consumption bundle: the quantity of the public commodity relative to t and the quantity of private good that is left after the individual tax associated to that allocation method is paid. It can be interpreted as an individual "free riding" level of x relative to one particular tax method and one amount of total tax. Clearly, it varies across individuals and depends upon t. The authors first find its minimum across individuals, $x^{\phi}(t)$, and then find the maximum for t. This means that a ϕ-EEA selects the highest of this "free riding" levels, which is compatible with the allocation generated by the tax allocation method ϕ.

The equity of a ϕ-EEA seems to derive from the fact that, given a tax allocation method, it maximizes the highest compatible level of free riding by choosing the appropriate level of total tax. In this sense, equity does not concern the way in which taxes are individually allocated.

The authors prove the existence of a ϕ-EEA, and its weak Pareto efficiency, under some assumptions on utility functions. The second part of the paper deal with the implementation of a type of equilibrium which depends on the allocations previously discussed.

3 Voting

This is not so much a summary of voting theory as a collection of its results that are more related to the problem of deciding the level of production of public goods according to the outcome of some form of ballot. Considering the provision of public goods by government, it is natural to assume that certain allocating decisions are made by political rather than market processes. On the other side, it may appear unrealistic to consider that choices made in the public domain are the outcome of some voting process; however, if political parties make choices so as to maximize the likelihood of being elected then it is possible to view the chosen options as being determined, although indirectly, by a voting (see, for example, Coughlin 1986).

The general problem is that of selecting one alternative out of a set of alternatives on the basis of the preferences of many individuals. In choosing among competing voting procedures it is useful to identify a set of desirable criteria and then determine which procedure uniquely meets them. Along this method May (1952) identified the following four attractive conditions and demonstrated that only majority rule satisfies them: -*Decisiveness* means that however people vote there is always a clear outcome; -*Anonymity* of voters says that it is not needed to know who cast which votes to determine the outcome; -*Neutrality* between alternatives requests that if the outcome were initially a tie, it would remain a tie; -*Positive responsiveness* says that if alternative A at least ties B and then someone makes his vote more favourable to A, eventually A wins.

It is important to make clear an economic observation about the selection of one alternative. First, there is no assurance that the selection is a Pareto improvement (that would request a unanimously approved selection). Second, assuming the possibility of some interpersonal comparison between states associated to different alternatives, there is no guarantee that the selection of one alternative satisfies the weak version of the compensation principle.

Even if some objections can be raised against voting based on majority rule, we can say that when there are no more than two alternatives it is essentially straightforward. On the other side, when the set of alternatives is larger many problems arise. There are many different types of voting procedures that are used in this case and all of them reduce to majority rule if there are just two alternatives: however, it is not clear which, if any, of these procedures is the more appropriate extension of simple majority rule. I simply recall here three classes of voting systems: aggregation procedures, elimination procedures and sequential binary procedures.

The first important problem is that for given voter preferences, different procedures may imply a different selected alternative. A second difficulty is that all of these voting procedures are sensitive to agenda manipulation: adding or deleting alternatives from the agenda of choices, even if they cannot themselves win, can influences the outcome.

The third major problem is more evident considering the application of simple majority rule to all pair of alternatives, that is, using the majority preference relation: it is called "voting paradox" or "Condorcet effect" and it consists of the absence of a social ordering. For example, if there are three alternatives it may be the case that in pairwise votes, A defeats B, B defeats C, and C defeats A. In this case we say that a majority voting equilibrium does not exists. Nowadays, it is clear that this kind of cyclical majority is avoided if preferences are single-peaked: this means that there is a single dimension to the decision being taken. For instance, if the voting is about the level of public spending, consider the utility of each individual as its function: under a certain level of the public good provision it may be the case that some individuals prefer the alternative private good. The graph of the utility has more than one local maximum and the single-peaked property is lost. A good example of double-peakedness of preferences is given by considering the case where a municipal body is deciding upon the level of public transport facilities. If an individuals owns a car then his preferences over the level of provision are likely to be single-peaked, and so they are if he does not own a car . But at some level of provision the individual might well switch from owning to not owning a car. We can generalize this notion of single-peakedness to a multidimensional space of alternatives, where each point in the space is associated to a combination of policies like taxes, import tariffs, and public spending. In this setting we can easily see that the single-peaked property is equivalent to the standard economic assumption that individual preferences on a space of commodity bundles are convex.

When the single-peaked preferences property holds, all voter ideal points can be rank ordered along one dimension and the *median voter theorem* tells us that the alternative corresponding to the median voter ideal point is the winner. The median voter model can be applied straightaway to public expenditure issues: assume, for example, that agents differ only in pre-tax income, and that the ideal point of each agent -the preferred level of public good- is a monotonic increasing function of income. Then, the majority voting equilibrium coincides with the level of public good preferred by the agent with median income and, for given preferences, it would vary with median income. It is evident that the tax system influences the equilibrium level of public good: each tax system places a different burden on the median agent and then also the voting outcome will change accordingly. However, the whole problem becomes more difficult in a multidimensional setting, since the existence of a point that is the median ideal point along all directions is quite unlikely.

A fourth major problem of voting is that it may be rational for voters not to reveal their true preferences; then, voting can be considered as a game of strategy. Among several questions that naturally occur eliminating the assumption of "honest revelation of preferences", the most important one is probably whether there exists a strategy-proof voting procedure, where the best and honest strategies always coincide for all voters. After many years of conjecturing about this question, it was Gibbard (1973) to demonstrate a negative answer. In particular, it has been shown that the conditions for a voting scheme to be strategy-proof are equivalent to the Arrow conditions.

It is now clear that the search for any solution of the majority voting paradox starts from a useful way of relaxing the Arrow conditions: for example, the property of single-peakedness, under which a voting equilibrium exists, is associated to a restriction of the range of preferences.

4 Voting over tax allocation systems.

Among several approaches to positive tax theory, I am interested here in that one which assumes the existence of one particular aspect of tax structure and allows voters to choose its relevant parameters through majority rule. It must be said that even if this method reaches a detailed analysis of that particular tax, it may lack generality, and it neither explains the lot of existing tax structures nor how different parts of each tax structure are related.

In the framework of a linear income tax function, Romer (1975) defines the following indirect utility function, where C_n^*, L_n^*, and (k,t) are respectively n-man' s utility maximizing consumption and labour choices, and tax parameters: $V_n(k,t) = U(C_n^*,L_n^*)$. He shows that without unemployment, each individual's maximized utility, as a function of tax parameters, monotonically increases with personal ability. Actually, the property that income (and the associated well-being) increases with ability holds generally for all income taxes under homogeneous preferences. Moreover, it is shown that if the lowest permissible tax rate is greater than a certain level $\bar{t}$, then the indirect utility function is single-peaked in t. This is important for the possibility of a majority voting equilibrium (MVE) over the tax system.

From this setting, Romer defines an individual most preferred tax rate t^* and constructs the function $t^*(n)$, which gives the value of the most preferred tax rate by an individual with ability n. For skill levels greater than some critical value $\tilde{n}$, the voter's preferred value of t will be the lowest possible one, $\tilde{t}$. His model yields the result that less skilled individuals have a higher most preferred tax rate than those with greater ability.

Given that each individual has a $t^*(n)$ that is different from the others, it is impossible to chose a tax rate that is everyone' s preferred tax (we do not consider the trivial case of equal ability): this point enlightens the type of social conflict involved in choosing particular values for k and t.

However, single-peakedness of preferences over the permissible range of tax parameters implies that there exists a value $\hat{t}$ of the tax rate that is stable against the rule of majority voting. It is easy to show that $\hat{t} = t^*(\hat{n})$, the last one being the most preferred tax rate of the median ability voter.

From this last result and assuming that the median ability is greater than the lowest one, it follows that $\hat{t}$ is lower than the tax rate that would be chosen using the Rawlsian maxi-min criterion.

The critical skill level $\tilde{t}$ depends both on the skill distribution ($\bar{n}$ and n_0) and on the amount of required revenue G. Two conclusions are prompted by considering two cases. If $\hat{n} > \tilde{n}$, then the MVE tax rate will be the lowest possible value, that is, $\hat{t} = \tilde{t}$. In this case $\hat{t}$ minimizes the after tax income of the relatively poorest individuals with the minimum skill level n_0. Moreover, if the per capita required revenue is larger than a positive n_0, then $\hat{t}$ will imply a positive lump-sum parameter: the *regressive* tax function selected by majority voting will make low pre-tax income individuals face a higher average tax rate than that of greater pre-tax income individuals. In the second case, that is $\hat{n} < \bar{n}$, the median skill level will lie below the mean skill level, and the skill distribution is skewed rightward. Even in this more realistic situation, in which the majority of voters have before tax incomes below the mean level, the majority voting equilibrium might well result in a regressive income tax function.

This contradicts a result by Foley (1967): an individual with an income y less than the mean income $\bar{y}$ will always prefer a higher to a lower tax rate and preferences will be reversed if y is greater than $\bar{y}$. If this happens, the consequences of majority voting are easily analysed. If the median income is less than the mean income, so that the income distribution is positively skewed, then majority voting will lead to the tax schedule with the highest marginal tax rate being adopted.

The part of Romer's work that deals with the existence of a majority voting equilibrium is very important. The general problem is given by the two well known results of Black (1958) and Arrow (1963): Black showed that if individual preferences are single-peaked then majority rule will produce an equilibrium outcome; on the other side, Arrow proved that if individual preferences are unrestricted then choice sets may fail to exist under many rules, and in particular, under majority rule. The question of whether preferences over income tax schedules are likely to be single-peaked is also addressed by Roberts (1977). I have already discussed this point and provided an example in section 3. However, the discussion of Roberts is somewhat different: there are cases in which individual preferences over goods and services are well behaved although preferences over options are not single-peaked. In particular, Roberts is concerned with not single-peaked preferences in relation to a linear income tax schedule that may be viewed as a distortionary tax on income which is used to finance a lump-sum subsidy: so now, the tax function has the form $T(y) = -k + ty$.

However, instead of investigating the likelihood of single-peakedness, Roberts is interested in finding whether the voting mechanism will give rise to a most preferred outcome. His main result is that a mild assumption on preferences is a sufficient condition for the existence of a most preferred outcome under a wide class of voting mechanisms that comprises the majority rule. Firstly, a set of standard assumptions (see Sen 1970) on the binary relation over a set of tax parameters (τ) provided by this wide class of voting mechanism is considered. Secondly, instead of studying individual preferences over tax parameters and single individual's utility function, Roberts imposes a condition which restricts the set of preferences that individuals may have relative to each other. This is the assumption of *Hierarchical adherence* (HA): it states that there exists an ordering of individuals such that the pre-tax income is monotonically increasing irrespective of the pair of tax parameters which is used.

Essentially, considering the provision of some public good, this assumption is equivalent to asking that the ordering of individuals in terms of their marginal preference for the public good is independent of the level of provision. Note that from the request of single-peakedness that there be an ordering of options, here we consider an assumption that requires that there be an ordering of individuals. Note also that if individuals have the same preferences over consumption and leisure time -as in Romer (1975)- but differ with regard to their personal wage rate, then a sufficient condition for HA to be satisfied is the following: the elasticity of labour supply (in terms of hours worked) is not less than minus unity, so that pre-tax income increases with the wage rate. Moreover, HA holds even when, due to the selected tax parameters, some individuals are idle: this is the "second general case" of Romer, in which he shows the related problems with single-peakedness. (It turns out the HA allows for Theorems that are quite interesting in welfare analysis).

The main result, as I said before, is that if the voting mechanism satisfies three assumptions on the binary relation over tax parameters, preferences satisfy HA, and τ is finite, then the choice set of τ under the voting mechanism is nonempty. Note that majority voting satisfies the three assumptions on the binary relation.

Kaneko, (1977 a, and 1977 b) applied the concept of ratio equilibrium and introduced voting into a public-goods economy. The assumptions of Mantel (1975) are broader than those of Kaneko, and it would be interesting to know whether the concept of ratio equilibrium, which is clearly related to minority rights protection, can be usefully extended to Mantel's model formulated as a cooperative game with voting.

Kaneko (1977 a) defines an economy with many public goods and only one private good called money, which is the only one owned by agents as an endowment. He develops the concept of a ratio equilibrium: it is a vector of quotas of the total tax collected by the public producers in order to cover costs of production, each one attached to a particular agent. Kaneko proves that such a ratio equilibrium exists. Then, he is left with two problems: the first is how to chose a ratio equilibrium, that means how much each agents contributes. The second is how to decide the output level of the public goods.

Kaneko presents a voting game $G(N,W,r)$ to solve the second problem, the ratio equilibrium being exogenously fixed by the producing bureaus. His results are the following:

- the core of $G(N,W,r)$ is nonempty if and only if the bureaus find an equilibrium ratio; then the voting game is stable in this sense.
- when an equilibrium ratio is known the ratio equilibrium can be achieved by the voting game.
- there remains the problem of how an equilibrium ratio is computed: this problem concerns the revelation of demands of individuals for public goods.
- the last problem is the possibility of extending the model to economies with more than one private good.

Earlier in the literature (see for example, Champsaur - Robert - Rosenthal, 1975) the impossibility of a generalization of the Debreu-Scarf theorem (the evidence that increasing the number of agents does not cause the core to shrink in public-goods economies) has been related to the presence of only one private good. Following this understanding, one would expect the effect of replicating an economy to be ambiguous: with many private goods there would be a tendency for the core to reduce, while the effect of the public goods would be to enlarge the core.

Kaneko constructed a taxation game with minority protection through ratio taxation; he is able to obtain a coincidence between the core and the Lindahl equilibrium. Examining their relations, however, it is clear that the former does depend on the characteristic function, whereas the latter does not. As we have seen, some voting rules lead to empty cores. Thus, any hope for a theorem on the equivalence of the core and the Lindahl equilibrium must rest on our being able to select a model with an appropriate rule for minority rights protection.

Kaneko (1977 b) presents a new voting game, $G = (N, W)$, in which no ratio equilibrium is a priori given: it is decided within the game. The rule is that if a winning coalition wants to improve an allocation x, then it must bear the cost in the proposed allocation at greater or equal proportion to x. However, a different setting is also conceivable for winning coalitions: the amount of tax to be paid by the coalitions is a constraint on their production possibilities. Then, it is plausible to think that each coalition will try to relax this constraint and to pay less taxes by voting for a different tax system.

PART 4:

AGGREGATION OF PREFERENCES AND RESTRICTIONS IN VOTING MODELS

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

 DOI: 10.2870/7482

Table of Contents

Introduction 1

1 Aggregation of preferences 3
- 1.1 Social functions of individual data 16
- 1.2 On convenient restrictions on sets of orderings 18
- 1.3 A "respect the minority" restriction on permissible sets of alternatives 24
- 1.4 Appropriate restrictions associated to public goods, and their effects on Arrow's impossibility results 27

2 On usual restrictions in voting models 29

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

Introduction

When the economic environment is competitive the allocation rule for private and public goods are well known: parametric prices equal marginal costs of private goods, personal Lindahl prices for public goods, exchange of information at no cost in the case of planning procedures. In this approach the government maximizes an objective function that depends on the utilities of consumers.

We can list three reasons to explain the departure of real economies from this model. Firstly, the political mechanism is much more complex and it is not clear whether the objective function of the government, and of public good producing firms, still depends on consumer preferences. Secondly, the administration is usually structured in many levels whose interrelations are difficult to generalize. Thirdly, simple majority rule as well as other allocative rules are far more important in practice than the Pareto unanimity rule: this last one is difficult to implement since it gives each individual a veto power which in turn increases negotiation costs.

Then, a positive approach to political economy stems from the recognition of these problems and goes even beyond second best analysis. The consequences of a non Pareto preference relation for the state are considered in the literature: - the public sector can be an instrument for the macro-economic policy when the state maximizes a function of the price system and of the level of public provision under resource constraints, such that of public good production technology, and a budget constraint (e.g., Bös (1978)); - the public choice approach explicitly considers bureaucratic models that emphasize both the limited control of the electorate over many aspects of public decision making and the particular goals of the public administration; - voting models view public choices as the outcome of different electoral processes, from the simple majority rule of a direct democracy to the formal analysis of political parties' behaviour in representative democracy; - interest group models reckon that the operation of both electoral process and administrative procedure is likely to be affected by special interest groups and by differential power (Foley (1978)).

More generally, all of these approaches deal with the problem of asymmetric information: basically, it is *the* obstacle to the realization of an optimal allocation of resources in both its forms of adverse selection and of moral hazard. The first term refers to hidden information, while the second points out the problem of predicting the behaviour of agents whose actions are not observable. In the last ten years, scholars in the field of welfare economics have increasingly become acquainted with a concept that unifies this general problem of finding allocation mechanisms that produce appropriate economic allocations even when agents seek to take advantage of it by misrepresenting their characteristics: this is the notion of *incentive compatibility*.

I am concerned here with a particular allocation mechanism which finds its theoretical environment in the literature on collective choices: the determination of the level and of the composition of public good bundles by means of voting systems[1]. I will consider existing voting models, and in particular, the relevance of the needed restrictions on the choice dominion and on the distribution of preferences to assure the existence of an equilibrium outcome. Before that, section 1 contains a summary on preference aggregation, which is the broadest approach to the problem of getting a social decision with regard to any issue from individual preferences over the same issue.

1 A *voting scheme* picks one social state x from a given set X of social states for any logically possible n-tuple of reported preference orderings. Formally, a voting scheme is a function $V:\{R_i\} \rightarrow X$ which is said to be *manipulable* if and only if for some profile there is at least one individual who can improve the final outcome for himself by reporting an untrue preference ordering when others report their true ones.

1 Aggregation of preferences

If all individual preferences over a set of different social states were identical, we would have no problem in aggregating them in order to get a social preference pattern. Difficulties arise because of the diversity in individual preferences. I will summarize the usual framework in which this problem is analysed as well as the major results obtained so far[2].

The classic way of welfare economics to deal with the problem of aggregation of preferences is based on the idea of ordering individual utility vectors defined on the set of alternatives. Section 1.1 clarifies the actual taxonomy of aggregation functions.

In general, given the difference of individual preferences over a set of alternatives, we can think of three different approach to the problem of their aggregation when the task is to get a social ordering of these alternatives:

a) first, one could take the view that a society is not different from individuals, and then the properties of individual preferences are the same of the collective ones. In particular, if we assume that individual preferences are transitive, then we require that transitivity also holds for social preferences.

b) Second, we can assume that a social criterion has to show a subset of alternatives that the collective body considers as the best ones: in this case we are possibly not disturbed by ties between several alternatives.

c) Third, we may require that the subset of best alternatives from the social point of view is a singleton: this case is similar to the second a part from the fact that a device to break ties has to be used in order for the process to end with only one chosen alternatives.

With approach a) we have a *social welfare function* in the sense of Arrow (1963): it maps the n-tuples of complete and transitive individual preference relations into the set of complete and transitive social preference relations. In other words, given the individual preferences R_i, the classic Arrow's problem is to find a social preference relation R that would satisfies the same conditions, namely completeness and transitivity.

Following approach b) one constructs *social choice functions*, that for each set of alternatives and each n-tuple of individual preferences return the subset of socially best alternatives. The range of a *SCF* is the Cartesian product of the set of alternatives and of the sets of individual preference relations. Sen (1970) imposes the

2 What follows is mainly based on Pattanaik (1971). I omit the proofs of the propositions since I just introduce minor modifications.

restriction of acyclicity on the social preference relation in order to get a non-empty choice set in all situations: this approach lies between a) and b).
Approach c) results from the restriction on the range of a *SCF*: it now consists only of the set of alternatives.

A *society* is represented by a set N of individuals indexed by subscript i. The set of alternatives that could possibly be presented for social choice will be indicated by $X \subset \Re^m$. The vectors of X, indicated by lower-case letters, have various interpretations. They may represent different candidates in an election or alternative policies. However, the most general point of view is that of considering them as alternative social states (or proposals), each one characterized by m dimensions: then, for each given $x \in X$, its k-th coordinate pin-points the social state's value along the k-th dimension.

Every individual has preferences over alternatives belonging to X. This simply means that every individual can pairwise correlate different alternatives and judge whether he prefers one of them or he is indifferent between them. Such preferences are *binary relations*, that is, they hold between two objects[3].

Definition 1. For any two alternatives x, y, we say that individual i *weakly prefers* x if and only if he considers x to be at least as good as y. A weak preference relation is denoted by R.

Definition 2. For any two alternatives x, y, we say that individual i *strictly prefers* x if and only if he considers x to be at least as good as y but not y to be at least as good as x. A strict preference is denoted by P and formally defined as follows:

$$\forall x, y \in X, \quad xPy \Leftrightarrow [xRy \quad \wedge \quad \neg\, (yRx)]$$

Definition 3. For any two alternatives x, y, we say that individual i is *indifferent* between them if and only if he considers x to be at least as good as y and also y to be at least as good as x. Indifference is then formally written:

$$\forall x, y \in X, \quad xIy \Leftrightarrow [xRy \quad \wedge \quad yRx]$$

Then, it is clear that R denotes the union of P and I.

3 A part from the usual notation, I will use the following symbols throughout: $\wedge$, $\vee$, and $\neg$ meaning respectively "and", "or", and "not".

 DOI: 10.2870/7482

I assume here that each R_i defined over X is known; this amounts both to avoid the complication of strategic individual behaviour and to postulate the existence of a practical procedure that precisely records individual preferences[4].

Let me recall in general some useful properties of a binary relation B.

Definition 4.1. B is *reflexive* over X iff $\forall x \in X$, xBx.

Definition 4.2. B is *connected* over X iff $\forall x,y \in X$, $x \neq y$, $xBy \vee yBx$.

Definition 4.3. B is *transitive*[5] over X iff $\forall x,y,z \in X$, $[(xBy \wedge yBz) \Rightarrow xBz]$.

Definition 4.4. B is *quasi-transitive* over X iff $\forall x,y,z \in X$, $\{[xBy \wedge \neg (yBx)] \wedge [yBz \wedge \neg (zBy)]\} \Rightarrow [xBz \wedge \neg (zBx)]$.

Definition 4.5. B is *asymmetrical* over X iff $\forall x,y \in X$, $[xBy \Rightarrow \neg (yBx)]$.

Definition 4.6. B is *anti-symmetric* over X iff $\forall x,y \in X$, $(xBy \wedge yBx) \Rightarrow (x = y)$.

Definition 4.7. A *B-cycle* exists in $A \subseteq X$ iff for some finite set of alternatives $\{a^1,..,a^n\} \subseteq A$ we have $(a^1Ba^2 \wedge \ldots \wedge a^{n-1}Ba^n \wedge a^nBa^1)$.

Definition 4.8. B is *acyclic* over $A \subseteq X$ iff no *B-cycle* occurs in A.

Definition 4.9. B is *founded* over $A \subseteq X$ iff there does not exist an infinitely long descending chain of the type $(\ldots \wedge a^3Ba^2 \wedge a^2Ba^1)$ where all $a^i \in A$.

It is easy to prove that foundedness implies acyclicity when A is not finite and that they are equivalent if A is finite.

Definition 4.10. B is a *complete ordering* over X iff it is reflexive, connected, and transitive over X.

4 On this, see the discussion on voting schemes in the following section. Note also that the n-tuples of individual preference orderings over X are often called *profiles* or *social profiles*.

5 It is well known (see for instance Sen (1970), p.47) that if transitivity holds for all triples, then it must hold for the whole set X.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

Definition 4.11. B is a *complete linear ordering* over X iff it is reflexive, connected, transitive, and anti-symmetric over X.

Note that connectedness is also called completeness, and however, they imply that there are not incomparable pairs. With preference orderings we say that $x, y \in X$ are incomparable if none of xPy, yPx, or xIy, holds. When there are incomparable pairs an ordering is called *partial*. It is also called *weak* (*strong*) when there are (not) indifferent and unequal pairs. In Debreu (1959), the names *quasiorder* and *preorder* are used for the R relation of an order that satisfies reflexivity and transitivity; when, in addition $xRy \wedge yRx \Rightarrow x = y$, he calls R an ordering. Nomenclature in this area is far from standardized. I will assume throughout that every weak preference relation R is a complete ordering (in short, an ordering) over X as defined in Definition 4.10 above.

A preference relation over different states is held by any individual in the society: R_i is intended to be individual *I's list of pairwise relations* between these social states. Then, considering the whole society, we have a set of N individual orderings $\{R_1,..,R_N\}$. From now on, let R, P, I stand respectively for the social weak preference, the social strict preference, and the social indifference relation over X. Since it is generally acknowledged that a social decision should be based on individual preferences, we face the problem of aggregating different R_i into a meaningful R. Let us denote by $\{R_i\}$ the class of ordered sets of individual weak preference relations over X, and by $\{R\}$ the set of social weak preference relations over X.[6]

In general a binary relation defined over a set X of n elements takes the form of a set of pairs of elements which are associated by the relation: if no restrictions are considered, then the maximum number of such pairs is n^2. Note that if we consider two binary relations over X, as in the case of strict and indifference preference relations, we have more elements in the set, since two alternatives can now be related by two relations (the element xPy is distinct from xIy). When we impose a restriction of the type defined above the binary relation induces a structure on this set according to the restriction: for example, transitivity over a triple rules out one pair of otherwise possibly related elements (with xBy and yBz we cannot have zBx); the same example shows that we get a smaller number of pairs of elements in the set. The restriction of being a reflexive

6 Given any list of individual orderings $(R_i)_{i \in N}$ the image $f((R_i))$ is the social ordering.

and connected preference ordering induces the following structure on the set of associated pairs: each pair, together with a suitable number of other pairs, forms a sequence which satisfy the restrictions. Each list contains all alternatives and it is different from the others in the way alternatives are related to each other.

To make this clear consider the case in which social preferences are determined by pairwise strict majority rule voting (see below for a formal definition of majority rule), and in which the set of alternatives is the triple $\{x,y,z\}$. An individual or social preference ordering over this triple is, for example, the sequence $<xIy,\ yPz,\ xPz>$. There are 27 logically possible complete preference relations over a triple; out of 27, 14 are transitive, 20 are quasi-transitive, 26 have no cycles and 2 have cycles over the alternatives. Under transitivity, a complete social ordering is determined; under quasi-transitivity there still exists an unbeaten alternative and the same is true under acyclicity. Note that while transitivity and quasi-transitivity over every triple of alternatives in a set X imply the same properties of preferences over the whole X[7], this is not true for acyclicity.

Definition 5. A *social decision rule* (SDR) is a one-to-one mapping from the class of ordered sets of individual weak preference relations over X to the set of social weak preference relations over X. Formally: $f:\{R_i\} \rightarrow \{R\}$.[8]

Since no restriction has been imposed on $\{R\}$ many problems are likely to show up: for example, some $R \in \{R\}$ may not be connected or may violate transitivity in the triple $\{x,y,z\}$, so that it is impossible to define a socially best alternative in the same set. To avoid these problems we may want a SDF to pick up socially best alternatives for every non-empty subset of X. Then, we firstly have to define the idea of a socially best alternative in a given set.

Definition 6.[9] For any given A, $A \subseteq X$, the *choice set* $C(A)$ is a subset of A such that every one of its elements is socially weakly preferred to every element in A. Formally:

$$\forall x\{[x \in C(A)] \Leftrightarrow [(x \in A) \wedge \forall y((y \in A) \Rightarrow (xRy))]\}$$

7 This is a result of Arrow (1951b): he provided a general formulation of single-peakedness as a property on "triples", i.e., on any three-element subset of the set of alternatives.

8 Note that $\{R_i\} = \times_{i \in N} R_i$, where R_i is the set of orderings which person i may have.

9 This definition is due to Arrow (1963)

It is easily checked that if R is reflexive and connected over A, then $C(A)$ equals the so called *maximal set*: the set of all alternatives in A such that there does not exist any socially better alternative in A.

It is clearly desiderable to have one SDR that defines socially best alternatives for every non-empty subset of X, i.e., every $f(\{R_i\}) \in \{R\}$ should generate a function that defines a non-empty choice set for every non-empty subset of A. Let us firstly define such a function.

Definition 7. A *social choice function* (SCF) over $A \subseteq X$, defines a non-empty choice set for every non-empty subset of A.

Clearly, we are interested in SCF defined over X. Eventually:

Definition 8. A *social decision function* (SDF) is a SDR such that, for every social weak preference relation in its range, there exists a SCF over X.

In short, we have done the following. We defined a SDR which gives social weak preference relations over a set of alternatives as a function of individuals ones. Then we defined a SCF as a function that gives the set of socially best alternatives for every subset of the option set. Thirdly, defining a SDF, we made ourselves sure that to every social weak preference relation there correspond a choice set for the set of alternatives. The crucial point is that R may fail to generate a SCF over X, that is, a SDF may not exist. The following results give conditions for assuring the existence of a SDF.

Theorem 1. A SDF exists iff R is reflexive and connected, and P is founded over X.

This means that R generates a SCF over X iff it is reflexive and connected, and P is founded over X. Moreover, we have the following:

Theorem 2. If X is finite, then a sufficient condition for the existence of a SDF is that R should be a complete ordering over X.

It is clear that if X is finite, then a SDR -the range of which is a set of complete social orderings- will necessarily be a SDF. We have:

Definition 9. A *social welfare function* (SWF) is a SDR the range of which is a set of complete social orderings.

If every R_i is a complete ordering the above definition is equivalent to Arrow's definition of a social welfare function (Arrow 1963). On the other hand, a Bergson social welfare function (BSWF) is a real valued function that maps each social states into a real number as the welfare index of that social state. For each preference profile there is a set of welfare indices corresponding to the set of alternative social states. It is however clear that each set of welfare indices implies a complete social ordering R whose existence is then a necessary (but not sufficient) condition for the existence of a BSWF. As an example of the existence of complete social ordering without the existence of a corresponding real valued function, consider lexicographic ordering of two-dimensional real space (Debreu (1959), pages 69-70)[10]

The notion of SDR is perfectly general since it does not specify the intrinsic nature of this functional relation: starting from the same profile of individual preferences the SDR can yield xPy, as well as yPx; it can result in xPy whenever a particular individual has xP_iy or generate yPx if and only if all individuals weakly prefer y to x. The problem of specifying the intrinsic nature of a SDR is twofold: there is a technical aspect which deals with the collection of individual preferences and the computation of the social preference according to the chosen algorithm; on the other side, there is the ethical problem of choosing the particular algorithm. Any solution to this last problem has to be searched using value judgements. I share the opinion that these two aspects are inseparable and that welfare economics cannot be built only on value-free propositions[11].

Since a SDR as such does not characterize any particular way of aggregating individual preferences into a pattern of social preferences we say that to specify one of these ways amounts to a restriction of the set of all the possible aggregation rules. One restriction on the SDR that has been extensively discussed -actually, a considerable volume of welfare economics has been built around it- concerns the pattern of social preference when individual preferences do not conflict. Indeed, this weak restriction correspond to a rather mild value judgment: if everyone in the society prefers x to y it will be normally agreed that x should be socially preferred to y.

This approach is the base of the so called *strict* and *weak Pareto Criteria*. I will not discuss them here, a part from recalling their definitions and from expressing two short comments: -in real life, conflicts of

---

10 With regard to the general framework for studying the problem of aggregation of individual preferences, different approaches and terminologies abound in the literature. Moreover, a certain part of the work is concerned with relations between different approaches, so that the same problem is considered from many points of view. As a matter of clarification I examine the guide-lines of alternative approaches in section 1.2 .

11 It is just the case to recall that the Pareto Criterion is nothing but a value judgement, though a weak one (see for example Sen (1970)).

individual strict preferences are rather common and the greater the degree of the conflict, the wider is the range over which the strict Pareto criterion[12] is non-applicable. No satisfactory theory of social choices can be built on such an implausible circumstance; -the strict Pareto criterion gives only a sufficient condition for social strict preferences and for social indifference: it is a property of a SDR rather the a SDR by itself: many SDR satisfy the strict Pareto criterion.

Under the *strict Pareto criterion* (SPC), if every individual is indifferent between x and y, then so is society; furthermore, if every one considers x at least as good as y and at least one individual strictly prefers x to y, the society strictly prefers x to y.

Under the *weak Pareto criterion* (WPC), if every individual is indifferent between x and y, then so is society; additionally, if every one strictly prefers x to y, then society strictly prefers x to y.[13]

To explore other appropriate restrictions on the SDR and to conceive an effective theory of social choice we must go beyond the Pareto criteria. However, we must be aware of the fact that the wider the acceptability of these restrictions the weaker they are, and that this makes a large number of SDR compatible with them, leaving us with a rather ambiguous characterization of SDR itself. On the other hand, one of the lessons from Arrow's possibility theorem is that, looking for the simultaneous fulfilment of stronger value judgement, we may be left without an "acceptable" SDR: certain highly appealing and jointly more demanding restrictions may be inconsistent[14].

I will not review here neither Arrow's possibility theorem (Arrow (1951b) and Arrow (1963)) nor the enormous literature it gave rise to; I shall rather consider some SDR in the light of its result. To do this I firstly recall some notions usually mentioned in that context (definitions 10 to 13) and secondly I define additional properties which a SDR should fulfil.

Definition 10. Positive association of social and individual values (PA) is a condition expressed as follows: let R and R' correspond to the two different profiles $\{R_N\}$ and $\{R_N'\}$. Assume that for all alternatives a, and b different from x it is true that:

12 The strict Pareto criterion implies the weak one

13 The *extended weak Pareto criterion* declares all x and y non-comparable under the WPC to be socially indifferent. So does the *extended strict Pareto criterion* with regard to the SPC. These rules satisfy all of the Arrow's conditions (see below) except for transitivity of R . However, the feature which accounts for their unsatisfactory nature is that they give a veto power to every individual, since every individual can always cause social indifference between two alternatives even if the rest of the society strictly prefers one to the other.

14 On the "way out" of impossibility, based on interpersonal comparison of utility see Sen (1985) as well as Hammond (1990b)

$$(aR_ib \Leftrightarrow aR_i'b) \qquad \forall i \in N$$

and that:

$$[(xP_ia \Rightarrow xP_i'a) \wedge (xI_ia \Rightarrow xR_i'a)] \qquad \forall a \in X, \quad \forall i \in N$$

Then,

$$[(xPy \Rightarrow xP'y) \wedge (xIy \Rightarrow xR'y)] \qquad \forall y \in X$$

In words, if the only change in individual orderings is that x rises in some persons' orderings, then x, when pairwise compared to any alternative, is not less preferred in the new social preference ordering.

Definition 11. Independence of irrelevant alternatives (IA) is a condition expressed as follows: let R and R' correspond to the two different profiles $\langle R_i \rangle$ and $\langle R_i' \rangle$. Let A be any subset of X and let $C(A)$ and $C'(A)$ be the choice sets respectively generated by R and R'. If, $\forall x, y \in A$, and $\forall i \in N$, $(xR_iy \Leftrightarrow xR_i'y)$ then $C(A) = C'(A)$.[15]

This condition ensures that a social decision between any two alternatives depends on, and only on, the individual orderings of those two alternatives.[16]

Definition 12. Citizens' sovereignty (CS) implies that $\forall x, y \in X$ there exists a set of individual orderings for which xPy.

Definition 13. Nondictatorship (ND) implies that there does not exists any individual i such that $\forall x, y \in X$ the SDR yields the outcome xPy whenever xP_iy, irrespective of the orderings of all other individuals.

The four conditions lastly defined, together with the following double condition *(1)* set up the framework for the possibility theorem. *Condition (1-i)* says that every R in the range of a SDR is a complete ordering

15 The purpose of this condition is to rule out interpersonal comparison of utility.

16 Define binariness of a SDR as follows: let $\{R_i\}$ and $\{R_i'\}$ be any two sets of individual orderings over X. Let a SDR map them respectively into R and R'. Then, the SDR is *binary* iff for all $x, y, \in X$, we have $(xR_iy \Leftrightarrow xR_i'y \wedge yR_ix \Leftrightarrow yR_i'x) \Rightarrow (xRy \Leftrightarrow xR'y \wedge yRx \Leftrightarrow yR'x)$ for all i.

(so that it is a SWF, see Definition 9 above and the following comment), and *condition (1-ii)* says that every logically possible set of individual orderings belongs to the domain of the SDR[17]. These five conditions form the set of value judgments that Arrow imposes on the SDR[18].

Condition (1-i) above implies that every weak social preference relation satisfies reflexivity, connectedness, and transitivity, that is, it is a complete ordering. It has been criticized for the reason that, unlike transitivity of individual preferences, transitivity of social preferences is not a value in itself and it has to be justified only in terms of its consequences. It is argued that the problem caused by intransitivity of social preferences is insignificant; furthermore, occasional inconsistency of majority decisions due to their intransitivity may protect minority interest. However, while this particular aspect of the problem is questionable, some results on transitivity are unequivocal.

Indeed, suppose to take a decision by a sequence of pairwise comparisons and suppose that an alternative which is beaten in any pairwise comparison is not allowed to enter the sequence again. Assume that the set of alternatives is $\{x,y,z\}$ and that we have xPy, yPz, and zPx. We first compare x to y and reject y; secondly we compare x to z and choose z as the final outcome without comparing it to y. In this case intransitivity of social preferences will not be revealed at all. Nevertheless, this example shows that unless transitivity is satisfied, the final outcome may depend on the order of the sequence itself: if we had compared first y to z the final outcome would have been x[19]. Now, add to this examples the three following individual orderings over $\{x,y,z\}$ and use the majority rule to aggregate them: xP_1yP_1z, yP_2zP_2x, zP_3xP_3y. We have, as before, $(xPy \wedge yPz \wedge zPx)$. It should be clearer the sense in which intransitivity may favour minorities: for example, if the sequence of pairwise comparison is $(xPy \rightarrow zPx \rightarrow yPz)$, then z will be chosen even if both individual *1* and *2* (the majority) strictly prefer y to z.

17 This condition is often called *Unrestricted domain*.

18 As is well known, the conclusion of Arrow's work is that, if there are at least three alternatives, then there is no SDR which satisfies the conditions stated above. It may be noted that these conditions are just a subset of the set of value judgements which an appealing SDR should satisfy. For example, the weak and justifiable value judgment inherent in the strict Pareto criterion neither is contained nor is implied by this collection. The negative conclusion of Arrow's possibility theorem is better understood if we consider that there is no point in adding more conditions to a set of them which have already been proved to be inconsistent. Blau (1957) showed that the original axioms were not quite sufficient. Arrow (1963) presented a corrected version of his theorem with four conditions: unrestricted domain, Pareto principle, independence of irrelevant alternatives, and nondictatorship. In this set-up the Weak Pareto criterion replaces both PA and CS (actually, Arrow introduced the Pareto criterion explicitly in his 1952 French paper). Hammond (1976) proved that introducing interpersonal comparisons which reflect some equity judgments, a generalized social welfare function can also be equitable.

19 In this example six distinct sequences are possible and, depending on them, each of the three alternatives can be chosen two times as the socially preferred outcome. Note that in general, with three alternatives in X and if R, P, and I are considered, then there are 13 possible types of individual preference orderings; this means that the number of permissible sets of orderings is $2^{13} = 8.192$.

Given that X is a finite set, the assumption that R is a complete ordering is sufficient to eliminate difficulties due to intransitivity: theorem 2 above tells us that in this case the SDR is a SDF, that is, it defines a non-empty choice set. In fact, whatever be the sequence of pairwise comparison, once an element of a choice set enters the sequence, it beats every other alternative till the end. Clearly, if R is a complete ordering, then the SDR is a SWF by definition.

However, theorem 1 above seems suggesting that to eliminate these problems we only need that R is reflexive and connected and that P is acyclic[20]. In this sense, *condition (1-i)* imposes an unnecessarily strong restriction on the SDR: the existence of a SDF solves the basic problem of rational social choice and we do not need the SDR to be also a SWF [21].

Assuming *condition (1-i)*, *condition (1-ii)* imposes the restriction that the SWF should have an unrestricted domain. This is also arguable since it seems reasonable to believe that certain patterns of individual preferences, even if logically possible, never arise in actual life.

After this short digression on the conditions that define the environment of Arrow's possibility theorem, a few other definitions are proposed before examining a particular social decision rule.

Definition 14. Decisiveness (D). A SDR is decisive iff for every logically possible set of individual orderings it defines a social weak preference relation satisfying reflexivity and connectedness.

Clearly, a SDR satisfying Arrow's condition (1) is also necessarily decisive. Assume now that $\{R_i\}$ and $\{R_i'\}$ $\forall i \in N$ are any two sets of individual orderings belonging to the domain of a SDR and let the SDR map these two sets into two different social weak preference relations R and R' belonging to $\{R\}$. Let $[x,y]$ and $[z,w]$ be any two ordered pairs of alternatives[22].

Definition 15. Anonymity (A). If for any $\{j,k\}$, with $j,k \in N$ $R_j = R_k'$ and $R_k = R_j'$, and if $\forall i, i \neq j,k$, we have $R_i = R_i'$, then $R = R'$.

20 See the comment to Definitions 4.8 and 4.9.

21 *Condition (1-i)* is also necessary and sufficient for a choice function generated by R to fulfil Arrow's *condition of rationality* (CR): for all subset A and B of X, with A contained in B, if some alternatives are chosen from B and then the range of alternatives is restricted to A still containing some of the alternatives chosen in B, no previously unchosen element becomes chosen and viceversa. Since there does not seem to be any intuitive justification for requiring a SCF to satisfy CR, even from this point of view *condition (1-i)* is stronger than actually needed.

22 The following three properties, (A), (N), and (PR) were introduced by May (1952) for the case of only two alternatives. Here they coincide with those given by Arrow (1963).

Assuming that the set of individual orderings belongs to the domain of the SDR both before and after a permutation of individual orderings has been made, such a permutation must leave the social ordering R unchanged. Anonymity rules out discrimination among individuals. Note that A implies Arrow's ND, and that it is a strengthening of it.

Definition 16 . Neutrality (N). Assume that $\forall a,b, a,b \neq x,y,z,w$, and that for all i we have:

- $(aR_ib \Leftrightarrow aR_i'b)$; $(xR_iy \Leftrightarrow zR_i'w) \wedge (yR_ix \Leftrightarrow wR_i'z)$; and
- $(xR_ia \Leftrightarrow zR_i'a) \wedge (aR_ix \Leftrightarrow aR_i'z) \wedge (yR_ia \Leftrightarrow wR_i'a) \wedge (aR_iy \Leftrightarrow aR_i'w)$;

then, $(xRy \Leftrightarrow zR'w)$.

Requiring that a permutation of the alternatives in everyone's preference should induce the same permutation in social preferences, neutrality rules out any bias among alternatives: for example the bias in favour of the status quo given by the rule that it must be changed if and only if a two-thirds majority strictly prefers the change, is not allowed by neutrality.

Definition 17. Positive responsiveness (PR). Assume that $\forall a,b, a,b \neq x$, and for all i we have $(aR_ib \Leftrightarrow aR_i'b)$. Assume that for all a and for all i we have $(xP_ia \Rightarrow xP_i'a) \wedge (xI_ia \Rightarrow xR_i'a)$, while for some i, either $(xI_iy \wedge xP_i'y)$ or $(yP_ix \wedge xR_i'y)$. Then, $(xRy \Rightarrow xP'y)$.

This property essentially requires that when an alternative x is socially at least as good as any of the alternatives in a certain set, if x rises in some individual orderings with respect to any of the other alternatives in that set, then x has to become socially preferred to it.

Definition 18. Non-negative responsiveness (NR). Assume that $\forall a,b, a,b \neq x$, and for all i, we have $(aR_ib \Leftrightarrow aR_i'b)$ and that for all a and i we have $(xP_ia \Rightarrow xP_i'a) \wedge (xI_ia \Rightarrow xR_i'a)$. Then, for all y, $(xPy \Rightarrow xP'y) \wedge (xIy \Rightarrow xR'y)$.

Like (PR), this property concerns changes in some individual orderings and the related change in the social ordering: in this case it is requested that x does not fall in the social ordering if it rises in some individual orderings. Note that PR implies NR and this, in turn, implies Arrow's PA.[23]

23 If a SDR is binary (see footnote 16), then D, N, and NA together imply that either the WPC is fulfilled or social indifference holds for every pair of alternatives. On the other hand, for a binary SDR, D, N, and PR together imply the SPC.

We are now ready to consider the most widely discussed social decision rule.

Definition 20. The *majority decision rule* (MD-rule). Let $N(aR_ib)$ denote the number of individuals that prefer any alternative a to any other alternative b in X. For all $x, y \in X$

$$xRy \Leftrightarrow [N(xR_iy) \geq N(yR_ix)]$$

The MD-rule satisfies decisiveness, anonymity, neutrality, positive responsiveness (then also non-negative responsiveness), and binariness. Consider that A implies ND, that PR implies PA, that binariness is equivalent to IA, and that -since D, N, and PR together imply the strict Pareto criterion- the MD-rule also satisfies CS. Therefore, by Arrow's general possibility theorem, it must violate condition 1. Actually, the MD-rule violates *condition (1-ii)* since it has a restricted domain as a SDF because for certain sets of individual orderings it does not satisfy acyclicity of P[24].

24 The famous voting paradox is an example of an MD-rule which violates acyclicity. When the number of alternatives is large the probability of acyclicity is low and then the probability of there being no majority winner is very high. On this see Sen (1982a) page 162 and the referred literature.

1.1 Social functions of individual data

Two different approaches to individual preference aggregation are considered in what follows. This material is well known and I present it without specific attribution: the mentioned papers give appropriate references.

Sen (1970) considered, for the same problem given by a set of alternatives X and the set of involved persons N, the notion of social welfare functionals (SWFL). A SWFL specifies exactly one social ordering R over X for any given n-tuple of real-valued personal welfare functions $W_i(\cdot)$ each defined over X, for each person in N. Formally, $R = F(\{W_i\})$.

It must be recalled that Roberts (1980) established the first important result on the relation between classical welfare economics and social choice theory: he was able to show that, under certain conditions, any SWFL admits a continuous and monotonic real valued representation whose arguments are individual utilities.

Blackorby, Donaldson and Weymark (1990) give an alternative proof of Arrow's Theorem. In doing this they elucidate the relations between their approach, the Arrowian one (the so-called social choice approach), and the Bergson-Samuelson framework for welfare economics. Their framework follows that of Sen.

The cardinality of the set of alternatives is assumed to be larger than three: $\#X \geq 3$. Each of the n individuals, $n \geq 2$, has a utility function defined over the set of alternatives: $u_i:X \rightarrow \Re$. The vector $u = (u_1, u_2, .., u_n)$ is called a *profile*; its image $u(x) = (u_1(x), u_2(x), .., u_n(x)) \in \Re$ is a vector of utility numbers. Let U denote the set of all *possible* profiles and $D \subsetneq U$ the set of *admissible* profiles. Let R be the family of all orderings of elements in X and R be one of these orderings.

The problem is to assign a social preference ordering to each admissible utility profile. Formally this means to construct a function $F:D \rightarrow \mathrm{R}$. Then, $F(u)\,(= R_u)$ is the social ordering of X associated to the profile u. R_u is called a *Social Evaluation Functional* (SEFL). There is no need to require that R_u be representable by a social utility function; clearly, P_u and I_u are the strict preference and the indifference relations corresponding to R_u.

Notice that a SEFL is an ordering of alternatives, while in much of welfare economics orderings of feasible utility n-tuples are considered. Indeed, let $\overline{U} \subset \Re^n$ be the set of feasible utility vectors; then a *social welfare ordering* is an ordering $\overline{R}$ of elements in $\overline{U}$. Formally, we construct a function $\overline{F}:\overline{U} \rightarrow \overline{\mathrm{R}}$. Moreover, if $\overline{R}$ is representable by a function $W:\overline{U} \rightarrow \Re$, then W is called a *Bergson-Samuelson social welfare function* (BSWF). There is a clear distinction between social orderings of alternatives (an Arrow-type SWF or a social evaluation) and social orderings of utility vectors (BSWF).

Now, let $\overline{U}_u$ be the set of utility n-tuples that can be achieved as the alternative x varies over X. There exists a social welfare ordering $\overline{R}_u$ of elements in $\overline{U}_u$: more precisely, the relation between a social welfare ordering with a social evaluation R_u is consistent when a SEFL satisfies the following condition:

Definition 1.1.1 A SEFL satisfies *Profile Dependent Welfarism* (PDW) iff for all $u \in D$ there exists a $\overline{R}_u$ such that

$$xR_u y \Leftrightarrow u(x)\overline{R}_u u(y) \qquad \forall x, y \in X$$

Let $\overline{U}_D$ be the set of all attainable utility n-tuples when all admissible utility profiles and all alternatives are considered. The condition called Welfarism requires that there exists a single social welfare ordering $\overline{R}$ on $\overline{U}_D$ consistent with every SEFL R_u.

Definition 1.1.2 A SEFL R_u satisfies Welfarism iff for all $u \in D$, there exists an $\overline{R}$ of $\overline{U}_D$ such that

$$xR_u y \Leftrightarrow u(x)\overline{R}u(y) \qquad \forall x, y \in X$$

Letting $\overline{R}_u$ be the restriction of $\overline{R}$ to $\overline{U}_u$ it is clear that Welfarism implies Profile Dependent Welfarism.

Notice that the SEFL makes no a priori assumption about the measurability and comparability of utilities, while Arrow's SWF, assigning a social preference ordering of X to each profile of individual orderings, implicitly assumes that utility is ordinally measurable and interpersonally noncomparable.

1.2 On convenient restrictions on sets of orderings

Discussing *condition (1-ii)*, we remarked that there is no reason why we should require a SDF to have for its domain all possible sets of individual orderings. If in actual life the set of individual orderings happens to reveal specific properties, then we can just require the domain of the SDF to contain all sets of individual orderings that have these properties. Assuming to follow this approach, we have to characterize the sets of individual orderings that actually belong to the domain of a SDF. Since the MD-rule has several desirable properties, I will focus my attention on it. We have seen that the MD-rule has a restricted domain as a SDF. Then, to find reasonable restrictions on the sets of individual orderings means to study the conditions under which a set of individual orderings belongs to the domain of a MD-rule viewed as a SDF. More precisely, we have to introduce a restriction on sets of individual orderings and to characterize the resulting domain of a SDF: this can be done by giving sufficient, necessary, and both sufficient and necessary conditions for a set of individual orderings to belong to the domain of a SDF[25]. However, before actually specifying these conditions, we have to analyse their general meaning. The following definition of such sufficient conditions seems most convenient; it is preceded by a terminology definition.

Definition 21. A set of individual orderings and a set of possible preference orderings *correspond* to each other iff the preference ordering implied by every individual ordering in the former set belongs to the latter set and every preference ordering belonging to the latter set is implied by some individual ordering in the former[26].

Definition 22. A *sufficient condition of Type I* for a set of individual orderings to be in the domain of a SDF specifies a class T^I of permissible sets of orderings such that any set of individual orderings corresponding to any set in T^I is in the domain of a SDF.

A sufficient condition of Type I specifies certain permissible sets of orderings and also says that any set of individual orderings which corresponds to any of these permissible sets belongs to the domain of a SDF. The implicit restriction imposed through the notion of "corresponding" to permissible sets is that there must be, for each one of the orderings they contains, at least an individual who has it.

25 What follows also applies to SWF. Note that we could also look at the actual practice to check whether a particular set of individual orderings shows up and to specify consistent restrictions. However, restrictions introduced in theory should be tested empirically.

26 This mean the following: let $\{\bar{R}_i\}$ be any set of individual orderings (e.g. $\{xP_1y, xP_2y, xI_3y\}$) and let $\{\bar{R}\}$ be any set of possible preference ordering over the same set of alternatives. They are said to *correspond* to each other iff each element in one set is equivalent to an element of the other set.

There is another definition which is provided for a similar scope: A *sufficient condition of Type II* for a set of individual orderings to be in the domain of a SDF specifies a class T^{II} of permissible sets of orderings such that any set of individual orderings corresponding to a subset of any set in T^{II} is in the domain of a SDF. The task of a sufficient condition of Type II, though a specification of permissible sets of ordering itself, is accomplished without restrictions on the distribution of individuals over the different orderings in the permissible sets: there may be no individual who has an ordering belonging to the permissible set, or more than one. Note that a sufficient condition of Type II implies a sufficient condition of Type I. Note also that if a permissible set of orderings $\overline{R}$ belongs to T^{II}, then any subset of $\overline{R}$ also belongs to T^{II}. This is not necessarily true in the case of a sufficient condition of Type I. Sufficient conditions of Type I seem more relevant in this context (see Pattanaik (1971), page 68).

The formulation of a necessary condition for a set of individual orderings to be in the domain of a SDF is more troublesome: a necessity condition would be naturally be defined as one which must be verified by every set of individual orderings in the domain of a SDF, and, as we have done in the case of a sufficient condition, it would be conceived in terms of restrictions on permissible sets of orderings. This aim turns out to be unachievable in most relevant cases. Consider the following example:

It is impossible to give a necessary condition that must be verified by every set of individual orderings in order for it to be in the domain of a MD-rule viewed as a SDF. In fact, a Lemma by Pattanaik (1971) (page 69) shows that the MD-rule yields transitive results if for every triple of alternatives more than half of the individuals concerned[27] with respect to the same triple have identical preference orderings. Now, assume that this condition is verified: this means that the set of individual orderings will be in the domain of the MD-rule viewed as a SDF (provided that the set of alternatives is finite). But the orderings of the individuals that do not belong to the majority can change and violate any restriction without altering the transitivity of the outcome obtained under the majority rule.

In the usual sense "necessary" would mean that the violation of the condition would imply that some set of individual orderings do not belong to the domain of a SDF. A different and more useful approach to the problem of formulating necessary conditions is the following[28]:

27 An individual is said to be *concerned* with respect to a set of alternatives if and only if he does not show indifference between all the alternatives in that set.

28 This notion corresponds to that of "minimal" condition as defined by Kramer (1973). See also section 2 below.

Definition 23. A *necessary condition of Type I* for a set of individual orderings to be in the domain of a SDF specifies a class T_I of permissible sets of orderings such that for any permissible set of orderings $\{\overline{R}_i\}$ not in that class, there exists a set of individual orderings which correspond to $\{\overline{R}_i\}$ and that does not belong to the domain of a SDF.

There is also a definition of a necessity condition of Type II: for a set of individual orderings to be in the domain of a SDF it specifies a class T_{II} of permissible sets of orderings such that for any permissible set $\{\overline{R}_i\}$ not in that class, there exists a set of individual orderings which correspond to a subset of $\{\overline{R}_i\}$ and that does not belong to the domain of a SDF. Clearly, a necessary condition of Type I implies a necessary condition of Type II.

Merging these definitions with those of sufficiency conditions we get necessary and sufficient conditions of Type I and of Type II. To understand the sense of these two types of conditions, we may look at the case of X containing three alternatives. There are 13 possible individual preference orderings over X, and 2^{13} permissible sets of orderings. Some of these sets are such that whenever a set of individual orderings correspond to one of them, it belongs to the domain of a SDF regardless of the distribution of individual orderings over the different orderings in the permissible set. Then, a necessary condition of Type I will ask to pick up from among the 8.192 permissible sets all the sets that have this characteristic. Analogously, the task of a necessary and sufficient condition of Type II is that of finding all the permissible sets of orderings having the characteristic that whenever a set of individual preferences orderings corresponds to a subset of one of them, then the set of individual orderings belongs to the domain of a SDF regardless of the distribution of individual orderings.

We can now turn our attention to the task of actually specifying some conditions under which the MD-rule defines a non-empty choice set for every non-empty subset of X[29]. All needed restrictions are imposed on a given set of preference orderings and have the following general form: let $\overline{R} = \{\overline{R}_1,..,\overline{R}_n\}$ be any set of preference orderings; if a preference pattern of a certain type p belongs to the given set, then a preference pattern of a certain type q does not belong to that set. All restrictions are conditions on $\overline{R}$ and refer to a given ordered triple of alternatives $\{x, y, z\}$. From now on R simply indicates a weak preference relation, without specifying whose preference it is.

29 This means to investigate the existence of a SCF under the MD-rule, i.e, the existence of a best alternative under the MD-rule for every non-empty subset of X.

Definition 24. For a given R, x is said to be given the *worst value* in the triple $\{x, y, z\}$ iff $(yRx \wedge zRx)$; it is given the *best value* iff $(xRy \wedge xRz)$; it is given the *medium value* iff $(yRx \wedge xRz) \vee (zRx \wedge xRy)$.

Definition 25. Value Restriction (VR): in the triple $\{x, y, z\}$ there is an alternative such that it is not given the worst value in any concerned R, or it is not given the best value in any concerned R, or it is not given the medium value in any concerned R[30].

Definition 26. Limited Agreement (LA): in the triple $\{x, y, z\}$ there exists an ordered pair of distinct alternatives such that in all R the first alternative in the ordered pair is considered to be at least as good as the second.

Definition 27. Extremal Restriction (ER): if there is an R which is antisymmetric over $\{x, y, z\}$, then for every other individual R^* different from R and belonging to $\overline{R}$, z is uniquely best iff x is uniquely worst[31].

Theorem 3. A sufficient condition of Type II for the MD-rule to yield transitive results over a triple (x,y,z) is that every permissible set of orderings satisfies ER over that triple[32].

Theorem 4. A necessary and sufficient condition of Type II for a set of individual orderings defined over a finite set of alternatives X to be in the domain of a MD-rule viewed as a SDF is that every permissible set of orderings must satisfy at least one of the conditions VR, LA, and ER over every triple of alternatives belonging to X.

It is possible to prove that in the special case of antisymmetric individual orderings, ER and LA both imply VR. In this case Theorem 4 takes the following form:

30 It can be easily seen that a non-worst value restriction on a permissible set of orderings is equivalent to saying that the orderings in the permissible set are single-peaked: there exists a linear ordering B of x, y, and z such that the curves representing the permissible set of orderings, and drawn with reference to B are all single-peaked. Single peakedness was first introduced by Black (1948)(*On the rationale of group decision making*, Journal of Political Economy 56, but see also Black (1958)), and Arrow (1951). For a formal definition see section 2 below.

31 This is equivalent to saying that if for some R_i x is preferred to y and y is preferred to z, then for all R_j such that z is preferred to x, z is also preferred to y and y is preferred to x. ER, first introduced by Sen and Pattanaik (1969) (reprinted in Sen (1982) as chapter 7), is the union of the conditions of *Dichotomous Preferences*, of *Echoic Preferences*, and of *Antagonistic Preferences* defined by Inada (1969).

32 Recall that a sufficient condition of Type II is also a sufficient condition of Type I. However, the equivalence does not hold for necessary and sufficient conditions of Type I and of Type II: then, the following Theorem 4 specifies only conditions of Type II. A similar result for conditions of Type I has not been proved by Pattanaik.

DOI: 10.2870/7482

Theorem 4'. If individual preference orderings are antisymmetric, then a necessary and sufficient condition of Type II for a set of individual orderings defined over a finite set of alternatives X to be in the domain of a MD-rule viewed as a SDF is that every permissible set of orderings must satisfy VR over every triple of alternatives belonging to X.

I consider now the existence of a best alternative under the MD-rule for any given subset of X. This is a weaker condition than that of requiring the existence of such a best alternative for every non-empty subset of X, i.e., of requiring the existence of a SCF under the MD-rule. Consequently, the needed restrictions turn out to be weaker.

Theorem 5. A sufficient condition of Type II for a finite set of alternatives A to have a non-empty choice set under the MD-rule is that every permissible set of orderings R should satisfy ER or VR over every triple of alternatives belonging to A_R^* [33].

The condition given in Theorem 5 on A cannot be a necessary one: for if R violates both ER and VR over a triple of alternatives in A_R^*, then for some set of individual orderings corresponding to a subset of R, the choice set for that triple will be empty. But, in general, this does not prevent A_R^* from having a non-empty choice set.

We can now turn our attention to the problem of finding some conditions for a SDR to generate SCFs satisfying Arrow's condition of rationality, that is, to generate a complete social ordering. The following theorem holds under the assumption that the number of concerned individuals is odd; however, in the general case where such a restriction is not imposed, the theorem still holds iff ER is satisfied. The necessary and sufficient condition is of the Type II.

Theorem 6. Given that the number of concerned individuals is odd for every triple of alternatives, a set of individual orderings is in the domain of a MD-rule viewed as a SFW iff every possible set of orderings satisfies VR or LA or ER over every triple of alternatives.

---

33 Let R be any set of preference orderings defined over a finite set of alternatives X. For any subset A of X, A_R^* is the set of Pareto optimal alternatives.

The assumption that the number of concerned individual is odd is clearly relevant when there are "few" concerned individuals: in fact, it is in this case that the probability of having an exact half-and-half division of the group in any pairwise comparison may be significant. When the number of concerned individuals is large the assumption on its oddness can be discarded without danger of the result.

1,213

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

1.3 A "respect the minority" restriction on permissible sets of alternatives

One of the usual characteristics of public decision making in democratic societies is its aptitude to compromise between different goals of various concerned interest groups. Compared to a situation in which there exists an absolute majority (>50%), compromising is more likely to arise when a relative majority forms on some decision to be taken. Compromising is also more probable when the difference between the size of majority and that of minority is not too large. Given the fact that many different interest groups are a datum in most modern societies, we can say that the tendency to compromise is a major characteristic of any democratic process. We can consider it as the manifestation of a general bias toward some form of social consensus and equity; this fact can be modelled as the simple problem of dividing a cake: if certain equity considerations are introduced, then the most appropriate analytical approach is that which leads to a value solution.

On the other hand, if we keep at a minimum the introduction of value judgements, the problem of dividing a cake by the rule of simple majority in a society formed by three individuals can be modelled by a voting game whose characteristic function depends only on the size of coalitions: this is a symmetric game. Now, assume that the feasible set of options consists only of the following allocations: a is the allocation when the cake is equally shared between the three voters, and b is that which gives to individuals i and j half of the cake each and nothing to individual k. Under the further assumption that voter are selfish, the allocation b is clearly a majority winner alternative. If it allows (as it does indeed in this case) for the complete exploitation of a group, the majority rule is undoubtedly anti-egalitarian[34]. As such, the game has no core since the two winners take all, and one of the conceivable coalitions -i.e., that of voter k alone - is left well short of its potential. It is well known that the model can be modified in order to describe a situation in which there exists a law designated to protect minority rights: this protection of minority rights allows for the existence of a non-empty core.

Following such a suggestion, note that the exploitation can only take place if the option of dividing the cake between i and j is in the set of permissible alternatives. If this particular option is not available, the majority rule has a restricted domain and it cannot yield such an anti-egalitarian outcome. However, this

34 This example also allows us to answer negatively a fundamental question: is the majority rule, in general, a plausible way of aggregating preferences when welfare-economic judgements enter the problem? This is essentially because majority rule is based only on individual orderings and it does not consider either magnitudes of gains and losses or rankings of well-being of different individuals. Notice that in the example there are only two alternatives, so that there is no problem with transitivity: the question of the egalitarian property of the majority rule arises before the difficulty of transitivity.

restriction of individual preference orderings is indirect: it rules out some of them simply because it removes those alternatives on which they are defined. What I am saying is that we should consider subsets of the set of logically possible social states, where these subsets are constructed in such a way that they do not contain "extreme" alternatives. Most of the work that has been done in the past has been in the context of unstructured sets of alternatives, and the exclusion conditions have been defined in terms only of triples of options. In fact, the usual economic assumptions of convexity for the feasible set of alternatives or continuity and quasi-concavity of individual preferences are absent.

For example, Schmeidler and Sonnenschein (1978) assume the following to provide an alternative proof of the Gibbard-Satterthwaite theorem[35]: if one changes any individual preference ordering by moving any pair of alternatives to the top of the initial preference ordering, then the new preference ordering is still admissible: this clearly violates continuity of preferences.

Now, assume that there are originally two proposed alternatives x and y out of a continuum of alternatives X. Suppose that a majority of individuals prefers x but that y is also preferred by a relatively large number of individuals. My idea is that a final social choice would be an alternative which is between x and y. One way to model this possibility amounts to restrict the set of possible alternatives to those alternatives that are not so "opposite" or so "distant" from each other. Whatever it be, the precise definition of this idea must grasp the fact that when the distribution of individual preferences is more dense corresponding to few options, the associated game is almost zero-sum: what a group gains from the choice of an option is a cost for another group.

In other words, I will assume that the social situation associated to the set of all logically options is such that the aggregation of individual preferences through the MD-rule would yield the following outcome: the chosen alternative leaves a significant minority very unsatisfied. Furthermore, also the alternative supported by the minority, if chosen, would leave the majority unsatisfied. Clearly, this is a situation of acute social contrast that may lack stability even considering the commitment to a legally accepted MD-rule.

35 Satterthwaite (1975) proved that, within a reasonable framework designing the problem of a committee which has to choose an alternative, a voting procedure is strategy-proof if and only if it is dictatorial. Notice that the definition of strategy-proofness used in his paper is quite exacting: a voting procedure is strategy-proof if it makes each voting agent reveal his preference sincerely.

Caplin and Nalebuff (1988) recognize the need to reduce social contrast in order to get transitive results, and impose appropriate conditions, without mentioning the implicit effect on fairness. In actual life many of these situations -but clearly not all- are avoided by proposing less drastic options to the electorate. In what follows I simply try to model this fact.

Definition A. For a given individual preference ordering R_i over a set of alternatives X, $\hat{x} \in X$ is said to be the *largest value* iff $\hat{x} R_i y \ \forall y \neq \hat{x}, y \in X$. $x' \in X$ is said to be the *smallest value* iff $y R_i x' \ \forall y \neq x', y \in X$.

Definition B. A subset X^* of a finite set of alternatives X, is said to be *fair* iff it does not contain the smallest value of any individual preference ordering.

It is interesting to see if the elimination of all smallest values from a given set of alternatives X influence the possibility of individual orderings to be in the domain of a MD-rule viewed as a SDF. We have to check if the conditions under which Theorem 4 holds are still valid on a fair subset of alternatives. The following theorem provides a positive answer.

Theorem A. A set of individual orderings defined over X^* is in the domain of a MD-rule viewed as a SDF if it is in the same domain when defined over X.

Proof. Assume that a set of individual orderings defined over X is in the domain of a MD-rule viewed as a SDF. This means that every permissible set of orderings satisfies at least one of the conditions VR, LA, and ER over every triple of alternatives in X. Clearly $X^* \subset X$, that is, every triple of alternatives in X^* is also a triple of alternatives in X and every ordering of alternatives in X^* is also an ordering of alternatives in X. Then, every permissible set of orderings satisfies at least one of the conditions VR, LA, ER, over every triple of alternatives in X^*; Q.E.D..

This simple theorem ensures that, provided that the original set of alternatives and the associated individual orderings admit a choice set under the MD-rule, the resulting fair subset of alternatives and subset of individual orderings do the same.

1.4 Appropriate restrictions associated to public goods, and their effects on Arrow's impossibility results

As we have seen, no nondictatorial SWF for a set of at least three alternatives exists which satisfies unanimity (U)[36] and IA. However, this result does not hold if the admissible set of preference orderings is sufficiently restricted: sections 1.2 and 2 investigate some positive solutions related to this approach. Nevertheless, most of these solutions describe properties of some MD-rule.

At a more general level the discussion starts from considering that most formulations of Arrow's Theorem require a SWF to be defined for all conceivable profiles of individual preferences. However, it is clear that a much smaller class of preference relations is usually relevant for economics considerations: for example, the set of alternatives is taken to be a set of distribution of commodities, and preferences are assumed to satisfy some restricting conditions like monotonicity and convexity. This recognition gave rise to the study of Arrow's conclusion in *economic domains*: this notion clearly refers to various relaxations of the Unrestricted Domain condition. Kalai, Muller, and Satterthwaite (1979) make clear that the negative conclusions of Arrow's work still hold for any general SWF when the space of alternatives contains public goods and each individual preference relation is restricted in the manner appropriate for this economic environment. In what follows, I briefly summarize their assumptions and the main result.

Elements of the set of alternatives $X = \Re^m_+$ are m-vectors of public goods of the type $x = (x^1, x^2, \ldots, x^m)$, where x^i is the quantity of the ith public good.

There are n individuals whose preference orderings over X are complete (transitive, connected, and reflexive). Each individual preference can be represented by a continuous utility function which is monotonic (insatiable individual). Individuals' indifference surfaces are convex from below.

An individual's preference relation R_i depends together on his tastes and on the intrinsic structure of X. The notion of "structure" of X, it is meant to account for any relation that links different elements of X. Then, as individual tastes limit the set of logically possible orderings, so does the structure of X: let $\{R_i\}$ be the set of all logically possible orderings, and let $\{\hat{R}_i\}$ represent its restriction to the set of admissible preference orderings. $\{\hat{R}_i\}$ is common to all individuals since the restriction that originates it depends on the structure of

36 Recall that Arrow (1963) replaced CS and PA by the Weak Pareto criterion. In this form the condition is called *unanimity*.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

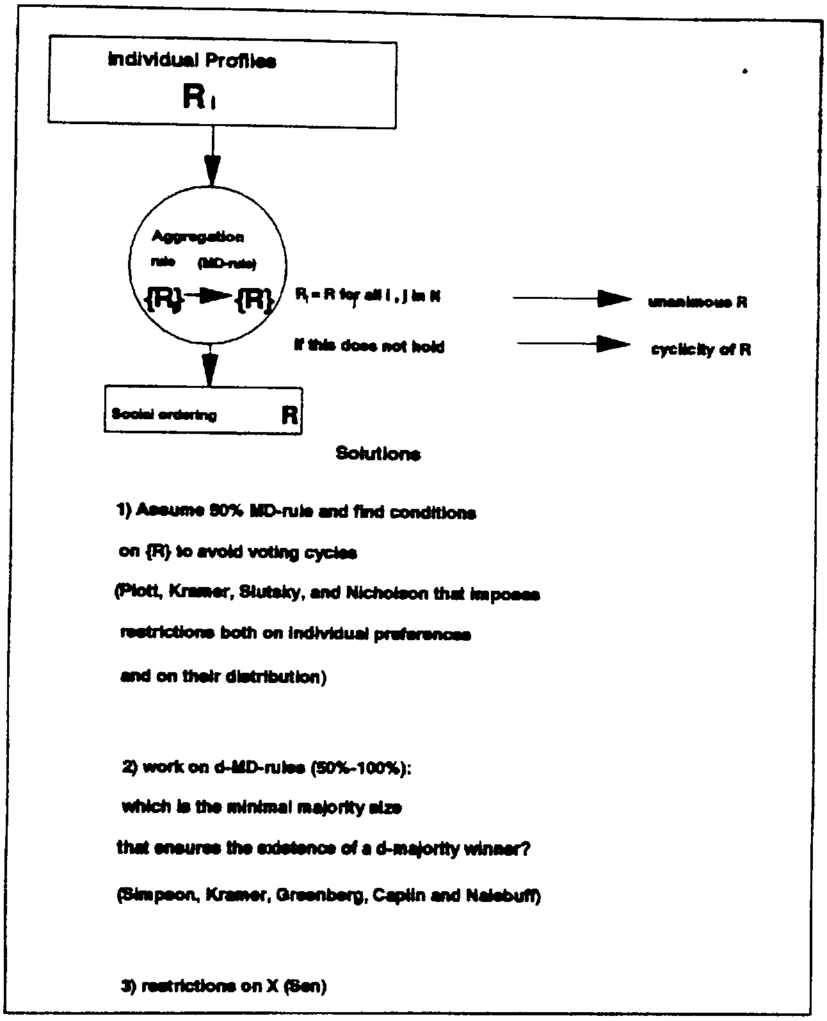
X which, for the case of public goods, is invariant across individuals[37]. A SWF is defined over the n- fold cartesian product of $\{\bar{R}_i\}$, $\{\bar{R}_i^n\}$, whose elements are profiles of individual orderings $< R_1, R_2, \ldots, R_n >$, and takes values in the set of complete social orderings *{R}* over X. A family of $\{\bar{R}_i\}$ is called dictatorship enforcing if every SWF that satisfies unanimity and IA on $\{\bar{R}_i^n\}$ has a dictator on the whole set of alternatives X. Denote by $\{R_m\}$ the family of preference relations defined on $\Re_+^m$. The main result of the paper is the following.

Theorem 1.3.1 The family $\{R_m^*\}$ of convex, strictly monotonic, continuous preference relations on $\Re_+^m$ is dictatorship enforcing for all $m \geq 1$.

37 However, the restriction also depends on individual tastes: I guess that they are implicitly considered equivalent with regard to the restriction.

2 On usual restrictions in voting models

The following picture summarizes the general problem.



Bowen (1943) examined the problem of allocation of resources between public and private use by voting and showed that when the distribution of the tax burden is assumed given, then there will be one resource allocation which is preferred to any other by a majority. In theory, majority voting is thought to be the primary instrument of political decisions: it is well known that this rule does not necessarily yield transitivity of social preferences. This is the core of the whole problem since Condorcet, for it is argued that if the pattern of social preferences has to be rational in any of its usual meanings, it must be transitive. However, this request is not obvious: transitivity in social choices is not a value per se as for individual choices.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

Black (1948)[38] generalized Bowen's result by demonstrating that the wider class of cases with single-peaked preferences is such that majority voting leads to transitivity of social preferences. There are two requirements for single peakedness: the social decision has to be one dimensional and voters' utility functions must be unimodal in this dimension. Each voter has one most preferred point, and the median voter's most preferred point guarantees the existence of a majority over all alternatives[39].

As we have seen in the foregoing section, Arrow[40] showed that if the voting model had to work with individual preferences that do not satisfy Black's conditions, than any "democratic" voting method could yield intransitivity of social choices. Most of the literature which followed Black (1948) deals with multidimensional analogues of the median voter result: a multidimensional median is established by imposing both type restrictions and symmetry conditions on the set of individual preferences. For example, Grandmont (1978) has the result that when the distribution of most preferred points is radially symmetric around a median voter, this voter's optimum will obtain a majority over all alternatives.

Nicholson (1965)[41] proposed another type of condition which is more than an exclusion restriction. It requires two things: first, it rules out individuals holding certain preference orderings; second, it imposes additional restrictions upon the distribution of individual having the non-excluded preferences.

Plott (1967), allowing only a finite number of individuals to be considered, gives conditions for the dominance of a single alternative and explores the problem of determining when a unique alternative can be certain to command a majority. He has individual differentiable utility functions defined over variables whose magnitude is to be decided by a majority of agents. Small changes of these variables are called "motions" and each individual reaction to a motion is analyzed by the associated gradient vector of his utility function: he favours the motion if it would increase his utility function.

An equilibrium for the decision process is reached if and only if there does not exist a motion that could be selected by a majority of individuals. The sufficient condition that a point is an equilibrium if the nonsatiated individuals can be paired in such a way that the individuals in each pair have gradients pointing

38 *On the rationale of group decision making*, Journal of Political Economy 56, 1948, 23-43. See also Black (1958).

39 Foley (1967) worked basically in this framework.

40 As far as I know, the first paper by Arrow dealing with impossibility results for social choices is *A difficulty in the concept of social welfare*, Journal of Political Economy 58, 1950, 328-346. This paper has been reprinted in Arrow-Scitovsky (1969). However, the standard texts on the matter are Arrow (1951) and its second edition in 1963.

41 *Conditions for the "Voting Paradox" in committee decisions*, Metroeconomica 42, 1965. I did not read this paper since it is not in the library. Note that the same basic idea of Nicholson is explored (without explicit reference) in Grandmont (1978).

in opposite directions is provided. It becomes also necessary if the number of satiated individuals is zero when the number of individuals is even, or is one when the total number is odd. Plott's conclusions were rather negative: there is definitely nothing inherent in utility theory that could warrant the existence of such an equilibrium. The necessary and sufficient condition for an equilibrium is considerably strong. In words, this condition, together with the assumptions, require that for every voter assigned a preference ordering of one type, another voter must be assigned an ordering of a complementary type: they are not exclusion conditions but they amount to demand a certain severe symmetry of the set of individual preferences.

The addition of a constraint -e.g. a budget constraint on the sum of resources available to produce priced variables[42]- do not change this situation if there are more than two variables to be determined. However, Plott discloses the possibility that a suitable restriction of the set of available motions, to be made via the actual decision process, could allow for more positive results: for example, a subcommittee can determine what motions can be voted on by a committee; then, the equilibrium conditions only apply to its members and in this context they may appear more plausible[43].

Davis, DeGroot, and Hinich (1972) consider social preference orderings and majority rule in a setting with unimodal and multidimensional utility functions. They give a rule of social preferences based on the notion of Euclidean distance between a most preferred point and all other alternatives[44]

The class of restrictions which the assumed utility functions satisfy does not coincide with conditions on triples of the type introduced by Black (1948) and Arrow (1963), so that they need not satisfy the formal definition of single-peakedness. Conditions for an alternative to be dominant refer to the probability distribution of most preferred point of the individuals, and a relation between dominance and the transitivity of the social preference ordering is established. Exploring this relation the authors are able to derive a transitive social preference ordering from given individual orderings. Eventually, they establish necessary and sufficient conditions for a transitive social preference ordering to be constructed from given individual orderings via the device of majority rule. However, the entire framework and the results do rest upon the assumed class of utility functions.

42 This amounts to bound the space of available alternatives.

43 This suggestion has the same meaning of the idea explored in section 1.2.

44 This notion of Euclidean preferences was introduced by Davis, O. and Hinich, M. in *"A mathematical model of policy formation in a democratic society"*, in Bernd, J. (ed) *"Mathematical applications in political sciences"*, Dallas, Southern Methodist University Press (1967). See below the part dedicated to Caplin and Nalebuff (1988).

All results reviewed so far suggest that in order to support simple majority rule very strong assumptions are needed. It was Kramer (1963) to show clearly this point.

He evidences that intransitivity of the MD-rule is implied by a very modest degree of heterogeneity of tastes when the voters have quasi-concave differentiable utility functions over a multidimensional choice space. He argues that the various equilibrium conditions for majority rule defined in the preceding literature are not quite less restrictive than the drastic condition of complete unanimity of individual preferences. However, Kramer himself notes that this implication may disappear when the problem is one of choosing from among a relatively small set of discrete alternatives. Since Kramer's work has been given much attention in the literature, I will analyse it accurately.

Denote by $\{R\}$ the set of all possible distinct preference orderings over a finite set of alternatives X. For each $R' \in \{R\}$, denote by $N(R')$ the number of voters who hold the preference ordering R'. A voting population is described by specifying $N(R)$ for each $R \in \{R\}$. Apart from those of Plott (1967), all conditions appeared in the literature reviewed so far consist of exclusion restrictions: they are restrictions imposed on $N(R)$ sufficient to ensure an equilibrium in any voting population described by a pattern of preferences satisfying the restriction[45]. An "equilibrium" means the existence of a SDF, or a SWF, or a stable outcome. A *stable outcome* under MD-rule exists when, for some option x, it follows that $xR_iy \; \forall y \in X, y \neq x$. There exists a *SDF* on X if every nonempty finite subset of X has a stable outcome. There exists a *SWF* on X if the weak social preference relation obtained by the MD-rule is transitive[46].

Let B be a *strong ordering* on X: that is, a binary relation which is connected and transitive[47]. Let $R_i = \{R_1, .., R_n\}$ denote a set of individual orderings on X and P_i denote an individual strict preference relation.

Definition 2.1. R_i is single-peaked (SP) over X when there exists a strong ordering B over X such that for any $x,y,z \in X$ if either $xByBz$ or $zByBx$, then for each voter i it follows that $xR_iz \Rightarrow yP_iz$ and that $zR_ix \Rightarrow yP_ix$.

---

45 This statement is not clear to me and it does appear to be false: I think that these restriction are imposed on the set of possible individual orderings. I can hardly see them as restriction on the real number $N(R)$. However, see also below the definition of minimal condition.

46 It is clear that a stable outcome belongs to a choice set; more precisely, the notion of choice set is a generalization of a stable outcome that does not specify the particular method of social decision (see Definition 6 on page 4). The same relation holds between Kramer's SDF and the SDF of Definition 8 above, as well as between his SWF, which he also calls *social ordering*, and the SWF of Definition 9. Note that both these SWFs are not Arrow's SWF.

47 Kramer does not explicitly require reflexivity.

In words, for a set of preference orderings to be single-peaked there must exist an ordering B of the alternatives such that whenever one alternative is between two others -according to B- each voter must strictly prefer it to at least one of the other two (see also Definition 25 and the associated Note). Single peakedness is viewed as a weak type of unanimity, a sort of social consensus that arises when individuals have an identical way of arraying their preferences which are different only in term of "intensity".

As an example consider three parties and order them according to a single ideological dimension: party L on the left, party C in the centre, and party R on the right. Then, single-peakedness rules out both $< LP_iRP_iC >$ and $< RP_iLP_iC >$.

After this definition, Kramer lists a set of sufficient conditions for the existence of an equilibrium for the MD-rule under the common label of "post single-peakedness conditions": among them, VR, LA, and ER that I recalled in Definition 25, 26, and 27. They all require that every triple of alternatives satisfy certain restrictions. For the special case of majority rule, Kramer summarizes the main results on the existence of an equilibrium employing these conditions. If any of the conditions is satisfied over all subset of X, then there exists a SDF on X; if any of VR, LA, and ER is satisfied, provided that the number of concerned individuals is odd, then there exists a SWF on X[48].

Note that ER is a "necessary" and sufficient condition: a necessary condition as defined in section 1.2 is called by Kramer "*minimal*". This notion is useful to understand that, when a minimal condition is found, there are no weaker and still undiscovered exclusion conditions for the existence of an equilibrium. Let $\{R'\} \subset \{R\}$ be a collection of orderings; as before, R is a single list of orderings on X; $N(R)$ denotes the number of voters who are assigned R and it is called an *assignment*. Then, an exclusion restriction specifies a collection $\{R'\}$ such that an assignment $N(R)$ satisfies the restriction when the collection of orderings, say $\{R^*\}$, for which $N(R)>0$ is contained in $\{R'\}$. Assume now to weaken the restriction by replacing $\{R'\}$ by some proper superset $\{\hat{R}\}$. If for every such restriction there is an assignment $N(R)$, satisfying the weakened restriction but otherwise unconstrained, for which MD-rule does not yield an equilibrium, then the original condition is said to be minimal for that type of equilibrium.

48 These results appear in section 1.1 above as theorems 3 to 6.

The next step is to set the framework for the main result of the paper. Each voter has a complete ordering[49] which is also *convex* in the following sense: $\forall x, x' \in X$ and $0 < \lambda < 1$, we have 1) $xP_ix' \Rightarrow \lambda x + (1-\lambda)x'P_ix'$ and 2) $xI_ix' \Rightarrow \lambda x + (1-\lambda)x'R_ix'$.

Each such preference ordering is representable by a differentiable ordinal utility function $u_i(x)$. Then, its gradient $\nabla_i(x)$[50] exists at all $x \in X$. Linear independence between gradient vectors of different voters clearly implies divergent preferences.

Theorem 2.1. If there exists a point $x \in X$ at which the gradient vectors of any three voters are linearly independent, the set of preference orderings does not satisfy any exclusion condition over X.

In the proof, a triple $\{x, y, z\}$ is found such that it fails to satisfy VR, LA, ER, and other exclusion conditions like single-peakedness and single-cavedness. It is interesting to remark that for the conditions to fail it is only needed that the voting population includes just three voters with divergent preferences; provided that this case occurs, the distributions of preferences among the rest of the voters is unimportant. It must be clear that an equilibrium may exist even when any of the conditions fails to hold. This is hardly surprising, since they are not necessary (in the usual sense) conditions: their violation does not imply non-existence of the equilibrium. Consider this simple example: let *{a,b,c}* be a triple of alternatives that, when ordered with regard to the preferences of three voters, fails to satisfy any of the exclusion conditions; now, add two voters whose preferences are identical to those of one of the original three voters; then MD-rule will clearly yield a social ordering, irrespective of the violation of the exclusion conditions.

Actually, each exclusion condition gives a set of preferences none of which may be held by any individual in the society: since they are not necessary we can find many societies in equilibrium with regard to the decision-making process. The notion of "minimality" clarifies that the set of conditions of this type is complete, that is, no others remain to be discovered.

Summing it up, Kramer shows that over a multidimensional space of alternatives with standard economic assumptions on individual preferences, all exclusion conditions will fail to be satisfied on some triple of

---

49 Kramer actually says "complete" and "transitive": referring to Definitions 4.10. this is misleading.

50 The vector of partial derivative with regard to all the variables evaluated at x.

 DOI: 10.2870/7482

alternatives unless individuals are virtually unanimous in their preferences. However, for the special case of a one-dimension-space of alternatives, the single-peakedness exclusion condition will be satisfied and MD-rule will yield consistent outcomes.

Slutsky (1979) generalizes Plott's conclusions by giving necessary and sufficient conditions on the set of gradients for a point in a multidimensional space to be a voting equilibrium. His assumption are similar to those of Plott: the individual preference relations over a multidimensional space of alternatives are reflexive, transitive and strictly convex; they also have a satiation point; utility functions are differentiable and, corresponding to non satiation points, the partial derivative with respect to some dimension must be different from zero. Slutsky defines a class of α-majority rules by the following equivalence relation between social preferences and the outcome of a voting procedure: $aR^{\alpha}b \Leftrightarrow N(aR_ib) \geq [(\alpha/(1-\alpha))N(bR_ia)]$ for all α such that $0 < \alpha < 1$ and all $a, b \in X$ (see Definition 20).

Two Lemmas show that the class of α rules can be devided into two sets, those with $\alpha < 1/2$ and those with $\alpha > 1/2$, whose corresponding preference relations are similar; equilibria are defined as points belonging to choice sets under R^{α}: $c(\alpha) = \{a \in X \mid aR^{\alpha}b, \forall b \in X\}$. It is interesting to note that $\alpha_1 > \alpha_2 \Rightarrow c(\alpha_1) \subseteq c(\alpha_2)$. A preliminary result shows that if $\alpha > 1/2$ (or if $\alpha = 1/2$ when the number of voters is odd), then the choice set consists of at most a single options.

The major result is a theorem that gives necessary and sufficient conditions for the existence of an equilibrium under both the sets of rules. The theorem specifies a set of pointed two dimensional convex cones upon which the given condition must be satisfied. Essentially the condition requires that on every triple in X which contains the equilibrium point -after eliminating the maximum number of pairs of individuals with opposite preferences- at least half of the remaining individuals agree that the equilibrium point is best. However, the likelihood of the existence of a majority equilibrium does not appear to be greater under the more general conditions requested by this theorem, than under Plott's special case: the two sets of conditions differ only when more than one voter is satiated at the same point. Unless some quantity parameters such as Lindahl taxes are introduced and can be adjusted to bring satiation points in line, it seems as unlikely for two individuals to be satiated at the same point as for the same individuals to have gradients pointing in opposite directions.

In a short paper Rubinstein (1979) develops a quite different approach. He starts from noting that the literature on equilibrium states under majority rule gives the impression that cases of existence of equilibrium points are rare. He then tries to measure this rareness. As usually, there is a set of voters N, a set $X \subset \Re^m$ of alternatives, individual preference relations R_i defined over X, and *n-tuples* $< R_1, .., R_n >$ which are called social profiles. The social preference relation, R, is defined as follows: for all a, b in X $aPb \Leftrightarrow N(aP_i b) > n/2$. The solution concept adopted is that of the core; given a social profile $< R_i >$ the core is the set $C(< R_i >) = \{a \in X \mid \neg \exists b \in X \mid bPa\}$. Then he defines a set θ of complete, transitive, reflexive, and continuous preference relations over X. Lastly, for every i, a topology $T_i = < \theta, F >$ is defined; F is the Kannai[51] topology on θ.

The main result asserts that for $n > 3$ the set of social profiles with non-empty core is a closed set with an empty interior in the Kannai topology. This means that it is a nowhere dense set, and this notion expresses its smallness from the topological point of view. It is not clear whether this conclusion has meaningful economic implications: it seems to be just a formal way of asserting the well known difficulties of getting equilibrium social choices.

It is interesting to compare Rubintein's result with those of Schofield (1983): briefly, using the Kannai topology you can consider two profiles close to each other even if their derivatives are not close, while under the Whitney C_∞ topology used by Schofield the same two profiles would not be considered close. This fact explains the difference between the results of Rubinstein and Schofield -that regard the existence of non-empty cores with majority rule- and in general reveal the sensitivity of this kind of analysis to the particular topology used.

In a neat paper Greenberg (1979) studies the problem of what majority rules give raise to an equilibrium, namely, a feasible alternative unbeaten by any other under the same majority rule. Given that the main approach to the problem -that of considering restrictions of the admissible set of preferences- has produced extremely severe conditions, it seems meaningful to address a different question: which is the smallest majority size, say d, needed to have an alternative which is weakly preferred by more than $(n\text{-}d)$ individuals?

51 Kannai, Y. (1970) ***Continuity properties of the core of market***, Econometrica 38, 791-815

In other words, having a set of conclusions on conditions which avoid voting cycles under simple majority rule, this approach is to consider the problem under super-majority rule. Intuitively, it seems that weaker assumptions are possible if we consider the generalization to super-majority rules. In fact, we have two extremes: with the drastic restriction of unanimity no problems arise for the aggregation of individual preferences into social ones; on the other hand, with 50%-majority rule we still need severe restrictions. It is plausible that we can find a value for super-majority rule such that the needed restrictions are more likely to be met in actual societies. Note that also this approach -which is somewhere called the Simpson-Kramer[52] min-max majority- goes back to Black[53]; Greenberg seems to be unaware of both these circumstances.

The general environment is straightforward (but the formal model is expressed in terms that are different from the above reviewed literature; I use here the more common setting): there is a set $X \subset \Re^m$ of alternatives, there are N voters with individual orderings on X, and $N(yR_ix)$ denotes the number of voters that prefer y to x. Moreover, let $d \leq n$ be a positive integer: a *majority rule* is a d that specifies the smallest number of voters that can enforce a change over a status quo. An alternative $x^* \in X$ is called a *d-majority equilibrium* if there is no alternative in X that is strictly preferred to x^* by at least d voters[54].

A majority rule d induces then a social ordering as follows: $xP^dy \Leftrightarrow N(xP_iy) \geq d$. The main result concerns a convex and compact set of alternatives of dimension m, and convex and continuous individual preferences relations[55]: for every profile of individual preferences relations there exists a d-majority equilibrium if and only if d is greater than $\lceil (m/(m+1))n \rceil$[56].

Note that when the dimensionality of the set of alternatives increases, so does the proportion of individuals needed to yield an equilibrium: if $m \to \infty$, then only unanimity can ensure an equilibrium.

52 Simpson, P.B. (1969) *On defining areas of voter choice*, Quarterly Journal of Economics 83, 478-490; and Kramer, G.H. (1977) *A dynamic model of political equilibrium*, Journal of Economic Theory 16, 310-334

53 Black, D. (1948) *The decisions of a committee using a special majority*, Econometrica 16, 245-261. See also Black (1958)

54 The set of $(n/2)$-majority equilibria coincides with the set of simple majority equilibria, and the set of n-majority equilibria is the set of the Pareto optima.

55 Individual preferences may be intransitive and/or not connected.

56 An equilibrium exists iff the proportion of individuals that can enforce a change exceeds $m/(m+1)$.

Using this result, Greenberg shows that, when the set of alternatives is finite with cardinality T, for every profile of individual preferences orderings[57] there exists a d-majority equilibrium if and only if d is greater than $\left[\left(\frac{(T-1)}{T}\right)n\right]$.

The literature on the super-majority method culminates in a paper by Caplin and Nalebuff (1988) in which the authors generalize preceding results and present a new approach to the whole problem. Basically, Arrow's idea on the importance of unanimity and of some sort of similarity of individual preferences is explored given a definition of social consensus; then a minimal (with regard to the social consensus) d-majority rule that avoids voting cycles, and whose value turns to be 64%, is looked for.

The general setting is the usual one: for a given proposal $x \in X \subset \Re^m$ its kth coordinate, x_k, reflects the proposal's point of view on the kth issue. However, individual preferences over X are now divided into types indexed by subscript i, $i \in I$: a voter of type i has preferences P_i. Then, the distribution of types, specified by the density function $f(i)$ on i, characterizes a given society. The definitions of a *social decision problem* and of the *mim-max majority* are straightforward:

Definition 2.2 A social decision problem is the triple $C = \{X, P_i, f(i)\}$.

Definition 2.3 For any C and $x, y \in X$,

- $N(x,y)$ is the fraction of the population for whom yP_ix;
- $N(x) = \sup_{y \in X} N(x,y)$ is the maximal fraction against x;
- $N^*(C) = \inf_{x \in X} N(x)$ is the min-max majority;
- all points x^* for which $N(x^*) = N^*(C)$ form the *min-max* set.

In words, given a status quo x, there are alternatives which are supported by fractions of the population; some of them have many supporters and some have few of them: corresponding to the most sponsored alternative, *N(x)* tells us exactly how many are its supporters. This value clearly varies with the particular status quo: $N^*(C)$ is then the minimal fraction of population that can be against the status quo. Note that a d-majority winner exists iff $d \geq N^*(C)$, and recall that $N^*(C)$ is bounded above by $m/(m+1)$ (Greenberg (1979)).

57 Note that now individual preference relations have to be reflexive, connected, and transitive.

Consider now that a voting rule specifies a majority size, *d*, required to overturn the status quo; to avoid voting cycles *d* has to be bigger than the min-max majority. The problem is that when preferences are polarized then unanimity (*d=1*) is required, anybody can veto change from the status quo. The notion of a social consensus ensures that voters are not so much polarized: it includes two restrictions on preferences which are called respectively the assumption of Euclidean Preferences (EP) and that of concavity of the distribution of voters' most preferred points. Under EP individuals rank options according to their distance from a most preferred point then the distribution of preference types can be represented by the density function of voters' most preferred points.

Assumption 2.1 (EP) Let x_i represent a type *i* individual's most preferred point, and let $\| \|$ indicate the Euclidean norm. Then, for a voter of type *i*

$$aP_ib \quad \Leftrightarrow \quad \| a-x_i \| < \| b-x_i \|$$

In one dimension Ass. 2.1 implies single-peakedness; in the light of the discussion of Kramer (1977) it is interesting that the authors prove that the restrictive implications in term of value restrictions on preference orderings associated to EP diminish as dimensionality increases. The significant property of EP is that voters that prefer different alternatives are divided by a hyperplane in the space of most preferred points.

Assumption 2.2 (C) Let $f(x)$ be the density of voters' most preferred points, and let S, a convex subset of $\Re^m$ with positive and finite volume, be its support. Then $f(x)$ is concave over S. That is, for $0 \le \lambda \le 1$ and $(x_1, x_2) \in S$, we have:

$$f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda) f(x_2)$$

A third assumption, called *Inclusivity* (I), is needed to simplifies some proofs: it says that the set X is compact and contains S. This means that there is a continuum of alternatives. However, the results extend to the case of a finite X.

Assumption 2.2 essentially requires a degree of consensus and thus implies a non polarized society. In one dimension C implies that voters rank alternatives according to their absolute difference from some most preferred option[58]. The results of the paper are not quite robust with regard to relaxations of C, a part from functions that are close to concave[59]; moreover, C becomes more restrictive as dimensionality increases. However, the authors claim, on empirical grounds, that many real situations match the condition requested by C.

The main result is provided by the following theorem:

Theorem 2.1 For any decision problem C in $\Re^m$ satisfying EP, C and I,

$$N^*(C) \le 1 - \left(\frac{m}{m+1}\right)^m$$

Note that the bound of the min-max majority varies with m, the dimension of the choice space: for $m=1$ it is 0.5, for $m=2$ it is 0.56, for $m=3$ it is 0.58 for $m=10$ it is 0.61. As the dimension tends to infinity the bound converges to 0.632.

Relaxing EP to the class of intermediate preferences presented by Grandmont (1978), the authors are able to get a similar result: as an example, they use CES utility function -that corresponds to preferences belonging to the class of intermediate preferences- to show that the bound is slightly inferior to that given by Theorem 2.1 for any value of m.

The important extension to large finite populations is established for the case in which the density of most preferred points satisfies C, and the population consists of a sample drawn from it. The sample min-max majority converges almost everywhere to the min-max majority relative to the density function.

Consider now that in actual life the size d of a majority rule varies with the concerned situation, and so does the min-max majority $N^*(C)$. When $d < N^*(C)$ voting cycles may occur. On the other side, if $d > N^*(C)$ no voting cycles are possible, but there are many d-majority winners, that is, an indeterminacy appears: once any of them becomes the status quo it cannot be overturned. Nonetheless, Caplin and Nalebuff prove that the

58 Note that the social consensus implied by single-peakedness is about the way voters rank alternatives. See Definition 2.1 and its comment.

59 Closeness is defined by means of a measure of the distance between two integrable functions.

set of d-majority winners shrinks uniformly to zero as d tends to $N^{*}(C)$. Then, the importance of these two problems is not symmetric around the min-max majority: voting cycles exist even for very small positive values of $N^{*}(C)-d$.

Since avoiding voting cycles seems to be more important than a small amount of indeterminacy, the authors argue that is better fixing the majority size above rather than below the average value of $N^{*}(C)$. They provide examples showing that the indeterminacy associated to a 64% majority rule decreases speedily with m . This is why the 64%-majority rule is chosen as the best rule for most social decision problems when there exists a certain social consensus on the issue.

PART 5:

THE MEDIAN VOTER THEOREM

1 Introduction

The median voter theorem rests on the assumption that a majority equilibrium exists. It is well known that single-peakedness is a sufficient condition for the existence of such an equilibrium.

Under the assumption of sincere voting and that of a single dimension of the decision to be taken, the median voter model provides strong predictions. In the case of choosing a provided level of a public good, it says that the majority voting equilibrium occurs when this level equals the quantity preferred by the median individual, and that corresponding to the voting equilibrium, the chosen output level is efficient.

Since its presentation by Bowen (1943), this theorem influenced many researches on spatial models of the electoral process: see for example, the works of Downs (1957) and Black (1958). This paper aims to give a slight generalization of the original theorem. It must be said that this task does not contradicts the basic criticism of Hinich (1977): he showed that the median is not an equilibrium when probabilistic voting is introduced. Probabilistic voting accounts for a certain amount of indeterminateness in voter choices when the difference between alternatives is small, and thus it introduces a quite realistic feature in the model.

Eventually, it is worth considering that Bowen's work is the second example of a partial equilibrium analysis of public goods, the first being that of Lindahl (1919). The first general equilibrium version came much later with Samuelson (1954).

2 The model

There are two basic desiderata in the solution of the problem of providing a public good: first, the efficient provision level should depend upon individual benefits from the public good. This means that sufficient private information must be available. Second, a tax mechanism has to be designed in such a way that it satisfies the minimal condition of a balanced budget.

The simplest way to solve the problem consists in assuming that benefits and taxes are distributed equally among agents: then, every one will obviously agree on the provision level, since the point at which marginal benefits and marginal taxes will be the same for all. The median voter theorem states that voting by majority rule will lead to an efficient provision level of the public good: it keeps the assumption about equally distributed taxes, but only requires that marginal benefits are distributed in such a way that mean and median marginal benefits are equal at the level of public good provision associated to the majority rule outcome.

Originally the median voter theorem was proved by Bowen (1943) in this form:

There is one public good to be produced: optimal [1] conditions of supply are sought, considering that benefits from the same consumed quantity differ across individuals. Voting is a substitute for the classic consumer choice in the private good case. Individuals have preferences that depend on two factors: personal benefits and personal share of production costs, that is, taxes: these factors are represented by individual curves of marginal substitution (*MS*) [2] between the public good and one private good, and by individual curves of marginal costs (*MC*). Each individual will vote for that quantity of the public good at which his marginal benefit is equal to his marginal cost.

Let subscripts i, j denote individuals in the set N; the total cost of producing the public good is T, T/x is the average cost, and T' denotes the total marginal cost curve. Let B' be the total marginal benefits[3] obtained by summing individual marginal benefits b'_i for each level of output. Assuming constant costs, marginal and average individual taxes will be identical. Figure 1 shows that the ideal output is $0x^*$, corresponding to the point of intersection between the curve of total marginal benefit B', and that of average cost T/x.

Let t indicate the distribution of individual taxes with $\bar{t}$ $(= T/xN)$ as its mean: depending on the total output x, it indicates the average tax paid by individuals for each unit of public good. Let $\bar{t}'$ $(= T'/N)$ be the mean marginal cost, that is the average marginal tax per person, and $\bar{b}'$ $(= B'/N)$ denote the curve of the mean marginal benefit per person. Moreover, let $\breve{b}'$ and $\hat{b}'$ be respectively the modal and the median marginal benefit curve. They indicate the mode and the median of the distribution of individual marginal benefits at each level of output. To be clear, I am saying that for any x the corresponding b'_i , one for each agent, are realizations of a random variable: thus, it make sense to speak of its moments. In this setting the median of b' is the smallest b'_i satisfying $F_b(b'_i) \geq .5$, where $F_b(\cdot)$ is the cumulative distribution function of b'. The mode of b' is that realization such that $f_b(b'_i)$ reaches its maximum, where $f_b(\cdot)$ is the density function of b'. Strictly speaking, b' is a discrete random variable, and assumption 4 by saying that it is normally distributed rests on

1 In the literature the use of both terms (*Pareto*) *optimal* and (*Pareto*) *efficient* generates some confusion. I would use *efficiency* referring to allocations or states in exchange economies, reserving *optimality* for a broader set of conditions in the case of production economies. A state is then Pareto optimal if the conditions of efficient exchanges, of efficient allocation of factors, and of efficient output choice hold together. Others reserve *optimality* when the model considers distributional issues, in connection with the use of social welfare functions (Musgrave (1984)). However, in what follows both terms are interchangeable.

2 Bowen used the notion of marginal rate of substitution to express individual benefits: I will use the more direct concept of marginal benefit.

3 In Bowen's paper B' is a curve of total marginal rate of substitution between the public and the private good. This is the sum (vertical addition) of individual marginal rates of substitution and expresses the amount of the private good that the society would be willing to give up in order to ge successive units of the public good.

the fact that there are as many agents as needed in order to give a statistically satisfying approximation. Figure 1 shows that ideal output corresponds to the point of intersection between the curve of the mean marginal benefit per person, and the curve of the mean marginal tax per person.

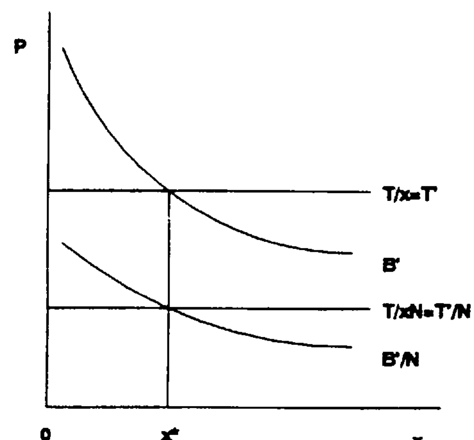


Figure 1

Four assumptions are needed:

Assumption 1. All N agents vote sincerely for their preferred quantities of public good: each agent reveals a preference which corresponds to his own interest;

Assumption 2. Production costs are known for each output quantity. The public good is produced under constant average costs[4];

Assumption 3. The tax allocation system is such that all agents pay the same tax: for all j, i, and for every quantity of public good, $t_i = t_j$. Then, all individual average cost curves and marginal cost curves coincide with the mean average-cost curve $\bar{t}$, and with the mean marginal-cost curve $\bar{t}'$;

Assumption 4. Individual marginal benefit curves are normally distributed, as shown in Figure 2 and 3[5]. Then, at any level of output the modal marginal benefit curve $\hat{b}'$ coincides with the mean marginal benefit curve $\bar{b}'$.

4 In the case of decreasing cost, the marginal cost curve T' lies below the average cost curve T/x so that the optimum level will require B' to be equal to T'.

5 It is arguable that the distribution must be truncated at zero, since negative marginal benefits are meaningless, at least if we exclude the case of public "bads".

In other words, the distribution of marginal benefits curves, b', is such that $\breve{b}' = \hat{b}' = \bar{b}'$ [6]. Bowen himself stresses that two conditions are necessary for this assumption to be met: first, individual tastes have to be distributed normally; second, all individuals have to be equally able to benefit from the public good.

The relevance of assumption 4 is that at any level of output x, the set of b'_i is symmetrically distributed around the mode $\breve{b}'$, so that its mode coincides with the curve of the mean marginal benefits $\bar{b}'$.

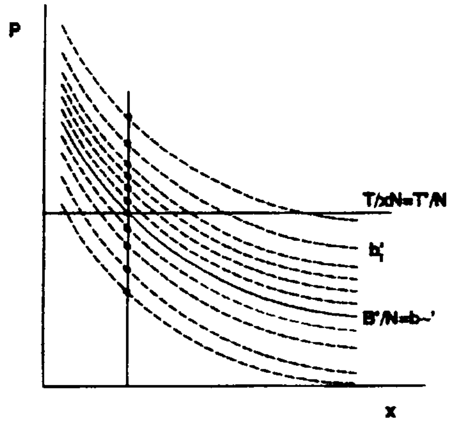


Figure 2

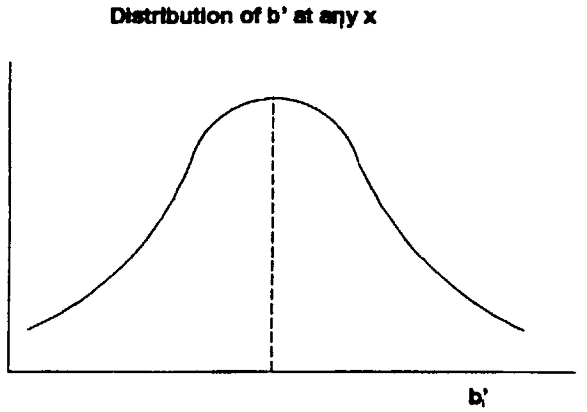


Figure 3

Theorem 1. Under assumptions *1* to *4*, the quantity of public good provided under majority rule is efficient.

This result can be explained as follows. Consider that, under assumption 2 , the optimum level of output x is such that $T/x(= T')$ is equal to B'; this is equivalent to require that $\bar{t}(= \bar{t}')$ is equal to $\bar{b}'$. That is, the mean individual tax (which equals the mean marginal individual tax, under Assumption 2), is equal to the mean individual marginal benefit.

Now, compare the $\bar{t}$ (and $\bar{t}'$) curve to the set of individual marginal benefit curves b'_i . Assumption *4* says that these last curves are distributed normally, thus the modal marginal curve $\breve{b}'$ coincides both with the median $\hat{b}'$ and with the mean $\bar{b}'$.

---

6 Notice that this does not imply $b_i = b_j$ for all j, i, that is, individual benefits need not be the same.

Each person will vote for that level of public good such that $b'_i = t'_i$, that is, his marginal benefits equals his marginal cost. An intermediate level will be voted for by more agents than any other level: this will be the level voted for by all those individuals whose marginal benefits are modal. The level of output corresponding to the equivalence between the modal marginal benefit curve and the mean marginal cost curve will be the level voted for by at least a relative majority of individuals. Assumption *4* and *2* ensures the uniqueness of this level, and assumption *4* also allows us to replace the ***modal*** with the ***median*** voter.

Then, on one side we have this condition for efficient output: $\bar{t}' = \bar{b}'$; on the other side, the majority equilibrium outcome x^* is such that $\tilde{b}' = \bar{t}'$. But we know that $\tilde{b}'$ is equal to $\bar{b}'$. Thus, the voting equilibrium takes place at a level of public good provision such that the individual with the median marginal benefit pays a marginal tax equal to his marginal benefit. A voting equilibrium is also efficient since by assumption, the median and the mean marginal taxes are equal.

In other words, under the foregoing assumptions, an alternative cannot be beaten by any other alternative under the majority rule, if and only if it is the most preferred choice of the median voter: this is a voter whose most preferred alternative is such that at least half of the voters' most preferred points equal it or are to the left of it, or such that at least half of these points are the same as it or to its right. Since in this setting the median voter necessarily exists, the theorem proves the existence of an unbeaten alternative, and identifies its location as an efficient one.

I claim that a more general version of this theorem can be proved by only assuming that the median $\hat{r}$ of the distribution of the difference between marginal benefits shares and marginal taxes shares, $r = [(b'_i/B') - (t'_i/T')]$, equals zero. Assumptions *1* and *2* remain the same.

Assumption 4'. The median of the distribution r of differences between the distribution of individual shares of total marginal benefits (b'_i/B') and the distribution of individual shares of total marginal taxes (t'_i/T'), at the output level x^* corresponding to the majority rule equilibrium is zero: $\hat{r} = 0$.

Notice that $b'_i/B' \neq b'_j/B' \quad \forall i, j \in N$. At each output level we have a set of $r_i = b'_i/B' - t'_i/T'$. See Figure 4, where the individual marginal tax share is considered the same for both i and j for simplicity:

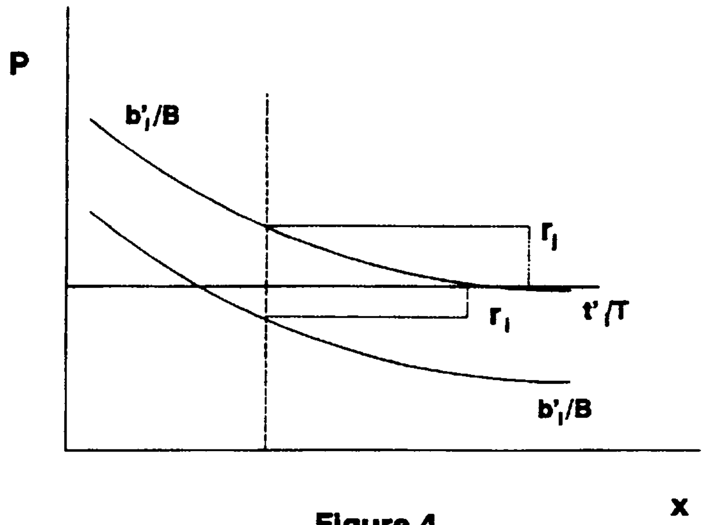


Figure 4

Assumption *4'* says that the distribution of r_i is such that its median is zero at the output level corresponding to the majority rule equilibrium. The difference between the two assumptions is the following: Assumption 4 holds that a random variable (whose realization are individual marginal benefits curves) has a normal distribution, while Assumption 4' only says that the median of the random variable r is zero at x^*. Notice that r_i can be smaller, larger or equal to zero; however, since both $\sum_i^N b'_i/B'$ and $\sum_i^N t'_i/T'$ equal 1, we have $\sum_i^N r_i = 0$. Moreover, a particular distribution of individual taxes, as in assumption 3, is not assumed.

Theorem 1'. Under assumptions *1*, *2*, and *4'*, the level of the public good output x^* provided at a majority rule equilibrium is efficient. That is, $\{x^* \mid B' = T'\}$.

Proof. We have to show that, under Assumptions *1*, *2*, and *4'* $\{\hat{r} = 0\} \Leftrightarrow \{B' = T'\}$. First, assume that $\hat{r} = 0$: we can write $m = b'_i - t'_i$ to denote the distribution of differences between individual marginal benefits and taxes which corresponds to an equilibrium level x^* of the public good output: then, the median $\hat{m}$ of m is zero. The distribution of differences between individual marginal benefit shares and taxes shares at the equilibrium level x^* is

$$r = \frac{b'_i}{B'} - \frac{t'_i}{T'} \qquad (1)$$

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

and, since $f = 0$ by assumption, also the median of

$$b'_i - t'_i \frac{B'}{T'} \tag{2}$$

is zero. The distribution *(2)* can be written as

$$(b'_i - t'_i) - t'_i \frac{B' - T'}{T'} \tag{3}$$

which has a zero median too. The distribution in brackets is m, whose median is zero. It differs from the whole *(3)* by the quantity $t'_i(B' - T')/T'$, and they have the same median. Since t'_i is positive, and the fraction is constant for the distribution, $(B' - T')$ must be zero, otherwise their median would differ. This means that the sum of marginal benefits equals the sum of marginal costs, which implies efficiency.

On the other side, we assume that $f \neq 0$ and show that this implies $B' \neq T'$. If the median of *(1)* is different from zero, then so it is the median of *(3)*. But the median of $(b'_i - t'_i)$ is zero. Again, to change the median of a distribution we need to subtract a non zero factor: since t'_i is positive, $(B' - T')$ must be different from zero. This implies $B' \neq T'$, which means that the level of output is inefficient.

PART 6:

HOW MUCH ROOM IS THERE REALLY FOR STRATEGIC VOTING ?

...perhaps what ought to happen depends not on what people actually prefer, but on what they would prefer if they were fully informed and clear-headed..[1]

1 I am sorry for not being able to give credit to the author of this sentence: it means that maybe the whole problem is ill specified!

 DOI: 10.2870/7482

Table of Contents

1 Introduction 1

2 An example of strategic voting 4

3 A combinatorial model 6
3.1 Computing the number of orderings and profiles 9

4 The Borda Rule as an aggregation rule 12
4.1 Some notions and definitions 15
4.2 How the Borda Rule works 23
4.3 The algorithm used in the analysis 28

5 Two examples and some results 29

6 Non manipulability from the set of total scores 37

7 A modified Borda Rule 42

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

1 Introduction

The problem of strategic voting has been given much attention in the literature. Strategic voting occurs when a person votes not in order to reveal his preferences honestly, but in order to advance most effectively his interests[2].

However, it is not clear how much this problem is actually relevant. Firstly, if the requested informations is anonymous, its revelation is not directly linked to the actions that the government will take toward a particular voter, so that there is no individual interest in cheating.

On the other hand, it may be the case that the possibility for individuals to misrepresent their preferences is large, also because these preferences are not easily related to other observable individual characteristics and actions.

To be clear, I do not deal here with the very special case in which individuals are requested to reveal their personal income in order to be taxed: in this case the messages between the government and individuals are not anonymous, in the sense that Mr. Smith sends an answer to the government that knows exactly who is the sender. When anonymity does not hold, the government has a certain capacity to verify the likelihood of the message: in general this capacity depends on the available technology as well as on the type of message. We can imagine, for example, a computer which records most of the trades between individuals in an economy and a big bureaucracy which verifies that 90% of these trades are recorded. Then, it would be virtually impossible for individuals to cheat.

I deal here with a different situation: there are some alternatives, individuals have preferences over each of them and they are asked to report their rankings. Then, following a rule that is understood by every individual, the government aggregates individual preferences to make a collective choice. This is an exercise of "interest-aggregation" in the sense of Sen (1977): different people's personal interests are aggregated and not different persons' judgements of what is good for the society[3].

2 A slightly different reason for not voting sincerely, which is not discussed here, is the following: in a relative majority procedure a voter may refrain from casting his vote for his first choice if he thinks that the option has no chance of success, for fear of wasting his vote.

3 However, Sen (1982b) remarks that the Borda rule, which is discussed below, was originally addressed to the problem of judgement-aggregation.

In such a setting the possibility of individuals to misrepresent their preferences is large, not only because messages are anonymous, but for the most important reason that these preferences are not related to other observable individual characteristics and actions. That is, individual reports are the only source of information.

A well-meaning government that tries to base its decisions on the preferences of the people involved may well not be able to learn what those preferences are. It can know them through plebiscites or public opinion polls. However, people who understand the system by which decisions are made may have an incentive to misreport their preferences. This is the standard way to state the problem and to introduce the discussion on strategic voting. Following the work of Vickrey (1960), Dummett and Farquharson (1961) conjectured that it is unlikely that there is any voting procedure in which it can never be advantageous for any voter to vote strategically. More recent papers by Gibbard (1973) and Satterthwaite (1975) gave a definitive answer to the problem: every non-dictatorial voting scheme with at least three distinct outcomes is manipulable.

Actually, Satterthwaite (1975) proved that every strategy-proof voting procedure is dictatorial and also that, deriving a voting procedure from a social welfare function that violates any one of Arrow's conditions, the same voting procedure is not strategy-proof. This last result establishes that strategy proofness of voting mechanisms corresponds to Arrow's conditions for SWF.

The Gibbard-Satterthwaite theorem would seem to undermine the very rationale of the voting procedures. However, this conclusion turns out to be unwarranted, since the theorem applies only to voting procedures that are single valued. It must be said that Gärdenfors (1976) proved a theorem which states that all anonymous and neutral voting procedures satisfying the Condorcet winner criterion are manipulable by individuals. More precisely, it says that if a voting procedure has the above three properties, then there are profiles under which an individual may benefit from voting as if his preferences were different from what they really are.

Pazner and Wesley (1977) established that when the set of voters is infinite, and the set of alternatives is finite, there exist a nonimposed and nondictatorial social choice function which is individually and coalitionally cheatproof. Actually, if the set of alternatives is countably infinite, then such a function exists also for a set of voters which is of measurable cardinality: however, apart from the mathematical problem of the existence of such sets, it has been shown (see their reference [12]) that they are too large for any practical purpose.

Following their previous work, Pazner and Wesley (1978) firstly construct some examples of social choice functions wich very nearly satisfy the conditions of being coalitionally-cheatproof, non-imposed, and non-dictatorial. They are all variations of majority rule. Secondly, they show that the limit of the ratio between the number of individually cheatproof profiles and the number of all possible profiles equals 1 when the number of voters goes to infinity. This is interesting, also because the result seems to be independent of the finite number of alternatives. However, there are two important remarks to be made.

First, they define the set of individual preferences as the set of "total, transitive, asymmetric (preference) orderings" over the set of alternatives: this means that individuals are not allowed to be indifferent between alternatives. Then, they define a social choice function as a function from the set of profiles to that of alternatives: each profile is associated to one alternative, the socially preferred one. This setting is different from the one developed here, since my aggregation rule chooses a social ordering of all alternatives for any possible profile[4]. When the aggregation rule is defined as an SCF that chooses only one alternative, we disregard most of the information provided by a profile. Moreover, by doing so, we do not take care of the problem that a social ordering may not belong to the set of individual orderings as defined above: in fact, a social indifference relation between pairs of alternatives might well arise from individual orderings without indifference relations, depending on the aggregation rule. With the non-positionalist majority rule this becomes the problem of the possible non transitivity of the social choice: they deal with it by assuming "a reasonable tie-breaking device". This is an oversimplification, since strategic voting is much easier in the case of ties.

The second remark concerns one assumption of their limit theorem about individual cheatproofness of majority rule: namely the assumption that all possible profiles are equally probable in the society: this can hardly be considered as a plausible assumption.

Lepelley and Mbih (1987) address the question of coalitional cheatproofness for majority rule when the number of alternatives is smaller than five. They actually compute the number of strategy proof profiles under majority rule for the case of three alternatives. Their results show that the possibility of strategic voting by coalitions is high even with small values of the number of voters, and that it increases with the number of alternatives.

---

4 Note however, that to simplify the analysis it is assumed that strategic voters are interested only in the top social alternative.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

In a recent paper, extending the work of Satterthwaite and Sonnenschein (1981), and of Border and Jordan (1983), Zhou (1989) addressed the question of wether the Gibbard-Satterthwaite theorem still holds under various restrictions on the domain of admissible preferences. In particular he considers an economy with pure public goods in which individual preferences are increasing over goods, continuous, and quasi-concave. His aggregation rule is a single-valued SCF, which maps a preference profile to an allocation. One of Zhou's results is that if the dimension of the set of feasible allocations is larger or equal to two, then any strategy-proof mechanism is dictatorial[5].

MacIntyre (1991) deals with strategic voting under a particular choice rule, which combines the Pareto criterion with the lexicographic maximin rule when individual preferences are strict. He shows that, starting from a profile of true orderings, every strategic misrevelation is an improvement for all voters. Furthermore, the existence of an equilibrium profile in which no voter has incentive to cheat is demonstrated.

In my opinion, however, the attention to the nonexistence of strategy-proof mechanisms is misplaced. First of all, given a voting mechanism, its ability to circumvent individual strategic voting depends also on the particular profile of preferences reported by individuals. Even if there is not a mechanism which is *always* not manipulable (with regard to the many possible profiles), it may be the case that the conditions for manipulation are exceptions. Strategy proofness might be a convenient guideline for judging voting mechanisms, and in this sense we could look for mechanisms which minimize the occurrence of strategic voting[6]. Secondly, it is apparent that a voter needs a huge amount of information even to assess if he has any chance of bettering his interest by misreporting his preferences. Thirdly, it is not clear that coalitional strategic voting has the same nature as individual strategic voting: under any mechanism the fact that a coalition of voters behaves in a certain way should be considered as a datum of the situation and not a negative effect per se.

In what follows an example of strategic voting is given and then the relevance of cheating is investigated with regard to the Borda Rule.

2 An example of strategic voting

Example 1. Consider four alternatives denoted by a_1, a_2, a_3, a_4 ; there are three individuals with preferences over these alternatives; the collective choice is made by the Borda rule: each alternative gets a value

5 A similar subject is addressed in a paper presented to ESEM, 1991, by Barbera and Jackson.

6 This is also suggested by Arrow (1963) page 7.

for its position in every individual ranking according to the rule that assigns $(4-1)$ to the first best choice, $(4-2)$ to the second, $(4-3)$ to the third and 0 to the last choice. Then, the points each alternative gets are added up and the alternative with the most points wins. In general, if there are alternatives which get the same score, then the decision is made by chance. The following scheme shows the orderings (true preferences) of the three individuals in the three left end columns, the way the rule applies, and the resulting social ordering:

i	j	k	Computation	Social ordering
a_1	a_3	a_3	$a_1 = 3+2+2 = 7$	a_1 (7)
a_2	a_1	a_1	$a_2 = 2+1+1 = 4$	a_3 (6)
a_4	a_2	a_2	$a_3 = 0+3+3 = 6$	a_2 (4)
a_3	a_4	a_4	$a_4 = 1+0+0 = 1$	a_4 (1)

Notice that individual k prefers a_3 to a_1 and this last alternative to a_2 ; however, the resulting collective choice does not satisfy him. Now, holding constant i's and j's revelations, assume that k misreports his preferences as follows, and let us compute again the final outcome:

i	j	k^*	Computation	Social ordering
a_1	a_3	a_3	$a_1 = 3+2+0 = 5$	a_3 (6)
a_2	a_1	a_2	$a_2 = 2+1+2 = 5$	a_1 (5)
a_4	a_2	a_4	$a_3 = 0+3+3 = 6$	a_2 (5)
a_3	a_4	a_1	$a_4 = 1+0+1 = 2$	a_4 (2)

Thus, we conclude, k actually gains by misreporting his preferences: he can alter the final outcome of the process in his favour. Intuitively, for this opportunity to occur some conditions must be satisfied: first, individual k should know in advance if not the preferences of all the others, at least the final outcome without his input. Second, the final outcome must be sensitive to his preference ordering, in the sense that k's false ordering can produce a switch in the social ranking of alternatives. The aim of this paper is to investigate when these conditions are met, that is, how relevant is this problem of strategic voting in real situations.

3 A combinatorial model

In the sequel I assume that an individual ordering does not include indifference relations between unequal pairs of alternatives. Thus, the set of possible individual orderings contains complete and strong orderings of unequal pairs. To be clear, a voter is not allowed to report an indifference relation between alternatives.

With n agents in the set N, and m alternatives in the set A, denote by O the set of all possible orderings; denote also by O' the set of n orderings actually set by voters when asked to report their preferences. In general we have $O' \subset O$ instead of $O' \subsetneq O$. There are two reasons which justify this statement. First, the cardinality of O is $m!$ and it may well be the case that ($m!$ =) $\#O$ is larger than $\#N$[7]. This implies that we can be sure that some of the possible orderings are not expressed simply because there are not enough voters. Second, even if there are more voters than possible orderings, it is clear that the case of each voter preferring a ranking different from that of any other voter is unlikely to happen: it suffices to consider any form of social consensus about some basic issue to see that in most cases groups of voters will agree on some orderings. Then, a profile, that is a sequence of n orderings one for each voter, may well comprehend many equal orderings. (Profiles with all different orderings give a hard time to any aggregation rule, indeed!). Eventually , I presume that in general not all elements of O are present in $O' \subset O$.

An election produces a profile: this is a matrix with n columns and m rows. Each column is an ordering belonging to O and the whole matrix is a n-collection of elements of O. In the matrix (a_{ji}) the i^{th}-column is the preference ordering of voter i, while the j^{th}-row lists the alternatives which fill the j^{th} place in all the n orderings.

Given the set O[8], we can investigate how many profiles can develop from it. This is useful because in general an aggregation rule is a function from the set of possible profiles to the set of possible social orderings.

We can have both *(i)* $m!>n$ and *(ii)* $m!<n$ (equality, albeit highly improbable, is significant from the computational point of view). Different voters can have the same orderings so that *(ii)* does not seems to be a limitation for individual choices.

7 In a committee this is clearly usual. For the case of a general election, consider for example that there are eleven parties represented in the Italian Parliament: this gives rise to 39.916.800 different rankings, which is a number larger than the number of voters.

8 Notice that each preference ordering is assumed to rank all available options: this is a *complete list* mechanism in the sense of Dummett (1984).

In the first case we have $m! > n$, that is, we have $m!$ possible orderings (while only n can actually be expressed and they can coincide). How many profiles of n columns can we form with $m!$ elements? The answer is $m!^n$ if we allow for the order of equal columns (preference orderings) to distinguish two profiles; this amounts to compute the number of possible ordered selections. However, *anonymity* would imply that two profiles are different if and only if they contain at least one different ordering. To see this, consider two equal profiles: then, in the second, interchange the first column with the third one; we still have two equal profiles since the social aggregation rule treats orderings per se and does not regard who holds them. If we use $m!^n$ to compute the number of possible n-profiles, then the imposition of anonymity does not make the aggregation rule to be a one-to-one function: some elements of the set of profiles have the same image in the set of social orders. So, we have the problem of computing the number of unordered selections with repetition. In general, when there are groups of n objects to be chosen from p of them, this number is

$$C^{(r)}_{p,n} = \frac{(p+n-1)!}{n!(p-1)!}$$

To fix ideas, consider that with 3 alternatives and 4 voters we have 6 different orderings and 126 different profiles. There are m alternatives in the set A and n voters in the set I. Then, we have $m!$ orderings in the set O. With $m=10$ there are 3,628,800 orderings.

We have seen that if $m!>n$ there is no problem: we just want to know how many ways there are to choose n objects (orderings) out of $m!$. This is a classical problem which in combinatorial analysis is known as "unordered selection without repetitions". If $m!<n$ we can solve the same problem allowing for repetitions: in any group (profile) an object can appear up to n times. Since in both situations we allow for repetitions of the same ordering in a profile - two profiles differ if and only if there is at least a pair (one ordering from each profile) of different orderings - we are going to compute the cardinality of the set of possible profiles using the formula of unordered selections with repetitions. This is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n+k-1-k)!} = \frac{(n+k-1)!}{k!(n-1)}$$

which in this framework becomes:

$$C'_{(m!,n)} = \frac{(m!+n-1)!}{n!(m!-1)!} \tag{1}$$

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

While the formula for unordered selection without repetitions would be

$$C_{(m!,n)} = \frac{(m!)!}{n!(m!-n)!}$$

Notice that since $(m!+n-1)! > (m!)!$ and $n!(m!-n)! < n!(m!-1)! \quad \forall n > 1$, we have $C'_{(m!,n)} > C_{(m!,n)}$.

The choice of working with USR instead of OSR needs some more justifications, apart from the notion of anonymity: it is indeed true that both profiles DEEFG and FGEED[9] should be considered equal by any "democratic" aggregation rule. However, it is also true that these two profiles can both occur since they represent two different situations with regard to individual preferences: as such, they are both part of the population of possible events and when our task is to count all of them and/or some particular subset it would seem correct to treat them as different[10].

Despite this remark, there are some major problems with OSR. First of all, there are much more OSR than USR for any *m* and *n*: for example, with *m=3* and *n=6* we have 462 USR but already 4.666.656 OSR. This means that the counting problem is more difficult from a practical point of view. Secondly, many permutations of a profile are indistinguishable. Consider the profile DBB: it is one of the 3! permutations of three elements and more precisely one of the 3 distinguishable permutations DBB, BDB, and BBD. Let subscripts indicate the type of voter: then, $D_1B_2B_3$ is equal to $D_1B_3B_2$ for what concerns our problem; actually, it would be a mistake to count both of them as possible outcomes. This amounts to overestimate the size of the set of possible outcomes, while by considering USR we underestimate it.

It is true that another, more complex method would be possible: we could consider OSR, compute for each profile the number of indistinguishable permutations, subtract them, and base our analysis on the resulting set. In fact, the formula for counting distinguishable permutations of *n* elements when each of them can belong to one of the t_k types (in our setting each type is clearly a different ordering) is available[11]. However, even if the counting problem seems solvable, we are left with the problem of listing the elements of our set in order to investigate whether they satisfy certain conditions. This turns out to be a difficult task since no algorithm is available.

9 I assume that the order of voters for both profiles is the same: so that, for example, DEEFG tells us that the first voter prefer the ordering D, while the second profiles tells us that the same voter prefers F.

10 I owe this remark to the participant to a seminar held at the IAS.

11 This formula is $P(n;t_1,t_2,\dots,t_k) = n!/t_1,t_2,\dots,t_k$

Finally, I think that we should evaluate the type of mistake we are making by considering USR in the light of our present goal: the main messsage of this paper is to show that even a well informed voter finds a lot of computational problems when he tries to act strategically. It does not matter so much whether he has to consider (with m=3 and n=50) $3.5 \cdot 10^6$ instead of $8 \cdot 10^{38}$ profiles.

3.1 Computing the number of orderings and profiles

The set of all possible profiles is denoted by *P*. This is a huge set, whose dimension depends both on *n* and *m*. We construct it answering the following question: how many groups (profiles) of *n* elements (orderings) can we form choosing from *m!* of them? Again, notice that two profiles which contain the same *n* orderings in different columns (for example, in the first profile voter *j* has the preference *A* and voter *i* has *B*, while in the second profile *i* has *A* and *j* has *B*) are actually the same profile. Anonymity requires that it does not matter who has a preference ordering. Then, two profiles differ iff they do not contain the same preference orderings: the order of listings in a profile does not matter in making a difference. Following the formula, the cardinality of *P* is $C'_{(m!,n)}$.

So, we first have a function *f(m)=m!* from the cardinality of *A* to that of *O*. Given a number of alternatives in *{a,b,c,..,p}* it returns the number of possible orderings in the set *{D,G,H,..}*, where for example *D=(b c p..a)*[12].

Secondly, we have a function $E(m,n) = C'_{(m!,n)}$ which maps the number of orderings into the number of possible profiles $\{\Gamma, \Delta, \Theta, ..\}$. Eventually, *E* is defined on $\#O \times \#I \subset \Re^2$ and takes values in $\Re$. Recall that *anonymity* requires that the order of orderings in profiles does not matter in making two profiles differ.

We have a set *O* which contains *m!* objects (orderings). With them we construct $C'_{(m!,n)}$ profiles: these are *n*-sets of orderings. The problem is to specify the correct functional form of *f* and *E*. We have seen above that given the number of available alternatives, *f* has to return the number of possible orderings. An ordering is a complete list of all the alternatives without repetitions. Two orderings differ if and only if they rank at least one of the alternatives in a different way. An ordering is then a permutation of the set of the alternatives;

12 Throughout I will omit the usual symbols for preference relations between alternatives: in any ordering represented in brackets an alternative is strictly preferred to any alternative on its right. Indifference relations in social orderings are represented either by underscoring the concerned alternatives, or by including them in brackets: *(eda...)* is equivalent to *(e(da)...)*.

we know from combinatorial mathematics that the number of possible permutations of m objects is $m!$. So, I conclude that $f(m) = m!$. In other words, given that $\#A = m$, then $\#O = m!$: O can be viewed as the set of all permutations of the set A; a permutation of A is defined to be a bijection from A to itself.

The functional form of E clearly concerns also the number of voters. We want to consider n-groups of orderings (profiles), an ordering for each voter. The orderings may well be the same within a group, so that repetitions are allowed. Anonymity requires that any social aggregation rule treats indifferently two profiles that contain the same orderings in a different order: a permutation of a profile is not in the set of all possible profiles. Thus, this is a problem of computing the number of unordered selections with repetitions. The fact that selections are unordered can be understood by considering the following example: if we have to choose five out of ten books, the only interest is in which five books are chosen and not in the order in which they are chosen. Eventually, the functional form of E is given in equation (1). The following table shows some of the values of E for $m \leq 8$ and $n \leq 1000$. Notice that the numbers in the first row equal those computed by $m!$. We can have $m!$ profiles of size 1 from m alternatives: one for each of their permutations. The numbers of the second row can also be computed by $\sum_{i=0}^{m!-1}(m!-i)$. Notice that the expression $x.zey$ stands for $x.z \cdot 10^y$

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

TABLE 1: $E(m,n) = C'_{(m!,n)}$

n \ m	1	2	3	4	5	6	7	8
1	1	2	6	24	120	720	5040	40320
2	1	3	21	300	7260	259560	1.3e7	8.1e8
3	1	4	56	2600	295240	6.2e7	2.1e10	1.1e13
4	1	5	126	17550	9.1e6	1.1e10	2.7e13	1.1e17
5	1	6	252	98280	2.2e8	1.6e12	2.7e16	8.8e20
6	1	7	462	475020	4.7e9	2.0e14	2.3e19	5.9e24
7	1	8	792	2.0e6	8.4e10	2.0e16	1.6e22	3.4e28
8	1	9	1287	7.8e6	1.3e12	1.8e18	1.0e25	1.7e32
9	1	10	2002	2.8e7	1.9e13	1.5e20	5.8e27	7.8e35
10	1	11	3003	9.2e7	2.4e14	1.1e22	2.9e30	3.1e39
20	1	21	53130	9.6e11	7.1e23	7.4e38	4.7e55	5.3e73
50	1	51	3.5e6	5.7e18	2.5e43	1.2e79	5.5e120	6.4e165
150	1	151	6.9e8	2.5e28	7.7e78	1.4e172	3.6e293	1.5e428
300	1	301	2.1e10	8.9e34	1.6e107	4.3e266	1.1e500	4.5e767
600	1	601	6.6e11	4.8e41	5.1e138	1.2e393	1.8e828	1.4e1357
1000	1	1001	8.4e12	5.1e46	1.7e163	5.5e505	1.9e1175	1.8e2043

The computation of the cardinality of P following the formula $C'_{(m!,n)}$ is an alternative to the method followed by Fishburn and Gehrlein (1976). They use the formula to compute the number of ordered selections with repetitions, *(m!)ⁿ*. I do not think that it fits our problem since it does not consider the fact that two profiles differ if and only if they have at least two different orderings. The following table collects some values of $E'(m,n) = (m!)^n$. Notice that the computation is extended to values of m larger than *8* simply because the limitations due to computer time use are less severe than before: this formula is easier to calculate[13].

13 Actually, this depends on the used algorithm: for example, the use of the formula (1) for $C'_{(m!,n)}$ makes the computation block for $m>7$ on most PC's. However, we can take advantage of the equivalence between (1) and $\binom{m!+n-1}{n}$: the direct computation of the binomial coefficient is much faster.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

TABLE 2: $E'(m,n) = (m!)^n$

n\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	6	24	120	720	5040	40320	3.6e5	3.6e6	3.9e7	4.8e8	6.2e9	8.7e10	1.3e12
2	1	4	36	576	14400	518400	2.5e7	1.6e9	1.3e11	1.3e13	1.6e15	2.3e17	3.9e19	7.6e21	1.7e24
3	1	8	216	13824	1.7e6	3.7e8	1.3e11	6.5e13	4.8e16	4.8e19	6.4e22	1.1e26	2.4e29	6.6e32	2.2e36
4	1	16	1296	331776	2.1e8	2.7e11	6.4e14	2.6e18	1.7e22	1.7e26	2.5e30	5.3e34	1.5e39	5.8e43	2.9e48
5	1	32	7776	7.9e6	2.5e10	1.9e14	3.2e18	1.1e23	6.3e27	1e38	2.5e43	9.4e48	5e54	5e54	3.8e60
6	1	64	46656	1.9e8	3e12	1.4e17	1.6e22	4.3e27	2.3e33	2.3e39	4e45	1.2e52	5.8e58	4.4e65	5e72
7	1	128	279936	4.6e9	3.6e14	1e20	8.3e25	1.7e32	8.3e38	8.3e45	1.6e53	5.8e60	3.6e68	3.8e76	6.5e84
8	1	256	1.7e6	1.1e11	4.3e16	7.2e22	4.2e29	7e36	3e44	3e52	6.4e60	2.8e69	2.3e78	3.3e87	8.6e96
9	1	512	1e7	2.6e12	5.2e18	5.2e25	2.1e33	2.8e41	1.1e50	1.1e59	2.6e68	1.3e78	1.4e88	2.9e98	1.1e109
10	1	1024	6e7	6.3e16	6.2e20	3.7e28	1.1e37	1.1e48	3.9e55	3.9e65	1e76	6.4e86	8.7e97	2.5e109	1.5e121
20	1	1e6	3.6e15	4e27	3.8e41	1.4e57	1.1e74	1.3e92	1.5e111	1.5e131	1e152	4e173	7.6e195	6.4e218	2.1e242
50	1	1.1e15	8e38	1e69	9.1e103	7.3e142	1.3e185	1.8e230	9.7e277	1.1e328	1.1e380	1e454	5.2e489	1.1e547	6.7e605
150	1	1.4e45	5.3e116	1.1e207	8.1e311	4.1e428	2.3e555	6.7e690	9.1e833	9.2e983	1.5e1140	1.1e1302	1.4e1469	1.2e1641	3.1e1817
300	1	2e90	2.8e233	1.2e414	5.7e623	1.6e857	5.4e1110	4.5e1381	8.5e1667	8.1e1967	2.2e2280	1.3e2604	1.9e2938	1.3e3282	9.1e3634
600	1	4.1e180	7.8e466	1.3e828	3.2e1247	2.5e1714	2.9e2221	2.1e2763	7.2e3335	7.2e3935	4.9e4560	1.6e5208	3.7e5876	1.8e6564	8.1e7269
1000	1	1.1e301	1.4e779	1.6e1380	1.5e2079	2.2e2857	2.7e3702	3.3e4606	5.8e5559	5.8e6559	1.4e7601	2.2e8680	1.9e9794	2.6e10940	3.2e12116

The first thing to notice in Table 1 is that E increases more rapidly with m than with n. The two rows and two columns corresponding to $n, m < 2$ are clearly not interesting: there are at least two alternatives and two voters in any appealing social problem. Notice also that for any pair (m,n) we have $E(m,n) < E'(m,n)$.

4 The Borda Rule as an aggregation rule

Any aggregation rule is supposed to give an answer to two problems: the first is that of what the outcome ought to be, given what the voters want. The phrase "what the voters want" stands here for some function of the set of individual preferences. This introduces the second problem: how to devise a voting procedure that will produce the outcome "wanted" by the voters.

The most important criterion for a good voting procedure is that it be *equitable*: this is a naive statement if we do not specify it by saying, for example, that a voting mechanism is equitable when it reflects as accurately as possible the preferences of the voters. Even this characterization is not adequate, since individual preferences are typically different: there is not a direct answer to the problem of finding a social ordering which reflects "as accurately as possible" a set of different individual orderings. Among others, Black (1958), and Dummett (1984), approached this problem by giving a list of desirable properties, and then examining how different procedures perform with regard to these properties. The most celebrated contribution in the field is probably that of Arrow ((1951) and (1963)): with regard to the theory of voting, the well known "paradox of voting" which is that the relation of majority preference may not be transitive, can be seen as a special case of Arrow's theorem.

It is not my aim to discuss the vast literature on the fairness of voting procedures. In what follows the problem of strategic voting is analyzed in the particular case of the Borda Rule, and I simply want to justify this choice. It is apparent that in many cases the fairest outcome cannot be judged from the voters' preference orderings alone, but the strengths of those preferences are relevant. The Borda Rule represents imperfectly individual preferences, since it only uses preference scales: there is no attempt to compare the strengths of one voter's preferences with those of another. However, we can say that the greater the number of alternatives to be ranked by a voter in order of preference, the better the position of any alternative in his ordering can serve as a substitute for a weighting of his preference. The Borda Rule treats the distance on a voter's preference ordering as a rough measure of the strength of his preference for the higher over the lower.

The Borda Rule is possibly the most known example of the class of *positional* aggregation functions: the position of an alternative in individual orderings play a crucial role in choosing the social outcome. Majority rule is the typical social choice function belonging to the class of *non-positional* aggregation rules. Together with IIA, the weak Condorcet (WC) condition is a typical non-positional condition: an aggregation rule satisfies WC when the fact that an alternative b has a strict simple majority over all other alternatives in a profile Γ implies that b is ranked first in the social ordering. Decision procedures used in parliaments are almost always based on majority rule: politicians seem to prefer the non-positional view.

In passing, it is interesting to underline that Arrow's principle of independence of irrelevant alternatives (IIA) rules out the possibility to make the position of any two alternatives in the social ordering depend upon the relative strength of voters' preferences between these two alternatives. In this sense that principle lacks the intuitive justification of the other principles used in Arrow's theorem: it discords with the idea that whether a would be a fairer social choice than b depends not only on how many voters prefer a to b, but also on how strong their preferences are.

A rule which chooses an alternative because a majority of voters prefer it to any other, uses less information than a rule that takes into account each individual ordering of all the available alternatives. The Borda Rule does exactly this: as any other rule based on preference scores, it attaches a value at any position of a preference ordering and then gets a total score for each alternative by summing over these values. A voter's contribution to the preference score of any alternative can be viewed as representing the degree of satisfaction he would obtain if it were successful. The Borda Rule gives equal weight to every preference of every voter, and picks out that social ordering which will give the greatest overall satisfaction to the voters. It seems to me that, reflecting more accurately the preferences of the voters, a rule based on preference scores is more equitable than a rule based on majority preferences.

Gärdenfors (1973) has a neat paper about aggregation rules which shows the many advantages of positional functions over non-positional ones. Some of his conclusions are collected in the sequel. Let O denote the set of all possible individual orderings, and A be the set of alternatives; then, a *representation function* is a real valued function defined on $O \times A$. It associates a real number $g(D_{i\Gamma}, e)$ to a pair of one individual ordering relative to a profile and of one alternative; clearly $g_\Gamma(e) = \sum_i g(D_{i\Gamma}, e)$ denotes the total score of alternative e under profile Γ. An aggregation rule is *representable* if and only if there exists a representation function g such that for any alternative b and e, b is socially preferred to e iff $g_\Gamma(b) \geq g_\Gamma(e)$.

This defines precisely an importantant subclass of positional aggregation rules: the Borda Rule, for example, is a representable function: it is not in general a majority procedure. With regard to the choice of the Borda Rule, Theorem 6.2 states that it is the only representable function satisfying neutrality (all alternatives have the same probability of being the social top under the rule), strong monotonicity[14] (which relates changes in the position of alternatives in individual orderings to their position in social orderings), and stability (a condition wich prevents drastic changes in the social ordering) when there are at least 3 alternatives and 3 voters. It is interesting to note that Gärdenfors (Theorem 4.4) shows that the non-positional view is inconsistent in a well defined sense: there exists no voting function which satisfies both IIA and WC when there are at least three alternatives.

Saari (1989a) shows and explains possible failures of positional voting. His main result is that any imaginable voting paradox can actually occurs. However, he also proves that the Borda count has properties considerably more favourable than any other aggregation rule, in the sense that it causes fewer paradoxes. The main result in connection with strategic voting (albeit abstention from voting is a particular form of it) is his

14 Any rule satisfying neutrality and strong monotonicity also satisfies strong Pareto optimality (Lemma 5.4).

Corollary 6.1: it proves that there are cases in which a voter induces a social outcome more favourable to him by abstaining than by reporting his true ordering. Properties of the Borda rule with regard to possible paradoxes are also discussed in Saari (1989b).

4.1 Some notions and definitions

The next step is the following: how many profiles are sensitive to strategic voting? Take the case of $m=3$ and $n=6$: Table 1 shows that we already have 462 profiles. How many of them present a structure such as that of the *Example 1* about the Borda rule? Is there a way to compute this number for any rule? If the answer is positive, then we have a method to judge the "quality" of each rule with regard to its sensitivity to strategic voting. In order to answer these questions it is useful to formalize the whole process of voting, from individual preferences to the social ordering[15].

I recall the assumption that voters can express only strict preference relations: thus an individual ordering does not include indifference relations. The resulting restricted set of possible orderings is denoted by $\bar{O}$; O denotes the set of possible orderings that include indifference relations. Clearly $\bar{O} \subset O$; even restricting the set of all possible individual orderings to $\bar{O}$, an aggregation rule might give rise to social indifference relations between alternatives.

Definition 1. An *individual choice* (*IC*) is a function which, given a set A of alternatives $\{a,b,c,d..\}$, and a set of voters I, returns one preference ordering in the set $\bar{O}$ for each voter.

Definition 2. An *election* (*EL*) is a function from the range of IC to the set of all possible profiles P.

Definition 3. An *aggregation rule (AR)* is a function from P to O: it associates one social ordering to each possible profile.

If viewed as a composed function the whole process is called a *voting mechanism* (*VM*): $VM = AR(EL(IC(a,b,c,..))) \in O$.

Definition 4[16]. A voter is said to *behave strategically* when he reports not his true preference ordering but one of its permutations.

15 Dummett (1984), in order to clarify the analysis, distinguishes between voting mechanism and method of assessment: the former step is my *IC* while the latter corresponds to my *AR*. By *voting mechanism* I mean the whole process.

16 This corresponds to the usual definition which states the following: a person who votes not in order to reveal his preferences honestly, but in order to advance his interests most effectively, is said to vote strategically. See, for example, Gibbard (1978).

Definition 5. An aggregation rule is said to take its *normal value* when it associates a social ordering to a profile of true preference orderings (a true profile).

With regard to Definition 4, notice that his false report could be any of the $(m!-1)$ permutations of the true preference ordering. For example, assume that $m=3$ and that the true preference ordering of voter i is $(a\ c\ b)$. Then, his false report can be one out of the following five: $(a\ c\ b)$, $(b\ a\ c)$, $(b\ c\ a)$, $(c\ a\ b)$, $(c\ b\ a)$.

I assume that there is no reason for a voter to announce an ordering which does not rank first his preferred option. This hypothesis deserves some justifications. First of all, we could allow a voter to behave strategically only when he is sure of being successful; second, I assume that voters are interested only in the top alternative of the resulting social ordering. Both these condition are examined in the next page. However, it is already clear that the first condition is quite strong: it implies that a voter anticipates other voters' orderings and that he can compute the resulting social ordering. There are various ways to avoid this assumption, as long as there are various quantities of informations that a voter can have. In fact, we could assume that a voter (who prefers option a) has enough informations to forecast that option a and another, say d, have some chance to be the first one in the social ordering. On the other side, we could assume that he knows that options d and c (his second best alternative) have some chance to be ranked first and that he knows that option a cannot be chosen. Now, the first assumption implies that he has not enough informations to report any ordering but one which ranks a first. The second assumption allows him to report c first instead of a, hoping that c can win its fight against d. This difference in the informations sets needed to justify two different behaviours seems to allow for a choice between the two assumptions: I assume here that a voter has just enough informations to cheat about all his rankings but the first.

This fact limits the number of orderings available to strategic voting. Let b be the alternative preferred by voter i, and a be the normal value of a true profile. There are $m!/m$ $(=(m-1)!)$ orderings which rank b first. So that, apart from his true ordering he has $(m!/m)-1$ *favourable* orderings. I notice that the quota of favourable orderings with regard to the number of orderings diminishes with the increase of m; the list of its percent values for m varying from 2 to 20 is the following: 0, 16.7, 20.8, 19.2, 16.5, 14.3, 12.5, 11.1, 9.9, 9.1, 8.3, 7.7, 7.1, 6.7, 6.2, 5.9, 5.5, 5.2, 5.0. In fact we have: $\lim_{m\to\infty}(((m!/m)-1)/m!) = \lim_{m\to\infty}(1/m - 1/m!) = 0$

When voter i decides to act strategically he will report an ordering in which b is ranked first and a is ranked lower than in his true ordering. If i knows exactly the social ordering associated to the true profile he might be able to alter it or not. Assume he is in the position to modify it[17]. Then, he could downgrade a just enough to change the social order: to do this, voter i has available the following series of scores for a: $(m-(j^*+l))_i^a$, $l=1,2,..,m-j^*$, where j^* denotes his true ranking of a. These scores respectively cause the following losses to the total score of a: $1,2,..,m-j^*$. However, its choice is bounded to be that of the highest loss for a when his information is not perfect: it is safer to assign to a the lowest rank.

Now, assume that i reports one of the orderings in which b is ranked first and a is ranked last: let us call them *safe-favourable* orderings for i. They are $(m-2)!$.

Notice that, depending on other orderings in the profile, some of these $(m-2)!$ safe-favourable orderings might make the social ordering rank first one of the alternatives involved in the permutation chosen by voter i. That is, a safe-favourable ordering is clearly successful for interchanging the rank of a and b in the social ordering; however, the simultaneous permutations of other alternatives might make one of them win. Eventually, the set of orderings available to strategic voting might be bounded to be a singleton or a subset of the safe-favourable orderings. This is one of the reasons behind the idea that the possibility of strategic voting does not increase with the number of alternatives[18].

I will now examine the conditions under which a rational voter behaves strategically.

A rational voter, denoted by i, will behave strategically when the following two conditions are satisfied:
(i) the normal value of *AR* is not i's preferred ordering,
(ii) *AR* switches to i's preferred ordering when, other things being equal, i reports (at least) one of his available (false) favourable orderings.

More formally, let G be i's true preference ordering: it ranks b first; let P_i denote the subset of profiles generated by i's available orderings when *I-{i}* voters report their true orderings: elements of P_i are denoted by Θ_k, $k=1,2,..,m!/m$. Let Θ_k^* contain G as its i^{th} column: Θ_k^* is a true profile. Then, conditions *(i)* and *(ii)* are expressed as follows:

17 The mechanism described here is the Borda Rule: see section 3.3.

18 However, *Proposition 3* below, not being true for m larger than 3 seems to support the idea that the possibility of strategic voting increases with m.

(i) $AR(\Theta_k^*) \neq G$

(ii) $AR(\Theta_k) = G$ for at least one k.

Notice that voter i knows *(i)* and *(ii)* if and only if he knows all orderings reported by other voters. Then, *(i)* and *(ii)* together are necessary and sufficient conditions for a successful voting strategy. However, there is the possibility that one voter knows only *(i)*: in fact, understanding the mechanism of the aggregation rule, a voter might conclude that a certain alternative is socially preferred without knowing all individual orderings: it suffices to know say, 80% of them. In this case he does not know whether *(ii)* holds, and *(i)* is necessary but not sufficient for a successful voting strategy. This is said to emphasize how the assumption of perfect information simplifies the analysis: if partial information is considered many different conditions for strategic voting are possible. Furthermore, if a voter is partially informed and votes strategically, then the aggregation rule returns a partially random social ordering: it will reflect not only the preferences of the voters, but their guesses about the preferences of other voters.

It is now clear that our voters' interest in the chosen social ordering is restricted to the alternative which is ranked first[19]. Thus, the aim of a strategic voter is that of getting a social ordering which ranks as the first alternative his own first option. First, this means that if a voter's true ordering is *(a b c)*, then he is indifferent between *(a b c)* and *(a c b)* : they are actually the ***same social ordering*** for him. Second, any false ordering which, if reported by a voter, leads to a social ordering that is not his preferred one, it is not conveyed by that voter. This last assumption is very demanding in terms of information available to the strategic voter. If we do make it, then we do not need part (**) of the following Definition 6.

Before giving a meaningful notion of strategy proof profiles (and of strategy proof aggregation rules on these profiles) we have to analyze all possible types of social orderings with regard to the individual ordering of one voter. Let voter i prefer alternative b, which is ranked first in any ordering of the type $G=(b...)$. The main distinction is provided by the fact that the true social ordering can be of type G or not. If it is of type G, then there is no need for voter i to vote strategically. These profiles are called ***stable profiles*** with regard to voter i (SP_i).

$$(1) \qquad BR(\Theta_k^*) = G = (b\ldots)$$

19 This is equivalent to leave the SWF approach and to adopt that of a single-valued Social Choice Function.

If the true profile is not of type G, then voter i might try to act strategically. The social ordering associated to the true profile can either rank first an alternative different from b *(2)*, or rank first two alternatives *(3)*.

(2) $$BR(\Theta_k^*) = F = (a\ldots)$$

The following classification is based on the criterion that his action can be successful or not.

(2.1) $$BR(\Theta_k) = F \qquad \forall k$$

(2.2) $$BR(\Theta_k) = F \quad \text{for some} \quad k, \quad \text{and}$$
$$BR(\Theta_k) = H = (c\ldots) \quad \text{for some} \quad k$$

(2.3) $$BR(\Theta_k) = F \quad \text{for some} \quad k, \quad \text{and}$$
$$BR(\Theta_k) = I = ((ca)g\ldots) \quad \text{for some} \quad k$$

These are all cases of ***non sensitive profiles*** (NSP_i). On the other side, the following is a case of ***sensitive profile*** (SP_i).

(2.4) $$BR(\Theta_k) = F \quad \text{for some} \quad k, \quad \text{and}$$
$$BR(\Theta_k) = G \quad \text{for some} \quad k$$

Case *(2.5)* is such that the strategic voter can induce at most a favourable indifference relation between two alternatives in the social ordering. Profiles of this kind are called ***indeterminate*** (IP_i).

(2.5) $$BR(\Theta_k) = F \quad \text{for some} \quad k, \quad \text{and}$$
$$BR(\Theta_k) = L = ((ba)f\ldots) \quad \text{for some} \quad k$$

When the true social ordering ranks first more than one alternative, we simply consider that a strategic voter might be able to induce a social ordering of type G or not. Let M be a true social ordering which ranks first more than one alternative.

(3) $$BR(\Theta_k^*) = M = ((ha)d\ldots)$$

In case *(3.1)* we have *indeterminate sensitive profiles* (*ISP$_i$*), while in case *(3.2)* we have *indeterminate non sensitive profiles* (*INSP$_i$*).

(3.1) $BR(\Theta_k) = M$ for some k, and

$BR(\Theta_k) = G$ for some k

(3.2) $BR(\Theta_k) = M$ for some k, and

$BR(\Theta_k) = N \neq G$ for some k

Armed with this somewhat cumbersome taxonomy, I give the following definition.

Definition 6. An aggregation rule is said to be *locally constant* with regard to one voter when, corresponding to profiles that differ from a true profile because that voter behaves strategically, it either
(*) always returns its normal value (which might be an indifference relation), or
(**) for some false orderings it returns its normal value and for others it selects a first alternative which is different from the alternative preferred by that voter, or it ranks first more than one alternative not preferred by him.

Now, compare Definition 6 with the classification of true profiles above: it is clear that an aggregation rule is locally constant when it takes values corresponding to one of the following types of profiles: stable (1), non sensitive (2.1, 2.2, 2.3), and indeterminate non sensitive (3.2).

Definition 7. An aggregation rule is said to be *almost locally constant* when, corresponding to the introduction of safe-favourable orderings, it returns a social ordering which ranks first both the alternative preferred by the voter acting strategically and one or more other alternatives.

Thus, we can say that an aggregation rule is almost locally constant when it maps indeterminate true profile. In this case we say that the *AR* returns an *indeterminate*[20] social ordering.

20 Some aggregation rules can give the same total scores to more than one alternative, thus leading to an indecisive outcome since it is not possible to rank one option over another. See example 2 below. An indeterminate social ordering is such that at least two alternatives are ranked first: it contains at least one indifference relation between alternatives. I use this term to stress the point that such orderings are useless from the point of view of the decision making. Usually, with preference score procedures a tie between two top alternatives is broken by majority rule: if *a* and *b* tie for the first place, *a* will be declared successful if more voters rank it higher than *b* than rank *b* higher than it.

Notice that an aggregation rule which is neither locally nor almost locally constant is taking values either on a sensitive true profile (2.4), or on an indeterminate sensitive true profile (3.2).

The problem of strategic voting can be stated as follows: how many times an *AR* is locally constant over subsets of its domain? How large is the change (if any) in *VM* when there is a change in *EL*? To see this consider the following:

The $(m-1)!$ Θ_k differ from each other only for the i^{th} column that can be either i's true ordering (in this case F, and thus we have Θ_k^*) or one of i's false and safe-favourable orderings. In the sense of Definition 6, we are saying that a voter will act strategically when the *AR* is not locally constant on the concerned profile: profiles "around" a true profile which *AR* maps to a social ordering, are not all mapped to the same social ordering. We can translate the negative result on the existence of strategy-proof voting mechanism by saying that there does not exists an *AR* which is locally constant for every true profile. Then, the problem could be stated as follows: can we find an *AR* which is locally constant for say, 90% of its true profiles?

As such, the locally constant property is quite difficult to evaluate when large profiles are involved: we deal with many social orderings that can be quite different from each other.

To see this, consider that the class of locally constant profiles is associated by an *AR* to a type of social ordering like, say, G which belongs to a subset of social orderings always satisfying the condition that they are values associated by an AR to profiles in P_i , and one of the following (these are the same conditions given in the definition): any of them is either the normal value of AR ((1) and (2.1)), or it ranks first an alternative which is not that preferred by voter i (2.2), or it ranks first more than one alternative not preferred by him ((2.3) and (3.2)).

In the following definition I formalize a concept that will be useful to study possible profiles: it represents a restriction of the idea of a locally constant aggregation rule. Let the alternative a be the normal value of a true profile Θ_k^*, and let F be any social ordering which ranks a first. Moreover, let voter i prefer alternative b.

Definition 8. An *AR* is *locally not manipulable* (LNM_i) by voter i on $\Theta_k^* \in P_i$ when $AR(\Theta_k^*) = F$, if and only if

$$AR(\Theta_k) = F\,, \quad \forall\, \Theta_k \in P_i$$

I also say that a profile is LNM$_i$ by voter i under some aggregation rule, and that the same profile is *strategy proof*. It is clear that the definition of a locally non manipulable profile is equivalent to that of a locally constant with the exclusion of indeterminate non sensitive profiles. That is, $\Theta_k^* \in P_i$ is either stable or non sensitive.

Similarly I define almost-LNM$_i$ corresponding to almost locally constant profiles: thus $\Theta_k^* \in P_i$ is an indeterminate true profile.

Summing up, local and almost-local non manipulability correspond to profiles in which a strategic voter respectively -cannot take advantage of the situation, and -can at most induce the choice of a social ordering in which his preferred alternative is ranked first together with the normal value of the associated true profile. It is useful to show that definition 8 can be re-phrased as follows, at least when the Borda Rule is the chosen AR. Let $r^h \quad \forall h \in A$ be the total score of alternative h calculated by the *BR*. Then, under the same assumption of Definition 8, we have:

Proposition 1. A BR is LNM$_i$ on $\Theta_k^* \in P_i$ when $BR(\Theta_k^*) = F$ if and only if

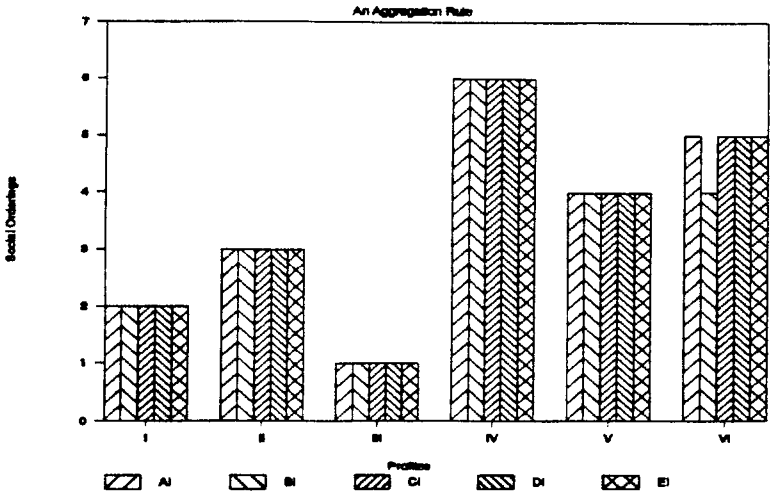
$$r^a - r^b \geq 1 \qquad \Theta_k \in P_i$$

Proof. By assumption F ranks a first for all $\Theta_k \in P_i$ applying the BR. By construction this means that $r^a > r^b \quad \forall \Theta_k \in P_i$. This is equivalent to $r^a - 1 \geq r^b$ since r^a and r^b are positive natural numbers.

It must be stressed that the existence of a *VM* which is strategy proof with regard to some profile does not contradict well known results such as those in Gibbard (1973) and Satterthwaite (1975). Indeed, what most of the impossibility-type results say is that there does not exist a *VM* which is strategy proof (or non manipulable) for all possible profiles. The present approach is different: it asks how many true profiles, out of the total number of them, are strategy proof. It seems clear that if there are many strategy proof profiles with regard to a specific *AR*, then the probability of cheating is very low.

In order to clarify the meaning of the preceding definitions, consider the following Figure 1: on the x-axis there are six true profiles, each of them having on the left and on the right four false profiles indicated by capital letters ($i=I,II,..,VI$ refers to the associated true profile). False profiles contain an ordering which is false because one of the voters is acting strategically: he reports a permutation of his true ordering. On the y-axis we have 7 social orderings. The aggregation rule maps each true profile into a social ordering; in this case it is locally constant on the first five true profiles: it maps all of the corresponding false profiles to the same social ordering. However, it is not locally constant on the sixth profile: one of its associated false profiles, B_{VI} , is mapped into social ordering number 4, which is different from its normal value.

Figure 1



4.2 How the Borda Rule works

It is evident that in order to study the probability of success of strategic voting when m and n are larger than 3, we need to approach the problem in a systematic way: the number of profiles becomes too large to allow for an inspection of each of them using Definition 8 or Proposition 1. For example, we could find a constraint which is satisfied by and only by the social ordering resulting from any non manipulable profile and then discover a shortcut to count the number of such profiles without being concerned by individual orderings. It is apparent that any such constraint will depend on the mechanism of the aggregation rule: thus, it is time to formalize the way the Borda Rule (BR) applies on a profile. This is done in three steps.

The first step starts from a profile and ends to a matrix of coefficients of alternatives. Assume that an election produced the following profile:

$$\Sigma = \begin{pmatrix} b & c & f & .. & a \\ a & d & c & .. & b \\ . & . & . & .. & . \\ . & . & . & .. & . \\ f & a & b & .. & c \end{pmatrix} = (h_{ji}) \quad i = 1,2,..,n \;, \quad j = 1,2,..,m$$

The BR consists of the following matrix or integers:

$$R' = \begin{pmatrix} (m-1)_1 & (m-1)_2 & .. & (m-1)_n \\ (m-2)_1 & (m-2)_2 & .. & (m-2)_n \\ . & . & .. & . \\ (m-m+1)_1 & (m-m+1)_2 & .. & (m-m+1)_n \\ 0_1 & 0_2 & .. & 0_n \end{pmatrix}$$

R' is associated to Σ using its elements as indices:

$$R = \begin{pmatrix} (m-1)_1^b & (m-1)_2^c & .. & (m-1)_n^a \\ (m-2)_1^a & (m-2)_2^d & .. & (m-2)_n^b \\ . & . & .. & . \\ (m-m)_1^f & (m-m)_2^a & .. & (m-m)_n^c \end{pmatrix} = ((m-j)_i^h) \quad h \in A$$

The second step consists of m sums of the elements of R over the alternatives:

$$(m-j)_1^h + (m-j)_2^h + .. + (m-j)_n^h = \sum_{i=1}^{i=n} (m-j)_i^h = r^h \quad \forall h \in A$$

Alternatively, let $q_i(h)$ be i's score of option h $\forall i \in N, \forall h \in A$. Clearly $q_i(h)$ can take any value from *(m-1)* to *0*. Then, $r^h = \sum_i q_i(h)$ is the total Borda score of option h. However, I will use the earlier notation, for it proved to be suitable to computer programming.

Finally, the BR chooses a social ordering $S \in O$ such that

$$aPb \Leftrightarrow r^a > r^b$$

$\forall a,b \in S$

$$aIb \Leftrightarrow r^a = r^b$$

Notice that an alternative gets at most $n(m-1)$ scores, and that the sum of elements of R is $n \sum_{j=1}^{m} m-j$.

Example 2. Assume that an election produced the following matrix R:

$$R = \begin{pmatrix} 3_1^b & 3_2^a & 3_3^a & 3_4^c \\ 2_1^a & 2_2^b & 2_3^c & 2_4^b \\ 1_1^c & 1_2^d & 1_3^b & 1_4^a \\ 0_1^d & 0_2^c & 0_3^d & 0_4^d \end{pmatrix}$$

I compute the score of each alternative as follows:

$$r^a = 2_1^a + 3_2^a + 3_3^a + 1_4^a = 9$$

$$r^b = 3_1^b + 2_2^b + 1_3^b + 2_4^b = 8$$

$$r^c = 1_1^c + 0_2^c + 2_3^c + 3_4^c = 6$$

$$r^d = 0_1^d + 1_2^d + 0_3^d + 0_4^d = 1$$

Here $r^a > r^b$, while voter 1 prefers b and gives a 2 scores. Notice that he could give a 0 scores, so that r^a would fell to 7 and b would win. This observation suggests that in order to avoid this situation we should have $r^a - r^b > m - 2_1^a$. Any profile which produce the same social ordering but satisfies this constraint will be not sensitive with regard to voter 1. If we do not know how voter 1 ranks the socially preferred alternative a, then we can assume the worst case: that is, a is ranked second by him so that, actually reporting it as his last choice, he can at most diminish its scores by $m - 2_1^a$. In this example we have *9-8=2*, and $m - 2_1^a = 2$, that is, the condition is not satisfied.

Notice that in general if $r^a - r^b > m - 2_1^a$, then the condition expressed by this constraint is satisfied for any ranking of alternative a different from the second place: $m - 2_1^a > m - j_1^a \quad \forall j = 3,4,..,m$. It will be shown in the sequel that a direct generalization of this condition is only sufficient for the local non manipulability.

It is not easy to check directly whether the condition for LNM_i holds for all profiles: we need to compute the BR for some of the permutations of an individual ordering starting from a true profile. However, it is simple to prove a proposition which states a necessary and sufficient condition for the non manipulability of a profile under the BR, whose validity is much easier to check. Before giving this proposition, I show how, given a true profile, the possibility of acting strategically depends on the true ordering of the concerned voter.

Assume that $r^a = \max\{r^h\}$, and let i denote the strategic voter who prefers b to a. Assume also that $(m - j^*)_i^a > 0$, otherwise voter i has no way to diminish r^a.

In what follows it is important to consider that the range of the Borda Rule is O: a social ordering can contain indifference relations between alternatives. If a voter i needs to act strategically (that is, the two conditions (*i*) and (*ii*) of section 4.1 are met), then he faces three possible cases:

(1) the true profile (with his true ordering, denoted respectively by Θ_k^* and F_k^*) is associated by the BR to a social ordering in which the difference between the score of the first alternative and his preferred one, say a and b, is larger than the score he himself assigns to a. This means $r^a - r^b > (m - j^*)_i^a$. By similar reasoning, we have the other two cases:

(2) $r^a - r^b = (m - j^*)_i^a$

(3) $r^a - r^b < (m - j^*)_i^a$

In the first case, even reporting an ordering in which he assigns 0 to a, making a lose $(m - j^*)_i^a$, voter i cannot reverse the social ordering. We always have $r^a - (m - j^*)_i^a > r^b$. When this condition holds we have a non sensitive profile.

In the second case, voter i makes an indifference relation between a and b appear in the social ordering. This means that Θ_k^* is an IP.

In the third case, since we have $r^a - (m - j^*)_i^a < r^b$, voter i can make the BR choose a different social order in which b is the first alternative. Then, we say that Θ_k^* is a SP.

Now, recall that we assume that Θ_k^* is such that there exists an alternative a such that $r^a - r^h \geq 1$ for all $h, a \in A$, $a \neq h$. That is, the normal value of Θ_k^* is a clear winning alternative.

Proposition 2. A profile $\Theta_k^* \in P_i$ is LNM by voter i under the BR if and only if $r^a - r^b > (m - j^*)_i^a$ for Θ_k^*.

Proof. I have to prove that

$$r^a - r^b \geq 1 \quad \forall \Theta_k \in P_i \qquad \Leftrightarrow \qquad r^a - r^b > (m - j^*)_i^a \quad \text{for} \quad \Theta_k^*$$

To prove sufficiency we show that the left disequations implies the disequation to the right. Assume $r^a - r^b < (m - j^*)_i^a$ for Θ_k^*. This means that there exists at least one Θ_k for which $r^a - (m - j^*)_i^a - r^b < 0$. Thus we have another total score for a relative to Θ_k, say r_k^a, such that $r_k^a - r^b < 0$ (notice that r^b does not change since b is i's most preferred alternative). But the last disequation is the negation of $r^a - r^b \geq 1 \quad \forall \Theta_k$ which is equivalent to $r^a - r^b > 0 \quad \forall \Theta_k$, since all total scores are natural numbers.

To prove necessity we show that $r^a - r^b > (m - j^*)_i^a$ for Θ_k^*, implies $r^a - r^b \geq 1 \; \forall \Theta_k \in P_i$. Assume there exists a $\Theta_{\bar{k}} \neq \Theta_k^*$ for which $r_{\bar{k}}^a - r^b < 1$. This means that $r^a - (m - j^*)_i^a - r^b < 1$, and that $r^a - r^b < (m - j^*)_i^a + 1$, and also that $r^a - r^b \leq (m - j^*)_i^a$ since scores are natural numbers. The last disequation is the negation of $r^a - r^b > (m - j^*)_i^a$.

Example 2 suggests that we can deduce a different condition for local non manipulability: this is $r^a - r^b > (m - 2)$. Consider that *(m-2)* is the largest loss a voter can cause to the score of an alternative in a social ordering: we could simply control the scores in the social ordering to see if the difference between some alternatives is smaller than *(m-2)*. When this is the case we could conclude that the profile is a sensitive one. This condition is much easier to check than the previous one, since we would not need to examine individual orderings. However, there is a problem: as such, this new condition is only sufficient for the local non manipulability of the profile. In fact, there are LNM profiles which do not satisfy it: I show this with an example.

Example 3. We have 5 alternatives and the profile

$$\Theta = \begin{pmatrix} d & a & a & e \\ c & b & b & d \\ e & c & c & a \\ a & d & d & b \\ b & e & e & c \end{pmatrix}$$

which produces the following social scores: $(11a, 7b, 7c, 9d, 6e)$. Here we have $r^a - r^d = 2 < m - 2 = 3$ so that the condition is not satisfied. However, we can't conclude that the profile is manipulable: voter 1 can report (d, c, e, b, a) making a lose 1 score, but this is not enough for d to win.

This sufficient condition is useful anyway: first, if we find that say, 35% of profiles satisfy it, we know that a larger quota of them is locally non manipulable. Second, it ensures that all profiles satisfying it prevent strategic actions of all voters.

Notice that there is not a free-lunch in the theory of information: to check the necessary and sufficient condition, which regards any single voter, we need to examine all individual orderings. On the other side the sufficient condition is easier to check since we only need to consider the final social ordering.

4.3 The algorithm used in the analysis

In order to understand how many profiles are LNM, it is worth to work out some examples. This turns out to be a difficult task: the complexity of the study is due to the fact that the number of profiles increases very fast with the number of voters and even faster with the number of alternatives, as Table 1 clearly shows. To approach the cases of 3 alternatives and 5 and 6 voters, where respectively 252 and 462 profiles are possible, it is no longer practical to work by hand. For that reason we developed a computer program to generate a list of possible profiles, and then to test them with regard to the sufficient condition for strategy proofness. The flowchart of the program is given in the figure below.

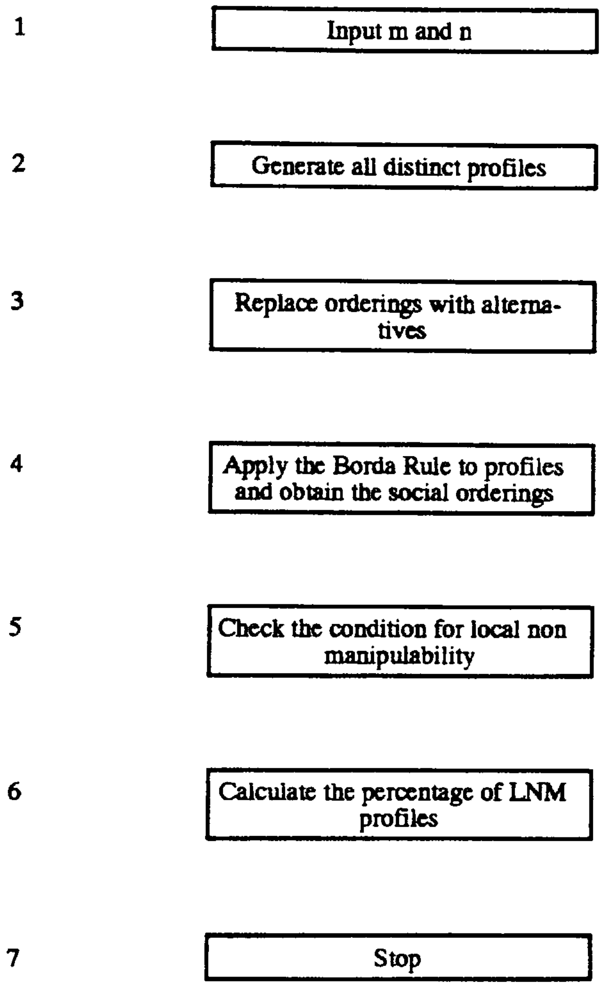


Figure 1: Flow chart of the program

5 Two examples and some results

Example 4. Assume that $A=\{a,b,c\}$, and $I=\{1,2,3\}$. Then, $\#O=6$ and $\#P=56$. Moreover, let $\Theta_1=((acb),(bac),(cab))$ be the true profile. Voter 3 prefers option c but $BR(\Theta_1)=(acb)$ respectively with 4, 3, and 2 scores. Let us write P_3.

$$P_3=\left\{\begin{pmatrix}abc\\caa\\bcb\end{pmatrix},\begin{pmatrix}abc\\cab\\bca\end{pmatrix},\begin{pmatrix}aba\\cab\\bcc\end{pmatrix},\begin{pmatrix}aba\\cac\\bcb\end{pmatrix},\begin{pmatrix}abb\\caa\\bcc\end{pmatrix},\begin{pmatrix}abb\\cac\\bca\end{pmatrix}\right\}$$

Let $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6,$ denote respectively each of the profiles in P_3. Simple arithmetic applying the Borda rule shows the following results: $BR(\Theta_1) = (acb)(432), BR(\Theta_2) = (abc)(333)$, $BR(\Theta_3) = (abc)(531), BR(\Theta_4) = (abc)(522), BR(\Theta_5) = (abc)(441), BR(\Theta_6) = (bac)(432)$. In this case condition *(i)* is satisfied: Θ_1^* contains *(c a b)* and $BR(\Theta_1^*) = (acb) \neq (cab)$. However, there is not a Θ_k such that $BR(\Theta_k) = (cab)$. In the true profile voter 3 prefers c but option a gets 4 and c gets 3 . Voter 3 ranks a before b, so that he can alter this order. Reporting *(c b a)* voter 3 gets at most an indeterminate outcome, since $BR(\Theta_2) = (abc)(333)$. In this example, we conclude, *VM* is almost locally non manipulable on P_3 .

If we look for a rule that favours the existence of such a P_i , it is clear from the beginning that this will depend on the functional form of *AR*; moreover, since the dimensions of P_i are given by m and n, the characteristic of being strategy proof should also depend on them.

For the case of Example 4 it is practical to extend the analysis to all possible true profiles, since they are 56, without the need for a more synthetic approach. The following is the set of possible orderings:

$$O = \begin{pmatrix} a & a & b & b & c & c \\ b & c & a & c & a & b \\ c & b & c & a & b & a \end{pmatrix}$$

Let *D*, *E*, *F*, *G*, *H*, and *I* denote each of these profiles from the left. Without losing generality we can study all the matter assuming that the voter who is thinking to vote strategically is the third one. The set *P* contains the following 56 elements:

DDD	DEE	DFF	DGG	DHH	DII
DDE	DEF	DFG	DGH	DHI	
DDF	DEG	DFH	DGI		
DDG	DEH	DFI			
DDH	DEI				
DDI					
EEE	EFF	EGG	EHH	EII	
EEF	EFG	EGH	EHI		
EEG	EFH	EGI			
EEH	EFI				
EEI					
FFF	FGG	FHH	FII		
FFG	FGH	FHI			
FFH	FGI				
FFI					
GGG	GHH	GII			
GGH	GHI				
GGI					
HHH	HII				
HHI					
III					

Applying the Borda Rule to these profiles we get the social orderings in Table 4. The header of each column indicates the social orderings resulting by profiles listed below it. For example, *(abc)* means that alternative *a* is preferred to *b* and *b* to *c*; when two or three alternatives are underscored this means that the social ordering is indifferent between them. The first, the second, and the third scores pertains respectively to alternative *a*, *b*, and *c*.

Table 4

(abc)	*(acb)*	*(bac)*	*(bca)*	*(cab)*	*(cba)*
630 DDD	603 EEE	360 FFF	063 GGG	306 HHH	036 III
621 DDE	612 DEE	261 FFG	162 FGG	216 HHI	126 HII
540 DDF	504 EEH	450 DFF	054 GGI	405 EHH	045 GII
531 DEF	513 DEH	351 DFG	153 FGI	315 EHI	135 GHI
432 DDI	423 EEG	342 EFG	243 DGI	324 FHH	234 FHI
432 DEG	423 EFH	342 DFI	243 EGG	324 DHI	234 DII
432 DFH	423 DEI	342 FFH	243 FGH	324 EGH	234 EGI
$(a\underline{bc})$	$(b\underline{ac})$	$(c\underline{ab})$	$(\underline{ab}c)$	$(\underline{ac}b)$	$(\underline{bc}a)$
522 DDH	252 FFI	225 GHH	441 EFF	414 EEI	144 FII
522 EEF	252 DGG	225 EII	441 DDG	414 DHH	144 GGH
$(\underline{abc})$					
333 DGH					
333 EFI					

Let us consider these as true profiles and see what happens when the third voter, not satisfied by the outcome, tries to vote strategically. We find all types of profiles described previously.

- first of all, it may be the case that there is no need for voter 3 to report a false ordering: for example, if $\Theta_1 = DGG$ we have $BR(\Theta_1)=$ $(b\underline{ca})$ *(522)*. So that voter 3 reporting his true ordering $G = (bca)$ is satisfied. These are stable profiles with regard to voter *i* (STP_i).

- Second, an unsatisfied voter can verify that it is impossible for him to report a false ordering such that the final outcome ranks first his best alternative. These are non sensitive profiles (NSP_i). For example, if $\Theta_1 = DEH$ we have $BR(\Theta_1) = (acb)(531)$. Voter 3 prefers *c* but if he reports *(cba)* instead of *(cab)* the aggregation rule still gives *(acb)* for *DEI* even if with different scores *(423)*.

-Third, a voter acting strategically can lead to an indeterminate outcome. I assume that such a case is solved in a way which is not under control of the strategic voter (for example by a random choice), so that this cannot be considered a fully successful strategy[21]. These were called cases of *indeterminate profiles* (IP_i). For example, if $\Theta_1 = DDF$ we have $BR(\Theta_1) = (abc)(540)$. Voter 3 prefers *(bac)* but he can report *(bca)*, give rise to *DDG*, and eventually he gets $BR(\Theta_2)$= *(abc) (441)*.

-Fourth, some true profiles can lead naturally to an indeterminate outcome. Now a strategic voter can take advantage of the situation: for suppose $\Theta_1 = DHH$ and thus $BR(\Theta_1) = (cab)(441)$. If voter 3 reports *(cba)* instead of *(cab)*, then the profiles becomes *DHI* and the aggregation rule switches to *(cab) (432)*. These were called *indeterminate-sensitive profiles* (ISP_i). Some indeterminate profiles are such that a strategic voter cannot take advantage of them: his available strategy (here a permutation of his second and third choice) makes the aggregation rule overpass the indeterminacy but the result is not favourable to him. These were called indeterminate non sensitive profiles ($INSP_i$).

-Fifth, we should have the so called sensitive profiles (SP_i): these are such that they lead to a determined social ordering which is not the one preferred by voter 3, and such that he can switch the ordering to one that ranks first his preferred alternative.

Recalling a previous definition, we say that the BR is LNM by voter 3 on stable and non sensitive profiles. The result from a careful examination of the 56 possible profiles can be summarized by a table. Horizontally I list the three possible situations that can be faced by voter 3 when he votes sincerely, before he consider whether to behave strategically: the situation is favourable when the social ordering lists first his preferred alternative; it is unfavourable when this is not the case; the situation is indeterminate when at least two alternatives (one of them is that preferred by voter 3) are ranked at the same level. Vertically we still have three types of situations that can emerge when voter 3 acts strategically. Clearly there is no need to act strategically when the social order is favourable to the voter (first column): the corresponding profiles are thus called stable profiles.

21 However, this is a partially successful strategy: any alternative mechanism can lead to an outcome which is not worse than the normal one had the voter voted sincerely.

Table 5

Situation ex ante	Favourable	Unfavourable	Indeterminate
Situation ex post			
Favourable	STP (31)	SP (0)	ISP (2)
Unfavourable		NSP (14)	INSP (4)
Indeterminate		IP (5)	

The most important thing to notice is that there are 19 unfavourable profiles: by acting strategically , however, voter 3 can never take a direct advantage. There are not sensitive profiles. Even starting from one of the 6 indeterminate profiles, strategic voting is successful only 2 times: furthermore the choice of the "good" false report is difficult, since it can lead 4 times out of 6 to unfavourable social orderings. Summing up, we can say that a voter finds reasons to act strategically in 25 situations out of 56; however, by doing so, he can take advantage of the mechanism only 2 times (ISP). We can consider final indeterminacy, as the outcome of strategic voting in reaction to unfavourable initial situations, as a good result: then, a voter can take advantage of the mechanism 7 out of 56 times. I conclude by saying that in this setting, with regard to the Borda rule, there are 49 strategy proof profiles out of 56.

Now we use the sufficient condition for local non manipulability, and compare our findings. Let t denote the top alternative and s the second one in any of the social orderings of Table 4. It is easy to check in how many profiles we have $r^t - r^s > (m-2) = 1$: there are 24 out of 56 profiles satisfying this condition, that is 42,85% of them. As expected, this figure is smaller than that determined by checking the necessary and sufficient condition.

It is worth to examine what is the common feature of *ISP*, *SP* and *IP* profiles under the Borda rule. *SP* profiles are not observed in this case. There are two *ISP* profiles: *DHH* and *EFF*. In both cases voter 3 ranks second the option which gets the same score as his preferred one. So that, un-sincerely ranking it third, he makes it loose one score.

The case of the 5 *IP* is slightly more complicated. With *EEH*, *GGI*, and *DDF*, voter 3 ranks second the socially preferred option, which happen to get just one score more than another one (this can be either voter 3's preferred outcome or not). So that, ranking it third, he makes it loose one score and gets an indeterminacy. With *DDI* and *DGI* voter 3 would rank second or third the option which has just one score less than his preferred one. It suffices that he actually ranks it first or second to obtain an indeterminacy.

If we are concerned by the problem of finding stable profiles, we should analyze all of them with regard to the possibility of strategic action by any voter. Given a profile, there might be many voters who, not satisfied by the foreseen social ordering, have an incentive to act strategically. This approach appears a natural extension of the study. However, it seems very complicated: the assumption that a strategic voter is informed about the individual preferences of other voters, which is needed anyway, is even less plausible when many voters are allowed to act strategically. In this case any strategic voter should also know the strategic reaction of all the others: each voter should update his choice of an ordering any time he understand that one of the others is going to change his own report. The present approach, which studies profiles with regard to only one voter, has not the same complexity and still makes good sense: it gives an answer to the question of in how many cases a single voter needs to act strategically and when his strategy is successful.

The generality of this approach clearly depends on this assumption: the quota of profiles which are LNM by a certain voter is the same for all voters. If this assumption was true, we would know that any voter faces the same probability of acting strategically and being successful, simply by checking profiles with regard to one voter. I cannot prove the plausibility of this assumption, but it seems that it should be true when the aggregation rule is symmetric in some sense.

There is, however, a good way to by-pass this problem. Our sufficient condition does not refer to any voter in particular, so that a profile which satisfies it is LNM by any voter.

Two questions arise now: can we modify the aggregation rule in order to further reduce the probability of success of strategic voting ? Does the probability of success increase with m and n? That is, does the quota of LNM profiles decrease with more alternatives and more voters?

Section 7 gives a positive answer to the first question. The second question is more difficult: recall that Example 1, in which a clear case of success for strategic voting is described, concerns 4 alternatives and 3 voters. On the other hand, Example 4 with 3 alternatives and 3 voters does not show sensitive profiles. Thus, also the answer to the second question seems to be positive. However, counter examples are possible, so that the conclusion is not germane. With m and n equal to three, take the true profile *DDD*, which returns *(abc)*; its 3 permutations *DDE*, *DDF*, and *DDI*, also return *(abc)* but *DDG* gives $(\underline{ab}c)$, and *DDH* gives $(a\underline{bc})$. On the other side, with $m=4$, all permutations of one voter starting from *DDDD* (*DDDE*, *DDDF*, ...) return *(abc)*.

In the next Example I study the case of 4 voters and 3 alternatives. For m and n larger than 3, we better make use of the sufficient condition for local non manipulability: this is a much more synthetic tool of inquiry than the condition of Proposition 2. It must be stressed that this sufficient condition is not useful for a strategic voter in order to simplify his computational needs: a rule that tells that strategic voting has, say, the 10 % of successful cases, is not useful for actually voting strategically. The voter still needs to know or to infer the orderings of the others, otherwise he can worsen his situation by misreporting his preferences.

Example 5. Consider 3 alternatives in the set $A=\{a,b,c\}$, and 4 voters in the set $I=\{i,j,k,l\}$. We have $\#O=6$ and $\#P=126$. The set of available orderings is

$$O=\begin{pmatrix} a & a & b & b & c & c \\ b & c & a & c & a & b \\ c & b & c & a & b & a \end{pmatrix}$$

For simplicity denote respectively from the left by *D, E, F, G, H, I* the six orderings of *O*.

<u>Step 1</u>: Input $m=3$ and $n=4$

<u>Step 2</u>: The program generates all 126 possible distinct profiles. They are listed below. First, there are 6 profiles with the same orderings:

DDDD	EEEE	FFFF	GGGG	HHHH	IIII

Second, we have a group of 30 profiles:

DDDE	EEED	FFFD	GGGD	HHHD	IIID
DDDF	EEEF	FFFE	GGGE	HHHE	IIIE
DDDG	EEEG	FFFG	GGGF	HHHF	IIIF
DDDH	EEEH	FFFH	GGGH	HHHG	IIIG
DDDI	EEEI	FFFI	GGGI	HHHI	IIIH

Third, a group of 15 mixed pairs:

DDEE				
DDFF	EEFF			
DDGG	EEGG	FFGG		
DDHH	EEHH	FFHH	GGHH	
DDII	EEII	FFII	GGII	HHII

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

Fourth, a group of 60 profiles:

DDEF	EEDF	FFDE	GGDE	HHDE	IIDE
DDEG	EEDG	FFDG	GGDF	HHDF	IIDF
DDEH	EEDH	FFDH	GGDH	HHDG	IIDG
DDEI	EEDI	FFDI	GGDI	HHDI	IIDH
DDFG	EEFG	FFEG	GGEF	HHEF	IIEF
DDFH	EEFH	FFEH	GGEH	HHEG	IIEG
DDFI	EEFI	FFEI	GGEI	HHEI	IIEH
DDGH	EEGH	FFGH	GGFH	HHFG	IIFG
DDGI	EEGI	FFGI	GGFI	HHFI	IIFH
DDHI	EEHI	FFHI	GGHI	HHGI	IIGH

Last, a group of 15 profiles:

DEFG	DEFH	DEFI	DEGH	DEGI	DEHI
DFGH	DFGI	DFHI	DGHI	EFGH	EFGI
EFHI	EGHI	FGHI			

Step 3: Each letter in these profiles represents the ordering of one of the four voters. In order to apply the Borda rule to the profiles, we substitute the capital letter with the explicit ordering of alternatives. For example, for GGDE we will get the list *{(b,c,a),(b,c,a),(a,b,c),(a,c,b)}*.

Step 4: Applying the Borda Rule to these profiles we get the social orderings in Table 6. As before, the header of each column indicates the ordering, that is, *(abc)* means that alternative *a* is preferred to *b* and *b* to *c*; when two or three alternatives are underscored this means that the social ordering is indifferent between them. The first, the second, and the third scores pertains respectively to alternative *a*, *b*, and *c*.

Table 6

(abc)	*(acb)*	*(bac)*	*(bca)*	*(cab)*	*(cba)*
840 DDDD	804 EEEE	480 FFFF	084 GGGG	408 HHHH	048 IIII
831 DDDE	813 EEED	381 FFFG	183 GGGF	318 HHHI	138 IIIH
750 DDDF	705 EEEH	570 FFFD	075 GGGI	507 HHHE	057 IIIG
741 DDEF	714 EEDH	471 FFDG	174 GGFI	417 HHEI	147 IIGH
732 DDDH	723 EEEF	372 FFFI	273 GGGD	327 HHHG	237 IIIE
732 EEDF	723 DDEH	372 GGDF	273 FFGI	327 IIEH	237 HHGI
651 DDDG	615 EEEI	561 FFFE	165 GGGH	516 HHHD	156 IIIF
651 FFDE	615 HHDE	561 DDFG	165 IIFG	516 EEHI	156 GGHI
642 DDDI	624 EEEG	462 FFFH	264 GGGE	426 HHHF	246 IIID
642 EEFF	624 DDHH	462 DDGG	264 FFII	426 EEII	246 GGHH
642 DDEG	624 EEDI	462 FFDI	264 GGDI	426 HHDI	246 IIEG
642 DDFH	624 EEFH	462 FFEG	264 GGFH	426 HHEG	246 IIFH
543 DDGH	534 DDHI	453 DDGI	354 FFHI	435 EEGI	345 GGEH
543 EEFG	534 EEFI	453 FFEI	354 GGDH	435 HHDG	345 HHFG
543 FFEH	534 HHDF	453 GGDE	354 IIDF	435 IIDE	345 IIEF
543 DEFI	534 DEGH	453 DFGH	354 EFGI	435 EFHI	345 DGHI

Table 6

(abc)	(bac)	(cab)	(abc)	(acb)	(bca)
822 DDEE	282 FFGG	228 HHII	660 DDFF	606 EEHH	066 GGII
633 DEFH	363 DFGI	336 EGHI	552 DDFI	525 EEGH	255 GGEI
633 DDEI	363 FFGH	336 HHFI	552 FFDH	525 HHEF	255 IIDG
633 EEDG	363 GGEF	336 IIDH	552 DEFG	525 DEHI	255 FGHI

(abc)
444 DDII
444 EEGG
444 FFHH
444 DFHI
444 DEGI
444 EFGH

Step 5: Our sufficient condition for local non manipulability is that $r' - r^{s} > m - 2 = 1$. This condition is checked for each profile in Table 6.

Step 6: Once the condition is checked, the percentage of LNM profiles is computed. In this case 62 out of 126 profiles satisfy it: this is 49,2% of the total. With 5 voters 60,32% of the 252 profiles satisfy the sufficient condition, while this quota increses to 66,24% in the case of 6 voters. The findings reached so far can be summarized in the following table:

Table 7: Quotas of strategic proof profiles for $m=3$ and different n

	$n=3$	$n=4$	$n=5$	$n=6$
Total # of profiles	56	126	252	462
Quota of stable profiles	42,85	49,2	60,32	66,24

6 Non manipulability from the set of total scores

Searching for a proof of the conjecture that the quota of non manipulable profiles increases with the number of agents we face two approaches. First, we can work with profile sets and try to discover some structure that could prove the conjecture. Second, we can analyse the sets of possible scores: each case of m options and n agents has a set of possible profiles, say $P(m,n)$, characterized by the fact the Borda rule assigns an m-tuple of scores to each of them. For example, with 3 options and 4 agents a score is one of 13 possible triples of the type *{8,4,0}*, *{8,3,1}*, all the way down until *{4,4,4}*. 5 of them are unstable, namely those in which the difference between the first and the second element is smaller than 2.

The first possibility has been explored without success. The second one looks much better, but it faces a major problem. In the following I will elaborate on this approach. Let us focus our attention to the case of 3 alternatives and of a variable number of agents. Here are the complete lists of possible scores for n equal to 3 and to 4.

For $n=3$: *{6,3,0}*, *{6,2,1}*, *{5,4,0}*, *{5,3,1}*, *{5,2,2}*, *{4,4,1}*, *{4,3,2}*, *{3,3,3}*. 4 of the 8 possible scores are unstable: this is 50% of them.

For $n=4$: *{8,4,0}*, *{8,3,1}*, *{8,2,2}*, *{7,5,0}*, *{7,4,1}*, *{7,3,2}*, *{6,6,0}*, *{6,5,1}*, *{6,4,2}*, *{6,3,3}*, *{5,5,2}*, *{5,4,3}*, *{4,4,4}*. 5 of the 13 possible scores are unstable: this is 38% of them. The idea is that the quota of unstable scores diminishes as n increases.

Let $M = \sum_j (m-j)n$ for $j \in [1,m]$. Let x_1, x_2, x_3 be the values of each element in the possible triples of scores. The number of scores depends on the maximum value of $\{x_i\}$, since starting with it the preferred alternative gets all values until $M/3$. It is possible to give a condition which states that the maximum value of x_i is different for unstable scores from that valid for any score without regard to manipulability. First we have to prove that the upper bound for elements in a generic triple is always larger than the upper bound of unstable triples. Second, we have to prove that the difference between the upper bound of generic triples and the upper bound of unstable scores increases with n. In fact, we have the following:

Proposition 3. For $m=2,3$, the upper bound of elements of unstable scores is always smaller than that of generic scores when n increases.

Proof. We can see that all triples satisfy the conditions to the left, while unstable triples satisfy those to the right.

$$x_1 + x_2 + x_3 = M$$
$$x_i \leq (m-1)n$$
$$x_i \geq 0 \qquad \forall i = 1,2,3$$

$$x_1 + x_2 + x_3 = M$$
$$x_i \leq \frac{M-1}{2} + 1 \quad \text{if} \quad M \quad \text{is odd}$$
$$x_i \leq \frac{M}{2} \quad \text{otherwise}$$
$$x_i \geq 0 \qquad \forall i = 1,2,3$$

Since $M/2 < (M-1)/2 + 1$, we have to prove the following:

$$\frac{M-1}{2} + 1 < (m-1)n \tag{0}$$

this is equivalent to the following:

$$\frac{(m-1)n + (m-2)n - 1}{2} + 1 < (m-1)n \tag{1}$$

To prove (1) assume the contrary, that is $(m-1)n + (m-2)n - 1 + 2 > 2((m-1)n)$. This leads to $mn - n + mn - 2n + 1 > 2mn - 2n$, to $-3n + 1 > -2n$, and to $1 > n$ which is false for all relevant cases. So that (1) is true and so it is (0).

Proposition 4. The difference between the upper bound for generic scores and the upper bound for unstable profiles increases with n, when $m=2,3$.

Proof. This is trivial:

$$mn - n - \frac{(m-1)n + (m-2)n - 1}{2} = 2mn - 2n - mn + n - mn + 2n + 1 = n + 1$$

and then $\lim_{n \to \infty} n + 1 = \infty$

These results mean that the quota of stable scores over the total possible scores increases with the number of agents. The major problem mentioned before is the following: the Borda rule is not a one-to-one function from the set of possible profiles to that of possible scores: different profiles are mapped to the same score or to any of its permutations. Suppose to order all scores in descending order from the left with regard

DOI: 10.2870/7482

to the value of $(r' - r'')$; then suppose to know the frequency with which profiles are mapped to each score. This would amount to be able to compute the average difference of scores between the socially preferred alternative and the second one. By understanding the way in which this distribution varies when n increases we could infer about the quota of manipulable profiles. Unfortunately, I could not provide a general answer to the question of how the distribution of profiles over scores depends on n.

Notice that Proposition 3 is not true for different m. Actually I guess that in order to deal with larger m the upper bounds of score elements take another form and that in this case a similar proposition will hold provided that n is above a certain threshold.

For the moment we have to content ourselves with results from examples with $m=3$ and $n \leq 6$. These results are condensed in the Table below: on the top we have all possible values for $(r' - r'')$ and its average, as a function of n

Table 8

9	8	7	6	5	4	3	2	1	0	mean $(r' - r'')$	n
3	6	18	33	30	65	67	83	103	54	2.768	6
		6	12	18	30	48	36	72	30	2.428	5
			3	6	18	15	30	36	18	1.976	4
					6	12	6	24	8	1.714	3

On the first row we have the possible values of $(r' - r'')$. The entries below show the frequencies of each of these values as a function of n. The last but one column lists corresponding values of the average of $(r' - r'')$. All profile which have $r' - r'' < 2$ are manipulable. This suggests that the average increases with n which implies that the quota of stables scores and of stable profiles increases with n too.

I would like to do a final remark. It is true that considerations about the set of scores do not in general imply conclusions about the set of profiles. However, it is also true that given a certain degree of social consensus (or of similarity of preferences) some profiles are more likely to show up than others. Given that m is sufficiently large there is one profile which contains all different rankings. A profile that arises from a population with similar preferences will to contain some subsets of many similar orderings. As long as this assumption is true, the resulting $(r' - r'')$ is going to be large enough to make strategic voting impossible.

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

7 A modified Borda Rule

Proposition 2 and the succeeding sufficient condition for local non manipulability of a profile under the Borda Rule, allowed us to study Example 5 without the need to analyse individual preference orderings. Furthermore, they suggest a simple modification of the Borda Rule which makes the set of local non manipulable profiles larger.

It is apparent that any aggregation rule which assigns a score of 1 to any alternative in the first place of individual orderings and 0 scores to all alternatives ranked after the first leaves no room for strategic voting: any voter that is not satisfied by the winning alternative does not rank it first, so that he can make it lose not even 1 score. The Borda rule can be viewed as an example of a class of aggregation rules in which voters are requested to report a strict preference ordering and in which rules differ from each other in the weights used to aggregate alternatives ranked differently. However, such an aggregation rule produces a social ordering in which one of the alternatives has a positive score (which is bounded by the total number of voters), and all the others have zero score. This means that a lot of information is lost in the process: we do not know, for example, by how much the second socially preferred alternative is behind the top one. Then, there is a trade off between the local non manipulability of all profiles and the amount of information that is lost and is not conveyed by the social ordering.

It suffices to introduce a slight discontinuity in the sequence of coefficients assigned to alternatives to get a dramatic increase in the quota of strategy proof profiles. We have seen that the Borda Rule gives the following sequence: $((m-1)(m-2)\ldots(m-m))$. We can define a modified Borda Rule (*MBR*) which assignees the sequence $((m)(m-2)\ldots(m-m))$. Table 8 below lists profiles relative to Example 5 in the same way as Table 6. Here the score of a social ordering is computed by the *MBR*. 90 profiles satisfy the sufficient condition for local non manipulability, that is, the 71,42% of the total; the same figure was the 49,2% with the *BR*.

Table 8

	(abc)		(acb)		(bac)		(bca)		(cab)		(cba)
1240	DDDD	1204	EEEE	4120	FFFF	0124	GGGG	4012	HHHH	0412	IIII
1231	DDDE	1213	EEED	3121	FFFG	1123	GGGF	3112	HHHI	1312	IIIH
1060	DDDF	1006	EEEH	6100	FFFD	0106	GGGI	6010	HHHE	0610	IIIG
1051	DDEF	1015	EEDH	5101	FFDG	1105	GGFI	5110	HHEI	1510	IIGH
1042	EEDF	1024	DDEH	4102	GGDF	2104	FFGI	4210	IIEH	2410	HHGI
961	DDDG	916	EEEI	691	FFFE	196	GGGH	619	HHHD	169	IIIF
952	DDEG	925	EEDI	592	FFEG	295	GGFH	529	HHDI	259	IIFH
943	DDDI	934	EEEG	493	GGEF	394	GGGE	439	HHHF	349	IIID
943	EEDG	934	DDEI	493	FFFH	394	FFGH	439	IIDH	349	HHFI
871	FFDE	817	HHDE	781	DDFG	187	IIFG	718	EEHI	178	GGHI
862	EEFF	826	DDHH	682	DDGG	286	FFII	628	EEII	268	GGHH
853	DDFH	835	EEFH	583	FFDI	385	GGDI	538	HHEG	358	IIEG
763	DDFI	736	DDHI	673	GGDE	376	FFHI	637	IIDE	367	HHFG
763	EEFG	736	EEGH	673	FFDH	376	GGEI	637	HHEF	367	IIDG
754	DDGH	745	DEGH	574	DFGH	475	GGDH	547	HHDG	457	DGHI
754	DEFI	745	EEFI	574	FFEI	475	EFGI	547	EFHI	457	IIEF
	(abc)		(bac)		(cab)		(abc)		(acb)		(bca)
1222	DDEE	2122	FFGG	2212	HHII	880	DDFF	808	EEHH	088	GGII
1033	DDDH	3103	FFFI	3310	HHHG	772	DEFG	727	DEHI	277	FGHI
1033	EEEF	3103	GGGD	3310	IIIE	664	DDGI	646	DDII	466	FFHH
844	DEFH	4845	DFGI	448	EGHI	664	EEGG	646	EEGI	466	GGEH
655	DEGI	565	EFGH	556	DFHI	664	FFEH	646	HHDF	466	IIDF

It is interesting to compare Table 8 to Table 6. In particular, we can check if the choice of a social ordering is sensitive to the change in the functional form of the aggregation rule. In fact, the use of the MBR introduces 6 types of change with respect to the BR. 30 profiles that are associated by the BR to some social orderings are associated by the MBR to different social orderings. I list these profiles in Table 9, each of them together with the social ordering resulting from the BR (on the left), and with that resulting from the MBR.

Table 9

(1)		(2)		(3)	
$abc \to a\underline{bc}$	DDDH	$abc \to \underline{ab}c$	FFEH	$a\underline{bc} \to acb$	DDEI
$acb \to a\underline{bc}$	EEEF	$acb \to \underline{ac}b$	HHDF	$a\underline{bc} \to abc$	EEDG
$bac \to b\underline{ac}$	FFFI	$bac \to \underline{ab}c$	DDGI	$b\underline{ac} \to bca$	FFGH
$bca \to b\underline{ac}$	GGGD	$bac \to \underline{bc}a$	IIDF	$b\underline{ac} \to bac$	GGEF
$cab \to c\underline{ab}$	HHHG	$cab \to \underline{ac}b$	EEGI	$c\underline{ab} \to cba$	HHFI
$cba \to c\underline{ab}$	IIIE	$cba \to \underline{bc}a$	GGEH	$c\underline{ab} \to cab$	IIDH

(4)		(5)		(6)	
$\underline{ab}c \to abc$	DDFI	$\underline{abc} \to \underline{ac}b$	DDII	$\underline{abc} \to c\underline{ab}$	DFHI
$\underline{ab}c \to bac$	FFDH		EEGG	$\underline{abc} \to a\underline{bc}$	DEGI
$\underline{ac}b \to acb$	EEGH		FFHH	$\underline{abc} \to b\underline{ac}$	EFGH
$\underline{ac}b \to cab$	HHEF				
$\underline{bc}a \to bca$	GGEI				
$\underline{bc}a \to cba$	IIDG				

In Table 8 the concerned profiles are grouped into 6 classes: class (1) is formed by profiles which result in social orderings with indifference relations between no top alternatives under the MBR. These social orderings have not indifference relations under the BR; as long as they concern no top alternatives there is no problem. In the case of class (3), the MBR breaks a tie between no top alternatives which exists under the BR. However, the MBR is not always able to solve the problem, as profiles of class (5) show. The situation under the MBR is worse than under the BR for the six profiles of class (2): here the MBR produces a tie between top alternatives. The MBR does well in the case of classes (4) and (6), which comprehend 9 profiles: it brokes a tie between top alternatives.

Summing up, from the point of view of the sensitivity of the outcome to the functional form of the aggregation rule, the introduction of the MBR is positive: the changes concern 30 profiles; these changes are not relevant in 15 cases, they are negative in 6 cases, and positive in 9 profiles. Such a judgement is clearly referred to the use of the BR as a bench-mark.

PART 7:

CONCLUSIONS AND FUTURE RESEARCH

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

 DOI: 10.2870/7482

Conclusions and future research

Most of the theoretical and experimental work in the literature on public economics concerns models in which a single pure public good is supplied. In contrast with the model that has received the most attention so far, the model in Part 1, section 3 considers that a consumer's utility depends not only on the aggregate amount of contribution, but also on his own contribution. Many theoretical papers deal with a Nash equilibrium in which each consumer assumes that the contribution of others will be independent of his own, while some experimental papers have been trying to verify whether behaviour is consistent with the Nash hypothesis.

In Part 2, I discussed two main points: the worth of each agent in a coalition and the process of coalition formation. Both these points have a direct influence on the non-emptiness and on the dimension of the related core, and I gave some alternatives definitions regarding them: so far, no experimental work has be done to test their relevance.

Developments of the idea that coalitions and not just individuals determine equilibrium conditions seem possible: a model based on a less individualistic basis could probably go much further than the proof of the equilibrium existence; it could explain uniqueness and stability without the need of assuming at the aggregate level that the economy behaves as an individual.

Probably, a good development of the ideas presented in Parts 2 and 3 would result from a different model of an economy with both public and private goods: many authors proved that the usual definition of the core with public goods (Foley, 1970) is not useful for capturing an intuitive notion of social stability. We can recall three ways of formulating a new definition that has to make improving easier for small coalitions. Rosenthal (1971) and Richter (1973) proposed a definition that recognizes that it may be advantageous for a complementary coalition to contribute to the provision of a public good; Ellickson (1973) and Roberts (1974) studied the case where the cost of producing the public good is a non-decreasing function of the size of the coalition; Champsaur, Roberts and Rosenthal (1975) presented a definition under which coalitions are allowed certain powers to tax their complements to help produce public goods. None of these ideas has been successful in using core-like notions to analyze large economies with public goods.

I thought for a while that if we wish to construct an institution-free model and we want to solve some of the problems with the threat structure, then the Harsanyi's modified value solution appears more promising than the core. My point of view is changing: even if most of the experiments confirm the likelihood of its axioms, the value solution has a very strong normative character. For it to be accepted we need a previous agreement on its axioms. In this sense the stability comes from the fact of agreeing about using the calculations needed to get a solution. So, a process of agreement is to be performed well before the solution is given: this is a costly and difficult task when many people are involved. On the other side, a core solution, when it exists, is stable in a well defined sense: the concept of "improving" that applies to allocations is obvious and natural and no previous agreement is needed. I would say that is acceptability is more incentive compatible. Generally speaking, a game that has a core has less potential for social conflict that one without a core, since every cooperative or collusive effective demand can be granted. On the other side, a coreless game must leave at least one coalition unsatisfied, that is, short of its potential.

As a result of these recognitions I am thinking about a model in which the core analysis applies to private goods allocations, while public goods decisions are decentralized by a voting system. An approach to unified models (realism in modeling seems to many scholars a chimera) is to consider games within games, each of them determining conditions for the others, trying to avoid circular networks. We might consider a vote to determine tax and subsidy rates prior to examining the economy with these rates as given. In such a model the level of produced public goods will depend on the outcome of a voting system through the balanced budget condition for the State.

In fact, it seems clear that if a core allocation is reached for what concerns private goods, then the associated coalition structure will have an influence on the decisions that are taken for public goods. This should be justified by two reasons:

Firstly, both the tax system and the availability of public goods are capable of changing the conditions under which a core allocation has been reached; the tax system influence the after tax income, which in turn gives individuals purchasing power for acquiring private goods; public goods also determine the overall welfare of individuals. Then, a coalition which receives a core allocation will try to control the public goods sector, if not to take advantage of new opportunities, at least to avoid a worsening of its own situation[1].

Secondly, there is a more subtle reason: probably the major feature of a core allocation, as long as my idea is concerned, is the associated notion of stability. Each conceivable coalition receives at least what it is worth and then it has no incentive to struggle anymore. The problem is that the worth of a coalition might

1 More generally, following Greenberg's (1975) suggestion, the conditions under which a core allocation is reached change because different tax systems make prices vary in a general equilibrium setting.

well be not clearly determined in real situations. It depends on the existing structure of property rights, for example. Moreover, an exact evaluation assume a complete knowledge of all the agents about the situation. Briefly, they have to construct a characteristic function and agree on it. In this process there is the possibility for some coalitions to make others underestimate their own potential, in order to select what we can call a "pseudo-core allocation": as long as the misjudgement is not corrected, the situation will be stable.

Now, the conditions for this to happen are well known and have been described in the literature: I just recall here the "easy riding" suffered by large coalitions, the fact that benefits from forming coalitions net of the administrative costs are typically larger the smaller the coalition, and problems of collecting and spreading information. What I am arguing is that the existence of a public sector (and indeed, the need for representative democracies due to large numbers of individuals) do increase the opportunity for such a mis-calculation to happen. Then, those coalitions that have the power to induce a core allocation, have even more incentives to try to control the public sector. The theory of *optimal obfuscation* of Magee-Brock-Young (1990) is on the same line of thought.

Another direction of research is worth to be explored: the existing literature deals in general with models in which it is assumed that individuals have equal weights in the decision making process for the production of public goods. Dropping this assumption, we are lead to examine the way in which one can model differences in power and the behaviour of interest groups.

Recalling the preceding discussion about voting on tax systems, we can sketch the consequence of the assumption of "differential" voting, where the effective voting power depends on economic and other variables. To the extent that eligibility and participation are correlated with endowments, the median of the effective voting population is likely to be at a higher level of income than the true median. This tends to bring the majority voting outcome, when it exists, up toward the mean of a positively skewed distribution. There are two major ways for a coalition to legally influence a ballot: the first one consists of altering the conditions of the ballot (for example participation) or of choosing favourable voting procedures. The second one consists of taking advantage of the asymmetry of information and of using advertising to gain votes. Having said all that, the question is clear: how could we put together both the evidence of lobbying coalitions that influence the public sector and the private sector of the economy as it is described in standard models?

DOI: 10.2870/7482

In Part 4, section 1.3, I presented a definition of a fair set of alternative choices. A simple theorem ensures that, provided that the original set of alternatives and the associated individual orderings admit a choice set under the MD-rule, the resulting fair subset of alternatives and subset of individual orderings do the same.

In Part 6, I analysed a particular problem of voting mechanisms: under the strong assumption that a voter knows the preferences of the others, strategic voting is usually considered a major problem in social choice theory. I argue that this is not an actual problem. After giving a simple example of a situation manipulable by a well informed voter, I clarified the dimensions of the sets of all possible individual and social orderings, and of all possible profiles as functions of the number of alternatives and of voters.

Then, I explained why the preference score procedure is preferable to a majority procedure, and I justified the use of the Borda Rule. Section 6.4.1 gives definitions about strategic voting and types of profiles, and it contains a necessary and sufficient condition for the strategy proofness of a profile with regard to a single voter. Section 6.4.2 defines formally the Borda Rule and gives a sufficient condition which is easier to check and which regards all voters. Examples 4 and 5 show the case of 3 alternatives and 3 and 4 voters: respectively almost 43% and 49% of profiles satisfy the sufficient condition. These examples and some other results relative to 5 and 6 voters suggest that at least with 3 alternatives the quota of stable profiles increases with the number of voters.

After that, a modified Borda Rule is defined and applied to Example 5: the quota of profiles satisfying the condition increases to 71.42%. I could not give a conclusion valid for any number of alternatives and of voters, since it seems that such a generalization can only be reached by numerical analysis: given the "big" numbers involved, this analysis faces computational problems. However, these are the same problems that a strategic voter should solve to be successful: the plausibility of the assumption of perfect information decreases with the increase of the number of voters and of alternatives.

Concerning further developments of the subject in Part 6, it is common sense to think of the following:
-an implementation of a Borda Rule-based telematic system to collect information from the electorate in polls and elections;
-the uncovering of internal patterns in the set of possible profiles would allow to find more useful necessary and sufficient conditions for local non manipulability, and would advance our knowledge about how to get an efficient democracy.

PART 8:

BIBLIOGRAPHY

Arrow, K.J. (1951a) *An extension of the basic theorems of classical welfare economics*, originally in **Neyman, J. (ed)** "Proceedings of the second Berkeley symposium on mathematical statistics and probability", University of California Press. Also available in **Arrow, K.J.** (1987) "Equilibrio, incertezza, scelta sociale", Il Mulino, Bologna, 75-112

---------- (1951b) "Social choice and individual values", Wiley, New York

---------- (1963) "Social choice and individual values", 2nd ed., Wiley, New York

---------- (1967) *Values and collective decision-making*, in **Laslett, P. - Runcimann, W.C. (eds)** "Philosophy, politics, and society", Basil Blackwell, 215-232

---------- (1969) *The organization of economic activity: issues pertinent to the choice of market versus non-market allocation*, in **Joint Economic Committee, U.S. Congress**, "The analysis and evaluation of public expenditure: the PBB System, vol.1", Government Printing Office, Washington, 47-64. Also available in **Arrow, K.J.** (1987) "Equilibrio, incertezza, scelta sociale", Il Mulino, Bologna, 287-313

---------- - **Scitovsky, T.** (1969) "Readings in welfare economics", Richard D. Irwin, Inc.

---------- - **Rader, T. - Mack, R.P.** (1970) *Discussion*, of **Rader, T.** (1970) *New Ideas in Pure Theory: problems in the theory of markets with uncertainty*, "Papers and Proceedings", "The American Economic Review" 40, 461-464

---------- (1971) *A utilitarian approach to the concept of equality in public expenditure*, "Quarterly Journal of Economics" 85, 409-415

---------- (1978a) *The future and the present in economic life*, "Economic Inquiry" 16, 157-170

---------- (1978b) *Nozick's entitlement theory of justice*, "Philosophia" 7, 265-279

---------- - **Raynaud, H.** (1986) "Social choice and multicriterion decision-making", The MIT Press

Aspremont, d', C. - Gérard-Varet, L.A. (1979) *Incentives and incomplete information*, "Journal of Public Economics" 11, 25-45

Atkinson, A.B. - Stiglitz, J.E. (1980) "Lectures in public economics", Mac Millan, New York

Auerbach, A.J.. - Feldstein, M. (eds) (1985) "Handbook of public economics", North Holland

Aumann, R.J. (1964) *Markets with continuum of traders*, ""Econometrica"" 32, 39-50

---------- - **Drèze, J.H.** (1975) *Cooperative games with coalition structures*, "International Journal of Game Theory" 3, 217-237

---------- - **Gardner, R.J. - Rosenthal, R.W.** (1977) *Core and value for a public goods economy: an example*, "Journal of Economic Theory" 15, 363-365

---------- - **Kurz, M.** (1978) *Power and taxes in a multi-commodity economy (updated)*, "Journal of Public Economics" 9, 139-161

Balasko, Y. (1988) "Foundations of the theory of general equilibrium", Academic Press, Inc., London

Bergstrom, T. (1975) *The core when strategies are restricted by law*, "Review of Economic Studies" 42, 249-257

---------- - **Cornes, R.C.** (1983) *Independence of allocative efficiency from distribution in the theory of public goods*, "Econometrica" 51, 1735-1765

---------- - **Blume, L. - Varian, H.** (1986) *On the private provision of public goods*], "Journal of Public Economics" 29, 25-49

Berliant, M. (1992) *On income taxation and the core*, "Journal of Economic Theory" 56, 121-141

Bernheim, D.B. - Peleg, B. - Whinston, M.D. (1987) *Coalition-proof Nash equilibria: I. Concepts*, "Journal of Economic Theory" 42, 1-12

Black, D. (1958) "The theory of committees and elections", Cambridge University Press

Blackorby, C. - Donaldson, D. - Weymark, J.A. (1990) *A welfarist proof of Arrow's theorem*, forthcoming in Recherches Economiques de Louvain

Blau, J.H. (1957) *The existence of social welfare functions*, "Econometrica" 25, 302-313

Boadway, R. - Bruce, N. (1984) "Welfare economics", Basil Blackwell

Böhm, V. (1972) *The limit of the core of an economy with production*, Discussion Paper n.37, University of Bonn, reprinted in "International Economic Review" 15, 143-148 (1974)

--------- (1974a) *The core of an economy with production*, "Review of Economic Studies" 41, 429-436

--------- (1974b) *On balanced games, cores and production*, CORE Discussion Paper n.7340

--------- (1988) *Returns to size vs returns to scale: the core with production revisited*, "Journal of Economic Theory" 46, 215-219

Burda, J.C. (1781) "Mémoire sur les élections Au scrutin", Histoire de l' Académie Royale des Sciences, Paris

Border, K.C. (1983) *Social welfare functions for economic environments with and without the Pareto principle*, "Journal of Economic Theory" 29, 205-216

--------- **- Jordan, J.S.** (1983) *Straightforward elections, unanimity and phantom agents*, "Review of Economic Studies" 50, 153-170

Bordes, G. - Le Breton, M. (1987) *Robert's theorem in economic environments: (2) the private goods case*, Discussion Paper 8703, Université de Rennes

--------- (1989) *Arrowian theorems with private alternatives domains and selfish individuals*, "Journal of Economic Theory" 47, 257-291

--------- **- Le Breton, M.** (1990a) *Arrowian theorems for economic domains. The case where there are simultaneously private and public goods*, "Social Choice and welfare" 7, 1-17

--------- - --------- (1990b) *Arrowian theorems for economic domains: assignments, matchings and pairings*, "Social Choice and welfare" 7, 193-208

Bös, D. (1978) *Cost of living indices and public pricing*, "Economica" 45, 59-69

Bowen, H.R. (1943) *The interpretation of voting in the allocation of economic resources*, "Quarterly Journal of Economics" 58, 27-48

Brandford, D.F. - Hildebrandt, G.G. (1977) *Observable preferences for public goods*, "Journal of Public Economics" 8, 111-131

Brams, S.J. - Fishburn, P.C. (1978) *Approval voting*, "American Political Sciences Review" 72, 831-847

--------- - --------- (1983) "Approval voting", Cambridge, Mass.

Buchanan, J.M. (1968) "The demand and supply of public goods", Rand Mac Nally & Company

Caplin, A. - Nalebuff, B. (1988) *On 64% majority rule*, "Econometrica" 56, 787-814

Castellan, M. (1970) *L'équilibre avec souscription dans la thérie des biens publics*, working paper of the Département de la Recherche, INSEE, Paris

Champsaur, P. - Roberts, D.J. - Rosenthal, R.W. (1975) *On cores in economies with public goods*, "International Economic Review" 16, 751-764

--------- (1975a) *How to share the cost of a public good*, "International Journal of Game Theory" 4, 113-129

---------- (1975b) *Cooperation vs competition*, "Journal of Economic Theory" 11, 394-417

---------- (1976) *Symmetry and continuity properties of Lindahl equilibria*, "Journal of Mathematical Economics" 3, 19-36

---------- - **Laroque, G.** (1980) *Fair allocations in large economies*, INSEE, Paris, Document de travail n.8004, reprinted in (1981) "Journal of Economic Theory" 25, 269-282

Chipman, J.S. - McFadden, D. - Richter, M.K. (eds) (1990) "Preferences, uncertainty, and optimality: essays in honour of Leonid Hurwicz", Westview Press

Condorcet, Marquis de (1785) "Essai sur l' application de l' analyse à la probabilité des décision rendues à la pluralité de voix", Paris, partially reprinted in English as *Essay on the application of mathematics to the theory of decision making*, in **Baker, K. (ed)** (1976) "Condorcet, selected writings", Bobbs-Merril

Corchón, L. (1988) *On Lindahl equilibria and incentive compatibility*, working paper n.29, Universidad de Alicante

Cornes, R. - Sandler, T. (1986) "The theory of externalities, public goods and club goods", Cambridge University Press

Coughlin, P.J. (1981) *Necessary and sufficient conditions for delta-relative majority voting equilibria*, "Econometrica" 49, 1223-1234

---------- (1986) *Elections and income redistribution*, "Public Choice" 50, 27-91

Crémer, J. - Riordan, M.H. (1985) *A sequential solution to the public goods problem*, "Econometrica" 53, 77-84

Dasgupta, P. - Hammond, P.J. - Maskin, E. (1979) *The implementation of social choice rules: some general results on incentive compatibility*, "Review of Economic Studies" 46, 181-216

Davis, O. - de Groot, M. - Hinich, M. (1972) *Social preference orderings and majority rule*, "Econometrica" 40, 147-157

Deaton, A. - Muellbauer, J. (1980) "Economics and consumer behaviour", Cambridge University Press

Debreu, G. (1959) "Theory of value", John Wiley & Sons, Inc.

---------- - **Scarf, H.** (1963) *A limit theorem on the core of an economy*, "International Economic Review" 4, 235-246

Diamond, P.A. - Mirrlees, J.A. (1973) *Aggregate production with consumption externalities*, "Quarterly Journal of Economics" Vol. LXXXVII No.I, 1-24

Dorfman, R. (1969) *General equilibrium with public goods*, in **Margolis, J. - Guitton, H. (eds)** (1969) "Public economics", mac Millan

Downs, A. (1957) "An economic theory of democracy", Harper, New York

Drèze, J.H. (1980) *Public goods with exclusion*, "Journal of Public Economics" 13, 15-24

Dummet, M. - Farquharson, R. (1961) *Stability in voting*, "Econometrica" 29, 33-43

---------- (1984) "Voting procedures", Clarendon Press, Oxford

Dutta, B. - Ray, D. - Sengupta, K. - Vohra, R. (1989) *A consistent bargaining set*, "Journal of Economic Theory" 49, 93-112

Edgeworth, F.Y. (1881) "Mathematical Psychics", Paul Kegan, London

Fishburn, P.C. - Gehrlein, W.V. (1976) *Borda's rule, positioning voting, and Condorcet's simple majority principle*, "Public Choice" 28, 79-88

Foley, D. (1967) *Resource allocation and the public sector*, "Yale Economic Essays" 7, 45-98

——— (1970) *Lindahl's solution and the core of an economy with public goods*, "Econometrica" 38, 66-72

——— (1978) *State expenditure from a Marxist perspective*, "Journal of Public Economics" 9, 221-238

Friedman, J.W. (1986) "Game theory with economic applications", Oxford University Press

Gabszewich, J. (1970) *Theories du noyau et de la concurrence imparfait*, Recherches Economique de Louvain n.1

Gärdenfors, P. (1973) *Positionalist voting functions*, "Theory and Decisions" 4, 1-24

Gehrlein, W.V. - Fishburn, P.C. (1976) *The probability of the paradox of voting: a computable solution*, "Journal of Economic Theory" 13, 217-228

——— (1990) *Probability calculations for transitivity of simple majority rule with anonymous voters*, "Public Choice" 66, 253-259

Gérard-Varet, L.A. (1987) *Quelques reflexions sue des problèmes actuels de la théorie economique: dynamique, jeu, information*, GREQE working paper n.8707, Marseille

Gibbard, A. (1973) *Manipulation of voting schemes: a general result*, "Econometrica" 41, 587-601

——— (1978) *Social decision, strategy behavior, and best outcomes*, in **Gottinger, H.W. - Leinfeller, W. (eds)** (1978) "Decision theory and social ethics", Reidel Publishing Company

Gillies, D.B. (1953) *Some theorems on n-persons games*, Ph.D. dissertation, Dpt of Mathematics, Princeton University

Grandmont, J.M. (1978) *Intermediate preferences and the majority rule*, "Econometrica" 46, 317-330

Grant, S. - Kajii, A. - Polak, B. (1992) *Many good choice axioms: when can many-good lotteries be treated as money lotteries?*, "Journal of Economic Theory" 56, 313-337

——— - ——— - ——— (1992) *Many good risks: an interpretation of multivariate risk and risk aversion without the independence axiom*, "Journal of Economic Theory" 56, 338-351

Green, J. - Laffont, J.J. (1979) *On coalition incentive compatibility*, "Review of Economic Studies" 46, 243-254

Greenberg, J. (1975) *Efficiency of tax systems financing public goods in general equilibrium analysis*,"Journal of Economic Theory" 11, 168-195

——— (1977) *Existence of an equilibrium with arbitrary tax schemes for financing local public goods*, "Journal of Economic Theory" 16, 137-150

——— (1979) *Consistent majority rules over compact sets of alternatives*, "Econometrica" 47, 627-636

——— - **Weber, S.** (1982) *The equivalence of superadditivity and balancedness in the proportional tax game*, "Economic Letters" 9, 113-117

——— - ——— (1985) *Strong Tiebout equilibrium under restricted preferences domain*, "Journal of Economic Theory" 38, 101-117

——— (1989a) "The theory of social situations", Cambridge University Press

——— (1989b) *Deriving strong and coalition-proof Nash equilibria from an abstract system*, "Journal of Economic Theory" 49, 195-202

Groves, J. - Ledyard, J. (1977) *Optimal allocation of public goods: a solution to the free rider problem*, "Econometrica" 45, 783-809

Guesnerie, R. (1974) *Monopoly, Syndicate and Shapley value: about some conjectures*, Institute for Mathematical Studies in the Social Sciences, working paper n. 49, Stanford

---------- (1974) "Modèles de l' économie publique", CNRS Edition, Paris

---------- - **Oddou, C.** (1979) *On economic games that are not necessarily superadditive*, "Economic Letters" 3, 301-306

---------- - ---------- (1981) *Second best taxation as a game*, "Journal of Economic Theory" 25, 67-91

---------- - ---------- (1988) *Increasing returns to size and their limits*, "Scandinavian Journal of Economics" 90(3), 259-273

Güth, W. - Hellwig, M. (1989) *The private supply of a public good*, "Journal of Economics" 5, 121-159

Hall, M. (1986) "Combinatorial theory", 2nd ed., Wiley, New York

Hammond, P.J. (1976) *Equity, Arrow's conditions and Rawl's difference principle*, "Econometrica" 44, 793-804

---------- (1979) *Symposium on incentive compatibility: Introduction*, "Review of Economic Studies" 46, 181-184

---------- (1979) *Straightforward individual incentive compatibility in large economies*, "Review of Economic Studies" 46, 263-282

---------- (1983) *Ex-post optimality as a dynamically consistent objective for collective choice under uncertainty*, in **Pattanaik, P.K. - Salles, M. (eds)** (1983) "Social choice and welfare", North-Holland

---------- (1985) *Welfare economics*, in **Feiwel, G.R. (ed)** (1985) "Issues in contemporary microeconomics and welfare", Chapter 13, State University of New York Press, Albany

---------- (1987) *Markets as Constraints: multilateral incentive compatibility in continuum economies*, "Review of Economic Studies" 54, 399-412

---------- (1988) *Consequentialist demographic norms and parenting rights*, "Social Choice and Welfare" 5, 127-145

---------- - **Kaneko, M. - Wooders, M.H.** (1989) *Continuum economies with finite coalitions: core, equilibrium and widespread externalities*, "Journal of Economic Theory" 49, 120-134

---------- (1990a) *A revelation principle for (bounded) Bayesian rationalizable strategies*, EUI working papers, ECO n.90/4

---------- (1990b) *Independence of irrelevant interpersonal comparisons*, "Social Choice and Welfare",

Hansen, T. - Gabszewicz, J.J. (1972) *Collusion of factor owners and distribution of social output*, "Journal of Economic Theory" 4, 1-18

Hare, P.G. (ed) (1988) "Surveys in public sector economics", Basil Blackwell

Harsany, J.C. (1959) *A bargaining model for the cooperative n-person game*, in **Tucker, A.W. - Luce, R.D. (eds)** (1959) "Contributions to the theory of games", vol.4 of Annals of Mathematical Studies n.40, Princeton University Press

---------- (1967) *Games with incomplete information played by "Bayesian" players, part I*, "Management Science" 14, 159-182

---------- (1968a) *Games with incomplete information played by "Bayesian" players, part II*, "Management Science" 14, 320-334

---------- (1968b) *Games with incomplete information played by "Bayesian" players, part III*, "Management Science" 14, 486-502

---------- (1977) "Rational behavior and bargaining equilibrium in games and in social situations", Cambridge University Press

Heller, W.P. - Starrett, D.A. (1976) *On the nature of externalities*, in **Lin, S.A.Y. (ed)** "Theory and measurement of economic externalities", Academic Press

Herrero, C. - Villar, A. (1989) *Egalitarian allocation in the presence of consumption externalities*, Paper presented at the Fourth Annual Congress of the EEA

--------- - --------- (1988) *Exploiting some properties of continuous mappings: Lindahl equilibria and welfare egalitaria allocations in presence of externalities*, working paper n.30, Universidad de Alicante

Hettich, C. - Winer, S.L. (1988) *Economic and political foundations of tax structure*, "The American Economic Review" 78, 701-712

Hildenbrand, W. (1968) *The Core of an Economy with a Measure Space of Economic Agents*, "Review of Economic Studies" 35, October, 443-452

--------- (1974) "Core and equilibria of a large economy", Princeton University Press

--------- (1982) *Core of an Economy*, in **Arrow, K.J. - Intriligator, M.D. (eds)** (1982) "Handbook of Mathematical Economics", North Holland

Hinich, M.J. (1977) *Equilibrium in spatial voting: the median voter theorem is an artifact*, "Journal of Economic Theory" 16, 208-219

Hirsh, M.W. - Smale, S. (1974) "Differential equations, dynamical systems, and linear algebra", page 17, chapter 8, and 9, Academic Press

Holler, M.J. - Host, V. (1989) *Maximin vs. Nash equilibrium: some empirical evidence*, Paper presented at the Fourth Annual Congress of the EEA

Ichiishi, T. (1977) *Coalition structure in a labor-managed market economy*, "Econometrica" 45, 341-360

-------- (1981) *A social coalitional equilibrium existence lemma*, "Econometrica" 49, 369-377

-------- (1982) *Management versus ownership, I*, "International Economic Review" 23, 323-336

-------- (1983) "Game theory for economic analysis", Academic Press

-------- (1985) *Management versus ownership, II*, "European Economic Review" 27, 115-138

Inada, K. (1969) *On the simple majority decision rule*, "Econometrica" 36,

Inarra, E. - Usategui, J. (1989) *The Shapley value, the core and average-convex games*, Paper presented at the Fourth Annual Congress of the EEA

Johansen, L. (1963) *Some notes on the Lindahl theory of determination of public expenditure*, "International Economic Review" 4, 346-358

-------- (1982) *On the status of the Nash type of non cooperative equilibrium theory*, "Scandinavian Journal of Economics" 84, 421-441

Kalay, E. - Muller, E. (1977) *Characterization of domains admitting nondictatorial social welfare functions and non-manipulable voting procedures*, "Journal of Economic Theory" 16, 456-469

--------- - --------- - **Satterthwaite, M.A.** (1979) *Social welfare functions when preferences are convex, strictly monotonic, and continuous*, "Public Choice" 34, 87-97

--------- - **Ritz, Z.** (1980) *Characterization of the private alternatives domains admitting Arrow social welfare functions*, "Journal of Economic Theory" 22, 23-36

Kaneko, M. (1977a) *The ratio equilibrium and a voting game in a public goods economy*, "Journal of Economic Theory" 16, 126-136

--------- (1977b) *The ratio equilibria and the core of the voting game G(N,W) in a public goods economy*, "Econometrica" 45, 1589-1594

--------- - **Wooders, M.H.** (1989) *The core of a continuum economy with widespread externalities and finite coalitions: from finite to continuum economies*, "Journal of Economic Theory" 49, 135-168

Khan, M.A. - **Vohra, R.** (1987) *On the existence of Lindahl - Hotelling equilibria*, "Journal of Public Economics" 34, 143-158

--------- - --------- (1987) *An extension of the second welfare theorem to economies with nonconvexities and public goods*, "Quarterly Journal of Economics" 102, 223-241

Kirman, A. (1982) *Measure theory with applications to economics*, in **Arrow, K.J.** - **Intriligator, M.D. (eds)** (1982) "Handbook of mathematical economics", Chapter 5, North-Holland

--------- - **Oddou, C.** - **Weber, S.** (1986) *Stochastic communication and coalition formation*, "Econometrica" 54, 129-138

Koopmans, T.C. (1957) "Three essays on the state of economic science", McGraw-Hill

Kolm, S.-C. (1968) "Prix publics optimaux", CNRS editions, Paris

Kramer, G. (1973) *On a class of equilibrium conditions for majority rule*, "Econometrica" 41, 285-297

--------- (1977) *A dynamic model of political equilibrium*, "Journal of Economic Theory" 16, 310-334

Laux-Meiselbach, W. (1988) *Impossibility of exclusion and characteristics of public goods*, "Journal of Public Economics" 36, 127-137

Le Breton, M. (1987) *Robert's theorem in economic environments: (1) the public goods case*, LEME discussion paper 8702, Université de Rennes 1

Lepelley, D. - **Mbih, B.** (1987) *The proportion of coalitionally unstable situations under the plurality rule*, "Economic Letters" 24, 311-315

Lindahl, E. (1919) *Positive Lösung, die Gerechtigkeit der Besteuerung*, translated as *Just taxation - a positive solution*, in **Musgrave, R.A.** - **Peacock, T. (eds)** (1958) "Classics in the theory of public finance, MacMillan

Liu, C.L. (1968) "Introduction to combinatorial mathematics", McGraw-Hill

Loher, W. - **Sandler, T.** (1978) "Public goods and public policy", Sage Publication

Luce, R.D. - **Raiffa, H.** (1957) "Games and decisions", Wiley

MacIntyre, I (1991) *The Pareto rule and strategic voting*, "Theory and Decision" 31, 1-19

Magee, S.P. - **Brock, W.A.** - **Young, L.** (1989) "Black hole tariffs and endogenous policy theory", Cambridge University Press

Makowski, L. (1980b) *Perfect competition, the profit criterion, and the organization of economic activity*, "Journal of Economic Theory" 22, 105-125

--------- (1980a) *A characterization of perfectly competitive economies with production*, "Journal of Economic Theory" 22, 208-221

--------- (1983) *Competition and unanimity revisited*, "The American Economic Review" 73, 329-339

--------- - **Ostroy, J.M.** (1991) *The margin of appropriation and an extension of the first theorem of welfare economics*, UCLA Department of Economics discussion paper n.629

Maler, K.G. (1970) *A method of estimating social benefits from pollution control*, "Sweedish Journal of Economics" 73, 121-133

Malinvaud, E. (1972) "Lectures in microeconomic theory", North Holland

Mantel, R.R. (1975) *General equilibrium and optimal taxes*, "Journal of Mathematical Economics" 2, 187-200

 DOI: 10.2870/7482

Mas-Colell, A. (1980) *Efficiency and decentralization in the pure theory of public goods*, "Quarterly Journal of Economics" 94, 625-641

---------- - **Silvestre, J.** (1989) *Cost share equilibria: a lindahlian approach*, "Journal of Economic Theory" 47, 239-256

---------- - ---------- (1991) *A note on cost-share equilibrium and owner-consumers*, "Journal of Economic Theory" 54, 204-214

Maskin, E. (1976) "Social welfare functions on restricted domains", Harvard University and Darwin College, Cambridge, MA

May, K. (1952) *A set of independent necessary and sufficient conditions for simple majority rule*, "Econometrica" 20, 680-684

Milleron, J.C. (1972) *Theory of value with public goods: a survey article*, "Journal of Economic Theory" 5, 419-477

Mishan, E.J. (1960) *A survey of welfare economics, 1939-59*, "The Economic Journal" June, 197-265

---------- (1971) *The postwar literature on externalities: an interpretative essay*, "Journal of Economic Literature" 9, 1-28

Moulin, H. (1982) *Voting with proportional veto power*, "Econometrica" 50, 145-162

Muench, T. (1972) *The core and the Lindhal equilibrium of an economy with a public good: an example*, "Journal of Economic Theory" 4, 241-255

Murakami, Y. (1966) *Formal structure of majority decisions*, "Econometrica" 34, 709-718

---------- (1968) "Logic and social choice", Dover, New York

Musgrave, R.A. - Musgrave, P.B. (1984) "Public finance in theory and practice", 4th ed., McGraw-Hill, Co., Singapore

Naylor, R. (1989) *Strikes, free riders, and social customs*, "Quarterly Journal of Economics" November, 771-785

Negishi, T. (1963) *On social welfare functions*, "Quarterly Journal of Economics" 77, 156-158

Nitzan, S. - Paroush, J. (1985) "Collective decision making", Cambridge University Press

Nozick, R. (1974) "Anarchy, state, and utopia", Basil Blackwell, Oxford

Nurmi, H. (1987) "Comparing voting systems", D. Reidel Publishing Company

Oakland, W.H. (1982) *Theory of public goods*, in **Arrow, K.J. - Intriligator, M.D. (eds)** (1982) "Handbook of mathematical economics", Chapter 9, North Holland

Oddou, C. (1976) *Théorèmes d'existence et d'équivalence pour des économies avec production*, "Econometrica" 44, 265-281

---------- (1988) *Jeux de mesure vectorielle et fonctions de production à rendements non constants*, in "Mélanges Economiques, Essais en l'honneur de Edmond Malinvaud", 145-155, Ed. Economica, Paris

Olson, M. (1971) "The logic of collective action", 2nd ed., Harvard University Press

Osana, H.(1972) *Externalities and the basic theorems of welfare economics*, "Journal of Economic Theory" 4, 401-414

Ostroy, J.M. (1980) *The no-surplus condition as a characterization of perfectly competitive equilibrium*, "Journal of Economic Theory" 22, 183-207

---------- (1984) *A reformulation of the marginal productivity theory of distribution*, "Econometrica" 52, 599-630

Owen, J. (1982) "Game theory", 2nd ed., Academic Press Inc.

Pattanaik, P.K. (1971) "Voting and collective choice", Cambridge University Press

---------- - **Salles, M.** (eds) (1983) "Social choice and welfare", North-Holland

Pazner, E. - Wesley, E. (1977) *Stability of social choices in infinitely large societies*, "Journal of Economic Theory" 14, 252-262

---------- - ---------- (1978) *Cheatproofness properties of the plurality rule in large societies*, "Review of Economic Studies" 45, 85-93

--------- - **Schmeidler, D.** (1978) *Egalitarian-equivalent allocations: a new concept of economic equity*, "Quarterly Journal of Economics" 92, 671-687

Piggot, J. - Whalley, J. (1990) *Public good provision rule and income distribution: some general equilibrium calculations*, in **Piggot, J. - Whalley, J.** (eds) (1990) "Applied general equilibrium", Physica Verlag, Heidelberg

Piskunov, N.S. (1977) "Differentsialnoe i integralnoe iscislenja", Italian edition: Editori Riuniti, Rome

Plott, C.R. (1977) *A notion of equilibrium and its possibility under majority rule*, "The American Economic Review" 57, 787-806

Quinzii, M. (1979) *An existence theorem for the core of a productive economy with increasing returns*, document de travail n.A2080979, Laboratoire d' Econometrie, Ecole Polytechnique, Paris

---------- (1982) *Core and competitive equilibria with indivisibilities*, document de travail n.A2521182, Laboratoire d' Econometrie, Ecole Polytechnique, Paris

Rader, R. (1970) *New ideas in pure theory: problems in the theory of markets with uncertainty*, "The American Economic Review" 40, Papers and Proceedings, 454-460

Rasmusen, E. (1989) "Games and information", Basil Blackwell

Rawls, J. (1971) "A theory of justice", Beknap Press of the Harvard University Press, Cambridge

Ray, D. (1989) *Credible coalitions and the core*, "International Journal of Game Theory" 18, 185-187

Richter, D.K. (1974) *The core of a public good economy*, "International Economic Review" 15, 131-142

Roberts, D.J. (1974) *The Lindahl solution for economies with public goods*, "Journal of Public Economics" 3, 23-42

Roberts, F.D. (1984) "Applied combinatorics", Prentice-Hall, Inc., Englewood Cliffs, New Jersey

Roberts, K.W.S. (1977) *Voting over income tax schedules*, "Journal of Public Economics" 8, 329-340

---------- (1980) *Interpersonal comparability and social choice theory*, "Review of Economic Studies" 47, 421-439

Romer, T. (1975) *Individual welfare, majority voting, and the properties of a linear income tax*, "Journal of Public Economics" 4, 163-185

Rosenthal, R.W. (1971) *External economies and cores*, "Journal of Economic Theory" 3, 182-188

---------- (1972) *Cooperative games in effectiveness form*, "Journal of Economic Theory" 5, 88-101

Rubinstein, A. (1979) *A note about the "nowhere denseness" of societies having an equilibrium under majority rule*, "Econometrica" 47, 511-514

---------- (1991) *Comments on the interpretation of game theory*, "Econometrica" 59, 909-924

Saari, D.G. (1985) *The optimal ranking method is the Borda count*, discussion paper n.636, Center for Mathematical Economics, Northwestern University

---------- - **Van Newenhizen, J.** (1988) *The problem of indeterminacy in approval, multiple, and truncated voting systems*, "Public Choice" 59, 101-120

--------- (1989a) *A dictionary for voting paradoxes*, "Journal of Economic Theory" 48, 443-475

--------- (1989b) *The Borda dictionary*, discussion paper n.821, Center for Mathematical Economics, Northwestern University

Samuelson, P.A. (1947) "Foundations of economic analysis", pages 92-120 of the Italian edition: Il Saggiatore (1973), Milan

--------- (1954) *The pure theory of public expenditure*, "Review of Economics and Statistics" 36, 387-389

--------- (1958) *Diagrammatic exposition of a theory of public expenditure*, "Review of Economics and Statistics" 40, 350-356

Satterthwaite, M.A. (1975) *Strategy proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions*, "Journal of Economic Theory" 10, 187-217

--------- - **Sonnenschein, H.** (1981) *Strategy proof allocation mechanisms at differentiable points*, "Review of Economic Studies" 48, 587-597

Scarf, H. (1971) *On the existence of a cooperative solution for a general class of n-person games*, "Journal of Economic Theory" 3, 169-181

Schmeidler, D. - Sonnenschein, H. (1978) *Two proofs of the Gibbard-Satterthwaite theorem on the possibility of a strategy-proof social choice function*, in **Gottinger, H.W. - Leinfeller, W. (eds)** (1978) "Decision theory and social ethics", Reidel Publishing Company

Schofield, N. (1983) *Generic instability of majority rule*, "Review of Economic Studies" 50, 695-705

Scitovsky, T. (1986) "Human desire and economic satisfaction", Wheatsheaf Books Ltd.

Sen, A.K. (1970) "Collective choice and social welfare", Holden Day, San Francisco

--------- (1977) *Social choice theory: a re-examination*, "Econometrica" 45, 53-89

--------- (1982a) "Choice welfare and measurement", Basil Blackwell

--------- (1982b) *Social choice theory*, in **Arrow, K.J. - Intriligator, M.D. (eds)** (1982) "Handbook of Mathematical Economics", Chapter 22, North Holland

--------- (1985) *Social choice and justice, a review article*, "Journal of Economic Literature" 23, 1764-1776

Shapley, L. (1953) *Open questions*, in "Report of an informal Conference on the theory of *n*-person games", mimeogrphed, Department of Mathematics, Princeton University

--------- - **Shubik, M.**(1966) *Quasi Cores in a Monetary Economy with Nonconvex Preferences*, "Econometrica" 34,

--------- - --------- (1969) *On the core of an economic system with externalities*, "The American Economic Review" 59, 678-684

Shih-Yen, Wu (1989) "Production, entrepreneurship, and profits", Basil Blackwell

Shitovitz, B. (1989) *The bargaining set and the core in mixed markets with atoms and an atomless sector*, "Journal of Mathematical Economics" 18, 377-383

Shubik, M. (1959) *Edgeworth Market Games*, in **Tucker, A.D. - Luce, R.D. (eds)** (1959) "Contributions to the theory of games", IV ed., Princeton Univ. Press

--------- (1982) "Game theory in the social science", The MIT Press

--------- (1984) "A game-theoretic approach to political economy", The MIT Press

Slutsky, S (1977a) *A characterization of societies with consistent majority decisions*, "Review of Economic Studies" 44, 211-225

---------- (1977b) *A voting model for the allocation of public goods: existence of an equilibrium*, "Journal of Economic Theory" 14, 299-325

---------- (1979) *Equilibrium under α-majority voting*, "Econometrica" 47, 1113-1125

Skiena, S. (1990) "Implementing discrete mathematics: combinatorics and graph theory with Mathematica", Addison-Wesley Publishing Co., Redwood City, CA

Spivak, M. (1965) "Calculus on manifolds", W.A. Benjamin Inc., chapters 4 and 5

Starret, D.A. (1972) *Fundamental nonconvexities in the theory of externalities*, "Journal of Economic Theory" 4,

Thrall, R.M. - Lucas, W.F. (1963) *n-person games in partition function form*, "Naval Research Logistic Quarterly" 10,

Tullock, G. (1959) *Problems on majority voting*, "Journal of Political Economy" 67, 571-579

Varian, H.R. (1984) "Microeconomic analysis", W.W. Norton & Co.

Vickrey, W. (1960) *Utility, strategy, and social decision rules*, "Quarterly Journal of Economics" 74, 507-535

---------- (1961) *Counterspeculation, auctions, and competitive sealed tenders*, "Journal of Finance" 16, 8-37

Vohra, R. (1983) "Increasing returns, public goods and general equilibrium theory", Ph.D. dissertation, Johns Hopkins Univ. 1983

Von Neumann, J. - Morgenstern, O. (1944) "Theory of games and economic behavior", 3rd ed. (1953), Princeton University Press, first Princeton paperback printing (1980)

Warr, P. (1983) *The Private Provision of a Public Good is Independent of the Distribution of Income*, "Economic Letters" 13, 207-211

Weber, S. - Wiesmeth, H. (1990) *On the theory of cost sharing*, "Journal of Economics" 52, 71-82

---------- - ---------- (1991) *The equivalence of core and cost share equilibria in an economy with a public good*, "Journal of Economic Theory" 54, 180-197

Wilson, R. (1969) *An axiomatic model of logrolling*, "The American Economic Review" 3, 331-341

---------- (1971) *A game-theoretic analysis of social choice*, in **Lieberman, B. (ed)** (1971) "Social choice", Gordon & Breach Science Publisher

Yablonsky, S.V. (1989) "Introduction to discrete mathematics", Mir Publishers, Moscow

Yen, Ping-Yuan (1989) *Egalitarian-equivalent cost-sharing and the M.D.P. Process for the provision of a public good*, CORE discussion paper n.8911, Université Catholique de Louvain

Young, H.P. (1985) "Cost allocation: methods, principles, applications", North Holland

-------- (1988) *Distributive justice in taxation*, "Journal of Economic Theory" 44, 321-355

Zhou, L. (1989) *Impossibility of strategy-proof mechanisms for economies with pure public goods*, Econometric Research Program, Research Memorandum n.343 May, Princeton University

 DOI: 10.2870/7482

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

Pasi, Luciano (1993), An essay on some aspects of the economic theory of public goods
European University Institute DOI: 10.2870/7482

DOI: 10.2870/7482