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OMAR LICANDRO
and LUIS A. PUCH

EUROPEAN UNIVERSITY INSTITUTE
Department of Economics

EUROPEAN UNIVERSITY INSTITUTE
DEPARTMENT OF ECONOMICS

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Is discrete time a good representation of continuous time?

Omar Licandro   Luis A. Puch*

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Abstract

Economists model time as continuous or discrete. The recent literature on continuous time models with delays should help to bridge the gap between these two families of models. In this note, we propose a simple time–to–build model in continuous time, and show that a discrete time version is a true representation of the continuous time problem under some sufficient conditions.

JEL codes: O40, E32, C61

Key words: Discrete Time, Continuous Time, Time–to–Build, Delay, DDEs

1 Introduction

The time dimension is of fundamental importance for macroeconomic theory, since most macroeconomic problems deal with intertemporal trade-offs. In modeling time, economists move from discrete to continuous time, as if both ways of representing time were equivalent. For example, growth theory is manly written in continuous time, but business cycle theory is in a large extend written in discrete time. However, they refer to each other as been two pieces of the same framework.

*Licandro, European University Institute and FEDEA; Puch, FEDEA, Universidad Complutense de Madrid and ICAE. We thank Raouf Boucekkine and Fabrice Collard for helpful comments. We acknowledge financial support of the Spanish Ministry of Sciences and Technology (SEJ2004-0459/ECON). Correspondence: Luis A. Puch, FEDEA, Jorge Juan 46, 28001 Madrid, Spain, e-mail: lpuch@fedea.es.

The view that continuous and discrete time representations are equivalent is manly supported by limit properties: the discrete time version of the standard dynamic general equilibrium model does converge to its continuous time representation when the period length tends to zero. However, this view hides a fundamental problem of timing. In continuous time, investment at time t becomes capital at time $t+\mathrm{d}t$. The discrete time equivalent is that period t investment transforms into capital at period $t+1$. Thus, the speed at which investment becomes capital depends directly on the length of the period. In this note, we show that the discrete time representation implicitly imposes a particular form of time–to–build to the continuous time representation.

Few papers have exploited this difference to study the properties of discrete versus continuous time models. Hintermaier (2003) is an exception. He proves that conditions for indeterminacy in a discrete time version of Benhabib and Farmer (1994) depend crucially on the frequency of the discrete time representation (see also Bambi and Licandro (2005)).

Optimal control theory with delays serves to characterize the gap between these two families of models.[1] We show that the discrete time representation of the standard optimal growth model is consistent with the continuous time representation under the additional assumption of time–to–build.[2]

The remaining of the paper is as follows. Section 2 introduces the model. Section 3 presents the main result. A last section concludes.

2 A continuous time model with time–to–build

Let us assume that time is continuous and introduce a simple time–to–build technology in an otherwise standard one-sector growth model. For simplicity, all variables are in per capita terms. Let $d > 0$ be the planed horizon of an investment project —*i.e.* the time–to–build delay. The technology to produce one unit of the investment good available at time $t + d$ requires a flow of $\frac{1}{d}$ units of the final good in the time interval $[t, t + d]$. Consequently, the only relevant decision at time t is the amount of *planed investment* $i(t)$, which will become operative at time $t + d$.

The stock of *planed capital* at time $t \geq -d$ is given by

$$k(t) = k\left(-d\right) + \int_{-d}^{t} i(s)\ \mathrm{d}s. \tag{1}$$

[1]See Kolmanovskii and Myshkis (1998), and recent applications by Boucekkine et al (2005) and others.

[2]Time–to-Build in discrete time is analyzed by Kydland and Prescott (1982). An alternative version of this assumption in continuous time is in Asea and Zak (1999) and Collard et al (2006).

The implicit assumption of zero depreciation makes $i(t)$ to be net investment. By definition of $i(t)$, $k(t)$ becomes operative at time $t+d$. Initial conditions need to be specified: $k(-d)=\bar{k}>0$ and $i(t)=i_0(t)\geq 0$ for all $t\in[-d,0[$. Consequently, $k(t)=k_0(t)$ for all $t\in[-d,0]$ is computed using (1).

Final output is produced using a standard neoclassical technology $f(k)$, assumed to be C^2, increasing and concave for $k>0$ and verifying Inada conditions. Operative capital at time t was already planed at time $t-d$, implying that production at time t is $f(k(t-d))$.

The production of the final good is allocated to consumption $c(t)$ and to net investment expenditures $x(t)$. At time $t\geq 0$, the amount of the final good employed in the production of investment goods is given by

$$x(t)=\frac{1}{d}\int_{t-d}^{t} i(s)\ \mathrm{d}s. \tag{2}$$

It corresponds to investment expenditures associated to all active investment projects. Under these assumptions, the feasibility constraint for $t\geq 0$ takes the following form:

$$f(k(t-d))=c(t)+x(t). \tag{3}$$

2.1 The planer's problem

Let a planer maximize the utility of the representative household

$$\max\int_0^{\infty} u(c(t))\ \mathrm{e}^{-\rho t} \tag{P}$$

subject to (3) and

$$\dot{x}(t)=\frac{1}{d}\left(i(t)-\ i(t-d)\right), \tag{4}$$

$$\dot{k}(t)=i(t). \tag{5}$$

Constraints (4) and (5) result from time differentiation of (2) and (1), respectively. The initial conditions are $x(0)=x_0=\frac{1}{d}\int_{-d}^{0} i_0(s)\ \mathrm{d}s$, $k(t)=k_0(t)$ and $i(t)=i_0(t)$ for all $t\in[-d,0[$, as specified previously. The instantaneous utility function $u(t)$ is C^2, increasing and concave for $c>0$, and verifies Inada conditions.

Using optimal control theory with delays,[3] necessary first-order-conditions for this problem are

$$u'(c(t))=\phi(t) \tag{6}$$

[3] Again, see Kolmanovskii and Myshkis (1998) and Boucekkine et al (2005).

$$\lambda(t) + \frac{1}{d}\mu(t) = \frac{1}{d}\mu(t+d)\ \mathrm{e}^{-\rho d} \tag{7}$$

$$-\phi(t+d) f'(k(t))\ \mathrm{e}^{-\rho d} = \dot{\lambda}(t) - \rho\lambda(t) \tag{8}$$

$$\phi(t) = \dot{\mu}(t) - \rho\mu(t) \tag{9}$$

and the transversality conditions

$$\lim_{t\to\infty} k(t)\lambda(t)\ \mathrm{e}^{-\rho t} = 0 \tag{10}$$

$$\lim_{t\to\infty} x(t)\mu(t)\ \mathrm{e}^{-\rho t} = 0. \tag{11}$$

The Lagrangian multiplier $\phi(t)$ is associated to constraint (3), and the co-states $\lambda(t)$ and $\mu(t)$ are associated to the states $k(t)$ and $x(t)$, respectively. Advanced terms appearing in (7) and (8), related to the delays in (4) and (5) – make explicit the trade-offs. Marginal investment at time t has three different effects on utility. Firstly, it increases planed capital, which marginal value is $\lambda(t)$. Second, it rises investment expenditures, with marginal costs $\frac{\mu(t)}{d}$. Finally, when the project will be finished at $t+d$, investment expenditures will.

3 Discrete Time as a Representation of Continuous Time

In this section, we study the relation between the proposed continuous time models with time–to–build and the discrete time representation of the neoclassical growth model. Let us assume that the initial function $i_0(t)$ is piecewise continuous, and that feasible trajectories $i(t)$, for $t \geq 0$, belong to the family of piecewise continuous functions.

Proposition 1 *Under $d = 1$, the optimal conditions (6) to (11) of problem (P) become*

$$k(t) - k(t-1) = f(k(t-1)) - c(t) \tag{12}$$

$$\frac{u'(c(t))}{u'(c(t+1))} = \beta(1 + f'(k(t))), \tag{13}$$

where $\beta \equiv e^{-\rho}$.

Proof. From (1) and (2), under $d = 1$, we get $x(t) = k(t) - k(t-1)$. The feasibility constraint (12) results from substituting the relation between x and k on equation (3). Differentiating (8), substituting $\dot{\lambda}$ and $\dot{\mu}$ by (9) and (10), after some rearrangements, we get (13). ■

The equilibrium path of the neoclassical growth model is represented by (12) and (13) for given initial conditions.

Corollary 2 *The steady state solution of (12) and (13) is saddle-path stable for* $t \geq 0$*.*

Corollary 2 implies that for every $s \in [0,1)$, the optimal sequence $\{c_{s+i}, k_{s+i}\}$, for $i = \{0,1,2,3...\}$, is the solution of the discrete time neoclassical growth model, given $k(-1) = k_0(-1)$. However, in continuous time, it involves the solution for all $s \in [0,1)$, which depends on the boundary function $k_0(t)$, for $t \in [-1,0)$, defining initial conditions.

Corollary 3 *Under* $d = 1$ *and* $k(t) = k_0 > 0$ *for* $t \in [-1,0)$*, the optimal solution* $k(t), c(t)$ *of problem (P) is constant in the interval* $[i-1,i)$ *for* $i = \{1,2,3...\}$ *and it corresponds to the stable brand of the discrete problem in Proposition 1.*

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